

## $\mathcal{N} = 1$ SCFTs from brane monodromy

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**ABSTRACT:** We present evidence for a new class of strongly coupled  $\mathcal{N} = 1$  superconformal field theories (SCFTs) motivated by F-theory GUT constructions. These SCFTs arise from D3-brane probes of tilted seven-branes which undergo monodromy. In the probe theory, this tilting corresponds to an  $\mathcal{N} = 1$  deformation of an  $\mathcal{N} = 2$  SCFT by a matrix of field-dependent masses with non-trivial branch cuts in the eigenvalues. Though these eigenvalues characterize the geometry, we find that they do not uniquely specify the holomorphic data of the physical theory. We also comment on some phenomenological aspects of how these theories can couple to the visible sector. Our construction can be applied to many  $\mathcal{N} = 2$  SCFTs, resulting in a large new class of  $\mathcal{N} = 1$  SCFTs.

**KEYWORDS:** F-Theory, Conformal Field Models in String Theory

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Geometric preliminaries</b>	<b>3</b>
<b>3</b>	<b>Probing a <math>D_4</math> singularity</b>	<b>6</b>
3.1	Mass deformations and the $\mathcal{N} = 1$ curve	7
3.2	$a$ -maximization and nilpotent mass deformations	8
3.3	Large $N$ limit	9
<b>4</b>	<b><math>\mathcal{N} = 1</math> deformations: generalities</b>	<b>11</b>
4.1	The $\mathcal{N} = 1$ curve and relative scaling dimensions	14
4.2	Characteristic polynomials in the infrared	14
4.3	Central charges and $a$ -maximization	16
<b>5</b>	<b>Probing an <math>E_n</math> singularity</b>	<b>17</b>
5.1	Nilpotent mass deformations	17
5.2	Large $N$ limit	20
5.3	Maximal monodromy	22
5.4	$\mathbb{Z}_2 \times \mathbb{Z}_2$ monodromy	22
5.5	$E_6$ and $\mathbb{Z}_2$ monodromy	24
<b>6</b>	<b>Stabilizing the D3-brane position</b>	<b>24</b>
<b>7</b>	<b>Coupling to the visible sector</b>	<b>26</b>
7.1	Unfolding $E_8$ to $SU(5)_{\text{GUT}}$	27
7.2	Coupling of the CFT to the visible sector	27
<b>8</b>	<b>Conclusions</b>	<b>28</b>
<b>A</b>	<b>Field theory tools</b>	<b>29</b>

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## 1 Introduction

The interplay between string theory and geometry provides a rich template for realizing many quantum field theories of theoretical and potentially experimental interest. A common theme is how geometric insights translate to non-trivial field theory statements, and conversely, how statements about the field theory allow us to probe details of the string geometry.

One class of theories which have recently been extensively studied is based on compactifications of F-theory, in part because such constructions combine the flexibility of intersecting D-brane configurations with the more attractive features of GUT models. See [1–5] and the references in [6] for a partial list of work on F-theory GUTs. E-type geometric singularities play an especially important role in realizing aspects of a GUT model such as the  $\mathbf{5} \times \mathbf{10} \times \mathbf{10}$  Yukawa coupling. In most cases, the focus of such models has been to realize weakly-coupled field theories which reproduce at least the qualitative features of the Standard Model. These models can also accommodate hidden sectors, which can be added as separate sectors.

D3-branes provide an additional set of ingredients which are present in such constructions. The presence of background fluxes often causes the D3-branes to be attracted to the E-type points of the geometry [7]. An important feature of such points is that the axio-dilaton  $\tau_{\text{IIB}}$  is of order one, and thus the D3-brane worldvolume theory is strongly coupled. Depending on the details of the geometry, we expect to realize a wide variety of possible strongly coupled quantum field theories. The study of such field theories is of independent interest, but is additionally exciting because the proximity to the visible sector suggests a phenomenologically novel way to extend the Standard Model at higher energy scales.

D3-brane probes of F-theory singularities have been considered in various works, for example [8–14]. In many cases of interest, the probe theory becomes an interacting superconformal field theory (SCFT). Geometrically, we engineer these SCFTs by considering the D3-brane probe of a parallel stack of seven-branes with gauge symmetry  $G$ . When the stack is flat, this provides a geometric realization of rank 1  $\mathcal{N} = 2$  SCFTs with flavor symmetry  $G$ , where  $G$  can be  $E_{6,7,8}$ . Denoting by  $z_1, z_2$  the coordinates parallel to the stack, and by  $z$  the coordinate transverse to the stack, in the probe theory  $z$  becomes a chiral superfield  $Z$  parameterizing the Coulomb branch, and  $z_{1,2}$  become a decoupled hypermultiplet  $Z_{1,2}$ . In addition, we have chiral operators  $\mathcal{O}$  in the adjoint representation of  $G$  parameterizing the Higgs branch. When we have a weakly coupled UV description of the theory, these operators can be written as composites made from quarks, e. g.  $\mathcal{O} \sim Q\bar{Q}$ .

We study  $\mathcal{N} = 1$  deformations of these theories by tilting the seven-branes. This tilting is described by activating a position-dependent vev for an adjoint-valued scalar  $\phi(z_1, z_2)$  on the stack. In the probe theory, this tilting corresponds to the superpotential deformation [7]:

$$\delta W = \text{Tr}_G(\phi(Z_1, Z_2) \cdot \mathcal{O}). \tag{1.1}$$

The eigenvalues of  $\phi$  specify the location of the seven-branes. Geometrically, this tilting process is known as “unfolding a singularity,” and is specified purely in terms of the Casimirs of  $\phi$ . A given matrix  $\phi$  will satisfy a characteristic equation of the form:

$$\phi^n + b_1(z_1, z_2)\phi^{n-1} + \dots + b_n(z_1, z_2) = 0 \tag{1.2}$$

where the  $b_i(z_1, z_2)$ ’s depend on the coordinates  $z_i$ . The most generic possibility is therefore that the eigenvalues for  $\phi$  will have branch cuts, a phenomenon known as “seven-brane monodromy”. Such monodromies are a natural ingredient for F-theory GUT models [15–19].

Although the unfolding is dictated purely by the Casimirs of  $\phi$ , we find that distinct  $\phi$ -deformations with the same Casimirs can produce strikingly different behavior in the IR.

In other words, the eigenvalues of  $\phi$  are *not* enough to specify the holomorphic data of the physical theory. This freedom opens a new avenue for realizing intersecting seven-brane configurations, which to this point appear to have been relatively unexplored.<sup>1</sup>

In this paper, our aim will be to elucidate these differences from the point of view of a probe D3-brane. Along the way, we will provide evidence for a large class of new  $\mathcal{N} = 1$  deformations of  $\mathcal{N} = 2$  theories.<sup>2</sup> We will provide various consistency checks that these deformations lead to new interacting  $\mathcal{N} = 1$  SCFTs. For example, assuming we realize an SCFT, we can use  $a$ -maximization [22] to determine the infrared R-symmetry. We can also check that the scaling dimensions of operators remain above the unitarity bound, and that the central charges of the SCFT decrease monotonically after further deformations of the theory. Moreover, in some cases we can argue that a further deformation induces a flow to a well-known interacting  $\mathcal{N} = 2$  SCFT. In such cases, the  $\phi$ -deformed theory can be viewed as an intermediate SCFT between the original  $\mathcal{N} = 2$  SCFT and another IR  $\mathcal{N} = 2$  SCFT.

The rest of the paper is organized as follows. In section 2 we introduce the brane setup for realizing the SCFTs of interest. As a first example, in section 3 we discuss the deformation of  $\mathcal{N} = 2$  SU(2) theory with four flavors, corresponding to a D3-brane probing a  $D_4$  singularity. Next, in section 4 we turn to  $\mathcal{N} = 1$  deformations of a broader class of non-Lagrangian theories, determining a general expression for the IR R-symmetry. We also discuss some of the geometric content associated with this class of deformations. Section 5 considers specific examples of  $\mathcal{N} = 1$  deformations. In section 6 we briefly consider further deformations of such theories by superpotential terms fixing the vev of the  $Z_i$  and  $Z$ , and in section 7 we indicate very briefly how the coupling of the SCFT sector to the visible sector works. Section 8 contains our conclusions. In appendix A we briefly review some standard tools from field theory.

## 2 Geometric preliminaries

In this section we review the general setup of D3-branes probing an F-theory singularity. Our aim here is to explain how the geometry of an F-theory compactification filters down to a D3-brane probe theory.

We are interested in F-theory singularities filling  $\mathbb{R}^{3,1}$ . The neighborhood of a singularity is a small patch in  $\mathbb{C}^3$  parameterized by  $z$ ,  $z_1$  and  $z_2$ . Over each point is an auxiliary elliptic curve, whose complex structure modulus is  $\tau_{\text{IIB}}$ . The elliptic curve is given in Weierstrass form by:

$$y^2 = x^3 + f(z_1, z_2, z)x + g(z_1, z_2, z), \quad (2.1)$$

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<sup>1</sup>Some discussion of the massless open string spectrum for “exotic” intersecting brane configurations based on a nilpotent Higgs field has appeared in [20]. Related issues in the context of F-theory compactifications will be discussed in [21].

<sup>2</sup>Though the setup is quite similar to that discussed in [9], here we emphasize the eight-dimensional gauge theory interpretation of these deformations, some aspects of which cannot be seen from the Calabi-Yau fourfold alone. Moreover, we find that the scaling dimensions of operators differ from what was found in [9], a point we shall comment more on in appendix A.

	$H_0$	$H_1$	$H_2$	$D_4$	$E_6$	$E_7$	$E_8$
$\Delta$	$6/5$	$4/3$	$3/2$	$2$	$3$	$4$	$6$

**Table 1.** The scaling dimension of the Coulomb branch parameter for the  $\mathcal{N} = 2$  SCFTs realized by a D3-brane probing an F-theory singularity at constant dilaton. Here, the  $H_i$  for  $i = 0, 1, 2$  respectively correspond to the Argyres-Douglas theories arising from an  $SU(2)$  gauge theory with  $i + 1$  flavors.

where the coefficients  $f$  and  $g$  fix  $\tau_{\text{IIB}}$ .

The locations of the seven-branes are specified by the zeros of the discriminant:

$$0 = \Delta(z_1, z_2, z) \equiv 4f^3 + 27g^2. \tag{2.2}$$

Each irreducible factor of  $\Delta$  determines a hypersurface in  $\mathbb{C}^3$ . Throughout this paper we shall be interested in the behavior of a D3-brane probing a seven-brane located originally at  $z = 0$ . In this convention, the coordinates  $z_1$  and  $z_2$  denote directions parallel to the seven-brane.

Away from the seven-branes, the worldvolume theory of a D3-brane is given by a  $U(1)$  gauge theory, with holomorphic gauge coupling  $\tau_{D3} = \tau_{\text{IIB}}$ , whose value is controlled by the position of the D3-brane. From the viewpoint of the open strings, the change in the coupling reflects the renormalization effects from the 3-7 strings. As the D3-brane moves close to a seven-brane, some of these states will become light, and in some cases we expect to recover an interacting SCFT.

When the seven-brane is flat, the probe theory is an  $\mathcal{N} = 2$  SCFT. These theories are described by a D3-brane probing a single parallel stack of seven-branes extending along  $z_{1,2}$ . The gauge symmetry on the seven-brane translates to a global symmetry  $G$  of the D3-brane probe theory. The probe theory has a Coulomb branch parameterized by  $z$ , the position of the D3-brane transverse to the seven-brane. The dimension of  $z$  is given in table 1. The Higgs branch of the probe theory corresponds to dissolving the D3-brane into the seven-brane stack as a gauge flux. Examples of such probe theories include  $SU(2)$  with four flavors [8], strongly-coupled theories of the type of Argyres and Douglas [23], and the E-type theories found in [24, 25].

Our main interest in this paper is to engineer  $\mathcal{N} = 1$  theories by considering more general seven-brane configurations. The positions of the seven-branes are dictated by a complex scalar  $\phi$  on the seven-brane stack taking values in the adjoint representation of  $G$ . Letting  $\phi$  depend on  $z_{1,2}$  corresponds to tilting the original stack of seven-branes. Geometrically, this corresponds to performing a deformation of the original Weierstrass model via:

$$y^2 = x^3 + (f_0 + \delta f(z_1, z_2, z))x + (g_0 + \delta g(z_1, z_2, z)). \tag{2.3}$$

Terms in  $\delta f$  and  $\delta g$  correspond naturally to expressions made from the Casimirs of  $G$  [26]. In the D3-brane probe theory this shows up as the superpotential deformation:

$$\delta W = \text{Tr}_G(\phi(Z_1, Z_2) \cdot \mathcal{O}) \tag{2.4}$$

where  $\phi$  and  $\mathcal{O}$  are both in the adjoint representation of  $G$ , and  $G$ -invariant information like the positions of the seven-branes is characterized by the Casimirs of  $\phi$ . Thus, from a given  $\phi$  one can naturally construct  $\delta f$  and  $\delta g$ .

