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Braiding properties of the $N = 1$ super-conformal blocks (Ramond sector)

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Abstract: Using a super scalar field representation of the chiral vertex operators we develop a general method of calculating braiding matrices for all types of $N = 1$ super-conformal 4-point blocks involving Ramond external weights. We give explicit analytic formulae in a number of cases.

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Contents

1 Introduction

One of the fundamental principles of the 2-dimensional conformal field theory (CFT) is the convergence of the operator product expansion [1]. It in particular implies that any 4-point function factorizes in three different ways corresponding to the scattering channels s, t, u. Equivalence of these decompositions is one of the basic consistency condition of the theory usually referred to as the crossing symmetry or the bootstrap equation.

Using the factorization and the conformal properties of 3-point functions one can express any 4-point correlator in terms of structure constants and holomorphic and anti-holomorphic 4-point conformal blocks [1]. In the rational CFT the crossing symmetry implies monodromy relations between conformal blocks in different channels [2–5]. Monodromy matrices between $s-t$, and $t-u$ channels, are called the fusion and the braiding matrices, respectively. As the spectrum of a rational CFT is finite they are finite-dimensional.

A well known example of the CFT with a continuous spectrum is the Liouville theory. In this case the fusion and braiding matrices were first calculated by Ponsot and Teschner using representations of $\mathrm{U}_q(\mathrm{sl}(2,\mathbb{R}))$ [6, 7]. These matrices can be also obtained calculating the exchange relations for the chiral operators in the scalar field representation [8, 9]. The explicite form of the integral kernels of the fusion and the braiding matrices was used in the analytic proof that the Liouville structure constants [10, 11] satisfy the bootstrap equation [6].

The structure of conformal blocks in the $N = 1$ superconformal theory is considerably more complicated. It has been recently analyzed in the context of recursion relations in a number of papers [12–19]. The form of the fusion matrix for the Neveu-Schwarz (NS) superconformal blocks was first proposed in [20] on the basis of the properties of super-symmetric extensions of b-hypergeometric functions. This result has been confirmed in [21] where the fusion matrix was derived from the exchange relations of the chiral operators in the super scalar free field representation. As in the case of the bosonic Liouville theory the explicit form of the fusion matrix was used to check the bootstrap equation in the NS sector of the $N = 1$ super-symmetric Liouville theory with the structure constants proposed in [22, 23].

The main aim of the present paper is to derive the integral kernels of the braiding matrices in the Ramond (R) sector of $N = 1$ SCFT by calculating the exchange relation for chiral vertex operators in the free super field representation. The extension of the theory by the Ramond sector leads to four types of chiral vertex operators. Their different compositions correspond to different 4-point blocks of the Neveu-Schwarz and Ramond fields [19]. We derive all technical ingredients necessary for calculating the braiding matrices for all types of $N = 1$ superconformal blocks involving external Ramond weights. Due to a proliferation of types of superconformal blocks we present detailed calculations only in a few cases. They were chosen to illustrate all the technicalities involved. The methods developed are general and can be easily applied to all other blocks.

Our first motivation was to complete the proof of the bootstrap equation in both sectors of the (GSO projected) $N = 1$ super Liouville theory. The matrices we are to calculate are however universal and can be used to check the bootstrap equation and to calculate 4-point correlation functions in any $N = 1$ SCFT with the central charge $c > \frac{3}{2}$. The second interesting problem is to find out if there is a supersymmetric counterpart of the relation between the fusion and the modular matrices recently found in the standard CFT [24]. Due to the technical complexity both problems are postponed to subsequent papers.

The paper is organized as follows. Following [12, 18, 19] we define in section 2 the Neveu-Schwarz (NS) and the Ramond (R) chiral vertex operators and analyze some of their properties. In section 3 the construction of chiral superscalar field space representation of the NS and the R vertex operators is described. In subsection 3.1 we derive braiding relations for the chiral fermion fields. Our method is to decompose the chiral fermion Fock space into Virasoro Verma modules and then use known braiding properties of the Ising model chiral vertices. In subsection 3.2 we introduce the chiral fields and clarify their relation to the chiral vertex operators of section 2. In subsection 3.3 the matrix elements of the chiral fields necessary for their normalization are calculated.

In section 4 we calculate the braiding matrices. In subsection 4.1 we derive the braiding kernel for the compositions of ordered exponentials and screening charges. This result along with the results of section 3 are used in subsection 4.2 to calculated the braiding kernel in several cases including pairs of NS-NS, R-NS and R-R chiral vertex operators.

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The paper is supplemented by a number of appendixes. Appendix A collects the properties of the chiral vertex operators of the Ising model we need in our construction of the chiral fermion fields. In appendix B we derive in some specific case the Ward identity for the fermionic current S in the presence of Ramond fields. Appendix C contains some relevant properties of the Barnes double gamma function. In appendix D we derive the orthogonality relations we use in subsection 4.1.

The paper is rather technical and some remarks concerning conventions and notations can be helpful. Let us first emphasize that the choice of chiral vertex operators in the Ramond sector is determined by the fact that the full theory is based on "small representations", i.e. irreducible representations of the left and the right $N = 1$ Ramond algebras extended only by the common parity operator [18, 19].

We shall adopt the symmetric form of the OPE of the fermionic current with the Ramond fields [26]:

$$S(z)R^{\pm}_{\beta}(w,\bar{w}) \sim \frac{i\beta \mathrm{e}^{\mp i\frac{\pi}{4}}}{(z-w)^{\frac{3}{2}}} R^{\mp}_{\beta}(w,\bar{w}) + \ldots, \tag{1.1}$$

and the standard normalization of the two-point function

$$\left\langle R^{+}_{\beta_2}(w_2,\bar{w}_2)R^{+}_{\beta_1}(w_1,\bar{w}_1)\right\rangle = \frac{\delta_{\beta_2+\beta_1,0}}{|w_2-w_1|^{2\Delta_{\beta_1}}}. \tag{1.2}$$

Formulae (1.1), (1.2) determine the braiding relations of the fermionic current with the Ramond fields up to a sign. This is in order related to the normalization of the two-point function of the R^{-}_{β} fields as can be easily verified analyzing analytic properties of the three-point functions[1]

$$\left\langle S(z)R^{\pm}_{\beta_2}(w_2,\bar{w}_2)R^{\mp}_{\beta_1}(w_1,\bar{w}_1)\right\rangle.$$

In the present paper we chose

$$\begin{gathered}\left\langle R^{-}_{\beta_2}(w_2,\bar{w}_2)R^{-}_{\beta_1}(w_1,\bar{w}_1)\right\rangle = \frac{\delta_{\beta_2+\beta_1,0}}{|w_2-w_1|^{2\Delta_{\beta_1}}},\\ \sqrt{z-w}\,S(z)R^{\pm}_{\beta}(w,\bar{w}) = \mp i\sqrt{w-z}\,R^{\pm}_{\beta}(w,\bar{w})S(z).\end{gathered} \tag{1.3}$$

The opposite convention

$$\begin{gathered}\left\langle R^{-}_{\beta_2}(w_2,\bar{w}_2)R^{-}_{\beta_1}(w_1,\bar{w}_1)\right\rangle = -\frac{\delta_{\beta_2+\beta_1,0}}{|w_2-w_1|^{2\Delta_{\beta_1}}},\\ \sqrt{z-w}\,S(z)R^{\pm}_{\beta}(w,\bar{w}) = \pm i\sqrt{w-z}\,R^{\pm}_{\beta}(w,\bar{w})S(z),\end{gathered} \tag{1.4}$$

is also possible. It is used for instance in [18, 19].

For the chiral Ramond fields we assume:

$$S(z)W^{\pm}_{\mathfrak{f}\,\beta}(w) \sim \frac{i\beta \mathrm{e}^{\mp i\frac{\pi}{4}}}{(z-w)^{\frac{3}{2}}} W^{\mp}_{\bar{\mathfrak{f}}\,\beta}(w) + \ldots, \tag{1.5}$$

[1] See appendix B for a similar analysis of chiral correlators.

where $\mathsf{f} = \mathrm{e}, \mathrm{o}$ is the parity index and $\bar{\mathsf{f}}$ denotes the parity opposed to f. Our convention for braiding (1.3) takes the form[2]

$$\begin{aligned} \sqrt{z-w}\, S(z) W^{\pm}_{\mathrm{e}\,\beta}(w) &= -\,i\,\sqrt{w-z}\, W^{\pm}_{\mathrm{e}\,\beta}(w) S(z), \\ \sqrt{z-w}\, S(z) W^{\pm}_{\mathrm{o}\,\beta}(w) &= +\,i\,\sqrt{w-z}\, W^{\pm}_{\mathrm{o}\,\beta}(w) S(z). \end{aligned} \tag{1.6}$$

Choosing the principal argument of a complex number ξ in the range $-\pi \leq \mathrm{Arg}\,\xi < \pi$ one can write (1.6) as

$$\begin{aligned} S(z) W^{\pm}_{\mathrm{e}\,\beta}(w) &= -\epsilon\, W^{\pm}_{\mathrm{e}\,\beta}(w) S(z) \\ S(z) W^{\pm}_{\mathrm{o}\,\beta}(w) &= +\epsilon\, W^{\pm}_{\mathrm{o}\,\beta}(w) S(z) \end{aligned}\,, \qquad \epsilon = \begin{cases} +1 & \text{for} \quad \mathrm{Arg}(z-w) > 0 \\ -1 & \text{for} \quad \mathrm{Arg}(z-w) < 0 \end{cases}. \tag{1.7}$$

while (1.3) reads

$$S(z) R^{\pm}_{\beta}(w,\bar{w}) = \mp\epsilon\, R^{\pm}_{\beta}(w,\bar{w}) S(z), \qquad \epsilon = \begin{cases} +1 & \text{for} \quad \mathrm{Arg}(z-w) > 0 \\ -1 & \text{for} \quad \mathrm{Arg}(z-w) < 0 \end{cases}. \tag{1.8}$$

Braiding properties (1.6), (1.7) are crucial for most of the calculations in the present paper.

Our notation for the chiral vertex operators and conformal blocks is organized as follows. In the NS sector the chiral vertex operators are denoted by

$$V_{\mathsf{f}}\!\left[{}^{\;\Delta_2}_{\Delta_3\,\Delta_1}\right]\!(z) \;:\; \mathcal{V}_{\Delta_1} \;\to\; \mathcal{V}_{\Delta_3}$$

where $\mathsf{f} = \mathrm{e}, \mathrm{o}$ is the parity index and the weights in the square brackets denote: Δ_2 — the weight of the vertex itself, Δ_1 — the weight of the source and Δ_3 — the weight of the target NS Verma module. The "star" vertices are defined by

$$V_{\mathrm{e}}\!\left[{}^{\;*\Delta_2}_{\Delta_3\,\Delta_1}\right]\!(z) = \left\{ S_{-\frac{1}{2}}, V_{\mathrm{o}}\!\left[{}^{\;\Delta_2}_{\Delta_3\,\Delta_1}\right]\!(z) \right\}, \qquad V_{\mathrm{o}}\!\left[{}^{\;*\Delta_2}_{\Delta_3\,\Delta_1}\right]\!(z) = \left[S_{-\frac{1}{2}}, V_{\mathrm{e}}\!\left[{}^{\;\Delta_2}_{\Delta_3\,\Delta_1}\right]\!(z) \right].$$

In the other three sectors the rules are similar but the vertices acquire an additional $\pm$ index:

$$V^{\pm}_{\mathsf{f}}\!\left[{}^{\;\Delta_2}_{\beta_3\,\beta_1}\right]\!(z) : \mathcal{W}_{\beta_1} \to \mathcal{W}_{\beta_3}\,, \quad V^{\pm}_{\mathsf{f}}\!\left[{}^{\;\beta_2}_{\Delta_3\,\beta_1}\right]\!(z) : \mathcal{W}_{\beta_1} \to \mathcal{V}_{\Delta_3}\,, \quad V^{\pm}_{\mathsf{f}}\!\left[{}^{\;\beta_2}_{\beta_3\,\Delta_1}\right]\!(z) : \mathcal{V}_{\Delta_1} \to \mathcal{W}_{\beta_3}\,.$$

This is related to the structure of Ward identities in these sectors. In contrast to the NS sector the 3-point conformal blocks are determined up to four rather then two structure constants. The $\pm$ values of the additional index correspond the choice of a basis of 3-point blocks required by the "small" representation mentioned above. The conformal weights of the Ramond modules are denoted by parameter β which emphasizes the sign dependence but also encodes information about the sectors. The notation of vertices is consistent with

[2]The consistency of (1.5) and (1.3) can be easily checked by explicit calculation in the representation we develop in section 3.

the notation of conformal blocks introduced in [18, 19].[3] One has for instance

$$
\begin{aligned}
\mathcal{F}^{\mathbf{f}}_{\Delta}\begin{bmatrix}\Delta_3 & \pm\beta_2\\ \Delta_4 & \beta_1\end{bmatrix}(z) &= \langle \nu_4|\, V_{\mathbf{f}}\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\Delta\end{smallmatrix}\right](1)V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\beta_2\\ \Delta\beta_1\end{smallmatrix}\right](z)\,\big|w_1^{+}\big\rangle\,,\\
\mathcal{F}^{\mathbf{f}}_{\Delta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \beta_4 & \beta_1\end{bmatrix}(z) &= \big\langle w_4^{+}\big|\, V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\beta_3\\ \beta_4\Delta\end{smallmatrix}\right](1)V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\beta_2\\ \Delta\beta_1\end{smallmatrix}\right](z)\,\big|w_1^{+}\big\rangle\,,\\
\mathcal{F}^{\mathbf{f}}_{\beta}\begin{bmatrix}\pm\beta_3 & \pm\beta_2\\ \Delta_4 & \Delta_1\end{bmatrix}(z) &= \langle \nu_4|\, V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\beta_3\\ \Delta_4\beta\end{smallmatrix}\right](1)V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\beta_2\\ \beta\Delta_1\end{smallmatrix}\right](z)\,|\nu_1\rangle\,,\\
\mathcal{F}^{\mathbf{f}}_{\beta}\begin{bmatrix}\pm\beta_3 & \Delta_2\\ \Delta_4 & \pm\beta_1\end{bmatrix}(z) &= \langle \nu_4|\, V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\beta_3\\ \Delta_4\beta\end{smallmatrix}\right](1)V^{\pm}_{\mathbf{f}}\left[\begin{smallmatrix}\Delta_2\\ \beta\beta_1\end{smallmatrix}\right](z)\,\big|w_1^{+}\big\rangle\,.
\end{aligned}
$$

Let us note that the $\pm$ in front of β-s in the symbol of a conformal block is related to the $\pm$ index of the corresponding vertex operator rather then an actual sign of this parameter. (When there are two β-s in a column we write the signs in front of the upper one.) According to these notational rules all braiding relations for the chiral vertex operators can be easily translated into analytic continuation formulae for corresponding 4-point blocks.

Although very economic for denoting vertices and blocks the Δ, β notation is not well suited for the analytic expressions for the braiding matrices. For this purposes we use in both sectors the α parametrization of conformal weights:

$$
\Delta_{\rm NS} = \frac{\alpha(Q-\alpha)}{2}, \qquad \Delta_{\rm R} = \frac{1}{16} + \frac{\alpha(Q-\alpha)}{2}.
$$

The relation α to β in the Ramond sector is straightforward $\alpha = \frac{Q}{2} - \sqrt{2}\beta$.

2 Chiral vertex operators

The relations of $N = 1$ superconformal algebra extended by the fermion parity operator $(-1)^F$ read

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n},\\
[L_m, S_k] &= \frac{m-2k}{2}S_{m+k},\\
\{S_k, S_l\} &= 2L_{k+l} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l},\\
[(-1)^F, L_m] &= \{(-1)^F, S_k\} = 0,
\end{aligned}
\tag{2.1}
$$

where $m, n \in \mathbb{Z}$ and $k, l \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz algebra sector and $k, l \in \mathbb{Z}$ in the Ramond algebra one.

The NS supermodule $\mathcal{V}_\Delta$ of the highest weight Δ and the central charge c is defined as a free vector space generated by all vectors of the form

$$
\nu_{\Delta,MK} \;=\; L_{-M}S_{-K}\nu_\Delta \;\equiv\; L_{-m_j}\ldots L_{-m_1}S_{-k_i}\ldots S_{-k_1}\nu_\Delta\,, \tag{2.2}
$$

where $K = \{k_1, k_2, \ldots, k_i\}$ and $M = \{m_1, m_2, \ldots, m_j\}$ are arbitrary ordered sets of indices

$$
k_i > \ldots > k_2 > k_1, \quad k_s \in \mathbb{N} - \frac{1}{2}, \qquad m_j \geqslant \ldots \geqslant m_2 \geqslant m_1, \quad m_r \in \mathbb{N}
$$

[3] The blocks themselves are different as in the present paper our conventions for braiding and hence the Ward identities are different.

and ν_Δ is the highest weight state with respect to the extended NS algebra:

$$L_0\nu_\Delta = \Delta\nu_\Delta, \quad (-1)^F\nu_\Delta = \nu_\Delta, \quad L_m\nu_\Delta = S_k\nu_\Delta = 0, \quad m\in\mathbb{N}, \;\; k\in\mathbb{N}-\frac{1}{2}. \tag{2.3}$$

In the Ramond sector the highest weight state is defined in a similar way

$$L_0 w^+_\beta = \Delta w^+_\beta, \quad (-1)^F w^+_\beta = w^+_\beta, \quad L_m w^+_\beta = S_k w^+_\beta = 0, \quad m\in\mathbb{N}, \;\; k\in\mathbb{N}. \tag{2.4}$$

A novel property is that the zero level subspace of the R supermodule $\mathcal{W}_\beta$ over w^+_β is 2-dimensional

$$S_0 w^\pm_\beta = i\mathrm{e}^{\mp i\frac{\pi}{4}}\beta w^\mp_\beta \quad \text{for} \quad \Delta = \frac{c}{24} - \beta^2 \neq 0. \tag{2.5}$$

Hermitian forms $\langle\,.\,,.\rangle_{c,\Delta}$ on $\mathcal{V}_\Delta$ and $\langle\,.\,,.\rangle_{c,\beta}$ on $\mathcal{W}_\beta$ are uniquely determined by the relations

$$(L_m)^\dagger = L_{-m}, \quad (S_k)^\dagger = S_{-k}, \quad \langle\nu_\Delta,\nu_\Delta\rangle = 1, \quad \langle w^+_\beta, w^+_\beta\rangle = 1, \quad \langle w^+_\beta, S_0\, w^+_\beta\rangle = 0. \tag{2.6}$$

They are block-diagonal with respect to the L_0- and $(-1)^F$-gradings.

Following [19] we introduce 3-point blocks as chiral 3-forms (anti-linear in the left argument and linear in the central and the right ones):

$$\begin{aligned}
\mathcal{V}_{\Delta_3}\times\mathcal{V}_{\Delta_2}\times\mathcal{V}_{\Delta_1} \ni (\xi_3,\xi_2,\xi_1) &\longrightarrow \varrho_{\rm NN}(\xi_3,\xi_2,\xi_1|z)\in\mathbb{C},\\
\mathcal{W}_{\beta_3}\times\mathcal{V}_{\Delta_2}\times\mathcal{W}_{\beta_1} \ni (\eta_3,\xi_2,\eta_1) &\longrightarrow \varrho_{\rm RR}(\eta_3,\xi_2,\eta_1|z)\in\mathbb{C},\\
\mathcal{V}_{\Delta_3}\times\mathcal{W}_{\beta_2}\times\mathcal{W}_{\beta_1} \ni (\xi_3,\eta_2,\eta_1) &\longrightarrow \varrho_{\rm NR}(\xi_3,\eta_2,\eta_1|z)\in\mathbb{C},\\
\mathcal{W}_{\beta_3}\times\mathcal{W}_{\beta_2}\times\mathcal{V}_{\Delta_1} \ni (\eta_3,\eta_2,\xi_1) &\longrightarrow \varrho_{\rm RN}(\eta_3,\eta_2,\xi_1|z)\in\mathbb{C},
\end{aligned}$$

satisfying the "bosonic" (with respect to L_n) and the "fermionic" (with respect to S_k) Ward identities. The "bosonic" identities are the same for 3-point blocks of all types. We shall not use them in the present discussion (see [19] for their explicit form).

The "fermionic" Ward identities for the NN type of 3-point block take the form [12]:

$$\begin{aligned}
\varrho_{\rm NN}(\xi_3,S_k\xi_2,\xi_1|z) &= \sum_{m=0}^{k+\frac{1}{2}}\binom{k+\frac{1}{2}}{m}(-z)^m\Big(\varrho_{\rm NN}(S_{m-k}\xi_3,\xi_2,\xi_1|z)\\
&\qquad - (-1)^{|\xi_1|+|\xi_3|}\,\varrho_{\rm NN}(\xi_3,\xi_2,S_{k-m}\xi_1|z)\Big), \qquad k\geqslant -\frac{1}{2},\\
\varrho_{\rm NN}(\xi_3,S_{-k}\xi_2,\xi_1|z) &= \sum_{m=0}^{\infty}\binom{k-\frac{3}{2}+m}{m} z^m\varrho_{\rm NN}(S_{k+m}\xi_3,\xi_2,\xi_1|z)\\
-\,(-1)^{|\xi_1|+|\xi_3|+k+\frac{1}{2}} &\sum_{m=0}^{\infty}\binom{k-\frac{3}{2}+m}{m} z^{-k-m+\frac{1}{2}}\varrho_{\rm NN}(\xi_3,\xi_2,S_{m-\frac{1}{2}}\xi_1|z), \qquad k>\frac{1}{2},\\
\varrho_{\rm NN}(S_{-k}\xi_3,\xi_2,\xi_1|z) &= (-1)^{|\xi_1|+|\xi_3|+1}\varrho_{\rm NN}(\xi_3,\xi_2,S_k\xi_1|z)\\
&\qquad + \sum_{m=-1}^{l(k-\frac{1}{2})}\binom{k+\frac{1}{2}}{m+1} z^{k-\frac{1}{2}-m}\varrho_{\rm NN}(\xi_3,S_{m+\frac{1}{2}}\xi_2,\xi_1|z).
\end{aligned} \tag{2.7}$$

The form $\varrho_{\rm NN}$ is determined by the Ward identities up to two independent constants

$$\begin{aligned}\varrho_{\rm NN}(\xi_3,\xi_2,\xi_1|z) &= \rho_{\rm NN}(\xi_3,\xi_2,\xi_1|z)\varrho_{\rm NN}(\nu_3,\nu_2,\nu_1|1)\\ &\quad+\rho^*_{\rm NN}(\xi_3,\xi_2,\xi_1|z)\varrho_{\rm NN}(\nu_3,*\nu_2,\nu_1|1)\end{aligned}$$

where $*\nu_i \equiv S_{-\frac{1}{2}}\nu_i$. For L_0-eingenstates, $L_0\,\xi_i = \Delta_i(\xi_i)\xi_i$

$$\begin{aligned}\rho_{\rm NN}(\xi_3,\xi_2,\xi_1|z) &= z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)}\rho_{\rm NN}(\xi_3,\xi_2,\xi_1|1),\\ \rho^*_{\rm NN}(\xi_3,\xi_2,\xi_1|z) &= z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)}\rho^*_{\rm NN}(\xi_3,\xi_2,\xi_1|1).\end{aligned}$$

Since the parity of the total number of fermionic excitations is preserved in identities (2.7):

$$\begin{aligned}\rho_{\rm NN}(S_I\nu_3,\nu_2,S_J\nu_1) = \rho^*_{\rm NN}(S_I\nu_3,*\nu_2,S_J\nu_1) = 0 \quad &\text{if} \quad |I|+|J| \in \mathbb{N}-\frac{1}{2},\\ \rho^*_{\rm NN}(S_I\nu_3,\nu_2,S_J\nu_1) = \rho_{\rm NN}(S_I\nu_3,*\nu_2,S_J\nu_1) = 0 \quad &\text{if} \quad |I|+|J| \in \mathbb{N}.\end{aligned}$$

The chiral vertex operators are defined by their matrix elements

$$\begin{aligned}\langle\xi_3|V_{\rm e}\big[{}^{\;\;\Delta_2}_{\Delta_3\,\Delta_1}\big](z)|\xi_1\rangle &= \rho_{\rm NN}(\xi_3,\nu_2,\xi_1|z),\\ \langle\xi_3|V_{\rm o}\big[{}^{\;\;\Delta_2}_{\Delta_3\,\Delta_1}\big](z)|\xi_1\rangle &= \rho^*_{\rm NN}(\xi_3,\nu_2,\xi_1|z),\\ \langle\xi_3|V_{\rm e}\big[{}^{\;*\Delta_2}_{\Delta_3\,\Delta_1}\big](z)|\xi_1\rangle &= \rho^*_{\rm NN}(\xi_3,*\nu_2,\xi_1|z),\\ \langle\xi_3|V_{\rm o}\big[{}^{\;*\Delta_2}_{\Delta_3\,\Delta_1}\big](z)|\xi_1\rangle &= \rho_{\rm NN}(\xi_3,*\nu_2,\xi_1|z).\end{aligned} \tag{2.8}$$

By the construction

$$V_{\rm e}\big[{}^{\;*\Delta_2}_{\Delta_3\,\Delta_1}\big] = \left\{S_{-\frac{1}{2}}, V_{\rm o}\big[{}^{\;\;\Delta_2}_{\Delta_3\,\Delta_1}\big]\right\}\;, \qquad V_{\rm o}\big[{}^{\;*\Delta_2}_{\Delta_3\,\Delta_1}\big] = \left[S_{-\frac{1}{2}}, V_{\rm e}\big[{}^{\;\;\Delta_2}_{\Delta_3\,\Delta_1}\big]\right].$$

