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Cornering the unphysical vertex

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Abstract: In the classical pure spinor worldsheet theory of $AdS_5 \times S^5$ there are some vertex operators which do not correspond to any physical excitations. We study their flat space limit. We find that the BRST operator of the worldsheet theory in flat space-time can be nontrivially deformed without deforming the worldsheet action. Some of these deformations describe the linear dilaton background. But the deformation corresponding to the nonphysical vertex differs from the linear dilaton in not being worldsheet parity even. The nonphysically deformed worldsheet theory has nonzero beta-function at one loop. This means that the classical Type IIB SUGRA backgrounds are not completely characterized by requiring the BRST symmetry of the classical worldsheet theory; it is also necessary to require the vanishing of the one-loop beta-function.

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1 Introduction

The pure spinor formalism for the classical Type IIB supergravity was developed in [1]. As typical for theories with extended supersymmetry, the formalism is technically challenging and involves many subtle geometrical constructions. Moreover, even the basic postulates of the formalism are not completely clear (at least to us). We would like to have some set of axioms which would allow us to encode the space-time dynamics (SUGRA) in terms of the worldsheet dynamics. Naively, the set of rules can be as follows:

"Postulate the action of the form:

$$\begin{aligned}\int d\tau^+ d\tau^- \, [& A_{mn}(x,\theta)\partial_+ x^m \partial_- x^n + A_{m\alpha}(x,\theta)\partial_+ x^m \partial_- \theta^\alpha + \\ & + A_{\alpha m}(x,\theta)\partial_+ \theta^\alpha \partial_- x^m + A_{\alpha\beta}(x,\theta)\partial_+\theta^\alpha\partial_-\theta^\beta + \\ & + w_{L+}(\partial_- + A_-(x,\theta))\lambda_L + w_{R-}(\partial_+ + A_+(x,\theta))\lambda_R + \\ & + < w_{L+} w_{R-} \lambda_L \lambda_R >]\end{aligned} \tag{1.1}$$

where λ and w are pure spinors and their conjugate momenta, and request that it satisfies the properties:

- Classical $2d$ conformal invariance
- Lagrangian is polynomial in λ and w
- Two separate conserved *ghost number* charges, left for λ_L , w_{L+} and right for λ_R , w_{R-}
- Nilpotent BRST symmetry

The constraints guarantee that these coupling constants $A_{MN}(x,\theta)$ encode a solution of the Type IIB SUGRA."

We believe that this is not very far from the truth, but there are subtleties.

In order to better understand the pure spinor formalism, it is useful to consider explicitly various specific examples beyond the flat space. The most symmetric non-flat example is $AdS_5 \times S^5$ which was constructed in [2]. In [3] we have discussed a special class of deformations of $AdS_5 \times S^5$ known as β-deformations. At the linearized level, we have explicitly constructed the corresponding deformations of the pure spinor action. They are described by the integrated vertex operators, which are products of two global symmetry currents with some constant coupling constant B^{ab}:

$$U = B^{ab} j_a \wedge j_b \tag{1.2}$$

1.1 Non-physical vertices

As was pointed out in [3], some apparently well-defined vertex operators of the form (1.2) do not correspond to any physical deformations of the $AdS_5 \times S^5$ background. They have:

$$B^{ab} f_{ab}{}^{c} \neq 0 \tag{1.3}$$

where $f_{ab}{}^{c}$ is the structure constants of the SUSY algebra $\mathbf{g} = \mathbf{psu}(2, 2|4)$. We will call such vertices "non-physical". Their appearence does not lead to any obvious contradiction, if one can either consistently throw them away, or perhaps learn to live with them. Throwing them away should presumably correspond to an additional restriction of the allowed BRST cochains, similar to the semi-relative cohomology of the bosonic string [4–6].

In this paper we will study the flat space limit of these unphysical vertices.

1.2 Flat space limit of SUGRA excitations

We will start by pointing out the following general fact about the flat space limit of SUGRA solutions.

Given a general nonlinear solution ("the background") of the Type IIB SUGRA we can consider the linear space of its infinitesimal deformations ("excitations"). Such excitations correspond to solutions of certain *linear* differential equations, namely the SUGRA equations of motion linearized around this background).

In particular, let us look at the flat space limit of the excitations of $AdS_5 \times S^5$. Both the flat space sigma-model and the sigma-model of $AdS_5 \times S^5$ are invariant under a parity symmetry. Therefore linearized excitations can be separated into parity-odd excitations and parity-even excitations. Let us restrict ourselves to the bosonic excitations. Those excitations which involve NSNS and RR B-fields (i.e. RR 3-form field strength) are parity-odd, while those which involve metric, dilaton, axion, and the RR 5-form field strength are parity-even.

Let us pick some particular excitation and look at its Taylor expansion around a fixed "marked point" x_*. Consider *only the leading term* in the Taylor expansion. It is a polynomial in $x - x_*$. We claim that:

$$\begin{array}{l}\text{The leading term of a parity-odd excitation of } AdS_5 \times S^5 \\ \text{is a polynomial solution of the flat space linearized SUGRA}\end{array} \tag{1.4}$$

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Proof. Equations of motion of Type IIB SUGRA are systematically reviewed in [7]. For the leading approximation to the flat space limit of $AdS_5 \times S^5$, we get the following linearized equations for H_{NSNS} and H_{RR}:

$$d * (H_{NSNS} + iH_{RR}) = -\frac{2}{3} i * \iota_{(H_{NSNS}+iH_{RR})} F_5 \tag{1.5}$$

where $\iota_{(H_{NSNS}+iH_{RR})}F_5$ is the substitution of the complex 3-form $H_{NSNS} + iH_{RR}$ into the RR 5-form field strength of $AdS_5 \times S^5$. We have to prove that the leading term of $H_{NSNS} + iH_{RR}$ is a solution of the linearized SUGRA in flat space. We expand (1.5) in Taylor series. For the term with the leading power of x, all that matters is the term with the maximal number of derivatives. It is the same as in flat space:

$$d *_{\text{flat}} (H_{NSNS} + iH_{RR}) = 0\,. \tag{1.6}$$

1.3 Flat space limit of non-physical vertices

Although the non-physical vertices do deform the AdS action consistently, and in a BRST-invariant way, they do not correspond to any linearized supergravity solution. We can see it in the flat space limit. We expand the vertex around a fixed "marked point" $x_* \in AdS_5 \times S^5$ and look at the leading term. We observe that the SUGRA fields read from the leading term do not solve the linearized SUGRA equations in flat space. This confirms the observation of [3] that the non-physical vertex does not correspond to any deformation of $AdS_5 \times S^5$. If the non-physical vertex corresponded to a valid deformation of $AdS_5 \times S^5$, then this would be in contradiction with (1.4).

Moreover, it turns out that there is an essential difference between the non-physicalness of the AdS deformation vertex (1.2) and its flat space limit. In case of AdS, the vertex given by eq. (1.2) at least deforms the worldsheet action in $AdS_5 \times S^5$ in a consistent way. Its flat space limit, however, does not even provide a consistent deformation of the flat space worldsheet action. How can it be?

1.4 Wild deformations of the BRST operator

The mechanism is the following. Remember that usually the BRST-invariant deformations of the worldsheet action are accompanied by the corresponding deformation of the BRST operator[1] Q. The deformations of the BRST structure are tied to the deformations of the action. But in the special case of flat space there are "wild" deformations of the BRST structure, which do not require the deformations of the action:

- We can deform the BRST structure keeping the action fixed.

We will call these deformations of Q "wild", in the sense that they are not tied to the deformations of the action. These "wild" deformations of the BRST structure play an important role in the flat space limit of the unphysical β-deformations. Let us consider a β-deformation of the AdS space and expand everything around flat space. If the expansion of the β-deformation vertex starts from R^{-3}, then the flat space limit is perfectly physical;

[1] Because the BRST-invariant integrated vertex is only BRST-invariant on-shell.

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it is just a constant RR 3-form field strength. But for some vertices (or, equivalently, for some choice of the expansion point $x_* \in AdS_5 \times S^5$) the expansion starts with R^{-4}. In this case we get:[2]

$$S = R^{-2} S_{\text{flat}} + \int R^{-3} U_{\text{AdS RR 5-form}} + \int \varepsilon R^{-4} U_\beta + \dots \tag{1.7}$$

Here R is the radius of AdS space, ε the small parameter measuring the strength of the β-deformation, $U_{\text{AdS RR 5-form}}$ is the integrated vertex corresponding to the deformation of flat space into AdS, and U_β is the leading term in the expansion of the β-deformation integrated vertex around the marked point. It turns out that the BRST operator of the unphysical β-deformation, in the flat space expansion, contains a *wild piece* at the lower order then one would expect:

$$Q = Q_{\text{flat}} + \varepsilon R^{-1} \Delta_{\text{wild}} Q + \dots \tag{1.8}$$

where $\Delta_{\text{wild}} Q$ is a wild deformation of Q_{flat}. Note that the BRST operator gets deformed at the order R^{-1}, although naively one would expect R^{-2}. Then we get:

$$(\Delta_{\text{wild}} Q)\ S_{\text{flat}} = 0 \tag{1.9}$$

$$(\Delta_{\text{wild}} Q)\ U_{\text{AdS RR 5-form}} = Q_{\text{flat}}\ U_\beta \tag{1.10}$$

This means that U_β is not even BRST closed.

In other words, when studying the flat space limit of this β-deformation, it only makes sense to consider the deforming vertex up to the relative order R^{-1}. But as we see in eq. (1.7), the beta-deformation starts only at the relative order R^{-2} (the term with U_β). In this sense, the flat space limit of our beta-deformation only affects the BRST operator *without touching the action.*

1.5 Deformations of the normal form of the action

However, as explained in [1], in order to read the SUGRA fields from the worldsheet action, we have to first bring the action to some special *normal form.* The definition of this normal form does depend on the BRST operator; therefore the *normal form* of the action does get deformed in the flat space limit. We will discuss this in section 6. We will find that the leading term in the near-flat space expansion of the nonphysical vertex would have resembled the linear dilaton, but differs from it in not being worldsheet parity invariant. This leads to the axial asymmetry of the vector components of the worldsheet Weyl connection, and consequently to the *anomaly* at the one-loop level.

Conclusion. A classical Type IIB background is not completely characterized by requiring the BRST symmetry of the classical worldsheet theory; it is also necessary to require the vanishing of the one-loop beta-function.

[2] Usually the action is defined with the overall coefficient R^2; then the flat space term is of the order 1. We prefer to define the action so that the flat space is of the order R^{-2}.

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Open question. It is not clear to us if there exists such nonphysical vertices in the backgrounds other than flat space and $AdS_5 \times S^5$. We suspect that, even forgetting about the quantum anomaly, the non-physical deformation of the classical sigma-model in curved space-time will be obstructed at the higher orders of the deformation parameter. (Although in flat space-time, section 4.1.6 shows that it is actually unobstructed.)

In the rest of the paper we will provide technical details.

2 $AdS_5 \times S^5$ and its β-deformations

2.1 Pure spinor formalism in $AdS_5 \times S^5$

2.1.1 The action

The action is:

$$S_{\text{AdS}} = \int d^2z \, \text{Str} \left(\frac{1}{2} J_{\bar{2}+} J_{\bar{2}-} + \frac{3}{4} J_{\bar{1}+} J_{\bar{3}-} + \frac{1}{4} J_{\bar{3}+} J_{\bar{1}-} + \texttt{[ghosts]} \right) \tag{2.1}$$

where the currents are $J = -dgg^{-1}$, $g = e^{\theta} e^{x}$, and the indices with the bar denote the $\mathbf{Z}_4$ grading.

2.1.2 Parity symmetry

There is a parity symmetry Σ:

$$\begin{aligned} \Sigma(\tau^{\pm}) &= \tau^{\mp} \\ \Sigma(g) &= SgS^{-1} \end{aligned} \tag{2.2}$$

where S is an element of $\text{PSU}(2,2|4)$ given by the following $(4|4) \times (4|4)$-matrix:

$$S = \text{diag}(e^{i\pi/4}, e^{i\pi/4}, e^{i\pi/4}, e^{i\pi/4}, e^{-i\pi/4}, e^{-i\pi/4}, e^{-i\pi/4}, e^{-i\pi/4}) \tag{2.3}$$

Under this symmetry:

$$\Sigma(J_{\bar{n}+}) = SJ_{(-\bar{n})-}S^{-1} \tag{2.4}$$

In particular:

$$\Sigma(J_{\bar{3}+}) = SJ_{\bar{1}-}S^{-1} \tag{2.5}$$

A generic string theory sigma-model does not have any parity symmetry. Parity invariance is a property of those backgrounds which only involve the metric, axion-dilaton and the RR 5-form field strength, but neither the B-field nor the RR 3-form. $AdS_5 \times S^5$ is one of such parity-invariant backgrounds.