This deformation breaks  $\mathcal{N} = 2$  supersymmetry to  $\mathcal{N} = 1$  when  $[\phi, \phi^\dagger] \neq 0$ , but is admissible as a background field configuration of the seven-brane once gauge fluxes are taken into account [2, 7]. Note that this deformation couples the original  $\mathcal{N} = 2$  theory to the hypermultiplet  $Z_{1,2}$ . The  $\mathcal{N} = 1$  theory then has a moduli space parameterized by  $z$  and  $z_{1,2}$ , on a generic point of which the low energy limit is just a  $U(1)$  theory. The physical coupling of this low-energy  $U(1)$  vector multiplet is holomorphic in  $z$  and  $z_{1,2}$ , and is given by a family of curves [27]. In this F-theoretic setup the required family of curves is exactly the elliptic fibration (2.3). It is worth noting that in contrast to the  $\mathcal{N} = 2$  case, this curve no longer describes a full solution of the low-energy theory because the behavior of the chiral multiplets is no longer controlled by the gauge coupling. The homogeneity of the curve can still be used to fix the relative scaling of further mass deformations and the Coulomb branch parameters. We shall meet examples of this analysis later on.

Although  $\phi$  should be holomorphic without branch cuts, its eigenvalues can have branch cuts as we vary  $z_i$ . In cases where such structures exist, we say these deformations exhibit “seven-brane monodromy.” A simple example is the matrix:

$$\phi = \begin{bmatrix} 0 & 1 \\ Z_1 & 0 \end{bmatrix} \tag{2.5}$$

which has eigenvalues  $\pm\sqrt{Z_1}$ . Closely related to seven-brane monodromy is the generic presence of nilpotent mass deformations. For example, in equation (2.5), the constant contribution is a mass matrix which is upper triangular, and thus nilpotent. This is the source of the monodromy after a further deformation by a lower triangular part proportional to  $Z_1$ .

Such nilpotent mass deformations are already of independent interest. Indeed, because all of the Casimirs of a nilpotent matrix vanish, the  $\mathcal{N} = 1$  curve of the deformed theory is identical to the original  $\mathcal{N} = 2$  curve. But this apparent invariance of the holomorphic geometry is deceptive. Clearly, we have added a mass term to the theory which breaks  $\mathcal{N} = 2$  supersymmetry in the probe theory and moreover gives a mass to some of the degrees of freedom of the original theory. Thus, we see that the Calabi-Yau fourfold alone does *not* fully specify the holomorphic data of the compactification. This does not immediately contradict the standard lore in much of the F-theory literature that the Calabi-Yau fourfold and flux data are enough to specify the compactification. The point is that Casimirs of  $\phi$  and of the gauge flux are not enough. This is quite exciting from the perspective of F-theory compactifications, because it points to a far greater degree of flexibility in the specification of a compactification, based on more holomorphic data than just the Casimirs of  $\phi$ .

The greater freedom in specifying  $\phi$  is also connected with the presence of singular fibers in the Calabi-Yau geometry. Indeed, in a compactification in which all singularities of the geometry have been deformed away, our general expectation is that there is no

ambiguity in reconstructing a unique choice of  $\phi$ . From the perspective of the seven-brane gauge theory this is equivalent to asking whether a given characteristic equation for  $\phi$  uniquely determines the physics of the seven-brane configuration. It would be interesting to study whether this natural physical expectation is always met for a general unfolding.

Before going further, let us point out that there is no distinction among  $z$ ,  $z_1$ , and  $z_2$  from the ten-dimensional point of view. For example, consider the configuration

$$y^2 = x^3 + z^5 + z_1. \tag{2.6}$$

This can be thought of as either a deformation of an  $E_8$  seven-brane at  $z = 0$ , or as a deformation of an  $H_0$  seven-brane at  $z_1 = 0$ . This suggests that the same  $\mathcal{N} = 1$  theory can be realized by deformations of two different  $\mathcal{N} = 2$  SCFTs. We hope to come back to this question in the future.

### 3 Probing a $D_4$ singularity

The cases of E-type flavor symmetry in which we are interested do not have an obvious Lagrangian description. As a warm-up, we start in this section by studying  $\mathcal{N} = 1$  deformations of a D3-brane probing a  $D_4$  singularity of F-theory; this setup leads to a Lagrangian theory. The weakly coupled theory of a D3-brane probing a  $D_4$  singularity is given by an  $SU(2)$  gauge theory with four quark flavors  $Q_i \oplus \tilde{Q}_{\bar{i}}$  for  $i = 1, \dots, 4$  [8]. The superpotential is dictated by  $\mathcal{N} = 2$  supersymmetry:

$$W = \sqrt{2} Q_i \varphi \tilde{Q}_{\bar{i}}. \tag{3.1}$$

The theory has an  $SO(8)$  flavor symmetry.

The moduli space is characterized in terms of gauge-invariant operators built from the elementary fields. The Coulomb branch of the theory is parameterized by the coordinate:

$$Z = \frac{1}{2} \text{Tr}_{SU(2)} \varphi^2. \tag{3.2}$$

Next consider the Higgs branch, touching the Coulomb branch at  $Z = 0$ . The Higgs branch is parameterized in terms of composite meson operators  $\mathcal{O}$  quadratic in the quarks. They transform in the adjoint representation of  $SO(8)$ . It is convenient to decompose them in terms of irreducible representations of  $U(4) \subset SO(8)$ :

$$\mathbf{16} : \mathcal{O}_{\bar{i}\bar{j}} = Q_i \tilde{Q}_{\bar{j}} \qquad \mathbf{6} : \mathcal{O}_{[ij]} = Q_{[i} Q_{j]} \qquad \bar{\mathbf{6}} : \mathcal{O}_{[\bar{i}\bar{j}]} = \tilde{Q}_{[\bar{i}} \tilde{Q}_{\bar{j}]} \tag{3.3}$$

Although at this stage we can write the  $\mathcal{O}$ 's in terms of the  $Q$ 's, when we later explore E-type theories, we will have nothing to work with except the analogous  $\mathcal{O}$  operators.

In addition to the degrees of freedom described above, there is a free hypermultiplet  $Z_1 \oplus Z_2$  representing the position of the D3-brane parallel to the seven-brane. Thus, we initially have two decoupled CFTs. Tilting the seven-branes to some new configuration couples these two CFTs, and generates a non-trivial flow to a new  $\mathcal{N} = 1$  theory.

In F-theory, the geometry of a  $D_4$  singularity is given by the Weierstrass equation:

$$y^2 = x^3 + Az^3 + xz^2 \tag{3.4}$$

where  $A$  is a free parameter. The modulus  $\tau$  of the torus (3.4), depending on  $A$ , gives the coupling of the  $SU(2)$  theory.

In the remainder of this section we study in greater detail  $\mathcal{N} = 1$  deformations of the  $D_4$  probe theory. In particular, our aim will be to present evidence that these theories realize interacting SCFTs in the IR. For the most part, we focus on nilpotent mass deformations such that  $\phi$  takes values in a single Jordan block of  $SU(n) \subset U(4) \subset SO(8)$ . Many of the checks we perform in the following subsections can be viewed as elucidating more details of the interacting SCFT.

### 3.1 Mass deformations and the $\mathcal{N} = 1$ curve

Mass deformations of the theory correspond to deformations of the form  $\text{Tr}_{SO(8)}(\phi \cdot \mathcal{O})$  for  $\phi$  independent of  $Z_1$  and  $Z_2$ . The Casimirs of  $\phi$  determine deformations of the original  $\mathcal{N} = 2$  curve:

$$y^2 = x^3 + Az^3 + xz^2 + (f_2z + f_4)x + g_4z + g_6 \tag{3.5}$$

where the  $f_i$ 's and  $g_i$ 's correspond to degree  $i$  polynomials in the masses  $m$  formed from expressions built from the Casimirs of  $\phi$ . Using the formulation in terms of the  $\mathcal{O}$ 's, these deformations can be written as:

$$\delta W = m_{j\bar{i}} \mathcal{O}_{i\bar{j}} + m_{[i\bar{j}]} \mathcal{O}_{[ij]} + m_{[ij]} \mathcal{O}_{[i\bar{j}]} \tag{3.6}$$

Additionally, we can consider field-dependent mass deformations which couple the  $\mathcal{N} = 2$   $D_4$  theory to the free hypermultiplet  $Z_1 \oplus Z_2$ . From the perspective of the geometry, the only change is that now the  $f_i$  and  $g_i$  in (3.5) can depend on the coordinates  $z_1$  and  $z_2$ .

Let us now consider a deformation by the nilpotent mass term:

$$\delta W = m_{1\bar{2}} \mathcal{O}_{2\bar{1}} = m_{1\bar{2}} Q_2 \tilde{Q}_{\bar{1}} \tag{3.7}$$

As the mass terms are nilpotent matrices, all Casimirs built from these operators are trivial, and the  $\mathcal{N} = 1$  curve is identical to the  $\mathcal{N} = 2$  curve.<sup>3</sup>

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<sup>3</sup>In the  $\mathcal{N} = 1$  theory, we have added mass terms for some of the quark flavors. As the theory flows from the UV to the new IR theory, the beta function of the gauge coupling is non-zero. This raises the question: Since we have initiated a flow of the gauge coupling, why does the  $\mathcal{N} = 1$  curve predict that on the Coulomb branch of the deformed theory there is no change to the value of  $\tau$ ?

To see what is happening, let us note that there are three mass scales of interest. First, there is the mass scale  $m$  associated with the nilpotent deformation. In addition, there is the scale  $m_W$  of the W-bosons of the Higgsed gauge theory. At scales  $m_W < \mu < m$ , we have integrated out a quark flavor from the  $SU(2)$  gauge theory, and the coupling increases as the theory flows to the IR. However, below the scale  $\mu < m_W$ , the W-bosons of the  $SU(2)$  gauge theory are also massive, and this in turn counters the effects of the initial decrease. In particular, we see that as the theory flows to the scale  $\mu = m_W^2/m < m_W < m$ , the value of the gauge coupling has flowed back to its original value. All of this behavior is automatically encoded in the geometry. See [28] for related discussions.



The existence of an  $\mathcal{N} = 1$  curve constrains the relative scaling dimensions of operators in the deformed theory, as in the  $\mathcal{N} = 2$  case. Since the  $\mathcal{N} = 1$  curve is no different from the  $\mathcal{N} = 2$  curve, this implies that the relative scalings of  $z$  and the mass deformations in the new  $\mathcal{N} = 1$  theory obey the same relations as in the original  $\mathcal{N} = 2$  theory. Let us now check how this works from the viewpoint of the Lagrangian.

To this end, consider the effective superpotential:

$$W_{\text{eff}} = \sum_{i=3,4} \sqrt{2} Q_i \varphi \tilde{Q}_i - 2 \frac{Q_1 \varphi \tilde{Q}_2}{m_{1\bar{2}}} \quad (3.8)$$

after integrating out  $Q_2$  and  $\tilde{Q}_{\bar{1}}$ .

Assuming we have flowed to an interacting CFT, let us now compute the relative scaling dimensions of the various mass deformations. For simplicity, we restrict attention to the mass parameters transforming in the adjoint representation of  $U(4)$ . Taking the superpotential of equation (3.8) to be marginal in the IR, we learn that the dimensions of the  $Q$ 's are related to the dimension of  $z$  by:

$$[Q_1 \tilde{Q}_2] = 3 - \Delta_{\text{IR}}, \quad [Q_1 \tilde{Q}_{\bar{1}}] = [Q_I \tilde{Q}_2] = 3 - \frac{3}{4} \Delta_{\text{IR}}, \quad [Q_I \tilde{Q}_{\bar{J}}] = 3 - \frac{1}{2} \Delta_{\text{IR}} \quad (3.9)$$

where  $\Delta_{\text{IR}}$  is the scaling dimension of  $z$  and  $I, J = 3, 4$ . The corresponding mass terms are

$$\delta W = m_{2\bar{1}} Q_1 \tilde{Q}_2 + m_{I\bar{1}} Q_1 \tilde{Q}_{\bar{1}} + m_{2\bar{I}} Q_I \tilde{Q}_2 + m_{J\bar{I}} Q_I \tilde{Q}_{\bar{J}} \quad (3.10)$$

where the dimension of the  $m$ 's can be easily obtained from the data given above. In order to match these mass deformations to quantities of the  $\mathcal{N} = 1$  curve, we must form invariants under the surviving flavor symmetries. We obtain four invariants with corresponding dimensions:

$$[m_{2\bar{1}}] = [m_{I\bar{J}} m_{J\bar{I}}] = \Delta_{\text{IR}}, \quad [m_{2\bar{I}} m_{I\bar{1}}] = \frac{3}{2} \Delta_{\text{IR}}, \quad [m_{2\bar{I}} m_{I\bar{J}} m_{J\bar{I}}] = 2 \Delta_{\text{IR}}. \quad (3.11)$$

We now compare these invariants to invariants of the  $\mathcal{N} = 2$  theory. First note that these expressions transform non-trivially under the (now broken) original flavor symmetry. Including appropriate factors of  $m_{1\bar{2}}$  to form flavor invariants of the original  $\mathcal{N} = 2$  theory, we see that these expressions descend from the  $\mathcal{N} = 2$  theory Casimirs:

$$[m_{1\bar{2}} m_{2\bar{1}}] = [m_{I\bar{J}} m_{J\bar{I}}] = \Delta^{\mathcal{N}=2}, \quad [m_{1\bar{2}} m_{2\bar{I}} m_{I\bar{1}}] = \frac{3}{2} \Delta^{\mathcal{N}=2}, \quad (3.12)$$

$$[m_{1\bar{2}} m_{2\bar{I}} m_{I\bar{J}} m_{J\bar{I}}] = 2 \Delta^{\mathcal{N}=2}. \quad (3.13)$$

Note that the same relative scalings are obtained once we set  $m_{1\bar{2}} = 1$ , as appropriate upon treating  $m_{1\bar{2}} Q_2 \tilde{Q}_{\bar{1}}$  as a marginal operator in the IR theory. Similar considerations hold for other group theory invariants, and for more general deformations as well.