The "fermionic" Ward identities for the RR 3-point block read [19]:

$$\begin{aligned}\varrho_{\rm RR}(S_{-n}\eta_3,\xi_2,\eta_1|z) &= (-1)^{|\eta_1|+|\eta_3|+1}\varrho_{\rm RR}(\eta_3,\xi_2,S_n\eta_1|z)\\ &\quad+\sum_{k=-\frac{1}{2}}^{\infty}\binom{n+\frac{1}{2}}{k+\frac{1}{2}}\,z^{n-k}\varrho_{\rm RR}(\eta_3,S_k\xi_2,\eta_1|z),\end{aligned} \tag{2.9}$$

$$\begin{aligned}\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\;z^{\frac{1}{2}-p}\;\varrho_{\rm RR}(\eta_3,S_{p-k}\xi_2,\eta_1|z) &= \sum_{p=0}^{\infty}\binom{\frac{1}{2}-k}{2}(-z)^p\;\varrho_{\rm RR}(S_{p+k-\frac{1}{2}}\eta_3,\xi_2,\eta_1|z)\\ &\quad-(-1)^{|\eta_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}-k}{2}(-z)^{\frac{1}{2}-k-p}\varrho_{\rm RR}(\eta_3,\xi_2,S_p\eta_1|z).\end{aligned}$$

The 3-form $\varrho_{\rm RR}(\eta_3,\xi_2,\eta_1|z)$ is determined up to four rather then two constants:

$$\begin{aligned}\varrho_{\rm RR}(\eta_3,\xi_2,\eta_1|z) = \;&\rho^{++}_{\rm RR}(\eta_3,\xi_2,\eta_1|z)\varrho_{\rm RR}(w_3^+,\nu_2,w_1^+|1)\\ &+\;\rho^{+-}_{\rm RR}(\eta_3,\xi_2,\eta_1|z)\varrho_{\rm RR}(w_3^+,\nu_2,w_1^-|1)\\ &+\;\rho^{-+}_{\rm RR}(\eta_3,\xi_2,\eta_1|z)\varrho_{\rm RR}(w_3^-,\nu_2,w_1^+|1)\\ &+\;\rho^{--}_{\rm RR}(\eta_3,\xi_2,\eta_1|z)\varrho_{\rm RR}(w_3^-,\nu_2,w_1^-|1)\,.\end{aligned}$$

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For L_0 eigenstates $\rho^{\imath\jmath}_{\mathrm{RR}}(\eta_3,\xi_2,\eta_1|z) = z^{\Delta_3(\eta_3)-\Delta_2(\xi_2)-\Delta_1(\eta_1)}\rho^{\imath\jmath}_{\mathrm{RR}}(\eta_3,\xi_2,\eta_1|1)$. As before the parity of the total number of fermionic excitations is preserved and therefore

$$\left.\begin{aligned} \rho^{++}_{\mathrm{RR}}(S_M\eta_3,\nu_2,S_N\eta_1) = \rho^{--}_{\mathrm{RR}}(S_M\eta_3,\nu_2,S_N\eta_1) = 0 \\ \rho^{+-}_{\mathrm{RR}}(S_M\eta_3,*\nu_2,S_N\eta_1) = \rho^{-+}_{\mathrm{RR}}(S_M\eta_3,*\nu_2,S_N\eta_1) = 0 \end{aligned}\right. \quad \text{if} \quad \#N + \#M \in 2\mathbb{N}+1,$$

$$\left.\begin{aligned} {}^{+-}_{\mathrm{RR}}(S_M\eta_3,\nu_2,S_N\eta_1) = \rho^{-+}_{\mathrm{RR}}(S_M\eta_3,\nu_2,S_N\eta_1) = 0 \\ \rho^{++}_{\mathrm{RR}}(S_M\eta_3,*\nu_2,S_N\eta_1) = \rho^{--}_{\mathrm{RR}}(S_M\eta_3,*\nu_2,S_N\eta_1) = 0 \end{aligned}\right. \quad \text{if} \quad \#N + \#M \in 2\mathbb{N}.$$

Using Ward identities (2.9) one can derive the relations

$$\begin{aligned} \rho^{++}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^+) &= \rho^{--}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^-), \\ \rho^{--}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^+) &= \rho^{++}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^-), \\ \rho^{+-}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^-) &= \rho^{-+}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^+), \\ \rho^{-+}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^-) &= -\rho^{+-}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^+), \end{aligned} \tag{2.10}$$

for the even number of fermionic operators $\#M + \#N \in 2\mathbb{N}$ and

$$\begin{aligned} \rho^{+-}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^-) &= -i\rho^{-+}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^+), \\ \rho^{-+}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^-) &= i\rho^{+-}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^+), \\ \rho^{++}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^+) &= -i\rho^{--}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^-), \\ \rho^{--}_{\mathrm{RR}}(S_M w_3^-,\nu_2,S_N w_1^+) &= -i\rho^{++}_{\mathrm{RR}}(S_M w_3^+,\nu_2,S_N w_1^-), \end{aligned} \tag{2.11}$$

for $\#M + \#N \in 2\mathbb{N}+1$. One also has

$$\begin{aligned} \rho^{++}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^+) &= (-1)^{\#J}\rho^{+-}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^-), \\ \rho^{--}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^+) &= i\,(-1)^{\#J}\rho^{-+}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^-), \\ \rho^{-+}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^+) &= (-1)^{\#J}\rho^{--}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^-), \\ \rho^{+-}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^+) &= i\,(-1)^{\#J}\rho^{++}_{\mathrm{RR}}(S_I w_3^+,\nu_2,S_J w_1^-). \end{aligned} \tag{2.12}$$

Identical relations hold for ν replaced by $*\nu$.

An appropriate basis for the 3-point blocks takes the form:

$$\rho^{(\pm)}_{\mathrm{RR,e}} = \rho^{++}_{RR} \pm \rho^{--}_{RR}\,, \qquad \rho^{(\pm)}_{\mathrm{RR,o}} = \rho^{+-}_{RR} \pm i\,\rho^{-+}_{RR}\,.$$

The corresponding chiral vertex operators are given by

$$\begin{aligned} \langle\eta_3|\, V^{\pm}_{\mathbf{f}}\big[{}^{\Delta_2}_{\beta_3\beta_1}\big](z)\,|\eta_1\rangle &= \rho^{(\pm)}_{\mathrm{RR,f}}(\eta_3,\nu,\eta_1|z), \\ \langle\eta_3|\, V^{\pm}_{\mathbf{f}}\big[{}^{*\Delta_2}_{\beta_3\beta_1}\big](z)\,|\eta_1\rangle &= \rho^{(\pm)}_{\mathrm{RR,\bar{f}}}(\eta_3,*\nu,\eta_1|z). \end{aligned} \tag{2.13}$$

In the mixed sectors the Ward identities read:

$$
\begin{aligned}
\sum_{p=0}^{\infty}\binom{n+\frac{1}{2}}{p}\; z^{n+\frac{1}{2}-p}\; \varrho_{\rm NR}(\xi_3, S_p\eta_2,\eta_1|z) &= \sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}(-z)^p\; \varrho_{\rm NR}(S_{p-n-\frac{1}{2}}\xi_3,\eta_2,\eta_1|z)\\
&\quad + i(-1)^{|\xi_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}(-1)^p\; z^{\frac{1}{2}-p}\varrho_{\rm NR}(\xi_3,\eta_2,S_{n+p}\eta_1|z),\\
\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}\; z^{\frac{1}{2}-p}\; \varrho_{\rm NR}(\xi_3, S_{p-n}\eta_2,\eta_1|z) &= \sum_{p=0}^{\infty}\binom{\frac{1}{2}-n}{p}(-z)^p\; \varrho_{\rm NR}(S_{p+n-\frac{1}{2}}\xi_3,\eta_2,\eta_1|z)\\
&\quad + i(-1)^{|\xi_3|+|\eta_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}-n}{p}(-1)^{n+p} z^{\frac{1}{2}-n-p}\varrho_{\rm NR}(\xi_3,\eta_2,S_p\eta_1|z),
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
\sum_{p=0}^{\infty}\binom{-n}{p}\; z^{-p-n}\; \varrho_{\rm RN}(\eta_3, S_p\eta_2,\xi_1|z) &= \sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}(-z)^p\; \varrho_{\rm RN}(S_{n+p}\eta_3,\eta_2,\xi_1|z)\\
&\quad + i(-1)^{|\eta_3|+|\xi_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}}{p}(-1)^p\; z^{\frac{1}{2}-p}\varrho_{\rm RN}(\eta_3,\eta_2,S_{p-n-\frac{1}{2}}\xi_1|z),\\
\varrho_{\rm RN}(\eta_3, S_{-n}\eta_2,\xi_1|z) &= \sum_{p=0}^{\infty}\binom{\frac{1}{2}-n}{p}(-z)^p\; \varrho_{\rm RN}(S_{p+n}\eta_3,\eta_2,\xi_1|z)\\
&\quad + i(-1)^{|\eta_3|+|\xi_1|+1}\sum_{p=0}^{\infty}\binom{\frac{1}{2}-n}{p}(-1)^{n+p}\, z^{\frac{1}{2}-n-p}\varrho_{\rm RN}(\eta_3,\eta_2,S_{p-\frac{1}{2}}\xi_1|z),
\end{aligned}
\tag{2.15}
$$

where $|\xi|, |\eta|$ denote parities of states $\xi \in \mathcal{V}_\Delta, \eta \in \mathcal{W}_\Delta$.[4] These relations along with the "bosonic" Ward identities determine each 3-point block up to four constants:

$$
\begin{aligned}
\varrho_{\rm NR}(\xi_3,\eta_2,\eta_1|z) &= \rho^{++}_{\rm NR}(\xi_3,\eta_2,\eta_1|z)\varrho_{\rm NR}(\nu_3,w_2^+,w_1^+|1)\\
&\quad +\rho^{+-}_{\rm NR}(\xi_3,\eta_2,\eta_1|z)\varrho_{\rm NR}(\nu_3,w_2^+,w_1^-|1)\\
&\quad +\rho^{-+}_{\rm NR}(\xi_3,\eta_2,\eta_1|z)\varrho_{\rm NR}(\nu_3,w_2^-,w_1^+|1)\\
&\quad +\rho^{--}_{\rm NR}(\xi_3,\eta_2,\eta_1|z)\varrho_{\rm NR}(\nu_3,w_2^-,w_1^-|1),\\
\varrho_{\rm RN}(\eta_3,\eta_2,\xi_1|z) &= \rho^{++}_{\rm RN}(\eta_3,\eta_2,\xi_1|z)\varrho_{\rm RN}(w_3^+,w_2^+,\nu_1|1)\\
&\quad +\rho^{+-}_{\rm RN}(\eta_3,\eta_2,\xi_1|z)\varrho_{\rm RN}(w_3^+,w_2^-,\nu_1|1)\\
&\quad +\rho^{-+}_{\rm RN}(\eta_3,\eta_2,\xi_1|z)\varrho_{\rm RN}(w_3^-,w_2^+,\nu_1|1)\\
&\quad +\rho^{--}_{\rm RN}(\eta_3,\eta_2,\xi_1|z)\varrho_{\rm RN}(w_3^-,w_2^-,\nu_1|1).
\end{aligned}
$$

For L_0-eingenstates, $L_0\,\xi_i = \Delta_i(\xi_i)\xi_i$, $L_0\,\eta_j = \Delta_j(\eta_j)\,\eta_j$ one has:

$$
\begin{aligned}
\rho^{\imath\jmath}_{\rm NR}(\xi_3,\eta_2,\eta_1|z) &= z^{\Delta_3(\xi_3)-\Delta_2(\eta_2)-\Delta_1(\eta_1)}\; \rho^{\imath\jmath}_{\rm NR}(\xi_3,\eta_2,\eta_1|1),\\
\rho^{\imath\jmath}_{\rm RN}(\eta_3,\eta_2,\xi_1|z) &= z^{\Delta_3(\eta_3)-\Delta_2(\eta_2)-\Delta_1(\xi_1)}\; \rho^{\imath\jmath}_{\rm RN}(\eta_3,\eta_2,\xi_1|1), \qquad \imath,\jmath = \pm\,.
\end{aligned}
$$

[4] Ward identities (2.14), (2.15) differ from the corresponding ones in [18, 19] by the sign in front of i. This comes from the opposite convention (1.4) which was (implicitly) assumed in [18, 19].

Using Ward identities (2.14), (2.15) one can derive the relations

$$\begin{aligned}
\rho^{+-}_{\rm NR}(S_I\nu, w_2^-, S_J w_1^+) &= i\,\rho^{--}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),\\
\rho^{-+}_{\rm NR}(S_I\nu, w_2^-, S_J w_1^+) &= \rho^{++}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),\\
\rho^{++}_{\rm NR}(S_I\nu, w_2^-, S_J w_1^+) &= i\,\rho^{-+}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),\\
\rho^{--}_{\rm NR}(S_I\nu, w_2^-, S_J w_1^+) &= \rho^{+-}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\rho^{+-}_{\rm RN}(S_J w_3^+, w_2^-, S_I\nu) &= \rho^{++}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),\\
\rho^{-+}_{\rm RN}(S_J w_3^+, w_2^-, S_I\nu) &= i\,\rho^{--}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),\\
\rho^{++}_{\rm RN}(S_J w_3^+, w_2^-, S_I\nu) &= i\,\rho^{+-}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),\\
\rho^{--}_{\rm RN}(S_J w_3^+, w_2^-, S_I\nu) &= \rho^{-+}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\rho^{+-}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^-) &= (-1)^{\#J}\,\rho^{++}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),\\
\rho^{-+}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^-) &= -i\,(-1)^{\#J}\,\rho^{--}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),\\
\rho^{++}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^-) &= -i\,(-1)^{\#J}\,\rho^{+-}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),\\
\rho^{--}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^-) &= (-1)^{\#J}\,\rho^{-+}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+),
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
\rho^{+-}_{\rm RN}(S_J w_3^-, w_2^+, S_I\nu) &= -i\,(-1)^{\#I}\,\rho^{--}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),\\
\rho^{-+}_{\rm RN}(S_J w_3^-, w_2^+, S_I\nu) &= (-1)^{\#I}\,\rho^{++}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),\\
\rho^{++}_{\rm RN}(S_J w_3^-, w_2^+, S_I\nu) &= -i\,(-1)^{\#I}\,\rho^{-+}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu),\\
\rho^{--}_{\rm RN}(S_J w_3^-, w_2^+, S_I\nu) &= (-1)^{\#I}\,\rho^{+-}_{\rm RN}(S_J w_3^+, w_2^+, S_I\nu).
\end{aligned} \tag{2.19}$$

For states with definite parities some forms identically vanish:

$$\begin{aligned}
\rho^{\pm\pm}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+) = \rho^{\pm\pm}_{\rm RN}(S_J w_3^+, w_2^-, S_I\nu) = 0 \quad &\text{if} \quad (2|I| + \#J) \in 2\mathbb{N}+1,\\
\rho^{\pm\mp}_{\rm NR}(S_I\nu, w_2^+, S_J w_1^+) = \rho^{\pm\mp}_{\rm RN}(S_J w_3^+, w_2^-, S_I\nu) = 0 \quad &\text{if} \quad (2|I| + \#J) \in 2\mathbb{N}.
\end{aligned}$$

The decomposition of the 4-point functions of Ramond fields into conformal blocks suggests the following convenient choice of a basis of the 3-point blocks [18, 19]

$$\begin{aligned}
\rho^{(\pm)}_{\rm NR,e} &= \rho^{++}_{\rm NR} \pm \rho^{--}_{\rm NR}\,, & \rho^{(\pm)}_{\rm RN,e} &= \rho^{++}_{\rm RN} \pm \rho^{--}_{\rm RN}\,,\\
\rho^{(\pm)}_{\rm NR,o} &= \rho^{+-}_{\rm NR} \pm i\rho^{-+}_{\rm NR}\,, & \rho^{(\pm)}_{\rm RN,o} &= \rho^{-+}_{\rm RN} \pm i\rho^{+-}_{\rm RN}\,.
\end{aligned} \tag{2.20}$$

The chiral vertex operators are then defined by their matrix elements as follows

$$\begin{aligned}
\langle\xi_3|V^{\pm}_{\mathbf{f}}\!\left[^{\ \beta_2}_{\Delta_3\,\beta_1}\right]\!(z)|\eta_1\rangle &= \rho^{(\pm)}_{\rm NR,\mathbf{f}}(\xi_3, w_2^+, \eta_1|z)\,,\\
\langle\eta_3|V^{\pm}_{\mathbf{f}}\!\left[^{\ \beta_2}_{\beta_3\,\Delta_1}\right]\!(z)|\xi_1\rangle &= \rho^{(\pm)}_{\rm RN,\mathbf{f}}(\eta_3, w_2^+, \xi_1|z)\,, \quad \mathbf{f} = {\rm e, o}.
\end{aligned} \tag{2.21}$$

3 Chiral superscalar

3.1 Chiral fermion

In the NS sector the chiral fermion field decomposes into half-integer modes:

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac12} \psi_r\, z^{-r-\frac12}, \quad \{\psi_r, \psi_s\} = \delta_{r+s}, \quad \{(-1)^F, \psi_s\} = 0, \quad \psi_r^\dagger = \psi_{-r}. \tag{3.1}$$

The algebra of modes is realized in the Fock space $\mathcal{F}_{\rm NS}$ generated out of the vacuum $|\Omega_{\rm F}\rangle$ satisfying $\psi_r|\Omega_{\rm F}\rangle = 0,\ r > 0,\ (-1)^F|\Omega_{\rm F}\rangle = |\Omega_{\rm F}\rangle,\ \langle\Omega_{\rm F}|\Omega_{\rm F}\rangle = 1.$

In the R sector $\psi(z)$ has the integer mode decomposition:

$$\psi(z) = \sum_{m\in\mathbb{Z}} \psi_m\, z^{-m-\frac{1}{2}}, \quad \{\psi_m, \psi_n\} = \delta_{m+n}, \quad \{(-1)^F, \psi_m\} = 0, \quad \psi_m^\dagger = \psi_{-m}, \tag{3.2}$$

and the vacuum state of the Fock space $\mathcal{F}_{\rm R}$ is doubly degenerated

$$\psi_0|\Omega^\pm\rangle \propto |\Omega^\mp\rangle,\ \ (-1)^F|\Omega^\pm\rangle = \pm|\Omega^\pm\rangle,\ \ \langle\Omega^\pm|\Omega^\pm\rangle = 1,\ \ \langle\Omega^\pm|\Omega^\mp\rangle = 0.$$

Both Fock spaces carry the $c = \frac{1}{2}$ Virasoro algebra representation

$$\begin{aligned} L_0 &= \sum_{k>0} \left(k + \frac{1}{2}\right) \psi_{-k}\psi_k + \begin{cases} 0 & \text{NS sector} \\ \frac{1}{16} & \text{R sector} \end{cases} \\ L_m &= \frac{1}{4}\sum_k (2k - m)\, \psi_{m-k}\psi_k\,, \end{aligned}$$

where $k \in \mathbb{Z} + \frac{1}{2}$ in the NS sector and $k \in \mathbb{Z}$ in the R sector. Each Fock space can be decomposed into Virasoro Verma modules which leads to the decomposition of the total Hilbert space of the chiral fermion into Virasoro Verma modules with conformal weights $0, \frac{1}{2}, \frac{1}{16}, \frac{1}{16}$:

$$\mathcal{H}_{\rm F} = \mathcal{F}_{\rm NS} \oplus \mathcal{F}_{\rm R} = \mathcal{U}_0 \oplus \mathcal{U}_{\frac{1}{2}} \oplus \mathcal{U}^+_{\frac{1}{16}} \oplus \mathcal{U}^-_{\frac{1}{16}}. \tag{3.3}$$

In this decomposition $\mathcal{U}_0 \oplus \mathcal{U}^+_{\frac{1}{16}}$ and $\mathcal{U}_{\frac{1}{2}} \oplus \mathcal{U}^-_{\frac{1}{16}}$ are the positive and the negative eigenspaces of the partity operator $(-1)^F$, respectively.

We introduce new chiral fields $\sigma^\pm(z)$ satisfying:

$$\begin{aligned} \psi(z)\sigma^\pm(w) &= \frac{{\rm e}^{\mp\frac{i}{4}\pi}\sigma^\mp(w)}{\sqrt{2(z-w)}} + \ldots\,, \\ \langle\Omega_{\rm F}\,|\sigma^\pm(z)\sigma^\pm(w)|\Omega_{\rm F}\,\rangle &= (z-w)^{-\frac{1}{8}}\,, \\ \psi(z)\sigma^\pm(w) &= \mp\epsilon\sigma^\pm(w)\psi(z), \qquad \epsilon = \begin{cases} +1 \text{ for } {\rm Arg}(z-w) > 0 \\ -1 \text{ for } {\rm Arg}(z-w) < 0 \end{cases}. \end{aligned} \tag{3.4}$$

The operators $\sigma^\pm$ are uniquely determined by the relations above. One could in principle calculate them in terms of modes. It is however more convenient to use decomposition (3.3) and to express all the fields in terms of the (Virasoro) chiral vertex operators:

$$\sigma^+(z) = \begin{pmatrix} 0 & 0 & V^{\sigma}_{1\,\sigma}(z) & 0 \\ 0 & 0 & 0 & \frac{e^{-\frac{i}{4}\pi}}{\sqrt{2}} V^{\sigma}_{\varepsilon\,\sigma}(z) \\ V^{\sigma}_{\sigma\,1}(z) & 0 & 0 & 0 \\ 0 & \frac{e^{\frac{i}{4}\pi}}{\sqrt{2}} V^{\sigma}_{\sigma\,\varepsilon}(z) & 0 & 0 \end{pmatrix}$$

$$\sigma^-(z) = \begin{pmatrix} 0 & 0 & 0 & V^{\sigma}_{1\,\sigma}(z) \\ 0 & 0 & \frac{e^{\frac{i}{4}\pi}}{\sqrt{2}} V^{\sigma}_{\varepsilon\,\sigma}(z) & 0 \\ 0 & \frac{e^{-\frac{i}{4}\pi}}{\sqrt{2}} V^{\sigma}_{\sigma\,\varepsilon}(z) & 0 & 0 \\ V^{\sigma}_{\sigma\,1}(z) & 0 & 0 & 0 \end{pmatrix} \tag{3.5}$$

$$\psi(z) = \begin{pmatrix} 0 & V^{\varepsilon}_{1\,\varepsilon}(z) & 0 & 0 \\ V^{\varepsilon}_{\varepsilon\,1}(z) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{e^{\frac{i}{4}\pi}}{\sqrt{2}} V^{\varepsilon}_{\sigma\,\sigma}(z) \\ 0 & 0 & \frac{e^{-\frac{i}{4}\pi}}{\sqrt{2}} V^{\varepsilon}_{\sigma\,\sigma}(z) & 0 \end{pmatrix}$$

where we have applied the standard notation $1, \varepsilon, \sigma$ for the Ising chiral fields with the weights $0, \frac{1}{2}, \frac{1}{16}$, respectively. The representation above can be easily verified using well known properties of the Ising chiral vertices summarized in appendix A.

In a similar way one can calculate the braiding relation we shall need in the following:

$$\begin{aligned}
\sigma^+(z)\sigma^+(w) &= \frac{e^{\frac{i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^+(w)\sigma^+(z) + \frac{e^{-\frac{3i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^-(w)\sigma^-(z), \\
\sigma^-(z)\sigma^-(w) &= \frac{e^{\frac{i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^-(w)\sigma^-(z) + \frac{e^{-\frac{3i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^+(w)\sigma^+(z), \\
\sigma^+(z)\sigma^-(w) &= \frac{e^{\frac{i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^+(w)\sigma^-(z) + i\frac{e^{-\frac{3i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^-(w)\sigma^+(z), \\
\sigma^-(z)\sigma^+(w) &= \frac{e^{\frac{i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^-(w)\sigma^+(z) - i\frac{e^{-\frac{3i\pi\epsilon}{8}}}{\sqrt{2}}\,\sigma^+(w)\sigma^-(z).
\end{aligned} \tag{3.6}$$

It follows from (3.5) that

$$\langle w^+ |\, \sigma^+(z) \,|\, \xi \rangle = \langle w^- |\, \sigma^-(z) \,|\, \xi \rangle, \quad \langle \xi \,|\, \sigma^+(z) \,| w^+ \rangle = \langle \xi \,|\, \sigma^-(z) \,| w^- \rangle, \tag{3.7}$$

for even $\xi \in \mathcal{V}$ and

$$i\,\langle w^+ |\, \sigma^-(z) \,|\, \xi \rangle = \langle w^- |\, \sigma^+(z) \,|\, \xi \rangle, \quad \langle \xi \,|\, \sigma^-(z) \,| w^+ \rangle = i\,\langle \xi \,|\, \sigma^+(z) \,| w^- \rangle, \tag{3.8}$$

for odd $\xi \in \mathcal{V}$. One also has

$$\langle w^+ |\, \psi(z) \,| w^- \rangle = i\,\langle w^- |\, \psi(z) \,| w^+ \rangle. \tag{3.9}$$

It is probably worth to mention that in the chiral fermion theory described above the algebra of chiral fields $1, \psi, \sigma^+, \sigma^-$ does not close with respect to OPE. For instance the OPE of σ^+ with itself contains a new local operator which is neither a conformal primary nor a descendent of a primary field. In this respect the chiral fermion with a partity operator in both sectors is not a complete chiral CFT. In consequence the corresponding superscalar model we shall describe in the next subsection is not a complete chiral $N = 1$ superconformal field theory. Nevertheless, it provides an appropriate representation of the chiral vertex operators in the $N = 1$ superconformal theory.