2.2 β-deformations

The β-deformations are the simplest deformations of the pure spinor action. The corresponding integrated vertex is just the exterior product of two global symmetry currents [3, 8]:

$$S_{\text{AdS}} \longrightarrow S_{\text{AdS}} + \int \varepsilon B^{ab} j_a \wedge j_b \tag{2.6}$$

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where ε is a small parameter measuring the strength of the deformation, and B^{ab} is a constant super-antisymmetric tensor with indices a, b enumerating the generators of the algebra of global symmetries $\mathbf{g} = psu(2,2|4)$. It turns out that when B is of the form $B^{ab} = f^{ab}{}_c A^c$ for some constant A^c, the deformation can be undone by a field redefinition. Therefore the space of linearized β-deformations is:

$$\mathcal{H} = (\mathbf{g} \wedge \mathbf{g})/\mathbf{g} \tag{2.7}$$

2.3 Physical and unphysical deformations

Physical β-deformations have zero internal commutator:

$$\mathcal{H}_{\text{phys}} = (\mathbf{g} \wedge \mathbf{g})_0/\mathbf{g} \tag{2.8}$$

Here $(\mathbf{g} \wedge \mathbf{g})_0$ means the subspace consisting of $\sum_i \xi_i \wedge \eta_i$ such that:

$$\sum_i [\xi_i, \eta_i] = 0 \tag{2.9}$$

Physical deformations describe solutions of linearized SUGRA on the background of $AdS_5 \times S^5$.

It was explained in [3] that the deformations which belong to the complement $\mathcal{H} \backslash \mathcal{H}_{\text{phys}}$ do not correspond to any SUGRA solutions. The spectrum of linearized excitations of SUGRA on $AdS_5 \times S^5$ does not contain states with such quantum numbers. Attempt to naively identify the supergravity fields gives the Ramond-Ramond field strength which is not closed: $dH_{RR} \neq 0$. This contradicts the SUGRA equations of motion.

For example, consider B of the form:

$$B^{ab} = \begin{cases} f^{ab}_c A^c & \text{if both } a \text{ and } b \text{ are even (bosonic) indices} \\ \quad 0 & \text{otherwise} \end{cases} \tag{2.10}$$

with some constant $A \in so(6) \subset \mathbf{psu}(2,2|4)$. The corresponding SUGRA solution would be constant in the AdS directions, and would transform in the adjoint representation of $so(6)$ (the rotations of the S^5). But there is no such state in the SUGRA spectrum [9].

Even without consulting [9], that there is no SUGRA solutions with such quantum numbers. Let us study the representations of SUGRA fields, even without equations of motion (off-shell). They are various tensor fields. A tensor field transforms in some representation ρ of the small algebra $so(5) \subset so(6)$ (we are looking only at the S^5 part). According to the Frobenius reciprocity, a representation of $so(6)$ enters as many times as ρ enters into its restriction on $so(5)$. In particular, the adjoint representation of $so(6)$ decomposes as follows:

$$\text{ad}_{so(6)} = \text{ad}_{so(5)} \oplus \text{Vec}_{so(5)} \tag{2.11}$$

But Type IIB SUGRA does not contain vectors, and the only 2-forms are: $*_5 H_{NSNS}$ and $*_5 H_{RR}$. In the space of 2-forms on S^5, the only subspace transforming in the adjoint of $so(6)$ are $dX_i \wedge dX_j$ where S^5 is parametrized by $X_1^2 + \ldots + X_6^2 = 1$. But H_{NSNS} and H_{RR} are closed 3-forms, while $*_5(dX_i \wedge dX_j)$ is not.

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3 Pure spinor formalism in flat space

3.1 Action, BRST transformation, supersymmetry and parity

The action in flat space is:

$$S_{\rm flat} = \int d\tau^+ d\tau^- \left[\frac{1}{2}\partial_+ x^m \partial_- x^m + p_+\partial_-\theta_L + p_-\partial_+\theta_R + \right. \\ \left. + \; w_+\partial_-\lambda_L + w_-\partial_+\lambda_R \right] \tag{3.1}$$

where $x, \theta_{L,R}$ are matter fields and λ are pure spinor ghosts, and $p_\pm, w_\pm$ are their conjugate momenta. The BRST transformation is generated by the BRST charge:

$$q_{\rm flat} = \int d\tau^+ \lambda_L d_+ + \int d\tau^- \lambda_R d_- \tag{3.2}$$

where $d_\pm$ is some composed field built from $p_\pm$, θ, $\partial_\pm x$, the explicit expressions are in section 5.2. The corresponding symmetry (called "BRST transformation") acts in the following way:

$$\begin{aligned}
\epsilon Q_{\rm flat}\ \theta_{L,R} &= \epsilon\lambda_{L,R} \\
\epsilon Q_{\rm flat}\ x^m &= \frac{1}{2}\left((\epsilon\lambda_L\Gamma^m\theta_L) + (\epsilon\lambda_R\Gamma^m\theta_R)\right) \\
\epsilon Q_{\rm flat}\ \lambda_{L,R} &= 0 \\
\epsilon Q_{\rm flat}\ w_\pm &= \epsilon d_\pm \\
\epsilon Q_{\rm flat}\ d_+ &= \Pi^m_+\Gamma_m\epsilon\lambda_L \\
\epsilon Q_{\rm flat}\ d_- &= \Pi^m_-\Gamma_m\epsilon\lambda_R
\end{aligned} \tag{3.3}$$

or in compact notations:

$$\epsilon Q_{\rm flat} = \epsilon\lambda_L\frac{\partial}{\partial\theta_L} + \epsilon\lambda_R\frac{\partial}{\partial\theta_R} + \frac{1}{2}\left((\epsilon\lambda_L\Gamma^m\theta_L) + (\epsilon\lambda_R\Gamma^m\theta_R)\right)\frac{\partial}{\partial x^m} + \\ + \epsilon d_+\frac{\partial}{\partial w_+} + \epsilon d_-\frac{\partial}{\partial w_-} + (\Pi^m_+\Gamma_m\epsilon\lambda_L)_{\hat\alpha}\frac{\partial}{\partial d_{\hat\alpha +}} + (\Pi^m_-\Gamma_m\epsilon\lambda_R)_\alpha\frac{\partial}{\partial d_{\alpha -}} \tag{3.4}$$

We will use the small-case q for both the conserved charge and the capital Q for the corresponding symmetry action. The BRST operator $\epsilon Q_{\rm flat}$ has the following key properties:

1. It is a symmetry of the action
2. It is nilpotent: $Q^2_{\rm flat} = 0$ (up to gauge tranformations)

Besides the BRST invariance, the flat space action is also invariant under the super-Poincare transformations. In particular, there are supersymmetries t^3_α and $t^1_{\dot\alpha}$ which act as follows:

$$\begin{aligned}
\kappa^\alpha_L t^3_\alpha &= \kappa^\alpha_L\frac{\partial}{\partial\theta^\alpha_L} - \frac{1}{2}(\kappa_L\Gamma^m\theta_L)\frac{\partial}{\partial x^m} \\
\kappa^{\hat\alpha}_R t^1_{\hat\alpha} &= \kappa^{\hat\alpha}_R\frac{\partial}{\partial\theta^{\hat\alpha}_R} - \frac{1}{2}(\kappa_R\Gamma^m\theta_R)\frac{\partial}{\partial x^m}
\end{aligned} \tag{3.5}$$

where κ^α_L and $\kappa^{\hat\alpha}_R$ are constant Grassmann numbers, enumerating the SUSY generators.

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The flat space theory has parity invariance, as eq. (2.2) of $AdS_5 \times S^5$. It exchanges τ^+ with τ^- and θ_L with θ_R.

3.2 Using AdS notations in flat space

Even in the strict flat space limit, it is still convenient to use the AdS notations. For example:

$$[\theta_L, \partial_+\theta_L]^m = (\theta_L \Gamma^n \partial_+ \theta_L) \tag{3.6}$$

$$[\theta_L, \theta_R]^{[mn]} = (\theta_L F^{mnpqr} \Gamma_{pqr} \theta_R) \tag{3.7}$$

$$[B_2, \theta_L]^{\hat{\alpha}} = \left(\widehat{F} B_2^m \Gamma_m \theta_L\right)^{\hat{\alpha}} \tag{3.8}$$

where F^{mnpqr} is the RR 5-form field strength of $AdS_5 \times S^5$ in the flat space limit. We will also put $\mathbf{Z}_4$ indices on the currents; the Lorentz currents will be denoted $j_{0\pm}$, the translations $j_{2\pm}$, and the supersymmetries $j_{3\pm}$ and $j_{1\pm}$.

4 Deformations of the flat space structures

4.1 Deforming Q_{flat} keeping S_{flat} undeformed

4.1.1 Construction of the deformation

Consider the following infinitesimal deformation of the BRST charge, parametrized by the constant bispinors $B_R^{\hat{\alpha}\hat{\beta}}$ and $B_L^{\alpha\beta}$:

$$\begin{aligned} \epsilon q_B &= \epsilon q_{\text{flat}} + \epsilon \Delta_{\text{wild}} q \\ \texttt{where } \epsilon \Delta_{\text{wild}} q &= \varepsilon \int \left((\theta_L \Gamma_m \epsilon \lambda_L) \Gamma^m_{\alpha\gamma} \theta_L^\gamma \right) B_L^{\alpha\beta} S_{\beta+} d\tau^+ + \\ &\quad + \varepsilon \int \left((\theta_R \Gamma_m \epsilon \lambda_R) \Gamma^m_{\hat{\alpha}\hat{\gamma}} \theta_R^{\hat{\gamma}} \right) B_R^{\hat{\alpha}\hat{\beta}} S_{\hat{\beta}-} d\tau^- \end{aligned} \tag{4.1}$$

Notations:

- q_{flat} is the standard flat space BRST charge (3.2).
- $B_R^{\hat{\alpha}\hat{\beta}}$ and $B_L^{\alpha\beta}$ are constant bispinors, $B_L^{\alpha\beta} = B_L^{\beta\alpha}$, $B_R^{\hat{\alpha}\hat{\beta}} = B_R^{\hat{\beta}\hat{\alpha}}$.
- ε is a small parameter, measuring the strength of the deformation; it should not be confused with ϵ — the formal Grassmann number. Note that ε is bosonic and ϵ is fermionic. To the first order in ε the deformed BRST operator is a new nilpotent symmetry of the action.
- $S_{\beta+}$ and $S_{\hat{\beta}-}$ are the holomorphic (left) and the antiholomorphic (right) supersymmetry charges[3] (see eqs. (5.31) and (5.32) for the explicit formulas)

[3]The fact that the supersymmetry charges are holomorphic or antiholomorphic is special to flat space, and is crucial for our construction.

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It follows from the definition that $\Delta_{\rm wild}q$ is a conserved charge. Indeed, on-shell $\partial_- S_{\beta+} = \partial_+ S_{\hat\beta -} = 0$ and $\partial_-\theta_L = \partial_+\theta_R = \partial_-\lambda_L = \partial_+\lambda_R = 0$.

The deformation $\Delta_{\rm wild}q$ consists of the "left" piece (proportional to B_L) and the "right" piece (proportional to B_R). These two pieces provide two separate deformations, the left one and the right one. They are separately well-defined.