### 3.2 $a$ -maximization and nilpotent mass deformations

Nilpotent mass deformations are of particular interest because although they constitute a non-trivial  $\mathcal{N} = 1$  deformation of the theory, they do not alter the geometry of the  $\mathcal{N} = 1$

curve. For simplicity, we again confine our analysis to nilpotent deformations where  $\phi$  takes values in  $SU(n) \subset U(4) \subset SO(8)$ . Explicitly, we consider the mass deformations:

$$\delta W = \sum_{k=1}^{n-1} m_{k\bar{k}+1} \mathcal{O}_{k+1\bar{k}}. \tag{3.14}$$

Let us first consider in detail the case  $n = 2$ . Upon integrating out the quarks  $Q_2 \oplus \tilde{Q}_1$ , we are left with an  $SU(2)$  gauge theory with chiral superfields  $\varphi$ ,  $Q_3 \oplus \tilde{Q}_3$ ,  $Q_4 \oplus \tilde{Q}_4$ , and  $Q_1 \oplus \tilde{Q}_2$  with an effective superpotential (3.8). By inspection, one finds

$$R(Q_3) = R(\tilde{Q}_3) = R(Q_4) = R(\tilde{Q}_4), \quad R(Q_1) = R(\tilde{Q}_2). \tag{3.15}$$

We require that the IR R-symmetry is non-anomalous, and that the two superpotential terms in equation (3.8) are marginal in the IR. We find that  $R(Q_3)$  and  $R(Q_1)$  can be expressed in terms of  $R(\varphi)$ , which can then be determined by  $a$ -maximization. The calculation is straightforward; we find a local maximum at  $R(\varphi) = (-9 + \sqrt{145})/6 \simeq 0.51$ . As can be checked, all gauge-invariant operators have dimensions above the unitarity bound.

Let us now generalize this to the cases  $n = 3, 4$ . Upon integrating out the heavy quarks, we are left with the effective superpotential:

$$W_{\text{eff}}^{(n=3)} = \sqrt{2} Q_4 \varphi \tilde{Q}_4 + 2\sqrt{2} \frac{Q_1 \varphi^3 \tilde{Q}_3}{m_{1\bar{2}} m_{2\bar{3}}}, \quad W_{\text{eff}}^{(n=4)} = -4 \frac{Q_1 \varphi^4 \tilde{Q}_4}{m_{1\bar{2}} m_{2\bar{3}} m_{3\bar{4}}}. \tag{3.16}$$

Again we impose the conditions that  $R_{\text{IR}}$  is non-anomalous and that the effective superpotential is marginal in the IR. For both  $n = 3, 4$  we find one undetermined parameter which we fix by  $a$ -maximization.

The behavior of the  $n = 2, 3$  theories is quite similar. The case of  $n = 4$  presents a new phenomenon, in that here it would appear that there are unitarity bound violations. Indeed, assigning R-charges to the operators  $Z$  and  $Q_1 \tilde{Q}_4$  then requires either that both operators saturate the unitarity bound, or that one operator violates this bound.

What are we to make of this case? One possibility is that this theory may not be an interacting conformal theory. This does not appear very plausible, because nothing drastic appears to be happening to the geometry. For example, we can still move onto the Coulomb branch and compute a non-trivial dependence of  $\tau$  on the parameters  $z_i$  and  $z$ . In what follows, we shall assume that much as in [29], an accidental symmetry appears which rescues only this individual operator.

We now recompute the dimensions for the quarks in the IR theory under the assumption that only  $Z$  decouples in the IR, with an associated emergent  $U(1)$  which only acts on  $Z$ . This emergent  $U(1)$  can be included in  $a$ -maximization via the procedure described in [30]. The scaling dimensions for the various fields are then given by the values shown in table 2. Note that although the dimensions of the  $Q$ 's are less than one, all of the composite operators built from two  $Q$ 's have dimension above the unitarity bound.

### 3.3 Large $N$ limit

In the previous section we discussed the specific case of a single D3-brane probing a  $D_4$  singularity. In the context of brane constructions, it is natural to consider the theory

	$R(\varphi)$	$[Z]$	$[Q_1]$	$[\tilde{Q}_4]$	$[\tilde{Q}_3]$	$[\tilde{Q}_2]$	$[\tilde{Q}_1]$	$[Q_4]$	$[Q_3]$	$[Q_2]$
$\mathcal{N} = 2$	2/3	2	1	1	1	1	1	1	1	1
$n = 2$	0.51	1.52	0.74	1.12	1.12	0.74	X	1.12	1.12	X
$n = 3$	0.36	1.07	0.69	1.23	0.69	X	X	1.23	X	X
$n = 4$	0.25	1	0.76	0.76	X	X	X	X	X	X

**Table 2.** Dimensions of the elementary fields obtained from nilpotent deformations of the probe  $D_4$  theory. The operators of the CFT are specified by gauge invariant expressions built from these elementary fields. Entries with an “X” indicate fields which have been integrated out of the low energy theory.

	$R(\phi)$	$[Z]$	$[P] = [Q]$	$[Q']$
$\mathcal{N} = 2$	$\frac{2}{3}$	2	1	1
$n = 2$	$\frac{2}{3} - \frac{2}{3N}$	$2 - \frac{2}{N}$	$1 + \frac{1}{2N}$	$\frac{1}{2} + \frac{1}{N}$
$n = 3$	$\frac{2}{3} - \frac{8}{3N}$	$2 - \frac{8}{N}$	$1 + \frac{2}{N}$	$\frac{6}{N}$
$n = 4$	$\frac{2}{3} - \frac{20}{3N}$	$2 - \frac{20}{N}$	$1 + \frac{5}{N}$	$-\frac{1}{2} + \frac{20}{N}$

**Table 3.** Dimensions of the elementary fields for the large  $N$  limit of the probe  $D_4$  theories with a nilpotent mass deformation turned on. Note that although the dimensions of some elementary fields fall far below the unitarity bound, the scaling dimensions of composite operators involving these fields can still remain above this bound.

obtained by  $N$  D3-branes probing the same configuration. The theory is given by an  $\mathcal{N} = 2$   $\text{USp}(2N)$  gauge theory with four quark flavors  $Q_i \oplus \tilde{Q}_i$  and an additional hypermultiplet  $P \oplus \tilde{P}$  in the two-index antisymmetric representation [10, 12]. The  $\mathcal{N} = 2$  superpotential for this theory is:

$$W = \sum_{i=1}^4 \sqrt{2} Q_i \varphi \tilde{Q}_i + \sqrt{2} P \varphi \tilde{P}. \tag{3.17}$$

This is an  $\mathcal{N} = 2$  SCFT. In addition, there is a free hypermultiplet  $Z_1 \oplus Z_2$  describing the motion of the center of mass for the configuration.

Let us now deform this theory by nilpotent masses. Assuming that the IR limit is an  $\mathcal{N} = 1$  SCFT without any accidental symmetry, we perform  $a$ -maximization and expand the result to first order in  $1/N$ . The R-charge assignment for  $\varphi$  and the scaling dimensions for the elementary fields are given in table 3.

In this table,  $Q$  indicates any flavors that couple to  $\varphi$  through  $Q\varphi\tilde{Q}$ , and  $Q'$  indicates flavors that couple through  $Q'\varphi^n\tilde{Q}'$ , which as in the previous section is present after integrating out the massive quarks. Looking at the table, we see that for  $n > 2$ , some of the operators quadratic in  $Q'$  will fall below the unitarity bound. As before, we can assume that these operators become free fields and decouple from the IR theory and re-do  $a$ -maximization. However, because in the large  $N$  limit the number of offending operators is  $\mathcal{O}(N^0)$ , there is no change at this order to the R-charge assignments, since  $a$  is  $\mathcal{O}(N^2)$ .

We still believe that this system flows to an IR SCFT. Consider the related deformation by  $\phi' = \phi + \phi^T$ , given by symmetrizing the original deformation. In the UV  $\mathcal{N} = 2$  theory,

this corresponds to a deformation which preserves  $\mathcal{N} = 2$  supersymmetry, and leads to the theory describing  $N$  D3-branes probing  $H_{0,1,2}$  singularities. If we now consider deforming first by  $\phi$ , and after some long RG time deforming further by  $\phi^T$ , we then flow to this same  $\mathcal{N} = 2$  theory. Assuming that this further deformation can be added at an arbitrarily late RG time, this strongly suggests that the above description of the  $\mathcal{N} = 1$  theory is legitimate, and that the further deformation by  $\phi^T$  induces a flow from the  $\phi$ -deformed theory to the  $\phi' = \phi + \phi^T$  deformed theory. The only concern here would be that somehow the further deformation by  $\phi^T$  is not valid in the deep IR, though this seems rather implausible. Furthermore, nothing dramatic seems to be happening to the F-theory geometry, so it seems reasonable to assume that the CFT is still present.

#### 4 $\mathcal{N} = 1$ deformations: generalities

In the previous section we studied the theory of a D3-brane probing a  $D_4$  singularity, relying on a Lagrangian formulation of the theory to analyze the effects of various deformations. In many cases of interest, however, we do not have a weakly coupled Lagrangian description, as when we have a flavor symmetry  $G = E_n$ .

In this section we consider superpotential deformations of the form:

$$\delta W = \text{Tr}_G(\phi(Z_1, Z_2) \cdot \mathcal{O}). \tag{4.1}$$

Our main assumption will be that we obtain a CFT in the infrared, and we shall present some consistency checks of this statement. Assuming we do flow to a new CFT, it is important to determine the scaling dimensions of operators, and the values of the various central charges in the infrared. This is of intrinsic interest, but is also of interest in potential model building applications where the degrees of freedom from the D3-brane couple to visible sector degrees of freedom associated with modes localized on seven-branes.

In the case where  $\phi$  has no constant terms, the results of [31] establish that this deformation is marginally irrelevant, and so induces a flow back to the original CFT [7]. For this reason, we shall focus on the case where  $\phi$  has a non-trivial constant part. Further, since we are interested in the structure of deformations where the geometric singularity is retained at  $z_i = 0$ , we also demand that all Casimirs of  $\phi$  vanish at  $z_i = 0$ . Hence, the constant part of  $\phi$  is a nilpotent matrix. This reinforces the point that nilpotent deformations go hand in hand with generic deformations of an F-theory singularity.

We now describe the general procedure for obtaining the R-charge assignments for the matter fields after turning on a combination of relevant and marginal deformations. We first catalogue the symmetries of the UV theory, and then the surviving symmetries compatible with the deformation  $\text{Tr}_G(\phi \cdot \mathcal{O})$ . The  $SU(2) \times U(1)$  R-symmetry of the  $\mathcal{N} = 2$  theory contains two  $U(1)$ 's, which we call  $R_{UV}$  and  $J_{\mathcal{N}=2}$ ; see appendix A for details. The remaining global symmetries are the non-abelian flavor symmetry  $G$  and the  $U(1)$  generators which rotate the fields  $Z_i$  of the free hypermultiplet. Since the infrared R-symmetry is a linear combination of abelian symmetries, it is enough to focus on the Cartan subalgebra  $U(1)^r$  of the flavor symmetries  $G$ , where  $r$  is the rank of  $G$ . We shall denote by  $F_i$  the corresponding generators, where  $i$  runs from 1 to  $r$ . Finally, we denote by

	$\mathcal{O}$	$Z$	$Z_j$
$R_{UV}$	$4/3$	$(2/3) \Delta_{UV}(Z)$	$2/3$
$J_{\mathcal{N}=2}$	$-2$	$2\Delta_{UV}(Z)$	$-1$
$U_i$	$0$	$0$	$\delta_{ij}$

**Table 4.** Assignment of UV charges where  $\Delta_{UV}(Z)$  denotes the dimension of  $Z$  in the UV. The charges refer to the bottom component of each supermultiplet.