3.2 Chiral fields

The chiral boson field with periodic boundary conditions can be defined in terms of the decomposition

$$\begin{aligned}\varphi(z) &= \mathsf{q} - i\ln(z)\,\mathsf{p} + \varphi_<(z) + \varphi_>(z),\\ \varphi_<(z) &= i\sum_{n=-\infty}^{-1}\frac{\mathsf{a}_n}{n}\,z^{-n}, \quad \varphi_>(z) \;=\; i\sum_{n=1}^{\infty}\frac{\mathsf{a}_n}{n}\,z^{-n},\end{aligned}$$

where the modes satisfy

$$[\mathsf{q},\mathsf{p}] = i, \quad [\mathsf{a}_m,\mathsf{a}_n] = m\delta_{m+n}, \quad \mathsf{p}^\dagger = \mathsf{p}, \quad \mathsf{q}^\dagger = \mathsf{q}, \quad \mathsf{a}_n^\dagger = \mathsf{a}_{-n}. \tag{3.10}$$

They are realized on the Hilbert space $\mathcal{H}_{\rm B} = L^2(\mathbb{R}) \otimes \mathcal{F}_{\rm B}$, where $\mathcal{F}_{\rm B}$ is the Fock space with the vacuum state $|\Omega_{\rm B}\rangle$ defined by the conditions $\mathsf{a}_n|\Omega_{\rm B}\rangle = 0,\ n > 0,\ \langle\Omega_{\rm B}|\Omega_{\rm B}\rangle = 1$. The superscalar Hilbert space is defined as a tensor product

$$\mathcal{H} = \mathcal{H}_{\rm B} \otimes \mathcal{H}_{\rm F} = \big(L^2(\mathbb{R}) \otimes \mathcal{F}_{\rm B} \otimes \mathcal{F}_{\rm NS}\big) \,\oplus\, \big(L^2(\mathbb{R}) \otimes \mathcal{F}_{\rm B} \otimes \mathcal{F}_{\rm R}\big)$$

and carries the representation of the $N = 1$ superconformal algebra with the central charge $c = \frac{3}{2} + 3Q^2$:

$$\begin{aligned}L_0 &= \frac{1}{8}Q^2 + \frac{1}{2}\mathsf{p}^2 + \sum_{m\geqslant 1}\mathsf{a}_{-m}\mathsf{a}_m + \sum_{k>0}\left(k+\frac{1}{2}\right)\psi_{-k}\psi_k + \begin{cases}0 & \text{NS sector}\\ \frac{1}{16} & \text{R sector}\end{cases},\\ L_n &= \left(\mathsf{p} + \frac{inQ}{2}\right)\mathsf{a}_n + \frac{1}{2}\sum_{m\neq 0,n}\mathsf{a}_{n-m}\mathsf{a}_m + \frac{1}{4}\sum_k(2k-m)\,\psi_{m-k}\psi_k\,,\\ S_k &= (\mathsf{p} + iQk)\psi_k + \sum_{m\neq 0}\mathsf{a}_m\psi_{k-m},\end{aligned} \tag{3.11}$$

where $k, l \in \mathbb{Z} + \frac{1}{2}$ in the NS sector and $k, l \in \mathbb{Z}$ in the R sector.

The superscalar Hilbert space $\mathcal{H}$ can be seen as a direct integral over the specturm of the operator p:

$$\mathcal{H} = \int_{\mathbb{R}}^{\oplus}(\mathcal{H}^p_{\rm NS} \oplus \mathcal{H}^p_{\rm R})\,dp$$

where $\mathcal{H}^p_{\rm NS} = |p\rangle \otimes \mathcal{F}_{\rm B} \otimes \mathcal{F}_{\rm NS}$, $\mathcal{H}^p_{\rm R} = |p\rangle \otimes \mathcal{F}_{\rm B} \otimes \mathcal{F}_{\rm NS}$, and $\mathsf{p}|p\rangle = p|p\rangle$. The representation (3.11) defines on $\mathcal{H}^p_{\rm NS}$ the structure of the NS Verma module $\mathcal{V}_\Delta$ with the conformal weight $\Delta = \frac{1}{8}Q^2 + \frac{1}{2}p^2$, and the structure of the R Verma module $\mathcal{W}_\Delta$ with the conformal weight $\Delta = \frac{1}{8}Q^2 + \frac{1}{16} + \frac{1}{2}p^2$ on $\mathcal{H}^p_{\rm R}$.

For our purposes it is convenient to work with the boson and the fermion fields on the unit circle, transformed back to the zero time slice of the infinite cylinder:

$$\varphi_<(\sigma) = i\sum_{n=-\infty}^{-1}\frac{\mathsf{a}_n}{n}\,\mathrm{e}^{-in\sigma},\quad \varphi_>(\sigma) = i\sum_{n=1}^{\infty}\frac{\mathsf{a}_n}{n}\,\mathrm{e}^{-in\sigma},\quad \psi(\sigma) = \sum_k \psi_k\,\mathrm{e}^{-ik\sigma},\quad \sigma^\pm(\sigma).$$

In terms of these fields we construct the ordered exponential

$$\mathsf{E}^\alpha(\sigma) = \mathrm{e}^{\frac{1}{2}\alpha\mathsf{q}}\,\mathrm{e}^{\alpha\varphi_<(\sigma)}\,\mathrm{e}^{\alpha\sigma\mathsf{p}}\,\mathrm{e}^{\alpha\varphi_>(\sigma)}\,\mathrm{e}^{\frac{1}{2}\alpha\mathsf{q}}, \tag{3.12}$$

and the screening charge operators in both sectors:

$$\mathsf{Q}(\sigma) = \int\limits_\sigma^{\sigma+2\pi} dx\;\psi(x)\mathsf{E}^b(x),\quad Q = b + \frac{1}{b},$$

satisfying

$$\begin{aligned}
[L_n, \mathsf{E}^\alpha(\sigma)] &= \mathrm{e}^{in\sigma}\left(-i\frac{d}{d\sigma} + n\Delta_\alpha\right)\mathsf{E}^\alpha(\sigma),\quad \Delta_\alpha = \frac{1}{2}\alpha(Q-\alpha),\\
[6pt]\, [S_k, \mathsf{E}^\alpha(\sigma)] &= -i\alpha\,\mathrm{e}^{ik\sigma}\,\psi(\sigma)\,\mathsf{E}^\alpha(\sigma),\\
[L_n, \mathsf{Q}(\sigma)] &= \{S_k, \mathsf{Q}(\sigma)\} \;=\; 0,
\end{aligned} \tag{3.13}$$

where $k \in \mathbb{Z}+\frac{1}{2}$ in the NS sector and $k \in \mathbb{Z}$ in the R sector. For real b the screening charge $\mathsf{Q}(\sigma)$ is hermitian, its square is positive and $\left[\mathsf{Q}(\sigma)^2\right]^t$ may be uniquely defined for complex t. Following [21] we define the "even" and the "odd" complex powers of the screening charge

$$\mathsf{Q}^s_{\rm e}(\sigma) = \left(\mathsf{Q}^2(\sigma)\right)^{\frac{s}{2}}\;,\quad \mathsf{Q}^s_{\rm o}(\sigma) = \mathsf{Q}(\sigma)\left(\mathsf{Q}^2(\sigma)\right)^{\frac{s-1}{2}}\;, \tag{3.14}$$

and the compositions

$$\mathsf{g}_{\mathfrak{f}}{}^{\alpha}_{s}(\sigma) = \mathsf{E}^\alpha(\sigma)\mathsf{Q}^s_{\mathfrak{f}}(\sigma). \tag{3.15}$$

The chiral NS fields are diagonal with respect to the sector decomposition

$$\begin{aligned}
\mathsf{V}_{\mathfrak{f}}{}^{\alpha}_{s}(\sigma) &= \begin{bmatrix}\mathsf{g}_{\mathfrak{f}}{}^{\alpha}_{s}(\sigma) & 0\\ 0 & \mathsf{g}_{\mathfrak{f}}{}^{\alpha}_{s}(\sigma)\end{bmatrix},\\
\mathsf{V}^*_{\mathfrak{f}}{}^{\alpha}_{s}(\sigma) &= \begin{bmatrix}[S_{-\frac{1}{2}}, \mathsf{E}^\alpha(\sigma)]\mathsf{Q}^s_{\bar{\mathfrak{f}}}(\sigma) & 0\\ 0 & [S_0, \mathsf{E}^\alpha(\sigma)]\mathsf{Q}^s_{\bar{\mathfrak{f}}}(\sigma)\end{bmatrix} =\\
&= \begin{bmatrix}-i\alpha\,\mathrm{e}^{-i\frac{\sigma}{2}}\psi(\sigma)\mathsf{g}_{\bar{\mathfrak{f}}}{}^{\alpha}_{s}(\sigma) & 0\\ 0 & -i\alpha\psi(\sigma)\mathsf{g}_{\bar{\mathfrak{f}}}{}^{\alpha}_{s}(\sigma)\end{bmatrix},
\end{aligned} \tag{3.16}$$

where $\mathfrak{f} = \mathrm{e}, \mathrm{o}$ and for $\mathfrak{f} = \mathrm{e}$ (resp. $\mathfrak{f} = \mathrm{o}$) we have $\bar{\mathfrak{f}} = \mathrm{o}$ (resp. $\bar{\mathfrak{f}} = \mathrm{e}$). The chiral R fields are off-diagonal:

$$\begin{aligned}
\mathsf{W}^{+}_{\mathfrak{f}}{}^{\beta}_{s}(\sigma) &= \sigma^+(\sigma)\mathsf{g}_{\mathfrak{f}}{}^{\frac{Q}{2}-\sqrt{2}\beta}_{s}(\sigma),\\
\mathsf{W}^{-}_{\mathfrak{f}}{}^{\beta}_{s}(\sigma) &= \sigma^-(\sigma)\mathsf{g}_{\bar{\mathfrak{f}}}{}^{\frac{Q}{2}-\sqrt{2}\beta}_{s}(\sigma).
\end{aligned} \tag{3.17}$$

The fields can be extended to Euclidean fields on the whole cylinder by the analytic continuation to the imaginary time

$$\begin{aligned} \mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(w) &= \mathrm{e}^{\tau L_0}\, \mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(\sigma)\, \mathrm{e}^{-\tau L_0}, \\ \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(w) &= \mathrm{e}^{\tau L_0}\, \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(\sigma)\, \mathrm{e}^{-\tau L_0}, \\ \mathsf{W}_{\mathsf{f}\,s}^{\pm\,\beta}(w) &= \mathrm{e}^{\tau L_0}\, \mathsf{W}_{\mathsf{f}\,s}^{\pm\,\beta}(\sigma)\, \mathrm{e}^{-\tau L_0}, \quad w = \tau + i\sigma. \end{aligned} \tag{3.18}$$

They are related to the fields on the complex plane $z = \mathrm{e}^{w}$ via

$$\begin{aligned} \mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(w) &= z^{\Delta_\alpha}\, \mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(z), \\ \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(w) &= z^{\Delta_\alpha+\frac{1}{2}}\, \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(z), \quad \Delta_\alpha = \frac{1}{2}\alpha(Q-\alpha), \\ \mathsf{W}_{\mathsf{f}\,s}^{\pm\,\beta}(w) &= z^{\Delta_\beta}\, \mathsf{W}_{\mathsf{f}\,s}^{\pm\,\beta}(z), \qquad \Delta_\beta = \frac{c}{24} - \beta^2. \end{aligned} \tag{3.19}$$

The chiral fields satisfy a simple braiding relation with functions of p:

$$\begin{aligned} \mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(w) f(\mathsf{p}) &= f\left(\mathsf{p} - i\left(\alpha + bs\right)\right) \mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(w), \\ \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(w) f(\mathsf{p}) &= f\left(\mathsf{p} - i\left(\alpha + bs\right)\right) \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(w), \\ \mathsf{W}_{\mathsf{f}\,s}^{\pm\,\beta}(w) f(\mathsf{p}) &= f\left(\mathsf{p} - i\left(\frac{Q}{2} - \sqrt{2}\beta + bs\right)\right) \mathsf{W}_{\mathsf{f}\,s}^{\pm\,\beta}(w). \end{aligned}$$

An important feature of the fields $\mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(z), \mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(z)$ is that for the conformal weights

$$\Delta_1 = \frac{Q^2}{8} + \frac{1}{2}p^2, \quad \Delta_2 = \frac{1}{2}\alpha(Q-\alpha), \quad \Delta_3 = \frac{Q^2}{8} + \frac{1}{2}\left(p - i\left[\alpha + bs\right]\right)^2,$$

there exists a unique form $\varrho_{\rm NN}(\xi_3, \xi_2, \xi_1|z) : \mathcal{V}_{\Delta_3} \times \mathcal{V}_{\Delta_2} \times \mathcal{V}_{\Delta_1} \to \mathbb{C}$, satisfying the Ward identities (2.7) such that

$$\left.\begin{aligned} \varrho_{\rm NN}(\xi_3, \nu_2, \xi_1|z) &= \langle \xi_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{\,\alpha}(z)\, |\xi_1\rangle \\ \varrho_{\rm NN}(\xi_3, *\nu_2, \xi_1|z) &= \langle \xi_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{*\,\alpha}(z)\, |\xi_1\rangle \end{aligned}\right. \quad \text{for } |\xi_3| + |\xi_1| \text{ even,}$$

$$\left.\begin{aligned} \varrho_{\rm NN}(\xi_3, \nu_2, \xi_1|z) &= \langle \xi_3 |\, \mathsf{V}_{\mathsf{o}\,s}^{\,\alpha}(z)\, |\xi_1\rangle \\ \varrho_{\rm NN}(\xi_3, *\nu_2, \xi_1|z) &= \langle \xi_3 |\, \mathsf{V}_{\mathsf{o}\,s}^{*\,\alpha}(z)\, |\xi_1\rangle \end{aligned}\right. \quad \text{for } |\eta_3| + |\eta_1| \text{ odd.}$$

It thus follows from definition (2.8) that one can use $\mathsf{V}_{\mathsf{f}\,s}^{\,\alpha}(z)$ and $\mathsf{V}_{\mathsf{f}\,s}^{*\,\alpha}(z)$ to represent the chiral vertex operators in the NS sector:

$$\begin{aligned} \langle \xi_3 | V_{\rm e}\big[{}_{\Delta_3}{}^{\Delta_2}{}_{\Delta_1}\big](z) |\xi_1\rangle &= \frac{\langle \xi_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{\,\alpha}(z)\, |\xi_1\rangle}{\langle \nu_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{\,\alpha}(1)\, |\nu_1\rangle}, \\ \langle \xi_3 | V_{\rm o}\big[{}_{\Delta_3}{}^{\Delta_2}{}_{\Delta_1}\big](z) |\xi_1\rangle &= \frac{\langle \xi_3 |\, \mathsf{V}_{\mathsf{o}\,s}^{\,\alpha}(z)\, |\xi_1\rangle}{\langle \nu_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{*\,\alpha}(1)\, |\nu_1\rangle}, \\ \langle \xi_3 | V_{\rm e}\big[{}_{\Delta_3}{}^{*\Delta_2}{}_{\Delta_1}\big](z) |\xi_1\rangle &= \frac{\langle \xi_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{*\,\alpha}(z)\, |\xi_1\rangle}{\langle \nu_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{*\,\alpha}(1)\, |\nu_1\rangle}, \\ \langle \xi_3 | V_{\rm o}\big[{}_{\Delta_3}{}^{*\Delta_2}{}_{\Delta_1}\big](z) |\xi_1\rangle &= \frac{\langle \xi_3 |\, \mathsf{V}_{\mathsf{o}\,s}^{*\,\alpha}(z)\, |\xi_1\rangle}{\langle \nu_3 |\, \mathsf{V}_{\mathsf{e}\,s}^{\,\alpha}(1)\, |\nu_1\rangle}\;. \end{aligned} \tag{3.20}$$

A similar property holds in the R sector. For the conformal weights

$$\Delta_1 = \frac{c}{24} + \frac{1}{2}p^2, \quad \Delta_2 = \frac{1}{2}\alpha(Q-\alpha), \quad \Delta_3 = \frac{c}{24} + \frac{1}{2}\big(p - i(\alpha + bs)\big)^2,$$

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there exists a unique form $\varrho_{\rm RR}(\eta_3,\xi_2,\eta_1|z):\mathcal{W}_{\Delta_3}\times\mathcal{V}_{\Delta_2}\times\mathcal{W}_{\Delta_1}\to\mathbb{C}$, satisfying the Ward identities (2.9) such that

$$\left.\begin{array}{rcl}\varrho_{\rm RR}(\eta_3,\nu_2,\eta_1|z) &=& \langle\eta_3|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(z)\,|\eta_1\rangle\\ \varrho_{\rm RR}(\eta_3,*\nu_2,\eta_1|z) &=& \langle\eta_3|\,\mathsf{V}^{*\,\alpha}_{\mathrm{e}\,s}(z)\,|\eta_1\rangle\end{array}\right.\quad\text{for }\ |\eta_3|+|\eta_1|\ \text{ even,}$$

$$\left.\begin{array}{rcl}\varrho_{\rm RR}(\eta_3,\nu_2,\eta_1|z) &=& \langle\eta_3|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(z)\,|\eta_1\rangle\\ \varrho_{\rm RR}(\eta_3,*\nu_2,\eta_1|z) &=& \langle\eta_3|\,\mathsf{V}^{*\,\alpha}_{\mathrm{o}\,s}(z)\,|\eta_1\rangle\end{array}\right.\quad\text{for }\ |\eta_3|+|\eta_1|\ \text{ odd.}$$

From definitions (3.16), (3.18), (3.19) and relation (3.9) one gets

$$\begin{aligned}\big\langle\, w^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(z)\,\big|\,w^+\big\rangle &= \big\langle\, w^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(z)\,\big|\,w^-\big\rangle\,,\\ \big\langle\, w^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(z)\,\big|\,w^-\big\rangle &= i\,\big\langle\, w^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(z)\,\big|\,w^+\big\rangle\,.\end{aligned}\tag{3.21}$$

Chiral vertices (2.13) can then be represented as follows

$$\begin{aligned}\langle\eta_3|V^{+}_{\mathrm{e}}\!\left[{}^{\Delta_2}_{\beta_3\beta_1}\right](z)|\eta_1\rangle &= \frac{\langle\eta_3|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(z)\,|\eta_1\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(1)\,\big|w_1^+\big\rangle},\\
\langle S_I w_3^+|V^{-}_{\mathrm{e}}\!\left[{}^{\Delta_2}_{\beta_3\beta_1}\right](z)|S_J w_1^+\rangle &= (-1)^{\#I}\frac{\big\langle S_I w_3^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(z)\,\big|S_J w_1^+\big\rangle}{\big\langle w_3^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(1)\,\big|w_1^+\big\rangle}\\
&= (-1)^{\#J}\frac{\big\langle S_I w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(z)\,\big|S_J w_1^-\big\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(1)\,\big|w_1^-\big\rangle},\\
\langle S_I w_3^+|V^{+}_{\mathrm{o}}\!\left[{}^{\Delta_2}_{\beta_3\beta_1}\right](z)|S_J w_1^+\rangle &= (-1)^{\#I}\frac{\big\langle S_I w_3^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(z)\,\big|S_J w_1^+\big\rangle}{\big\langle w_3^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(1)\,\big|w_1^-\big\rangle}\\
&= i(-1)^{\#J}\frac{\big\langle S_I w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(z)\,\big|S_J w_1^-\big\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(1)\,\big|w_1^+\big\rangle},\\
\langle\eta_3|V^{-}_{\mathrm{o}}\!\left[{}^{\Delta_2}_{\beta_3\beta_1}\right](z)|\eta_1\rangle &= \frac{\langle\eta_3|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(z)\,|\eta_1\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(1)\,\big|w_1^-\big\rangle},\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}\langle\eta_3|V^{+}_{\mathrm{o}}\!\left[{}^{*\Delta_2}_{\beta_3\beta_1}\right](z)|\eta_1\rangle &= \frac{\langle\eta_3|\,\mathsf{V}^{*\,\alpha}_{\mathrm{o}\,s}(z)\,|\eta_1\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(1)\,\big|w_1^+\big\rangle},\\
\langle S_I w_3^+|V^{-}_{\mathrm{e}}\!\left[{}^{*\Delta_2}_{\beta_3\beta_1}\right](z)|S_J w_1^+\rangle &= (-1)^{\#I}\frac{\big\langle S_I w_3^-\big|\,\mathsf{V}^{*\,\alpha}_{\mathrm{o}\,s}(z)\,\big|S_J w_1^+\big\rangle}{\big\langle w_3^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(1)\,\big|w_1^+\big\rangle}\\
&= (-1)^{\#J}\frac{\big\langle S_I w_3^+\big|\,\mathsf{V}^{*\,\alpha}_{\mathrm{o}\,s}(z)\,\big|S_J w_1^-\big\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(1)\,\big|w_1^-\big\rangle},\\
\langle S_I w_3^+|V^{+}_{\mathrm{o}}\!\left[{}^{*\Delta_2}_{\beta_3\beta_1}\right](z)|S_J w_1^+\rangle &= (-1)^{\#I}\frac{\big\langle S_I w_3^-\big|\,\mathsf{V}^{*\,\alpha}_{\mathrm{e}\,s}(z)\,\big|S_J w_1^+\big\rangle}{\big\langle w_3^-\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(1)\,\big|w_1^-\big\rangle}\\
&= i(-1)^{\#J}\frac{\big\langle S_I w_3^+\big|\,\mathsf{V}^{*\,\alpha}_{\mathrm{e}\,s}(z)\,\big|S_J w_1^-\big\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{e}\,s}(1)\,\big|w_1^+\big\rangle},\\
\langle\eta_3|V^{-}_{\mathrm{e}}\!\left[{}^{*\Delta_2}_{\beta_3\beta_1}\right](z)|\eta_1\rangle &= \frac{\langle\eta_3|\,\mathsf{V}^{*\,\alpha}_{\mathrm{e}\,s}(z)\,|\eta_1\rangle}{\big\langle w_3^+\big|\,\mathsf{V}^{\;\alpha}_{\mathrm{o}\,s}(1)\,\big|w_1^-\big\rangle}.\end{aligned}\tag{3.23}$$

One can repeat the above considerations for the R chiral fields. For the conformal weights

$$\Delta_1\;=\;\frac{c}{24}+\frac{1}{2}p^2,\quad \Delta_2\;=\;\frac{c}{24}-\beta^2,\quad \Delta_3\;=\;\frac{Q^2}{8}+\frac{1}{2}\left(p-i\left[\frac{Q}{2}-\sqrt{2}\beta+bs\right]\right)^2,$$

there exists a unique form $\varrho_{\rm NR}(\xi_3,\eta_2,\eta_1|z):\mathcal{V}_{\Delta_3}\times\mathcal{W}_{\Delta_2}\times\mathcal{W}_{\Delta_1}\to\mathbb{C}$, satisfying the Ward identities (2.14) such that

$$\left.\begin{array}{rcl}\varrho_{\rm NR}(\xi_3,w_2^+,\eta_1|z) &=& \langle\xi_3|\,\mathsf{W}^{+\,\beta}_{\rm e\,s}(z)\,|\eta_1\rangle\\ \varrho_{\rm NR}(\xi_3,w_2^-,\eta_1|z) &=& \langle\xi_3|\,\mathsf{W}^{-\,\beta}_{\rm e\,s}(z)\,|\eta_1\rangle\end{array}\right.\quad\text{for } |\xi_3|+|\eta_1| \text{ even,}$$

$$\left.\begin{array}{rcl}\varrho_{\rm NR}(\xi_3,w_2^+,\eta_1|z) &=& \langle\xi_3|\,\mathsf{W}^{+\,\beta}_{\rm o\,s}(z)\,|\eta_1\rangle\\ \varrho_{\rm NR}(\xi_3,w_2^-,\eta_1|z) &=& \langle\xi_3|\,\mathsf{W}^{-\,\beta}_{\rm o\,s}(z)\,|\eta_1\rangle\end{array}\right.\quad\text{for } |\xi_3|+|\eta_1| \text{ odd.}$$

In a similar way, for the conformal weights

$$\Delta_1 = \frac{Q^2}{8}+\frac{1}{2}p^2,\quad \Delta_2 = \frac{c}{24}-\beta^2,\quad \Delta_3 = \frac{c}{24}+\frac{1}{2}\left(p-i\left[\frac{Q}{2}-\sqrt{2}\beta+bs\right]\right)^2,$$

the relations

$$\left.\begin{array}{rcl}\varrho_{\rm RN}(\eta_3,w_2^+,\xi_1|z) &=& \langle\eta_3|\,\mathsf{W}^{+\,\beta}_{\rm e\,s}(z)\,|\xi_1\rangle\\ \varrho_{\rm RN}(\eta_3,w_2^-,\xi_1|z) &=& \langle\eta_3|\,\mathsf{W}^{-\,\beta}_{\rm e\,s}(z)\,|\xi_1\rangle\end{array}\right.\quad\text{for } |\eta_3|+|\xi_1| \text{ even,}$$

$$\left.\begin{array}{rcl}\varrho_{\rm RN}(\eta_3,w_2^+,\xi_1|z) &=& \langle\eta_3|\,\mathsf{W}^{+\,\beta}_{\rm o\,s}(z)\,|\xi_1\rangle\\ \varrho_{\rm RN}(\eta_3,w_2^-,\xi_1|z) &=& \langle\eta_3|\,\mathsf{W}^{-\,\beta}_{\rm o\,s}(z)\,|\xi_1\rangle\end{array}\right.\quad\text{for } |\eta_3|+|\xi_1| \text{ odd,}$$

uniquely define the form $\varrho_{\rm RN}(\eta_3,\eta_2,\nu_1|z):\mathcal{W}_{\Delta_3}\times\mathcal{W}_{\Delta_2}\times\mathcal{V}_{\Delta_1}\to\mathbb{C}$ satisfying the Ward identities (2.15). From definitions (3.17), (3.18), (3.19) and relations (3.7), (3.8) one gets

$$\begin{aligned}\langle\,\nu\,|\,\mathsf{W}^{+\,\beta}_{\rm e\,s}(z)\,|\,w^+\rangle &= \langle\,\nu\,|\,\mathsf{W}^{-\,\beta}_{\rm o\,s}(z)\,|\,w^-\rangle\,,\\ \langle\,\nu\,|\,\mathsf{W}^{+\,\beta}_{\rm o\,s}(z)\,|\,w^-\rangle &= i\,\langle\,\nu\,|\,\mathsf{W}^{-\,\beta}_{\rm e\,s}(z)\,|\,w^+\rangle\,,\\ \langle w^+|\,\mathsf{W}^{+\,\beta}_{\rm e\,s}(z)\,|\,\nu\,\rangle &= \langle w^-|\,\mathsf{W}^{-\,\beta}_{\rm o\,s}(z)\,|\,\nu\,\rangle\,,\\ \langle w^-|\,\mathsf{W}^{+\,\beta}_{\rm o\,s}(z)\,|\,\nu\,\rangle &= i\,\langle w^+|\,\mathsf{W}^{-\,\beta}_{\rm e\,s}(z)\,|\,\nu\,\rangle\,.\end{aligned}\tag{3.24}$$

Using identities (2.16), (2.17) and (3.21) one can express universal forms $\rho^{\pm\pm},\rho^{\pm\mp}$ and then the matrix elements of chiral vertex operators (2.21) in terms of matrix elements of operators $\mathsf{W}^{\pm\,\beta}_{\mathsf{f}\,s}(z)$. The result is:

$$\begin{aligned}\langle\xi_3|V^{\pm}_{\rm e}\big[{}^{\;\beta_2}_{\Delta_3\beta_1}\big](z)|\eta_1\rangle &= \frac{\langle\xi_3|\,\mathsf{W}^{\pm\,\beta_2}_{\rm e\,s}(z)\,|\eta_1\rangle}{\langle\nu_3|\,\mathsf{W}^{\pm\,\beta_2}_{\rm e\,s}(z)\,|w_1^+\rangle},\\ \langle\xi_3|V^{\pm}_{\rm o}\big[{}^{\;\beta_2}_{\Delta_3\beta_1}\big](z)|\eta_1\rangle &= \frac{\langle\xi_3|\,\mathsf{W}^{\mp\,\beta_2}_{\rm o\,s}(z)\,|\eta_1\rangle}{\langle\nu_3|\,\mathsf{W}^{\mp\,\beta_2}_{\rm o\,s}(z)\,|w_1^-\rangle},\\ \langle\eta_3|V^{\pm}_{\rm e}\big[{}^{\;\beta_2}_{\beta_3\Delta_1}\big](z)|\xi_1\rangle &= \frac{\langle\eta_3|\,\mathsf{W}^{\pm\,\beta_2}_{\rm e\,s}(z)\,|\xi_1\rangle}{\langle w_3^+|\,\mathsf{W}^{\pm\,\beta_2}_{\rm e\,s}(z)\,|\nu_1\rangle},\\ \langle\eta_3|V^{\pm}_{\rm o}\big[{}^{\;\beta_2}_{\beta_3\Delta_1}\big](z)|\xi_1\rangle &= \frac{\langle\eta_3|\,\mathsf{W}^{\mp\,\beta_2}_{\rm o\,s}(z)\,|\xi_1\rangle}{\langle w_3^-|\,\mathsf{W}^{\mp\,\beta_2}_{\rm o\,s}(z)\,|\nu_1\rangle}.\end{aligned}\tag{3.25}$$

3.3 Matrix elements

In this subsection we shall calculate the matrix elements

$$\langle\nu_3|\mathsf{W}^{\pm\,\beta}_{\mathsf{f}\,s}(1)|w_1\rangle,\qquad \langle w_3|\mathsf{W}^{\pm\,\beta}_{\mathsf{f}\,s}(1)|\nu_1\rangle\quad\text{and}\quad \langle w_3|\mathsf{V}^{-\,\alpha}_{\mathsf{f}\,s}(1)|w_1\rangle,$$

using a suitable modification of the procedure proposed in [9] and adapted to the NS sector of the $N = 1$ superconformal theory in [21]. The idea is to find an explicit form of an appropriate four-point, chiral correlator containing a degenerate Ramond field and then, by studying its different limits, to express the matrix elements we are after through the matrix elements of the chiral NS field computed in [21].