4.1.2 Proof that $\Delta_{\rm wild}q$ anticommutes with $q_{\rm flat}$

We will prove this using the Hamiltonian formalism. Let us calculate the Poisson bracket:

$$\{q_{\rm flat},\ \Delta_{\rm wild}q\} = Q_{\rm flat}\ \Delta_{\rm wild}q \tag{4.2}$$

Notice the descent relation for the density of $\Delta_{\rm wild}q$:

$$\begin{aligned}&\epsilon Q_{\rm flat}\left(((\theta_L\Gamma_m\epsilon'\lambda_L)\Gamma^m_{\alpha\gamma}\theta^\gamma_L)\ B_L^{\alpha\beta}S_{\beta+}\right) = \\ &= \partial_+\left(\frac{1}{6}\left((\theta_L\Gamma_m\epsilon'\lambda_L)\Gamma^m_{\alpha\gamma}\theta^\gamma_L\right)\ B_L^{\alpha\beta}\left((\theta_L\Gamma_m\epsilon\lambda_L)\Gamma^m_{\beta\delta}\theta^\delta_L\right)\right)\end{aligned} \tag{4.3}$$

which follows from the descent of the SUSY current:

$$\epsilon Q_{\rm flat}S_{\alpha+} = \partial_+\left(\frac{1}{3}(\theta_L\Gamma_m\epsilon\lambda_L)\Gamma^m_{\alpha\gamma}\theta^\gamma_L\right) \tag{4.4}$$

which can be derived by an explicit calculation, or as a limit of the similar relation in the $AdS_5\times S^5$ sigma-model derived in [10] and reviewed in [3]. Let us introduce the notation:

$$v_{L\alpha}(\epsilon) = \left((\theta_L\Gamma_m\epsilon\lambda_L)\Gamma^m_{\alpha\gamma}\theta^\gamma_L\right) \tag{4.5}$$

With this notations we have:

$$\epsilon Q_{\rm flat}\left(v_{L\alpha}(\epsilon')B_L^{\alpha\beta}S_{\beta+}\right) = \frac{1}{6}\partial_+\left(v_{L\alpha}(\epsilon')B_L^{\alpha\beta}v_{L\beta}(\epsilon)\right) \tag{4.6}$$

There is a similar descent relation for the charge density of the right deformation. Eq. (4.6) means that the $Q_{\rm flat}$-variation of the density of $\Delta_{\rm wild}q$ is a total derivative, and this implies:

$$\{q_{\rm flat},\ \Delta_{\rm wild}q\} = Q_{\rm flat}\ \Delta_{\rm wild}q = 0 \tag{4.7}$$

4.1.3 Deformation of the BRST transformation

This deformation of the BRST charges corresponds to the following deformation of the BRST transformation:

$$\epsilon Q_B = \epsilon Q_{\rm flat} + \Delta_{\rm wild}Q \tag{4.8}$$

$$\begin{aligned}\texttt{where}\ \Delta_{\rm wild}Q = &\ \varepsilon B_R^{\hat\alpha\hat\beta}\left((\theta_R\Gamma_m\epsilon\lambda_R)\Gamma^m_{\hat\alpha\hat\gamma}\theta^{\hat\gamma}_R\right)\ t^1_{\hat\beta} + \varepsilon B_L^{\alpha\beta}\left((\theta_L\Gamma_m\epsilon\lambda_L)\Gamma^m_{\alpha\gamma}\theta^\gamma_L\right)\ t^3_\beta + \\ &+ k_{\alpha+}\frac{\partial}{\partial p_{\alpha+}} + l_{\alpha+}\frac{\partial}{\partial w_{\alpha+}} + k_{\hat\alpha-}\frac{\partial}{\partial p_{\hat\alpha-}} + l_{\hat\alpha-}\frac{\partial}{\partial w_{\hat\alpha-}}\end{aligned} \tag{4.9}$$

where $t^1_{\hat\beta}$ and t^3_β are the right and left SUSY generators given by eq. (3.5), and $k_{\alpha+}, l_{\alpha+}, k_{\hat\alpha-}, l_{\hat\alpha-}$ define some infinitesimal shifts of the momenta $p_\pm, w_\pm$. We will not need the explicit formula for these shifts; they are canonically defined in terms of the shifts of x and θ generated by $t^1_{\hat\beta}$ and t^3_β.

4.1.4 When such a deformation can be undone by a field redefinition?

Sufficient condition. Consider the special case when B_L satisfies:

$$\Gamma^m_{\alpha\beta} B_L^{\alpha\beta} = 0 \tag{4.10}$$

In this case exists W_L:

$$v_{L\alpha}(\epsilon')B_L^{\alpha\beta}v_{L\beta}(\epsilon) = \epsilon Q_{\text{flat}}\left(\epsilon' W_L\right) \tag{4.11}$$

The structure of W_L is $[\theta_L^5 \lambda_L]$. This implies:

$$Q_{\text{flat}}\left(v_{L\alpha}(\epsilon')B_L^{\alpha\beta}S_{\beta+} - \partial_+(\epsilon' W_L)\right) = 0 \tag{4.12}$$

Because the cohomology in conformal dimension 1 is trivial, this implies the existence of y_{L+}:

$$v_{L\alpha}(\epsilon')B_L^{\alpha\beta}S_{\beta+} = \partial_+(\epsilon' W_L) + \epsilon' Q_{\text{flat}} y_{L+} \tag{4.13}$$

(See the discussion in appendix A.) We observe that $\partial_- y_{L+} \underset{\text{flat}}{\simeq} 0$. Thus y_{L+} is a conserved current of the flat space theory generating some transformation Y_L. We have therefore:

$$Q_B = Q_{\text{flat}} + [Y_L, Q_{\text{flat}}] \tag{4.14}$$

Therefore if (4.10) then the deformation $Q_B \to Q_{\text{flat}}$ is trivial.

Necessary condition. Let us assume that exists a vector field Y_L satisfying eq. (4.14). Let us assume that Y_L is a symmetry of the S_{flat}; in the next section 4.1.5 we will give a proof without this assumption. Then the conserved current $v_{L\alpha}B_L^{\alpha\beta}S_{\beta+}$ corresponding to $Q_B^{(1)}$ satisfies:

$$v_{L\alpha}B_L^{\alpha\beta}S_{\beta+} = Y_L j_{\text{flat BRST}+} + \partial_+\phi \tag{4.15}$$

for some holomorphic ϕ. Using that $Q_{\text{flat}} j_{\text{flat BRST}+} = 0$, this implies:

$$Q_{\text{flat}}\left(v_{L\alpha}B_L^{\alpha\beta}S_{\beta+}\right) = \partial_+\left(Q_{\text{flat}}\phi\right) \tag{4.16}$$

Therefore:

$$v_{L\alpha}B_L^{\alpha\beta}v_{L\beta} = Q_{\text{flat}}\phi \tag{4.17}$$

In the rest of this paragraph we will prove that this is only possible when (4.10). Indeed, suppose that (4.10) is not satisfied. Without loss of generality, we can assume: $B_L^{\alpha\beta} = B^m\Gamma_m^{\alpha\beta}$. We want to prove that $(v_L \hat{B} v_L)$ represents a nonzero cohomology class of Q_{flat}. Remember that Q_{flat} is defined in (3.3). Let us formally split x into x_L and x_R:

$$x^m = x_L^m + x_R^m \tag{4.18}$$

$$\epsilon Q_{\text{flat}} x_L^m = \frac{1}{2}(\epsilon\lambda_L\Gamma^m\theta_L) \tag{4.19}$$

$$\epsilon Q_{\text{flat}} x_R^m = \frac{1}{2}(\epsilon\lambda_R\Gamma^m\theta_R) \tag{4.20}$$

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Let us extend the BRST complex[4] by including functions of x_L and x_R (and not just of their sum). Then $(v_L \hat{B} v_L)$ is BRST trivial:

$$(v_L \hat{B} v_L) = Q_{\text{flat}} \mathcal{A} \tag{4.21}$$

$$\text{where } \mathcal{A} = A_m(x_L)(\theta_L \Gamma^m \lambda_L) + (dA)_{mn}[\theta_L^3 \lambda_L]^{[mn]} + \dots \tag{4.22}$$

where $A_m(x_L)$ is such that:

$$d * dA = *B \tag{4.23}$$

In other words, A is the Maxwell field created by the constant charge density B. The question is:

- Is it possible to correct $\mathcal{A}$ by adding to it something Q_{flat}-closed, so that the corrected $\mathcal{A}$ depends on x_L and x_R only through $x = x_L + x_R$?

If this is possible then $(v_L \hat{B} v_L)$ is Q_{flat} exact. We will now prove that it is not possible to make such a correction of $\mathcal{A}$, and therefore $(v_L \hat{B} v_L)$ is cohomologically nontrivial.

A function of $x_L, x_R, \theta_L, \theta_R, \lambda_L, \lambda_R$ can be written in terms of $x, \theta_L, \theta_R, \lambda_L, \lambda_R$ if and only if it is annihilated by $y^m \left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m}\right)$ for any constant vector y^m. Notice that:

$$Q_{\text{flat}} \left[y^m \left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m} \right) \mathcal{A} \right] = 0 \tag{4.24}$$

— this is because $y^m \left(\frac{\partial}{\partial x_L^m} - \frac{\partial}{\partial x_R^m}\right)$ commutes with Q_{flat} and annihilates $(v_L \hat{B} v_L)$. Let us consider the following solution of (4.23):

$$A_\mu = \frac{1}{18} x^2 B_\mu \tag{4.25}$$

Then $F_{\mu\nu} = \frac{1}{9}(x_\mu B_\nu - x_\nu B_\mu)$. We see that $(y\partial_{x_L} - y\partial_{x_R})\mathcal{A}$ represents a nontrivial cohomology class of Q_{flat}, corresponding to the Maxwell field of the constant field strength $y \wedge B$. Now the question is:

- Is it possible to obtain this cohomology class by acting with $(y\partial_{x_L} - y\partial_{x_R})$ on some cohomology class $\mathcal{Z}$ of Q_{flat}?

In other words, is it possible that exists $\mathcal{Z}$ such that:

$$(y\partial_{x_L} - y\partial_{x_R})\mathcal{A} = (y\partial_{x_L} - y\partial_{x_R})\mathcal{Z} \tag{4.26}$$

$$Q_{\text{flat}} \mathcal{Z} = 0 \tag{4.27}$$

(such a $\mathcal{Z}$ will necessarily be nontrivial in the cohomology of Q_{flat})? If and only if this were possible, then we could modify $\mathcal{A}$ by subtracting from it a representative of $\mathcal{Z}$ (and since $\mathcal{Z}$ is closed, this will not change the defining property (4.21)) so that the modified $\mathcal{A}$ depends on x_L and x_R through $x = x_L + x_R$. Then eq. (4.21) would have implied that $(v_L \hat{B} v_L)$ is BRST exact. We will now prove that this is impossible.

[4] I want to thank M. Movshev for teaching me this trick.

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Suppose that exists $\mathcal{Z}$ such that (4.26) and (4.27). As we already said, since the Q_{flat}-cohomology class of $(y\partial_{x_L} - y\partial_{x_R})\mathcal{A}$ is nontrivial, $\mathcal{Z}$ should be also nontrivial in Q_{flat}-cohomology. Modulo Q_{flat}-exact terms $\mathcal{Z}$ has to be of the following form:

$$\mathcal{Z} = Z_{Lm}(x_L, x_R)(\theta_L \Gamma^m \lambda_L) + Z_{Rm}(x_L, x_R)(\theta_R \Gamma^m \lambda_R) + [x\lambda\theta^3] + [\lambda\theta^5] \tag{4.28}$$

where Z_{Lm} and Z_{Rm} are quadratic in x. For (4.28) to be Q_{flat}-closed we need:

$$\partial_{x_R^n} Z_{Lm} = \partial_{x_L^m} Z_{Rn} \tag{4.29}$$

Since both Z_{Lm} and Z_{Rn} are quadratic polynomials in (x_L, x_R), let us introduce the notations:

$$\begin{aligned} Z_{Lm} &= Z_{Lm,LL} + Z_{Lm,LR} + Z_{Lm,RR} \\ Z_{Rm} &= Z_{Rm,LL} + Z_{Rm,LR} + Z_{Rm,RR} \end{aligned} \tag{4.30}$$

where e.g. $Z_{Rn,LL}$ is the term with $x_L x_L$ in Z_{Rm}, etc. Eq. (4.29) implies that the term with $x_R x_L$ in Z_{Lm} and the term with $x_L x_L$ in Z_{Rm} can be gauged away by $Q_{\text{flat}}(2Z_{Rn,LL}x_R^n)$:

$$Z_{Lm,LR}(x_L, x_R)(\theta_L \Gamma^m \lambda_L) + Z_{Rm,LL}(x_L, x_R)(\theta_R \Gamma^m \lambda_R) \ = Q_{\text{flat}}(2Z_{Rn,LL}x_R^n)$$

Similarly, the terms with $x_R x_R$ in Z_{Lm} plus terms with $x_L x_R$ in Z_{Rm} are $Q_{\text{flat}}(2Z_{Ln,RR}x_L^n)$, where $Z_{Ln,RR}$ is the coefficient of $x_R x_R$ in Z_{Ln}. After such a gauge transformation, we are left with:

$$\mathcal{A}' = Z_{Lm}(x_L)(\theta_L \Gamma^m \lambda_L) + Z_{Rm}(x_R)(\theta_R \Gamma^m \lambda_R) + [x\lambda\theta^3] + [\lambda\theta^5] \tag{4.31}$$

Now we observe that this corresponds to a pair of Maxwell fields with the field strength linearly dependent on the spacetime coordinates. One of these two Maxwell fields corresponds to Z_{Lm}, and another to Z_{Rm}. Up to gauge transformations, both transform in the traceless part of the $\begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$ of $so(1,9)$. At the same time, the cohomology class of $(y\partial_{x_L} - y\partial_{x_R})\mathcal{A}$ is parametrized by the vector B, therefore it transforms in a vector (i.e. $\square$) of $so(1,9)$. This implies that (4.26) is impossible.

