

S^3/\mathbb{Z}_n partition function and dualities

Yosuke Imamura and Daisuke Yokoyama

*Department of Physics, Tokyo Institute of Technology,
Tokyo 152-8551, Japan*

E-mail: imamura@phys.titech.ac.jp, d.yokoyama@th.phys.titech.ac.jp

ABSTRACT: We investigate S^3/\mathbb{Z}_n partition function of $\mathcal{N} = 2$ supersymmetric gauge theories. A gauge theory on the orbifold has degenerate vacua specified by the holonomy. The partition function is obtained by summing up the contributions of saddle points with different holonomies. An appropriate choice of the phase of each contribution is essential to obtain the partition function. We determine the relative phases in the holonomy sum in a few examples by using duality to non-gauge theories. In the case of odd n the phase factors can be absorbed by modifying a single function appearing in the partition function.

KEYWORDS: Supersymmetric gauge theory, Duality in Gauge Field Theories

ARXIV EPRINT: [1208.1404](https://arxiv.org/abs/1208.1404)

Contents

1	Introduction	1
2	The $\mathcal{S}^3/\mathbb{Z}_n$ partition function	2
2.1	The \mathcal{S}^3 partition function	2
2.2	\mathbb{Z}_n orbifolding	4
3	Dualities in $\mathcal{S}^3/\mathbb{Z}_n$	7
3.1	$\mathcal{N} = 2$ SQED and XYZ model	7
3.2	SU(2) gauge theory and a chiral multiplet	10
3.3	$\mathcal{S}^3/\mathbb{Z}_{2k+1}$	13
4	Derived dualities	14
4.1	$\mathcal{N} = 4$ SQED and hypermultiplet	14
4.2	$\mathcal{N} = 2$ SQED with $N_f \geq 2$ and quiver gauge theory	15
4.3	ABJM model and its mirror	18
5	Conclusions and discussions	21

1 Introduction

For last few years, there was large progress in supersymmetric field theories in three-dimension. Many physical quantities in strongly coupled field theories have been computed by using localization method. For example, partition functions of $\mathcal{N} = 2$ supersymmetric field theories in \mathcal{S}^3 [1–3] and $\mathcal{S}^2 \times \mathcal{S}^1$ [4, 5] are computed exactly, and they have been used to study non-perturbative aspects of three-dimensional field theories, such as dualities among three-dimensional field theories [6–9] and relation to M-theory via AdS/CFT correspondence [4, 10, 11]

In this paper, we investigate the partition function of three-dimensional $\mathcal{N} = 2$ supersymmetric field theories in the orbifold $\mathcal{S}^3/\mathbb{Z}_n$ [12, 13]. Due to the non-trivial homotopy of the orbifold, $\pi_1(\mathcal{S}^3/\mathbb{Z}_n) = \mathbb{Z}_n$, a gauge theory defined in it has degenerate vacua specified by the holonomy associated with the gauge symmetry. Their contributions are summed up to obtain the total partition function. In general, the partition function of a Euclidean theory is complex. We usually focus only on its absolute value and the phase is disregarded. This is, however, not allowed when we compute the partition functions of different sectors which are summed up. Even when we are interested only in the absolute value of the total partition function, we need to care about the relative phase of each contribution. The purpose of this paper is to determine appropriate phase factors in the holonomy sum in some gauge theories and look for a general rule for these phases. We consider two gauge

theories which are known to have dual field theories without vector multiplet. On one side of the dualities, in the non-gauge theories, we can compute the absolute value of the partition function up to overall constant factor independent of parameters. By comparing the partition functions of gauge and non-gauge theories in each dual pair, we infer the relative phases in the holonomy sum in the gauge theories.

This paper is organized as follows. In section 2, we summarize a general formula of the orbifold partition function. It is written by using orbifold extension of the double sign function, which we denote by $s_{b,h}(z)$. In section 3, we consider two dual pairs and determine the phase factors in the holonomy sum in the gauge theories so that the sum agrees with the partition function of the dual non-gauge theories. We find that when the order n of the orbifold group is odd the phase factor is absorbed in the definition of the function $s_{b,h}(z)$. In section 4 we consider more dual pairs which are derived from one of the dualities studied in section 3. The last section is devoted to the conclusions and discussions.

2 The S^3/\mathbb{Z}_n partition function

2.1 The S^3 partition function

Let us first summarize the partition function on S^3 without orbifolding [1–3, 14, 15]. We consider the squashed S^3 with the metric

$$ds^2 = r^2 \left[(\mu^1)^2 + (\mu^2)^2 + \frac{1}{v^2} (\mu^3)^2 \right], \tag{2.1}$$

where μ^a ($a = 1, 2, 3$) are the left-invariant differentials on $S^3 \sim \text{SU}(2)$. We set $r = 1$ in the following. v is the squashing parameter. We also define parameters u and b by

$$u = \pm \sqrt{v^2 - 1}, \quad b = \frac{1 + iu}{v}, \tag{2.2}$$

for later use. The isometry of the round sphere $\text{SU}(2)_L \times \text{SU}(2)_R$ is broken by the squashing to

$$\text{SU}(2)_L \times \text{U}(1)_r. \tag{2.3}$$

We consider an $\mathcal{N} = 2$ supersymmetric gauge theory on this manifold. The supercharges belong to the representation $\mathbf{2}_0$ of (2.3).

A general formula for the partition function for the squashed sphere is given in [15]. The same partition function is first obtained in [14] for a different deformation of S^3 . See also [16–18] for its relation to 4d superconformal index. The formula is

$$Z = \int [d\lambda] e^{-S_0(\lambda)} Z^{1\text{-loop}}, \tag{2.4}$$

where $Z^{1\text{-loop}}$ is the one-loop determinant

$$Z^{1\text{-loop}} = \frac{\prod_{\alpha \in \Delta} s_b \left(\alpha(\lambda) - \frac{i}{v} \right)}{\prod_I s_b \left(\rho_I(\lambda) - \frac{i(1-\Delta_I)}{v} \right)}. \tag{2.5}$$

The index I labels chiral multiplets, and ρ_I and Δ_I are the weight vectors and the Weyl weights of the chiral multiplets. The integration variable λ is an element of the Cartan subalgebra of the gauge group G parameterizing the Coulomb branch. If G is $U(N)$ or product of $U(N)$, the integration measure $[d\lambda]$ is defined by

$$[d\lambda] = \frac{1}{|W|} \prod_{a=1}^{\text{rank}G} d\lambda_a, \quad (2.6)$$

where $|W|$ is the order of the Weyl group of G , and λ_a are the diagonal components in the fundamental representation of λ . Different normalizations are also used in the literature. The normalization (2.6) is chosen so that theories in the dual pairs we will consider in section 3 have the same partition functions.¹

$S_0(\lambda)$ is the classical action. If the theory has the Chern-Simons term

$$S_{\text{CS}} = \frac{ik}{4\pi} \int \text{tr}_{\text{fund}} \left(AdA - \frac{2i}{3} A^3 \right), \quad (2.7)$$

and the Fayet-Iliopoulos term

$$S_{\text{FI}} = -\frac{\zeta}{2\pi} \int \sqrt{g} D_{U(1)} d^3x, \quad (2.8)$$

then the supersymmetric completion of these actions contributes to the classical action by

$$S_0(\lambda) = \pi i k \text{tr}_{\text{fund}}(\lambda^2) + 2\pi i \zeta \lambda_{U(1)}. \quad (2.9)$$

$s_b(z)$ is the double sine function defined by²

$$s_b(z) = \prod_{p,q=0}^{\infty} \frac{b(q + \frac{1}{2}) + b^{-1}(p + \frac{1}{2}) - iz}{b(p + \frac{1}{2}) + b^{-1}(q + \frac{1}{2}) + iz}. \quad (2.10)$$

The formula (2.5) is derived by using localization, which reduces the path integral to Gaussian integral for infinite number of non-zero modes and finite dimensional integral for zero modes. The Gaussian integral is performed by using spherical harmonics expansion. For a vector multiplet with weight vector α , the Gaussian integral of modes with specific $SU(2)_R$ quantum numbers (j, m) gives

$$\frac{2j - 2imu - iv\alpha(\lambda)}{2j + 2 + 2imu + iv\alpha(\lambda)}. \quad (2.11)$$

Notice that m dependence appears only when S^3 is squashed. For round sphere with $u = 0$, there is no m dependence due to the unbroken $SU(2)_R$ symmetry. If we set

$$j = \frac{p+q}{2}, \quad m = \frac{p-q}{2}, \quad (2.12)$$

¹The normalization of λ we use in this paper is different from that in [15]. λ here is related to the constant mode σ_0 of scalar field defined in [15] by $\lambda = r\sigma_0/v$.

² $s_b(z)$ is directly related to the non-compact quantum dilogarithm $\varphi_b(z)$ [19–21] by $s_b(z) = \exp[-\pi i(\frac{z^2}{2} + \frac{b^2+b^{-2}}{24})]\varphi_b(z)$. See also [22, 23] for more details on $s_b(z)$.

(2.11) becomes the factor in the definition (2.10) of the double sine function $s_b(z)$ with argument $z = \alpha(\lambda) - i/v$, and by taking the product over p and q , we obtain the double sine function appearing in the numerator in (2.5). The denominator in (2.5) also arises from the Gaussian integral of non-zero modes of chiral multiplets.

Note that variables p and q appear differently in the numerator and the denominator in (2.10). In the numerator p and q appear in the coefficients of b^{-1} and b , respectively, and in the denominator, the relation is reversed. Because they are dummy variables we can exchange them in the numerator or in the denominator so that they appear in the same manner. However, the relation to the $SU(2)_R$ quantum numbers (2.12) holds only when we write the infinite product as in (2.10). This becomes important when we consider the orbifolding by $\mathbb{Z}_n \subset SU(2)_R$ in the next subsection.