For $\beta = \beta_+ \equiv \frac{1}{2\sqrt{2}}\left(b^{-1} + 2b\right)$ the Ramond supermodule $\mathcal{W}_{\beta_+}$ is degenerate (with respect to scalar product (2.6)). The vector

$$\chi_+ \;=\; \left(\kappa_+ L_{-1} - S_{-1}S_0\right) w_{\beta_+}, \qquad \kappa_+ = \sqrt{2}b^{-1}\beta_+$$

is orthogonal to all vectors in $\mathcal{W}_{\beta_+}$. Using (2.5) we can rewrite the condition that χ_+ is null in the form of a pair of operator equations

$$\frac{\sqrt{2}}{b}\frac{\partial}{\partial z}\mathsf{W}^{+\,\beta_+}_{\mathrm{e}\,s}(z) = i\,\mathrm{e}^{-\frac{i\pi}{4}}S_{-1}\mathsf{W}^{-\,\beta_+}_{\mathrm{o}\,s}(z), \qquad \frac{\sqrt{2}}{b}\frac{\partial}{\partial z}\mathsf{W}^{-\,\beta_+}_{\mathrm{o}\,s}(z) = i\,\mathrm{e}^{\frac{i\pi}{4}}S_{-1}\mathsf{W}^{+\,\beta_+}_{\mathrm{e}\,s}(z), \tag{3.26}$$

which should hold in arbitrary correlation function. Introducing

$$\begin{aligned}
f_+(z) &= \langle\nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\,s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle\,,\\
f_-(z) &= \langle\nu_4|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\,s_3}(1)\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)\,|\nu_1\rangle\,,\\
g_+(z) &= \mathrm{e}^{\frac{i\pi}{4}}\,\langle\nu_4|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\,s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)|S_{-\frac12}\nu_1\rangle,\\
g_-(z) &= \mathrm{e}^{-\frac{i\pi}{4}}\,\langle\nu_4|\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\,s_3}(1)\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)|S_{-\frac12}\nu_1\rangle,
\end{aligned} \tag{3.27}$$

we get in particular

$$\begin{aligned}
&\left(\frac{\sqrt{2}}{b}\frac{\partial}{\partial z} - \frac{\frac12\beta_+}{z-1}\right)f_+(z) + \frac{\beta_3}{z-1}f_-(z) = -\frac{g_-(z)}{\sqrt{z(1-z)}},\\
&\left(\frac{\sqrt{2}}{b}\frac{\partial}{\partial z} - \frac{\frac12\beta_+}{z-1}\right)f_-(z) + \frac{\beta_3}{z-1}f_+(z) = \frac{g_+(z)}{\sqrt{z(1-z)}},\\
&\left(\frac{\sqrt{2}}{b}\frac{\partial}{\partial z} - \frac{\frac12\beta_+}{z-1}\right)g_+(z) + \frac{\beta_3}{z-1}g_-(z) = -\frac{1}{\sqrt{z(1-z)}}\left[\frac{\Delta_1}{z} + \Delta_{2+3-4} + (z-1)\frac{\partial}{\partial z}\right]f_-(z),\\
&\left(\frac{\sqrt{2}}{b}\frac{\partial}{\partial z} - \frac{\frac12\beta_+}{z-1}\right)g_-(z) + \frac{\beta_3}{z-1}g_+(z) = \frac{1}{\sqrt{z(1-z)}}\left[\frac{\Delta_1}{z} + \Delta_{2+3-4} + (z-1)\frac{\partial}{\partial z}\right]f_+(z).
\end{aligned} \tag{3.28}$$

For the new function $h(z)$:

$$f_+(z) + f_-(z) = (1-z)^A\,h(z)\;, \quad A = \frac{b\alpha_3}{2} - \frac18\,,$$

we obtain from (3.28)

$$\frac{\partial^2 h(z)}{\partial z^2} + \frac12\left(\frac{1-b^2}{z} + \frac{2\mathfrak{A}}{z-1}\right)\frac{\partial h(z)}{\partial z} = \frac{b^2}{2z(z-1)}\left[\frac{\Delta_1}{z} + \Delta_{2+3-4} + A\right]h(z), \tag{3.29}$$

where $\mathfrak{A} = b\alpha_3 - \frac12 b^2$.

The solution of (3.29) corresponding to the sum of correlators $f_+(z) + f_-(z)$ can be singled out by its leading behavior at $z \to 0$. It follows from the momentum conservation that all the states obtained by the action of $\mathsf{W}^{\pm\,\beta_+}_{\mathsf{f}}(z)$ on the vector ν_1 have momenta equal to

$$q = p_1 + \frac{ib}{2}. \tag{3.30}$$

The small z behavior of $f_\pm(z)$ can thus be calculated by inserting a projection on the highest weight states of $\mathcal{W}_q$ with an appropriately chosen parity. This gives

$$\begin{aligned} \langle \nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle &= \langle \nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)|w_q^+\rangle\, z^{\frac{b\alpha_1}{2}}\Big(1+\mathcal{O}(z)\Big),\\ \langle \nu_4|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)\,|\nu_1\rangle &= \langle \nu_4|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)|w_q^-\rangle\, z^{\frac{b\alpha_1}{2}}\Big(1+\mathcal{O}(z)\Big), \end{aligned} \tag{3.31}$$

where we have used the identities

$$\begin{aligned} \langle w_q^+|\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(1)|\nu_1\rangle &= \langle q|\mathsf{E}^{-\frac{b}{2}}(1)|p_1\rangle\,\langle \sigma^+|\sigma^+(1)|0\rangle = 1,\\ \langle w_q^-|\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(1)|\nu_1\rangle &= \langle q|\mathsf{E}^{-\frac{b}{2}}(1)|p_1\rangle\,\langle \sigma^-|\sigma^-(1)|0\rangle = 1. \end{aligned}$$

It follows from (3.21) that the asymptotics in (3.31) are identical, hence

$$\begin{aligned} f_+(z) + f_-(z) &= \langle \nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)|w_q^+\rangle\, z^{\frac{b\alpha_1}{2}}\,(1-z)^{\frac{b\alpha_3}{2}-\frac{1}{8}}\\ &\quad\times {}_2F_1\left(\frac{b}{4}(2a_1+2\alpha_3-2\alpha_4-b),\frac{b}{4}(2a_1+2\alpha_3-2\bar{\alpha}_4-b);\frac{1}{2}(1-b^2+2b\alpha_1);z\right), \end{aligned} \tag{3.32}$$

where ${}_2F_1$ denotes the hypergeometric function.

In a similar way one can deduce from (3.28) the differential equation satisfied by the difference $f_+(z) - f_-(z)$. Extracting a suitable power of $(1-z)$ from $f_+(z) - f_-(z)$ we obtain (3.29) with different values of the parameters $A, \mathfrak{A}$. Equations (3.31) and (3.21) then give

$$f_+(z) - f_-(z) = 0$$

and consequently

$$\begin{aligned} \langle \nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle &= \langle \nu_4|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)\,|\nu_1\rangle\\ &= \langle \nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)|w_q^+\rangle\, z^{\frac{b\alpha_1}{2}}\,(1-z)^{\frac{b\alpha_3}{2}-\frac{1}{8}}\\ &\quad\times {}_2F_1\left(\frac{b}{4}(2a_1+2\alpha_3-2\alpha_4-b),\frac{b}{4}(2a_1+2\alpha_3-2\bar{\alpha}_4-b);\frac{1}{2}(1-b^2+2b\alpha_1);z\right). \end{aligned} \tag{3.33}$$

We shall now determine $\langle \nu_4|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)|w_q^+\rangle$ by comparing the leading behavior of the left and the right hand side of (3.33) for $z \to 1$. We start by calculating the leading term of the OPE of $\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(z_3) = \sigma^+(z_3)\mathsf{E}^{\alpha_3}(z_3)\mathsf{Q}^{s_3}_{\mathrm{e}}(z_3)$ and $\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z_2) = \sigma^+(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2)$. Since

$$\mathsf{Q}^{s_3}_{\mathrm{e}}(z_3)\sigma^+(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2) = \mathsf{Q}^{s_3}_{\mathrm{e}}(z_2)\sigma^+(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2) + \left(\mathsf{Q}^{s_3}_{\mathrm{e}}(z_3) - \mathsf{Q}^{s_3}_{\mathrm{e}}(z_2)\right)\sigma^+(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2)$$

we have for $z_3 \to z_2$:

$$\mathsf{Q}^{s_3}_{\mathrm{e}}(z_3)\sigma^+(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2) \sim \mathsf{Q}^{s_3}_{\mathrm{e}}(z_2)\sigma^+(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2) = \sigma^+(z_2)\mathsf{Q}^{s_3}_{\mathrm{e}}(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2)$$

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where the equality follows form the definition of the screening charge Q, its even power (3.14) and braiding property (3.4). Further, from the definition of the screening charge and braiding properties of the normal ordered exponentials (see the next section for details) one has,

$$\mathsf{Q}_{\rm e}^{s_3}(z_2)\mathsf{E}^{-\frac{b}{2}}(z_2) = \mathrm{e}^{\frac{i\pi}{2}b^2 s_3}\,\mathsf{E}^{-\frac{b}{2}}(z_2)\mathsf{Q}_{\rm e}^{s_3}(z_2),$$

hence

$$\mathsf{W}^{+\,\beta_3}_{\rm e\;s_3}(z_3)\mathsf{W}^{+\,\beta_+}_{\rm e}(z_2) \;\sim\; \mathrm{e}^{\frac{i\pi}{2}b^2 s_3}\,\sigma^+(z_3)\sigma^+(z_2)\mathsf{E}^{\alpha_3}(z_3)\mathsf{E}^{-\frac{b}{2}}(z_2)\mathsf{Q}_{\rm e}^{s_3}(z_2).$$

Finally, from (3.5) and (3.12):

$$\sigma^+(z_3)\sigma^+(z_2) \;\sim\; (z_3-z_2)^{-\frac{1}{8}}, \qquad \mathsf{E}^{\alpha_3}(z_3)\mathsf{E}^{-\frac{b}{2}}(z_2) \;\sim\; (z_3-z_2)^{\frac{b\alpha_3}{2}}\,\mathsf{E}^{\alpha_3-\frac{b}{2}}(z_2)$$

and we get

$$\mathsf{W}^{+\,\beta_3}_{\rm e\;s_3}(z_3)\mathsf{W}^{+\,\beta_+}_{\rm e}(z_2) \sim \mathrm{e}^{\frac{i\pi}{2}b^2 s_3}\,(z_3-z_2)^{\frac{b\alpha_3}{2}-\frac{1}{8}}\,\mathsf{V}_{\rm e\;s_3}^{\;\alpha_3-\frac{b}{2}}(z_2),$$

so that

$$\begin{aligned} \mathsf{W}^{+\,\beta_3}_{\rm e\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\rm e}(z) &\sim \mathrm{e}^{\frac{i\pi}{2}b^2 s_3}\,(1-z)^{\frac{b\alpha_3}{2}-\frac{1}{8}}\,\mathsf{V}_{\rm e\;s_3}^{\;\alpha_3-\frac{b}{2}}(z) \\ &\sim \mathrm{e}^{\frac{i\pi}{2}b^2 s_3}\,(1-z)^{\frac{b\alpha_3}{2}-\frac{1}{8}}\,\mathsf{V}_{\rm e\;s_3}^{\;\alpha_3-\frac{b}{2}}(1). \end{aligned} \tag{3.34}$$

In conclusion, for $z\to 1$:

$$\langle\nu_4|\,\mathsf{W}^{+\,\beta_3}_{\rm e\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\rm e}(z)\,|\nu_1\rangle \;\sim\; \mathrm{e}^{\frac{i\pi}{2}b^2 s_3}\,(1-z)^{\frac{b\alpha_3}{2}-\frac{1}{8}}\;\langle\nu_4|\,\mathsf{V}_{\rm e\;s_3}^{\;\alpha_3-\frac{b}{2}}(1)\,|\nu_1\rangle \tag{3.35}$$

where the matrix element on the r.h.s. was calculated in [21] and reads:

$$\langle\nu_3|\,\mathsf{V}_{\rm e\;s}^{\;\alpha_2}(1)\,|\nu_1\rangle \;=\; \mathcal{M}^{\rm N\,N\,N}_{\alpha_3,\alpha_2,\alpha_1}, \tag{3.36}$$

$$\mathcal{M}^{\rm N\,N\,N}_{\alpha_3,\alpha_2,\alpha_1} \equiv \mathcal{N}_{\alpha_3\alpha_2\alpha_1}\frac{\Gamma_{\rm NS}(Q+\alpha_{1-2-3})\Gamma_{\rm NS}(\alpha_{1+3-2})\Gamma_{\rm NS}(Q+\alpha_{3-1-2})\Gamma_{\rm NS}(2Q-\alpha_{1+2+3})}{\Gamma_{\rm NS}(Q)\Gamma_{\rm NS}(2\alpha_1)\Gamma_{\rm NS}(2\alpha_2)\Gamma_{\rm NS}(2Q-2\alpha_3)},$$

with $\alpha_3 = \alpha_1+\alpha_2+bs$, $\alpha_{1-2-3}\equiv\alpha_1-\alpha_2-\alpha_3$ etc. and

$$\mathcal{N}_{\alpha_3\alpha_2\alpha_1} \;=\; \left[\frac{1}{2}\Gamma\left(\frac{bQ}{2}\right)b^{-\frac{bQ}{2}}\right]^{\frac{\alpha_3-\alpha_2-\alpha_1}{b}}\,\mathrm{e}^{\frac{i\pi}{2}(\alpha_3-\alpha_2-\alpha_1)(Q-\alpha_3+\alpha_2-\alpha_1)}. \tag{3.37}$$

On the other hand, we can analyze the $z\to 1$ behavior of the r.h.s. of (3.33). Using the analytic continuation formula for the hypergeometric function,

$$\begin{aligned} {}_2F_1(a,b;c;z) \;=\; &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;1+a+b-c;1-z) \\ &+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;1+c-a-b;1-z). \end{aligned} \tag{3.38}$$

we obtain a coefficient in front of $(1-z)^{\frac{b\alpha_3}{2}-\frac{1}{8}}$. It has the form

$$\frac{\Gamma\left(-\frac{b^2}{2}+b\alpha_1+\frac{1}{2}\right)\Gamma\left(\frac{b^2}{2}-b\alpha_3+1\right)}{\Gamma\left(\frac{1}{2}+\frac{b}{4}\left(2\alpha_{1+4-3}-b\right)\right)\Gamma\left(1+\frac{b}{4}\left(2\alpha_{1-4-3}+b\right)\right)}\,\langle\nu_4|\;\mathsf{W}^{+\,\beta_3}_{\rm e\;s_3}(1)|w_q^+\rangle, \tag{3.39}$$

with q given by (3.30). Comparing (3.39) with the corresponding coefficient in (3.35) we arrive at the identity

$$\begin{aligned}&\langle\nu_4|\ \mathsf{W}^{+\,\beta_3}_{\mathrm{e}\,s_3}(1)|w_q^+\rangle =\\ &\quad \mathrm{e}^{\frac{i\pi}{2}bs_3}\frac{\Gamma\left(\frac{1}{2}+\frac{b}{4}\left(2\alpha_{1+4-3}-b\right)\right)\Gamma\left(1+\frac{b}{4}\left(2\alpha_{1-4-3}+b\right)\right)}{\Gamma\left(-\frac{b^2}{2}+b\alpha_1+\frac{1}{2}\right)\Gamma\left(\frac{b^2}{2}-b\alpha_3+1\right)}\langle\nu_4|\,\mathsf{V}^{\alpha_3-\frac{b}{2}}_{\mathrm{e}\,s_3}(1)\,|\nu_1\rangle\end{aligned}\tag{3.40}$$

Using (3.36) and the "shift identities" for the Barnes functions (see appendix C) we finally get

$$\langle\nu_3|\ \mathsf{W}^{+\,\beta_2}_{\mathrm{e}\,s}(1)|w_1^+\rangle = \langle\nu_3|\ \mathsf{W}^{-\,\beta_2}_{\mathrm{o}\,s}(1)|w_1^-\rangle \ = \ \mathcal{M}^{\mathrm{R\,R\,N}}_{\alpha_3,\alpha_2,\alpha_1},\tag{3.41}$$

$$\mathcal{M}^{\mathrm{R\,R\,N}}_{\alpha_3,\alpha_2,\alpha_1} \equiv \mathcal{N}_{\alpha_3\alpha_2\alpha_1}\frac{\Gamma_{\mathrm{R}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{R}}(\alpha_{1+3-2})\Gamma_{\mathrm{NS}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{NS}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{R}}(2\alpha_1)\Gamma_{\mathrm{R}}(Q-2\alpha_2)\Gamma_{\mathrm{NS}}(2Q-2\alpha_3)}$$

where the first relation of (3.24) has been added for completeness.

The same procedure can be applied to other matrix elements. For instance, in order to compute $\langle\nu_3|\ \mathsf{W}^{+\,\beta_2}_{\mathrm{o}\,s}(1)|w_1^-\rangle$ we start from the function $\langle\nu_4|\ \mathsf{W}^{+\,\beta_3}_{\mathrm{o}\,s_3}(1)\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)|\nu_1\rangle$. The correlators

$$\langle\nu_4|\ \mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\mathsf{W}^{+\,\beta_2}_{\mathrm{e}\,s_2}(1)|\nu_1\rangle \qquad \text{and} \qquad \langle\nu_4|\ \mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)\mathsf{W}^{+\,\beta_2}_{\mathrm{o}\,s_2}(1)|\nu_1\rangle,$$

give the formulae for $\langle w_3^+|\,\mathsf{W}^{+\,\beta_2}_{\mathrm{e}\,s}(1)\,|\nu_1\rangle$ and $\langle w_3^-|\,\mathsf{W}^{+\,\beta_2}_{\mathrm{o}\,s}(1)\,|\nu_1\rangle$, respectively, and the matrix elements $\langle w_3^+|\,\mathsf{V}^{\alpha_2}_{\mathrm{e}\,s}(1)|w_1^+\rangle$ and $\langle w_3^-|\,\mathsf{V}^{\alpha_2}_{\mathrm{o}\,s}(1)|w_1^+\rangle$ can be obtained from the correlators

$$\langle w_4^+|\,\mathsf{V}^{\alpha_3}_{\mathrm{e}\,s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)|\nu_1\rangle \qquad \text{and} \qquad \langle w_4^-|\,\mathsf{V}^{\alpha_3}_{\mathrm{o}\,s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)|\nu_1\rangle.$$

The result reads

$$\begin{aligned}\langle\nu_3|\ \mathsf{W}^{+\,\beta_2}_{\mathrm{o}\,s}(1)|w_1^-\rangle &= -i\,\langle\nu_3|\ \mathsf{W}^{-\,\beta_2}_{\mathrm{e}\,s}(1)|w_1^+\rangle = \mathcal{M}^{\mathrm{N\,N\,R}}_{\alpha_3,\alpha_2,\alpha_1}\,,\\ \langle w_3^+|\,\mathsf{W}^{+\,\beta_2}_{\mathrm{e}\,s}(1)\,|\nu_1\rangle &= \quad\ \langle w_3^-|\,\mathsf{W}^{-\,\beta_2}_{\mathrm{o}\,s}(1)\,|\nu_1\rangle = \mathcal{M}^{\mathrm{R\,N\,N}}_{\alpha_3,\alpha_2,\alpha_1}\,,\\ \langle w_3^-|\,\mathsf{W}^{+\,\beta_2}_{\mathrm{o}\,s}(1)\,|\nu_1\rangle &= -i\langle w_3^+|\,\mathsf{W}^{-\,\beta_2}_{\mathrm{e}\,s}(1)\,|\nu_1\rangle = \mathcal{M}^{\mathrm{N\,R\,R}}_{\alpha_3,\alpha_2,\alpha_1}\,,\\ \langle w_3^+|\,\mathsf{V}^{\alpha_2}_{\mathrm{e}\,s}(1)|w_1^+\rangle &= \quad\ \langle w_3^-|\,\mathsf{V}^{\alpha_2}_{\mathrm{e}\,s}(1)|w_1^-\rangle = \mathcal{M}^{\mathrm{N\,R\,N}}_{\alpha_3,\alpha_2,\alpha_1}\,,\\ \langle w_3^-|\,\mathsf{V}^{\alpha_2}_{\mathrm{o}\,s}(1)|w_1^+\rangle &= -i\langle w_3^+|\,\mathsf{V}^{\alpha_2}_{\mathrm{o}\,s}(1)|w_1^-\rangle = \mathcal{M}^{\mathrm{R\,N\,R}}_{\alpha_3,\alpha_2,\alpha_1}\,,\end{aligned}\tag{3.42}$$

where

$$\begin{aligned}\mathcal{M}^{\mathrm{N\,N\,R}}_{\alpha_3,\alpha_2,\alpha_1} &\equiv \frac{\mathrm{e}^{-\frac{3i\pi}{4}}\mathcal{N}_{\alpha_3\alpha_2\alpha_1}}{\sqrt{2}}\frac{\Gamma_{\mathrm{NS}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{NS}}(\alpha_{1+3-2})\Gamma_{\mathrm{R}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{R}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{R}}(2\alpha_1)\Gamma_{\mathrm{R}}(Q-2\alpha_2)\Gamma_{\mathrm{NS}}(2Q-2\alpha_3)},\\ \mathcal{M}^{\mathrm{R\,N\,N}}_{\alpha_3,\alpha_2,\alpha_1} &\equiv \mathcal{N}_{\alpha_3\alpha_2\alpha_1}\frac{\Gamma_{\mathrm{R}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{NS}}(\alpha_{1+3-2})\Gamma_{\mathrm{NS}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{R}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{NS}}(2\alpha_1)\Gamma_{\mathrm{R}}(Q-2\alpha_2)\Gamma_{\mathrm{R}}(2Q-2\alpha_3)},\\ \mathcal{M}^{\mathrm{N\,R\,R}}_{\alpha_3,\alpha_2,\alpha_1} &\equiv \frac{\mathrm{e}^{-\frac{i\pi}{4}}\mathcal{N}_{\alpha_3\alpha_2\alpha_1}}{\sqrt{2}}\frac{\Gamma_{\mathrm{NS}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{R}}(\alpha_{1+3-2})\Gamma_{\mathrm{R}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{NS}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{NS}}(2\alpha_1)\Gamma_{\mathrm{R}}(Q-2\alpha_2)\Gamma_{\mathrm{R}}(2Q-2\alpha_3)},\\ \mathcal{M}^{\mathrm{N\,R\,N}}_{\alpha_3,\alpha_2,\alpha_1} &\equiv \mathcal{N}_{\alpha_3\alpha_2\alpha_1}\frac{\Gamma_{\mathrm{NS}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{R}}(\alpha_{1+3-2})\Gamma_{\mathrm{NS}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{R}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{R}}(2\alpha_1)\Gamma_{\mathrm{NS}}(Q-2\alpha_2)\Gamma_{\mathrm{R}}(2Q-2\alpha_3)},\end{aligned}$$

$$\mathcal{M}^{\mathrm{R\,N\,R}}_{\alpha_3,\alpha_2,\alpha_1} \equiv \frac{\mathrm{e}^{-\frac{i\pi}{4}}\mathcal{N}_{\alpha_3\alpha_2\alpha_1}}{\sqrt{2}} \frac{\Gamma_{\mathrm{R}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{NS}}(\alpha_{1+3-2})\Gamma_{\mathrm{R}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{NS}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{R}}(2\alpha_1)\Gamma_{\mathrm{NS}}(Q-2\alpha_2)\Gamma_{\mathrm{R}}(2Q-2\alpha_3)}. \tag{3.43}$$

Matrix elements (3.36), (3.41), (3.43), supplemented by the matrix element

$$\begin{aligned} \langle \nu_3 |\, \mathsf{V}^{*\,\alpha_2}_{\mathrm{e}\;\,s}(1)\, |\nu_1\rangle \;&=\; \mathcal{M}^{\mathrm{R\,R\,R}}_{\alpha_3,\alpha_2,\alpha_1} \qquad (3.44)\\ &\equiv\; 2\mathcal{N}_{\alpha_3\alpha_2\alpha_1} \frac{\Gamma_{\mathrm{R}}(Q+\alpha_{1-2-3})\Gamma_{\mathrm{R}}(\alpha_{1+3-2})\Gamma_{\mathrm{R}}(Q+\alpha_{3-1-2})\Gamma_{\mathrm{R}}(2Q-\alpha_{1+2+3})}{\Gamma_{\mathrm{NS}}(Q)\Gamma_{\mathrm{NS}}(2\alpha_1)\Gamma_{\mathrm{NS}}(Q-2\alpha_2)\Gamma_{\mathrm{NS}}(2Q-2\alpha_3)} \end{aligned}$$

calculated in [21] form a complete set of eight independent normalizations required for representation of all chiral vertices in both sectors.