4.1.5 Another proof of the necessary condition for triviality

Let us take $B^{\alpha\beta} = B^m \Gamma_m^{\alpha\beta}$. Suppose that exists an infinitesimal field redefinition Y_L such that (4.14). Let us study the action of Y_L on λ_L. We observe:

$$Q_{\text{flat}}\theta_L = \lambda_L \tag{4.32}$$

$$(Q_B - Q_{\text{flat}})\theta_L = \hat{B}\Gamma^m \theta_L(\theta_L \Gamma^m \lambda_L) \tag{4.33}$$

Therefore in order to satisfy (4.14) we should have:

$$Y_L \lambda_L = \hat{B}\Gamma^m \theta_L(\theta_L \Gamma^m \lambda_L) + Q_{\text{flat}}\Xi \tag{4.34}$$

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for some Ξ (we have $\Xi = Y_L\theta^\alpha_L$). Moreover, we should satisfy the pure spinor constraint:

$$(\lambda_L\Gamma^k Y_L\lambda_L) = 0 \tag{4.35}$$

Notice that $Y_L\lambda_L$ is necessarily Q_{flat}-closed, and that Ξ is necessarily of the form $[\theta^3 B]$. The only expression of the form $[\theta^2\lambda B]$ which satisfies (4.35) would have been:

$$Y_L\lambda_L = \Gamma^{mn}\lambda B^l(\theta\Gamma_{lmn}\theta) \tag{4.36}$$

but this is not BRST closed and therefore is not of the form (4.34).

Comment. But when B is a 5-form rather than a vector, see eq. (A.19).

Conclusion. Eq. (4.10) is a necessary and sufficient condition for the triviality of the deformation. In other words, the deformation of the flat space BRST operator parametrized by $B_L^{\alpha\beta}$ can be undone by a symmetry of the action if and only if (4.10).

4.1.6 Extension to higher orders

It should be possible to extend the deformation (4.8) to higher orders in ε. Let us for now put $B_R = 0$ in (4.8); that is, restrict ourselves to the "left" deformations only. We get:

$$\begin{aligned}\{Q_B, Q_B\} &= \varepsilon^2\left((\theta\Gamma^m\lambda)\left(\theta\Gamma^m B\frac{\partial}{\partial\theta}\right)\right)^2 = \\ &= \varepsilon^2(\theta\Gamma^m\lambda)(\theta\Gamma^m B\Gamma^n\lambda)\left(\theta\Gamma^n B\frac{\partial}{\partial\theta}\right) - \\ &\quad - \varepsilon^2(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)\left(\theta\Gamma^m B\Gamma^n B\frac{\partial}{\partial\theta}\right)\end{aligned} \tag{4.37}$$

If B is a 5-form, then one can see that this is BRST exact; but in fact we have already seen in section 4.1.4 that in this case Q_B is a trivial deformation of Q_{flat}. If B is a 1-form, then the obstacle is proportional to B_mB_m. To calculate the coefficient, we observe:

$$(\theta\Gamma^m\lambda)(\theta\Gamma^m B\Gamma^n\lambda)\theta\Gamma^n = \frac{1}{2}(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)\theta\Gamma^m B\Gamma^n + Q_{\text{flat}}(\ldots) \tag{4.38}$$

This means:

$$\{Q_B, Q_B\} = -\frac{1}{2}\varepsilon^2(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)\left(\theta\Gamma^m B\Gamma^n B\frac{\partial}{\partial\theta}\right) + [Q_{\text{flat}}, \ldots] \tag{4.39}$$

In Q_{flat} cohomology this is proportional to B_mB_m. To calculate the coefficient of proportionality we can substitute $B\otimes B = \Gamma^k\otimes\Gamma^k$. We get:

$$\{Q_B, Q_B\} = \frac{2}{5}|B|^2\varepsilon^2(\theta\Gamma^m\lambda)(\theta\Gamma^n\lambda)\left(\theta\Gamma^{mn}\frac{\partial}{\partial\theta}\right) + [Q_{\text{flat}}, \ldots] \tag{4.40}$$

Where $|B|^2 = B_mB_m$. When B is a lightlike vector, we can construct $Q_B^{(2)}$ such that the operator:

$$Q'_B = Q_{\text{flat}} + \varepsilon(\theta\Gamma^m\lambda)\left(\theta\Gamma^m B\frac{\partial}{\partial\theta}\right) + \varepsilon^2 Q_B^{(2)} \tag{4.41}$$

which is nilpotent up to the terms of the order ε^3. One can continue this procedure to higher orders in ε. The only invariant which can arise is $|B|^2$. Therefore we conclude that the deformation $Q_{\text{flat}} \to Q_B$ is unobstructed when B is lightlike, i.e. $|B|^2 = 0$.

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4.1.7 Relation to β-deformation

The deformation of the AdS action given by (2.6) preserves the BRST invariance of the action, but actually changes the action of the BRST transformation. Indeed, the deforming vertex is only BRST-closed on-shell:

$$Q_{\text{AdS}}\left(\int B^{ab} j_a \wedge j_b\right) \underset{\text{AdS}}{\simeq} 0 \tag{4.42}$$

where $\underset{\text{AdS}}{\simeq}$ means "up to the equations of motion of the AdS σ-model". Because (4.42) only holds on-shell, the deformed action is not invariant under the orginal BRST transformation, but instead under a deformed BRST transformation. The necessary deformation of the BRST transformation was constructed in [3], where it was called Q_1:

$$\epsilon Q_1 = 4\left(g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g\right)_a \; B^{ab}\, t_b \tag{4.43}$$

Here t_b are generators of $\mathbf{g} = \mathbf{psu}(2,2|4)$. Expanding $\left(g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g\right)_{\bar{1}}$ in powers of x and θ, we get:

$$\left(g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g\right)_{\bar{1}} = \epsilon Q\Psi_1 - \frac{4}{3}[\theta_L,[\theta_L,\epsilon\lambda_L]] \; + \;\ldots \tag{4.44}$$

$$\texttt{where}\;\; \Psi_1 = -\theta_R - [x,\theta_L] + \frac{1}{3}[\theta_R,[\theta_R,\theta_L]] \tag{4.45}$$

where dots stand for the higher order terms. Similarly:

$$\left(g^{-1}(\epsilon\lambda_3 - \epsilon\lambda_1)g\right)_{\bar{3}} = \epsilon Q\Psi_3 + \frac{4}{3}[\theta_R,[\theta_R,\epsilon\lambda_R]] \; + \;\ldots \tag{4.46}$$

$$\texttt{where}\;\; \Psi_3 = \theta_L + [x,\theta_R] - \frac{1}{3}[\theta_L,[\theta_L,\theta_R]] \tag{4.47}$$

We conclude that:

- up to a BRST exact expression ϵQ_1 is identical to ϵQ_B of (4.8). This means that the leading effect in the flat space limit of this particular nonphysical β-deformation is to deform the BRST structure of the flat space action as in Eq (4.8).

4.1.8 Field reparametrization K

Let us consider a particular example of B^{ab}, when the only nonzero component has both upper indices a and b in $\mathbf{g}_1$, and B has the form:

$$B^{\hat{\alpha}\hat{\beta}} = f^{\hat{\alpha}\hat{\beta}}{}_m B^m_{L2} \tag{4.48}$$

In this case:

$$\epsilon Q_1 = -\frac{16}{3}[\, B_{L2}\;,\; [\theta_L,[\theta_L,\epsilon\lambda_L]]\,]^\alpha\; t^3_\alpha + [\epsilon Q\;,\; K_L] \tag{4.49}$$

$$\texttt{where}\;\; K_L = 4[B_{L2},\Psi_1]^\alpha\; t^3_\alpha \tag{4.50}$$

This means that ϵQ_1 is of the form (4.8) after a field reparametrization specified by the vector field K_L.

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Similarly, consider the case when the only nonzero components of B^{ab} are the following:

$$B^{\alpha\beta} = f^{\alpha\beta}{}_m B^m_{R2} \tag{4.51}$$

In this case:

$$\epsilon Q_1 = \frac{16}{3}[\ B_{R2}\ ,\ [\theta_R, [\theta_R, \epsilon\lambda_R]]\ ]^{\hat{\alpha}}\ t^1_{\hat{\alpha}} + [\epsilon Q\ ,\ K_R] \tag{4.52}$$

$$\texttt{where}\ \ K_R = 4[B_{R2}, \Psi_3]^{\hat{\alpha}}\ t^1_{\hat{\alpha}} \tag{4.53}$$

Action of K on S_{AdS}.

$$KS_{\mathrm{AdS}} = -\int d^2\tau\ \mathrm{Str}\left(\partial_+[B_2, \Psi_1]\ j_{1-} - \partial_-[B_2, \Psi_1]\ j_{1+}\right) \tag{4.54}$$

Observe that $j_{1-} = -\partial_-\theta_R + \ldots$ and $j_{1+} = 3\,\partial_+\theta_R + \ldots$ With our definition of $j_\pm$ we have:

$$\xi.S_{\mathrm{AdS}} = -\frac{1}{4}\int d^2\tau\ \mathrm{Str}\left(\partial_+\xi\ j_- - \partial_-\xi\ j_+\right) \tag{4.55}$$

4.2 Deforming $(S_{\mathrm{flat}},\ Q_{\mathrm{flat}})$ to $(S_{\mathrm{AdS}},\ Q_{\mathrm{AdS}})$

Going from flat space to AdS changes the action, by turning on the RR five-form field strength. To describe the corresponding deformation of the action it is useful to introduce a small parameter $1/R$, which corresponds to the inverse radius of the AdS space. The scaling of the basic fields is as follows:

$$\begin{aligned} &x \simeq R^{-1}, \qquad &&\theta_{L,R} \simeq R^{-1/2}, \qquad &&p_\pm \simeq d_\pm \simeq R^{-3/2}, \\ &\lambda_{L,R} \simeq R^{-1/2}, \qquad &&w_\pm \simeq R^{-3/2} \end{aligned} \tag{4.56}$$

With these notations the flat action (3.1) is of the order R^{-2}. (Usually there is an overall coefficient R^{-2} in front of the action, then the action is of the order 1. But we will prefer to omit this overall coefficient.)

The RR five-form deforms the action as follows:

$$S_{\mathrm{flat}} \to S_{\mathrm{flat}} + \int F^{\alpha\hat{\beta}} d_\alpha d_{\hat{\beta}} + \ldots \tag{4.57}$$

where $\ldots$ is for terms containing θ. We observe that the deformation term is of the order R^{-3} (while the S_{flat} is of the order R^{-2}).

We will denote the AdS deformation vertex U_{AdS}:

$$U_{\mathrm{AdS}} = F^{\alpha\hat{\beta}} d_\alpha d_{\hat{\beta}} +\ [\text{terms with }\theta] \tag{4.58}$$

(The complete formula is (5.33).) Once again, observe that the flat space action is of the order R^{-2}, and the deformation U_{AdS} is of the order R^{-3}.

