

Wilson loops @ 3-loops in special kinematics

Paul Heslop^a and Valentin V. Khoze^b

^a*Department of Mathematical Sciences, Durham University,
Durham, DH1 3LE, United Kingdom*

^b*Institute for Particle Physics Phenomenology, Department of Physics, Durham University,
Durham, DH1 3LE, United Kingdom*

E-mail: paul.heslop@durham.ac.uk, valya.khoze@durham.ac.uk

ABSTRACT: We obtain a compact expression for the octagon MHV amplitude/Wilson loop at 3 loops in planar $\mathcal{N} = 4$ SYM and in special 2d kinematics in terms of 7 unfixed coefficients. We do this by making use of the cyclic and parity symmetry of the amplitude/Wilson loop and its behaviour in the soft/collinear limits as well as in the leading term in the expansion away from this limit. We also make a natural and quite general assumption about the functional form of the result, namely that it should consist of weight 6 polylogarithms whose symbol consists of basic cross-ratios only (and not functions thereof). We also describe the uplift of this result to 10 points.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry, Duality in Gauge Field Theories

ARXIV EPRINT: [1109.0058](https://arxiv.org/abs/1109.0058)

Contents

1	Introduction	1
2	Background material	3
2.1	Special kinematics	3
2.2	Collinear limits	4
2.3	Symbols	5
2.4	The integrability constraint	6
3	Fundamental assumption: the symbol contains u's only	7
4	The one- and two-loop remainder functions revisited	8
4.1	Non-existence of a 1 loop 8-point remainder	8
4.2	Uniqueness of the 2-loop 8-point remainder	9
4.3	Lifting to n -point functions at two-loops	10
5	The 3-loop octagon	11
6	Lifting the 3-loop octagon to higher polygons	15
6.1	Constructing the decagon: part 1	15
6.2	Constructing the decagon: part 2. Collinear-vanishing 10-point functions	17
7	Conclusions	19

1 Introduction

Our understanding of perturbative scattering amplitudes in planar $\mathcal{N} = 4$ SYM is currently increasing at a rather rapid rate. Indeed in just the last year or so the fruitful duality between MHV amplitudes and Wilson loops [1–3] has been formally extended to arbitrary amplitudes [4, 5] once issues of regularisation are properly understood [6]. A new duality between correlation functions and both Wilson loops and amplitudes has been found [7–12] and this has already proved useful in both directions, obtaining previously unknown correlation functions using known amplitudes as well as providing new insights into amplitudes themselves [13]. And a loop-level integrand version [14, 15] of the BCFW recursion relation [16] has enabled one to find arbitrary loop level amplitude integrands from purely algebraic methods [15, 17].

The above impressive results have been largely formulated at the level of the integrand. Of course ultimately we are interested in the amplitudes themselves, the result of having performed the integration of these integrands. Much progress has also been made here but as yet at a somewhat more modest level, and most of the developments [18–28] are still driven by the original MHV amplitude/Wilson loop duality, and result from the fact that

the Wilson loop integrals are simpler than the amplitude ones. A major new mathematical tool, arising from this is the notion of the “symbol” [29, 30]. This allows one to map highly complicated polylogarithmic functions to tensors involving rational functions. In this way obscure polylogarithmic identities become manifest algebraic identities satisfied by this tensor. This allowed the authors of [30] to reduce the huge formula arising from the impressive direct computation of the hexagon Wilson loop at two-loops [23, 24] to a single line [30]. Indeed the most recent results concerning amplitudes at the *integral* level have actually been given as symbols rather than the functions themselves [5, 28]. For example, very recently the symbol of the 3-loop hexagon Wilson loop was derived up to two unfixed coefficients in [28].

Another new tool for analytic amplitude computations is the OPE/near collinear limit [31–34] allowing an expansion around the collinear limit to be understood in terms of an OPE expansion. At the moment there is an obstruction to going beyond the next to leading term in this expansion, but even at this level we obtain important information about the amplitude which we will make use of here.

In order to investigate further perturbative amplitudes without doing a direct computation, we will restrict ourselves to the so-called AdS₃ special kinematics, first introduced in [35] in the strong coupling context. This corresponds to assuming that all the external momenta live in 1 + 1 dimensions rather than the full 3 + 1 dimensions. These provide a nice arena for studying non-trivial high loop order amplitudes/Wilson loops whilst avoiding some of the kinematical complications of the full amplitudes.

In [26] we were able to take the 2-loop result for the 8-point Wilson loop in special kinematics, computed directly in [25] and extend it to all (even) n -points, using symmetry and collinear limits as well as a simple assumption about its structure. The assumption was that at 2-loops the conformal part of the answer should depend only on logarithms of x space cross-ratios

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2}. \quad (1.1)$$

We then verified that our analytic expressions for all n agreed with numerical computations carried out following the numerical algorithm developed in [22] and further used in [36].

However the logs-only structure of the answer cannot be expected to hold beyond 2-loops since the OPE implies the presence of polylogarithms at 3 loop level [31]. The crucial insight which enables us to go further in this sector then, is our expectation that, despite the known complicated variables which occur in MHV amplitudes at two-loops and beyond for general kinematics, we expect that these all simplify in special kinematics. Indeed all expected variables in general kinematics (for example those given in [37]) reduce to simple cross-ratios. So we will assume in this paper that the symbol takes values only over the standard x -space cross-ratios (1.1). In other words we relax the assumption we made at two loops that the amplitude depends only on logarithms, but we maintain the assumption that the arguments of the symbol should be simple cross-ratios only.

So then using this assumption together with cyclic and parity symmetry of the Wilson loop/MHV amplitude, and the important restriction that the symbol should arise from

a function (the so called integrability constraint) we can firstly derive the 8 point 2-loop result of [25] without computation (with one unfixed coefficient), and prove that the uplift to n -points found in [26] is in fact the unique solution of our constraints. At 3-loops we can restrict the 3-loop 8-point amplitude down to just 13 unfixed coefficients. The further constraints arising from the OPE/collinear limit then reduces this to 7 unfixed coefficients. At higher n we are able to uplift the result to 10 points, albeit with the introduction of 12 new unfixed coefficients. The uplift to 12 points can also be performed, but again there will be further new unfixed coefficients introduced. However, the uplift from 12 points to 14 points and beyond is then unique at 3 loops within our ansatz.

More generally, at l loops, once the $4l$ -point function is known the uplift via inverse soft/triple collinear limits is unique.

Although we initially perform all this analysis at the level of the symbol, we are able to invert the symbol and obtain the functions themselves. Indeed although the corresponding symbols become quickly very large indeed with increasing n , the functions themselves can be written fairly compactly.

In section 2 we review the set up and some background material we will need. In section 3 we discuss further our assumption that only u_{ij} 's should appear in the symbol. Section 4 reviews the remainder function at one- and two-loops from this perspective. In section 5 we determine the octagon 3-loop amplitude as far as we can and in section 6 we discuss the uplift to higher points at 3-loops.

2 Background material

MHV amplitudes and null polygonal Wilson loops in planar $\mathcal{N} = 4$ SYM are traditionally characterised by the remainder function \mathcal{R}_n which is defined as the difference between the logarithm of the Wilson loop W_n and the known BDS expression of [38, 39],

$$\mathcal{R}_n = \log(W_n) - (\text{BDS})_n^{\text{WL}}. \tag{2.1}$$

\mathcal{R}_n is a conformally-invariant function and thus depends only on conformally-invariant cross-ratios [2, 19]. For a polygonal contour with n light-like edges, in general, there are $n(n-5)/2$ independent conformal cross-ratios (if we do not, as in [22], impose the Gram determinant constraints). A basis for the cross ratios is provided by u_{ij} , defined in (1.1).

2.1 Special kinematics

In this paper we will be restricting our attention exclusively to the case of special kinematics, first introduced in [35], where the external momenta lie entirely in $1+1$ dimensions. For the Wilson loop contour to be embeddable into two space-time dimensions the number of edges n must be even and the number of independent cross-ratios reduces and they have to satisfy the following conditions,

$$\begin{aligned} u_{i, i+\text{odd}} &= 1 \\ u_{2i+1, 2j+1} &= u_{ij}^+ & 2 \leq (i-j) \bmod n/2 \leq n/2 - 2 \\ u_{2i, 2j} &= u_{ij}^- \end{aligned} \tag{2.2}$$

Here the vertices of the contour have the following simple light-cone representation:

$$x_{2i} = (x_i^+, x_i^-), \quad x_{2i+1} = (x_i^+, x_{i+1}^-), \quad i = 1, \dots, n, \quad (2.3)$$

and the cross-ratios u_{ij}^\pm appearing on the right hand side of (2.2) are functions of only either x^+ or x^- light-cone coordinates:

$$u_{ij}^+ := \frac{x_{ij+1}^+ x_{i+1j}^+}{x_{ij}^+ x_{i+1j+1}^+}, \quad u_{ij}^- := \frac{x_{ij+1}^- x_{i+1j}^-}{x_{ij}^- x_{i+1j+1}^-}. \quad (2.4)$$

As such, these cross-ratios are essentially made from one-dimensional distances. This results in the following simple identity

$$(1 - u_{ij+1}^\pm)(1 - u_{i+1j}^\pm) = (1 - 1/u_{ij}^\pm)(1 - 1/u_{i+1j+1}^\pm) \quad (2.5)$$

$$u_{i,i+1} = u_{i+1,i} = 0 \quad u_{i,i} = \infty, \quad (2.6)$$

which is precisely the AdS₃ Y-system equation of [40], where the Y 's of [40] (evaluated at $\zeta = 0$) are associated with the cross-ratios as

$$u_{k,-k-1}^+ = \frac{Y_{2k}}{1 + Y_{2k}} \quad u_{k,-k-2}^- = \frac{Y_{2k+1}}{1 + Y_{2k+1}}. \quad (2.7)$$

We will thus refer to (2.5) as the Y-system from now on.