4 Braiding relations

4.1 Braiding of normal ordered exponentials and screening charges

In this subsection we shall calculate the braiding matrix for operators (3.15). We follow the procedure proposed in [9] and extended to the NS sector of the N=1 superconformal theory in [21]. Let us assume the existence of a braiding relation of the form

$$\mathsf{g}_{\mathsf{f}_3\,s_3}^{\;\alpha_3}(\sigma_3)\mathsf{g}_{\mathsf{f}_2\,s_2}^{\;\alpha_2}(\sigma_2) = \sum_{\vec{\mathsf{g}}} \int d\mu(\vec{t})\, B^{\epsilon}_{\natural}(\vec{\alpha};\vec{s},\vec{\mathsf{f}};\vec{t},\vec{\mathsf{h}})\mathsf{g}_{\mathsf{g}_2\,t_2}^{\;\alpha_2}(\sigma_2)\mathsf{g}_{\mathsf{g}_3\,t_3}^{\;\alpha_3}(\sigma_3), \tag{4.1}$$

where $\epsilon = \mathrm{sgn}(\sigma_3-\sigma_2)$, $\vec{\mathsf{f}} = (\mathsf{f}_2,\mathsf{f}_3)$, $\vec{s} = (s_2,s_3)$ etc. The sum over the indices $\vec{\mathsf{g}}$ is restricted by the parity conservation which holds in the chiral superscalar model and the integration measure $d\mu(\vec{t})$ is proportional (by the momentum conservation) to the Dirac delta $\delta(t_2+t_3-s_2-s_3)$. The additional subscript $\natural = \mathrm{R},\mathrm{NS}$ denotes the sector in which relation (4.1) is considered.

It follows from commutation relations (3.10) that the ordered exponentials satisfy the braiding relation:

$$\mathsf{E}^{\alpha}(x)\mathsf{E}^{\beta}(y) = \mathrm{e}^{-i\pi\alpha\beta\,\mathrm{sgn}(x-y)}\,\mathsf{E}^{\beta}(y)\mathsf{E}^{\alpha}(x). \tag{4.2}$$

Let, for $\sigma_3 > \sigma_2$, $I = [\sigma_2,\sigma_3]$, $I_c = [\sigma_3,\sigma_2+2\pi]$, $I' = [\sigma_2+2\pi,\sigma_3+2\pi]$ and define:

$$\mathsf{Q}_{\mathrm{I}} = \int_I dx\; \mathsf{E}^b(x)\psi(x), \qquad \mathsf{Q}^c_{\mathrm{I}} = \int_{I^c} dx\; \mathsf{E}^b(x)\psi(x), \qquad \mathsf{Q}'_{\mathrm{I}} = \int_{I'} dx\; \mathsf{E}^b(x)\psi(x),$$

so that

$$\mathsf{Q}(\sigma_2) = \mathsf{Q}^c_{\mathrm{I}} + \mathsf{Q}_{\mathrm{I}}, \qquad \mathsf{Q}(\sigma_3) = \mathsf{Q}^c_{\mathrm{I}} + \mathsf{Q}'_{\mathrm{I}}.$$

Using (4.2) we thus get

$$\begin{aligned} \mathsf{Q}(\sigma_3)\mathsf{E}^{\alpha_2}(\sigma_2) \;&=\; \mathrm{e}^{-i\pi b\alpha_2}\mathsf{E}^{\alpha_2}(\sigma_2)\left(\mathsf{Q}^c_{\mathrm{I}} + \mathrm{e}^{-2i\pi b\alpha_2}\mathsf{Q}'_{\mathrm{I}}\right),\\ \mathsf{Q}(\sigma_2)\mathsf{E}^{\alpha_3}(\sigma_3) \;&=\; \mathrm{e}^{-i\pi b\alpha_3}\mathsf{E}^{\alpha_3}(\sigma_3)\left(\mathsf{Q}^c_{\mathrm{I}} + \mathrm{e}^{2i\pi b\alpha_3}\mathsf{Q}_{\mathrm{I}}\right), \end{aligned}$$

and consequently

$$\mathsf{g}_{\mathsf{f}_3\, s_3}^{\ \alpha_3}(\sigma_3)\mathsf{g}_{\mathsf{f}_2\, s_2}^{\ \alpha_2}(\sigma_2) = \mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_3}(\sigma_3)\, \mathrm{e}^{-i\pi\alpha_2\alpha_3 - i\pi\alpha_2 b s_3}\left(\mathsf{Q}_{\rm I}^c + \mathrm{e}^{-2i\pi b\alpha_2}\mathsf{Q}_{\rm I}'\right)_{\mathsf{f}_3}^{s_3}\left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}\right)_{\mathsf{f}_2}^{s_2},$$
$$\mathsf{g}_{\mathsf{g}_2\, t_2}^{\ \alpha_2}(\sigma_2)\mathsf{g}_{\mathsf{g}_3\, t_3}^{\ \alpha_3}(\sigma_3) = \mathsf{E}^{\alpha_2}(\sigma_2)\mathsf{E}^{\alpha_3}(\sigma_3)\, \mathrm{e}^{-i\pi\alpha_3 b t_2}\left(\mathsf{Q}_{\rm I}^c + \mathrm{e}^{2i\pi b\alpha_3}\mathsf{Q}_{\rm I}\right)_{\mathsf{g}_2}^{t_2}\left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}'\right)_{\mathsf{g}_3}^{t_3}. \tag{4.3}$$

Braiding relation (4.1) is then equivalent to the equation

$$\left(\mathsf{Q}_{\rm I}^c + \mathrm{e}^{-2i\pi b\alpha_2}\mathsf{Q}_{\rm I}'\right)_{\mathsf{f}_3}^{s_3}\left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}\right)_{\mathsf{f}_2}^{s_2} = \sum_{\vec{\mathsf{g}}}\int d\mu(\vec{t})\, \tilde{B}_{\natural}^{+}(\vec{\alpha};\vec{s},\vec{\mathsf{f}};\vec{t},\vec{\mathsf{g}})\left(\mathsf{Q}_{\rm I}^c + \mathrm{e}^{2i\pi b\alpha_3}\mathsf{Q}_{\rm I}\right)_{\mathsf{g}_2}^{t_2}\left(\mathsf{Q}_{\rm I}^c + \mathsf{Q}_{\rm I}'\right)_{\mathsf{g}_3}^{t_3}, \tag{4.4}$$

where

$$\tilde{B}_{\natural}^{+}(\vec{\alpha};\vec{s},\vec{\mathsf{f}};\vec{t},\vec{\mathsf{g}}) = \mathrm{e}^{i\pi\alpha_2\alpha_3 + i\pi\alpha_2 b s_3 - i\pi\alpha_3 b t_2} B_{\natural}^{+}(\vec{\alpha};\vec{s},\vec{\mathsf{f}};\vec{t},\vec{\mathsf{g}}).$$

If the product of fields in (4.3) act on the NS state, the fermion field $\psi(x)$ appearing in the definition of the screening charges is anti-periodic, $\psi(x+2\pi) = -\psi(x)$, and

$$\mathsf{Q}_{\rm I}' = -\mathrm{e}^{i\pi b^2}\,\mathrm{e}^{2\pi b\mathsf{p}}\,\mathsf{Q}_{\rm I}.$$

On the other hand, while acting on the Ramond state the fermion field is periodic, $\psi(x+2\pi) = \psi(x)$. In this case

$$\mathsf{Q}_{\rm I}' = \mathrm{e}^{i\pi b^2}\,\mathrm{e}^{2\pi b\mathsf{p}}\,\mathsf{Q}_{\rm I}.$$

It follows from the definition of the screening charges and from braiding relation (4.2) that the operators $\mathsf{Q}_{\rm I}$, $\mathsf{Q}_{\rm I}^c$ and $\mathrm{e}^{\pi b\mathsf{p}}$ satisfy the Weyl-type algebra

$$\mathsf{Q}_{\rm I}\,\mathsf{Q}_{\rm I}^c = -\,\mathrm{e}^{i\pi b^2}\,\mathsf{Q}_{\rm I}^c\,\mathsf{Q}_{\rm I}, \qquad \mathsf{Q}_{\rm I}^c\,\mathrm{e}^{\pi b\mathsf{p}} = \mathrm{e}^{i\pi b^2}\,\mathrm{e}^{\pi b\mathsf{p}}\,\mathsf{Q}_{\rm I}^c, \qquad \mathsf{Q}_{\rm I}\,\mathrm{e}^{\pi b\mathsf{p}} = \mathrm{e}^{i\pi b^2}\,\mathrm{e}^{\pi b\mathsf{p}}\,\mathsf{Q}_{\rm I}. \tag{4.5}$$

Formula (4.4) can be seen as a relation in the algebra generated by the elements $\mathsf{Q}_{\rm I}, \mathsf{Q}_{\rm I}^c$ and p, satisfying (4.5). To compute the braiding matrix $\tilde{B}_{\natural}^{+}$ we shall choose a convenient representation of this algebra. Let us introduce an auxiliary Hilbert space $\mathcal{H}_{\rm aux} = \mathbb{C}^2 \otimes L^2(\mathbb{R}^2)$ and consider operators $\mathsf{p}, \mathsf{x}, \mathsf{t} \in \mathrm{End}(\mathcal{H}_{\rm aux})$ satisfying commutation relations

$$[\mathsf{p},\mathsf{x}] = -i, \qquad [\mathsf{p},\mathsf{t}] = [\mathsf{x},\mathsf{t}] = 0, \tag{4.6}$$

together with conjugation properties $\mathsf{p}^\dagger = \mathsf{p}^\dagger$, $\mathsf{x}^\dagger = \mathsf{x}^\dagger$ and $\mathsf{t}^\dagger = -\mathsf{t}$. One easily checks that the operators $\tilde{\mathsf{Q}}_{\rm I}^c, \tilde{\mathsf{Q}}_{\rm I} \in \mathrm{End}(\mathcal{H}_{\rm aux})$, defined by

$$\tilde{\mathsf{Q}}_{\rm I}^c = \tau_1\,\mathrm{e}^{b\mathsf{x}}\,\mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}, \qquad \tau_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right),$$
$$\tilde{\mathsf{Q}}_{\rm I} = \tau_2\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{-\pi b\mathsf{p}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{\frac{1}{2}i\pi b\mathsf{t}}, \quad \tau_2 = \left(\begin{smallmatrix}0 & i\\ -i & 0\end{smallmatrix}\right), \tag{4.7}$$

form a representation of (4.5). In this representation

$$\tilde{\mathsf{Q}}_{\rm I}' = \eta_{\natural}\,\tau_2\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{\pi b\mathsf{p}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{\frac{1}{2}i\pi b\mathsf{t}}, \quad \eta_{\rm NS} = -1, \quad \eta_{\rm R} = +1.$$

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Representation (4.7) allows to transform the powers of sums of $\tilde{\mathsf{Q}}$−s into the normal ordered form, with x operator to the left of p and t operators. Using the shift property of the Barnes functions:

$$G_{\mathrm{NS}}(y+b) = \left(1+\mathrm{e}^{i\pi by}\right) G_{\mathrm{R}}(y), \qquad G_{\mathrm{R}}(y+b) = \left(1-\mathrm{e}^{i\pi by}\right) G_{\mathrm{NS}}(y),$$

and

$$\mathrm{e}^{\alpha\mathsf{x}}\, f(\mathsf{p}) = f(\mathsf{p}+i\alpha)\,\mathrm{e}^{\alpha\mathsf{x}},$$

one has

$$\begin{aligned}
\mathrm{e}^{b\mathsf{x}}\,\mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}} + i\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{-\pi b\mathsf{p}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{\frac{1}{2}i\pi b\mathsf{t}} &= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\left(1+\mathrm{e}^{i\pi b\left(ip+t+\frac{1}{2}b^{-1}\right)}\right)\mathrm{e}^{\frac{1}{2}b\mathsf{x}} \\
&= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\frac{G_{\mathrm{NS}}\left(ip+t+\frac{1}{2}b^{-1}+b\right)}{G_{\mathrm{R}}\left(ip+t+\frac{1}{2}b^{-1}\right)}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}} \\
&= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,G_{\mathrm{NS}}\left(ip+t+\frac{1}{2}Q\right)\,\mathrm{e}^{b\mathsf{x}}\,\frac{1}{G_{\mathrm{R}}\left(ip+t+\frac{1}{2}Q\right)}
\end{aligned}$$

with $Q = b + b^{-1}$ and similarly

$$\begin{aligned}
\mathrm{e}^{b\mathsf{x}}\,\mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}} - i\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{-\pi b\mathsf{p}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\mathrm{e}^{\frac{1}{2}i\pi b\mathsf{t}} &= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\left(1-\mathrm{e}^{i\pi b\left(ip+t+\frac{1}{2}b^{-1}\right)}\right)\mathrm{e}^{\frac{1}{2}b\mathsf{x}} \\
&= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}}\,\frac{G_{\mathrm{R}}\left(ip+t+\frac{1}{2}b^{-1}+b\right)}{G_{\mathrm{NS}}\left(ip+t+\frac{1}{2}b^{-1}\right)}\,\mathrm{e}^{\frac{1}{2}b\mathsf{x}} \\
&= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,G_{\mathrm{R}}\left(ip+t+\frac{1}{2}Q\right)\,\mathrm{e}^{b\mathsf{x}}\,\frac{1}{G_{\mathrm{NS}}\left(ip+t+\frac{1}{2}Q\right)}\,.
\end{aligned}$$

Introducing a matrix notation

$$\mathbb{G}_{\mathrm{NS}}(z) \;=\; \begin{pmatrix} G_{\mathrm{NS}}(z) & 0 \\ 0 & G_{\mathrm{R}}(z) \end{pmatrix}, \qquad \mathbb{G}_{\mathrm{R}}(z) \;=\; \begin{pmatrix} G_{\mathrm{R}}(z) & 0 \\ 0 & G_{\mathrm{NS}}(z) \end{pmatrix} \;=\; \tau_1\cdot\mathbb{G}_{\mathrm{NS}}(z)\cdot\tau_1,$$

one can present the result of the calculations above in the compact form

$$\tilde{\mathsf{Q}}^c_{\mathrm{I}} + \tilde{\mathsf{Q}}_{\mathrm{I}} \;=\; \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathbb{G}_{\mathrm{NS}}\left(i\mathsf{p}+\mathsf{t}+\frac{1}{2}Q\right)\,\tau_1\,\mathrm{e}^{b\mathsf{x}}\,\mathbb{G}^{-1}_{\mathrm{NS}}\left(i\mathsf{p}+\mathsf{t}+\frac{1}{2}Q\right).$$

In a similar way one obtains

$$\begin{aligned}
\tilde{\mathsf{Q}}^c_{\mathrm{I}} + \mathrm{e}^{2\pi i b\alpha_3}\tilde{\mathsf{Q}}_{\mathrm{I}} &= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathbb{G}_{\mathrm{NS}}\left(i\mathsf{p}+\mathsf{t}+2\alpha_3+\frac{1}{2}Q\right)\,\tau_1\,\mathrm{e}^{b\mathsf{x}}\,\mathbb{G}^{-1}_{\mathrm{NS}}\left(i\mathsf{p}+\mathsf{t}+2\alpha_3+\frac{1}{2}Q\right), \\
\tilde{\mathsf{Q}}^c_{\mathrm{I}} + \tilde{\mathsf{Q}}'_{\mathrm{I}} &= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathbb{G}^{-1}_{\natural}\left(-i\mathsf{p}+\mathsf{t}+\frac{1}{2}Q\right)\,\tau_1\,\mathrm{e}^{b\mathsf{x}}\,\mathbb{G}_{\natural}\left(-i\mathsf{p}+\mathsf{t}+\frac{1}{2}Q\right), \\
\tilde{\mathsf{Q}}^c_{\mathrm{I}} + \mathrm{e}^{-2\pi i b\alpha_2}\tilde{\mathsf{Q}}'_{\mathrm{I}} &= \mathrm{e}^{-\frac{1}{2}i\pi b\mathsf{t}}\,\mathbb{G}^{-1}_{\natural}\left(-i\mathsf{p}+\mathsf{t}-2\alpha_2+\frac{1}{2}Q\right)\,\tau_1\,\mathrm{e}^{b\mathsf{x}}\,\mathbb{G}_{\natural}\left(-i\mathsf{p}+\mathsf{t}-2\alpha_2+\frac{1}{2}Q\right).
\end{aligned}$$

Defining an even and an odd complex power of τ_1

$$(\tau_1)^s_\rho \;\equiv\; \mathbf{1}_\rho \;=\; \begin{cases} \mathbf{1}, & \rho = \mathrm{e}, \\ \tau_1, & \rho = \mathrm{o}, \end{cases}$$

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we get:

$$(\tilde{\mathsf{Q}}_{\rm I}^c + \mathrm{e}^{-2i\pi b\alpha_2}\tilde{\mathsf{Q}}_{\rm I}')_{\mathbf{f}_3}^{s_3}(\tilde{\mathsf{Q}}_{\rm I}^c + \tilde{\mathsf{Q}}_{\rm I})_{\mathbf{f}_2}^{s_2} = \mathrm{e}^{b(s_2+s_3)\mathsf{x}}\,\mathrm{e}^{-\frac{1}{2}i\pi b(s_2+s_3)\mathsf{t}}$$
$$\times\ \mathbb{G}_\natural^{-1}\Big(-i\mathsf{p}+\mathsf{t}-2\alpha_2-bs_2-bs_3+\frac{1}{2}Q\Big)\ \mathbf{1}_{\mathbf{f}_3}\ \mathbb{G}_\natural\Big(-i\mathsf{p}+\mathsf{t}-2\alpha_2-bs_2+\frac{1}{2}Q\Big)$$
$$\times\ \mathbb{G}_{\rm NS}\Big(i\mathsf{p}+\mathsf{t}+bs_2+\frac{1}{2}Q\Big)\ \mathbf{1}_{\mathbf{f}_2}\ \mathbb{G}_{\rm NS}^{-1}\Big(i\mathsf{p}+\mathsf{t}+\frac{1}{2}Q\Big),$$
$$(\tilde{\mathsf{Q}}_{\rm I}^c + \mathrm{e}^{2i\pi b\alpha_3}\tilde{\mathsf{Q}}_{\rm I})_{\mathbf{g}_2}^{t_2}(\tilde{\mathsf{Q}}_{\rm I}^c + \tilde{\mathsf{Q}}_{\rm I}')_{\mathbf{g}_3}^{t_3} = \mathrm{e}^{b(t_2+t_3)\mathsf{x}}\,\mathrm{e}^{-\frac{1}{2}i\pi b(t_2+t_3)\mathsf{t}} \tag{4.8}$$
$$\times\ \mathbb{G}_{\rm NS}\Big(i\mathsf{p}+\mathsf{t}+2\alpha_3+bt_2+bt_3+\frac{1}{2}Q\Big)\ \mathbf{1}_{\mathbf{g}_2}\ \mathbb{G}_{\rm NS}^{-1}\Big(i\mathsf{p}+\mathsf{t}+2\alpha_3+bt_3+\frac{1}{2}Q\Big)$$
$$\times\ \mathbb{G}_\natural^{-1}\Big(-i\mathsf{p}+\mathsf{t}-bt_3+\frac{1}{2}Q\Big)\ \mathbf{1}_{\mathbf{g}_3}\ \mathbb{G}_\natural\Big(-i\mathsf{p}+\mathsf{t}+\frac{1}{2}Q\Big).$$

Evaluated on a common eigenstate of operators p and t,

$$\mathsf{p}\,|p,\tau\rangle = p\,|p,\tau\rangle\,, \qquad \mathsf{t}\,|p,\tau\rangle = \tau\,|p,\tau\rangle\,, \quad \tau\in i\mathbb{R},$$

the right hand sides of formulae (4.8) take the form of analytic functions of p and τ, multiplied by the operator

$$\mathrm{e}^{(s_1+s_2)\mathsf{x}} = \mathrm{e}^{(t_1+t_2)\mathsf{x}}.$$

Stripping off this factor from both sides of formula (4.4) one gets a relation between analytic functions of p and τ. In terms of the Barnes S functions[5]

$$\mathbb{S}_{\rm NS}\,(z) = \begin{pmatrix} S_{\rm NS}(z) & 0 \\ 0 & S_{\rm R}(z)\end{pmatrix}, \qquad \mathbb{S}_{\rm R}\,(z) = \begin{pmatrix} S_{\rm R}(z) & 0 \\ 0 & S_{\rm NS}(z)\end{pmatrix} = \tau_1\cdot\mathbb{S}_{\rm NS}\,(z)\cdot\tau_1,$$

this relation takes the form

$$\mathrm{e}^{i\chi}\mathbb{S}_\natural\Big(\tfrac{Q}{2}+iA_2-\tau\Big)\big(F^\natural_{\mathbf{f}_3}\big)^T\mathbb{S}_\natural^{-1}(Q-iC_2+ip_s-\tau)\mathbb{S}_{\rm NS}^{-1}(Q-iC_2-ip_s-\tau)F^{\rm NS}_{\mathbf{f}_2}\,\mathbb{S}_{\rm NS}\Big(\tfrac{Q}{2}+iB_2-\tau\Big)$$
$$= \sum_{\vec{\mathbf{g}}}\int d\mu(t_3)\ \mathrm{e}^{-\frac{i\pi}{2}(bt_3)^2+\pi p_1 bt_3}\ B_\natural(\vec{\alpha};\vec{s},\vec{\mathbf{f}};s-t_3,t_3,\vec{\mathbf{g}}) \tag{4.9}$$
$$\times\,\mathbb{S}_{\rm NS}\Big(\tfrac{Q}{2}+iA_3+\tau\Big)F^{\rm NS}_{\mathbf{g}_2}\mathbb{S}_{\rm NS}^{-1}(Q-iC_3+ip_u+\tau)\mathbb{S}_\natural^{-1}(Q-iC_3-ip_u+\tau)\big(F^\natural_{\mathbf{g}_3}\big)^T\mathbb{S}_\natural\Big(\tfrac{Q}{2}+iB_3+\tau\Big),$$

where $i\chi = \frac{i\pi b^2}{2}(s_2^2-s^2)+\pi bp_1s_2 - i\pi\alpha_2(\alpha_3+2bs_3)$, $s=s_2+s_3$,

$$F^{\rm NS}_{\rm e} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \quad F^{\rm NS}_{\rm o} = \begin{pmatrix} 0 & \mathrm{e}^{\frac{i\pi}{4}} \\ \mathrm{e}^{-\frac{i\pi}{4}} & 0\end{pmatrix}, \qquad F^{\rm R}_{\mathbf{f}} = \big(F^{\rm NS}_{\mathbf{f}}\big)^T,$$

and

$$A_2 = p_1-2i\alpha_2-ibs,\ \ B_2=-p_1,\ \ C_2=i(\alpha_2-Q/2),\ \ p_s=p_1-i(\alpha_2+bs_2),$$
$$A_3 = p_1-2i\alpha_3-ibs,\ \ B_3=-p_1,\ \ C_3=i(\alpha_3-Q/2),\ \ p_u=p_1-i(\alpha_3+bt_3). \tag{4.10}$$

[5]See appendix C for the relation between the Barnes G and S functions.

The Barnes functions in the integrand of (4.9) depend on the integration variable t_3 only via the parameter p_u. Multiplying both sides of equation (4.9) by $\mathbb{S}_{\rm NS}\left(\frac{Q}{2}+iA_3+\tau\right)^{-1} = \mathbb{S}_{\rm NS}\left(\frac{Q}{2}-iA_3-\tau\right)$ (from the left) and by $\mathbb{S}_\natural\left(\frac{Q}{2}+iB_3+\tau\right)^{-1} = \mathbb{S}_\natural\left(\frac{Q}{2}-iB_3-\tau\right)$ (from the right) and choosing the integration measure to be $d\mu(t_3) = \theta(p_u)\,dp_u$ one gets

$$
\begin{aligned}
&\mathrm{e}^{i\chi}\,\frac{\mathbb{S}_\natural\left(\frac{Q}{2}+iA_2-\tau\right)}{\mathbb{S}_{\rm NS}\left(\frac{Q}{2}+iA_3+\tau\right)}\left(F^\natural_{\mathbf{f}_3}\right)^T\frac{\mathbb{S}_{\rm NS}(ip_s+\tau+iC_2)}{\mathbb{S}_\natural(Q+ip_s-\tau-iC_2)}F^{\rm NS}_{\mathbf{f}_2}\frac{\mathbb{S}_{\rm NS}\left(\frac{Q}{2}+iB_2-\tau\right)}{\mathbb{S}_\natural\left(\frac{Q}{2}+iB_3+\tau\right)} \qquad (4.11)\\
&=\sum_{\vec{\mathbf{g}}}\int_0^\infty dp_u\;\mathrm{e}^{-\frac{i\pi}{2}(bt_3)^2+\pi p_1 bt_3}\;B^+_\natural(\vec{\alpha};\vec{s},\vec{\mathbf{f}};s-t_3,t_3,\vec{\mathbf{g}})\;F^{\rm NS}_{\mathbf{g}_2}\frac{\mathbb{S}_\natural(ip_u-\tau+iC_3)}{\mathbb{S}_{\rm NS}(Q+ip_u+\tau-iC_3)}\left(F^\natural_{\mathbf{g}_3}\right)^T
\end{aligned}
$$

where $\frac{\mathbb{S}_\natural\left(\frac{Q}{2}+iA_2-\tau\right)}{\mathbb{S}_{\rm NS}\left(\frac{Q}{2}+iA_3+\tau\right)}$ stands for $\mathbb{S}_\natural\left(\frac{Q}{2}+iA_2-\tau\right)\mathbb{S}^{-1}_{\rm NS}\left(\frac{Q}{2}+iA_3+\tau\right)$ etc.