From the definition (2.10) we can easily show the following relations for $s_b(z)$.

- Self-duality and reflection property

$$s_b(z) = s_{b^{-1}}(z) = \frac{1}{s_b(-z)}. \tag{2.13}$$

- Functional equations

$$\begin{aligned} \frac{s_b(z + \frac{ib}{2})}{s_b(z - \frac{ib}{2})} &= \frac{1}{2 \cosh(\pi bz)}, \\ \frac{s_b(z + \frac{ib^{-1}}{2})}{s_b(z - \frac{ib^{-1}}{2})} &= \frac{1}{2 \cosh(\pi b^{-1}z)}, \\ \frac{s_b(z + \frac{i}{v})}{s_b(z - \frac{i}{v})} &= \frac{1}{[2 \sinh(\pi bz)][2 \sinh(\pi b^{-1}z)]}. \end{aligned} \tag{2.14}$$

2.2 \mathbb{Z}_n orbifolding

We consider the left-invariant orbifold $\mathcal{S}^3/\mathbb{Z}_n$ with $\mathbb{Z}_n \subset U(1)_r \subset SU(2)_R$. The partition function on the orbifold is obtained in [12] for theories without matter fields in a general Lens space $L(p, q)$ without squashing. It is extended to theories with chiral multiplets in background with nontrivial squashing parameter in [13]. Our orbifold corresponds to $L(n, -1)$.

Because supercharges are $U(1)_r$ neutral, the orbifolding by $\mathbb{Z}_n \subset U(1)_r$ does not break any supersymmetry, and we can define $\mathcal{N} = 2$ supersymmetric theories on the orbifold. A gauge theory in this orbifold has degenerate vacua specified by the holonomy

$$m = \frac{n}{2\pi} \oint_C A, \tag{2.15}$$

where C is the generator of the fundamental group $\pi_1(\mathcal{S}^3/\mathbb{Z}_n) = \mathbb{Z}_n$. The consistency to $nC = 0$ requires $e^{2\pi im} = 1$. (Note that we define m with the factor n in (2.15).) The holonomy can be turned on for both global and gauge symmetries. The holonomy for gauge symmetries should be summed up in the path integral. The partition function is given by

$$Z(m_{\text{global}}) = \sum_{m_{\text{local}}} \int [d\lambda] e^{-S_0(\lambda, m)} Z^{1\text{-loop}}(\lambda, m), \tag{2.16}$$

where $S_0(\lambda, m)$ and $Z^{1\text{-loop}}(\lambda, m)$ are the classical action and the one-loop determinant. The summation is taken over the holonomy associated with gauge symmetry, which is denoted by m_{local} in (2.16). The holonomy for global symmetry m_{global} is not summed, and the partition function Z depends on m_{global} .

The integration measure $[d\lambda]$ is defined by

$$[d\lambda] = \frac{1}{|W|} \prod_{a=1}^{\text{rank}G} \frac{d\lambda_a}{n}. \tag{2.17}$$

We introduce the factor $1/n$ for each integration variable for later convenience.

One may think that the classical action for $\mathbf{S}^3/\mathbb{Z}_n$ is obtained by dividing that for \mathbf{S}^3 by n . This naive expectation is not correct. The classical action $S_0(\lambda, m)$ consists of two parts;

$$S_0^{\mathbf{S}^3/\mathbb{Z}_n}(\lambda, m) = \frac{1}{n} S_0^{\mathbf{S}^3}(\lambda) - i\Phi(m). \tag{2.18}$$

One is $1/n$ of the classical action for \mathbf{S}^3 , and has the same origin as the \mathbf{S}^3 case. The other part comes from the Chern-Simons term. Due to the non-trivial topology of $\mathbf{S}^3/\mathbb{Z}_n$, the Chern-Simons term gives non-vanishing contribution even for a flat gauge connection [12, 24, 25];

$$\Phi = \frac{\pi k}{n} \text{tr}_{\text{fund}}(m^2). \tag{2.19}$$

This phase plays an important role in dualities in $\mathbf{S}^3/\mathbb{Z}_n$. The factor $e^{i\Phi}$ may be ill-defined depending on the coefficient. If nk is odd, the holonomies $m = \text{diag}(\dots, h, \dots)$ and $m = \text{diag}(\dots, h+n, \dots)$, which are identified in \mathbb{Z}_n , give different phases. We will meet such an ambiguity in the example in section 3.2, and there we will give an additional rule to fix the ambiguity.

The one-loop partition function for the orbifold can be obtained by projecting out the factors in (2.10) which originate from \mathbb{Z}_n -variant modes. Let φ be a field with a weight vector ρ . On \mathbf{S}^3 it is Fourier expanded as

$$\varphi(\psi) = \sum_{m \in \mathbb{Z}/2} \varphi_m e^{im\psi}, \tag{2.20}$$

where $0 \leq \psi < 4\pi$ is the coordinate along the Hopf fiber of \mathbf{S}^3 , and m is the $\text{SU}(2)_R$ magnetic quantum number. After \mathbb{Z}_n orbifolding, the field must satisfy the boundary condition

$$\varphi\left(\psi + \frac{4\pi}{n}\right) = e^{2\pi i \frac{\rho(h)}{n}} \varphi(\psi), \tag{2.21}$$

and only modes φ_m with the index m satisfying

$$2m = p - q = \rho \cdot h \pmod{n} \tag{2.22}$$

survive after the orbifold projection. We define $s_{b,h}(z)$ as the function obtained from (2.10) by restricting the product over (p, q) by (2.22). This restricted product is realized by substituting

$$p = np' + [k+h]_n, \quad q = nq' + k, \tag{2.23}$$

to (2.10), and perform the product with respect to non-negative integers p' and q' , and $k = 0, 1, \dots, n-1$. $[m]_n$ represents the remainder when m is divided by n . It is convenient to introduce notation $\langle \dots \rangle_n$ defined by

$$\langle m \rangle_n = \frac{1}{n} \left([m]_n + \frac{1}{2} \right) - \frac{1}{2}. \tag{2.24}$$

This satisfies the relations

$$\langle m + an \rangle_n = \langle m \rangle_n \quad (a \in \mathbb{Z}), \quad \langle -1 - m \rangle_n = -\langle m \rangle_n. \tag{2.25}$$

We rewrite the numerator in (2.10) as

$$\begin{aligned} & b \left(p + \frac{1}{2} \right) + b^{-1} \left(q + \frac{1}{2} \right) - iz \\ &= n \left[b \left(p' + \langle k + h \rangle_n + \frac{1}{2} \right) + b^{-1} \left(q' + \langle k \rangle_n + \frac{1}{2} \right) - i \frac{z}{n} \right]. \end{aligned} \tag{2.26}$$

The denominator in (2.10) is also rewritten in a similar way, and we obtain

$$\begin{aligned} s_{b,h}(z) &= \prod_{k=0}^{n-1} \prod_{p',q'=0}^{\infty} \frac{b(q' + \frac{1}{2}) + b^{-1}(p' + \frac{1}{2}) + b\langle k \rangle_n + b^{-1}\langle k + h \rangle_n - i \frac{z}{n}}{b(p' + \frac{1}{2}) + b^{-1}(q' + \frac{1}{2}) + b\langle k + h \rangle_n + b^{-1}\langle k \rangle_n + i \frac{z}{n}} \\ &= \prod_{k=0}^{n-1} \prod_{p',q'=0}^{\infty} \frac{b(q' + \frac{1}{2}) + b^{-1}(p' + \frac{1}{2}) + b\langle k \rangle_n + b^{-1}\langle k + h \rangle_n - i \frac{z}{n}}{b(p' + \frac{1}{2}) + b^{-1}(q' + \frac{1}{2}) - b\langle k \rangle_n - b^{-1}\langle k + h \rangle_n + i \frac{z}{n}}. \end{aligned} \tag{2.27}$$

In the second line we replaced k in the denominator by $-1 - k - h$ and used the second relation in (2.25). In the final expression the product with respect to p' and q' has the same form as that in the definition (2.10) of the double sine function $s_b(z)$, and we obtain

$$s_{b,h}(z) = \prod_{k=0}^{n-1} s_b \left(\frac{z}{n} + ib\langle k \rangle_n + ib^{-1}\langle k + h \rangle_n \right). \tag{2.28}$$

By definition, the product of $s_{b,h}(z)$ over all h reproduces the original double sine function;

$$s_b(z) = \prod_{h=0}^{n-1} s_{b,h}(z). \tag{2.29}$$

The function $s_{b,h}(z)$ satisfies the following formulae, which are analogs of (2.13) and (2.14).