For the two lowest- n cases, the octagon and the decagon, all the cross-ratios different from 1 in (2.2) are of the form $u_{i,i+4}$, with $i = 1, \dots, 4$ for the octagon, and $i = 1, \dots, 10$ for the decagon. To simplify notation in these two cases, we define $u_i := u_{i,i+4}$. Similarly, for their decomposition into \pm components, we will often use $u_j^\pm := u_{j,j+2}^\pm$.

Clearly, the cross-ratios u_i are not all independent, as we have seen above, they are further constrained by the Y-system equations, leaving $n - 6$ (i.e. 2 for the octagon and 4 for the decagon) independent solutions. Nevertheless, as in our earlier work, [26] we will use the full set of u_i as the set of variables appearing in all expressions.

More details of the special kinematics in this context can be found in [26].

2.2 Collinear limits

The collinear limits which allow us to remain in the special kinematics have to reduce the number of edges (number of external momenta for amplitudes) by an even number. The minimal such limit is the triple-collinear limit in which three consecutive edges become collinear.¹ For concreteness, consider the limit of \mathcal{R}_n in which edges $n - 2$, $n - 1$ and n become collinear (and in which in fact edge $n - 1$ becomes soft). In this case one has

$$u_{i,n-1} \rightarrow 1, \quad u_{1,n-3} \rightarrow 0, \quad (2.8)$$

¹A more appropriate way to visualise this limit in the way which is consistent with the zig-zag construction of the polygon, is in terms of the collinear-soft-collinear limit. In this case the middle edge becomes soft and the two edges, one on the left and one on the right of it, are collinear to each other; thus the three edges are reduced to one.

while the remaining cross-ratios $u_{i,j}$ remain unchanged. The remainder function \mathcal{R}_n reduces in this limit to \mathcal{R}_{n-2} plus a correction \mathcal{R}_6 arising from the triple-collinear splitting function² (in our special kinematics $\mathcal{R}_6 = \text{const}$). Specifically, one has [20, 22, 26]

$$\mathcal{R}_n(u_{i,j}) \rightarrow \mathcal{R}_{n-2}(\hat{u}_{i,j}) + \mathcal{R}_6. \tag{2.9}$$

Here the $(n - 2)$ -point cross-ratios $\hat{u}_{i,j}$ are defined in terms of the n -point cross-ratios in the collinear limit as

$$\hat{u}_{i,n-2} = u_{i,n-2} u_{i,n}, \quad \hat{u}_{i,j} = u_{i,j} \quad i, j \neq n - 2. \tag{2.10}$$

In particular, for the octagon we have $\mathcal{R}_8(u_i) \rightarrow 2 \mathcal{R}_6 = \text{const}$, and for the decagon,

$$\mathcal{R}_{10}(u_i) \rightarrow \mathcal{R}_8(\hat{u}_i) + \mathcal{R}_6 \quad \text{where} \quad \hat{u}_4 = u_4 u_{10}. \tag{2.11}$$

In the above equation the u 's solve the 10-point Y-system equation (2.5) and the \hat{u} 's then automatically solve the 8-point Y-system equation. From now on we will always refer to these (triple) collinear-soft limits as collinear limits. For more detail on collinear limits in special kinematics we refer the reader to [26].

2.3 Symbols

The ‘‘symbol’’ is an important new mathematical tool, introduced in the context of particle physics in [30], and already proving highly useful in $\mathcal{N} = 4$ SYM amplitudes, but which should also be relevant more generally in particle physics (see for example [41]).

The symbol associates to any (generalised) polylogarithm, a tensor whose entries are rational functions of the arguments. The rank of the tensor is equal to the weight of the polylogarithm. For example $\log x$ has weight 1 and gives rise to a 1-tensor

$$\mathcal{S}(\log x) = x \tag{2.12}$$

whereas the classical polylogarithms have symbol given as

$$\mathcal{S}(\text{Li}_w(x)) = -(1 - x) \otimes \overbrace{x \otimes \dots \otimes x}^{w-1}. \tag{2.13}$$

The symbol has the following properties inherited from the logarithm

$$\begin{aligned} \dots \otimes x y \otimes \dots &= \dots \otimes x \otimes \dots + \dots \otimes y \otimes \dots \\ \dots \otimes 1/x \otimes \dots &= - \dots \otimes x \otimes \dots \end{aligned} \tag{2.14}$$

from which follows the important property that the symbol vanishes when any entry equals unity

$$\dots \otimes 1 \otimes \dots = 0. \tag{2.15}$$

²An especially nice feature of the remainder function is that the contribution of the splitting function to the collinear limit is itself a remainder function. Indeed, in the limit where m adjacent momenta become collinear, the contribution of the splitting function is \mathcal{R}_{m-3} and one has $\mathcal{R}_n \rightarrow \mathcal{R}_{n+1-m} + \mathcal{R}_{m-3}$ (see [22, 26]).

It is also blind to multiplication by constants. The final property of the symbol we need is the symbol of products of functions. This is given by taking the shuffle product of the symbol of each function

$$\mathcal{S}(fg) = \mathcal{S}(f) \text{III} \mathcal{S}(g). \tag{2.16}$$

For example

$$\begin{aligned} \mathcal{S}(\text{Li}_2(x) \log y) &= (-(1-x) \otimes x) \text{III} y \\ &= -(1-x) \otimes x \otimes y - (1-x) \otimes y \otimes x - y \otimes (1-x) \otimes x, \end{aligned} \tag{2.17}$$

or for three log functions we have,

$$\begin{aligned} \mathcal{S}(\log(x) \log(y) \log(z)) &= x \text{III} y \text{III} z = (x \otimes y + y \otimes x) \text{III} z \\ &= x \otimes y \otimes z + x \otimes z \otimes y + z \otimes x \otimes y + y \otimes x \otimes z + y \otimes z \otimes x + z \otimes y \otimes x. \end{aligned} \tag{2.18}$$

The symbol can be defined recursively. One can write the total derivative of any weight w generalised polylogarithm (here by this we mean any function with a well-defined rank- w symbol) as follows

$$df = \sum_i g_i d \log(x_i) \tag{2.19}$$

where the g_i are weight $w - 1$ polylogarithms. Then the corresponding symbol is given as

$$\mathcal{S}(f) = \sum_i \mathcal{S}(g_i) \otimes x_i. \tag{2.20}$$

This definition (together with (2.12)) gives all the above properties.

The symbol is incredibly useful since it trivialises otherwise complicated identities involving polylogarithms. The most spectacular example of such a simplification is the reduction of the formula found for the hexagon two-loop Wilson loop in [23, 24] to the single line formula in [30]. However the inverse process of finding the function from the symbol is far from straightforward to do in practice. Indeed the symbol is often much more complicated and longer than the actual functions which produce it due to the shuffle product for example. The symbol is also non-unique. It is equivalent to the “maximally transcendental” piece of the function, but all information about lower weight terms is lost in the symbol.

The great advantage of the special kinematics we consider here is that the functions that occur will turn out to be relatively simple and after obtaining the symbol we will be able to reconstruct the functional form in section 5.

2.4 The integrability constraint

The fact that $d^2 f = 0$ together with its recursive definition (2.19), (2.20) give non-trivial and powerful constraints on symbols of functions. Namely for a function of weight w , which will have a symbol in the form

$$\mathcal{S}(f) = \sum_{i_1, \dots, i_w} \text{const}_{i_1 \dots i_w} x_{i_1} \otimes \dots \otimes x_{i_w} \tag{2.21}$$

we obtain the $w - 1$ integrability equations

$$\sum_{i_1, \dots, i_a, i_{a+1}, \dots, i_w} d \log x_{i_a} \wedge d \log x_{i_{a+1}} x_{i_1} \otimes \dots \otimes x_{i_{a-1}} \otimes x_{i_{a+2}} \otimes \dots \otimes x_{i_w} = 0, \quad (2.22)$$

true for any value of $a = 1 \dots w - 1$, with no sum over i_a and i_{a+1} in the second line. We will make extensive use of this constraint in deriving the 8-point 3-loop remainder function.

3 Fundamental assumption: the symbol contains u 's only

In the rest of this paper we will attempt to constrain, as far as possible, the analytic form of the remainder functions using symmetries and collinear limits. In order to do this we make one fundamental assumption which makes this possible. Namely we assume that the function has a symbol whose entries can always be taken from the basis of cross-ratios u_{ij} . In other words, the symbol is made of sums of the tensor products of u_{ij} 's, and no functions of the cross-ratios should appear in the symbol. This is certainly not the case in general kinematics where, for example at 6-points one can have entries $1 - u$ as well as functions involving square roots of combinations of u 's. However in the special kinematics we consider, we expect these will always reduce to u 's.