In order to calculate the braiding kernel from (4.11) one can then use the orthogonality relation:[6]

$$
\begin{aligned}
&\int_{i\mathbb{R}}\frac{d\tau}{i}\mathrm{Tr}\left\{\left[F^{\rm NS}_{\mathbf{h}_2}\frac{\mathbb{S}_\natural(ip_u-\tau+iC_3)}{\mathbb{S}_{\rm NS}(Q+ip_u+\tau-iC_3)}\left(F^\natural_{\mathbf{h}_3}\right)^T\right]\left[F^{\rm NS}_{\mathbf{g}_2}\frac{\mathbb{S}_\natural(ip'_u-\tau+iC_3)}{\mathbb{S}_{\rm NS}(Q+ip'_u+\tau-iC_3)}\left(F^\natural_{\mathbf{g}_3}\right)^T\right]^\dagger\right\}\\
&=\mathcal{N}^{-1}_\natural(p_u)\delta_{\vec{\mathbf{h}},\vec{\mathbf{g}}}\,\delta(p_u-p'_u), \qquad p_u,p'_u\in\mathbb{R}_+, \qquad (4.12)
\end{aligned}
$$

where

$$
\mathcal{N}_{\rm NS}(p_u) = \sinh\pi bp_u\sinh\frac{\pi p_u}{b}, \qquad \mathcal{N}_{\rm R}(p_u) = \cosh\pi bp_u\cosh\frac{\pi p_u}{b}. \qquad (4.13)
$$

This yields

$$
\begin{aligned}
&B^+_\natural(\vec{\alpha};\vec{s},\vec{\mathbf{f}};s-t_3,t_3,\vec{\mathbf{g}}) = \mathcal{N}_\natural(p_u)\mathrm{e}^{i\chi+\frac{i\pi}{2}(bt_3)^2-\pi p_1 bt_3}\\
&\quad\times\int_{i\mathbb{R}}\frac{d\tau}{i}\;\mathrm{Tr}\left\{\frac{\mathbb{S}_\natural(Q/2+iA_2-\tau)}{\mathbb{S}_{\rm NS}(Q/2+iA_3+\tau)}\left(F^\natural_{\mathbf{f}_3}\right)^T\frac{\mathbb{S}_{\rm NS}(ip_s+\tau+iC_2)}{\mathbb{S}_\natural(Q+ip_s-\tau-iC_2)}F^{\rm NS}_{\mathbf{f}_2}\right. \qquad (4.14)\\
&\qquad\left.\frac{\mathbb{S}_\natural(Q/2-iB_3-\tau)}{\mathbb{S}_{\rm NS}(Q/2-iB_2+\tau)}\left[F^{\rm NS}_{\mathbf{g}_2}\frac{\mathbb{S}_\natural(ip_u-\tau+iC_3)}{\mathbb{S}_{\rm NS}(Q+ip_u+\tau-iC_3)}\left(F^\natural_{\mathbf{g}_3}\right)^T\right]^\dagger\right\}.
\end{aligned}
$$

For further applications it is convenient to regard the braiding kernel as a function of new variables:

$$
\alpha_\imath = \frac{Q}{2}+ip_\imath, \quad \imath = 1,s,u, \qquad \alpha_4 = \alpha_3+\alpha_2+\alpha_1+bs. \qquad (4.15)
$$

In order to make it more readable we introduce the notation

$$
\mathbb{B}^{\epsilon,\natural}_{\alpha_s\alpha_u}\left[{}^{\alpha_3\;\alpha_2}_{\alpha_4\;\alpha_1}\right]^{\mathbf{g}_2\,\mathbf{g}_3}_{\mathbf{f}_3\,\mathbf{f}_2} \equiv B^\epsilon_\natural(\vec{\alpha};\vec{s},\vec{\mathbf{f}};s-t_3,t_3,\vec{\mathbf{g}}) \qquad (4.16)
$$

[6] A derivation of (4.12) is presented in appendix D.

so that

$$\mathbb{B}^{+,\natural}_{\alpha_s\alpha_u}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathbf{g_2}\,\mathbf{g_3}}_{\mathbf{f_3}\,\mathbf{f_2}} = \frac{1}{4}\mathrm{e}^{\delta}\, S_\natural(2\alpha_u)S_\natural(2Q-2\alpha_u) \times \int_{i\mathbb{R}} \frac{d\tau}{i}\ \mathrm{Tr}\left\{\frac{\mathbb{S}_\natural(\alpha_4-\alpha_3+\alpha_2-\tau)}{\mathbb{S}_{\mathrm{NS}}(\alpha_4+\alpha_3-\alpha_2+\tau)}\left(F^{\natural}_{\mathbf{f}_3}\right)^T \frac{\mathbb{S}_{\mathrm{NS}}(\alpha_s-\alpha_2+\tau)}{\mathbb{S}_\natural(\alpha_s+\alpha_2-\tau)}F^{\mathrm{NS}}_{\mathbf{f}_2} \frac{\mathbb{S}_\natural(\alpha_1-\tau)}{\mathbb{S}_{\mathrm{NS}}(\alpha_1+\tau)}\left[F^{\mathrm{NS}}_{\mathbf{g}_2}\frac{\mathbb{S}_\natural(\alpha_u-\alpha_3-\tau)}{\mathbb{S}_{\mathrm{NS}}(\alpha_u+\alpha_3+\tau)}\left(F^{\natural}_{\mathbf{g}_3}\right)^T\right]^\dagger\right\}, \tag{4.17}$$

where

$$\begin{aligned}\delta &= i\chi + \frac{i\pi}{2}(bt_3)^2 - \pi p_1 bt_3\\ &= \frac{i\pi}{2}\Big[\alpha_4(Q-\alpha_4)+\alpha_1(Q-\alpha_1)-\alpha_u(Q-\alpha_u)-\alpha_s(Q-\alpha_s)\\ &\qquad +2\alpha_3(\alpha_4-\alpha_u)-2\alpha_2(\alpha_4-\alpha_s)\Big].\end{aligned}$$

Repeating the steps above for $\epsilon = \mathrm{sgn}(\sigma_3-\sigma_2) < 0$ one gets

$$\mathbb{B}^{-,\natural}_{\alpha_s\alpha_u}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathbf{g_2}\,\mathbf{g_3}}_{\mathbf{f_3}\,\mathbf{f_2}} = \frac{1}{4}\mathrm{e}^{\delta^{(-)}}\, S_\natural(2\alpha_u)S_\natural(2Q-2\alpha_u) \times \int_{i\mathbb{R}} \frac{d\tau}{i}\ \mathrm{Tr}\left\{\frac{\mathbb{S}_{\mathrm{NS}}(\alpha_4-\alpha_3+\alpha_2-\tau)}{\mathbb{S}_\natural(\alpha_4+\alpha_3-\alpha_2+\tau)}F^{\mathrm{NS}}_{\mathbf{f}_3} \frac{\mathbb{S}_\natural(\alpha_s-\alpha_2+\tau)}{\mathbb{S}_{\mathrm{NS}}(\alpha_s+\alpha_2-\tau)}\left(F^{\natural}_{\mathbf{f}_2}\right)^T \frac{\mathbb{S}_{\mathrm{NS}}(\alpha_1-\tau)}{\mathbb{S}_\natural(\alpha_1+\tau)}\left[\left(F^{\natural}_{\mathbf{g}_2}\right)^T\frac{\mathbb{S}_{\mathrm{NS}}(\alpha_u-\alpha_3-\tau)}{\mathbb{S}_\natural(\alpha_u+\alpha_3+\tau)}F^{\mathrm{NS}}_{\mathbf{g}_3}\right]^\dagger\right\}, \tag{4.18}$$

where

$$\delta^{(-)} = \delta - i\pi\big(\alpha_4(Q-\alpha_4)+\alpha_1(Q-\alpha_1)-\alpha_u(Q-\alpha_u)-\alpha_s(Q-\alpha_s)\big).$$

The explicit form of the braiding matrix $\mathbb{B}$ can be read off directly from formula (4.14). For instance

$$\begin{aligned}\mathbb{B}^{+,\mathrm{NS}}_{\alpha_s\alpha_u}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{e}\,\mathrm{e}}_{\mathrm{e}\,\mathrm{e}} &= \frac{1}{4}\mathrm{e}^{\delta}\, S_{\mathrm{NS}}(2\alpha_u)S_{\mathrm{NS}}(2Q-2\alpha_u)\\ &\times\int_{i\mathbb{R}}\frac{d\tau}{i}\left[\frac{S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\bar\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4-\alpha_3+\alpha_2)S_{\mathrm{NS}}(\tau+\bar\alpha_4-\alpha_3+\alpha_2)}{S_{\mathrm{NS}}(\tau+\alpha_s+\alpha_2)S_{\mathrm{NS}}(\tau+\bar\alpha_s+\alpha_2)S_{\mathrm{NS}}(\tau+\alpha_u+\bar\alpha_3)S_{\mathrm{NS}}(\tau+\bar\alpha_u+\bar\alpha_3)}\right.\\ &\quad\left.+\frac{S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\bar\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4-\alpha_3+\alpha_2)S_{\mathrm{R}}(\tau+\bar\alpha_4-\alpha_3+\alpha_2)}{S_{\mathrm{R}}(\tau+\alpha_s+\alpha_2)S_{\mathrm{R}}(\tau+\bar\alpha_s+\alpha_2)S_{\mathrm{R}}(\tau+\alpha_u+\bar\alpha_3)S_{\mathrm{R}}(\tau+\bar\alpha_u+\bar\alpha_3)}\right],\end{aligned} \tag{4.19}$$

$$\begin{aligned}\mathbb{B}^{+,\mathrm{NS}}_{\alpha_s\alpha_u}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\mathrm{o}\,\mathrm{o}}_{\mathrm{e}\,\mathrm{e}} &= \frac{1}{4}\mathrm{e}^{-\frac{i\pi}{2}}\,\mathrm{e}^{\delta}\, S_{\mathrm{NS}}(2\alpha_u)S_{\mathrm{NS}}(2Q-2\alpha_u)\\ &\times\int_{i\mathbb{R}}\frac{d\tau}{i}\left[\frac{S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\bar\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4-\alpha_3+\alpha_2)S_{\mathrm{NS}}(\tau+\bar\alpha_4-\alpha_3+\alpha_2)}{S_{\mathrm{NS}}(\tau+\alpha_s+\alpha_2)S_{\mathrm{NS}}(\tau+\bar\alpha_s+\alpha_2)S_{\mathrm{R}}(\tau+\alpha_u+\bar\alpha_3)S_{\mathrm{R}}(\tau+\bar\alpha_u+\bar\alpha_3)}\right.\\ &\quad\left.-\frac{S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\bar\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4-\alpha_3+\alpha_2)S_{\mathrm{R}}(\tau+\bar\alpha_4-\alpha_3+\alpha_2)}{S_{\mathrm{R}}(\tau+\alpha_s+\alpha_2)S_{\mathrm{R}}(\tau+\bar\alpha_s+\alpha_2)S_{\mathrm{NS}}(\tau+\alpha_u+\bar\alpha_3)S_{\mathrm{NS}}(\tau+\bar\alpha_u+\bar\alpha_3)}\right],\end{aligned} \tag{4.20}$$

or

$$\mathbb{B}^{+,\mathrm{R}}_{\alpha_s\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\mathrm{o\,e}}_{\mathrm{e\,o}} = \frac{1}{4}\mathrm{e}^{\frac{i\pi}{2}}\,\mathrm{e}^{\delta}\, S_{\mathrm{R}}(2\alpha_u)S_{\mathrm{R}}(2Q-2\alpha_u) \tag{4.21}$$

$$\times \int\limits_{i\mathbb{R}} \frac{d\tau}{i}\left[\frac{S_{\mathrm{NS}}(\tau+\alpha_1)S_{\mathrm{R}}(\tau+\bar\alpha_1)S_{\mathrm{R}}(\tau+\alpha_4-\alpha_3+\alpha_2)S_{\mathrm{NS}}(\tau+\bar\alpha_4-\alpha_3+\alpha_2)}{S_{\mathrm{R}}(\tau+\alpha_s+\alpha_2)S_{\mathrm{NS}}(\tau+\bar\alpha_s+\alpha_2)S_{\mathrm{NS}}(\tau+\alpha_u+\bar\alpha_3)S_{\mathrm{R}}(\tau+\bar\alpha_u+\bar\alpha_3)}\right.$$
$$\left. - \frac{S_{\mathrm{R}}(\tau+\alpha_1)S_{\mathrm{NS}}(\tau+\bar\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4-\alpha_3+\alpha_2)S_{\mathrm{R}}(\tau+\bar\alpha_4-\alpha_3+\alpha_2)}{S_{\mathrm{NS}}(\tau+\alpha_s+\alpha_2)S_{\mathrm{R}}(\tau+\bar\alpha_s+\alpha_2)S_{\mathrm{R}}(\tau+\alpha_u+\bar\alpha_3)S_{\mathrm{NS}}(\tau+\bar\alpha_u+\bar\alpha_3)}\right].$$

The other cases differ from the expressions above only by constant phases and NS/R indices of the Barnes functions.

4.2 Braiding of chiral vertex operators

In this subsection we shall derive the braiding properties of the chiral vertex operators. Rather than presenting the general formula (which would be quite clumsy due to a plethora of indices) we will discuss several examples choosing vertex operators from different sectors. All other cases can be easily obtained in a similar way.

4.2.1 NS-NS braiding

The braiding properties of the Neveu-Schwarz vertex operators in the NS sector were already discussed in [21]. In this subsection we shall calculate two examples of braiding matrices for the NS operators in the Ramond sector. Let us first consider the composition

$$V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_3\\ \beta_4\,\beta_s\end{smallmatrix}\right](z_3)V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_2\\ \beta_s\,\beta_1\end{smallmatrix}\right](z_2):\ \mathcal{W}_{\beta_1}\to\mathcal{W}_{\beta_4}\,,$$

which in representation (3.22) takes the form

$$V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_3\\ \beta_4\,\beta_s\end{smallmatrix}\right](z_3)V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_2\\ \beta_s\,\beta_1\end{smallmatrix}\right](z_2) = \frac{\mathsf{V}_{\mathrm{e}}{}^{\alpha_3}_{s_3}(z_3)\mathsf{V}_{\mathrm{e}}{}^{\alpha_2}_{s_2}(z_2)\big|_{\mathrm{R}}}{\left\langle w_4^+\right|\mathsf{V}_{\mathrm{e}}{}^{\alpha_3}_{s_3}(1)\left|w_s^+\right\rangle\left\langle w_s^+\right|\mathsf{V}_{\mathrm{e}}{}^{\alpha_2}_{s_2}(1)\left|w_1^+\right\rangle}.$$

The notation $\big|_{\mathrm{R}}$ indicates that the product of chiral fields $\mathsf{V}_{\mathrm{e}}{}^{\alpha_3}_{s_3}(z_3)\mathsf{V}_{\mathrm{e}}{}^{\alpha_2}_{s_2}(z_2)$ acts on the states from the Ramond sector. It follows from (3.16) that for this product one can apply braiding relation (4.1) derived in the previous subsection. In notation (4.16) we get

$$\mathsf{V}_{\mathrm{e}}{}^{\alpha_3}_{s_3}(z_3)\mathsf{V}_{\mathrm{e}}{}^{\alpha_2}_{s_2}(z_2)\big|_{\mathrm{R}} = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\sum_{\vec{\mathbf{g}}}\mathbb{B}^{\epsilon,\mathrm{R}}_{\alpha_s\alpha_u}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\mathbf{g_2 g_3}}_{\mathrm{e\;e}}\ \mathsf{V}_{\mathbf{g_2}}{}^{\alpha_2}_{t_2}(z_2)\mathsf{V}_{\mathbf{g_3}}{}^{\alpha_3}_{t_3}(z_3)\big|_{\mathrm{R}}.$$

Using (3.22) one can express the r.h.s. in terms of chiral vertex operators

$$\mathsf{V}_{\mathrm{e}}{}^{\alpha_2}_{t_2}(z_2)\mathsf{V}_{\mathrm{e}}{}^{\alpha_3}_{t_3}(z_3)\big|_{\mathrm{R}} = \left\langle w_4^+\right|\mathsf{V}_{\mathrm{e}}{}^{\alpha_2}_{t_2}(1)\left|w_u^+\right\rangle\left\langle w_u^+\right|\mathsf{V}_{\mathrm{e}}{}^{\alpha_3}_{t_3}(1)\left|w_1^+\right\rangle V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_2\\ \beta_4\,\beta_u\end{smallmatrix}\right](z_2)V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_3\\ \beta_u\,\beta_1\end{smallmatrix}\right](z_3),$$

$$\mathsf{V}_{\mathrm{o}}{}^{\alpha_2}_{t_2}(z_2)\mathsf{V}_{\mathrm{o}}{}^{\alpha_3}_{t_3}(z_3)\big|_{\mathrm{R}} = \left\langle w_4^+\right|\mathsf{V}_{\mathrm{o}}{}^{\alpha_2}_{t_2}(1)\left|w_u^-\right\rangle\left\langle w_u^+\right|\mathsf{V}_{\mathrm{o}}{}^{\alpha_3}_{t_3}(1)\left|w_1^-\right\rangle V^{-}_{\mathrm{o}}\left[\begin{smallmatrix}\Delta_2\\ \beta_4\,\beta_u\end{smallmatrix}\right](z_2)V^{-}_{\mathrm{o}}\left[\begin{smallmatrix}\Delta_3\\ \beta_u\,\beta_1\end{smallmatrix}\right](z_3).$$

Thus, if we define the braiding matrix by the relation

$$V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_3\\ \beta_4\,\beta_s\end{smallmatrix}\right](z_3)V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_2\\ \beta_s\,\beta_1\end{smallmatrix}\right](z_2) = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\left(\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ +\beta_4 & +\beta_1\end{smallmatrix}\right]^{\mathrm{e\,e}}_{\mathrm{e\,e}} V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_2\\ \beta_4\,\beta_u\end{smallmatrix}\right](z_2)V^{+}_{\mathrm{e}}\left[\begin{smallmatrix}\Delta_3\\ \beta_u\,\beta_1\end{smallmatrix}\right](z_3)\right.$$
$$\left. + \mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\left[\begin{smallmatrix}\Delta_3 & \Delta_2\\ +\beta_4 & +\beta_1\end{smallmatrix}\right]^{\mathrm{o\,o}}_{\mathrm{e\,e}} V^{-}_{\mathrm{o}}\left[\begin{smallmatrix}\Delta_2\\ \beta_4\,\beta_u\end{smallmatrix}\right](z_2)V^{-}_{\mathrm{o}}\left[\begin{smallmatrix}\Delta_3\\ \beta_u\,\beta_1\end{smallmatrix}\right](z_3)\right),$$

then

$$
\begin{aligned}
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} \Delta_3 & \Delta_2 \\ +\beta_4 & +\beta_1 \end{bmatrix}^{\rm e\,e}_{\rm e\,e} &= \frac{\mathcal{M}^{\rm N\,R\,N}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\rm N\,R\,N}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\rm N\,R\,N}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\rm N\,R\,N}_{\alpha_s,\alpha_2,\alpha_1}}\;\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}^{\rm e\,e}_{\rm e\,e}, \\
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} \Delta_3 & \Delta_2 \\ +\beta_4 & +\beta_1 \end{bmatrix}^{\rm o\,o}_{\rm e\,e} &= -\frac{\mathcal{M}^{\rm R\,N\,R}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\rm R\,N\,R}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\rm N\,R\,N}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\rm N\,R\,N}_{\alpha_s,\alpha_2,\alpha_1}}\;\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}^{\rm o\,o}_{\rm e\,e}.
\end{aligned}
\tag{4.22}
$$

The braiding of all other pairs of the NS chiral vertex operators can be calculated in an essentially the same way. For instance, for the composition

$$
V^{+}_{\rm o}\left[{}^{\Delta_3}_{\beta_4\beta_s}\right](z_3)V^{-}_{\rm e}\left[{}^{\Delta_2}_{\beta_s\beta_1}\right](z_2) = \frac{i\,\mathsf{V}_{{\rm e}\,s_3}^{\;\alpha_3}(z_3)\mathsf{V}_{{\rm o}\,s_2}^{\;\alpha_2}(z_2)\big|_{\rm R}}{\left\langle w_4^+\right|\mathsf{V}_{{\rm e}\,s_3}^{\;\alpha_3}(1)\left|w_s^+\right\rangle\left\langle w_s^-\right|\mathsf{V}_{{\rm o}\,s_2}^{\;\alpha_2}(1)\left|w_1^+\right\rangle}
\tag{4.23}
$$

the relevant braiding relation (4.1) reads

$$
\mathsf{V}_{{\rm e}\,s_3}^{\;\alpha_3}(z_3)\mathsf{V}_{{\rm o}\,s_2}^{\;\alpha_2}(z_2)\big|_{\rm R} = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\sum_{\vec{\mathbf{g}}}\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}^{\mathbf{g_2 g_3}}_{\rm e\,o}\;\mathsf{V}_{\mathbf{g_2}\,t_2}^{\;\alpha_2}(z_2)\mathsf{V}_{\mathbf{g_3}\,t_3}^{\;\alpha_3}(z_3)\big|_{\rm R}.
\tag{4.24}
$$

As before, one can express the r.h.s. of (4.23) in terms of chiral vertex operators:

$$
\begin{aligned}
\mathsf{V}_{{\rm e}\,t_2}^{\;\alpha_2}(z_2)\mathsf{V}_{{\rm o}\,t_3}^{\;\alpha_3}(z_3)\big|_{\rm R} &= -i\left\langle w_4^+\right|\mathsf{V}_{{\rm e}\,t_2}^{\;\alpha_2}(1)\left|w_u^+\right\rangle\left\langle w_u^-\right|\mathsf{V}_{{\rm o}\,t_3}^{\;\alpha_3}(1)\left|w_1^+\right\rangle V^{+}_{\rm o}\left[{}^{\Delta_2}_{\beta_4\beta_u}\right](z_2)V^{-}_{\rm e}\left[{}^{\Delta_3}_{\beta_u\beta_1}\right](z_3), \\
\mathsf{V}_{{\rm o}\,t_2}^{\;\alpha_2}(z_2)\mathsf{V}_{{\rm e}\,t_3}^{\;\alpha_3}(z_3)\big|_{\rm R} &= i\left\langle w_4^-\right|\mathsf{V}_{{\rm o}\,t_2}^{\;\alpha_2}(1)\left|w_u^+\right\rangle\left\langle w_u^-\right|\mathsf{V}_{{\rm e}\,t_3}^{\;\alpha_3}(1)\left|w_1^-\right\rangle V^{-}_{\rm e}\left[{}^{\Delta_2}_{\beta_4\beta_u}\right](z_2)V^{+}_{\rm o}\left[{}^{\Delta_3}_{\beta_u\beta_1}\right](z_3).
\end{aligned}
\tag{4.25}
$$

Equations (4.23), (4.24) and (4.25) suggest the following definition of the braiding matrix

$$
\begin{aligned}
V^{+}_{\rm o}\left[{}^{\Delta_3}_{\beta_4\beta_s}\right](z_3)V^{-}_{\rm e}\left[{}^{\Delta_2}_{\beta_s\beta_1}\right](z_2) = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\Bigg(&\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} \Delta_3 & \Delta_2 \\ +\beta_4 & -\beta_1 \end{bmatrix}^{\rm o\,e}_{\rm o\,e} V^{+}_{\rm o}\left[{}^{\Delta_2}_{\beta_4\beta_u}\right](z_2)V^{-}_{\rm e}\left[{}^{\Delta_3}_{\beta_u\beta_1}\right](z_3) \\
&+\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} \Delta_3 & \Delta_2 \\ +\beta_4 & -\beta_1 \end{bmatrix}^{\rm e\,o}_{\rm o\,e} V^{-}_{\rm e}\left[{}^{\Delta_2}_{\beta_4\beta_u}\right](z_2)V^{+}_{\rm o}\left[{}^{\Delta_3}_{\beta_u\beta_1}\right](z_3)\Bigg).
\end{aligned}
$$

Then the results above yield

$$
\begin{aligned}
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} \Delta_3 & \Delta_2 \\ +\beta_4 & -\beta_1 \end{bmatrix}^{\rm o\,e}_{\rm o\,e} &= \frac{\mathcal{M}^{\rm N\,R\,N}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\rm R\,N\,R}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\rm N\,R\,N}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\rm R\,N\,R}_{\alpha_s,\alpha_2,\alpha_1}}\;\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}^{\rm e\,o}_{\rm e\,o}, \\
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} \Delta_3 & \Delta_2 \\ +\beta_4 & -\beta_1 \end{bmatrix}^{\rm e\,o}_{\rm o\,e} &= -\frac{\mathcal{M}^{\rm R\,N\,R}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\rm N\,R\,N}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\rm N\,R\,N}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\rm R\,N\,R}_{\alpha_s,\alpha_2,\alpha_1}}\;\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}^{\rm o\,e}_{\rm e\,o}.
\end{aligned}
\tag{4.26}
$$

4.2.2 NS-R braiding

As an example of the braiding between the NS and the R fields consider the composition

$$
V^{-}_{\rm o}\left[{}^{\Delta_3}_{\beta_4\beta_s}\right](z_3)V^{-}_{\rm e}\left[{}^{\beta_2}_{\beta_s\Delta_1}\right](z_2):\;\mathcal{V}_{\alpha_1}\to\mathcal{W}_{\beta_4}\,,
$$

which can be represented, (3.22), (3.25), as:

$$
V^{-}_{\rm o}\left[{}^{\Delta_3}_{\beta_4\beta_s}\right](z_3)V^{-}_{\rm e}\left[{}^{\beta_2}_{\beta_s\Delta_1}\right](z_2) = \frac{\mathsf{V}_{{\rm o}\,s_3}^{\;\alpha_3}(z_3)\mathsf{W}^{-\,\beta_2}_{{\rm e}\,s_2}(z_2)\big|_{\rm NS}}{\left\langle w_4^+\right|\mathsf{V}_{{\rm o}\,s_3}^{\;\alpha_3}(1)\left|w_s^-\right\rangle\left\langle w_s^+\right|\mathsf{W}^{-\,\beta_2}_{{\rm e}\,s_2}(1)\left|\nu_1\right\rangle}.
$$