- Self-duality and reflection property

$$s_{b,h}(z) = s_{b^{-1},-h}(z) = \frac{1}{s_{b,-h}(-z)} = \frac{1}{s_{b^{-1},h}(-z)}. \tag{2.30}$$

- Functional equations

$$\begin{aligned}
 \frac{s_{b,h+1}(z + \frac{ib}{2})}{s_{b,h}(z - \frac{ib}{2})} &= \frac{1}{2 \cosh\left(\frac{\pi bz}{n} + \pi i \langle h \rangle\right)}, \\
 \frac{s_{b,h-1}(z + \frac{ib^{-1}}{2})}{s_{b,h}(z - \frac{ib^{-1}}{2})} &= \frac{1}{2 \cosh\left(\frac{\pi b^{-1}z}{n} + \pi i \langle -h \rangle\right)}, \\
 \frac{s_{b,h}(z + \frac{i}{v})}{s_{b,h}(z - \frac{i}{v})} &= \frac{1}{[2 \sinh\left(\frac{\pi bz + \pi ih}{n}\right)][2 \sinh\left(\frac{\pi b^{-1}z - \pi ih}{n}\right)]}. \tag{2.31}
 \end{aligned}$$

The one-loop determinant for S^3/\mathbb{Z}_n is obtained simply by replacing $s_b(z)$ in (2.5) by $s_{b,h}(z)$.

$$Z^{1\text{-loop}}(\lambda, m) = \frac{\prod_{\alpha \in \Delta} s_{b,\alpha(m)}\left(\alpha(\lambda) - \frac{i}{v}\right)}{\prod_I s_{b,\rho_I(m)}\left(\rho_I(\lambda) - \frac{i(1-\Delta_I)}{v}\right)}. \tag{2.32}$$

The 1-loop determinant of vector multiplets can be rewritten in terms of elementary functions.

$$\begin{aligned}
 Z_{\text{vector}}^{1\text{-loop}}(\lambda, m) &= \prod_{\alpha \in \Delta} s_{b,\alpha(m)}\left(\alpha(\lambda) - \frac{i}{v}\right) \\
 &= \prod_{\alpha \in \Delta_+} \left[2 \sinh \frac{\pi}{n} (b\alpha(\lambda) + i\alpha(m))\right] \left[2 \sinh \frac{\pi}{n} (b^{-1}\alpha(\lambda) - i\alpha(m))\right]. \tag{2.33}
 \end{aligned}$$

When $b = 1$, this agree with the partition function in the lens space $L(n, -1)$ given in [12].

3 Dualities in S^3/\mathbb{Z}_n

A gauge theory in S^3/\mathbb{Z}_n has degenerate vacua labeled by holonomies associated with the gauge symmetry. The contributions of these vacua should be summed up to obtain the total partition function. In this section we consider two dual pairs and confirm that the partition functions of theories dual to each other agree if appropriate phase factors are inserted in the holonomy sum.

3.1 $\mathcal{N} = 2$ SQED and XYZ model

We first consider the mirror symmetry between an $\mathcal{N} = 2$ SQED and the XYZ model [33]. On one side of the duality, we consider $\mathcal{N} = 2$ SQED with two chiral multiplets q and \tilde{q} with U(1) charge +1 and -1, respectively. We assume that q and \tilde{q} have the Weyl weight Δ . The mirror theory, the XYZ model, consists of three chiral multiplets Q , \tilde{Q} and S interacting through the superpotential

$$W = \tilde{Q}SQ. \tag{3.1}$$

Although three fields S , Q , and \tilde{Q} in this model are symmetric, we treat Q and \tilde{Q} as a quark and an antiquark, and S as a neutral field, because the mirror pair considered here

	q	\tilde{q}	m	\tilde{m}	Q	\tilde{Q}	S
$U(1)_V$	0	0	1	-1	1	-1	0
$U(1)_A$	1	1	0	0	-1	-1	2

Table 1. The charge assignment of the global symmetry $U(1)_V \times U(1)_A$ of SQED and XYZ model which are mirror to each other. m and \tilde{m} are the monopole and anti-monopole operators.

is a special case of a series of mirror pairs, which are studied in 4.2, and in a general mirror pair Q and \tilde{Q} are replaced by charged multiplets and S by neutral ones. By the operator relation $S = \tilde{q}q$ and the marginality of the superpotential (3.1), we can determine the Weyl weights of the fields in this theory as

$$\Delta_S = 2\Delta, \quad \Delta_Q = \Delta_{\tilde{Q}} = 1 - \Delta. \tag{3.2}$$

Although the correct value of Δ at the infra-red fixed point is $\Delta = 1/3$, the equality of partition functions holds regardless of Δ [9], and we leave Δ unfixed.

The global symmetry which is the same for these two theories is $U(1)_V \times U(1)_A$. The charge assignments are shown in table 1. We introduce real mass parameters ζ and μ for $U(1)_V$ and $U(1)_A$, respectively. $U(1)_A$ symmetry in SQED is the topological $U(1)$ symmetry acting on monopole operators, and the corresponding mass parameter ζ is the Fayet-Iliopoulos parameter. The \mathcal{S}^3 partition functions of these theories are

$$\begin{aligned} Z^{\text{SQED}} &= \int \frac{e^{-2\pi i \zeta \lambda}}{s_b(\lambda + \mu - \frac{i(1-\Delta)}{v})s_b(-\lambda + \mu - \frac{i(1-\Delta)}{v})} d\lambda, \\ Z^{\text{XYZ}} &= \frac{1}{s_b(\zeta - \mu - \frac{i\Delta}{v})s_b(-\zeta - \mu - \frac{i\Delta}{v})s_b(2\mu - \frac{i(1-2\Delta)}{v})}. \end{aligned} \tag{3.3}$$

Because two theories are mirror to each other, the partition functions should agree. This agreement is confirmed by using the pentagon relation of the double sine function [26]:³

$$\int \frac{s_b(x+r)}{s_b(x+s)} e^{-2\pi i t x} dx = e^{\pi i t(r+s)} \frac{s_b(t - \frac{r}{2} + \frac{s}{2} + \frac{i}{v})}{s_b(t + \frac{r}{2} - \frac{s}{2} - \frac{i}{v})} s_b(r - s - \frac{i}{v}). \tag{3.4}$$

By substituting

$$x = \lambda, \quad r = -\mu + \frac{i(1-\Delta)}{v}, \quad s = \mu - \frac{i(1-\Delta)}{v}, \quad t = \zeta, \tag{3.5}$$

to the pentagon relation (3.4), we obtain $Z^{\text{XYZ}} = Z^{\text{SQED}}$. Note that for the agreement of the two partition functions, the integration measure should be chosen as in (2.6).

Let us generalize this to the theories on the orbifold $\mathcal{S}^3/\mathbb{Z}_n$. On the SQED side, we need to sum up the contribution of n saddle points specified by the holonomy h of the

³The pentagon relation usually refers to the operator equation $\varphi_b(\hat{P})\varphi_b(\hat{X}) = \varphi_b(\hat{X})\varphi_b(\hat{X} + \hat{P})\varphi_b(\hat{P})$, where \hat{X} and \hat{P} are operators satisfying $[\hat{P}, \hat{X}] = 1/2\pi i$. This relation is proved in [21, 27]. This operator equation is equivalent to (3.4) [21, 28, 29], which we refer to as the pentagon relation.

U(1) gauge symmetry. We can also introduce holonomies h_V and h_A for $U(1)_V$ and $U(1)_A$ global symmetries as non-dynamical background gauge potentials. Because $U(1)_V$ current in SQED is the field strength of the dynamical gauge field A , the $U(1)_V$ holonomy is realized by the Chern-Simons term

$$S = \frac{i}{2\pi} \int V dA, \quad (3.6)$$

where V is the non-dynamical $U(1)_V$ background gauge field. In the orbifold $\mathcal{S}^3/\mathbb{Z}_n$, this term gives rise to the non-trivial phase factor

$$\Phi = 2\pi \frac{h_V h}{n}. \quad (3.7)$$

Taking account of this phase factor, the partition function for each holonomy is

$$Z^{\text{SQED}}(h, h_V, h_A) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i \zeta \lambda / n} e^{2\pi i h_V h / n}}{s_{b, h_A + h}(\mu + \lambda - \frac{i(1-\Delta)}{v}) s_{b, h_A - h}(\mu - \lambda - \frac{i(1-\Delta)}{v})} \frac{d\lambda}{n}. \quad (3.8)$$

On the other hand, the partition function of the XYZ model is

$$Z^{\text{XYZ}}(h_V, h_A) = \frac{1}{s_{b, -h_A + h_V}(-\mu + \zeta - \frac{i\Delta}{v}) s_{b, -h_A - h_V}(-\mu - \zeta - \frac{i\Delta}{v}) s_{b, 2h_A}(2\mu - \frac{i(1-2\Delta)}{v})}. \quad (3.9)$$

Naive expectation is that these are related by

$$Z^{\text{XYZ}}(h_V, h_A) = \sum_{h=0}^{n-1} Z^{\text{SQED}}(h, h_V, h_A). \quad (3.10)$$

This is actually the case when $h_A = 0$. We confirmed this relation numerically up to $n = 10$. Again, the choice of the integration measure (2.17) is essential for the equality in (3.10).

The relation (3.10), however, does not hold if we turn on the holonomy h_A for $U(1)_A$ symmetry. Instead, we found that the relation

$$Z^{\text{XYZ}}(h_V, h_A) = \sum_h \sigma(h, h_V, h_A) Z^{\text{SQED}}(h, h_V, h_A) \quad (3.11)$$

hold if we choose an appropriate sign function $\sigma(h, h_V, h_A) = \pm 1$. The analysis for $h_A = 0$ implies

$$\sigma(h, h_V, 0) = 1. \quad (3.12)$$

We can determine $\sigma(h, h_V, h_A)$ for general h_A by the numerical analysis. For $n = 2, 3, 4$, we obtained

$$\begin{aligned} \sigma_1^{(2)} &= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, & \sigma_1^{(3)} = \sigma_2^{(3)} &= \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix}, \\ \sigma_1^{(4)} = \sigma_3^{(4)} &= \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}, & \sigma_2^{(4)} &= \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \end{aligned} \quad (3.13)$$

where we express the function in the matrix form

$$(\sigma_{h_A}^{(n)})_{h,h_V} = \sigma(h, h_V, h_A). \tag{3.14}$$

We determined the signs up to $n = 10$, and found the general form

$$\sigma(h, h_V, h_A) = (-1)^{f(h_A)+g(h_A,h)+g(h_A,h_V)}, \tag{3.15}$$

where

$$f(h) = \min(|h + n\mathbf{Z}|), \quad g(h, h') = \min(f(h), f(h')). \tag{3.16}$$

3.2 SU(2) gauge theory and a chiral multiplet

As the second example, we consider the duality proposed by Jafferis and Yin in [30]. The theory on one side is SU(2) Chern-Simons theory with level $k = 1$ coupling to one adjoint chiral multiplet Φ . It is dual to the theory consisting of a single chiral multiplet X . These theories have global symmetry $U(1)_A$ rotating Φ and X with charges 1 and 2, respectively.

