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Massive type IIA string theory cannot be strongly coupled

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Abstract: Understanding the strong coupling limit of massive type IIA string theory is a longstanding problem. We argue that perhaps this problem does not exist; namely, there may be no strongly coupled solutions of the massive theory. We show explicitly that massive type IIA string theory can never be strongly coupled in a weakly curved region of space-time. We illustrate our general claim with two classes of massive solutions in $\mathrm{AdS}_4 \times \mathbb{CP}^3$: one, previously known, with $\mathcal{N} = 1$ supersymmetry, and a new one with $\mathcal{N} = 2$ supersymmetry. Both solutions are dual to $d = 3$ Chern-Simons-matter theories. In both these massive examples, as the rank N of the gauge group is increased, the dilaton initially increases in the same way as in the corresponding massless case; before it can reach the M-theory regime, however, it enters a second regime, in which the dilaton decreases even as N increases. In the $\mathcal{N} = 2$ case, we find supersymmetry-preserving gauge-invariant monopole operators whose mass is independent of N. This predicts the existence of branes which stay light even when the dilaton decreases. We show that, on the gravity side, these states originate from D2-D0 bound states wrapping the vanishing two-cycle of a conifold singularity that develops at large N.

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Contents

1 Introduction and summary of results

One of the most striking aspects of string theory is its uniqueness, realized by the famous "web of dualities" that interconnect its various perturbative realizations. A famous thread in this web connects weakly coupled, perturbative type IIA string theory with its strong coupling limit, M theory (which reduces at low energies to eleven-dimensional supergravity).

It has been known for a while, however, that this duality does not work when the Romans mass parameter F_0 [1], which can be thought of as a space-filling Ramond-Ramond

(RR) 10-form flux, is switched on. There is no candidate parameter in eleven-dimensional supergravity to match with F_0, unlike for all the other fluxes; nor is there any massive deformation of the eleven-dimensional theory [2–4].[1] And, from the type IIA point of view, the D0-branes which give rise to the momentum modes in the eleventh dimension at strong coupling do not exist in the massive theory (as there is a tadpole for their worldvolume gauge field). This would then appear to be an imperfection in our understanding of string duality: it would be one string theory whose strong coupling limit is not known.

In this paper, we will argue that this strong coupling limit may not exist, and we will show this explicitly at least at the level of weakly curved solutions. In general these are the only solutions we have any control over, unless we have a large amount of supersymmetry; one can separately consider cases with a large amount of supersymmetry, and none of them seem to lead to strong coupling either. (The type I' theory of [7] contains in some of its vacua strongly coupled regions of massive type IIA string theory, but these regions have a varying dilaton and their size is never larger than the string scale.) Thus, we claim that there is no reason to believe that any strongly coupled solutions exist (with the exception of solutions with small strongly coupled regions), and we conjecture that there are none. This is consistent with the fact that no suggestion for an alternative description of the massive theory at strong coupling is known.

In section 2 we provide a simple argument that the string coupling g_s in massive type IIA string theory must be small, if the curvature is small. Generically, we find that $g_s \lesssim l_s/R$, where R is a local radius of curvature. The argument just uses the supergravity equations of motion and flux quantization.

This result is in striking contrast with what happens in the massless case. In the ten-dimensional massless vacuum, for example, the dilaton is a free parameter, and in particular it can be made large, resulting in the M-theory phase mentioned earlier. The massive theory has no such vacua.

It is of interest to consider examples with AdS$_4$ factors, where we can take advantage of a dual field theory interpretation via the AdS/CFT correspondence, which also provides a non-perturbative definition for the corresponding string theory backgrounds. In particular, it is natural to consider solutions like the $\mathcal{N}=6$ supersymmetric solution AdS$_4 \times \mathbb{CP}^3$ of the massless type IIA string theory [8–10]. In this solution, the dilaton is determined by the internal flux integers $k \propto \int_{\mathbb{CP}^1} F_2$ and $N \propto \int_{\mathbb{CP}^3} F_6$, $g_s \sim N^{1/4}/k^{5/4}$, whereas the curvature radius $R/l_s \sim N^{1/4}/k^{1/4}$. In particular, for $N \gg k^5$ one has a large dilaton with small curvature. In this limit, the solution is better described as the AdS$_4 \times S^7/Z_k$ M-theory background. The dual field theory has been identified in [11] as the $\mathcal{N}=6$ superconformal Chern-Simons-matter theory with gauge group $\mathrm{U}(N)\times\mathrm{U}(N)$ and Chern-Simons couplings k and $-k$.