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By definitions (3.16), (3.17)

$$\mathsf{V}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3)\mathsf{W}_{\mathrm{e}\,s_2}^{-\,\beta_2}(\sigma_2)\big|_{\mathrm{NS}} = \mathsf{g}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3)\,\sigma^-(\sigma_2)\mathsf{g}_{\mathrm{o}\,s_2}^{\,\alpha_2}(\sigma_2)\big|_{\mathrm{NS}},$$

where $\alpha_2 = \frac{Q}{2} - \sqrt{2}\beta_2$. From (3.4) one gets the braiding of $\mathsf{g}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3)$ and $\sigma^-(\sigma_2)$

$$\mathsf{g}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3)\sigma^-(\sigma_2) = \mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\sigma^-(\sigma_2)\mathsf{g}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3).$$

Then using braiding relation (4.1) and notation (4.16) we obtain

$$\begin{aligned}\mathsf{g}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3)\,\sigma^-(\sigma_2)\mathsf{g}_{\mathrm{o}\,s_2}^{\,\alpha_2}(\sigma_2)\big|_{\mathrm{NS}} &= \mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\sigma^-(\sigma_2)\mathsf{g}_{\mathrm{o}\,s_3}^{\,\alpha_3}(\sigma_3)\,\mathsf{g}_{\mathrm{o}\,s_2}^{\,\alpha_2}(\sigma_2)\big|_{\mathrm{NS}}\\ &= \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\sum_{\vec{\mathbf{g}}}\mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\mathbf{g_2 g_3}}_{\mathrm{o}\ \mathrm{o}}\ \sigma^-(\sigma_2)\mathsf{g}_{\mathbf{g_2}\,t_2}^{\,\alpha_2}(\sigma_2)\,\mathsf{g}_{\mathbf{g_3}\,t_3}^{\,\alpha_3}(\sigma_3)\big|_{\mathrm{NS}}\end{aligned}$$

and

$$\mathsf{V}_{\mathrm{o}\,s_3}^{\,\alpha_3}(z_3)\mathsf{W}_{\mathrm{e}\,s_2}^{-\,\beta_2}(z_2)\big|_{\mathrm{NS}} = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\sum_{\vec{\mathbf{g}}}\mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\mathbf{g_2 g_3}}_{\mathrm{o}\ \mathrm{o}}\ \mathsf{W}_{\bar{\mathbf{g}}_2\,t_2}^{-\,\beta_2}(z_2)\mathsf{V}_{\mathbf{g_3}\,t_3}^{\,\alpha_3}(z_3)\big|_{\mathrm{NS}}.$$

By formulae (3.20) and (3.25) the r.h.s. can be expressed in terms of chiral vertex operators

$$\mathsf{W}_{\mathrm{o}\,t_2}^{-\,\beta_2}(z_2)\mathsf{V}_{\mathrm{e}\,t_3}^{\,\alpha_3}(z_3)\big|_{\mathrm{NS}} = \langle w_4^-|\,\mathsf{W}_{\mathrm{o}\,t_2}^{-\,\beta_2}(1)\,|\nu_u\rangle\,\langle\nu_u|\,\mathsf{V}_{\mathrm{e}\,t_3}^{\,\alpha_3}(1)\,|\nu_1\rangle\ V_{\mathrm{o}}^{+}\!\left[{}_{\beta_4}{}^{\beta_2}{}_{\Delta_u}\right](z_2)V_{\mathrm{e}}\!\left[{}_{\Delta_u}{}^{\Delta_3}{}_{\Delta_1}\right](z_3),$$

$$\mathsf{W}_{\mathrm{e}\,t_2}^{-\,\beta_2}(z_2)\mathsf{V}_{\mathrm{o}\,t_3}^{\,\alpha_3}(z_3)\big|_{\mathrm{NS}} = \langle w_4^+|\,\mathsf{W}_{\mathrm{e}\,t_2}^{-\,\beta_2}(z_2)\,|\nu_u\rangle\,\langle\nu_u|\,\mathsf{V}_{\mathrm{e}\,t_3}^{*\,\alpha_3}(1)\,|\nu_1\rangle\ V_{\mathrm{e}}^{-}\!\left[{}_{\beta_4}{}^{\beta_2}{}_{\Delta_u}\right](z_2)V_{\mathrm{o}}\!\left[{}_{\Delta_u}{}^{\Delta_3}{}_{\Delta_1}\right](z_3).$$

This leads to the braiding relation

$$\begin{aligned}V_{\mathrm{o}}^{-}\!\left[{}_{\beta_4}{}^{\Delta_3}{}_{\beta_s}\right](z_3)V_{\mathrm{e}}^{-}\!\left[{}_{\beta_s}{}^{\beta_2}{}_{\Delta_1}\right](z_2) = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\Bigg(&\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\!\begin{bmatrix}\Delta_3 & -\beta_2\\ -\beta_4 & \Delta_1\end{bmatrix}^{\mathrm{o\,e}}_{\mathrm{o\,e}} V_{\mathrm{o}}^{+}\!\left[{}_{\beta_4}{}^{\beta_2}{}_{\Delta_u}\right](z_2)V_{\mathrm{e}}\!\left[{}_{\Delta_u}{}^{\Delta_3}{}_{\Delta_1}\right](z_3)\\ &+\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\!\begin{bmatrix}\Delta_3 & -\beta_2\\ -\beta_4 & \Delta_1\end{bmatrix}^{\mathrm{e\,o}}_{\mathrm{o\,e}} V_{\mathrm{e}}^{-}\!\left[{}_{\beta_4}{}^{\beta_2}{}_{\Delta_u}\right](z_2)V_{\mathrm{o}}\!\left[{}_{\Delta_u}{}^{\Delta_3}{}_{\Delta_1}\right](z_3)\Bigg),\end{aligned}$$

where

$$\begin{aligned}\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\!\begin{bmatrix}\Delta_3 & -\beta_2\\ -\beta_4 & \Delta_1\end{bmatrix}^{\mathrm{o\,e}}_{\mathrm{o\,e}} &= -i\mathrm{e}^{\frac{i\pi\epsilon}{2}}\,\frac{\mathcal{M}^{\mathrm{R\,N\,N}}_{\alpha_4,\alpha_2,\alpha_u}\ \mathcal{M}^{\mathrm{N\,N\,N}}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\mathrm{R\,N\,R}}_{\alpha_4,\alpha_3,\alpha_s}\ \mathcal{M}^{\mathrm{N\,R\,R}}_{\alpha_s,\alpha_2,\alpha_1}}\ \mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\mathrm{e\,e}}_{\mathrm{o\,o}},\\ \mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\!\begin{bmatrix}\Delta_3 & -\beta_2\\ -\beta_4 & \Delta_1\end{bmatrix}^{\mathrm{e\,o}}_{\mathrm{o\,e}} &= \mathrm{e}^{-\frac{i\pi\epsilon}{2}}\,\frac{\mathcal{M}^{\mathrm{N\,R\,R}}_{\alpha_4,\alpha_2,\alpha_u}\ \mathcal{M}^{\mathrm{R\,R\,R}}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\mathrm{R\,N\,R}}_{\alpha_4,\alpha_3,\alpha_s}\ \mathcal{M}^{\mathrm{N\,R\,R}}_{\alpha_s,\alpha_2,\alpha_1}}\ \mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\!\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]^{\mathrm{o\,o}}_{\mathrm{o\,o}}.\end{aligned} \tag{4.27}$$

4.2.3 R-R braiding

We shall start from the composition of Ramond vertex operators in the NS sector

$$V_{\mathrm{e}}^{+}\!\left[{}_{\Delta_4}{}^{\beta_3}{}_{\beta_s}\right](z_3)V_{\mathrm{o}}^{+}\!\left[{}_{\beta_s}{}^{\beta_2}{}_{\Delta_1}\right](z_2):\quad \mathcal{V}_{\alpha_1}\to\mathcal{V}_{\alpha_4}.$$

In representation (3.22):

$$V_{\mathrm{e}}^{+}\!\left[{}_{\Delta_4}{}^{\beta_3}{}_{\beta_s}\right](z_3)V_{\mathrm{o}}^{+}\!\left[{}_{\beta_s}{}^{\beta_2}{}_{\Delta_1}\right](z_2) = \frac{\mathsf{W}_{\mathrm{e}\,s_3}^{+\,\beta_3}(z_3)\mathsf{W}_{\mathrm{o}\,s_2}^{-\,\beta_2}(z_2)\big|_{\mathrm{NS}}}{\langle\nu_4|\,\mathsf{W}_{\mathrm{e}\,s_3}^{+\,\beta_3}(1)\,|w_s^+\rangle\,\langle w_s^-|\,\mathsf{W}_{\mathrm{o}\,s_2}^{-\,\beta_2}(1)\,|\nu_1\rangle}.$$

From definition (3.17) and braiding properties (3.4), (3.6) one gets

$$\begin{aligned}
\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\ s_3}(\sigma_3)\mathsf{W}^{-\,\beta_2}_{\mathrm{o}\ s_2}(\sigma_2)\big|_{\mathrm{NS}} &= \sigma^+(\sigma_3)\mathsf{g}^{\ \alpha_3}_{\mathrm{e}\,s_3}(\sigma_3)\sigma^-(\sigma_2)\mathsf{g}^{\ \alpha_2}_{\mathrm{e}\,s_2}(\sigma_2)\big|_{\mathrm{NS}}\\
&= \sigma^+(\sigma_3)\sigma^-(\sigma_2)\mathsf{g}^{\ \alpha_3}_{\mathrm{e}\,s_3}(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{e}\,s_2}(\sigma_2)\big|_{\mathrm{NS}}\\
&= \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{i\pi}{8}}\left(\sigma^-(\sigma_2)\sigma^+(\sigma_3)+\sigma^+(\sigma_2)\sigma^-(\sigma_3)\right)\mathsf{g}^{\ \alpha_3}_{\mathrm{e}\,s_3}(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{e}\,s_2}(\sigma_2)\big|_{\mathrm{NS}}.
\end{aligned}$$

It the present case the braiding relation of g fields we need reads

$$\mathsf{g}^{\ \alpha_3}_{\mathrm{e}\,s_3}(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{e}\,s_2}(\sigma_2)\big|_{\mathrm{NS}} = \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\sum_{\vec{\mathbf{g}}}\mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\Big[{}^{\alpha_3\ \alpha_2}_{\alpha_4\ \alpha_1}\Big]^{\mathbf{g_2 g_3}}_{\mathrm{e\ e}}\ \mathsf{g}_{\mathbf{g_2}\,t_2}^{\ \ \alpha_2}(\sigma_2)\mathsf{g}_{\mathbf{g_3}\,t_3}^{\ \ \alpha_3}(\sigma_3)\big|_{\mathrm{NS}}.$$

Using braiding relations (3.4) and representation (3.22) one gets:

$$\begin{aligned}
\sigma^-(\sigma_2)\sigma^+(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{e}\,t_2}(\sigma_2)\mathsf{g}^{\ \alpha_3}_{\mathrm{e}\,t_3}(\sigma_3)\big|_{\mathrm{NS}} &= \mathsf{W}^{-\,\beta_2}_{\mathrm{o}\ t_2}(\sigma_2)\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\ t_3}(\sigma_3)\big|_{\mathrm{NS}}\\
&= \langle\nu_4|\,\mathsf{W}^{+\,\beta_2}_{\mathrm{e}\ t_2}(1)\,|w_u^+\rangle\,\langle w_u^+|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\ t_3}(1)\,|\nu_1\rangle\ V^+_{\mathrm{o}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^+_{\mathrm{e}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3),\\
\sigma^+(\sigma_2)\sigma^-(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{e}\,t_2}(\sigma_2)\mathsf{g}^{\ \alpha_3}_{\mathrm{e}\,t_3}(\sigma_3)\big|_{\mathrm{NS}} &= \mathsf{W}^{+\,\beta_2}_{\mathrm{e}\ t_2}(\sigma_2)\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\ t_3}(\sigma_3)\big|_{\mathrm{NS}}\\
&= \langle\nu_4|\,\mathsf{W}^{+\,\beta_2}_{\mathrm{e}\ t_2}(1)\,|w_u^+\rangle\,\langle w_u^+|\,\mathsf{W}^{+\,\beta_3}_{\mathrm{e}\ t_3}(1)\,|\nu_1\rangle\ V^+_{\mathrm{e}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^+_{\mathrm{o}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3),
\end{aligned}$$

$$\begin{aligned}
\sigma^-(\sigma_2)\sigma^+(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{o}\,t_2}(\sigma_2)\mathsf{g}^{\ \alpha_3}_{\mathrm{o}\,t_3}(\sigma_3)\big|_{\mathrm{NS}} &= \mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\mathsf{W}^{-\,\beta_2}_{\mathrm{e}\ t_2}(\sigma_2)\mathsf{W}^{+\,\beta_3}_{\mathrm{o}\ t_3}(\sigma_3)\big|_{\mathrm{NS}}\\
&= i\mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\langle\nu_4|\,\mathsf{W}^{-\,\beta_2}_{\mathrm{e}\ t_2}(1)\,|w_u^+\rangle\,\langle w_u^+|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{e}\ t_3}(1)\,|\nu_1\rangle\ V^-_{\mathrm{e}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^-_{\mathrm{o}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3),\\
\sigma^+(\sigma_2)\sigma^-(\sigma_3)\mathsf{g}^{\ \alpha_2}_{\mathrm{o}\,t_2}(\sigma_2)\mathsf{g}^{\ \alpha_3}_{\mathrm{o}\,t_3}(\sigma_3)\big|_{\mathrm{NS}} &= \mathrm{e}^{\frac{i\pi}{2}(\epsilon+1)}\,\mathsf{W}^{+\,\beta_2}_{\mathrm{o}\ t_2}(\sigma_2)\mathsf{W}^{-\,\beta_3}_{\mathrm{e}\ t_3}(\sigma_3)\big|_{\mathrm{NS}}\\
&= -i\mathrm{e}^{\frac{i\pi}{2}(\epsilon-1)}\,\langle\nu_4|\,\mathsf{W}^{-\,\beta_2}_{\mathrm{e}\ t_2}(1)\,|w_u^+\rangle\,\langle w_u^+|\,\mathsf{W}^{-\,\beta_3}_{\mathrm{e}\ t_3}(1)\,|\nu_1\rangle\ V^-_{\mathrm{o}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^-_{\mathrm{e}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3).
\end{aligned}$$

We are thus lead to following form of the braiding relation

$$\begin{aligned}
&V^+_{\mathrm{e}}\big[{}^{\ \beta_3}_{\Delta_4\beta_s}\big](z_3)V^+_{\mathrm{o}}\big[{}^{\ \beta_2}_{\beta_s\Delta_1}\big](z_2) =\\
&= \int\limits_{\frac{Q}{2}+i\mathbb{R}}\frac{d\alpha_u}{2i}\Bigg[\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix}+\beta_3 & +\beta_2\\ \Delta_4 & \Delta_1\end{bmatrix}^{++}_{\mathrm{e\,o}}\left(V^+_{\mathrm{e}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^+_{\mathrm{o}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3)+V^+_{\mathrm{o}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^+_{\mathrm{e}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3)\right)\\
&\qquad -\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix}+\beta_3 & +\beta_2\\ \Delta_4 & \Delta_1\end{bmatrix}^{--}_{\mathrm{e\,o}}\left(V^-_{\mathrm{e}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^-_{\mathrm{o}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3)-V^-_{\mathrm{o}}\big[{}^{\ \beta_2}_{\Delta_4\beta_u}\big](z_2)V^-_{\mathrm{e}}\big[{}^{\ \beta_3}_{\beta_u\Delta_1}\big](z_3)\right)\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix}+\beta_3 & +\beta_2\\ \Delta_4 & \Delta_1\end{bmatrix}^{++}_{\mathrm{e\,o}} &= \frac{1}{\sqrt{2}}\,\mathrm{e}^{\frac{i\pi}{8}}\,\frac{\mathcal{M}^{\mathrm{RRN}}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\mathrm{RNN}}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\mathrm{RRN}}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\mathrm{RNN}}_{\alpha_s,\alpha_2,\alpha_1}}\ \mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\Big[{}^{\alpha_3\ \alpha_2}_{\alpha_4\ \alpha_1}\Big]^{\mathrm{ee}}_{\mathrm{ee}},\\
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix}+\beta_3 & +\beta_2\\ \Delta_4 & \Delta_1\end{bmatrix}^{--}_{\mathrm{e\,o}} &= \frac{1}{\sqrt{2}}\,\mathrm{e}^{\frac{i\pi}{8}+\frac{i\pi\epsilon}{2}}\,\frac{\mathcal{M}^{\mathrm{NNR}}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\mathrm{NRR}}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\mathrm{RRN}}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\mathrm{RNN}}_{\alpha_s,\alpha_2,\alpha_1}}\ \mathbb{B}^{\epsilon,\,\mathrm{NS}}_{\alpha_s\alpha_u}\Big[{}^{\alpha_3\ \alpha_2}_{\alpha_4\ \alpha_1}\Big]^{\mathrm{oo}}_{\mathrm{ee}}.
\end{aligned}$$

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As our last example we present the result for a composition of two Ramond vertex operators in the Ramond sector,

$$
\begin{aligned}
&V_{\rm e}^{+}\left[{}_{\beta_4}{}^{\beta_3}_{\Delta_s}\right](z_3)V_{\rm o}^{-}\left[{}_{\Delta_s}{}^{\beta_2}_{\beta_1}\right](z_2) \;= \\
&= \int\limits_{\frac{Q}{2}+i\mathbb{R}} \frac{d\alpha_u}{2i}\left[\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} +\beta_3 & -\beta_2 \\ \beta_4 & \beta_1\end{bmatrix}^{+-}_{\rm e\,o}\left(V_{\rm e}^{+}\left[{}_{\beta_4}{}^{\beta_2}_{\Delta_u}\right](z_2)V_{\rm o}^{-}\left[{}_{\Delta_u}{}^{\beta_3}_{\beta_1}\right](z_3) - V_{\rm o}^{+}\left[{}_{\beta_4}{}^{\beta_2}_{\Delta_u}\right](z_2)V_{\rm e}^{-}\left[{}_{\Delta_u}{}^{\beta_3}_{\beta_1}\right](z_3)\right)\right.\\
&\qquad\left. + \,\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} +\beta_3 & -\beta_2 \\ \beta_4 & \beta_1\end{bmatrix}^{-+}_{\rm e\,o}\left(V_{\rm e}^{-}\left[{}_{\beta_4}{}^{\beta_2}_{\Delta_u}\right](z_2)V_{\rm o}^{+}\left[{}_{\Delta_u}{}^{\beta_3}_{\beta_1}\right](z_3) + V_{\rm o}^{-}\left[{}_{\beta_4}{}^{\beta_2}_{\Delta_u}\right](z_2)V_{\rm e}^{+}\left[{}_{\Delta_u}{}^{\beta_3}_{\beta_1}\right](z_3)\right)\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} +\beta_3 & -\beta_2 \\ \beta_4 & \beta_1\end{bmatrix}^{+-}_{\rm e\,o} &= -\frac{i}{\sqrt{2}}\mathrm{e}^{\frac{i\pi}{8}}\,\frac{\mathcal{M}^{\rm RNN}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\rm NNR}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\rm RNN}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\rm NNR}_{\alpha_s,\alpha_2,\alpha_1}}\;\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\rm e\,o}_{\rm e\,o},\\
\mathsf{B}^{\epsilon}_{\alpha_s\alpha_u}\begin{bmatrix} +\beta_3 & -\beta_2 \\ \beta_4 & \beta_1\end{bmatrix}^{-+}_{\rm e\,o} &= \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{i\pi}{8}-\frac{i\pi\epsilon}{2}}\,\frac{\mathcal{M}^{\rm NRR}_{\alpha_4,\alpha_2,\alpha_u}\,\mathcal{M}^{\rm RRN}_{\alpha_u,\alpha_3,\alpha_1}}{\mathcal{M}^{\rm RNN}_{\alpha_4,\alpha_3,\alpha_s}\,\mathcal{M}^{\rm NNR}_{\alpha_s,\alpha_2,\alpha_1}}\;\mathbb{B}^{\epsilon,\rm R}_{\alpha_s\alpha_u}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^{\rm o\,e}_{\rm e\,o}.
\end{aligned}
$$

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A Properties of Ising chiral vertex operators

Properties of the chiral vertex operators in the Ising model can be easily derived from well known analytic properties of the 4-point conformal blocks [1, 25]. The fusion matrix yields the coefficients of the chiral OPE:

$$
\begin{aligned}
&V^{\,\varepsilon}_{1\,\varepsilon}(z_3)V^{\,\sigma}_{\varepsilon\,\sigma}(z_2) \sim \frac{1}{\sqrt{z_{32}}}V^{\,\sigma}_{1\,\sigma}(z_2), &&V^{\,\varepsilon}_{\varepsilon\,1}(z_3)V^{\,\sigma}_{1\,\sigma}(z_2) \sim \frac{1}{2\sqrt{z_{32}}}V^{\,\sigma}_{\varepsilon\,\sigma}(z_2),\\
&V^{\,\varepsilon}_{\sigma\,\sigma}(z_3)V^{\,\sigma}_{\sigma\,\varepsilon}(z_2) \sim -\frac{1}{\sqrt{z_{32}}}V^{\,\sigma}_{\sigma\,\varepsilon}(z_2), &&V^{\,\varepsilon}_{\sigma\,\sigma}(z_3)V^{\,\sigma}_{\sigma\,1}(z_2) \sim \frac{1}{\sqrt{z_{32}}}V^{\,\sigma}_{\sigma\,1}(z_2),\\
&V^{\,\sigma}_{\varepsilon\,\sigma}(z_3)V^{\,\sigma}_{\sigma\,1}(z_2) \sim z_{32}^{\frac{3}{8}}V^{\,\varepsilon}_{\varepsilon\,1}(z_2), &&V^{\,\sigma}_{1\,\sigma}(z_3)V^{\,\sigma}_{\sigma\,\varepsilon}(z_2) \sim z_{32}^{\frac{3}{8}}V^{\,\varepsilon}_{1\,\varepsilon}(z_2),\\
&V^{\,\sigma}_{1\,\sigma}(z_3)V^{\,\sigma}_{\sigma\,1}(z_2) \sim z_{32}^{-\frac{1}{8}}V^{\,1}_{1\,1}(z_2), &&V^{\,\sigma}_{\varepsilon\,\sigma}(z_3)V^{\,\sigma}_{\sigma\,\varepsilon}(z_2) \sim 2z_{32}^{-\frac{1}{8}}V^{\,1}_{\varepsilon\,\varepsilon}(z_2),\\
&V^{\,\sigma}_{\sigma\,1}(z_3)V^{\,\sigma}_{1\,\sigma}(z_2) \sim \frac{1}{\sqrt{2}}z_{32}^{-\frac{1}{8}}\,V^{\,1}_{\sigma\,\sigma}(z_2) + \frac{1}{2\sqrt{2}}z_{32}^{\frac{3}{8}}\,V^{\,\varepsilon}_{\sigma\,\sigma}(z_2), \\
&V^{\,\sigma}_{\sigma\,\varepsilon}(z_3)V^{\,\sigma}_{\varepsilon\,\sigma}(z_2) \sim \sqrt{2}z_{32}^{-\frac{1}{8}}\,V^{\,1}_{\sigma\,\sigma}(z_2) - \frac{1}{\sqrt{2}}z_{32}^{\frac{3}{8}}\,V^{\,\varepsilon}_{\sigma\,\sigma}(z_2).
\end{aligned}
\tag{A.1}
$$

The braiding matrix gives the braiding relations:

$$
\begin{aligned}
&V_{1\,\varepsilon}^{\,\varepsilon}(z_3)V_{\varepsilon\,\sigma}^{\,\sigma}(z_2) = \mathrm{e}^{-\frac{i}{2}\pi\epsilon}V_{1\,\sigma}^{\,\sigma}(z_2)V_{\sigma\,\sigma}^{\,\varepsilon}(z_3), \quad V_{\varepsilon\,1}^{\,\varepsilon}(z_3)V_{1\,\sigma}^{\,\sigma}(z_2) = \frac{1}{2}\mathrm{e}^{\frac{i}{2}\pi\epsilon}V_{\varepsilon\,\sigma}^{\,\sigma}(z_2)V_{\sigma\,\sigma}^{\,\varepsilon}(z_3),\\
&V_{\sigma\,\sigma}^{\,\varepsilon}(z_3)V_{\sigma\,\varepsilon}^{\,\sigma}(z_2) = 2\mathrm{e}^{\frac{i}{2}\pi\epsilon}V_{\sigma\,1}^{\,\sigma}(z_2)V_{1\,\varepsilon}^{\,\varepsilon}(z_3), \quad V_{\sigma\,\sigma}^{\,\varepsilon}(z_3)V_{\sigma\,1}^{\,\sigma}(z_2) = \mathrm{e}^{-\frac{i}{2}\pi\epsilon}V_{\sigma\,\varepsilon}^{\,\sigma}(z_2)V_{\varepsilon\,1}^{\,\varepsilon}(z_3),\\
&V_{1\,\sigma}^{\,\sigma}(z_3)V_{\sigma\,1}^{\,\sigma}(z_2) = \mathrm{e}^{-\frac{i}{8}\pi\epsilon}V_{1\,\sigma}^{\,\sigma}(z_2)V_{\sigma\,1}^{\,\sigma}(z_3), \quad V_{\varepsilon\,\sigma}^{\,\sigma}(z_3)V_{\sigma\,\varepsilon}^{\,\sigma}(z_2) = \mathrm{e}^{-\frac{i}{8}\pi\epsilon}V_{\varepsilon\,\sigma}^{\,\sigma}(z_2)V_{\sigma\,\varepsilon}^{\,\sigma}(z_3),\\
&V_{\varepsilon\,\sigma}^{\,\sigma}(z_3)V_{\sigma\,1}^{\,\sigma}(z_2) = \mathrm{e}^{\frac{i3}{8}\pi\epsilon}V_{\varepsilon\,\sigma}^{\,\sigma}(z_2)V_{\sigma\,1}^{\,\sigma}(z_3), \quad V_{1\,\sigma}^{\,\sigma}(z_3)V_{\sigma\,\varepsilon}^{\,\sigma}(z_2) = \mathrm{e}^{\frac{i3}{8}\pi\epsilon}V_{1\,\sigma}^{\,\sigma}(z_2)V_{\sigma\,\varepsilon}^{\,\sigma}(z_3),\\
&V_{\sigma\,1}^{\,\sigma}(z_3)V_{1\,\sigma}^{\,\sigma}(z_2) = \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{i}{8}\pi\epsilon}\; V_{\sigma\,1}^{\,\sigma}(z_2)V_{1\,\sigma}^{\,\sigma}(z_3) + \frac{1}{2\sqrt{2}}\mathrm{e}^{-\frac{i3}{8}\pi\epsilon}\; V_{\sigma\,\varepsilon}^{\,\sigma}(z_2)V_{\varepsilon\,\sigma}^{\,\sigma}(z_3),\\
&V_{\sigma\,\varepsilon}^{\,\sigma}(z_3)V_{\varepsilon\,\sigma}^{\,\sigma}(z_2) = \sqrt{2}\mathrm{e}^{-\frac{i3}{8}\pi\epsilon}\; V_{\sigma\,1}^{\,\sigma}(z_2)V_{1\,\sigma}^{\,\sigma}(z_3) + \frac{1}{\sqrt{2}}\mathrm{e}^{\frac{i}{8}\pi\epsilon}\; V_{\sigma\,\varepsilon}^{\,\sigma}(z_2)V_{\varepsilon\,\sigma}^{\,\sigma}(z_3).
\end{aligned}
\tag{A.2}
$$