Let us first compute the \mathcal{S}^3 partition function of the SU(2) theory. We parameterize the SU(2) Cartan algebra by

$$\lambda = xT_3, \quad T_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \tag{3.17}$$

and we adopt the integration measure

$$[d\lambda] = \frac{dx}{2\sqrt{2}}, \tag{3.18}$$

where the factor $1/2$ comes from the order of the Weyl group of SU(2), and $1/\sqrt{2}$ from the normalization of the SU(2) generators $\text{tr}T_aT_b = (1/2)\delta_{ab}$. The classical value of the Chern-Simons term with level $k = 1$ is

$$S_0 = \pi i \text{tr}(\lambda^2) = \frac{\pi i}{2} x^2. \tag{3.19}$$

The partition function of the SU(2) theory is

$$Z^{\text{SU}(2)} = \int \frac{e^{-\frac{\pi}{2}ix^2} s_b(x - \frac{i}{v}) s_b(-x - \frac{i}{v})}{s_b(x - \frac{i(1-\Delta)}{v}) s_b(-\frac{i(1-\Delta)}{v}) s_b(-x - \frac{i(1-\Delta)}{v})} \frac{dx}{2\sqrt{2}}, \tag{3.20}$$

where we denote the Weyl weight of the adjoint chiral multiplet by Δ . If we turn on the real mass parameter μ for $U(1)_A$ the weight Δ is replaced by $\Delta - iv\mu$.

The dual theory contains a single chiral multiplet X . This corresponds to the gauge invariant operator $\text{tr}\Phi^2$ in the SU(2) gauge theory, and has Weyl weight 2Δ . The \mathcal{S}^3 partition function is

$$Z^X = \frac{1}{s_b(-\frac{i(1-2\Delta)}{v})}. \tag{3.21}$$

We can easily check numerically that these two partition functions coincide up to a phase factor.

$$Z^{\text{SU}(2)} = e^{i\phi} Z^X, \quad \phi = -\pi \left(\frac{1}{4} + \frac{2\Delta + \Delta^2}{2v^2} \right). \quad (3.22)$$

This relation is confirmed numerically in [30] and analytically in [31] for the round sphere. The coincidence of the absolute value is due to our choice of the integration measure. In [30] different measure is used and extra numerical factor arises. We do not argue about this point, and focus only on the phases. For the round sphere, the phase factor

$$e^{i\phi} = \exp \left[\pi i \left(\frac{1}{4} - \frac{(1 + \Delta)^2}{2} \right) \right] = \int e^{\pi i t^2 - \sqrt{2}\pi i(1+\Delta)t} dt, \quad (3.23)$$

is interpreted in [30] as the contribution of a decoupled topological sector. For squashed \mathbf{S}^3 , there seems no such a simple explanation for this factor.

We would like to extend this relation to the orbifolds. In the introduction of holonomy, we should note that the gauge group is, precisely speaking, not $\text{SU}(2)$ but $\text{SU}(2)/\mathbb{Z}_2 = \text{SO}(3)$. The allowed holonomies are

$$\exp \left(2\pi i \frac{h}{n} T_3 \right), \quad h = 0, \dots, n-1. \quad (3.24)$$

(If the gauge group were $\text{SU}(2)$, 2π in the exponent in (3.24) should be replaced by 4π .) For the flat connection specified by the holonomy h , the classical Chern-Simons action gives the phase factor

$$e^{i\Phi} = e^{\frac{\pi i}{2n} h^2}. \quad (3.25)$$

This is not well defined as a map from \mathbb{Z}_n to \mathbb{C} . This gives different phases for h and $h+n$, which are identical in \mathbb{Z}_n . We will fix this ambiguity later by an additional rule.

The orbifold partition function of the $\text{SU}(2)$ theory is obtained from the \mathbf{S}^3 partition function (3.20) by

- replacing each $s_b(z)$ by $s_{b,h}(z)$ with an appropriate holonomy,
- replacing the measure dx by dx/n ,
- replacing the classical action S_0 in (3.19) by S_0/n ,
- and introducing the phase factor $e^{\pi i h^2/2n}$.

We obtain

$$Z^{\text{SU}(2)}(h, h_A) = \int \frac{e^{\pi i \frac{h^2}{2n}} e^{-\frac{\pi}{2n} i x^2} s_{b,h}(x - \frac{i}{v}) s_{b,-h}(-x - \frac{i}{v})}{s_{b,h_A+h}(x - \frac{i(1-\Delta)}{v}) s_{b,h_A}(-\frac{i(1-\Delta)}{v}) s_{b,h_A-h}(-x - \frac{i(1-\Delta)}{v})} \frac{dx}{2\sqrt{2n}}. \quad (3.26)$$

The partition function of the chiral multiplet X is

$$Z^X(h_A) = \frac{1}{s_{b,2h_A}(-\frac{i(1-2\Delta)}{v})}. \quad (3.27)$$

We consider two cases with even n and odd n separately. Let us first consider the case with odd n . In this case, (3.25) defines double-valued map from \mathbb{Z}_n to \mathbb{C} . For h and $h+n$, which are identified in \mathbb{Z}_n , the phase factor takes different values whose phases always differ by $\pi/2$. We denote these two phase factors by $(e^{\frac{\pi i}{2n}h^2})_{\pm}$. The subscript \pm is chosen so that the two phases satisfy $(e^{\frac{\pi i}{2n}h^2})_+ = i(e^{\frac{\pi i}{2n}h^2})_-$. Corresponding to these two choices of the phase factor, we define two partition functions $Z_{\pm}^{\text{SU}(2)}(h, h_A)$.

We take the ansatz

$$\sum_{h=0}^{n-1} \sigma(h, h_A) e^{\mp \frac{\pi i}{4}} Z_{\pm}^{\text{SU}(2)}(h, h_A) = e^{i\phi} Z^X(h_A), \tag{3.28}$$

between the partition functions of the dual theories. $\sigma(h, h_A)$ is an unknown phase function depending on the $\text{SU}(2)$ holonomy h and $\text{U}(1)_A$ holonomy h_A , and $e^{i\phi}$ is a phase factor independent of holonomies. The double signs on the left hand side are in the same order. The factor $e^{\mp \frac{\pi i}{4}}$ is inserted to cancel the difference of $Z_+^{\text{SU}(2)}$ and $Z_-^{\text{SU}(2)}$. Although we can choose one of signs as a convention and absorb this factor by $\sigma(h, h_A)$ or $e^{i\phi}$, we separate this factor for later convenience. We carried out the numerical analysis up to $n = 29$, and we found

$$\begin{aligned} \sigma(h, h_A) &= (-1)^{g(h_A, h)} \exp \left[i\pi \frac{f(h_A)(f(h_A) + n)}{2n} \right], \\ \phi(h_A) &= -\pi \frac{\Delta^2 + 2\Delta}{2nv^2}, \end{aligned} \tag{3.29}$$

make the equation (3.28) hold, where f and g are the functions defined in (3.16).

Let us turn to the case with even n . In this case we divide n possible holonomies to the $n/2$ satisfying

$$h - \frac{n}{2} \in 2\mathbb{Z}_n, \tag{3.30}$$

and the others. The phase factor (3.25) is well-defined for holonomies satisfying (3.30), while (3.25) has the sign ambiguity for the other holonomies. With the numerical analysis up to $n = 30$, we found that $Z^X(h_A)$ can be given as a linear combination of only $Z^{\text{SU}(2)}(h, h_A)$ with h satisfying (3.30),

$$\sum_{h-n/2 \in 2\mathbb{Z}_n} \sigma(h, h_A) \sqrt{2} Z^{\text{SU}(2)}(h, h_A) = e^{i\phi} Z^X(h_A), \tag{3.31}$$

where $\sigma(h, h_A)$ and $\phi(h_A)$ are functions defined in (3.29). Comparing this to (3.28), we notice that the phase factor $e^{\pm \frac{\pi i}{4}}$ is replaced by $\sqrt{2} = e^{\frac{\pi i}{4}} + e^{-\frac{\pi i}{4}}$. Although this factor depends on the choice of the integration measure and this may not have physical significance, it may be interesting to discuss what this factor implies under the assumption that our choice of the integration measure is an appropriate one. One possible interpretation is as follows. In the theory of the chiral multiplet X , the $\text{U}(1)_A$ holonomy appear only through $2h_A$. When n is even, there are two holonomies which gives the same $2h_A$. Let h_A be one of them, and $h'_A = h_A + n/2$ the other. It is natural to sum up the contribution of these

two holonomies on the $SU(2)$ theory side. If we introduce different phase factors $e^{+\frac{\pi i}{4}}$ and $e^{-\frac{\pi i}{4}}$ for h_A and h'_A in this summation, we obtain the following relation similar to (3.28).

$$\begin{aligned}
 e^{i\phi} Z^X(h_A) = e^{i\phi} Z^X(h'_A) &= \sum_{h-n/2 \in 2\mathbb{Z}_n} \sigma(h, h_A) e^{\frac{\pi i}{4}} Z^{SU(2)}(h, h_A) \\
 &+ \sum_{h-n/2 \in 2\mathbb{Z}_n} \sigma(h, h'_A) e^{-\frac{\pi i}{4}} Z^{SU(2)}(h, h'_A). \quad (3.32)
 \end{aligned}$$