$U_i$  the generator under which  $Z_i$  has charge  $+1$ . Under  $R_{UV}$ ,  $J_{\mathcal{N}=2}$  and the  $U_i$ , the charges of the original operators are given in table 4.

The  $\mathcal{N} = 1$  deformation explicitly breaks some of these flavor symmetries. The infrared R-symmetry will then be given by a linear combination of the UV symmetries, and possibly some additional emergent flavor symmetries. In what follows, we shall assume that there are no emergent abelian symmetries in the infrared. Then the R-symmetry is given by

$$R_{IR} = R_{UV} + \left(\frac{t}{2} - \frac{1}{3}\right) J_{\mathcal{N}=2} + \sum_{i=1}^r t_i \cdot F_i + u_1 U_1 + u_2 U_2. \tag{4.2}$$

The coefficient of  $J_{\mathcal{N}=2}$  has been chosen for later convenience.

Let us determine which symmetries are left unbroken by the original deformation. At first, it may appear that no solution is available which is compatible with a deformation of the form given by equation (4.1). Indeed, though there are at most  $r + 3$  flavor symmetries, there will typically be far more independent entries in the matrix  $\phi$ . However, in the infrared, our expectation is that some of these deformations will become irrelevant operators. For example, if one of the entries of  $\phi$  contains a term of the form  $Z_1^{100}$ , we expect this term to be irrelevant in the infrared. This also matches with geometric expectations. The geometry is well-approximated to leading order by the lowest degree polynomials in the  $Z_i$ . Higher order polynomials correspond to subleading features of the geometry. Since we are only interested in a small neighborhood of the region where  $Z_1 = Z_2 = 0$ , much of this information is washed out in the infrared. We shall return to this theme later when we discuss the UV and IR behavior of the characteristic polynomial for  $\phi$ .

The flow to a new CFT is dominated by the operators of lowest scaling dimension. In the UV, the most relevant terms are the constant matrices. By assumption, the constant matrix is nilpotent, and so by a unitary change of basis, we can present it as an upper triangular matrix. For simplicity, in what follows we assume that the constant part of  $\phi$  denoted by  $\phi_0$  decomposes as a collection of  $n_a \times n_a$  blocks, each of which corresponds to an upper triangular matrix:

$$\phi_0 = \bigoplus_{a=1}^k J^{(a)}. \tag{4.3}$$

We assume that the upper triangular matrices  $J^{(a)}$  are generic in the sense that the first superdiagonal has only nonzero entries.

Associated with each block is an  $SU(2)$  subalgebra of the original flavor symmetry group  $G$  with generators  $T_{\pm}^{(a)}$  and  $T_3^{(a)}$  in the spin  $j_{(a)} = (n_{(a)} - 1)/2$  representation,

satisfying

$$[T_+^{(a)}, T_-^{(a)}] = 2T_3^{(a)} \qquad [T_3^{(a)}, T_\pm^{(a)}] = \pm T_\pm^{(a)}. \qquad (4.4)$$

In this basis, the  $T_3^{(a)}$  generator is:

$$T_3^{(a)} = \text{diag}(j_{(a)}, j_{(a)} - 1, \dots, 1 - j_{(a)}, -j_{(a)}). \qquad (4.5)$$

Most of the data associated with each block  $\phi_0^{(a)}$  of the decomposition in equation (4.3) drops out in the infrared. To see how this comes about, consider the entries of the upper triangular block  $J^{(a)}$ . Along each superdiagonal of the matrix, the value of the  $T_3^{(a)}$  charge is the same. Moving out from the diagonal, the entries of  $\phi_0^{(a)}$  on the first superdiagonal have charge +1, the second have +2, and so on until the upper righthand entry which has charge  $n_{(a)} - 1$ . The operators  $\mathcal{O}^{(a)}$  which pair with  $\phi_0^{(a)}$  in the deformation have respective  $T_3^{(a)}$  charges  $-1$  down to  $-(n_{(a)} - 1)$ . Since all operators on the same superdiagonal have the same  $T_3^{(a)}$  charge, we see that the one with the charge of smallest norm will dominate the flow. In other words, the first superdiagonal of  $\phi_0^{(a)}$  dominates the flow. In the following we assume  $\phi_0^{(a)}$  takes the form of a nilpotent Jordan block from the start.

Then the operators  $\mathcal{O}^{(a)} = \text{Tr}_G(\phi_0^{(a)} \cdot \mathcal{O})$  have  $T_3^{(a)}$  charge  $-1$ . The requirement that  $\mathcal{O}^{(a)}$  is marginal for all  $a$  in the IR can now be satisfied by the choice

$$R_{\text{IR}} = R_{\text{UV}} + \left(\frac{t}{2} - \frac{1}{3}\right) J_{\mathcal{N}=2} - tT_3 + u_1U_1 + u_2U_2 \qquad (4.6)$$

where

$$T_3 = \sum_a T_3^{(a)} \qquad (4.7)$$

is the generator of the diagonal SU(2) subalgebra. The coefficients  $u_i$  are still undetermined. We can now organize the operators  $\mathcal{O}$  into representations of this diagonal SU(2). We denote by  $\mathcal{O}_s$  an operator with spin  $s$  under this SU(2).

To fix the value of the  $u_i$ 's, we need to know which of the remaining operator deformations are most relevant in the IR. In the case of a deformation by a constant  $\phi$ , the free hypermultiplet decouples, and we can neglect the  $U_i$ 's. We therefore focus on the additional effects of  $Z_i$ -dependent deformations. Unitarity dictates that  $\mathcal{O}$  and the  $Z_i$  have dimensions greater than or equal to one. This means that if two or more  $Z_i$ 's multiply an operator  $\mathcal{O}$ , their product will be irrelevant. Hence, it is enough to focus on contributions which are linear in the  $Z_i$ .

Most of the deformations linear in the  $Z_i$  will also be irrelevant. Given two operators  $Z_i \times \mathcal{O}_s$  and  $Z_i \times \mathcal{O}'_{s'}$  which have different  $T_3$  charges  $s$  and  $s'$ , the operator with the larger charge will have lower dimension, and will therefore dominate the flow.

Since the IR behavior is dictated by the operators  $\mathcal{O}_s$  with the highest values of  $s$ , it is enough to consider the deformation by just these highest values. Let  $\mathcal{O}_{S_1}$  and  $\mathcal{O}_{S_2}$  denote the operators which respectively multiply  $Z_1$  and  $Z_2$ . The parameters  $u_i$  are now fixed by

requiring that these deformations have R-charge 2 in the IR. In terms of  $S_1$ ,  $S_2$  and  $t$ , this constraint yields:

$$u_i = (S_i + 3/2)t - 1 \equiv \mu_i t - 1 \quad \text{where} \quad \mu_i = S_i + 3/2. \quad (4.8)$$

We now see that for a given  $\mu_1, \mu_2$ , the only free parameter in  $R_{\text{IR}}$  is  $t$ .

#### 4.1 The $\mathcal{N} = 1$ curve and relative scaling dimensions

Now let us study to what extent we can read off properties of the  $\mathcal{N} = 1$  deformed theory without determining  $t$ . As we have already mentioned, on the Coulomb branch of the  $\mathcal{N} = 1$  theory, we can read off the U(1) coupling from the  $\mathcal{N} = 1$  curve, which is the F-theory geometry (2.3). Homogeneity of this  $\mathcal{N} = 1$  curve then predicts the relative scaling dimensions of the mass deformations to that of the Coulomb branch parameter.

The form of the infrared R-symmetry (4.6) implies

$$\Delta_{\text{IR}}(Z) = \frac{3}{2}t \times \Delta_{\text{UV}}(Z). \quad (4.9)$$

Therefore the unknown parameter  $t$  can be eliminated in favor of the ratio  $\rho = \Delta_{\text{IR}}(Z)/\Delta_{\text{UV}}(Z)$ , and we find

$$\Delta_{\text{IR}}(Z_i) = \left(S_i - \frac{1}{2}\right)\rho, \quad \Delta_{\text{IR}}(\mathcal{O}_s) = 3 - (s + 1)\rho. \quad (4.10)$$

The dimension of the mass parameter  $m_{\mathcal{O}_s}$  associated with an operator  $\mathcal{O}_s$  is then:

$$\Delta_{\text{IR}}(m_{\mathcal{O}_s}) = 3 - \Delta_{\text{IR}}(\mathcal{O}_s) = (s + 1)\rho. \quad (4.11)$$

In particular, when we form flavor invariants out of the mass parameters  $m$  as in section 3.1, their ratio in the IR is the same in the UV, because the total spin  $s$  of the flavor invariants is zero.

As a passing comment, let us also note that the value of the IR central charge  $k_{\text{IR}}$  agrees with the computation in [7]: using (A.5), we easily find

$$k_{\text{IR}} = \rho k_{\text{UV}}. \quad (4.12)$$

#### 4.2 Characteristic polynomials in the infrared

Some aspects of the deformation  $\phi$  are irrelevant in the IR. To study the possibilities, we can consider choices for  $\phi$  which in the IR induce a flow to the same theory as the original  $\phi$ . To indicate the UV and IR behavior we write  $\phi_{\text{UV}}$  and  $\phi_{\text{IR}}$ .

The matrix  $\phi_{\text{IR}}$  is fully characterized by terms which are at most linear in the  $Z_i$ . Indeed, as the D3-brane only probes a small patch of the geometry, it is insensitive to higher order terms in the geometry, which are effectively gone in the deep infrared. Of course these further effects can still be probed by moving a finite distance onto the Coulomb branch.

Given two different  $\phi$ 's, linearizing in the  $Z_i$  can produce the same IR behavior for  $\phi$ . For example, the characteristic equations

$$\phi^5 + z_1 = 0, \tag{4.13}$$

$$\phi^5 + z_2^w \phi + z_1 = 0 \tag{4.14}$$

respectively define solvable and unsolvable quintics. For  $w > 1$ , however, the term linear in  $\phi$  drops out in the infrared. Thus, the UV and IR behavior of  $\phi$  can be different.

For the more mathematically inclined reader, we note that the “seven-brane monodromy group” corresponds to the Galois group for the characteristic equation for  $\phi$ . The monodromy group acts by permuting the roots of the polynomial, and is indicated by the specific branch cut structure present in the eigenvalues of  $\phi$ . Here we see that the infrared monodromy groups which can be realized are of quite limited type.

A polynomial of the form:

$$\phi^n + b_2 \phi^{n-2} + \dots + b_n = 0 \tag{4.15}$$

will generically have maximal Galois group given by  $S_n$ , the symmetric group on  $n$  letters. In particular, we can take the  $b_i$  to admit a power series expansion in the  $Z_i$ . By a general coordinate redefinition of the geometry, and a field redefinition in the CFT, we see that generically, we can take the leading order behavior of the lowest coefficients to be  $b_n = Z_1$  and  $b_{n-1} = Z_2$ .

Finding a representative  $\phi$  with the corresponding characteristic equation is also straightforward. To illustrate the main points, let us focus on the case of  $\phi$  given by a  $5 \times 5$  matrix. A representative  $\phi$  with characteristic equation as in (4.15) can be taken in the form:

$$\phi = \begin{bmatrix} 0 & 1 & & & \\ -c_2^{(2,1)} & 0 & 1 & & \\ -c_3^{(3,1)} & -c_2^{(3,2)} & 0 & 1 & \\ -c_4^{(4,1)} & -c_3^{(4,2)} & -c_2^{(4,3)} & 0 & 1 \\ -c_5^{(5,1)} & -c_4^{(5,2)} & -c_3^{(5,3)} & -c_2^{(5,4)} & 0 \end{bmatrix} \tag{4.16}$$

for appropriate  $c_n^{(i,j)}$ . In the infrared, the relevant deformation by  $\phi$  is:

$$\phi_{\text{IR}} = \begin{bmatrix} & & & 1 & \\ & & & & 1 \\ & & & & & 1 \\ -\alpha Z_2 & & & & & & 1 \\ -Z_1 & -\beta Z_2 & & & & & & \end{bmatrix} \tag{4.17}$$

for some coefficients  $\alpha$  and  $\beta$ . The characteristic equation for  $\phi_{\text{IR}}$  is:

$$\phi_{\text{IR}}^5 + (\alpha + \beta) Z_2 \phi_{\text{IR}} + Z_1 = 0. \tag{4.18}$$

As can be checked, the monodromy group for this degree five polynomial is again  $S_5$ .



### 4.3 Central charges and $a$ -maximization

We now fix the infrared R-symmetry using  $a$ -maximization. The trial central charge  $a_{\text{IR}}(t)$  can be computed using 't Hooft anomaly matching between the UV and IR theories. Thus  $a_{\text{IR}}(t)$  depends on  $a_{\text{UV}}$ ,  $c_{\text{UV}}$  and  $k_{\text{UV}}$ , as well as the details of the Jordan block structure associated with the deformation  $\text{Tr}_G(\phi \cdot \mathcal{O})$ .