B Conformal Ward Identities for the fermionic current $S(\xi)$

Correlation functions of the fermionic current $S(\xi)$ in the presence of the Ramond fields are no longer single-valued and derivation of their form, even if standard, is subtle. As as example we shall discuss the correlator

$$\langle\nu_4|\, \mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)S(\xi)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle\,.$$

Due to OPE (1.1) it has square root branch cuts at $\xi = 1$ and $\xi = z$. The function

$$s(\xi) = \sqrt{1-\xi}\sqrt{\xi-z}\,\langle\nu_4|\, \mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)S(\xi)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle \tag{B.1}$$

should then be single-valued and analytic on the complex ξ plane save the simple poles at $\xi = 0, z, 1$. With the principal argument of a complex number ξ in the range $-\pi \leq \operatorname{Arg}\xi < \pi$ one has:

$$\sqrt{1-\xi} = \mathrm{e}^{-\frac{i\pi}{2}\mathrm{sgn}\,\mathrm{Arg}(\xi-1)}\sqrt{\xi-1}, \qquad \sqrt{\xi-z} = \mathrm{e}^{\frac{i\pi}{2}\mathrm{sgn}\,\mathrm{Arg}(\xi-z)}\sqrt{z-\xi}. \tag{B.2}$$

Braiding relations (1.7) yield

$$
\begin{aligned}
s(\xi) &= i\sqrt{\xi-1}\sqrt{\xi-z}\,\langle\nu_4|\, S(\xi)\mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle\\
&= -i\sqrt{1-\xi}\sqrt{z-\xi}\,\langle\nu_4|\, \mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)S(\xi)\,|\nu_1\rangle\,.
\end{aligned}
\tag{B.3}
$$

Note that in (B.3) the multi-valuednes of braiding relation (1.7) is compensated by the signs coming from (B.2). It then follows from OPE-s (1.1) and $S(\xi)\,|\nu_1\rangle \sim \frac{1}{\xi}S_{-\frac{1}{2}}\,|\nu_1\rangle$ that

$$
s(\xi) \;\sim\; \begin{cases}
-\dfrac{i\sqrt{z}}{\xi}\,\langle\nu_4|\, \mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)S_{-\frac{1}{2}}\,|\nu_1\rangle & \text{for} \quad \xi\to 0,\\[2ex]
\dfrac{\sqrt{1-z}}{\xi-z}\,\mathrm{e}^{\frac{i\pi}{4}}\beta_+\,\langle\nu_4|\, \mathsf{W}^{-\,\beta_3}_{\mathrm{o}\;s_3}(1)\mathsf{W}^{-\,\beta_+}_{\mathrm{o}}(z)\,|\nu_1\rangle & \text{for} \quad \xi\to z,\\[2ex]
-\dfrac{\sqrt{1-z}}{\xi-1}\,\mathrm{e}^{\frac{i\pi}{4}}\beta_3\,\langle\nu_4|\, \mathsf{W}^{+\,\beta_3}_{\mathrm{e}\;s_3}(1)\mathsf{W}^{+\,\beta_+}_{\mathrm{e}}(z)\,|\nu_1\rangle & \text{for} \quad \xi\to 1.
\end{cases}
$$

This, together with the condition $s(\xi) \to 0$ for $\xi \to \infty$, completely determines the form of the function $s(\xi)$ and yields

$$
\begin{aligned}
\langle \nu_4 | \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_3}_{s_3}(1) S(\xi) \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_+}(z) |\nu_1\rangle &= \frac{1}{\sqrt{1-\xi}\sqrt{\xi - z}} \left[-\frac{i\sqrt{z}}{\xi} \langle \nu_4 | \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_3}_{s_3}(1) \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_+}(z) S_{-\frac{1}{2}} |\nu_1\rangle \right. \\
&\left. + \frac{\sqrt{1-z}}{\xi - z} \mathrm{e}^{\frac{i\pi}{4}} \beta_+ \langle \nu_4 | \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_3}_{s_3}(1) \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_+}(z) |\nu_1\rangle - \frac{\sqrt{1-z}}{\xi - 1} \mathrm{e}^{\frac{i\pi}{4}} \beta_3 \langle \nu_4 | \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_3}_{s_3}(1) \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_+}(z) |\nu_1\rangle \right].
\end{aligned}
\tag{B.4}
$$

Expanding both sides of this equation around $\xi = z$ and equating the coefficients at $(\xi - z)^{-\frac{1}{2}}$ we get

$$
\begin{aligned}
\langle \nu_4 | \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_3}_{s_3}(1) S_{-1} \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_+}(z) |\nu_1\rangle &= \frac{-i}{\sqrt{1-z}\sqrt{z}} \langle \nu_4 | \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_3}_{s_3}(1) \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_+}(z) S_{-\frac{1}{2}} |\nu_1\rangle \qquad \text{(B.5)} \\
&+ \frac{\mathrm{e}^{\frac{i\pi}{4}}}{1-z} \left[\frac{1}{2} \beta_+ \langle \nu_4 | \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_3}_{s_3}(1) \mathsf{W}^{-}_{\mathrm{o}}{}^{\beta_+}(z) |\nu_1\rangle + \beta_3 \langle \nu_4 | \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_3}_{s_3}(1) \mathsf{W}^{+}_{\mathrm{e}}{}^{\beta_+}(z) |\nu_1\rangle \right].
\end{aligned}
$$

Formula (B.5) is used in a derivation of differential equation (3.28).

C Some properties of special functions related to the Barnes double gamma

For $\Re x > 0$ the Barnes double function $\Gamma_b(x)$ has an integral representation of the form:

$$
\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left[\frac{\mathrm{e}^{-xt} - \mathrm{e}^{-Qt/2}}{\left(1 - \mathrm{e}^{-tb}\right)\left(1 - \mathrm{e}^{-t/b}\right)} - \frac{(Q/2 - x)^2}{2\mathrm{e}^t} - \frac{Q/2 - x}{t} \right].
$$

With a help of relations

$$
\Gamma_b(x + b) = \frac{\sqrt{2\pi}\, b^{bx - \frac{1}{2}}}{\Gamma(bx)} \Gamma_b(x), \qquad \Gamma_b\left(x + b^{-1}\right) = \frac{\sqrt{2\pi}\, b^{-\frac{x}{b} + \frac{1}{2}}}{\Gamma(\frac{x}{b})} \Gamma_b(x), \tag{C.1}
$$

one can continue $\Gamma_b(x)$ analytically to a meromorphic function of $x \in \mathbb{C}$ with no zeroes and with simple poles located at $x = -mb - nb^{-1}$, $m, n \in \mathbb{N}$.

Borrowing the notation from [14] we define

$$
S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q - x)}, \qquad G_b(x) = \mathrm{e}^{-\frac{i\pi}{2} x(Q - x)} S_b(x), \tag{C.2}
$$

and

$$
\begin{aligned}
\Gamma_{\mathrm{NS}}(x) &= \Gamma_b\left(\tfrac{x}{2}\right) \Gamma_b\left(\tfrac{x+Q}{2}\right), & \Gamma_{\mathrm{R}}(x) &= \Gamma_b\left(\tfrac{x+b}{2}\right) \Gamma_b\left(\tfrac{x+b^{-1}}{2}\right), \\
S_{\mathrm{NS}}(x) &= S_b\left(\tfrac{x}{2}\right) S_b\left(\tfrac{x+Q}{2}\right), & S_{\mathrm{R}}(x) &= S_b\left(\tfrac{x+b}{2}\right) S_b\left(\tfrac{x+b^{-1}}{2}\right), \\
G_{\mathrm{NS}}(x) &= G_b\left(\tfrac{x}{2}\right) G_b\left(\tfrac{x+Q}{2}\right), & G_{\mathrm{R}}(x) &= G_b\left(\tfrac{x+b}{2}\right) G_b\left(\tfrac{x+b^{-1}}{2}\right).
\end{aligned}
\tag{C.3}
$$

Using relations (C.1) and definitions (C.2), (C.3) one can easily establish some basic properties of these functions.

- Relations between S and G functions:

$$G_{\mathrm{NS}}(x) = \zeta_0\, \mathrm{e}^{-\frac{i\pi}{4}x(Q-x)} S_{\mathrm{NS}}(x), \qquad G_{\mathrm{R}}(x) = \mathrm{e}^{-\frac{i\pi}{4}} \zeta_0\, \mathrm{e}^{-\frac{i\pi}{4}x(Q-x)} S_{\mathrm{R}}(x), \tag{C.4}$$

where $\zeta_0 \;=\; \mathrm{e}^{-\frac{i\pi Q^2}{8}}$.

- Shift relations:

$$G_{\mathrm{NS}}(x+b^{\pm 1}) = \left(1+\mathrm{e}^{i\pi b^{\pm 1}x}\right) G_{\mathrm{R}}(x), \qquad G_{\mathrm{R}}(x+b^{\pm 1}) = \left(1-\mathrm{e}^{i\pi b^{\pm 1}x}\right) G_{\mathrm{NS}}(x). \tag{C.5}$$

- Reflection properties:

$$S_{\mathrm{NS}}(x) S_{\mathrm{NS}}(Q-x) \;=\; S_{\mathrm{R}}(x) S_{\mathrm{R}}(Q-x) \;=\; 1. \tag{C.6}$$

- Locations of zeroes and poles:

$$\begin{aligned}
S_{\mathrm{NS}}(x) &= 0 \quad \Leftrightarrow \quad x = Q + mb + nb^{-1}, \quad && m,n \in \mathbb{Z}_{\geq 0},\; m+n \in 2\mathbb{Z},\\
S_{\mathrm{R}}(x) &= 0 \quad \Leftrightarrow \quad x = Q + mb + nb^{-1}, \quad && m,n \in \mathbb{Z}_{\geq 0},\; m+n \in 2\mathbb{Z}+1,\\
S_{\mathrm{NS}}(x)^{-1} &= 0 \quad \Leftrightarrow \quad x = -mb - nb^{-1}, \quad && m,n \in \mathbb{Z}_{\geq 0},\; m+n \in 2\mathbb{Z},\\
S_{\mathrm{R}}(x)^{-1} &= 0 \quad \Leftrightarrow \quad x = -mb - nb^{-1}, \quad && m,n \in \mathbb{Z}_{\geq 0},\; m+n \in 2\mathbb{Z}+1.
\end{aligned}$$

- Basic residue:

$$\lim_{x\to 0}\; x\, S_{\mathrm{NS}}(x) \;=\; \frac{1}{\pi}. \tag{C.7}$$

D Orthogonality relations

For $\xi \in i\mathbb{R}_+$ we define

$$\begin{aligned}
\langle \tau \,|^{\mathrm{N}}_{\mathrm{N}}|\, \xi \rangle &= \frac{1}{S_{\mathrm{NS}}(Q+\tau+\xi-0^+) S_{\mathrm{NS}}(Q+\tau-\xi-0^+)} \;=\; \frac{S_{\mathrm{NS}}(\xi-\tau)}{S_{\mathrm{NS}}(Q+\tau+\xi-0^+)},\\
\langle \tau \,|^{\mathrm{R}}_{\mathrm{N}}|\, \xi \rangle &= \frac{1}{S_{\mathrm{NS}}(Q+\tau+\xi-0^+) S_{\mathrm{R}}(Q+\tau-\xi)} \;=\; \frac{S_{\mathrm{R}}(\xi-\tau)}{S_{\mathrm{NS}}(Q+\xi+\tau-0^+)},
\end{aligned}$$

etc.

The orthogonality relations

$$\begin{aligned}
\int_{i\mathbb{R}} \frac{d\tau}{i} \;\left[\langle \tau \,|^{\mathrm{N}}_{\mathrm{N}}|\, ip_2 \rangle^* \langle \tau \,|^{\mathrm{N}}_{\mathrm{N}}|\, ip_1 \rangle + \langle \tau \,|^{\mathrm{R}}_{\mathrm{R}}|\, ip_2 \rangle^* \langle \tau \,|^{\mathrm{R}}_{\mathrm{R}}|\, ip_1 \rangle \right] &= \mathcal{N}^{-1}_{\mathrm{NS}}(p_2)\, \delta(p_2-p_1),\\
\int_{i\mathbb{R}} \frac{d\tau}{i} \;\left[\langle \tau \,|^{\mathrm{N}}_{\mathrm{N}}|\, ip_2 \rangle^* \langle \tau \,|^{\mathrm{R}}_{\mathrm{R}}|\, ip_1 \rangle - \langle \tau \,|^{\mathrm{R}}_{\mathrm{R}}|\, ip_2 \rangle^* \langle \tau \,|^{\mathrm{N}}_{\mathrm{N}}|\, ip_1 \rangle \right] &= 0,
\end{aligned} \tag{D.1}$$

where $p_1, p_2 \in \mathbb{R}_+$, were derived in [20]. They followed from the "Saalschutz summation formulae"

$$\begin{aligned}
&\int_{-i\infty}^{i\infty} \frac{d\tau}{i}\; \mathrm{e}^{i\pi\tau Q} \left[\frac{G_{\mathrm{NS}}(\tau+a) G_{\mathrm{NS}}(\tau+b)}{G_{\mathrm{NS}}(\tau+d) G_{\mathrm{NS}}(\tau+Q)} + \frac{G_{\mathrm{R}}(\tau+a) G_{\mathrm{R}}(\tau+b)}{G_{\mathrm{R}}(\tau+d) G_{\mathrm{R}}(\tau+Q)} \right]\\
&\qquad = 2\zeta_0^{-3}\, \mathrm{e}^{\frac{i\pi}{2}d(Q-d)} \frac{G_{\mathrm{NS}}(a) G_{\mathrm{NS}}(b) G_{\mathrm{NS}}(Q+a-d) G_{\mathrm{NS}}(Q+b-d)}{G_{\mathrm{NS}}(Q+a+b-d)},
\end{aligned} \tag{D.2}$$

and

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \mathrm{e}^{i\pi\tau Q} \left[\frac{G_{\mathrm{NS}}(\tau+a)G_{\mathrm{NS}}(\tau+b)}{G_{\mathrm{R}}(\tau+d)G_{\mathrm{NS}}(\tau+Q)} + \frac{G_{\mathrm{R}}(\tau+a)G_{\mathrm{R}}(\tau+b)}{G_{\mathrm{NS}}(\tau+d)G_{\mathrm{R}}(\tau+Q)}\right]$$
$$= 2i\zeta_0^{-3}\, \mathrm{e}^{\frac{i\pi}{2}d(Q-d)} \frac{G_{\mathrm{NS}}(a)G_{\mathrm{NS}}(b)G_{\mathrm{R}}(Q+a-d)G_{\mathrm{R}}(Q+b-d)}{G_{\mathrm{R}}(Q+a+b-d)}, \qquad \text{(D.3)}$$

which were also derived in that paper. For our present purposes we need another two Saalschutz summation formulae, which may be derived following the steps that lead to (D.2), (D.3). They read

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \mathrm{e}^{i\pi\tau Q} \left[\frac{G_{\mathrm{NS}}(\tau+a)G_{\mathrm{R}}(\tau+b)}{G_{\mathrm{R}}(\tau+d)G_{\mathrm{NS}}(\tau+Q)} + \frac{G_{\mathrm{R}}(\tau+a)G_{\mathrm{NS}}(\tau+b)}{G_{\mathrm{NS}}(\tau+d)G_{\mathrm{R}}(\tau+Q)}\right]$$
$$= 2i\zeta_0^{-3}\, \mathrm{e}^{\frac{i\pi}{2}d(Q-d)} \frac{G_{\mathrm{NS}}(a)G_{\mathrm{R}}(b)G_{\mathrm{NS}}(Q+a-d)G_{\mathrm{R}}(Q+b-d)}{G_{\mathrm{NS}}(Q+a+b-d)}, \qquad \text{(D.4)}$$

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i}\, \mathrm{e}^{i\pi\tau Q} \left[\frac{G_{\mathrm{R}}(\tau+a)G_{\mathrm{NS}}(\tau+b)}{G_{\mathrm{R}}(\tau+d)G_{\mathrm{NS}}(\tau+Q)} + \frac{G_{\mathrm{NS}}(\tau+a)G_{\mathrm{R}}(\tau+b)}{G_{\mathrm{NS}}(\tau+d)G_{\mathrm{R}}(\tau+Q)}\right]$$
$$= 2i\zeta_0^{-3}\, \mathrm{e}^{\frac{i\pi}{2}d(Q-d)} \frac{G_{\mathrm{R}}(a)G_{\mathrm{NS}}(b)G_{\mathrm{NS}}(Q+a-d)G_{\mathrm{R}}(Q+b-d)}{G_{\mathrm{NS}}(Q+a+b-d)}. \qquad \text{(D.5)}$$

Using the relations between the $S_\natural(x)$ and $G_\natural(x)$ functions and substituting in (D.4)

$$a = 2\varepsilon + \xi_2 - \xi_1, \qquad b = 2\varepsilon - \xi_2 - \xi_1, \qquad d = Q - 2\xi_1$$

we get

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left[\langle \tau - \epsilon\, |{}^{\mathrm{R}}_{\mathrm{N}}|\, \xi_2\rangle^* \langle \tau - \epsilon\, |{}^{\mathrm{R}}_{\mathrm{N}}|\, \xi_1\rangle + \langle \tau - \epsilon\, |{}^{\mathrm{N}}_{\mathrm{R}}|\, \xi_2\rangle^* \langle \tau - \epsilon\, |{}^{\mathrm{N}}_{\mathrm{R}}|\, \xi_1\rangle\right] =$$
$$= 2i\zeta_0^{-3}\mathrm{e}^{\frac{i\pi}{2}(\xi_1^2-\xi_2^2)-2i\pi\xi_1^2}\, \frac{G_{\mathrm{NS}}(2\epsilon+\xi_-)G_{\mathrm{NS}}(2\epsilon-\xi_-)G_{\mathrm{R}}(2\epsilon-\xi_+)G_{\mathrm{R}}(2\epsilon+\xi_+)}{G_{\mathrm{NS}}(4\epsilon)},$$

where $\xi_- = \xi_2 - \xi_1,\ \xi_+ = \xi_2 + \xi_1$.

For $x \to 0$ we have from (C.4) and (C.7):

$$G_{\mathrm{NS}}(x) = \frac{\zeta_0}{\pi x} + \mathcal{O}(1)$$

so that

$$\lim_{\epsilon\to 0} \frac{G_{\mathrm{NS}}(2\epsilon+ip_2-ip_1)G_{\mathrm{NS}}(2\epsilon-ip_2+ip_1)}{G_{\mathrm{NS}}(4\epsilon)} = \frac{\zeta_0}{\pi} \lim_{\epsilon\to 0} \frac{4\epsilon}{(2\epsilon)^2+(p_2-p_1)^2} = 2\zeta_0\delta(p_2-p_1).$$

This finally gives

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left[\langle \tau\, |{}^{\mathrm{R}}_{\mathrm{N}}|\, ip_2\rangle^* \langle \tau\, |{}^{\mathrm{R}}_{\mathrm{N}}|\, ip_1\rangle + \langle \tau\, |{}^{\mathrm{N}}_{\mathrm{R}}|\, ip_2\rangle^* \langle \tau\, |{}^{\mathrm{N}}_{\mathrm{R}}|\, ip_1\rangle\right] \qquad \text{(D.6)}$$
$$= 4i\zeta_0^{-2}\mathrm{e}^{2i\pi p_1^2}G_{\mathrm{R}}(-2ip_1)G_{\mathrm{R}}(2ip_1)\delta(p_2-p_1) = \mathcal{N}_{\mathrm{R}}^{-1}\,\delta(p_2-p_1).$$

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Similarly, with a help of formula (D.5), on gets

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left[\langle \tau - \epsilon \, |{}^{\rm N}_{\rm R}| \, \xi_2 \rangle^* \langle \tau - \epsilon \, |{}^{\rm R}_{\rm N}| \, \xi_1 \rangle + \langle \tau - \epsilon \, |{}^{\rm R}_{\rm N}| \, \xi_2 \rangle^* \langle \tau - \epsilon \, |{}^{\rm N}_{\rm R}| \, \xi_1 \rangle \right] =$$

$$= 2i\zeta_0^{-3} \mathrm{e}^{\frac{i\pi}{2}(\xi_1^2 - \xi_2^2) - 2i\pi\xi_1^2} \frac{G_{\rm NS}(2\epsilon + \xi_+) G_{\rm NS}(2\epsilon - \xi_+) G_{\rm R}(2\epsilon - \xi_-) G_{\rm R}(2\epsilon + \xi_-)}{G_{\rm NS}(4\epsilon)}.$$

Now, since $\xi_+ = i(p_1 + p_2)$ does not vanish for $p_i \in \mathbb{R}_+$, we do not have a singular contribution from

$$G_{\rm NS}(2\epsilon + \xi_+) G_{\rm NS}(2\epsilon - \xi_+)$$

in the limit $\epsilon \to 0$. Moreover, $G_{\rm R}(x)$ is regular in the vicinity of the imaginary x axis and

$$\lim_{\epsilon \to 0} G_{\rm R}(2\epsilon + \xi_-) G_{\rm R}(2\epsilon - \xi_-)$$

is finite. This gives

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left[\langle \tau \, |{}^{\rm N}_{\rm R}| \, ip_2 \rangle^* \langle \tau \, |{}^{\rm R}_{\rm N}| \, ip_1 \rangle + \langle \tau \, |{}^{\rm R}_{\rm N}| \, ip_2 \rangle^* \langle \tau \, |{}^{\rm N}_{\rm R}| \, ip_1 \rangle \right] = 0. \tag{D.7}$$

Using the reflection formula for the Barnes $S-$functions, eq. (C.6), the fact that $S_\natural(x)$ are real analytic and that $p_i \in \mathbb{R}$, $\tau \in i\mathbb{R}$, we can rewrite (D.1), (D.6) and (D.7) in the form

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left\{ \frac{S_{\rm NS}(ip_2 + \tau)}{S_{\rm NS}(Q + ip_2 - \tau)} \frac{S_{\rm NS}(ip_1 - \tau)}{S_{\rm NS}(Q + ip_1 + \tau)} + \frac{S_{\rm R}(ip_2 + \tau)}{S_{\rm R}(Q + ip_2 - \tau)} \frac{S_{\rm R}(ip_1 - \tau)}{S_{\rm R}(Q + ip_1 + \tau)} \right\}$$
$$= \mathcal{N}_{\rm NS}^{-1} \, \delta(p_2 - p_1), \tag{D.8}$$

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left\{ \frac{S_{\rm R}(ip_2 + \tau)}{S_{\rm NS}(Q + ip_2 - \tau)} \frac{S_{\rm NS}(ip_1 - \tau)}{S_{\rm R}(Q + ip_1 + \tau)} + \frac{S_{\rm NS}(ip_2 + \tau)}{S_{\rm R}(Q + ip_2 - \tau)} \frac{S_{\rm R}(ip_1 - \tau)}{S_{\rm NS}(Q + ip_1 + \tau)} \right\}$$
$$= \mathcal{N}_{\rm R}^{-1} \, \delta(p_2 - p_1),$$

and

$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left\{ \frac{S_{\rm NS}(ip_2 + \tau)}{S_{\rm NS}(Q + ip_2 - \tau)} \frac{S_{\rm R}(ip_1 - \tau)}{S_{\rm R}(Q + ip_1 + \tau)} - \frac{S_{\rm R}(ip_2 + \tau)}{S_{\rm R}(Q + ip_2 - \tau)} \frac{S_{\rm NS}(ip_1 - \tau)}{S_{\rm NS}(Q + ip_1 + \tau)} \right\} = 0,$$
$$\int_{i\mathbb{R}} \frac{d\tau}{i} \left\{ \frac{S_{\rm R}(ip_2 + \tau)}{S_{\rm NS}(Q + ip_2 - \tau)} \frac{S_{\rm R}(ip_1 - \tau)}{S_{\rm NS}(Q + ip_1 + \tau)} + \frac{S_{\rm NS}(ip_2 + \tau)}{S_{\rm R}(Q + ip_2 - \tau)} \frac{S_{\rm NS}(ip_1 - \tau)}{S_{\rm R}(Q + ip_1 + \tau)} \right\} = 0, \tag{D.9}$$

were for brevity we have not written the "-0^+" prescription explicitly. It is now easy to check that for every choice of the parity indices $\mathtt{h}_2, \mathtt{h}_3$, and $\mathtt{g}_2, \mathtt{g}_3$, the orthogonality formula (4.12) reduces to one of the equations (D.8), (D.9).

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