3.3 S^3/\mathbb{Z}_{2k+1}

In the previous subsections, we found that we need non-trivial phase factors to match the partitions functions of dual theories in two examples. For odd n , in fact, we can express these phase factors in a unified way. Let us define σ_h by

$$\sigma_h = (-1)^{[h]_n([h]_n - (-1)^{(n-1)/2})/2}. \quad (3.33)$$

When n is odd, this takes values ± 1 depending on $h \in \mathbb{Z}_n$. We can represent $(-1)^{f(h)}$ and $(-1)^{g(h, h')}$ with this function by

$$(-1)^{f(h)} = \sigma_{2h}, \quad (-1)^{g(h, h')} = \sigma_{h+h'} \sigma_{h-h'}. \quad (3.34)$$

Therefore, the sign function (3.15) in the first example can be given as the product of five σ_h ;

$$\sigma(h, h_V, h_A) = \sigma_{h-h_A} \sigma_{h+h_A} \sigma_{h_V+h_A} \sigma_{h_V-h_A} \sigma_{2h_A}. \quad (3.35)$$

The indices of five σ_h coincide up to sign with the holonomy indices of the functions $s_{b,h}(z)$ appearing in the mirror relation (3.11). Because $\sigma_h = \sigma_{-h}$ and the sign of the index of σ_h does not matter, the phases can be absorbed into the definition of the function $s_{b,h}(z)$. Namely, if we define \widehat{Z}^{SQED} and \widehat{Z}^{XYZ} from Z^{SQED} and Z^{XYZ} , respectively, by replacing $s_{b,h}(z)$ in these partition functions by $\widehat{s}_{b,h}(z)$ defined by

$$\widehat{s}_{b,h}(z) = \sigma_h s_{b,h}(z), \quad (3.36)$$

the relation

$$\widehat{Z}^{XYZ}(h_V, h_A) = \sum_{h=0}^{n-1} \widehat{Z}^{SQED}(h, h_V, h_A) \quad (3.37)$$

holds without the extra sign factors. This is actually the case in the second example. Because the phase function can be written as

$$\sigma(h, h_A) = \sigma_{h_A+h} \sigma_{h_A-h} \exp \left[i\pi \frac{f(h_A)(f(h_A) + n)}{2n} \right], \quad (3.38)$$

$\widehat{Z}^{SU(2)}$ and \widehat{Z}^X defined with $\widehat{s}_{b,h}(z)$ satisfy the relation

$$\sum_{h=0}^{n-1} e^{\mp \frac{\pi i}{4}} \widehat{Z}_{\pm}^{SU(2)}(h, h_A) = \omega(h_A) e^{i\phi} \widehat{Z}^X(h_A), \quad (3.39)$$

where $\omega(h_A)$ is a certain factor depending only on h_A .

In the two examples, we found that if we replace $s_{b,h}(z)$ by $\widehat{s}_{b,h}(z)$ the duality relations hold without introducing non-trivial relative phase factors in the holonomy sums. This is simple enough for us to expect that this rule is universal. It would be interesting to check whether this rule really holds for other examples of dual pairs.

	q	\tilde{q}	\tilde{S}	m	\tilde{m}	Q	\tilde{Q}
$U(1)_V$	0	0	0	1	-1	1	-1
$U(1)_A$	1	1	-2	0	0	-1	-1

Table 2. Global symmetries for $\mathcal{N} = 4$ SQED and the hypermultiplets. m and \tilde{m} are again (anti-)monopole operators.

4 Derived dualities

In this section we discuss three more dualities which can be derived from the mirror symmetry studied in section 3.1.

4.1 $\mathcal{N} = 4$ SQED and hypermultiplet

It is known that the $\mathcal{N} = 4$ SQED with one flavor is mirror to a hypermultiplet [32]. This mirror pair is obtained from the $\mathcal{N} = 2$ mirror pair in section 3.1 by adding a chiral multiplet \tilde{S} on the both sides of the duality. On the SQED side, the new chiral multiplet \tilde{S} couples to the system through the superpotential $W = \tilde{q}\tilde{S}q$. This corresponds to the mass term $W = \tilde{S}S$ on the other side of the duality, and we can integrate out S and \tilde{S} to obtain the system with a hypermultiplet (Q, \tilde{Q}) . The global symmetry of the resulting mirror pair is $U(1)_V \times U(1)_A$ with the charge assignment summarized in table 2. We again introduce the mass parameters ζ and μ for $U(1)_V$ and $U(1)_A$, respectively. We denote the Weyl weights of q and \tilde{q} by Δ . Then the Weyl weight of \tilde{S} is $1 - 2\Delta$. The introduction of \tilde{S} changes the partition functions by the factor

$$\frac{1}{s_{b,-2h_A}(-2\mu + \frac{i(1-2\Delta)}{v})} = s_{b,2h_A}(2\mu - \frac{i(1-2\Delta)}{v}). \quad (4.1)$$

The partition function of two theories are given by

$$Z^{\mathcal{N}=4}(\zeta, \mu; h, h_V, h_A) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i \zeta \lambda / n} e^{2\pi i h_V h / n}}{s_{b,h_A+h}(\mu + \lambda - \frac{i(1-\Delta)}{v})} \frac{d\lambda}{n}, \quad (4.2)$$

$$\times s_{b,h_A-h}(\mu - \lambda - \frac{i(1-\Delta)}{v})$$

$$\times s_{b,-2h_A}(-2\mu - \frac{i(2\Delta-1)}{v})$$

$$Z^{\text{hyper}}(\zeta, \mu; h_V, h_A) = \frac{1}{s_{b,-h_A+h_V}(-\mu + \zeta - \frac{i\Delta}{v}) s_{b,-h_A-h_V}(-\mu - \zeta - \frac{i\Delta}{v})}. \quad (4.3)$$

Because the factor (4.1) does not depend on h , it is rather trivial that the partition functions match if we use the same sign function (3.15) as in the $\mathcal{N} = 2$ case. Namely, the following relation holds.

$$Z^{\text{hyper}}(\zeta, \mu; h_V, h_A) = \sum_{h=0}^{n-1} \sigma(h, h_V, h_A) Z^{\mathcal{N}=4}(\zeta, \mu; h, h_V, h_A). \quad (4.4)$$

4.2 $\mathcal{N} = 2$ SQED with $N_f \geq 2$ and quiver gauge theory

The mirror symmetry between $\mathcal{N} = 2$ SQED with $N_f \geq 2$ and a quiver gauge theory [33] can be derived from the $\mathcal{N} = 4$ mirror symmetry in the previous subsection. This fact is used in [34] to prove the partition functions of the mirror theories coincide to each other in the case of round \mathbf{S}^3 . In this subsection, we generalize this to $\mathbf{S}^3/\mathbb{Z}_n$.

We start with N_f copies of the mirror pairs constructed in the previous subsection, and gauge the diagonal subgroup $U(1)_{\text{diag}}$ of N_f $U(1)_V$ symmetries. Let V_i ($i = 1, \dots, N_f - 1$) be the background gauge fields corresponding to the $U(1)_V$ symmetries. The gauging of $U(1)_{\text{diag}}$ is realized by the replacement

$$V_i \rightarrow V'_i + V_{\text{diag}}, \quad (4.5)$$

where V_{diag} is the dynamical gauge field of $U(1)_{\text{diag}}$. On one side of the duality, we have N_f pairs of the chiral multiplets (Q_i, \tilde{Q}_i) and the $U(1)_{\text{gauge}}$ vector multiplet. These form $\mathcal{N} = 2$ SQED with N_f flavors.

On the other side, we have N_f copies of SQED containing fields $(A_i, q_i, \tilde{q}_i, \tilde{S}_i)$ ($i = 1, \dots, N_f$) and the $U(1)_{\text{gauge}}$ vector multiplet V_{gauge} . These fields form a $U(1)^{N_f+1}$ gauge theory. Because each copy of SQED has the Chern-Simons term (3.6), the gauging of $U(1)_{\text{gauge}}$ by the replacement (4.5) induces the Chern-Simons term

$$S = \frac{i}{2\pi} \int V_{\text{diag}} dA_D, \quad (4.6)$$

where $A_D = \sum_{i=1}^{N_f} A_i$ is the gauge field of the diagonal subgroup $U(1)_D$ of the N_f $U(1)$ gauge symmetries. The equation of motion of V_{diag} gives the constraint

$$A_D = 0 \quad (4.7)$$

on the gauge fields A_i . We can solve this by

$$A_i = \tilde{A}_i - \tilde{A}_{i-1}, \quad (4.8)$$

where \tilde{A}_i ($i = 1, \dots, N_f - 1$) are independent dynamical gauge fields and $\tilde{A}_0 = \tilde{A}_{N_f} = 0$. As the result we have $U(1)^{N_f-1}$ quiver gauge theory represented by the quiver diagram in figure 1.

Let us confirm that the partition functions of the SQED and the $U(1)^{N_f-1}$ quiver gauge theory agree. What is non-trivial is how we should choose the relative phases in the holonomy sum for the newly introduced $U(1)_{\text{diag}}$ gauge symmetry. In the following, we will find that we need a non-trivial sign factor depending on the $U(1)_{\text{gauge}}$ holonomy for the reduction of the gauge group from $U(1)^{N_f+1}$ to $U(1)^{N_f-1}$ works in the context of the $\mathbf{S}^3/\mathbb{Z}_n$ partition function.