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4.3 Interplay between the two deformations

We have considered two deformations of the flat space superstring: the deformation (4.8) which leaves the action invariant and only changes the BRST structure, and the deformation from flat space to $AdS_5 \times S^5$. Let us look at the interplay between these two deformations. The action of Q_{flat} on U_{AdS} is a total derivative on the equations of motion of S_{flat}. But the deformed Q generally speaking acts nontrivially:

$$Q_{\text{flat}} \int U_{\text{AdS}} \underset{\text{flat}}{\simeq} 0 \tag{4.59}$$

$$Q_B \int U_{\text{AdS}} \underset{\text{flat}}{\simeq} R^{-4} \tag{4.60}$$

where $\underset{\text{flat}}{\simeq}$ means equality up to the equations of motion of flat space. In the next section we will see that (4.60) is important for understanding the flat space limit.

5 Flat space limit of the β-deformation vertices

5.1 Flat space limit of the $AdS_5 \times S^5$ sigma-model

5.1.1 Coset space and BRST operator

We choose the following parametrization of the PSU$(2,2|4)/(\text{SO}(1,4)\times\text{SO}(5))$ coset space:

$$g = e^{\theta} e^{X} \tag{5.1}$$

The action of the BRST operator on the matter fields:

$$\epsilon Q g \;=\; \epsilon(\lambda_L + \lambda_R) g + \omega(\epsilon) g \tag{5.2}$$

where $\omega(\epsilon)$ is some compensating $\text{SO}(1,4) \times \text{SO}(5)$ gauge transformation.

In terms of θ and x:

$$\begin{aligned}
\epsilon Q = \epsilon\lambda_L \frac{\partial}{\partial\theta_L} + \epsilon\lambda_R \frac{\partial}{\partial\theta_R} + \frac{1}{2}\left([\epsilon\lambda_L, \theta_L] + [\epsilon\lambda_R, \theta_R]\right) \frac{\partial}{\partial X} -& \\
- \frac{1}{6}[\theta_L, [\theta_L, \epsilon\lambda_R]] \frac{\partial}{\partial\theta_L} - \frac{1}{6}[\theta_L, [\theta_R, \epsilon\lambda_L]] \frac{\partial}{\partial\theta_L} +& \\
+ \frac{1}{3}[\theta_R, [\theta_L, \epsilon\lambda_L]] \frac{\partial}{\partial\theta_L} + \frac{1}{3}[\theta_R, [\theta_R, \epsilon\lambda_R]] \frac{\partial}{\partial\theta_L} +& \\
- \frac{1}{6}[\theta_R, [\theta_R, \epsilon\lambda_L]] \frac{\partial}{\partial\theta_R} - \frac{1}{6}[\theta_R, [\theta_L, \epsilon\lambda_R]] \frac{\partial}{\partial\theta_R} +& \\
+ \frac{1}{3}[\theta_L, [\theta_R, \epsilon\lambda_R]] \frac{\partial}{\partial\theta_R} + \frac{1}{3}[\theta_L, [\theta_L, \epsilon\lambda_L]] \frac{\partial}{\partial\theta_R} +&
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
+ \frac{1}{24}[\theta_L, [\theta_L, [\theta_R, \epsilon\lambda_L]]] \frac{\partial}{\partial X} + \frac{1}{24}[\theta_L, [\theta_R, [\theta_L, \epsilon\lambda_L]]] \frac{\partial}{\partial X} +& \\
+ \frac{1}{24}[\theta_R, [\theta_L, [\theta_L, \epsilon\lambda_L]]] \frac{\partial}{\partial X} + \frac{1}{24}[\theta_R, [\theta_R, [\theta_R, \epsilon\lambda_L]]] \frac{\partial}{\partial X} +& \\
+ \frac{1}{24}[\theta_R, [\theta_R, [\theta_L, \epsilon\lambda_R]]] \frac{\partial}{\partial X} + \frac{1}{24}[\theta_R, [\theta_L, [\theta_R, \epsilon\lambda_R]]] \frac{\partial}{\partial X} +& \\
+ \frac{1}{24}[\theta_L, [\theta_R, [\theta_R, \epsilon\lambda_R]]] \frac{\partial}{\partial X} + \frac{1}{24}[\theta_L, [\theta_L, [\theta_L, \epsilon\lambda_R]]] \frac{\partial}{\partial X} + \ldots&
\end{aligned} \tag{5.4}$$

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In this formula, the first line is of the order 1, and the following lines are of the order R^{-1}, and the dots stand for the terms of the order $O(R^{-2})$. The currents:

$$-J = dgg^{-1} = e^{\theta}(de^{X}e^{-X})e^{-\theta} + de^{\theta}e^{-\theta} = \tag{5.5}$$
$$= e^{\theta}\left(dX + \frac{1}{2}[X, dX]\right)e^{-\theta} + d\theta + \frac{1}{2}[\theta, d\theta] + \frac{1}{6}[\theta,\theta,d\theta] + \frac{1}{24}[\theta,\theta,\theta,d\theta] + \dots$$

$$-J_{\bar{3}} = d\theta_L + [\theta_R, dX] + \frac{1}{6}[\theta,\theta,d\theta]_L + \dots \tag{5.6}$$

$$-J_{\bar{2}} = dX + \frac{1}{2}[\theta, d\theta]_{\bar{2}} + \frac{1}{2}[\theta,[\theta,dX]]_{\bar{2}} + \frac{1}{24}[\theta,\theta,\theta,d\theta]_{\bar{2}} + \dots \tag{5.7}$$

$$-J_{\bar{1}} = d\theta_R + [\theta_L, dX] + \frac{1}{6}[\theta,\theta,d\theta]_R + \dots \tag{5.8}$$

The action (2.1) modulo terms of the order R^{-3} and higher is:

$$S = \int d^2\tau \left[R^{-1}\partial_+\theta_R\partial_-\theta_L \; + \; \frac{1}{2}R^{-2}\partial_+ x\partial_- x + R^{-2}\,(\,L_3 + L_4) + \dots\right] \tag{5.9}$$

where:

$$L_3 = -\frac{1}{2}\,(\,[\theta_R, \partial_+\theta_R]\,,\,\partial_- x\,) - \frac{1}{2}\,(\,\partial_+ x\,,\,[\theta_L, \partial_-\theta_L]\,) \tag{5.10}$$

$$L_4 = -\frac{1}{24}([\theta_L,\partial_+\theta_L],[\theta_L,\partial_-\theta_L]) - \frac{1}{24}([\theta_R,\partial_+\theta_R],[\theta_R,\partial_-\theta_R]) + \tag{5.11}$$

$$-\frac{1}{12}([\theta_R,\partial_+\theta_R],[\theta_L,\partial_-\theta_L]) - \tag{5.12}$$

$$-\frac{1}{6}([\theta_R,\partial_+\theta_L],[\theta_R,\partial_-\theta_L]) - \frac{1}{6}([\theta_L,\partial_+\theta_R],[\theta_L,\partial_-\theta_R]) - \tag{5.13}$$

$$-\frac{1}{3}([\theta_L,\partial_+\theta_R],[\theta_R,\partial_-\theta_L]) \tag{5.14}$$

5.1.2 First order formalism

We get rid of the leading term $R^{-1}\partial_+\theta_R\partial_-\theta_L$ using the first order formalism:

$$S = \int d^2\tau \left[R^{-2}(\tilde{p}_{1+}\partial_-\theta_L) + R^{-2}(\tilde{p}_{3-}\partial_+\theta_R) - R^{-3}(\tilde{p}_{1+}\tilde{p}_{3-}) + \right.$$
$$\left. + \frac{1}{2}R^{-2}\partial_+ x\partial_- x + R^{-2}\,(\,L_3 + L_4) + \dots\right] \tag{5.15}$$

where dots stand for the terms of the higher order in R^{-1} (including terms the order R^{-3}, of which the one which depends on $\tilde{p}$, namely $R^{-3}(\tilde{p}_{1+}\tilde{p}_{3-})$, we put explicitly on the first line). Integrating out $\tilde{p}_{\pm}$:

$$\tilde{p}_{1+} = R\partial_+\theta_R\;, \quad \tilde{p}_{3-} = R\partial_-\theta_L \tag{5.16}$$

generates $R^{-1}\partial_+\theta_R\partial_-\theta_L$ and brings us back to (5.9).

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Importantly, we can remove the leading nonlinear terms $R^{-2}(L_3+L_4)$ by a redefinition of $\tilde{p}$. (Otherwize the flat space limit would not have been a free theory.) It is done as follows:

$$\begin{aligned} p_{1+} =& \tilde{p}_{1+} + \frac{1}{2}[\theta_L, \partial_+ x] + \\ &+ \frac{1}{24}[\theta_L,[\theta_L,\partial_+\theta_L]] + \frac{1}{24}[\theta_L,[\theta_R,\partial_+\theta_R]] + \\ &+ \frac{1}{6}[\theta_R,[\theta_R,\partial_+\theta_L]] + \frac{1}{6}[\theta_R,[\theta_L,\partial_+\theta_R]] \end{aligned} \tag{5.17}$$

$$\begin{aligned} p_{3-} =& \tilde{p}_{3-} + \frac{1}{2}[\theta_R, \partial_- x] + \\ &+ \frac{1}{24}[\theta_R,[\theta_R,\partial_-\theta_R]] + \frac{1}{24}[\theta_R,[\theta_L,\partial_-\theta_L]] + \\ &+ \frac{1}{6}[\theta_L,[\theta_L,\partial_-\theta_R]] + \frac{1}{6}[\theta_L,[\theta_R,\partial_-\theta_L]] \end{aligned} \tag{5.18}$$

After these changes of variables, the leading terms in the action are:

$$S = \int d^2\tau \, \left[R^{-2}(p_{1+}\partial_-\theta_L) + R^{-2}(p_{3-}\partial_+\theta_R) + \frac{1}{2}R^{-2}\partial_+ x \partial_- x \right] \tag{5.19}$$

5.2 Relation between $J_\pm$ and $d_\pm$

We observe that in the flat space limit J_{3-} and J_{1+} go like $R^{-3/2}$. We identify:

$$J_{1+} = -\, d_+ + O(R^{-5/2}) \tag{5.20}$$

$$J_{3-} = -\, d_- + O(R^{-5/2}) \tag{5.21}$$

In terms of x and θ, at the order $R^{-3/2}$:

$$\begin{aligned} J_{1+} =& -\partial_+\theta_R - [\theta_L, \partial_+ x] - \\ &- \frac{1}{6}[\theta_L,[\theta_L,\partial_+\theta_L]] - \frac{1}{6}[\theta_R,[\theta_R,\partial_+\theta_L]] = \\ =& -p_{1+} - \frac{1}{2}[\theta_L,\partial_+ x] - \frac{1}{8}[\theta_L,[\theta_L,\partial_+\theta_L]] \end{aligned} \tag{5.22}$$

5.3 Global symmetry currents

The matter contribution into the global symmetry currents:

$$-j_+ = g^{-1}\left(J_{\bar{3}+} + 2J_{\bar{2}+} + 3J_{\bar{1}+}\right) g \tag{5.23}$$

$$j_- = g^{-1}\left(3J_{\bar{3}-} + 2J_{\bar{2}-} + J_{\bar{1}-}\right) g \tag{5.24}$$

For example consider the global symmetry currents j_{3+} and j_{3-}.

Up to $O(R^{-7/2})$ and up to terms which do not contain $\partial_+\theta_R$:

$$\begin{aligned} j_{3+} &= \partial_+\Psi_3 + 4[\partial_+\theta_R, x] + 2[\theta_L,[\theta_L,\partial_+\theta_R]] + \frac{2}{3}[\theta_R,[\theta_R,\partial_+\theta_R]] + \ldots \\ \texttt{where}\ \Psi_3 &= \theta_L + [x,\theta_R] - \frac{1}{3}[\theta_L,[\theta_L,\theta_R]] \end{aligned} \tag{5.25}$$

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Up to $O(R^{-5/2})$:

$$\begin{aligned} j_{3-} &= \partial_-\Psi_3 - 4\partial_-\theta_L - \frac{2}{3}[\theta_L,[\theta_L,\partial_-\theta_R]] + \dots = \\ &= \partial_-\Psi_3 - 4d_{3-} + 4[\theta_R,\partial_- x] + \frac{2}{3}[\theta_R,[\theta_R,\partial_-\theta_R]] + \dots \end{aligned} \tag{5.26}$$

Similarly:

$$j_{1+} = \partial_+\Psi_1 + 4d_{1+} - 4[\theta_L,\partial_+ x] - \frac{2}{3}[\theta_L,[\theta_L,\partial_+\theta_L]] + \dots \tag{5.27}$$

$$j_{1-} = \partial_-\Psi_1 - 4[\partial_-\theta_L, x] - 2[\theta_R,[\theta_R,\partial_-\theta_L]] - \frac{2}{3}[\theta_L,[\theta_L,\partial_-\theta_L]] + \dots \tag{5.28}$$

where Ψ_1 is given by (4.44). The density of a local conserved charge is defined up to a total derivative.