For example twistor brackets, in terms of which remainder function symbols seem to be naturally given (see for example [15, 30, 37, 42]) always reduce in special kinematics to simple products of x 's. So for example in a conformally invariant expression a four-bracket of two even and two odd twistors reduces as

$$\langle 2i2j(2k-1)(2l-1) \rangle \rightarrow x_{ij}^+ x_{kl}^-, \quad (3.1)$$

with any other possibility vanishing, whereas more complicated twistor invariants which should appear reduce similarly, e.g.

$$\langle X_{2i} \bar{Z}_{2j} \cap \bar{Z}_{2k-1} \rangle \sim x_{ij}^+ x_{j+1}^- x_{k-1k}^+ x_{ik}^-. \quad (3.2)$$

Furthermore it is always possible to rewrite $1 - u$ in terms of product of u 's using the Y-system equations (2.5). Indeed one can check that

$$1 - u_{ij}^\pm = \prod_{k=i+1}^{j-1} \prod_{l=j+1}^{i-1} u_{kl}^\pm. \quad (3.3)$$

Clearly inside a symbol, using (2.14), this can then be written in terms of a sum of terms involving u_{ij} 's.

For the case of octagon, (3.3) collapses to

$$n = 8 : \quad 1 - u_1 = u_3, \quad 1 - u_2 = u_4, \quad (3.4)$$

and for the decagon we have

$$n = 10 : \quad 1 - u_i^\pm = u_{i-1}^\pm u_{i+1}^\pm \quad i = 1, \dots, 5. \quad (3.5)$$

In summary the natural assumption that the entries in the symbol are always u_{ij} 's in special kinematics, is consistent with all expectations for general kinematics.

4 The one- and two-loop remainder functions revisited

In this section we revisit the two-loop n -point remainder functions in $1 + 1$ dimensional kinematics. The 8-point remainder function was first obtained by a direct computation of the Wilson loop in [25]. It can be written as

$$\mathcal{R}_8^{(2)} = -\frac{1}{2} \log(u_1) \log(u_2) \log(u_3) \log(u_4) - \frac{\pi^4}{18}. \quad (4.1)$$

where $u_i := u_{ii+4}$. We then uplifted this in [26] to give the two-loop remainder function for any n (in $1+1$ dimensions) in the remarkably concise form

$$\mathcal{R}_n^{(2)} = -\frac{1}{2} \left(\sum_{\mathcal{S}} \log(u_{i_1 i_5}) \log(u_{i_2 i_6}) \log(u_{i_3 i_7}) \log(u_{i_4 i_8}) \right) - \frac{\pi^4}{72} (n - 4), \quad (4.2)$$

where the sum runs over the set

$$\mathcal{S} = \{i_1, \dots, i_8 : 1 \leq i_1 < i_2 < \dots < i_8 \leq n, \quad i_k - i_{k-1} = \text{odd}\}. \quad (4.3)$$

This uplift from 8 points to n points was done by considering collinear limits alone. We found functions satisfying these and we then checked the result using the numerical code constructed in [22].

This result was derived in [26] following the assumption (based on the explicit form of the 8-point 2-loop result as well as the n -point 1-loop results) that only logs of cross-ratios can appear. This is correct at 2-loops, but at 3-loops the OPE analysis suggests that one needs to consider more general functions than simple logarithms [32]. To find the strategy which works at all loops, we are thus lead to re-derive the two-loop results (4.1) and (4.2) from a weaker assumption. In this paper therefore we will instead make the much less restrictive assumption (motivated in section 3) that the function has a symbol as a sum of tensor products of basis cross-ratios, u_{ij} 's. As we shall see, this weaker assumption, together with collinear limits, and cyclic and parity symmetry implies the appearance of $\log(u)$'s only at two loops.

It turns out that under this simple and natural assumption, we can both rule out the existence of a 1 loop remainder function and derive, without any direct computations, the 8-point 2-loop remainder function (up to 1 unfixed constant). We will also show that under this assumption the uplift to the n -point 2-loop remainder (found in [26]) is unique (but not the 3-loop uplift which will be constructed in a later section).

4.1 Non-existence of a 1 loop 8-point remainder

The n -point remainder function at any loop order must reduce under the collinear limit to the $n - 2$ -point remainder function plus the 6-point remainder (which is a constant in the $1 + 1$ dimensional kinematics). So we can consider

$$\tilde{\mathcal{R}}_n = \mathcal{R}_n - \frac{1}{2} (n - 4) \mathcal{R}_6 \quad (4.4)$$

which simply reduces as $\tilde{\mathcal{R}}_n \rightarrow \tilde{\mathcal{R}}_{n-2}$ in the collinear limit. In particular $\tilde{\mathcal{R}}_6 = 0$ and so $\tilde{\mathcal{R}}_8 \rightarrow 0$ in the collinear limit.

Now at 1 loop one can quickly see that there is no weight-2 symbol (i.e. no 2-tensor) we can write down which will vanish in all collinear limits. The collinear limit is $u_4 \rightarrow 1$, $u_2 \rightarrow 0$ but with $u_1 = 1 - u_3$ left arbitrary. So in order for a tensor involving u 's only to vanish in this collinear limit, all terms must therefore contain a u_4 . But cyclic symmetry ensures that this can never be the case. We therefore immediately rule out a 1-loop 8-point collinear vanishing remainder.

Similar considerations rule out the 1-loop n -point remainder function.

4.2 Uniqueness of the 2-loop 8-point remainder

Let us now consider therefore the most general possible collinear vanishing 2-loop remainder function. This will give a nice illustration of the technique we will implement later in more general cases.

We wish to write down the most general 2-loop (i.e. weight 4) symbol which has dihedral symmetry (cyclic + parity) and vanishes in any soft/triple collinear limit. In order for the symbol to vanish in any collinear limit, each term in the symbol must contain all four cross-ratios u_1, u_2, u_3, u_4 . Indeed if a term contains just 3 out of the four u 's, then this will never vanish under the particular collinear limit which has the remaining $u \rightarrow 1$. For example if one chooses a term to be $u_1 \otimes u_1 \otimes u_2 \otimes u_3$ then under the collinear limit $u_4 \rightarrow 1$, $u_2 \rightarrow 0$, this will not vanish (indeed it will diverge). Furthermore this can never be compensated by a similar non-vanishing term in the symbol. We therefore consider all $4!$ terms in the symbol which contain all 4 cross-ratios, $u_1 \otimes u_2 \otimes u_3 \otimes u_4$ together with permutations. Now we impose dihedral symmetry generated by

$$u_1 \rightarrow u_2 \rightarrow u_3 \rightarrow u_4 \rightarrow u_1 \quad \text{and} \quad u_1 \leftrightarrow u_4, u_2 \leftrightarrow u_3. \quad (4.5)$$

In this way we obtain just three independent symbols

$$\begin{aligned} \mathcal{R}_8^{(2)} &= a\mathcal{R}_{8;a}^{(2)} + b\mathcal{R}_{8;b}^{(2)} + c\mathcal{R}_{8;c}^{(2)} + 2\mathcal{R}_6^{(2)} & (4.6) \\ \mathcal{S}(\mathcal{R}_{8;a}^{(2)}) &= u_1 \otimes u_2 \otimes u_3 \otimes u_4 + 7 \text{ terms related by dihedral symmetry} \\ \mathcal{S}(\mathcal{R}_{8;b}^{(2)}) &= u_1 \otimes u_2 \otimes u_4 \otimes u_3 + 7 \text{ terms related by dihedral symmetry} \\ \mathcal{S}(\mathcal{R}_{8;c}^{(2)}) &= u_1 \otimes u_3 \otimes u_2 \otimes u_4 + 7 \text{ terms related by dihedral symmetry.} & (4.7) \end{aligned}$$

All three terms separately vanish in the collinear limit, and are symmetric under the full dihedral symmetry. However they are *not* necessarily symbols of functions. The integrability constraint, $d^2\mathcal{R}_8^{(2)} = 0$ imposes constraints on the allowed symbols as described in section 2.4. We get three equations from the derivatives hitting the first two entries, the second and third entries or the third and fourth entries respectively in the symbol:

$$\begin{aligned} a \frac{du_1 \wedge du_2}{u_1 u_2} u_3 \otimes u_4 + b \frac{du_1 \wedge du_2}{u_1 u_2} u_4 \otimes u_3 + c \frac{du_1 \wedge du_3}{u_1 u_3} u_2 \otimes u_4 + \text{dihedral} &= 0 \\ a \frac{du_2 \wedge du_3}{u_2 u_3} u_1 \otimes u_4 + b \frac{du_2 \wedge du_4}{u_2 u_4} u_1 \otimes u_3 + c \frac{du_3 \wedge du_2}{u_2 u_3} u_1 \otimes u_4 + \text{dihedral} &= 0 \\ a \frac{du_3 \wedge du_4}{u_3 u_4} u_1 \otimes u_2 + b \frac{du_4 \wedge du_3}{u_3 u_4} u_1 \otimes u_2 + c \frac{du_2 \wedge du_4}{u_2 u_4} u_1 \otimes u_3 + \text{dihedral} &= 0. \end{aligned} \quad (4.8)$$