Let us write down the orbifold partition functions. Corresponding to (4.5) we replace the $U(1)_V$ holonomies h_V^i and $U(1)_V$ mass parameters ζ_i by

$$\zeta_i = \zeta'_i + x, \quad h_V^i = h_V^i + h', \quad (4.9)$$

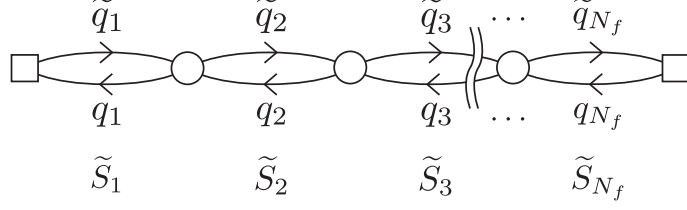


Figure 1. Quiver diagram of the mirror of $\mathcal{N} = 2$ SQED with N_f flavors. Circles and squares represent the $U(1)$ gauge groups and the $U(1)$ flavor symmetry, respectively. q_i and \tilde{q}_i represented by arrows are bifundamental fields, and \tilde{S}_i are neutral fields.

where x and h' are the $U(1)_{\text{diag}}$ modulus and the $U(1)_{\text{diag}}$ holonomy. On the side of N_f pairs of chiral multiplets (Q_i, \tilde{Q}_i) we obtain

$$\begin{aligned} Z^{N_f}(\zeta'_i, \mu_i; \vec{h}'_V, \vec{h}_A) &= \sum_{h'=0}^{n-1} \tau(h', \vec{h}'_V, \vec{h}_A) \int \frac{dx}{n} \prod_{i=1}^{N_f} Z^{\text{hyper}}(x + \zeta'_i, \mu_i; h' + h_V^i, h_A^i) \\ &= \sum_{h'=0}^{n-1} \tau(h', \vec{h}'_V, \vec{h}_A) \int \frac{dx}{n} \prod_{i=1}^{N_f} \frac{1}{s \cdots s}, \end{aligned} \quad (4.10)$$

where $s \cdots s$ represents the product of $s_{b,h}(z)$ coming from Z^{hyper} . We use arrows to represent sets of N_f parameters; $\vec{h}'_V = (h_V^1, \dots, h_V^{N_f})$ etc. We introduced unknown sign function $\tau(h', \vec{h}'_V, \vec{h}_A)$ depending on the holonomies.

Application of the same prescription on the side of N_f copies of $\mathcal{N} = 4$ SQED gives the partition function

$$\begin{aligned} Z^{\text{quiver}}(\zeta'_i, \mu_i; \vec{h}'_V, \vec{h}_A) &= \sum_{h'=0}^{n-1} \tau(h', \vec{h}'_V, \vec{h}_A) \int \frac{dx}{n} \prod_{i=1}^{N_f} Z^{\mathcal{N}=4}(x + \zeta'_i, \mu_i; h' + h_V^i, h_A^i) \\ &= \sum_{h'=0}^{n-1} \tau(h', \vec{h}'_V, \vec{h}_A) \int \frac{dx}{n} \\ &\quad \prod_{i=1}^{N_f} \left[\sum_{h_i=0}^{n-1} \sigma(h_i, h' + h_V^i, h_A^i) \int_{-\infty}^{\infty} \frac{d\lambda_i}{n} \frac{e^{-2\pi i(x+\zeta'_i)\lambda_i/n} e^{2\pi i(h'+h_V^i)h_i/n}}{s \cdots s} \right]. \end{aligned} \quad (4.11)$$

Again, we represent the product of $s_{b,h}(z)$ coming from $Z^{\mathcal{N}=4}$ by $s \cdots s$. By definition this is the same as (4.10). What is non-trivial is if this partition function can be regarded as that for quiver gauge theory with the reduced gauge group $U(1)^{N_f-1}$. For this to be the case, the summation over the holonomies h' and \vec{h} and integral over the parameters x and $\vec{\lambda}$ should reduce to those over $N_f - 1$ parameters corresponding to the $N_f - 1$ dynamical gauge fields.

Let us first look at the x -integral in (4.11). The relevant part is

$$\int \frac{dx}{n} \prod_{i=1}^{N_f} e^{-2\pi i(x+\zeta'_i)\lambda_i/n} = \delta \left(\sum_{i=1}^{N_f} \lambda_i \right) \exp \left(-\frac{2\pi i}{n} \sum_{i=1}^{N_f} \zeta'_i \lambda_i \right) \quad (4.12)$$

The delta function reduces the dimension of the λ_i integral by one, and imposes the constraint

$$\sum_{i=1}^{N_f} \lambda_i = 0. \quad (4.13)$$

We can solve this by

$$\lambda_i = \tilde{\lambda}_i - \tilde{\lambda}_{i-1}, \quad \tilde{\lambda}_0 = \tilde{\lambda}_{N_f} = 0. \quad (4.14)$$

The $N_f - 1$ parameters $\tilde{\lambda}_i$ correspond to the $N_f - 1$ dynamical vector multiplets introduced in (4.8).

Concerning the summation with respect to h' , the relevant part in (4.11) is

$$\begin{aligned} & \sum_{h'=0}^{n-1} \tau(h', \vec{h}'_V, \vec{h}_A) \prod_{i=1}^{N_f} \left(\sigma(h_i, h' + h_V^i, h_A^i) e^{2\pi i (h' + h_V^i) h_i / n} \right) \\ &= \sum_{h'=0}^{n-1} \left(\tau(h', \vec{h}'_V, \vec{h}_A) \prod_{i=1}^{N_f} \sigma(h_i, h' + h_V^i, h_A^i) \right) e^{\left(\frac{2\pi i}{n} h' \sum_{i=1}^{N_f} h_i \right)} e^{\left(\frac{2\pi i}{n} \sum_{i=1}^{N_f} h_V^i h_i \right)} \end{aligned} \quad (4.15)$$

For (4.11) to be regarded as the partition function of the $U(1)^{N_f-1}$ quiver gauge theory, the condition

$$\sum_{i=1}^{N_f} h_i = 0, \quad (4.16)$$

corresponding to (4.7) and (4.13) should arise. Namely, the summation over h' in (4.15) should give the Kronecker's delta imposing the condition (4.16). For this to happen the sign factor in front of the exponentials in the summand must be h' independent. Because $\tau(h', \vec{h}'_V, \vec{h}_A)$ is h_i -independent, this is possible only when h_i dependence and h' dependence of $\sigma(h_i, h' + h_V^i, h_A^i)$ are factorized. This is indeed the case as (3.15) shows, and we can make the coefficients in (4.15) h' independent by choosing the function $\tau(h', \vec{h}'_V, \vec{h}_A)$ as

$$\tau(h', \vec{h}'_V, \vec{h}_A) = \prod_{i=1}^{N_f} (-1)^{f(h_A^i) + g(h_A^i, h' + h_V^i)}. \quad (4.17)$$

With this choice the summation over h' gives

$$\begin{aligned} & \sum_{h'=0}^{n-1} \tau(h', \vec{h}'_V, \vec{h}_A) \prod_{i=1}^{N_f} \left(\sigma(h_i, h' + h_V^i, h_A^i) e^{2\pi i (h' + h_V^i) h_i / n} \right) \\ &= n v(\vec{h}, \vec{h}_A) \delta \left(\sum_{i=1}^{N_f} h_i \right) \exp \left(\frac{2\pi i}{n} \sum_{i=1}^{N_f} h_V^i h_i \right) \end{aligned} \quad (4.18)$$

where $\delta(*)$ in this equation is the Kronecker's delta, and we defined the sign function

$$v(\vec{h}, \vec{h}_A) = \prod_{i=1}^{N_f} (-1)^{g(h_A^i, h_i)}. \quad (4.19)$$

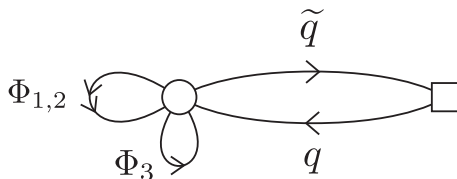


Figure 2. Quiver diagram of the mirror ABJM

The constraint imposed on h_i by the Kronecker's delta in (4.18) can be solved by

$$h_V^i = \tilde{h}_i - \tilde{h}_{i-1}, \quad \tilde{h}_i = \tilde{h}_{N_f} = 0. \quad (4.20)$$

\tilde{h}_i are the holonomies of the $U(1)^{N_f-1}$ gauge symmetry. We can rewrite (4.11) as

$$Z^{\text{quiver}}(\vec{h}'_V, \vec{h}_A) = \prod_{i=1}^{N_f-1} \left(\sum_{\tilde{h}_i} \int \frac{d\tilde{\lambda}_i}{n} \right) v(\vec{h}, \vec{h}_A) \frac{e^{-2\pi i \vec{\zeta}' \cdot \vec{\lambda}/n} e^{2\pi i \vec{h}'_V \cdot \vec{h}/n}}{s \cdots s}. \quad (4.21)$$

This is precisely the partition function of the $U(1)^{N_f-1}$ quiver gauge theory.