Therefore, let us redefine $j_\pm \to S_\pm$, by removing total derivatives:

$$\begin{aligned} j_{3\pm} &= \partial_\pm\Psi_3 + S_{3\pm} \\ j_{1\pm} &= \partial_\pm\Psi_1 + S_{1\pm} \end{aligned} \tag{5.29}$$

In the flat space expansion:

$$S_{1+} \simeq R^{-3/2},\ S_{1-} \simeq R^{-5/2},\ S_{3-} \simeq R^{-3/2},\ S_{3+} \simeq R^{-5/2} \tag{5.30}$$

We should identify S_{1+} and S_{3-} with the supersymmetry currents of the flat space superstring. Explicitly we have:

$$\begin{aligned} S_{1+} &= 4\left(p_{1+} - \frac{1}{2}[\theta_L,\partial_+ x] - \frac{1}{24}[\theta_L,[\theta_L,\partial_+\theta_L]]\right) = \\ &= 4\left(d_{1+} - [\theta_L,\partial_+ x] - \frac{1}{6}[\theta_L,[\theta_L,\partial_+\theta_L]]\right) \end{aligned} \tag{5.31}$$

$$\begin{aligned} -S_{3-} &= 4\left(p_{3-} - \frac{1}{2}[\theta_R,\partial_- x] - \frac{1}{24}[\theta_R,[\theta_R,\partial_-\theta_R]]\right) = \\ &= 4\left(d_{3-} - [\theta_R,\partial_- x] - \frac{1}{6}[\theta_R,[\theta_R,\partial_-\theta_R]]\right) \end{aligned} \tag{5.32}$$

$\mathbf{U_{AdS}}$ in terms of the global currents. Now we can write eq. (4.58) precisely, including the terms with θ:

$$U_{\rm AdS} = \mathrm{Str}(S_{1+}S_{3-})\,. \tag{5.33}$$

5.4 Unphysical vertex of the order R^{-3}

Let us consider the following example of the unphysical vertex:

$$U_{\bar 0} = [j_{\bar 1+}, j_{\bar 3-}] + [j_{\bar 3+}, j_{\bar 1-}] = [S_{\bar 1+}, S_{\bar 3-}] \simeq R^{-3} \tag{5.34}$$

In this case the flat space limit of the unphysical vertex appears to be perfectly physical, and in fact corresponds to turning on the constant RR 3-form field strength. Indeed, there is a term of the type d_+d_- plus terms containing θ's:

$$U_{\bar 0} = [d_+, d_-] + \dots \tag{5.35}$$

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A careful analysis of the index structure shows that this actually corresponds to the constant RR 3-form field strength.

The flat space limit of the vertex operator for the beta-deformation is generally speaking of the order εR^{-3}. It typically starts with $xdx \wedge dx$, plus terms of the type $d_+ d_-$ (which are also of the order R^{-3}, since $d_\pm$ are of the order $R^{-3/2}$). Plus terms with θ. The leading bosonic term $xdx \wedge dx$ describes a NSNS B_{NSNS}-field. At the order εR^{-3} we can only see the constant NSNS field strength H_{NSNS}. The terms with $d_+ d_-$ describe the constant RR field strength H_{RR}. We conclude that we see some constant H_{NSNS} and some constant H_{RR}. This is nice.

But let us expand it at a different point in AdS, the point at which the field strengths are zero. Then the leading terms in the vertex will be of the order R^{-4}.

5.5 Unphysical vertex of the order R^{-4}

5.5.1 Definition of the vertex and how the descent procedure does not work

Consider another example of the unphysical vertex:

$$U_{\bar{2}} = \frac{1}{2}\,\mathrm{Str}\,(\;[B_2, j_1] \wedge j_1 \;+\; [B_2, j_3] \wedge j_3\;) \;= \tag{5.36}$$

$$= \mathrm{Str}\,(\;B_{\bar{2}}\;[j_{\mathrm{odd}}\;,\;j_{\mathrm{odd}}]) \tag{5.37}$$

The flat space limit of an unintegrated unphysical vertex was derived in [3]:

$$V_{\bar{2},\,\mathrm{flat}} = [[\theta_R, [\theta_R, \epsilon\lambda_R]], [\theta_R, [\theta_R, \epsilon\lambda_R]] + [[\theta_L, [\theta_L, \epsilon\lambda_L]], [\theta_L, [\theta_L, \epsilon\lambda_L]] \tag{5.38}$$

What happens if we apply to it the flat space descent procedure? Observe:

$$\partial_-[\theta_R, [\theta_R, \epsilon\lambda_R]] = Q(3S_{3-}) \tag{5.39}$$

Notice that in flat space the supersymmetry current S_{3-} is holomorphic. Therefore the second step of the descent procedure is zero:

$$\partial_+[\;[\theta_R, [\theta_R, \epsilon\lambda_R]]\;,\;S_{3-}\;] = 0 \tag{5.40}$$

This means that the corresponding integrated vertex, defined by the descent procedure, is zero. (If it were not zero, it would have been of the order R^{-3}.)

Conclusion. The leading flat space limit of (5.36) is *not* related to $V_{\bar{2},\,\mathrm{flat}}$ by a descent procedure.

5.5.2 Explicit formula for the vertex in flat space

We observe:

$$\epsilon Q \int U_{\bar{2}} = -\int \mathrm{Str}\left(\;\left[\;B_2\;,\;g^{-1}(\epsilon\lambda_L - \epsilon\lambda_R)g\right]\;(dj_1 + dj_3)\;\right) \tag{5.41}$$

The variation is proportional to the equation of motion $dj_1 = 0$, $dj_3 = 0$. To compensate this variation we need the field redefinition:

$$\epsilon Q_1 = 4\left[\;B_2\;,\;g^{-1}(\epsilon\lambda_L - \epsilon\lambda_R)g\;\right]_3^{\alpha} t_{\alpha}^3 + 4\left[\;B_2\;,\;g^{-1}(\epsilon\lambda_L - \epsilon\lambda_R)g\;\right]_1^{\hat{\alpha}} t_{\hat{\alpha}}^1 \tag{5.42}$$

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Then the deformed action:

$$S_{\rm AdS} + \int {\rm Str}\,(\, B_2\, j_{\rm odd} \wedge j_{\rm odd}) \tag{5.43}$$

is invariant under the deformed BRST transformation $\epsilon(Q + Q_1)$.

To get the expression starting with R^{-4}, we do the field redefinition with the vector field K given by (4.50) plus (4.53). Then the deformed action

$$S_{\rm AdS} + K S_{\rm AdS} + \int {\rm Str}\,(B_2\, j_{\rm odd} \wedge j_{\rm odd}) \tag{5.44}$$

is invariant under the deformed BRST transformation:

$$\begin{aligned} & \epsilon Q + \epsilon Q_1' \\ \texttt{where}\ & \epsilon Q_1' = \epsilon Q_1 + [K, \epsilon Q] = \\ & = -\frac{16}{3}[B_2,\ [\theta_L, [\theta_L, \epsilon\lambda_L]]]^{\alpha} t^3_{\alpha} + \frac{16}{3}[B_2,\ [\theta_R, [\theta_R, \epsilon\lambda_R]]]^{\hat\alpha} t^1_{\hat\alpha} \end{aligned} \tag{5.45}$$

Using (4.54) we get:

$$\begin{aligned} & S_{\rm AdS} + K S_{\rm AdS} + \int {\rm Str}\,(\, B_2\, j_1 \wedge j_1) + \int {\rm Str}\,(\, B_2\, j_3 \wedge j_3) = \\ & = S_{\rm AdS} + \Bigg(- \int d^2\tau\ {\rm Str}\,(\partial_+ [B_2, \Psi_1]\ j_{1-} - \partial_- [B_2, \Psi_1]\ j_{1+})\ + \\ & \qquad + \int d^2\tau\ {\rm Str}([B_2, j_{1+}]\ j_{1-}) + (1 \to 3) \Bigg)\ = \\ & = S_{\rm AdS} + \left(\int d^2\tau\ {\rm Str}\,([B_2\ ,\ (j_{1+} - \partial_+ \Psi_1)]\ \ (j_{1-} - \partial_- \Psi_1)\ ) + (1 \to 3) \right)\ = \\ & = S_{\rm AdS} + \int d^2\tau\ {\rm Str}\,(B_2\, [S_{1+}, S_{1-}] + B_2\, [S_{3+}, S_{3-}]) \end{aligned} \tag{5.46}$$

Now formulas of section 5.3 imply that the flat space limit is of the order R^{-4}:

$$\begin{aligned} U_{\bar 2,\,\rm flat}\ =\ {\rm Str}\Bigg(& B_2 \Bigg[4[d_{1+}, x] + 2[\theta_L, [\theta_L, d_{1+}]] + \frac{2}{3}[\theta_R, [\theta_R, d_{1+}]]\ , \\ & \qquad - 4 d_{3-} + 4[\theta_R, \partial_- x] + \frac{2}{3}[\theta_R, [\theta_R, \partial_- \theta_R]]\ \Bigg] + \\ & + B_2 \Bigg[4 d_{1+} - 4[\theta_L, \partial_+ x] - \frac{2}{3}[\theta_L, [\theta_L, \partial_+ \theta_L]]\ , \qquad (5.47) \\ & \qquad - 4[d_{3-}, x] - 2[\theta_R, [\theta_R, d_{3-}]] - \frac{2}{3}[\theta_L, [\theta_L, d_{3-}]] \Bigg] \Bigg) \qquad (5.48) \end{aligned}$$

where … stand for the terms of the same order R^{-4} containing higher number of thetas. Also the ghosts contribute:

$$U_{\bar 2,\,\rm flat,\,gh} = 4\,[[\theta_L, \{w_{1+}, \lambda_L\}]\ ,\ S_{3-}] \tag{5.49}$$

but their contribution will not be very important here.

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We observe that there is the term xd_+d_-, more precisely:

$$16\,\mathrm{Str}\,([B_2,x][d_{1+},d_{3-}]) \tag{5.50}$$

which usually corresponds to the Ramond-Ramond field. Since it is odd under the worldsheet parity (i.e. under the exchange $d_+ \leftrightarrow d_-$) we should have concluded that it corresponds to the Ramond-Ramond 3-form field strength H. But we also find that $dH \neq 0$. In the usual notations (5.50) would correspond to $H = \iota_{B_2\wedge x}F$, where F is the leading flat space limit of the RR field of $AdS_5 \times S^5$. This is not a closed form. Naively this is in contradiction with [1], as $dH = 0$ is one of the SUGRA equations of motion. The resolution is, as explained in section 1.4, that $U_{\bar{2},\,\mathrm{flat}}$ is actually not annihilated by Q_{flat}.

5.6 Demonstration of the l.h.s. of (1.10) being nonzero

Let us calculate the variation of the AdS action along the vector field (4.8). We get the following expression of the order R^{-4}:

$$\left([B_2,[\theta_L,[\theta_L,\epsilon\lambda_L]]]^{\hat\alpha}\, t^1_{\hat\alpha}\right) S_{\mathrm{AdS}} = \tag{5.51}$$

$$= \int d^2\tau\; \mathrm{Str}\,(\partial_-[\theta_L,[\theta_L,\epsilon\lambda_L]]\; S_{1+} - \partial_+[\theta_L,[\theta_L,\epsilon\lambda_L]]\; S_{1-}) \tag{5.52}$$

The term with $\partial_-[\theta_L,[\theta_L,\epsilon\lambda_L]]\; S_{1+}$ generates:

$$\int d^2\tau\; \mathrm{Str}\,([d_{3-},[\theta_L,\epsilon\lambda_L]]\; d_{1+} + [\theta_L,[d_{3-},\epsilon\lambda_L]]\; d_{1+}) \tag{5.53}$$

which does not have anything to cancel with. This demonstrates that the l.h.s. of (1.10) is nonzero.