Here “+ dihedral” signifies the addition of all terms related by dihedral transformations. Now we must consider the wedge terms. Since $u_1 = 1 - u_3$, and $u_2 = 1 - u_4$, we have $du_1 = -du_3$ and $du_2 = -du_4$. The minus sign disappears at the level of the symbol (since it is blind to multiplication by constants) and so there is only one independent wedge product, $du_1 \wedge du_2$. For example we have:

$$du_1 \wedge du_3 = du_2 \wedge du_4 = 0 \quad du_1 \wedge du_4 = du_1 \wedge du_2 \quad du_2 \wedge du_3 = -du_1 \wedge du_2 \quad \text{etc.} \quad (4.9)$$

So (4.8a) becomes

$$(a - b) du_1 \wedge du_2 \left(\frac{u_3 \otimes u_4 - u_4 \otimes u_3}{u_1 u_2} + \frac{u_1 \otimes u_2 - u_2 \otimes u_1}{u_3 u_4} \right) + (a - c) du_1 \wedge du_2 \left(\frac{u_1 \otimes u_4 - u_4 \otimes u_1}{u_2 u_3} + \frac{u_3 \otimes u_2 - u_2 \otimes u_3}{u_2 u_3} \right) = 0 \quad (4.10)$$

and the other two equations are similar. The integrability constraint therefore fixes $a = b = c$. This then yields the symbol of the function $\log u_1 \log u_2 \log u_3 \log u_4$. So we conclude that the 2-loop 8-point function (4.6) is fixed to be

$$\begin{aligned} \mathcal{R}_8^{(2)} &= a (\mathcal{R}_{8;a}^{(2)} + \mathcal{R}_{8;b}^{(2)} + \mathcal{R}_{8;c}^{(2)}) + 2\mathcal{R}_6^{(2)} \\ &= a \log(u_1) \log(u_2) \log(u_3) \log(u_4) + 2\mathcal{R}_6^{(2)}, \end{aligned} \quad (4.11)$$

in agreement with the computed result (4.1), with $a = -1/2$ and $\mathcal{R}_6^{(2)} = -\pi^4/36$. We could thus have derived the 2-loop 8-point result in this case with these reasonable assumptions, up to two unfixed constants, one of which is simply $\mathcal{R}_6^{(2)}$.

4.3 Lifting to n -point functions at two-loops

In the previous subsection we were able to derive the form of the 8-point 2-loop remainder function using some basic assumptions only (dihedral symmetry, collinear limits, symbol made out of u 's). We now wish to consider the lift to higher point functions. We find that the result found in [26] is the unique function satisfying these assumptions. In [26] we assumed the result consisted of logs of u only, but now we can derive the same result without this assumption.

Let us then analyse the most general possible 10-point function. This must be symmetric under dihedral symmetry and reduce to the 8-point function under collinear limit. The most general solution of this constraint is a “particular solution” together with the most general dihedrally symmetric 10-point function which *vanishes* under the collinear limit (the “homogeneous solution”). So we have

$$\mathcal{R}_{10}^{(2)} = \mathcal{R}_{10;\text{PS}} + \mathcal{R}_{10;\text{HS}}^{(2)} \quad (4.12)$$

$$\mathcal{R}_{10;\text{PS}} = -\frac{1}{2} (\log(u_1) \log(u_2) \log(u_3) \log(u_4) + \text{cyclic}) - \frac{\pi^4}{12}, \quad (4.13)$$

where we have $u_i := u_{i+4}$. Here $\mathcal{R}_{10;\text{PS}}$ is a particular solution of the collinear limit constraint. Indeed it is the known 10-point result from [26]. If we can show that $\mathcal{R}_{10;\text{HS}}^{(2)}$

vanishes, then the solution is unique. Now $\mathcal{R}_{10;\text{HS}}^{(2)}$ is a dihedrally symmetric function which vanishes in any collinear limit. In fact it is quite straightforward to see that no symbol exists with these properties at 2 loops. All terms in the symbol of $\mathcal{R}_{10;\text{HS}}^{(2)}$ involves four u 's and so have the form $u_{i_1 i_2} \otimes u_{i_3 i_4} \otimes u_{i_5 i_6} \otimes u_{i_7 i_8}$. Now consider an edge j where $j \notin \{i_1, \dots, i_8\}$ (clearly such an edge exists at 10 or more points, but not at 8-points). Now consider the collinear limit occurring when $p_j \rightarrow 0$. This implies that $u_{j j+4} \rightarrow 1$, $u_{j j-4} \rightarrow 1$ and $u_{j-2 j+2} \rightarrow 0$ with all other u 's unconstrained (apart from via the Y-system.) We can see that since $u_{j j+4}$ and $u_{j j-4}$ are not in our tensor, it will not vanish in this collinear limit. Furthermore there is no way for different terms to combine to give vanishing contributions either. We conclude that we can not obtain a collinear vanishing term at 10 points.

So the 2-loop 10 point function found in [26] is the unique function whose symbol has cross-ratios as entries, satisfying the correct collinear limits and dihedral symmetry.

The same analysis can be performed at all higher points and we thus find that the solution (4.2) found in [26] is the unique n -point 2-loop result satisfying our assumptions.

5 The 3-loop octagon

In this section we describe our technique for applying constraints on the form of the 8-point function at 3-loops in special kinematics.

In the following we will write the 8-point remainder as

$$\mathcal{R}_8^{(3)} = F_8^{(3)} + 2\mathcal{R}_6^{(3)} \tag{5.1}$$

where $\mathcal{R}_6^{(3)}$ is constant in $1 + 1$ dimensions. Then according to the analysis in section 2.2 the function $F_8^{(3)}$ will vanish in the collinear limit.

All the 8-point remainder functions we have found can be written in the ‘sum of products’ form

$$F_8^{(3)}(u_1, u_2, u_3, u_4) = \sum_i \text{const}_i [f_i(u_1, u_3)g_i(u_2, u_4) + g_i(u_1, u_3)f_i(u_2, u_4)]. \tag{5.2}$$

Cyclic symmetry implies that f and g are symmetric functions

$$f_i(u_1, u_3) = f_i(u_3, u_1) \quad g_i(u_1, u_3) = g_i(u_3, u_1) \tag{5.3}$$

and collinear limits imply that

$$f_i(0, 1) = 0 \quad g_i(0, 1) = 0. \tag{5.4}$$

Since in the octagon case $u_3 = 1 - u_1$ and $u_4 = 1 - u_2$, the functions f and g are really functions of a single argument. We thus will use a dual notation: when discussing symbols of f and g , we will talk of $f(u, v)$ and $g(u, v)$, where u and v are cross-ratios which satisfy $u + v = 1$. On the other hand, when we reconstruct the actual functions we can choose to use the more appropriate single-argument definition:

$$f_i(u) := f_i(u, 1 - u), \quad g_i(u) := g_i(u, 1 - u) \tag{5.5}$$

with (from (5.3) $f_i(u) = f_i(1-u)$ and $g_i(u) = g_i(1-u)$, and (from the collinear limits (5.4)) $f_i(0) = 0 = f_i(1)$ and $g_i(0) = 0 = g_i(1)$.

The characteristic feature of the expression on the r.h.s. of (5.2) is that for each term in the sum the u^+ cross ratios, u_1, u_3 factorise from the u^- cross ratios, u_2, u_4 .

To arrive at (5.2) we have started by writing down a general symbol which by construction is a linear combination of weight-6 tensor products of the cross-ratios u_1, u_2, u_3 and u_4 (and not functions thereof as explained in section 3):

$$\mathcal{S}[F_8^{(3)}(u_1, u_2, u_3, u_4)] = \sum_{i_1 \dots i_6} \text{const}_{i_1 \dots i_6} \cdot u_{i_1} \otimes u_{i_2} \otimes u_{i_3} \otimes u_{i_4} \otimes u_{i_5} \otimes u_{i_6}. \quad (5.6)$$

Next we imposed the requirement that the corresponding function should not explode (and in fact must vanish) in any of the collinear limits, i.e. where $u_i \rightarrow 0$ and $u_{i+2} \rightarrow 1$. This automatically requires that each tensor product must contain all four cross-ratios u_1, u_2, u_3 and u_4 . Indeed, to survive the collinear limit, $u_1 \rightarrow 0$ and $u_3 \rightarrow 1$ for example, whenever u_1 is present, there should also be a u_3 to regulate it, and the same applies for u_2 and u_4 for the limit $u_2 \rightarrow 0$ and $u_4 \rightarrow 1$ or vice versa. The second requirement is that the symbol in (5.6) should respect cyclic symmetry and parity, generated by (4.5) which are the symmetries of the amplitude/Wilson loop. With these requirements the number of different constants $\text{const}_{i_1 \dots i_6}$ in (5.6) reduced to 195. The final requirement we have imposed on (5.6) is that it must be a symbol of a local function. This is known as the $d^2 = 0$ or integrability constraint, and described in section 2.4. It implies that:

$$\sum_{i_1 \dots i_6} \text{const}_{i_1 \dots i_6} d \log(u_{i_k}) \wedge d \log(u_{i_{k+1}}) u_{i_1} \dots \otimes u_{i_{k-1}} \otimes u_{i_{k+2}} \dots \otimes u_{i_6} = 0. \quad (5.7)$$

for each k . We found that implementing this constraint reduces the number of independent constants down to 13, and at the same time imposes the ‘sum of products’ functional form given by (5.2).