Plugging (4.6) into (A.3) and rewriting it using (A.7)–(A.9), we obtain the value of the IR central charges as follows:

$$a_{\text{IR}} = \frac{3}{32} \left[ \left( 36a_{\text{UV}} - 27c_{\text{UV}} - \frac{9k_{\text{UV}}r}{4} \right) t^3 + \left( -72a_{\text{UV}} + 36c_{\text{UV}} + \frac{9}{4}(u_1 + u_2) \right) t^2 + \left( 48a_{\text{UV}} - 12c_{\text{UV}} - \frac{9}{2}(u_1^2 + u_2^2) \right) t + \left( -(u_1 + u_2) + 3(u_1^3 + u_2^3) \right) \right] \quad (4.19)$$

$$c_{\text{IR}} = \frac{1}{32} \left[ \left( 108a_{\text{UV}} - 81c_{\text{UV}} - \frac{27k_{\text{UV}}r}{4} \right) t^3 + \left( -216a_{\text{UV}} + 108c_{\text{UV}} + \frac{27}{4}(u_1 + u_2) \right) t^2 + \left( 96a_{\text{UV}} + 12c_{\text{UV}} - \frac{27}{2}(u_1^2 + u_2^2) \right) t + \left( -5(u_1 + u_2) + 9(u_1^3 + u_2^3) \right) \right] \quad (4.20)$$

$$k_{\text{IR}} = \frac{3}{2}t \times k_{\text{UV}} \quad (4.21)$$

where in the above, we have introduced the parameter  $r$  which measures the sizes of the nilpotent block:

$$r \equiv 2 \text{Tr} (T_3 T_3). \quad (4.22)$$

We need to find the local maximum of  $a_{\text{IR}}$  given in (4.19) to find the value  $t$ . There are two cases of interest, which we analyze separately. The first case corresponds to deformations where  $\phi$  is a constant nilpotent matrix. In this case, we formally set  $u_1 = u_2 = 0$ . In addition, we must remember that the free hypermultiplet  $Z_1 \oplus Z_2$  decouples, and in particular does not contribute to the central charges  $a_{\text{UV}}$  and  $c_{\text{UV}}$ . The other case corresponds to the more generic geometry in which  $\phi$  has some linear dependence in both  $Z_1$  and  $Z_2$ . In this case, the contribution from the hypermultiplet must be included in the values of  $a_{\text{UV}}$  and  $c_{\text{UV}}$ . These values are tabulated in appendix A.

**Nilpotent mass case.** First consider the case where  $\phi$  is a constant nilpotent matrix. Setting  $u_1 = u_2 = 0$  in (4.19),  $a$ -maximization yields an extremum at:

$$t_* = \frac{4}{3} \times \frac{8a_{\text{UV}} - 4c_{\text{UV}} - \sqrt{4c_{\text{UV}}^2 + (4a_{\text{UV}} - c_{\text{UV}})k_{\text{UV}}r}}{16a_{\text{UV}} - 12c_{\text{UV}} - k_{\text{UV}}r}. \quad (4.23)$$

with  $r$  as in equation (4.22). Note that for  $r = 0$ , we recover  $t_* = 2/3$ , corresponding to the correct branch of solutions to the quadratic equation.

**Monodromic case.** Next consider the case of position-dependent  $\phi(Z_1, Z_2)$  where a term linear in each  $Z_i$  appears in the deformation  $\text{Tr}_G(\phi \cdot \mathcal{O})$ . Applying (4.8) in (4.19) and performing  $a$ -maximization, we find

$$t_* = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \quad (4.24)$$

where:

$$A = \frac{3}{4}(48a_{UV} - 36c_{UV} - 3k_{UV}r + 3\mu_1 + 3\mu_2 - 6\mu_1^2 - 6\mu_2^2 + 4\mu_1^3 + 4\mu_2^3) \quad (4.25)$$

$$B = -3 - 48a_{UV} + 24c_{UV} + 6\mu_1 + 6\mu_2 - 6\mu_1^2 - 6\mu_2^2 \quad (4.26)$$

$$C = -3 + 16a_{UV} - 4c_{UV} + \frac{8}{3}\mu_1 + \frac{8}{3}\mu_2. \quad (4.27)$$

The choice of branch cut in equation (4.24) is fixed as in the nilpotent case.

## 5 Probing an $E_n$ singularity

Having given a general analysis of the expected IR R-symmetry, we now turn to some examples. In fact the  $D_4$  case analyzed in section 3 falls within the analysis presented in the last section. Here we will study the  $E_n$  case where a weakly-coupled UV description is not available.

We first consider nilpotent mass deformations of the  $\mathcal{N} = 2$   $E_8$  SCFT. We find a consistent structure of flows between various deformations of this theory. We also study the large  $N$  limit of such probe theories. After this analysis, we turn to the more generic case of deformations which include a  $Z_i$ -dependent contribution. In F-theory, allowing a position-dependent profile for the field  $\phi$  corresponds to tilting the configuration of the seven-branes. Finally, we consider some particular examples which are of interest for F-theory GUTs.

### 5.1 Nilpotent mass deformations

We now turn to deformations of the  $E_8$  theory by  $\phi$  valued in  $SU(n) \subset SU(9) \subset E_8$ .<sup>4</sup> The adjoint of  $E_8$  decomposes under  $SU(9)$  into the adjoint representation, and a three index antisymmetric tensor as:

$$\mathbf{248} \rightarrow \mathbf{80} + \mathbf{84} + \overline{\mathbf{84}}. \quad (5.1)$$

Hence, the operators  $\mathcal{O}$  initially transforming in the adjoint representation of  $E_8$  will decompose into singlets, and one-, two-, and three-index tensor representations of  $SU(n)$ . Another feature of interest is that this also suggests a natural split between the cases of  $n \leq 5$  and  $n > 5$ . For  $n \leq 5$ , the three index representation is already the dual representation of a representation with a smaller number of indices, while for  $n > 5$ , no such redundancy is present.

For simplicity we confine our analysis to deformations where  $\phi$  is given by a single  $n \times n$  Jordan block. The parameter  $r$  introduced in (4.22) is then given by

$$r = (n^3 - n)/6. \quad (5.2)$$

Let us now comment on the representation content of the operators  $\mathcal{O}$ . Under the  $SU(2)$  subalgebra specified by the Jordan block, the fundamental representation becomes

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<sup>4</sup>This decomposition makes the representation content under  $G_{\text{GUT}}$  less manifest, but for our present purposes this is not necessary.

$n$	$E_8$	2	3	$E_7$	4	$E_6$	5	6	7	8	9
$t_*$	X	0.54	0.40	X	0.29	X	0.23	0.18	0.15	0.12	0.10
$a_{\text{IR}}$	3.96	3.42	2.69	2.46	2.09	1.71	1.66	1.34	1.11	0.94	0.81
$c_{\text{IR}}$	5.17	4.40	3.40	3.17	2.62	2.17	2.07	1.67	1.38	1.16	1.00
$k_{\text{IR}}$	12	9.73	7.14	8	5.31	6	4.09	3.25	2.66	2.22	1.88
$[Z]$	6	4.86	3.57	4	2.65	3	2.04	1.63	1.33	1.11	1
$[\mathcal{O}_{\text{GUT}}]$	2	2.19	2.41	2	2.56	2	2.66	2.73	2.78	2.81	2.84
$[\mathcal{O}_{ij}^{\text{min}}]$	X	1.38	1.22	X	1.23	X	1.30	1.37	1.45	1.52	1.59
$[\mathcal{O}_i^{\text{min}}]$	X	1.78	1.81	X	1.89	X	1.98	2.05	2.11	2.17	2.22
$[\mathcal{O}_{ij}^{\text{min}}]$	X	X	X	X	1.67	X	1.64	1.65	1.67	1.70	1.75
$[\mathcal{O}_{ijk}^{\text{min}}]$	X	X	X	X	X	X	X	1.51	1.45	1.43	1.43

**Table 5.** Central charges and operator scaling dimensions for the  $\mathcal{N} = 1$  SCFTs realized by nilpotent  $\phi$ -deformations of the  $\mathcal{N} = 2$   $E_8$  SCFT. An “X” indicates that this entry has no meaning for the specified deformation.

a spin  $j = (n - 1)/2$  irreducible representation of  $SU(2)$ . For the higher tensor index structures, the indices are free to range over the spin  $j$  irreducible representation, subject to appropriate anti-symmetry or tracelessness conditions. Since the dimension of the operators  $\mathcal{O}$  is specified by its spin content, and thus its tensor structure in  $SU(n)$ , we shall denote by  $\mathcal{O}_{\text{GUT}}$  the singlets,  $\mathcal{O}_i$  an operator in the fundamental of  $SU(n)$ ,  $\mathcal{O}_{ij}$  an operator in the two-index antisymmetric, and so on. We denote by  $\mathcal{O}_i^{\text{min}}$  the operator in the fundamental with the lowest scaling dimension, with similar notation for the other  $\mathcal{O}$ 's. The scaling dimension of  $\mathcal{O}^{\text{min}}$  is then given by (4.10) for appropriate  $s$ .

Using equation (4.23) and the expressions for the operator scaling dimensions and the values of the central charges obtained in section 4.3, we find the values for the various parameters given in table 5. In the table, we have ordered the entries according to decreasing values of  $a_{\text{IR}}$ . We have also included the corresponding  $\mathcal{N} = 2$  SCFT values. Note that increasing  $n$  always decreases  $a_{\text{IR}}$ , in accord with the expectation that we lose degrees of freedom as we continue to flow to the IR. Further, the theories with  $\mathcal{N} = 2$  supersymmetry and  $E$ -type flavor symmetry have smaller central charges than their nilpotent counterparts with the same non-abelian flavor symmetries ( $n = 2$  for  $E_7$  and  $n = 3$  for  $E_6$ ). Physically this is reasonable, as we have given a mass deformation to a smaller number of 3-7 strings in the case of nilpotent deformations. Finally, in all cases but the last with  $n = 9$ , all of the original operators remain above the unitarity bound. In this one case, we find that a first application of  $a$ -maximization yields a value for the dimension of  $Z$  which falls below the unitarity bound. In the above, we have assumed that there is an emergent  $U(1)$  which only acts on  $Z$ , so that  $Z$  decouples as a free field. Recomputing the value of the parameter  $t$  and the associated dimensions yields the corresponding values for  $n = 9$ . Note that this behavior is quite similar to what we observed in the case of the  $D_4$  probe theory. In that case, we observed that small nilpotent deformations produced a self-consistent picture for

operator dimensions, while for the  $4 \times 4$  Jordan block, there was an apparent violation of the dimension for the operator  $Z$ . There, this was ascribed to integrating out so many quarks, so presumably a similar phenomenon is present in the E-type case as well.

The theories defined by different  $n$ 's are all connected by further deformations. Mathematically, starting from the deformation defined by an  $n \times n$  nilpotent Jordan block, there is a deformation we can perform by enlarging  $\phi$  to an  $(n + 1) \times (n + 1)$  nilpotent Jordan block. This corresponds to a further deformation by a relevant operator. For example, starting from the  $n = 2$  theory, adding the operator  $O_i^{\min}$  corresponds to adding the next entry of the  $3 \times 3$  nilpotent block, inducing a deformation to the  $n = 3$  theory. Note that from table 5 each such operator is relevant in the corresponding theory, so it will indeed induce a flow to a new theory. See figure 1 for a depiction of these flows.

Further deformations of the  $\phi$ -deformed theories can also induce flows back to an  $\mathcal{N} = 2$  theory. For example, the  $n = 2$  theory is specified by deforming the  $E_8$   $\mathcal{N} = 2$  theory by an operator  $\mathcal{O}_-$  dotted into the matrix:

$$\phi_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{5.3}$$

In the IR theory we can consider further deformations with opposite  $T_3$  charge which we denote by  $\mathcal{O}_+$ . In the UV theory, this would correspond to adding the deformation:

$$\phi = \phi_0 + \phi_0^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.4}$$

This deformation preserves  $\mathcal{N} = 2$  supersymmetry since  $[\phi, \phi^\dagger] = 0$ , and induces a flow to the  $E_7$   $\mathcal{N} = 2$  SCFT. This is consistent with the fact that the  $n = 2$  theory has a larger central charge. Note also that the non-abelian E-type flavor symmetries agree.