Massive type IIA solutions are also known on AdS$_4 \times \mathbb{CP}^3$, and it is natural to compare

[1] See [5, 6] for some attempts to lift massive type IIA string theory to eleven dimensions.

JHEP11(2010)047

their behavior to the massless case. For example, some solutions with $\mathcal{N} = 1$ supersymmetry are known explicitly [12, 13]; they contain the $\mathcal{N} = 6$ solution as a particular case. The field theory duals are Chern-Simons-matter theories whose levels do not sum up to zero. Even though F_0 is quantized as $n_0/(2\pi l_s)$, one might think that introducing the smallest quantum of it, say $n_0 = 1$, should have little effect on the solutions, if the other flux integers k and N are already very large. It would seem, then, difficult to understand how a massless solution with large dilaton can suddenly turn into a massive solution with small dilaton when n_0 is turned on.

As we will see in section 3, in general this "small deformation" intuition is flawed. When trying to express the dilaton in terms of the flux parameters, in the massive case one ends up with expressions in which F_0 multiplies other, large flux parameters. Hence F_0 can have a large effect on the behavior of the solutions even if it is the smallest allowed quantum. As it turns out, as we increase N, the dilaton does start growing as $g_s \sim N^{1/4}/k^{5/4}$, as in the massless case. But, before it can become large, g_s enters a second phase, where it starts decreasing with N. Specifically, for N larger than the "critical value" k^3/n_0^2, we have $g_s \sim N^{-1/6} n_0^{-5/6}$. Both behaviors are visible in figure 2.

Notice that what happens for these $\mathcal{N} = 1$ solutions is not entirely a consequence of the general argument in section 2. One could have found, for example, that for large N the radius of curvature became small in string units. In such a situation, our supergravity argument would not have been able to rule out a large dilaton; even worse, it would actually generically predict it to be large. It is interesting to ask whether there are situations where that happens. Of course, one would not trust such strongly-curved, strongly-coupled solutions, since we have no control over them; but, if they existed, they would suggest that perhaps strongly coupled solutions do exist and need to be understood.

To look for such a different behavior, we turn to a second class of massive solutions, still on $\mathrm{AdS}_4 \times \mathbb{CP}^3$, but this time with $\mathcal{N} = 2$ supersymmetry. Such gravity solutions were predicted to exist via AdS/CFT [14], and found as first-order perturbations in F_0 of the $\mathcal{N} = 6$ solution in [11]. The field theory duals are again Chern-Simons-matter theories whose levels do not sum up to zero. In section 4 we point out that these theories have certain gauge-invariant monopole operators, whose mass (which is protected by supersymmetry) is independent of the rank N. This suggests the existence of wrapped branes that remain light in the large N limit. This cannot happen for backgrounds which are both weakly-coupled and weakly-curved.

To see what happens at large N, in section 5 we find these $\mathcal{N} = 2$ gravity solutions, generalizing the construction in [15] (see also [16]). We reduce the equations of motion and supersymmetry equations to a system of three ODEs for three functions, which we study numerically. As in section 3, we then study the behavior of g_s as a function of the flux integers. We find exactly the same phenomenon as in section 3: g_s follows initially the same growth observed for the $\mathcal{N} = 6$ solutions, and departs from that behavior before it can get large. The existence of the light states found in section 4 is not a consequence of strong coupling, but is instead explained by the fact that the internal space develops a conifold singularity where branes can wrap a small cycle. We compute numerically the mass of D2-D0 bound states wrapping the vanishing cycle, and we reproduce very accurately the mass predicted in section 4 from AdS/CFT.