We now come back to our starting point (5.2) in order to describe the 13 functions explicitly. The functions f and g must have transcendental weight 2 or more (since it must contain both u_1 and u_3 in its symbol in order not to vanish in the collinear limit) and the product fg sum must have weight 6. So we either have g with weight 2 and f with weight 4 or both f, g have weight 3 each. Notice that in the case when g has weight 2, the symmetry in (5.3) implies that the only possibility is $g(u, v) = \log(u) \log(v)$.

Type a. This type has g with weight 2 and $f(u, v)$ of weight 4, consisting of 3 u ’s and 1 v in the symbol or vice versa. There are four different possibilities, given by

$$g(u, v) = \log(u) \log(v)$$

$$\mathcal{S}[f(u, v)] = \begin{cases} \mathcal{S}[f_{a1}(u, v)] := u \otimes u \otimes u \otimes v + v \otimes v \otimes v \otimes u \\ \mathcal{S}[f_{a2}(u, v)] := u \otimes u \otimes v \otimes u + v \otimes v \otimes u \otimes v \\ \mathcal{S}[f_{a3}(u, v)] := u \otimes v \otimes u \otimes u + v \otimes u \otimes v \otimes v \\ \mathcal{S}[f_{a4}(u, v)] := v \otimes u \otimes u \otimes u + u \otimes v \otimes v \otimes v \end{cases} \quad (5.8)$$

Type b. This type has g with weight 2 again and $f(u, v)$ of weight 4, but this time consisting of 2 u 's and 2 v 's in the symbol. There are only three different possibilities this time, given by

$$g(u, v) = \log(u) \log(v)$$

$$\mathcal{S}[f(u, v)] = \begin{cases} \mathcal{S}[f_{b1}(u, v)] := u \otimes u \otimes v \otimes v + v \otimes v \otimes u \otimes u \\ \mathcal{S}[f_{b2}(u, v)] := u \otimes v \otimes u \otimes v + v \otimes u \otimes v \otimes u \\ \mathcal{S}[f_{b3}(u, v)] := u \otimes v \otimes v \otimes u + v \otimes u \otimes u \otimes v \end{cases} \quad (5.9)$$

Type c. Finally we have type c functions in which both f and g have weight 3. There are three possibilities for both f and g , given by

$$\mathcal{S}[f(u, v)] \text{ or } \mathcal{S}[g(u, v)] = \begin{cases} \mathcal{S}[f_{c1}(u, v)] := u \otimes u \otimes v + v \otimes v \otimes u \\ \mathcal{S}[f_{c2}(u, v)] := u \otimes v \otimes u + v \otimes u \otimes v \\ \mathcal{S}[f_{c3}(u, v)] := u \otimes v \otimes v + v \otimes u \otimes u \end{cases} \quad (5.10)$$

yielding 6 possible functions

$$\begin{aligned} & f_{c1}(u_1, u_3) f_{c1}(u_2, u_4); \\ & f_{c1}(u_1, u_3) f_{c2}(u_2, u_4) + f_{c2}(u_1, u_3) f_{c1}(u_2, u_4); \\ & f_{c1}(u_1, u_3) f_{c3}(u_2, u_4) + f_{c3}(u_1, u_3) f_{c1}(u_2, u_4); \\ & f_{c2}(u_1, u_3) f_{c2}(u_2, u_4); \\ & f_{c2}(u_1, u_3) f_{c3}(u_2, u_4) + f_{c3}(u_1, u_3) f_{c2}(u_2, u_4); \\ & f_{c3}(u_1, u_3) f_{c3}(u_2, u_4). \end{aligned} \quad (5.11)$$

At this point we have in total 13 combinations: 4 from type-a, 3 from type-b and 6 from type-c above. We have already imposed the dihedral symmetry on the answer and have built into it the requirement that our 8-point expression must vanish in the collinear limit as required in the special kinematics and these are of course the 13 functions we found using the computer based method described around (5.7).

However, we have not yet checked that all the functions we have constructed so far vanish *sufficiently slowly* in the collinear limit. We will show now that three of our structures, f_{a1} , f_{a2} and f_{b1} , are actually more singular in the collinear limit than allowed, and will have to be discarded, reducing the number of allowed combinations by 3. In fact, using the near-collinear OPE, the authors [32] have deduced the leading behaviour of the three-loop result,

$$\lim_{u_1 \rightarrow 0} F_8^{(3)}(u_1, u_2, u_3, u_4) = \log^2(u_1) \log(u_3) \cdot F_3(u_2, u_4) + O(\log(u_1)). \quad (5.12)$$

where $F_3(u_2, u_4)$ is known and was written in [32] in the form

$$\begin{aligned} F_3(u_2, u_4) = & -2\text{Li}_3(1 - 1/u_4) + \log(u_2/u_4)\text{Li}_2(1 - 1/u_4) + \frac{4}{3} \log^3(u_4) \\ & + 2 \log(u_2/u_4) \log^2(u_4) + \frac{1}{2} \log^2(u_2/u_4) \log(u_4) + \frac{\pi^2}{6} \log(u_4). \end{aligned} \quad (5.13)$$

We will return to the function $F_3(u_2, u_4)$ below, but first we concentrate on the u_1 -, u_3 -dependence in (5.12). This equation implies that the answer $\propto \log^2(u_1) \log(u_3)$ in the limit $u_1 \rightarrow 0$, $u_3 := 1 - u_1 \rightarrow 1$. This functional form rules out f_{a1} and f_{a2} since

$$\begin{aligned} \lim_{u_1 \rightarrow 0} \mathcal{S}^{-1}(u_1 \otimes u_1 \otimes u_1 \otimes u_3) &\sim \log^3(u_1) \log(u_3), \\ \lim_{u_1 \rightarrow 0} \mathcal{S}^{-1}(u_1 \otimes u_1 \otimes u_3 \otimes u_1) &\sim -\log^2(u_1) \text{Li}_2(u_1), \end{aligned}$$

giving functions with the wrong asymptotic properties. The function f_{b1} is ruled out for the same reason.

Explicit expressions for the remaining seven functions

$$f_{ai}, f_{bi}, f_{ci}(u) := f_{ai}, f_{bi}, f_{ci}(u, v \equiv 1 - u) \quad (5.14)$$

can now be straightforwardly reconstructed from their symbols (5.8)–(5.10) by taking into account the constraint on the variables $v = 1 - u$, and the properties of the symbol. We find

$$\begin{aligned} f_{a3}(u, v) &= 3\text{Li}_4(u) - \text{Li}_3(u) \log(u) + 3\text{Li}_4(v) - \text{Li}_3(v) \log(v) - \frac{\pi^4}{30}, \\ f_{a4}(u, v) &= -\text{Li}_4(u) - \text{Li}_4(v) + \frac{\pi^4}{90}, \\ f_{b2}(u, v) &= (\text{Li}_3(u) - \zeta_3) \log(v) - \text{Li}_2(u) \text{Li}_2(v) + \log^2(u) \log^2(v) + (\text{Li}_3(v) - \zeta_3) \log(u), \\ f_{b3}(u, v) &= -(\text{Li}_3(u) - \zeta_3) \log(v) + \text{Li}_2(u) \text{Li}_2(v) - \frac{1}{2} \log^2(u) \log^2(v) - (\text{Li}_3(v) - \zeta_3) \log(u), \\ f_{c1}(u, v) &= -\text{Li}_3(u) - \left(\text{Li}_2(v) - \frac{\pi^2}{6} \right) \log(u) - \frac{1}{2} \log(v) \log^2(u) \\ &\quad - \text{Li}_3(v) - \left(\text{Li}_2(u) - \frac{\pi^2}{6} \right) \log(v) - \frac{1}{2} \log(u) \log^2(v) + \zeta_3, \\ f_{c2}(u, v) &= 2\text{Li}_3(u) + \left(\text{Li}_2(v) - \frac{\pi^2}{6} \right) \log(u) + \log(v) \log^2(u) \\ &\quad + 2\text{Li}_3(v) + \left(\text{Li}_2(u) - \frac{\pi^2}{6} \right) \log(v) + \log(u) \log^2(v) - 2\zeta_3, \\ f_{c3}(u, v) &= -\text{Li}_3(v) - \text{Li}_3(u) + \zeta_3, \end{aligned} \quad (5.15)$$

where the constants on the r.h.s. are determined from the requirement that all functions must vanish in the collinear limit.