4.3 ABJM model and its mirror

The ABJM model [35] is an $U(N) \times U(N)$ Chern-Simons theory with four bi-fundamental chiral multiplets A_i and B_i ($i = 1, 2$). A_i and B_i belong to $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$, respectively. The theory has the superpotential

$$W = \epsilon_{ik} \epsilon_{jl} \text{tr}(A_i B_j A_k B_l), \quad (4.22)$$

and all chiral multiplets have Weyl weight $1/2$. When the Chern-Simons levels are 1 and -1 , the ABJM model is known to be mirror to the $\mathcal{N} = 4$ $U(N)$ gauge theory with a fundamental hypermultiplet (q, \tilde{q}) and an adjoint hypermultiplet (Φ_1, Φ_2) [6, 35, 36]. In terms of $\mathcal{N} = 2$ language this theory has the superpotential

$$W = \text{tr} \Phi_3 (q\tilde{q} + [\Phi_1, \Phi_2]), \quad (4.23)$$

where Φ_3 is the chiral multiplet which form the $\mathcal{N} = 4$ vector multiplet together with the $\mathcal{N} = 2$ vector multiplet. The quiver diagram of this theory is shown in figure 2. The Weyl weights of the chiral multiplets are

$$\Delta_q = \Delta_{\tilde{q}} = \Delta_{\Phi_1} = \Delta_{\Phi_2} = \frac{1}{2}, \quad \Delta_{\Phi_3} = 1. \quad (4.24)$$

These theories are expected to have $SO(8)$ global symmetry, which are not manifest in the Lagrangians. We only focus on the Cartan subalgebra

$$U(1)_A \times U(1)_B \times U(1)_T \times U(1)_R \subset SO(8), \quad (4.25)$$

where $U(1)_R$ is the R-symmetry acting on the $\mathcal{N} = 2$ supercharges. The charge assignments in the two theories are listed in table 3 and 4. $U(1)_B$ in the mirror theory is a topological $U(1)$ symmetry coupling to monopole operators, and the corresponding mass parameter is the Fayet-Iliopoulos parameter.

	A_1	A_2	B_1	B_2
$U(1)_A$	+1	-1	0	0
$U(1)_B$	0	0	+1	-1
$U(1)_T$	+1	+1	-1	-1

Table 3. Global symmetries of ABJM model

	Φ_1	Φ_2	Φ_3	m	\tilde{m}	q	\tilde{q}
$U(1)_A$	+1	-1	0	0	0	0	0
$U(1)_B$	0	0	0	+1	-1	0	0
$U(1)_T$	+1	+1	-2	0	0	+1	+1

Table 4. Charge assignments in the mirror theory of the ABJM model. m and \tilde{m} are monopole and anti-monopole fields.

Although this mirror symmetry holds for an arbitrary rank N of the gauge groups, we only consider Abelian ($N = 1$) case for simplicity. Let us first consider the orbifold partition function in the mirror theory. When the gauge group is $U(1)$, the system consists of two sectors decoupled from each other: the neutral hypermultiplet and the $\mathcal{N} = 4$ SQED with one flavor. In the SQED we need to carry out the holonomy sum to obtain the partition function, and it is non-trivial how we should choose the relative phases. We here take the sign function (3.15) used in section 4.1. We will see shortly that the partition function obtained with this phase factor is indeed reproduced in the ABJM model, too, with an appropriate choice of the phases in the holonomy sum in the ABJM model. The partition function of the mirror theory is

$$Z^{\text{mirror}}(h_A, h_B, h_T) = \sum_h \sigma(h, h_B, h_T) Z_{\text{mirror}}(h, h_A, h_B, h_T) \quad (4.26)$$

with the contribution of each holonomy sector

$$\begin{aligned} & Z_{\text{mirror}}(h, h_A, h_B, h_T) \\ &= \frac{1}{s_{b, h_A + h_T} \left(m_A + m_T - \frac{i}{2v} \right) s_{b, -h_A + h_T} \left(-m_A + m_T - \frac{i}{2v} \right) s_{b, -2h_T} (-2m_T)} \\ & \quad \times \int \frac{dx}{n} \frac{e^{2\pi i \frac{h h_B}{n}} e^{-2\pi i \frac{m_B x}{n}}}{s_{b, h + h_T} \left(x + m_T - \frac{i}{2v} \right) s_{b, -h + h_T} \left(-x + m_T - \frac{i}{2v} \right)} \\ &= Z^{\text{hyper}}(m_A, -m_T; h_A, -h_T)|_{\Delta=\frac{1}{2}} Z^{\mathcal{N}=4}(m_B, m_T; h, h_B, h_T)|_{\Delta=\frac{1}{2}}, \end{aligned} \quad (4.27)$$

where we introduce mass parameters (m_A, m_B, m_T) and holonomies (h_A, h_B, h_T) corresponding to the flavor symmetry $U(1)_A \times U(1)_B \times U(1)_T$. h is the holonomy for the gauge symmetry. With the relation (4.4), we can rewrite the partition function (4.26) as the product of two Z^{hyper} ;

$$Z^{\text{mirror}}(h_A, h_B, h_T) = Z^{\text{hyper}}(m_A, -m_T; h_A, -h_T)|_{\Delta=\frac{1}{2}} Z^{\text{hyper}}(m_B, m_T; h_B, h_T)|_{\Delta=\frac{1}{2}}. \quad (4.28)$$

On the ABJM side, we need to sum up n^2 contributions parameterized by a pair of holonomies (h_1, h_2) for the gauge group $U(1)_1 \times U(1)_2$. The partition function of the sector specified by (h_1, h_2) is

$$\begin{aligned}
 Z^{\text{ABJM}}(h_1, h_2, h_A, h_B, h_T) &= e^{i\Phi(h_1, h_2)} \int \frac{d\lambda}{n} \frac{d\tilde{\lambda}}{n} \frac{\exp\left[-\frac{\pi i}{n}(\lambda^2 - \tilde{\lambda}^2)\right]}{s_{b, h_A+h_T+h_1-h_2}(m_A + m_T + \lambda - \tilde{\lambda} - \frac{i}{2v})} \\
 &\quad \times s_{b, -h_A+h_T+h_1-h_2}(-m_A + m_T + \lambda - \tilde{\lambda} - \frac{i}{2v}) \\
 &\quad \times s_{b, h_B-h_T-h_1+h_2}(m_B - m_T + \tilde{\lambda} - \lambda - \frac{i}{2v}) \\
 &\quad \times s_{b, -h_B-h_T-h_1+h_2}(-m_B - m_T + \tilde{\lambda} - \lambda - \frac{i}{2v})
 \end{aligned} \tag{4.29}$$

A question is if it is possible to choose an appropriate phases in the holonomy sum. The answer is rather simple. We do not need any non-trivial phases in this sum. Let us confirm this by summing up (4.29) over holonomies h_1 and h_2 . If we define $h_{12} \equiv h_1 - h_2$ and replace h_1 by $h_{12} + h_2$, h_2 appears only in the phase factor

$$\Phi = \frac{\pi}{n}(h_1^2 - h_2^2) = \frac{\pi}{n}(h_{12}^2 + 2h_2h_{12}). \tag{4.30}$$

The summation with respect to h_2 gives non-vanishing result only when $h_{12} = 0$, and we obtain

$$\begin{aligned}
 Z^{\text{ABJM}}(h_A, h_B, h_T) &= \sum_{h_1, h_2=0}^{n-1} Z^{\text{ABJM}}(h_1, h_2, h_A, h_B, h_T) \\
 &= \sum_{h=0}^{n-1} Z^{\text{ABJM}}(h, h, h_A, h_B, h_T) = n Z^{\text{ABJM}}(0, 0, h_A, h_B, h_T) \\
 &= n \int \frac{d\lambda}{n} \frac{d\tilde{\lambda}}{n} \frac{\exp\left[-\frac{\pi i}{n}(\lambda^2 - \tilde{\lambda}^2)\right]}{s_{b, h_A+h_T}(m_A + m_T + \lambda - \tilde{\lambda} - \frac{i}{2v})} \\
 &\quad \times s_{b, -h_A+h_T}(-m_A + m_T + \lambda - \tilde{\lambda} - \frac{i}{2v}) \\
 &\quad \times s_{b, h_B-h_T}(m_B - m_T + \tilde{\lambda} - \lambda - \frac{i}{2v}) \\
 &\quad \times s_{b, -h_B-h_T}(-m_B - m_T + \tilde{\lambda} - \lambda - \frac{i}{2v})
 \end{aligned} \tag{4.31}$$

We can easily perform the integral and have

$$\begin{aligned}
 Z^{\text{ABJM}}(h_A, h_B, h_T) &= \frac{1}{s_{b, h_A+h_T}(m_A + m_T - \frac{i}{2v}) s_{b, -h_A+h_T}(-m_A + m_T - \frac{i}{2v})} \\
 &\quad \times s_{b, h_B-h_T}(m_B - m_T - \frac{i}{2v}) s_{b, -h_B-h_T}(-m_B - m_T - \frac{i}{2v}) \\
 &= Z^{\text{hyper}}(m_A, -m_T; h_A, -h_T)|_{\Delta=\frac{1}{2}} Z^{\text{hyper}}(m_B, m_T; h_B, h_T)|_{\Delta=\frac{1}{2}}.
 \end{aligned} \tag{4.32}$$

This result precisely agrees with the partition function of the mirror theory (4.28).

5 Conclusions and discussions

We investigated relative phases in the holonomy sum, which is necessary to obtain the partition functions of gauge theories in S^3/\mathbb{Z}_n . We used dualities between gauge theories and non-gauge theories to determine the phases.

We first consider mirror symmetry between $\mathcal{N} = 2$ SQED with one flavor and the XYZ model containing three chiral multiplets. We showed that with the appropriate choice of the phases in the holonomy sum the partition functions of these theories coincides. Furthermore, we found that when n is odd, the phase factor is absorbed by the redefinition of the single function $s_{b,h}(z)$, the orbifold extension of the double sine function. We also consider the duality between $SU(2)$ gauge theory and a chiral multiplet proposed by Jafferis and Yin. We could again find phase factors which makes the duality relation hold. When n is odd the phases are absorbed by redefining the function $s_{b,h}(z)$ in the same way as in the first example.