5.7 Parity even physical vertex

It is also interesting to consider the following *physical* vertex:

$$U_{\bar{2},\,\mathrm{phys}} = \frac{1}{2}\,\mathrm{Str}\,(\;[B_2,j_1]\wedge j_1 \;-\; [B_2,j_3]\wedge j_3\;) \tag{5.54}$$

It differs from (5.36) by the relative sign of the two terms. Unlike (5.36), this vertex does satisfy the physical condition (2.9), and does correspond to a meaningful excitation of $AdS_5 \times S^5$. Notice that $U_{\bar{2},\,\mathrm{phys}}$ is parity-even, therefore it should correspond to either a metric, or a dilaton, or a RR 1-form, or a RR 5-form.

As becomes clear from section 6, the flat space limit of the parity even vertex is the linear dilaton background. (Whereas the parity odd vertex is unphysical and does not correspond to anything.)

6 Bringing the action to the normal form of [1]

This section was added in the revised version of the paper.

Generally speaking, given a sigma-model, we can always rewrite it in many different forms using field redefinitions, introducing Lagrange multipliers, alternative gauge fixings,

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etc. In order to make contact with the spacetime description in terms of Type IIB SUGRA fields, the authors of [1] used a special "normal form" of the sigma-model action. The definition of this normal form depends on how the BRST symmetry acts. Although in our case the action of the sigma-model does not change, but the BRST operator does get deformed. Therefore, the *normal form* of the action does get deformed. We will now study the deformation of the normal form of the action. We will show that it leads to the nontrivial spin connection. It turns out that the vector components of the left and right spin connections do not coincide (contrary to what was conjectured in [1]); this is why the deformation is nonphysical.

We will use the notations of [1]; we also recommend [11] for the detailed explanations of the formalism. We will continue using the flat space notations (with Γ-matrices) and the AdS notations (commutators and Str) intermittently, as explained in section 3.2.

6.1 Action in terms of $d_\pm$

As we explained, the action is undeformed:

$$S_{\text{flat}} = \int d\tau^+ d\tau^- \left[\frac{1}{2}\partial_+ x^m \partial_- x^m + p_{\alpha+}\partial_-\theta_L^\alpha + p_{\hat\alpha-}\partial_+\theta_R^{\hat\alpha} + \right. \tag{6.1}$$

$$\left. + w_{\alpha+}\partial_-\lambda_L^\alpha + w_{\hat\alpha-}\partial_+\lambda_R^{\hat\alpha} \right] = \tag{6.2}$$

$$= \int d\tau^+ d\tau^- \ \mathrm{Str}\ \left[\frac{1}{2}\partial_+ x_2 \partial_- x_2 + p_{1+}\partial_-\theta_L + p_{3-}\partial_+\theta_R + \right.$$
$$\left. + w_{1+}\partial_-\lambda_L + w_{3-}\partial_+\lambda_R \right] \tag{6.3}$$

(Eq. (6.1) uses traditional notations, while eq. (6.3) uses AdS notations.) The deformation only touches the BRST operator. In order to bring the action to the form of [1], we need to trade $p_\pm$ for $d_\pm$, where $d_\pm$ is defined as the density of the BRST charge:

$$Q_{L|R} = \oint \lambda_{L|R} d_\pm \tag{6.4}$$

In the undeformed theory, the relation between $d_\pm$ and $p_\pm$ is given by eqs. (5.31), (5.32):

$$p_{1+} = d_{1+} - \frac{1}{2}[\theta_L,\ \partial_+ x] - \frac{1}{8}[\theta_L,\ [\theta_L,\ \partial_+\theta_L]] \tag{6.5}$$

$$p_{3-} = d_{3-} - \frac{1}{2}[\theta_R,\ \partial_- x] - \frac{1}{8}[\theta_R,\ [\theta_R,\ \partial_-\theta_R]] \tag{6.6}$$

After the deformation, this relation is modified. Let us consider the case when $B_R = 0$ (only the left deformation):

$$p_{1+} = d_{1+} - \frac{1}{2}[\theta_L,\ \partial_+ x] - \frac{1}{8}[\theta_L,\ [\theta_L,\ \partial_+\theta_L]] +$$
$$+ [\theta_L, [\theta_L,\ [B_2,\ S_{1+}]]] \tag{6.7}$$

$$p_{3-} = d_{3-} - \frac{1}{2}[\theta_R,\ \partial_- x] - \frac{1}{8}[\theta_R,\ [\theta_R,\ \partial_-\theta_R]] \tag{6.8}$$

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Let us substitute S_{1+} from (5.31) into (6.7):

$$p_{1+} = d_{1+} - \frac{1}{2}[\theta_L,\ \partial_+ x] - \frac{1}{8}[\theta_L,\ [\theta_L,\ \partial_+\theta_L]] + \\ + 4\left[\theta_L, \left[\theta_L, \left[B_2\ , \left(d_{1+} - [\theta_L,\ \partial_+ x] - \frac{1}{6}[\theta_L,\ [\theta_L,\ \partial_+\theta_L]]\right)\right]\right]\right] \tag{6.9}$$

Therefore, we get the following formula for the action, which at this point is almost in the normal form of [1]:

$$S = \int d\tau^+ d\tau^-\ \mathrm{Str}\left(\frac{1}{2}\partial_+ x_2 \partial_- x_2 + d_{1+}\partial_-\theta_L + d_{3-}\partial_+\theta_R - \right. \\ - \frac{1}{2}[\theta_L,\ \partial_+ x]\partial_-\theta_L - \frac{1}{8}[\theta_L,\ [\theta_L,\ \partial_+\theta_L]]\partial_-\theta_L - \\ - \frac{1}{2}[\theta_R,\ \partial_- x]\partial_+\theta_R - \frac{1}{8}[\theta_R,\ [\theta_R,\ \partial_-\theta_R]]\partial_+\theta_R + \\ + w_{1+}\partial_-\lambda_3 + w_{3-}\partial_+\lambda_R + \\ \left. + 4\left[B_2\ , \left(d_{1+} - [\theta_L,\ \partial_+ x] - \frac{1}{6}[\theta_L,\ [\theta_L,\ \partial_+\theta_L]]\right)\right]\ [\theta_L, [\theta_L,\ \partial_-\theta_L]]\right) \tag{6.10}$$

6.2 B-field

In particular this allows us to read the B-field part:

$$B_{MN}\ dZ^M \wedge dZ^N = \mathrm{Str}\Big(- \frac{1}{2}[\theta_L,\ dx_2]d\theta_L - \frac{1}{8}[\theta_L,\ [\theta_L,\ d\theta_L]]\ d\theta_L - \\ - \frac{1}{2}[\theta_R,\ dx_2]d\theta_R - \frac{1}{8}[\theta_R,\ [\theta_R,\ d\theta_R]]\ d\theta_R - \\ - 4\left[B_2, \left([\theta_L,\ dx_2] + \frac{1}{6}[\theta_L,\ [\theta_L,\ d\theta_L]]\right)\right]\ [\theta_L\ , [\theta_L,\ d\theta_L]]\Big) \tag{6.11}$$

The 3-form field strength $H = dB$ is:

$$H = \mathrm{Str}\Big(- \frac{1}{2}[d\theta_L\ ,\ dx_2]d\theta_L + \frac{1}{4}[d\theta_L\ ,\ d\theta_L]\ [\theta_L\ ,\ d\theta_L] - \\ - \frac{1}{2}[d\theta_R\ ,\ dx_2]d\theta_R + \frac{1}{4}[d\theta_R\ ,\ d\theta_R]\ [\theta_R\ ,\ d\theta_R] - \\ - 4\left[B_2\ , \left([d\theta_L,\ dx_2] + \frac{1}{4}[\theta_L,\ [d\theta_L,\ d\theta_L]]\right)\right]\ [\theta_L\ , [\theta_L,\ d\theta_L]] + \\ + 6\left[B_2\ , \left([\theta_L,\ dx_2] + \frac{1}{6}[\theta_L,\ [\theta_L,\ d\theta_L]]\right)\right]\ [\theta_L\ , [d\theta_L,\ d\theta_L]]\Big) \tag{6.12}$$

For example, let us demonstrate that:

$$H_{\alpha\beta m}\lambda^\alpha\lambda^\beta = 0 \tag{6.13}$$

in accordance with [1]. The last row in (6.12) does not contribute, because $\{\lambda_L\ ,\ \lambda_L\} = 0$. In the previous rows, the terms containing $dx d\theta_L d\theta_L$ combine into:

$$\frac{1}{2}\mathrm{Str}\Big(dx_2\ \Big[\ d\theta_L - 4[B_2, [\theta_L, [\theta_L, d\theta_L]]]\ \ ,\ \ d\theta_L - 4[B_2, [\theta_L, [\theta_L, d\theta_L]]]\ \Big]\Big) \tag{6.14}$$

Notice that $\epsilon Q\theta_L = \epsilon\lambda_L + 4\ [B_2, [\theta_L, [\theta_L, \epsilon\lambda_L]]]$ and (6.13) follows.

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6.3 Torsion

The action (6.10) is almost in the normal form, but not completely. To complete the procedure described in [1] we have to eliminate some components of the torsion, namely $T_{\alpha\beta}{}^{\gamma}$. Let us therefore study the torsion.

The 16-beins E^{α} and $E^{\hat{\alpha}}$ are defined as the coefficients of $d_{\pm}$ in the worldsheet action (6.10):

$$E^{\alpha} = E^{\alpha}_{M} dZ^{M} = d\theta^{\alpha}_{L} - 4\,[B_2\ ,\ [\theta_L\ ,\ [\theta_L\ ,\ d\theta_L]]]^{\alpha} \tag{6.15}$$

$$E^{\hat{\alpha}} = E^{\hat{\alpha}}_{M} dZ^{M} = d\theta^{\hat{\alpha}}_{R} \tag{6.16}$$

Notice that the pure spinor terms in the action (6.10) are the same as in flat space, therefore $\Omega_{M}{}^{\alpha}_{\beta} = \widehat{\Omega}_{M}{}^{\hat{\alpha}}_{\hat{\beta}} = 0$. Therefore the torsion is defined as in flat space: $T^{\alpha} = T^{\alpha}_{MN} dZ^{M} dZ^{N} = dE^{\alpha}$, $T^{\hat{\alpha}} = T^{\hat{\alpha}}_{MN} dZ^{M} dZ^{N} = dE^{\hat{\alpha}}$. In particular:

$$T^{\alpha} = -\,6\,[B_2\ ,\ [\theta_L\ ,\ [d\theta_L\ ,\ d\theta_L]]] \tag{6.17}$$

$$\texttt{in other words}\ \ T^{\gamma}_{\alpha\beta} = -\,6\,\Gamma^{n}_{\alpha\beta}(\overline{B}^{m}_{2}\Gamma_m\Gamma_n\theta_L)^{\gamma} \tag{6.18}$$

Here the notation $\overline{B}^{m}$ stands for: B^{m} for $m \in \{0, 1, \ldots 4\}$ and $-B^{m}$ for $m \in \{5, \ldots, 9\}$. The difference between B and $\overline{B}$ does not play any role in our discussion here; it is an artifact of notations in section 3.2.