We can now further constrain 3 more coefficients of our general expression by making use of the function $F_3(u_2, u_4)$. First we find another equivalent form for the function in (5.13) so that its arguments on the r.h.s. are just the cross-ratios u_2 and u_4 :

$$\begin{aligned} F_3(u_2, u_4) &= 2\text{Li}_3(u_2) + \left(\text{Li}_2(u_4) - \frac{\pi^2}{6} \right) \log(u_2) + \frac{3}{2} \log(u_4) \log^2(u_2) \\ &\quad + 2\text{Li}_3(u_4) + \left(\text{Li}_2(u_2) - \frac{\pi^2}{6} \right) \log(u_4) + \frac{3}{2} \log(u_2) \log^2(u_4) - 2\zeta_3 \end{aligned} \quad (5.16)$$

This function has the same symbol as (5.13) (note that as always $u_2 + u_4 = 1$ at 8-points) and moreover we checked that the two functions agree numerically. Now we notice that

$F_3(u_2, u_4)$ in (5.16) is just a linear combination of our functions f_{c1} , f_{c2} and f_{c3} . In other words, the GMSV condition takes the form

$$\lim_{u_1 \rightarrow 0} F_8^{(3)}(u_1, u_2, u_3, u_4) = \log^2(u_1) \log(u_3) [f_{c1}(u_2, u_4) + 2f_{c2}(u_2, u_4) + f_{c3}(u_2, u_4)] + O(\log(u_1)). \quad (5.17)$$

We conclude that the coefficients in front of 3 of the 6 c -type functions listed in (5.11) are fixed. We note that the fact that the r.h.s. of eq. (5.16) can be presented entirely in terms of simple functions of cross-ratios (u_2, u_4) , and more specifically that the symbol of $F_3(u_2, u_4)$ is the tensor product of u variables, gives a self-consistency check on our fundamental assumption that the symbol of the full answer is made out of u 's.

We can now write the most general function consistent with all available conditions:

$$\begin{aligned} F_8^{(3)}(u_1, u_2, u_3, u_4) = & \log u_1 \log u_3 [\alpha_1 f_{a3}(u_2, u_4) + \alpha_2 f_{a4}(u_2, u_4) + \alpha_3 f_{b2}(u_2, u_4) + \alpha_4 f_{b3}(u_2, u_4)] \\ & + \alpha_5 f_{c2}(u_1, u_3) f_{c2}(u_2, u_4) + \alpha_6 f_{c2}(u_1, u_3) f_{c3}(u_2, u_4) + \alpha_7 f_{c3}(u_1, u_3) f_{c3}(u_2, u_4) \\ & + f_{c1}(u_1, u_3) \left[\frac{1}{2} f_{c1}(u_2, u_4) + 2f_{c2}(u_2, u_4) + f_{c3}(u_2, u_4) \right] \\ & + (u_1 \leftrightarrow u_2, u_3 \leftrightarrow u_4) \end{aligned} \quad (5.18)$$

Thus we have obtained an analytic expression for the 3-loop contribution to the 8-point amplitude which contains 7 free constants α_i . It is remarkable that the 3-loop octagon in special 2d kinematics can be written in such a compact form and involving only classical polylogarithms of degree ≤ 4 and logarithms. It is clearly important to further constrain at least some of the yet undetermined 7 coefficients in the expression above. It would be interesting to investigate whether one can fix some of the α 's by going to the BFKL limit of the 8-point amplitude in the special kinematics — for the lower hexagon case, this procedure has reduced the number of free constants at 3-loops in general kinematics, as was shown very recently in [28]. We have not attempted to generalise their approach to the octagon case considered here.

In the following section we will outline the procedure of finding the general uplift to 10-points. This approach is general and conceptually there are no restrictions for continuing to an arbitrary high number of 2n-points.

Finally, the fact that not just the symbol, but the functional form of the 3-loop 8-point result is now known, one would be able to determine the coefficients and check the validity of the above approach against numerical results at a few fixed values of the cross-ratios, whenever these results become available.

6 Lifting the 3-loop octagon to higher polygons

6.1 Constructing the decagon: part 1

We will now show how to uplift the 8-point function to 10 points guided by the collinear limits. Here we will construct a ‘particular solution’ for the 10-point polygon remainder,

which is just consistent with the collinear limits. In the following subsection we will obtain the general solution by determining all 10-point structures which vanish in the collinear limit. Similarly to 8-points, we will write

$$\mathcal{R}_{10}^{(3)} = F_{10}^{(3)} + 3\mathcal{R}_6^{(3)} \tag{6.1}$$

so that under the collinear limits described in section 2.2 we have simply $F_{10}^{(3)} \rightarrow F_8^{(3)}$. We will then consider the various contributions to $F_{10}^{(3)}$.

To begin with we consider the 8-point type-a and type-b functions. As explained in section 5, they are of the form,

$$F_{ab8} = \log(u_1) \log(u_3) f_{ab}(u_2, u_4) + \log(u_2) \log(u_4) f_{ab}(u_1, u_3), \tag{6.2}$$

where

$$f_{ab}(u_2, u_4) := \alpha_1 f_{a3}(u_2, u_4) + \alpha_2 f_{a4}(u_2, u_4) + \alpha_3 f_{b2}(u_2, u_4) + \alpha_4 f_{b3}(u_2, u_4), \tag{6.3}$$

as can be seen from the first line on the r.h.s. of (5.18). We now find the lift of this expression to 10-points, it turns out that this is quite straightforward.

The 10-point function is supposed to reduce under the collinear limit $u_5 \rightarrow 1, u_7 \rightarrow 0, u_9 \rightarrow 1$ to the corresponding 8-point function with u_4 replaced by $u_4 u_{10}$. So in this case we are supposed to get

$$\log(u_1) \log(u_3) f_{ab}(u_2, u_4 u_{10}) + \log(u_2) \log(u_4 u_{10}) f_{ab}(u_1, u_3) \tag{6.4}$$

in the collinear limit.

To achieve an uplift consider the function

$$\log(u_1) \log(u_3) f_{ab}(u_2, u_4 u_{10}) + \text{cyclic}. \tag{6.5}$$

One can easily check, using (5.3) and (5.4) that this function reduces under the collinear limit correctly to (6.4). Indeed, the three terms, corresponding to $i = 1, i = 2$ and $i = 10$ of

$$F_{ab10} = \sum_{i=1}^{10} \log(u_i) \log(u_{i+2}) f_{ab}(u_{i+1}, u_{i+3} u_{i-1}), \tag{6.6}$$

combine to the two terms in (6.4). All the remaining terms in the sum in (6.6) vanish in this limit.

Note that in the octagon case, the functions $f_{ab}(u_2, u_4)$ in (6.3) were in fact functions of a *single* variable u_2 , since for the octagon $u_4 = 1 - u_2$ (and $u_3 = 1 - u_1$). Hence it would have been more appropriate to define $f_{ab}(u_2) := f_{ab}(u_2, 1 - u_2)$. The question arises as what is the meaning of the function $f_{ab}(u_2, u_4 u_{10})$ appearing in the decagon case in (6.5) and (6.6)? In fact it is the same function of a single variable u_2 just as for $n = 8$. The Y-system for the decagon (3.5) allows us to rewrite the products as $u_4 u_{10} = 1 - u_2$, and since

$$f_{ab}(u_2, u_4 u_{10}) = f_{ab}(u_2, 1 - u_2) := f_{ab}(u_2). \tag{6.7}$$

Furthermore, since the symbol for each $f_{ab}(u)$ in (5.8)–(5.10) involves only u and $1 - u$ (which can be rewritten as a product of u 's) then making use of the product rule for symbols (2.14), we see that the symbols of these functions are indeed made out of tensor products of the u 's alone, with no functions of u appearing, as required by our fundamental assumption in special kinematics. We can write

$$F_{ab10} = \log(u_1) \log(u_3) f_{ab}(u_2) + \text{cyclic}. \quad (6.8)$$

So in this way we can immediately uplift the type-a and type-b 8-point functions to 10 points. To find the general solution to the 10-point structure one would need to add to eq. (6.8) (and to eq. (6.10) below) also the general set of functions which vanish in the collinear limit (this is a new possibility at 3 loops which couldn't occur at 2 loops as detailed in section 4.3). This will be done in the following subsection.

Now consider the type-c 8-point functions. These are of the form

$$F_{c8} = \sum f_c(u_1, u_3) g_c(u_2, u_4) + g_c(u_1, u_3) f_c(u_2, u_4), \quad (6.9)$$

with f and g each of weight-3. Equation (6.9) corresponds to the second-through-last lines on the r.h.s. of (5.18).

The corresponding 10-point function we are trying to obtain is therefore supposed to reduce under the collinear limit, $u_5 \rightarrow 1$, $u_7 \rightarrow 0$, $u_9 \rightarrow 1$ to $f_c(u_1, u_3) g_c(u_2, u_4 u_{10}) + g_c(u_1, u_3) f_c(u_2, u_4 u_{10})$.

Again this is fairly straightforward to achieve, we simply take

$$F_{c10} = \frac{1}{2} f_c(u_1) (g_c(u_2) - g_c(u_4) + g_c(u_6) - g_c(u_8) + g_c(u_{10})) + \text{cyclic}. \quad (6.10)$$

where we have once again defined the single-argument functions,

$$f_c(u) := f_c(u, 1 - u), \quad g_c(u) := g_c(u, 1 - u), \quad (6.11)$$

so that $f_c(u) = f_c(1 - u)$ and $g_c(u) = g_c(1 - u)$.