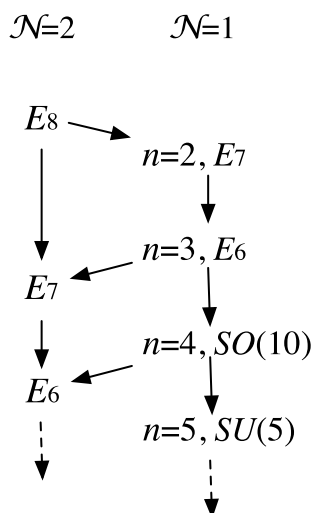
It is also possible to perform a further deformation of the nilpotent deformed theories to an  $\mathcal{N} = 2$  theory with a *larger* non-abelian flavor symmetry. For example, a similar argument to that given for the  $n = 2$  theory establishes that in the  $n = 3$  theory, we can perform a flow to an  $\mathcal{N} = 2$  theory associated with the  $\phi$ -deformation:

$$\phi = \phi_0 + \phi_0^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{5.5}$$

This induces a flow back to the  $E_7$  theory which has a *larger* E-type flavor symmetry than the  $n = 3$  theory! Indeed, by a unitary change of basis we can write  $\phi \simeq \text{diag}(+\sqrt{2}, -\sqrt{2}, 0)$ , which has commutant  $E_7$ . Note that this is also consistent with the fact that the central charge of the  $n = 3$  theory is greater than that of the  $\mathcal{N} = 2$   $E_7$  theory.

Similar considerations hold for the  $n = 4$  theory, and deformations to the  $E_6$  theory. Indeed, starting from

$$\phi_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.6}$$



**Figure 1.** Depiction of the various theories obtained by performing a single  $n \times n$  nilpotent Jordan block deformation of the  $E_8$   $\mathcal{N} = 2$  SCFT. Also indicated is the associated non-abelian flavor symmetry of each  $\mathcal{N} = 1$  theory. As summarized in table 5, increasing the size of the block leads to a further flow, and a decrease in the central charge  $a_{\text{IR}}$ . In the  $n = 3$  theory the non-abelian flavor symmetry is  $E_6$ , but it is nevertheless possible to deform the theory to the  $\mathcal{N} = 2$   $E_7$  SCFT. Similarly, the  $n = 4$  theory with non-abelian flavor symmetry  $SO(10)$  can be deformed to the  $\mathcal{N} = 2$   $E_6$  SCFT.

a further deformation by a lower triangular matrix will have rank three or higher.<sup>5</sup> Thus, the  $E$ -type non-abelian flavor symmetry is at most  $E_6$ . This is also consistent with the fact that the central charge of the  $n = 4$  theory is lower than the  $\mathcal{N} = 2$   $E_8$  and  $E_7$  theories, but is higher than the  $E_6$  theory; the  $E_6$  theory can indeed be realized by an appropriate deformation of the  $n = 4$  theory. Finally, let us note that for the  $n = 5$  theory, similar considerations establish that adding a further deformation to  $\phi_0$  can allow the non-abelian flavor symmetry to increase its rank by at most one unit, so that it is at most  $SO(10)$ . Note that this is consistent with the fact that the central charge of this theory is *below* that of the  $\mathcal{N} = 2$   $E_6$  theory. All of these checks suggest a highly non-trivial structure, providing further evidence for the existence of the  $\phi$ -deformed theories. See figure 1 for a schematic presentation of how these theories are connected by further deformations.

## 5.2 Large $N$ limit

It is quite natural to also consider the limit with a large number of D3-branes located at the same point. Note that if we move all the D3-branes together away from the seven-branes, the probe theory is  $\mathcal{N} = 4$   $U(N)$  gauge theory, so we expect an interacting SCFT at the origin of the Coulomb branch.

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<sup>5</sup>To see this, note that in the further deformation by such a  $\phi$ , the columns of this matrix provide three linearly independent vectors in the image space of  $\phi$ .

An interesting feature of the large  $N$  case is that for  $n \geq 3$ , we find that some of the operators  $\mathcal{O}$  will naively violate the unitarity bound. To see the apparent violations of the unitarity bound, we can compute the value of  $t_*$  in the large  $N$  limit, to find:

$$t_* = \frac{2}{3} - \frac{2\alpha}{3} \frac{1}{N} + \mathcal{O}(1/N^2) \tag{5.7}$$

where  $\alpha$  is an order one parameter which depends on the details of the model. The coefficient  $\alpha$  is positive, because the dimension of the Coulomb branch parameter decreases along the flow. Indeed, the IR dimension of  $Z$  is:

$$\Delta_{\text{IR}}(Z) = \frac{3}{2}t \times \Delta_{\text{UV}}(Z) = \Delta_{\text{UV}}(Z) - \frac{\alpha}{N} + \mathcal{O}(1/N^2). \tag{5.8}$$

Further, the dimension of the operators  $\mathcal{O}_s$  are:

$$\Delta_{\text{IR}}(\mathcal{O}_s) = 3 - (s + 1) + \frac{\alpha}{N} \times (s + 1) + \mathcal{O}(1/N). \tag{5.9}$$

Thus, we see that although  $Z$  essentially maintains its UV value, the operators  $\mathcal{O}_s$  will generically fall below the unitarity bound for sufficiently high values of  $s$ . Since  $\alpha > 0$ , the operators which violate the unitarity bound satisfy:

$$s > 1. \tag{5.10}$$

As the size of  $n$  increases, the available spins  $s$  will also increase, and the number of operators falling below the unitarity bound will increase. For example, starting from the  $E_8$   $\mathcal{N} = 2$  SCFT, consider adding a deformation by the  $3 \times 3$  Jordan block  $\phi_0$ :

$$\phi_0 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}. \tag{5.11}$$

In this theory, there is an operator with spin  $s = +2$  corresponding to the mass parameter in the lower left corner of (5.11). The dimension of this operator is order  $1/N$ , and in particular, below the unitarity bound.

Note, however, that we expect this deformation to induce a flow to a CFT. Indeed, in the theory deformed by just the operators  $\mathcal{O}_-$  of  $T_3$  charge  $-1$ , the operators  $\mathcal{O}_+$  of  $T_3$  charge  $+1$  have dimension:

$$[\mathcal{O}_+] = 1 + \frac{\alpha}{N} \tag{5.12}$$

which is slightly above the unitarity bound if we trust our naive calculation. It seems therefore consistent to perform a further deformation by this operator, leading to a flow to the large  $N$   $\mathcal{N} = 2$   $E_7$  theory. Performing  $a$ -maximization, we find to leading order in the  $1/N$  expansion:

$$a_{\mathcal{N}=2, E_8} = \frac{3}{2}N^2 + \frac{5}{2}N > a_{\mathcal{N}=1, E_8, \phi} = \frac{3}{2}N^2 - \frac{1}{2}N \gg a_{\mathcal{N}=2, E_7} = N^2 + \frac{3}{2}N. \tag{5.13}$$

How then do we interpret the fact that the operator with  $SU(2)$  spin  $+2$  drops below the unitarity bound? A self-consistent possibility is that as the dimension of the offending operator decreases and passes to the unitarity bound, an additional  $U(1)$  emerges,

and the offending operator decouples as a free field, with dimension 1. Re-performing  $a$ -maximization with this extra  $U(1)$  included, we can then read off the new IR  $R$ -symmetry. Note, however, that since only  $O(1)$  operators fall below the unitarity bound, this is a second order effect in a  $1/N$  expansion. Working to first order in a  $1/N$  expansion, we can therefore ignore this effect. Finally, because the operators  $\mathcal{O}_i^{\text{min}}$  generically decouple as free fields in the IR, it also follows that as opposed to the case  $N = 1$ , the natural extension of flows from an  $n \times n$  nilpotent Jordan block to an  $(n + 1) \times (n + 1)$  nilpotent Jordan block deformation is now obstructed. Nevertheless, we find that increasing  $n$  always decreases the value of the central charge  $a_{\text{IR}}$ . Further, we find that all  $\phi$ -deformed  $E_8$  theories have central charge above that of the  $\mathcal{N} = 2$   $E_7$  SCFT.

### 5.3 Maximal monodromy

As our first example of non-trivial seven-brane monodromy, we consider a  $\phi$  taking values in  $SU(n) \subset SU(9) \subset E_8$ . Moreover, in this section we assume that the unfolding is “generic” in the sense that the characteristic polynomial for  $\phi$  has Galois group  $S_n$ .<sup>6</sup>

Using the general results of section 4, and the values of the UV central charges (including the contribution from the free hypermultiplet) we can extract the infrared values of the central charges and infrared scaling dimensions. To illustrate the general pattern, we now present a general table of the IR values for  $n = 2, \dots, 9$  for  $\phi_{\text{IR}}$  satisfying the characteristic equation:

$$\phi_{\text{IR}}^n + z_2 \phi_{\text{IR}} + z_1 = 0 \tag{5.14}$$

for  $n > 2$ . The UV inputs are quite similar to the case of the nilpotent deformations, though we also need to specify the values of the parameters  $\mu_1$  and  $\mu_2$ . In the case of a single Jordan block, we have:

$$\mu_1 = (n - 1) + \frac{3}{2}, \quad \mu_2 = (n - 2) + \frac{3}{2}. \tag{5.15}$$

The IR values of the various dimensions and central charges are presented in table 6. In comparison with the case of nilpotent deformations, we see that the central charges and scaling dimensions shift very little. In the case  $n = 9$ , we find that a first application of  $a$ -maximization yields a dimension for  $Z$  below the unitarity bound. Applying the prescription in [30], we assume that this field decouples in the IR when its dimension saturates the unitarity bound. Another curious feature of the above examples is that in the case  $n = 3$ , the cubic anomaly  $a_{\text{IR}}$  is quadratic in  $t$  rather than cubic. This means that the values of  $t_*$  in this case will be rational numbers, and all operator dimensions will also be rational. It would be interesting to see whether there are any additional properties associated with this behavior.

### 5.4 $\mathbb{Z}_2 \times \mathbb{Z}_2$ monodromy

As a simple case of potential phenomenological relevance, we now consider an unfolding of  $E_8$  down to  $SU(5)_{\text{GUT}} \times SU(5)_{\perp}$ . Specifically, we consider a Dirac neutrino scenario of [18],

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<sup>6</sup>Let us note that in the specific context of F-theory GUTs where  $\phi$  takes values in the  $SU(5)_{\perp}$  factor of  $SU(5)_{\text{GUT}} \times SU(5)_{\perp} \subset E_8$ , this choice is unacceptable, because it means there is one curve for all fields in the  $\bar{\mathbf{5}}$  and  $\mathbf{5}$  of  $SU(5)_{\text{GUT}}$ , and in particular no distinction between the Higgs and matter fields.

$n$	$E_8$	2	3	$E_7$	4	$E_6$	5	6	7	8	9
$t_*$	X	0.53	0.39	X	0.29	X	0.22	0.17	0.14	0.12	0.10
$a_{\text{IR}}$	3.96	3.41	2.69	2.46	2.09	1.71	1.66	1.35	1.13	0.96	0.83
$c_{\text{IR}}$	5.17	4.38	3.40	3.17	2.61	2.17	2.05	1.67	1.39	1.19	1.03
$k_{\text{IR}}$	12	9.60	7	8	5.15	6	3.93	3.10	2.52	2.10	1.76
$[Z]$	6	4.80	3.5	4	2.58	3	1.96	1.55	1.26	1.05	1
$[Z_1]$	1	1.60	1.75	1	1.72	1	1.64	1.55	1.47	1.40	1.32
$[Z_2]$	1	1	1.17	1	1.29	1	1.31	1.29	1.26	1.22	1.17
$[\mathcal{O}_{\text{GUT}}]$	2	2.20	2.42	2	2.57	2	2.67	2.74	2.79	2.83	2.85
$[\mathcal{O}_{ij}^{\text{min}}]$	X	1.40	1.25	X	1.28	X	1.36	1.45	1.53	1.60	1.68
$[\mathcal{O}_i^{\text{min}}]$	X	1.80	1.83	X	1.93	X	2.02	2.10	2.16	2.21	2.23
$[\mathcal{O}_{ij}^{\text{min}}]$	X	X	X	X	1.71	X	1.69	1.71	1.74	1.78	1.83
$[\mathcal{O}_{ijk}^{\text{min}}]$	X	X	X	X	X	X	X	1.58	1.53	1.51	1.53

**Table 6.** Central charges and operator scaling dimensions for the  $\mathcal{N} = 1$  SCFTs realized by the  $\phi$ -deformed  $E_8$  probe theory with maximal monodromy. An “X” indicates that this entry has no meaning for the specified deformation.

where  $\phi \in \text{SU}(5)_\perp$  exhibits  $\mathbb{Z}_2 \times \mathbb{Z}_2$  monodromy. In this case, the monodromy group acts by interchanging two pairs of eigenvalues for  $\phi$  independently. To illustrate the main point, we consider a configuration with eigenvalues:

$$\text{Eigenvalues}(\phi) = \{a + \sqrt{z_1}, a - \sqrt{z_1}, b + \sqrt{z_2}, b - \sqrt{z_2}, -2a - 2b\} \quad (5.16)$$

where  $a$  and  $b$  are generic linear expressions in the  $z_i$ . A matrix representative composed of two  $2 \times 2$  blocks and one  $1 \times 1$  block is:

$$\phi_{\text{UV}} = \begin{pmatrix} & 1 \\ Z_1 - a^2 & 2a \end{pmatrix} \oplus \begin{pmatrix} & 1 \\ Z_2 - b^2 & 2b \end{pmatrix} \oplus (-2a - 2b). \quad (5.17)$$

In the infrared, the deformation is characterized by:

$$\phi_{\text{IR}} = \begin{pmatrix} & 1 \\ Z_1 & \end{pmatrix} \oplus \begin{pmatrix} & 1 \\ Z_2 & \end{pmatrix} \oplus (0). \quad (5.18)$$

Note that in the infrared, the non-abelian flavor symmetry is  $\text{SO}(10)$  rather than  $\text{SU}(5)$ . Further, the characteristic equation for  $\phi_{\text{IR}}$  is:

$$(\phi_{\text{IR}}^2 - Z_1)(\phi_{\text{IR}}^2 - Z_2)\phi_{\text{IR}} = 0. \quad (5.19)$$

As for the case of maximal monodromy, the UV inputs are quite similar to the case of the nilpotent deformations, though we also need to specify the values of the parameters  $\mu_1$  and  $\mu_2$ . In the case of the two Jordan blocks, we have  $r = 2$  and:

$$\mu_1 = \mu_2 = \frac{5}{2} \quad (5.20)$$

The IR values of the various central charges and dimensions are shown in table 7.