Removing $T^{\gamma}_{\alpha\beta}$. As instructed in [1], we have to remove $T^{\gamma}_{\alpha\beta}$ by a special field redefinition which at the same time modifies the spin connection $\Omega^{[mn]}_{\alpha}$ and $\Omega^{(s)}_{\alpha}$. This is done in the following way. Notice that the following *field redefinition* $d \to \tilde{d}$, parametrized by $h^{a\alpha}(Z)$:

$$d_{\alpha+} = \tilde{d}_{\alpha+} + h^{b\beta}\Gamma^{k}_{\alpha\beta}(w_{+}\Gamma_b\Gamma_k\lambda) \tag{6.19}$$

does not change the expression (6.4) for the BRST current, and therefore is a residual field redefinition preserving the normal form of [1] of the worldsheet action/BRST structure. This field redefinition changes the string worldsheet action by adding to it the term:

$$\partial_{-}Z^{M} E^{\alpha}_{M}\Gamma^{k}_{\alpha\beta}h^{b\beta}(w_{+}\Gamma_b\Gamma_k\lambda) \tag{6.20}$$

which encodes the modification of the left connection Ω_{α}:

$$\Omega^{(s)}_{\alpha} = \Gamma^{k}_{\alpha\beta}h^{k\beta}\ , \qquad \Omega^{[mn]}_{\alpha} = \Gamma^{[m}_{\alpha\beta}h^{n]\beta} \tag{6.21}$$

This changes the T^{α}_{MN}:

$$T^{\alpha}_{MN} \to T^{\alpha}_{MN} + 2E^{\beta}_{(M}\Omega_{N)}{}^{\alpha}_{\beta} = T^{\alpha}_{MN} + E^{\alpha'}_{M}E^{\beta'}_{N}\Gamma^{b}_{\alpha'\beta'}\Gamma^{b}_{\gamma'\delta'}\Gamma^{\alpha\gamma'}_{c}h^{c\delta'} \tag{6.22}$$

Taking $h^{a\alpha}$ as follows:

$$h^{a\alpha} = 6\overline{B}^{a}_{2}\theta^{\alpha}_{L} \tag{6.23}$$

we get rid of $T^{\gamma}_{\alpha\beta}$ (i.e. the $T^{\gamma}_{\alpha\beta}$ calculated with this new Ω is zero) at the price of generating $\Omega^{(s)}_{\alpha}$ and $\Omega^{[mn]}_{\alpha}$ given by (6.21). Notice that $\Omega^{(s)}_{\hat{\alpha}} = 0$, as it should be. Also notice that the

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right connection remains zero, both $\hat{\Omega}^{(s)}_{\hat{\alpha}}$ and $\hat{\Omega}^{(s)}_{\alpha}$. According to [1] we should then be able to solve the equations $(D_\alpha + \Omega^{(s)}_\alpha)\Phi = 0$ and $(D_{\hat{\alpha}} + \hat{\Omega}^{(s)}_{\hat{\alpha}})\Phi = 0$ which imply:

$$\left(\frac{\partial}{\partial\theta^\alpha_L} + \Gamma^m_{\alpha\beta}\theta^\beta_L\frac{\partial}{\partial x^m} + 6\overline{B}^m_2\Gamma^m_{\alpha\beta}\theta^\beta_L\right)\Phi = 0 \tag{6.24}$$

$$\left(\frac{\partial}{\partial\theta^{\hat{\alpha}}_R} + \Gamma^m_{\hat{\alpha}\hat{\beta}}\theta^{\hat{\beta}}_R\frac{\partial}{\partial x^m}\right)\Phi = 0 \tag{6.25}$$

The first of these equations can be solved by the *linear dilaton*:[5]

$$\Phi = -6\overline{B}^m_2 x^m + \text{const} \tag{6.26}$$

but this *does not satisfy the second equation* (6.25). In fact, (6.25) immediately implies that $\Phi = \text{const}$. This result can be also formulated in the following way:

- it is not true in this case that $\Omega^{(s)}_m = \widehat{\Omega}^{(s)}_m$

Notice that the equality of the vector component of the left and right spin connections was only conjectured (but not proven) in [1]; our construction provides a counter-example to this conjecture.

We feel that this problem only arizes for the states of low momentum, although it is not very clear what "low momentum" would mean in a generic background. Perhaps the non-physical vertex only exists in AdS and flat space, and the corresponding deformation is obstructed at the higher orders of the deformation parameter. In any case, as was demonstrated in [3], the non-physical vertices go away if, in addition to the BRST invariance, we also impose the 1-loop conformal invariance. This suggests that a modification of the BRST complex, taking into account the additional structure provided by the b-ghost [4–6], would take care of the problem.

A Vector field Y_L

A.1 Ansatz for y_+

It is usually assumed that the pure spinor BRST cohomology at the positive conformal dimension is trivial. We do not have a general proof of this fact. Let us consider a particular example which we needed in section 4.1.4:

$$\begin{aligned} & Q_{\text{flat}} M_+ = 0 \\ \texttt{where } & M_+ = (\theta\Gamma^m\lambda)(\theta\Gamma_m)_\alpha B^{\alpha\beta}_L S_{\beta+} - \partial_+(\epsilon' W_L) \end{aligned} \tag{A.1}$$

We want to prove that exists such y_+ that $M_+ = Q_{\text{flat}} y_+$. *We do not have the complete proof*, but only a schematic expression:

$$y_+ = [\theta_L\theta_L N_+] + [\theta_L\theta_L\theta_L d_+] + [\theta^5_L\partial_+\theta_L] + [\theta^4_L\partial_+ x] \tag{A.2}$$

[5]It is not surprizing that the linear dilaton is involved. In the case of bosonic string, also the linear dilaton background does not deform the worldsheet action on a flat worldsheet, but does deform the BRST trasnformation. We would like to thank Nathan Berkovits for suggesting to look at it from this angle.

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where $N_{[mn]+} = (\lambda_L \Gamma_{mn} w_+)$ is the contributions of the pure spinors to the Lorentz current. The term with $[\theta_L \theta_L N_+]$ is necessary because $S_{\beta+}$ contains $d_{\beta+}$, and its coefficient in M_+ (which is $(\theta_L \Gamma^m \lambda_L)(\theta_L \Gamma_m)_\alpha B_L^{\alpha\beta}$) is not Q_{flat}-exact. Such term can only come from the BRST variation of something of the type $[\theta_L \theta_L N_+]$. In the next section we will discuss the structure of this term.

A.2 The term $\theta\theta N_+$

In order to obtain the term $(\theta_L \Gamma^m \lambda_L)(\theta_L \Gamma_m)_\alpha B_L^{\alpha\beta} d_+$, we need the first term $[\theta_L \theta_L N_+]$ in (A.2) of the form:

$$[\theta_L \theta_L N_+] \simeq B_{lmnpq}(\theta_L \Gamma^{lmn} \theta_L)(\lambda_L \Gamma^{pq} w_+) \tag{A.3}$$

where B_{lmnpq} is a self-dual antisymmetric tensor defined so that:

$$B_{lmnpq} \Gamma^{\alpha\beta}_{lmnpq} = B^{\alpha\beta} \tag{A.4}$$

We observe that Q_{flat} of so defined $[\theta_L \theta_L N_+]$ does not contain w_+:

$$B_{lmnpq}(\theta_L \Gamma^{lmn} \lambda_L)(\lambda_L \Gamma^{pq} w_+) = 0 \tag{A.5}$$

Let us prove (A.5). This is equivalent to:

$$B_{lmnpq}(\theta_L \Gamma^{lmn} \widehat{\mathcal{F}}\, \Gamma^{pq} w_+) = 0 \tag{A.6}$$

for any self-dual 5-forms $\mathcal{F}$ and B, with $\widehat{\mathcal{F}} = \mathcal{F}_{ijklm} \Gamma^{ijklm}$. To prove (A.6), we consider particular values for $\widehat{\mathcal{F}}$ and B. Let us work in the *Euclidean signature*: $\Gamma_0^2 = \Gamma_1^2 = \ldots = 1$. Modulo SO(10) rotations, there are exactly 3 cases to consider.

Case 0.

$$\widehat{\mathcal{F}} = \widehat{B} = \Gamma^{01234} + i\Gamma^{56789} \tag{A.7}$$

In order to calculate $B_{lmijk}(w_+ \Gamma^{lm} \widehat{\mathcal{F}}\, \Gamma^{ijk} \theta_L)$, we need:

$$\begin{aligned} &\Gamma^{[01|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|234]} + i\Gamma^{[56|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|789]} = \\ &\quad = (\Gamma^{01234} + i\Gamma^{56789})^2 = 0 \end{aligned} \tag{A.8}$$

Case 1.

$$\widehat{\mathcal{F}} = \Gamma^{01234} + i\Gamma^{56789} \tag{A.9}$$

$$\widehat{B} = \Gamma^{01235} - i\Gamma^{46789} \tag{A.10}$$

To calculate $B_{lmijk}(w_+ \Gamma^{lm} \widehat{\mathcal{F}}\, \Gamma^{ijk} \theta_L)$, consider:

$$\begin{aligned} &120 \left(\Gamma^{[01|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|235]} - i\Gamma^{[46|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|789]} \right) = \\ &= 72\, \Gamma^{[01|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|23]5} + 48\, \Gamma^{5[0|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|123]} - \\ &\quad - 48\, i\, \Gamma^{4[6|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|789]} - 72\, i\, \Gamma^{[67|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|89]4} = \\ &= 0 \end{aligned} \tag{A.11}$$

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Case 2.

$$\widehat{\mathcal{F}} = \Gamma^{01234} + i\Gamma^{56789} \tag{A.12}$$
$$\widehat{B} = \Gamma^{01256} + i\Gamma^{34789} \tag{A.13}$$

In order to calculate $B_{lmijk}(w_+\Gamma^{lm}\widehat{\mathcal{F}}\,\Gamma^{ijk}\theta_L)$, we consider:

$$\begin{aligned}
120 &\left(\Gamma^{[01|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|256]} + i\Gamma^{[34|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|789]}\right) = \\
&= 36\,\Gamma^{[01|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|2]56} + 12\,\Gamma^{56}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{201} - \\
&\quad - 36\,\Gamma^{5[2|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|01]6} + 36\,\Gamma^{6[2|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|01]5} + \\
&\quad + 12\,i\,\Gamma^{34}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{789} + 36\,i\,\Gamma^{[89|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{7]34} - \\
&\quad - 36\,i\,\Gamma^{3[7|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|89]4} + 36\,i\,\Gamma^{4[7|}(\Gamma^{01234} + i\Gamma^{56789})\Gamma^{|89]3} = \\
&= 0
\end{aligned} \tag{A.14}$$

Therefore, in this case also $B_{lmijk}(w_+\Gamma^{lm}\widehat{\mathcal{F}}\,\Gamma^{ijk}\theta_L) = 0$. This concludes the proof of (A.5).

Proof that $B_{lmnpq}(\theta\Gamma^{lmn}\theta)\lambda\Gamma^{pq}$ is not BRST-exact. The only possibility for it to be BRST-exact would be:

$$B_{lmnpq}(\theta_L\Gamma^{lmn}\theta_L)\lambda_L\Gamma^{pq} \overset{?}{\simeq} Q\left(B_{lmnpq}(\theta_L\Gamma^{lmn}\theta_L)\theta_L\Gamma^{pq}\right) \tag{A.15}$$

The r.h.s. is a linear combination of two BRST-closed expressions:

$$B_{lmnpq}(\theta_L\Gamma^{lmn}\theta_L)\lambda_L\Gamma^{pq} \quad \texttt{and} \quad B_{lmnpq}(\theta_L\Gamma^{lmn}\lambda_L)\theta_L\Gamma^{pq} \tag{A.16}$$

These expressions are linearly independent. Indeed, we have:

$$B_{lmnpq}(\theta_L\Gamma^{lmn}\theta_L)(\lambda_L\Gamma^{pq}\,\Gamma^k\lambda_L) = 0 \tag{A.17}$$
$$B_{lmnpq}(\theta_L\Gamma^{lmn}\lambda_L)(\theta_L\Gamma^{pq}\,\Gamma^k\lambda_L) \neq 0 \tag{A.18}$$

Therefore $\left(Q\left(B_{lmnpq}(\theta_L\Gamma^{lmn}\theta_L)\theta_L\Gamma^{pq}\right)\,\Gamma^k\lambda_L\right)$ is nonzero.

But $B_{lmnpq}(\theta_L\Gamma^{lmn}\theta_L)(\lambda_L\Gamma^{pq}\,\Gamma^k\lambda_L)$ is zero. This implies that (A.15) is false.

A.3 Pure spinor redefinition

Therefore the vector field Y_L of section 4.1.4 involves an infinitesimal redefinition of the pure spinor field:

$$Y_L\lambda_L^\alpha = B_{lmnpq}(\theta\Gamma^{lmn}\theta)(\lambda\Gamma^{pq})^\alpha \tag{A.19}$$

which preserves the pure spinor condition: $(\lambda_L\Gamma^k Y_L\lambda_L) = 0$.

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