To see that the r.h.s. of (6.10) reduces to the desired expression in the collinear limit, we note that when we take $u_5 \rightarrow 1$, $u_7 \rightarrow 0$, $u_9 \rightarrow 1$ we also have automatically $u_3 = 1 - u_1$, as must be the case for the octagon. Thus in the collinear limit, $f_c(u_1) = f_c(u_3)$ so that the first and the third terms in the cyclic permutation have $g_c(u_4)$, $g_c(u_6)$, $g_c(u_8)$ and $g_c(u_{10})$ cancelled and amount to $f_c(u_1) g_c(u_2)$, while the second term produces the other required factor, $f_c(u_2) g_c(u_1)$. The remaining cyclic permutations in (6.10) vanish in the limit. Furthermore, as before, the functions $f_c(u)$ and $g_c(u)$ are made out of tensor products of u 's alone.

6.2 Constructing the decagon: part 2. Collinear-vanishing 10-point functions

We have uplifted the 8-point function to 10 points, but to what extent is this unique? There exist collinear-vanishing 3-loop functions at 10-points and these can never be detected by this uplift. The most general possible 10-point function is the function uplifted from 8 points plus the most general collinear vanishing 10-point function.

We approach the problem of finding the most general collinear vanishing 10-point function in two independent ways. Firstly we work systematically: using a computer, we write down the most general cyclic and parity symmetric, collinear vanishing symbol made of tensor products of u 's. Then we impose the integrability constraint (2.22). This gives 888 constraints thus leaving just 12 collinear vanishing functions.

The second method starts with the assumption that the collinear vanishing function has the form

$$f(u_i^+)g(u_i^-) + \text{cyclic} + \text{parity}. \tag{6.12}$$

Now we analyse the possible functions f, g . These functions must themselves vanish in any collinear limit. To do this they must have weight 3 or more and each term must contain 3 consecutive u_i^\pm e.g. $u_1^+, u_2^+, u_3^+ = u_1, u_3, u_5$. So since the same conditions are true for both functions f, g , and the total weight is 6, they must both have weight 3. Now writing out the most general such symbol for f (or equivalently g) and imposing the integrability constraint we find there are just 11 possibilities which come in 3 types. These are not too hard to find analytically:

$$\begin{aligned} f_1(u_1^\pm, u_2^\pm, u_3^\pm) &= \log(u_1^\pm) \log(u_2^\pm) \log(u_3^\pm) \\ f_2(u_1^\pm, u_2^\pm, u_3^\pm) &= \log(u_2^\pm) (\text{Li}_2(u_1^\pm) - \text{Li}_2(1 - u_2^\pm) + \text{Li}_2(u_3^\pm) - \pi^2/6) \\ f_3(u_i^\pm) &= \sum_{i=1}^5 (\text{Li}_3(u_i^\pm) - \text{Li}_3(1 - u_i^\pm)) - \zeta_3. \end{aligned} \tag{6.13}$$

Here f_1 and f_2 give 5 independent functions via cyclic permutations of the arguments, whereas f_3 is cyclically symmetric giving only 1 independent function, thus we have 11 functions in total. We can now combine these together to obtain a total of 12 independent weight 6 collinear vanishing 10 point function as follows:

$$\begin{aligned} &f_1(u_1, u_3, u_5) f_1(u_2, u_4, u_6) + \text{cyclic} + \text{parity} \\ &f_1(u_1, u_3, u_5) f_1(u_4, u_6, u_8) + \text{cyclic} + \text{parity} \\ &f_1(u_1, u_3, u_5) f_1(u_6, u_8, u_{10}) + \text{cyclic} + \text{parity} \\ &f_1(u_1, u_3, u_5) f_2(u_2, u_4, u_6) + \text{cyclic} + \text{parity} \\ &f_1(u_1, u_3, u_5) f_2(u_4, u_6, u_8) + \text{cyclic} + \text{parity} \\ &f_1(u_1, u_3, u_5) f_2(u_6, u_8, u_{10}) + \text{cyclic} + \text{parity} \\ &f_2(u_1, u_3, u_5) f_2(u_2, u_4, u_6) + \text{cyclic} + \text{parity} \\ &f_2(u_1, u_3, u_5) f_2(u_4, u_6, u_8) + \text{cyclic} + \text{parity} \\ &f_2(u_1, u_3, u_5) f_2(u_6, u_8, u_{10}) + \text{cyclic} + \text{parity} \\ &f_1(u_1, u_3, u_5) f_3(u_i^-) + \text{cyclic} + \text{parity} \\ &f_2(u_1, u_3, u_5) f_3(u_i^-) + \text{cyclic} + \text{parity} \\ &f_3(u_i^+) f_3(u_i^-) + \text{cyclic} + \text{parity} \end{aligned} \tag{6.14}$$

Our expression for the decagon remainder function is obtained by adding together equations (6.8), (6.10) (with constants $\alpha_1, \dots, \alpha_7$) and 12 contributions from (6.14). We have

thus constructed the general analytic expression for the decagon which contains 19 as yet undetermined constant coefficients.

One obvious question is if the current understanding of the near-collinear OPE at 10-points could restrict the function further. Unfortunately this is not the case. None of the collinear vanishing terms found here contribute to the OPE (at the order at which this is currently understood) and nor does the function $F_{ab\ 10}$. Indeed only those functions in $F_{c\ 10}$ whose coefficients have already been fixed by the 8-point OPE are detectable by the 10-point OPE, thus providing a consistency check, but no new information.

7 Conclusions

The main results of this paper are derived from a single fundamental assumption of what are the correct variables of the Wilson loop symbol in the special kinematics. We have postulated that these variables are given by the conformal cross-ratios u_{ij} so that the symbol is a sum of tensor products of u_{ij} .

Based on this constraint on the symbol, and using the symmetries of the system together with collinear limits, we have re-derived the 2-loop n -point analytic expressions for general (even) n in agreement with the previously known results of [25, 26]. Our purpose was to achieve this without performing the direct perturbative computation (which was carried out in [25] for $n = 8$), whilst making a weaker assumption than was made in [26] that only $\log(u)$ can appear in the two-loop answer.

We then applied this strategy at 3-loops in section 5 where we have determined the functional form of the 8-point Wilson loop answer. Our analytic result has a very compact form and is expressed in terms of logarithms and classical polylogarithms of cross-ratios only. After imposing the constraint arising from the near-collinear OPE of [32] we ended up with 7 so far undetermined constant coefficients $\alpha_1, \dots, \alpha_7$. Our final result for the octagon at 3-loops is given by

$$\begin{aligned}
 F_8^{(3)} = & \log u_1 \log(1 - u_1) [\alpha_1 f_{a3}(u_2) + \alpha_2 f_{a4}(u_2) + \alpha_3 f_{b2}(u_2) + \alpha_4 f_{b3}(u_2)] \\
 & + \alpha_5 f_{c2}(u_1) f_{c2}(u_2) + \alpha_6 f_{c2}(u_1) f_{c3}(u_2) + \alpha_7 f_{c3}(u_1) f_{c3}(u_2) \\
 & + f_{c1}(u_1) \left[\frac{1}{2} f_{c1}(u_2) + 2 f_{c2}(u_2) + f_{c3}(u_2) \right] \\
 & + (u_1 \leftrightarrow u_2)
 \end{aligned} \tag{7.1}$$

with the f_a , f_b and f_c functions defined in (5.15).

Our strategy also works for higher polygons. Following the uplift of the 8-point answer, we have constructed the general analytic expression for the 3-loop decagon. Our result for the 3-loop decagon is given by

$$\begin{aligned}
 F_{10}^{(3)} = & \sum_{k=1}^{12} \beta_k \phi_k + \log(u_1) \log(u_3) \tilde{f}_{ab}(u_2) \\
 & + \frac{1}{2} \sum_{i=1}^3 f_{c_i}(u_1) (\tilde{f}_{c_i}(u_2) - \tilde{f}_{c_i}(u_4) + \tilde{f}_{c_i}(u_6) - \tilde{f}_{c_i}(u_8) + \tilde{f}_{c_i}(u_{10})) + \text{cyclic}
 \end{aligned} \tag{7.2}$$

where ϕ_k are the 12 combinations on the r.h.s. of (6.14) and

$$\begin{aligned}\tilde{f}_{ab}(u) &= \alpha_1 f_{a3}(u) + \alpha_2 f_{a4}(u) + \alpha_3 f_{b2}(u) + \alpha_4 f_{b3}(u), \\ \tilde{f}_{c_1}(u) &= \frac{1}{2} f_{c_1}(u) + 2 f_{c_2}(u) + f_{c_3}(u), \\ \tilde{f}_{c_2}(u) &= \alpha_5 f_{c_2}(u) + \alpha_6 f_{c_3}(u), \\ \tilde{f}_{c_3}(u) &= \alpha_7 f_{c_3}(u),\end{aligned}\tag{7.3}$$

with the f_a , f_b and f_c functions collected in (5.15). The 19 free constants are $\beta_1, \dots, \beta_{12}$ and $\alpha_1, \dots, \alpha_7$.

One should also bear in mind that one can always add to any 3-loop remainder π^2 times the 2-loop remainder (which as has been seen is uniquely fixed by our considerations). Such a possibility can never be ruled out from our considerations, since we know that this satisfies all the requirements we are insisting upon. so in other words we can always add

$$\mathcal{R}_n^{(3)} \rightarrow \mathcal{R}_n^{(3)} + k \pi^2 \mathcal{R}_n^{(2)}.\tag{7.4}$$

But this is the only possible lower transcendental function we can add.