	$t_*$	$a_{\text{IR}}$	$c_{\text{IR}}$	$k_{\text{IR}}$	$[Z]$	$[Z_1]$	$[Z_2]$	$[\mathcal{O}_{\text{GUT}}]$	$[\mathcal{O}_{ij}^{\text{min}}]$	$[\mathcal{O}_i^{\text{min}}]$	$[\mathcal{O}_{ij}^{\text{min}}]$
$E_8$ with $\mathbb{Z}_2 \times \mathbb{Z}_2$	0.46	3.11	3.95	8.32	4.16	1.39	1.39	2.31	1.61	1.96	1.61
$E_6$ with $\mathbb{Z}_2$	0.51	1.45	1.80	4.58	2.29	1.53	1	2.24	1.47	1.85	X

**Table 7.** Central charges and dimensions of the cases with monodromy. The first row is for the deformation of the  $E_8$  theory with  $\mathbb{Z}_2 \times \mathbb{Z}_2$  monodromy, discussed in section 5.4. The second row is for the  $E_6$  theory with  $\mathbb{Z}_2$  monodromy, discussed in section 5.5.

### 5.5 $E_6$ and $\mathbb{Z}_2$ monodromy

The minimal requirement for a large top quark Yukawa coupling is an  $E_6$  point. Additionally, for one heavy generation, we require the unfolding to  $\text{SU}(5)_{\text{GUT}}$  to have  $\mathbb{Z}_2$  monodromy [15]. To explicitly see the effects of the monodromy group, we consider the breaking pattern:

$$E_6 \supset \text{SU}(5) \times \text{U}(1) \times \text{SU}(2) \tag{5.21}$$

$$78 \rightarrow (\mathbf{24}_0, \mathbf{1}) + (\mathbf{5}_6, \mathbf{1}) + (\overline{\mathbf{5}}_{-6}, \mathbf{1}) + (\mathbf{1}_0, \mathbf{3}) + (\mathbf{10}_{-3}, \mathbf{2}) + (\overline{\mathbf{10}}_{+3}, \mathbf{2}). \tag{5.22}$$

The monodromy group  $\mathbb{Z}_2$  is the Weyl group of  $\text{SU}(2)$ , which acts by interchanging the two components of an  $\text{SU}(2)$  doublet. A matrix  $\phi$  taking values in  $\text{SU}(2) \times \text{U}(1)$  which accomplishes this is:

$$\phi = \begin{pmatrix} & 1 \\ Z_1 & \end{pmatrix} \oplus (Z_2). \tag{5.23}$$

In particular, using the general result of [31], we see that the  $z_2$ -dependent contribution drops out and only the  $2 \times 2$  block of  $\text{SU}(2)$  dictates the flow in the IR. In the IR, the non-abelian flavor symmetry of the CFT is  $\text{SU}(6)$ , and the dimension of  $Z_2$  remains one. The value of the parameter  $\mu_1$  is:

$$\mu_1 = \frac{5}{2}. \tag{5.24}$$

Computing the IR values of the various central charges and operator dimensions, we find the values shown in table 7.

## 6 Stabilizing the D3-brane position

In much of this paper we have focused on the effects of the deformation:

$$\delta W = \text{Tr}_G(\phi(Z_1, Z_2) \cdot \mathcal{O}). \tag{6.1}$$

From the perspective of the field theory, additional deformations can be built purely from  $Z$ ,  $Z_1$  and  $Z_2$ . In the context of a string compactification, such superpotentials serve to stabilize the position of the D3-brane. These position-dependent superpotential terms can be expanded in a power series in the  $Z_i$ 's:

$$W_{\text{position}}(Z_1, Z_2, Z) = F_i Z_i + M_{ij} Z_i Z_j + \lambda_{ijk} Z_i Z_j Z_k + \dots \tag{6.2}$$

where we have set  $Z_3 = Z$ , and the  $F$ ,  $M$  and  $\lambda$  correspond to constants of the theory, which in a string compactification would be moduli-dependent parameters. Such terms are expected to be generated in the presence of appropriate fluxes, as studied for example in [32] and [33].

This is already quite interesting for the purposes of model building because superficially the term (6.2) suggests the superpotential of an O’Raifeartaigh model. Caution is warranted, however, because the fields  $Z_i$  do not have a canonical Kähler potential. It is nevertheless tempting to speculate that the D3-brane could naturally provide a source of supersymmetry breaking. Further, the fact that there are operators with Standard Model gauge quantum numbers also suggests that such a sector could naturally communicate supersymmetry breaking to the visible sector via gauge mediation effects. A full analysis of these possibilities is beyond the scope of the present work, and so in the remainder of this section we confine our analysis to the question of which deformations lead to another interacting CFT, when combined with the deformation of equation (6.1).

In principle, we can consider either continuous or discrete symmetries which exclude all such contributions, thus retaining the original form of the CFT induced by just the deformation  $\text{Tr}_G(\phi \cdot \mathcal{O})$ . For an appropriate discrete subgroup of the remaining flavor symmetries of the system, it is immediate that we can exclude such linear and quadratic deformations from appearing in equation (6.2). On the other hand, for appropriate choices of a discrete symmetry such as  $\mathbb{Z}_2$ , we can also consider cases where for example  $Z_1$  is excluded, but  $Z_1^2$  is allowed.

**Linear term in  $Z_1$  and  $Z_2$ .** Let us now suppose that at least some of the terms of  $W_{\text{position}}$  are not forbidden by a discrete symmetry. We can see that some of these terms could be relevant deformations because the deformation  $\text{Tr}_G(\phi(Z_1, Z_2) \cdot \mathcal{O})$  increases the dimension of  $Z_1$  and  $Z_2$ , but decreases the dimension of  $Z$ . Note that deformations linear in  $Z_1$  and  $Z_2$  do not induce a flow to a CFT. The reason is that if we demand such operators are marginal in the IR, then the operator  $\mathcal{O}_{(i)}$  multiplying  $Z_i$  in the superpotential deformation would have dimension zero. Indeed, in the UV theory it is clear that adding this term simply enforces the condition that the vev of  $\mathcal{O}_{(i)}$  is non-zero. This is also what is expected based on the brane construction. Viewing the probe D3-brane as an instanton, the linear terms in the  $Z_i$  fix some of its moduli.

**Linear term in  $Z$ .** Let us next consider a term linear in  $Z$  in the  $D_4$  probe theory. With notation as in section 3, this corresponds to a mass term for the adjoint scalar  $\varphi$  which tends to attract the D3-brane to the seven-brane. In the absence of the  $\text{Tr}_{\text{SO}(8)}(\phi \cdot \mathcal{O})$  deformation, deforming by  $Z$  is the standard adjoint mass deformation. In the IR, the field  $\varphi$  has been integrated out, leading to an infrared marginal quartic interaction between the quarks. In the presence of the further deformations by  $\text{Tr}_{\text{SO}(8)}(\phi \cdot \mathcal{O})$ , we cannot simultaneously demand that both  $Z$  and these deformations are marginal in the IR. To see this, consider again the case of constant nilpotent  $\phi$  studied in section 3.2. Integrating out the massive quarks requires that the operator  $Q_1 \varphi^{n+1} \tilde{Q}_{\bar{n}}$  be marginal in the IR. Note that this is incompatible with the condition that  $Z$  is also marginal. Thus, starting from the original  $\mathcal{N} = 2$  theory, we can either deform by  $\text{Tr}_{\text{SO}(8)}(\phi \cdot \mathcal{O})$  or by  $Z$ , leading to two

different interacting CFTs. Adding both terms simultaneously, we see also that we cannot simultaneously enforce that both deformations are marginal in the IR.

Next consider adding a linear term in  $Z$  to the  $\phi$ -deformed E-type theories. In the original  $\mathcal{N} = 2$   $E_6$  theory,  $Z$  is dimension three but is irrelevant in the IR [31]. In the  $E_7$  and  $E_8$  theories,  $Z$  has dimension 4 and 6 respectively, and so is also irrelevant. In the  $\phi$ -deformed theories, however, the dimension of  $Z$  can be significantly lower. This in turn means that in some circumstances it is indeed appropriate to treat it as a potentially marginal operator in the IR. For simplicity, let us consider the case of deformations by a nilpotent mass deformation taking values in an  $SU(n)$  subblock, and consider the further effect of deforming the superpotential by a term linear in  $Z$ .

We now study whether such a deformation induces a flow to a new CFT. To this end, we assume that the deformation linear in  $Z$  is marginal in the IR, and deduce whether this leads to a consistent picture of possible flows. In the absence of the term linear in  $Z$ , the IR R-symmetry was given in (4.6). Requiring that  $Z$  is also marginal in the IR determines  $t$  via:

$$t = \frac{2}{\Delta_{UV}(Z)}. \quad (6.3)$$

Let us demand that there is no violation of the unitarity bound for operators  $\mathcal{O}_{i\bar{j}}$ . The  $T_3$  charge of the lowest spin component is  $n - 1$ , which in turn means:

$$R_{IR}(\mathcal{O}_{i\bar{j}}^{\min}) = 2 - \frac{2}{\Delta_{UV}(Z)} - \frac{2}{\Delta_{UV}(Z)}(n - 1). \quad (6.4)$$

Requiring  $R_{IR}(\mathcal{O}_{i\bar{j}}^{\min}) > 2/3$  implies:

$$n < \frac{2\Delta_{UV}(Z)}{3}. \quad (6.5)$$

Note in particular that for the  $D_4$  and  $E_6$  theories,  $\Delta_{UV}(Z) = 2$  and 3 respectively, and so this condition is not satisfied. For the  $E_7$  theory  $\Delta_{UV}(Z) = 4$  and we can deform by an  $SU(2)$  nilpotent subblock, while for the  $E_8$  theory  $\Delta_{UV}(Z) = 6$  and we can deform by an  $SU(3)$  or smaller subblock. In these cases, however, we observe that the dimension of  $Z$  in the original  $\phi$ -deformed theory is above three, and so the corresponding deformation is irrelevant.

When (6.5) is not satisfied, it is not clear whether to expect an interacting CFT in the IR; the endpoint might instead be a massive theory. It would be interesting to classify the available IR phases from the combined deformations induced by  $\text{Tr}_G(\phi \cdot \mathcal{O})$  and  $W_{\text{position}}$ .

## 7 Coupling to the visible sector

In much of this paper we have focused on the dynamics of the CFT sector, providing evidence that close to the visible sector, there could be an interacting  $\mathcal{N} = 1$  CFT. A full analysis of the various consequences for phenomenology is beyond the scope of the present paper, and so in this section we shall only make some general comments.

## 7.1 Unfolding $E_8$ to $SU(5)_{\text{GUT}}$

In this subsection we discuss in more practical terms the deformations of an  $E_8$  singularity down to an  $SU(5)_{\text{GUT}}$  singularity. In practice, extracting the explicit form of an unfolding based on the local form of  $\phi$  can be somewhat cumbersome, in part because the expressions for the primitive Casimir invariants of the E-type algebras are quite unwieldy (see [26] for their explicit forms). In the special case where  $\phi$  takes values in the  $SU(5)_{\perp}$  factor of the subalgebra  $SU(5)_{\text{GUT}} \times SU(5)_{\perp} \subset E_8$ , a significant simplification in the form of the unfolding occurs.

In this case,  $\phi$  has a characteristic equation of the form:

$$b_0\phi^5 + b_2\phi^3 + b_3\phi^2 + b_4\phi + b_5 = 0 \tag{7.1}$$

where the  $b_i$  are holomorphic  $z_i$ -dependent coefficients. As explained in [17, 34] (see also [35]), a local unfolding of  $E_8$  down to  $SU(5)_{\text{GUT}}$  can then be written as:

$$y^2 = x^3 + b_0z^5 + b_2xz^3 + b_3yz^2 + b_4x^2z + b_5xy. \tag{7.2}$$

From this family of curves we can read off the value of  $\tau$  on the Coulomb branch, and also the relative scaling dimensions of mass deformations to the dimension of the Coulomb branch parameters. We emphasize, however, that knowing the coefficients  $b_i$  is *not* enough to reconstruct a unique choice of  $\phi$ .