Hence, in both examples we examined, the curvature stays bounded almost everywhere, and the dilaton does not become strongly coupled.[2] Our argument in section 2 does not rule out the possibility of solutions with large curvature and large dilaton, and it would be nice to find a way to rule them out. In general, such solutions would not be trustworthy, but in some situations one might understand them via chains of dualities. For example, in some cases it might be possible to T-dualize to a massless solution with small curvature, which in turn might be liftable to M-theory, along the lines of [18]. The behavior found in the two examples analyzed in this paper may not be universal, and we expect the AdS/CFT correspondence to be very helpful in any further progress.

One motivation for understanding the strong coupling limit of massive type IIA string theory is the Sakai-Sugimoto model [19] of holographic QCD, which has N_f D8-branes separating a region of space with $F_0 = 0$ from a region with $F_0 = N_f/(2\pi l_s)$. The solution of this model is known in the IR, where it is weakly coupled and weakly curved and the D8-branes may be treated as probes; but it is not clear what happens in the UV, where, before putting in the D8-branes, the coupling became large (see [20] for an analysis of the leading order back-reaction of the D8-branes in this model). Our analysis rules out the possibility that the region of massive type IIA string theory between the D8-branes becomes strongly coupled while remaining weakly curved in the UV. It would be interesting to understand whether there is a sensible UV completion of this model, and, if so, what it looks like.

2 A general bound on the dilaton

In this section, we will find a bound for the dilaton for type IIA solutions with non-zero 0-form flux $F_0 \neq 0$, assuming that the ten-dimensional curvature is small.

The argument is simply based on the equations of motion of type IIA supergravity. Note that due to supersymmetry, these equations are actually exact (at two-derivative order) and can be trusted even when the coupling constant becomes large.

The Einstein equations of motion in the string frame take the form

$$e^{-2\phi}\left(R_{MN} + 2\nabla_M\nabla_N\phi - \frac{1}{4}H_M{}^{PQ}H_{NPQ}\right) = \sum_{k=0,2,4} T^{F_k}_{MN}\,, \tag{2.1}$$

where

$$T^{F_k}_{MN} = \frac{1}{2(k-1)!}F_M{}^{M_2\dots M_k}F_{NM_2\dots M_k} - \frac{1}{4k!}F_{M_1\dots M_k}F^{M_1\dots M_k}g_{MN}\ . \tag{2.2}$$

[2] A correlation between the string coupling and the curvature in massive type IIA string theory was noticed recently in [17].

The equations (2.1) are valid at every point in spacetime, away from possible branes or orientifolds. On such objects, we would need to include further localized terms, but they will not be needed in what follows. In fact, all we need is a certain linear combination: let us multiply (2.1) by ${e_0}^M {e_0}^N$, where e are the inverse vielbeine; 0 is a frame index in the time direction. We can now use frame indices to massage T_{00} on the right hand side:

$$
\begin{aligned}
2T^{F_k}_{00} &= \frac{1}{(k-1)!} {F_0}^{A_2\ldots A_k} F_{0A_2\ldots A_k} - \eta_{00}\left(\frac{1}{2(k-1)!} F^{0A_2\ldots A_k} F_{0A_2\ldots A_k} + \frac{1}{2k!} F^{A_1\ldots A_k} F_{A_1\ldots A_k}\right) \\
&= \frac{1}{2(k-1)!} {F_0}^{A_2\ldots A_k} F_{0A_2\ldots A_k} + \frac{1}{2k!} F^{A_1\ldots A_k} F_{A_1\ldots A_k} \equiv \frac{1}{2}(F^2_{0,k-1} + F^2_{\perp,k}) \ . \qquad (2.3)
\end{aligned}
$$