We also confirmed that it is possible to find phase factors in three more examples of dual pairs, which are derived from the mirror symmetry we studied first.

When n is odd, in all these examples, the phase factors can be absorbed by the definition of the function $s_{b,h}(z)$. This fact strongly suggests that the modified function $\hat{s}_{b,h}$ in (3.36) always gives a “correct” partition function in some sense. It would be interesting to check whether the modified function gives the same partition functions for theories in dual pairs we did not studied in this paper.

For the purpose of obtaining non-trivial evidences for dualities, it is desirable that we first fix the relative phases in the holonomy sum in each theory without relying on dualities. This should be possible at least for saddle points belonging to the same component of the configuration space. In this case we should obtain the relative phases by analyzing carefully the behavior of the integration measure in the path integral under adiabatic deformations connecting saddle points. It is also interesting to search for guiding principles to fix the ambiguity for the relative phases for the contribution of topologically disconnected sectors.

The large N limit of the partition function of the ABJM model on the orbifold is considered in [37], and the coincidence of the free energy calculated from the ABJM model and that from the gravity dual is confirmed. Though many sectors with different holonomies contribute to the partition function the authors found that in the large N limit it is dominated by the specific contribution with a certain holonomy configuration and the relative phases are not important. However, when one consider the next leading order of $1/N$ the other contributions become significant, therefore, the phase plays an important role there.

Acknowledgments

We are grateful to M. Honda for valuable discussions. The work of Y.I. was partially supported by Grant-in-Aid for Scientific Research (C) (No.24540260), Ministry of Education, Science and Culture, Japan. D.Y. acknowledges the financial support from the Global Center of Excellence Program by MEXT, Japan through the “Nanoscience and Quantum Physics” Project of the Tokyo Institute of Technology.

References

- [1] A. Kapustin, B. Willett and I. Yaakov, *Exact results for Wilson loops in superconformal Chern-Simons theories with matter*, *JHEP* **03** (2010) 089 [[arXiv:0909.4559](#)] [[INSPIRE](#)].
- [2] D.L. Jafferis, *The exact superconformal R-symmetry extremizes Z*, *JHEP* **05** (2012) 159 [[arXiv:1012.3210](#)] [[INSPIRE](#)].
- [3] N. Hama, K. Hosomichi and S. Lee, *Notes on SUSY gauge theories on three-sphere*, *JHEP* **03** (2011) 127 [[arXiv:1012.3512](#)] [[INSPIRE](#)].
- [4] S. Kim, *The complete superconformal index for $N = 6$ Chern-Simons theory*, *Nucl. Phys. B* **821** (2009) 241 [Erratum *ibid.* **B 864** (2012) 884] [[arXiv:0903.4172](#)] [[INSPIRE](#)].
- [5] Y. Imamura and S. Yokoyama, *Index for three dimensional superconformal field theories with general R-charge assignments*, *JHEP* **04** (2011) 007 [[arXiv:1101.0557](#)] [[INSPIRE](#)].
- [6] A. Kapustin, B. Willett and I. Yaakov, *Nonperturbative tests of three-dimensional dualities*, *JHEP* **10** (2010) 013 [[arXiv:1003.5694](#)] [[INSPIRE](#)].
- [7] A. Kapustin, B. Willett and I. Yaakov, *Tests of Seiberg-like duality in three dimensions*, [arXiv:1012.4021](#) [[INSPIRE](#)].
- [8] A. Kapustin, *Seiberg-like duality in three dimensions for orthogonal gauge groups*, [arXiv:1104.0466](#) [[INSPIRE](#)].
- [9] B. Willett and I. Yaakov, *$N = 2$ dualities and Z extremization in three dimensions*, [arXiv:1104.0487](#) [[INSPIRE](#)].
- [10] N. Drukker, M. Mariño and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [[arXiv:1007.3837](#)] [[INSPIRE](#)].
- [11] C.P. Herzog, I.R. Klebanov, S.S. Pufu and T. Tesileanu, *Multi-matrix models and tri-Sasaki Einstein spaces*, *Phys. Rev. D* **83** (2011) 046001 [[arXiv:1011.5487](#)] [[INSPIRE](#)].
- [12] D. Gang, *Chern-Simons theory on $L(p, q)$ lens spaces and localization*, [arXiv:0912.4664](#) [[INSPIRE](#)].
- [13] F. Benini, T. Nishioka and M. Yamazaki, *4D index to 3D index and 2d TQFT*, *Phys. Rev. D* **86** (2012) 065015 [[arXiv:1109.0283](#)] [[INSPIRE](#)].
- [14] N. Hama, K. Hosomichi and S. Lee, *SUSY gauge theories on squashed three-spheres*, *JHEP* **05** (2011) 014 [[arXiv:1102.4716](#)] [[INSPIRE](#)].
- [15] Y. Imamura and D. Yokoyama, *$N = 2$ supersymmetric theories on squashed three-sphere*, *Phys. Rev. D* **85** (2012) 025015 [[arXiv:1109.4734](#)] [[INSPIRE](#)].
- [16] F. Dolan, V. Spiridonov and G. Vartanov, *From 4d superconformal indices to 3D partition functions*, *Phys. Lett. B* **704** (2011) 234 [[arXiv:1104.1787](#)] [[INSPIRE](#)].
- [17] A. Gadde and W. Yan, *Reducing the 4d index to the S^3 partition function*, [arXiv:1104.2592](#) [[INSPIRE](#)].
- [18] Y. Imamura, *Relation between the 4d superconformal index and the S^3 partition function*, *JHEP* **09** (2011) 133 [[arXiv:1104.4482](#)] [[INSPIRE](#)].
- [19] L. Faddeev, *Current-like variables in massive and massless integrable models*, [hep-th/9408041](#) [[INSPIRE](#)].

- [20] L. Faddeev, *Discrete Heisenberg-Weyl group and modular group*, *Lett. Math. Phys.* **34** (1995) 249 [[hep-th/9504111](#)] [[INSPIRE](#)].
- [21] L. Faddeev, R. Kashaev and A.Y. Volkov, *Strongly coupled quantum discrete Liouville theory. 1. Algebraic approach and duality*, *Commun. Math. Phys.* **219** (2001) 199 [[hep-th/0006156](#)] [[INSPIRE](#)].
- [22] S. Kharchev, D. Lebedev and M. Semenov-Tian-Shansky, *Unitary representations of $U(q)\text{sl}(2, \mathbb{R})$, the modular double and the multiparticle q deformed Toda chains*, *Commun. Math. Phys.* **225** (2002) 573 [[hep-th/0102180](#)] [[INSPIRE](#)].
- [23] A.G. Bytsko and J. Teschner, *Quantization of models with Non-compact quantum group symmetry: modular XXZ magnet and lattice sinh-Gordon model*, *J. Phys. A* **39** (2006) 12927 [[hep-th/0602093](#)] [[INSPIRE](#)].
- [24] S.K. Hansen and T. Takata, *Reshetikhin-Turaev invariants of Seifert 3-manifolds for classical simple Lie algebras and their asymptotic expansions*, [math/0209403](#).
- [25] L. Griguolo, D. Seminara, R.J. Szabo and A. Tanzini, *Black holes, instanton counting on toric singularities and q -deformed two-dimensional Yang-Mills theory*, *Nucl. Phys. B* **772** (2007) 1 [[hep-th/0610155](#)] [[INSPIRE](#)].
- [26] V.V. Bazhanov, V.V. Mangazeev and S.M. Sergeev, *Faddeev-Volkov solution of the Yang-Baxter equation and discrete conformal symmetry*, *Nucl. Phys. B* **784** (2007) 234 [[hep-th/0703041](#)] [[INSPIRE](#)].
- [27] S.L. Woronowicz, *Quantum exponential function*, *Rev. Math. Phys.* **12** (2000) 873.
- [28] B. Ponsot and J. Teschner, *Clebsch-Gordan and Racah-Wigner coefficients for a continuous series of representations of $U(q)(\text{SL}(2, \mathbb{R}))$* , *Commun. Math. Phys.* **224** (2001) 613 [[math/0007097](#)] [[INSPIRE](#)].
- [29] R.M. Kashaev, *The non-compact quantum dilogarithm and the Baxter equations*, *J. Statist. Phys.* **102** (2001) 923.
- [30] D. Jafferis and X. Yin, *A duality appetizer*, [arXiv:1103.5700](#) [[INSPIRE](#)].
- [31] P. Agarwal, A. Amariti and M. Siani, *Refined checks and exact dualities in three dimensions*, *JHEP* **10** (2012) 178 [[arXiv:1205.6798](#)] [[INSPIRE](#)].
- [32] K.A. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, *Phys. Lett. B* **387** (1996) 513 [[hep-th/9607207](#)] [[INSPIRE](#)].
- [33] J. de Boer, K. Hori, Y. Oz and Z. Yin, *Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three-dimensions*, *Nucl. Phys. B* **502** (1997) 107 [[hep-th/9702154](#)] [[INSPIRE](#)].
- [34] A. Kapustin and B. Willett, *Generalized superconformal index for three dimensional field theories*, [arXiv:1106.2484](#) [[INSPIRE](#)].
- [35] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[INSPIRE](#)].
- [36] F. Benini, C. Closset and S. Cremonesi, *Chiral flavors and M2-branes at toric CY4 singularities*, *JHEP* **02** (2010) 036 [[arXiv:0911.4127](#)] [[INSPIRE](#)].
- [37] L.F. Alday, M. Fluder and J. Sparks, *The large- N limit of M2-branes on lens spaces*, *JHEP* **10** (2012) 057 [[arXiv:1204.1280](#)] [[INSPIRE](#)].