## 7.2 Coupling of the CFT to the visible sector

Now, note that the full system described by the CFT and the visible sector will no longer be a CFT. Indeed, upon compactifying to four dimensions, the flavor symmetry will be weakly gauged, and conformality will be lost. The matter fields of the visible sector can either localize on matter curves of the compactification, or propagate in the bulk worldvolume of the seven-brane. Thus, we can in principle study the effects of first compactifying the matter curves, and then consider the additional effects of compactifying the remaining directions of the seven-brane.

Matter fields  $\psi_R$  transforming in a representation  $R$  of  $SU(5)_{\text{GUT}}$  couple to operators  $\mathcal{O}_{R^*}$  of the CFT transforming in the dual representation via:

$$\int d^2\theta \psi_R^{(i)} \cdot f_{(i)}(Z_{\parallel}) \mathcal{O}_{R^*} \tag{7.3}$$

where  $i = 1, \dots, 3$  is a generational index for chiral matter, and  $f_{(i)}(Z_{\parallel})$  is a function of the local coordinate  $Z_{\parallel}$  for the matter curve. Though the specific details of the couplings depend on seven-brane monodromy, the main point is that this adds another class of interactions to consider, which it would be interesting to analyze further. Since the matter field wave functions have different profiles near a Yukawa point, the order of vanishing near this point will dictate the relevance of the coupling to the visible sector. For example the coupling to the third (heavy) generation quarks will be most relevant, while the coupling to the first generation will be least relevant.

The CFT also possesses a large number of states which are charged under the GUT group  $SU(5)_{\text{GUT}}$ . These states will in turn affect the running. As explained in [7], the contribution to the running is essentially fixed by the scaling of the Coulomb branch parameter  $\Delta_{\text{IR}}(z)$ . More directly, the contribution to the one-loop running effects of the  $SU(5)$  GUT coupling from the CFT is the same as  $N_{\mathbf{5} \oplus \bar{\mathbf{5}}}$  pairs of  $\mathbf{5} \oplus \bar{\mathbf{5}}$  chiral multiplets, where

$$N_{\mathbf{5} \oplus \bar{\mathbf{5}}} = \frac{k_{\text{IR}}}{2}. \tag{7.4}$$

Scanning over the values of  $k_{\text{IR}}$  we have already computed, we see that in the case of the  $\mathcal{N} = 2$   $E_8$  theory, this has the effect of six  $\mathbf{5} \oplus \bar{\mathbf{5}}$ 's. On the other hand, in the case of larger deformations down to  $SU(5)$ , we see what would appear as an irrational number of  $\mathbf{5} \oplus \bar{\mathbf{5}}$ 's, with the net effect on the order of two  $\mathbf{5} \oplus \bar{\mathbf{5}}$ 's.

The study of how this sector couples to the visible sector likely has a rich phenomenology which could be studied further for various model building applications.

## 8 Conclusions

Recent work has shown that compactifications of F-theory provide a natural arena for engineering gauge theories of potential phenomenological interest. In this paper we have found a new class of SCFTs which arise as the worldvolume theories of D3-branes probing F-theory seven-branes, which in appropriate circumstances can couple to the visible sector of the Standard Model. These SCFTs are characterized in terms of deformations of an  $\mathcal{N} = 2$  system. In many cases, we have argued that the resulting deformation induces a flow to a new interacting SCFT. We have also seen that while the geometry of the seven-branes of F-theory is able to capture a great deal of information about such theories, in particular through the  $\mathcal{N} = 1$  curve, additional input from  $a$ -maximization is often necessary to fully specify the infrared R-symmetry. These CFTs are also of potential phenomenological relevance, as the states of the D3-brane theory can couple to the Standard Model. In the rest of this section we discuss some future avenues of potential investigation.

One of the central themes of this work has been the role of backgrounds in which  $[\phi, \phi^\dagger] \neq 0$ . This suggests a sense in which the seven-branes of F-theory could “puff up” to non-commutative nine-branes. Non-commutativity in F-theory compactifications has recently been discussed in [33, 36]. It would be worthwhile to develop a more uniform treatment of F-theory from the non-commutative viewpoint.

In much of this work, we have only been able to provide various consistency checks that the  $\mathcal{N} = 1$  theories flow to an interacting SCFT. It would be interesting to develop further consistency checks of these statements. Along these lines, it would be useful to develop a holographic dual description of these SCFTs in the large  $N$  limit. In the  $\mathcal{N} = 2$  setting, holographic duals are available which have been studied for example in [13]. In addition, we have argued that further deformations can restore the system to an  $\mathcal{N} = 2$  system which also admits a holographic dual. It would be quite instructive to study whether there is an interpolating  $\mathcal{N} = 1$  geometry which connects these  $\mathcal{N} = 2$  theories.

Another potential avenue would be to search for possible field theory duals of the theories considered here. Indeed, some notable examples for related  $\mathcal{N} = 2$  theories have

been studied for example in [37, 38], and it would be interesting to see whether  $\mathcal{N} = 1$  analogues of these duals could be constructed.

Aside from providing further consistency checks, it would also be enlightening to further study the structure of these new SCFTs. For example, determining the chiral ring for these theories, or even the number of independent generators for the chiral ring (perhaps along the lines of [39]) would be quite helpful. Determining an index similar to the one recently computed in [40] for related  $\mathcal{N} = 2$  theories would also be of interest.

Finally, though our main focus in this paper has been the study of the associated SCFTs, we have also seen that some of the main ingredients present in the D3-brane probe theory could potentially be used for breaking supersymmetry. Further, since the CFT comes equipped with fields charged under the visible sector gauge group, it is quite natural to speculate that the D3-brane already contains all the ingredients to realize a self-contained gauge mediation sector.

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## A Field theory tools

Here we collect standard facts on four-dimensional SCFTs. Detailed discussions can be found in references.

First, there is a lower bound to the dimensions  $\Delta$  of the operators of a unitary CFT. For example, all spin zero gauge-invariant operators which are not free fields must satisfy  $\Delta > 1$ . For scalar chiral primaries of a SCFT, the dimension of an operator is related to its R-charge  $R$  through the relation

$$\Delta = \frac{3}{2}R. \tag{A.1}$$

To extract the scaling dimensions of operators, it is therefore of interest to determine the infrared R-charge  $R_{\text{IR}}$  of the operators. Note that finding  $R_{\text{IR}} < 2/3$  would imply that we had made an incorrect assumption. Ensuring all operators are above the unitarity bound therefore provides a basic check on our analysis.

In the context of  $\mathcal{N} = 2$  theories, it is often possible to fix  $R_{\text{IR}}$  by using the  $\mathcal{N} = 2$  Seiberg-Witten curve, see [23, 41]. For example, using this procedure it is possible to fix the scaling of the Coulomb branch parameter for the  $\mathcal{N} = 2$  SCFTs realized by a D3-brane

probing an F-theory singularity at constant axio-dilaton. For  $\mathcal{N} = 1$  theories the situation is more complicated because we no longer have the analogue of the  $\mathcal{N} = 2$  BPS bound.<sup>7</sup> To fix the scaling dimensions we use  $a$ -maximization [22].

Let  $R_0$  denote an R-symmetry, i.e. a symmetry under which the supercharge has charge 1. Let  $F_I$  be the generators of the abelian flavor symmetries. Then  $R_{\text{IR}}$ , the R-symmetry of the IR theory in the superconformal algebra can be written as

$$R_{\text{IR}} = R_0 + \sum t_I F_I. \tag{A.2}$$

The central charges  $a$  and  $c$  are then given in terms of 't Hooft anomalies by

$$a_{\text{IR}} = \frac{3}{32} [3 \text{Tr} R_{\text{IR}}^3 - \text{Tr} R_{\text{IR}}], \quad c_{\text{IR}} = \frac{1}{32} [9 \text{Tr} R_{\text{IR}}^3 - 5 \text{Tr} R_{\text{IR}}]. \tag{A.3}$$

The  $a$ -maximization procedure states that the  $t_I$  can be determined by first promoting  $a_{\text{IR}}$  to a function of  $t_I$  by putting (A.2) into (A.3), and then finding the unique local maximum of  $a_{\text{IR}}(t_I)$ .

A practical difficulty which is always encountered is to list all the infrared flavor symmetries  $F_I$ , which might include emergent symmetries in the IR. Assuming there is no such emergent symmetry, the procedure is to find a candidate  $R_0$  by requiring the vanishing of the gauge anomaly, and then to demand that any operators used to deform the superpotential become marginal in the IR and thus have  $R_{\text{IR}} = 2$ , with which we can extract relations between the parameters  $t_I$ . The remaining parameters are then determined by  $a$ -maximization.

When the resulting maximum leads to one or more gauge-invariant operators of dimension less than one, this means our original assumption is wrong. One interpretation is that an emergent U(1) appears in the IR which acts only on the operator which seems to violate the unitarity bound, making it saturate the unitarity bound and decouple as a free field instead [30] (see also [43]). The procedure is to perform  $a$ -maximization again, but with the contribution from this operator removed:

$$a_{\text{new}} = a_{\text{old}} - \frac{3}{32} [3(r-1)^3 - (r-1)] + \frac{1}{48}. \tag{A.4}$$

where  $r$  is the R-charge of the operator computed with respect to the old R-charge assignments.

We can also check if additional deformations of the CFT decrease the value of  $a_{\text{IR}}$ . Physically, this is a reasonable condition to hope for, as the central charges can be viewed as roughly counting the number of degrees of freedom of the CFT.<sup>8</sup>

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<sup>7</sup>In [9] it was suggested that the special geometry of the Calabi-Yau of F-theory could be used to fix the scaling dimensions of operators. As can be shown, this is equivalent to demanding the Gukov-Vafa-Witten flux induced superpotential of [42] has dimension exactly three. However, this superpotential also significantly alters the theory, rendering this method quite suspect.

<sup>8</sup>Violations of the a-theorem were observed in [44], associated to flows initiated by moving along the moduli space. In this paper we only consider flows associated to the deformation of the superpotential, for which no violation is known. We indeed will not see a violation in this paper.



In addition to the central charges  $a$  and  $c$ , there are central charges associated with flavor symmetries. Given flavor symmetry currents  $J_A$  and  $J_B$ , with  $A, B$  indices labelling the generators of the non-abelian flavor symmetry, the cubic anomaly

$$\text{Tr}(R_{\text{IR}} J_A J_B) = -\frac{k_{\text{IR}}}{6} \delta_{AB} \tag{A.5}$$

determines the effect of the CFT on the running of the holomorphic gauge coupling of a weakly gauged flavor symmetry group.

In any  $\mathcal{N} = 2$  conformal theory, there is an R-symmetry  $\text{SU}(2) \times \text{U}(1)$ . Denote by  $I_3$  the Cartan generator of the  $\text{SU}(2)$  factor, and  $R_{\mathcal{N}=2}$  the generator of the abelian factor. One linear combination of these generators is the  $\mathcal{N} = 1$  R-symmetry  $R_{\mathcal{N}=1}$  and another corresponds to a flavor symmetry  $J_{\mathcal{N}=2}$  as an  $\mathcal{N} = 1$  SCFT:

$$R_{\mathcal{N}=1} = \frac{1}{3} R_{\mathcal{N}=2} + \frac{4}{3} I_3, \quad J_{\mathcal{N}=2} = R_{\mathcal{N}=2} - 2I_3. \tag{A.6}$$

$\mathcal{N} = 2$  supersymmetry relates the anomalies with the central charges as follows:

$$\text{Tr}(R_{\mathcal{N}=2}^3) = \text{Tr}(R_{\mathcal{N}=2}) = 48(a_{\text{UV}} - c_{\text{UV}}), \tag{A.7}$$

$$\text{Tr}(R_{\mathcal{N}=2} I_3 I_3) = 4a_{\text{UV}} - 2c_{\text{UV}}, \tag{A.8}$$

$$\text{Tr}(R_{\mathcal{N}=2} J^A J^B) = -\frac{k_{\text{UV}}}{2} \delta^{AB}. \tag{A.9}$$

The central charges for the  $\mathcal{N} = 2$  SCFTs realized by  $N$  D3-branes probing an F-theory singularity with constant axio-dilaton have been determined [45, 46]:

$$a = \frac{1}{4} N^2 \Delta + \frac{1}{2} N (\Delta - 1) - \frac{1}{24}, \tag{A.10}$$

$$c = \frac{1}{4} N^2 \Delta + \frac{3}{4} N (\Delta - 1) - \frac{1}{12}, \tag{A.11}$$

$$k = 2N\Delta. \tag{A.12}$$

where  $\Delta$  is the dimension of the Coulomb branch parameter  $Z$  (see table 1). Note that in the above formulae for  $a$  and  $c$  the contribution from the hypermultiplet  $Z_1 \oplus Z_2$  has been subtracted off.

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