In principle, there are no obstacles in continuing to uplift these 3-loop results to higher points. An important point here is that there are no collinear vanishing functions beyond 12 points within our ansatz (and more generally at l loops beyond $4l$ points). This can be seen easily from the point of view of the symbol, there are too many edges (compared with the rank of the symbol-tensor) for one to ensure that each term in the tensor always contains a cross-ratio approaching unity in the limit, which is the only way to kill this term. We conclude that once the 12 point 3-loop remainder is known, and more generally the $4l$ point l -loop remainder, the uplift to all higher points is unique.

It will be also interesting to continue this programme to higher loops.

Acknowledgments

We would like to thank Herbert Gangl and Claude Duhr for stimulating discussions. We would also like to thank Timothy Goddard for a careful reading of the manuscript. VVK would like to thank the organisers and participants of the Amplitudes programme at KITP and the hospitality of the Aspen Center for Physics.

References

- [1] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [[arXiv:0705.0303](#)] [[INSPIRE](#)].
- [2] G.P. Korchemsky, J.M. Drummond and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys.* **B 795** (2008) 385 [[arXiv:0707.0243](#)] [[INSPIRE](#)].
- [3] A. Brandhuber, P. Heslop and G. Travaglini, *MHV amplitudes in $N = 4$ super Yang-Mills and Wilson loops*, *Nucl. Phys.* **B 794** (2008) 231 [[arXiv:0707.1153](#)] [[INSPIRE](#)].

- [4] L.J. Mason and D. Skinner, *The complete planar S-matrix of $N = 4$ SYM as a Wilson loop in twistor space*, *JHEP* **12** (2010) 018 [[arXiv:1009.2225](#)] [[INSPIRE](#)].
- [5] S. Caron-Huot, *Notes on the scattering amplitude/Wilson loop duality*, *JHEP* **07** (2011) 058 [[arXiv:1010.1167](#)] [[INSPIRE](#)].
- [6] A.V. Belitsky, G.P. Korchemsky and E. Sokatchev, *Are scattering amplitudes dual to super Wilson loops?*, *Nucl. Phys. B* **855** (2012) 333 [[arXiv:1103.3008](#)] [[INSPIRE](#)].
- [7] L.F. Alday, B. Eden, G.P. Korchemsky, J.M. Maldacena and E. Sokatchev, *From correlation functions to Wilson loops*, *JHEP* **09** (2011) 123 [[arXiv:1007.3243](#)] [[INSPIRE](#)].
- [8] B. Eden, G.P. Korchemsky and E. Sokatchev, *From correlation functions to scattering amplitudes*, [arXiv:1007.3246](#) [[INSPIRE](#)].
- [9] B. Eden, G.P. Korchemsky and E. Sokatchev, *More on the duality correlators/amplitudes*, [arXiv:1009.2488](#) [[INSPIRE](#)].
- [10] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *The super-correlator/super-amplitude duality: part I*, [arXiv:1103.3714](#) [[INSPIRE](#)].
- [11] T. Adamo, M. Bullimore, L. Mason and D. Skinner, *A proof of the supersymmetric correlation function/Wilson loop correspondence*, *JHEP* **08** (2011) 076 [[arXiv:1103.4119](#)] [[INSPIRE](#)].
- [12] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *The super-correlator/super-amplitude duality: part II*, [arXiv:1103.4353](#) [[INSPIRE](#)].
- [13] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Hidden symmetry of four-point correlation functions and amplitudes in $N = 4$ SYM*, [arXiv:1108.3557](#) [[INSPIRE](#)].
- [14] R.H. Boels, *On BCFW shifts of integrands and integrals*, *JHEP* **11** (2010) 113 [[arXiv:1008.3101](#)] [[INSPIRE](#)].
- [15] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The all-loop integrand for scattering amplitudes in planar $N = 4$ SYM*, *JHEP* **01** (2011) 041 [[arXiv:1008.2958](#)] [[INSPIRE](#)].
- [16] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [[hep-th/0501052](#)] [[INSPIRE](#)].
- [17] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo and J. Trnka, *Local integrals for planar scattering amplitudes*, [arXiv:1012.6032](#) [[INSPIRE](#)].
- [18] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, *Nucl. Phys. B* **795** (2008) 52 [[arXiv:0709.2368](#)] [[INSPIRE](#)].
- [19] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, *Nucl. Phys. B* **826** (2010) 337 [[arXiv:0712.1223](#)] [[INSPIRE](#)].
- [20] Z. Bern et al., *The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. D* **78** (2008) 045007 [[arXiv:0803.1465](#)] [[INSPIRE](#)].
- [21] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude*, *Nucl. Phys. B* **815** (2009) 142 [[arXiv:0803.1466](#)] [[INSPIRE](#)].
- [22] C. Anastasiou et al., *Two-loop polygon Wilson loops in $N = 4$ SYM*, *JHEP* **05** (2009) 115 [[arXiv:0902.2245](#)] [[INSPIRE](#)].
- [23] V. Del Duca, C. Duhr and V.A. Smirnov, *An analytic result for the two-loop hexagon Wilson loop in $N = 4$ SYM*, *JHEP* **03** (2010) 099 [[arXiv:0911.5332](#)] [[INSPIRE](#)].

- [24] V. Del Duca, C. Duhr and V.A. Smirnov, *The two-loop hexagon Wilson loop in $N = 4$ SYM*, *JHEP* **05** (2010) 084 [[arXiv:1003.1702](#)] [[INSPIRE](#)].
- [25] V. Del Duca, C. Duhr and V.A. Smirnov, *A two-loop octagon Wilson loop in $N = 4$ SYM*, *JHEP* **09** (2010) 015 [[arXiv:1006.4127](#)] [[INSPIRE](#)].
- [26] P. Heslop and V.V. Khoze, *Analytic results for MHV Wilson loops*, *JHEP* **11** (2010) 035 [[arXiv:1007.1805](#)] [[INSPIRE](#)].
- [27] S. Caron-Huot, *Superconformal symmetry and two-loop amplitudes in planar $N = 4$ super Yang-Mills*, [arXiv:1105.5606](#) [[INSPIRE](#)].
- [28] L.J. Dixon, J.M. Drummond and J.M. Henn, *Bootstrapping the three-loop hexagon*, *JHEP* **11** (2011) 023 [[arXiv:1108.4461](#)] [[INSPIRE](#)].
- [29] A.B. Goncharov, *A simple construction of Grassmannian polylogarithms*, [arXiv:0908.2238](#).
- [30] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical polylogarithms for amplitudes and Wilson loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [[arXiv:1006.5703](#)] [[INSPIRE](#)].
- [31] L.F. Alday, D. Gaiotto, J.M. Maldacena, A. Sever and P. Vieira, *An operator product expansion for polygonal null Wilson loops*, *JHEP* **04** (2011) 088 [[arXiv:1006.2788](#)] [[INSPIRE](#)].
- [32] D. Gaiotto, J.M. Maldacena, A. Sever and P. Vieira, *Bootstrapping null polygon Wilson loops*, *JHEP* **03** (2011) 092 [[arXiv:1010.5009](#)] [[INSPIRE](#)].
- [33] D. Gaiotto, J.M. Maldacena, A. Sever and P. Vieira, *Pulling the straps of polygons*, [arXiv:1102.0062](#) [[INSPIRE](#)].
- [34] A. Sever, P. Vieira and T. Wang, *OPE for super loops*, *JHEP* **11** (2011) 051 [[arXiv:1108.1575](#)] [[INSPIRE](#)].
- [35] L.F. Alday and J.M. Maldacena, *Null polygonal Wilson loops and minimal surfaces in Anti-de-Sitter space*, *JHEP* **11** (2009) 082 [[arXiv:0904.0663](#)] [[INSPIRE](#)].
- [36] A. Brandhuber, P. Heslop, V.V. Khoze and G. Travaglini, *Simplicity of polygon Wilson loops in $N = 4$ SYM*, *JHEP* **01** (2010) 050 [[arXiv:0910.4898](#)] [[INSPIRE](#)].
- [37] V. Del Duca et al., *The one-loop six-dimensional hexagon integral with three massive corners*, *Phys. Rev. D* **84** (2011) 045017 [[arXiv:1105.2011](#)] [[INSPIRE](#)].
- [38] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [[hep-th/0309040](#)] [[INSPIRE](#)].
- [39] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev. D* **72** (2005) 085001 [[hep-th/0505205](#)] [[INSPIRE](#)].
- [40] L.F. Alday, D. Gaiotto and J.M. Maldacena, *Thermodynamic bubble ansatz*, *JHEP* **09** (2011) 032 [[arXiv:0911.4708](#)] [[INSPIRE](#)].
- [41] S. Buehler and C. Duhr, *CHAPLIN — complex harmonic polylogarithms in Fortran*, [arXiv:1106.5739](#) [[INSPIRE](#)].
- [42] M. Spradlin and A. Volovich, *Symbols of one-loop integrals from mixed Tate motives*, [arXiv:1105.2024](#) [[INSPIRE](#)].