We have defined the decomposition $F_k = e^0 \wedge F_{0,k-1} + F_{\perp,k}$. (In particular, $F_{\perp,0}$ is simply F_0.) Applying this to (2.1), we get

$$
e^{-2\phi}\left[{e_0}^M {e_0}^N \left(R_{MN} + 2\nabla_M \nabla_N \phi - \frac{1}{4} {H_M}^{PQ} H_{NPQ}\right)\right] = \frac{1}{4}\left(\sum_{k=2,4} F^2_{0,k-1} + \sum_{k=0,2,4} F^2_{\perp,k}\right) . \qquad (2.4)
$$

Again, this is satisfied at every spacetime point (away from possible sources): there is no integral in (2.4). R_{MN} needs to be small in the supergravity approximation. In fact, all the remaining terms in the parenthesis on the left-hand side need to be small too: they are all two-derivative NS-NS terms. If any of them is large in string units, we cannot trust the two-derivative action any more; hence that parenthesis needs to be $\ll l_s^{-2}$.

On the other hand, when $F_0 \neq 0$, the right-hand side of (2.4) is at least of order one in string units. To see this, recall that RR fluxes are quantized, in appropriate sense. The F_k are actually not closed under d, but under $(d - H\wedge)$. However, the fluxes

$$
\tilde{F}_k = \left[e^{-B}(F_0 + F_2 + F_4 + F_6 + F_8 + F_{10})\right]_k \qquad (2.5)
$$

are closed; when integrated over a closed space-like cycle C_a, they satisfy the quantization law

$$
\int_{C_a} \tilde{F}_k = n_k (2\pi l_s)^{k-1} \, , \qquad (2.6)
$$

where n_k are integers. In particular, $F_0 = n_0/(2\pi l_s)$. Since the right-hand side of (2.4) is a sum of positive terms, we get that it is $> 1/l_s^2$ (up to irrelevant order one factors).

Let us now put these remarks together. Since the parenthesis on the left-hand side is $\ll 1/l_s^2$, and the right-hand side is $> 1/l_s^2$, we have

$$
e^{\phi} \ll 1 \ . \qquad (2.7)
$$

For generic solutions, the parenthesis on the left hand side of (2.4) will be of order $1/R^2$, where R is a local radius of curvature. In that case, we can estimate, then,

$$
e^{\phi} \lesssim \frac{l_s}{R} \, , \qquad (2.8)
$$

which of course agrees with (2.7).

JHEP11(2010)047

When $F_0 = 0$, the conclusion (2.7) is not valid because all the remaining terms on the right hand side can be made small, in spite of flux quantization. For example, assume all the components of the metric are of the same order $1/R^2$ everywhere, and that $H = 0$. Then, the integral of F is an integer n_k, but the value of F_k^2 at a point will be of order $(n_k/R^k)^2$ (in string units). At large R, this can be made arbitrarily small. This is what happens in most type IIA flux compactifications with $F_0 = 0$; the dilaton can then be made large, and the limit $\phi \to \infty$ reveals a new phase of string theory, approximated by eleven-dimensional supergravity.

To summarize, we have shown that $F_0 \neq 0$ implies that the dilaton is small (2.7), as long as the two-derivative action (the supergravity approximation) is valid.

3 The $\mathcal{N} = 1$ solutions

In this section, we will see how the general arguments of section 2 are implemented in the $\mathcal{N} = 1$ vacua of [12].

3.1 The $\mathcal{N} = 1$ solutions

We recall here briefly the main features of the $\mathcal{N} = 1$ solutions in [12] on $\mathrm{AdS}_4 \times \mathbb{CP}^3$.

The metric is simply a product:

$$ds^2_{\mathcal{N}=1} = ds^2_{\mathrm{AdS}_4} + ds^2_{\mathbb{CP}^3,\mathcal{N}=1}\,. \tag{3.1}$$

Topologically, $\mathbb{CP}^3$ is an S^2 fibration over S^4. We use this fact to write the internal metric as

$$ds^2_{\mathbb{CP}^3,\mathcal{N}=1} = R^2\left(\frac{1}{8}\left(dx^i + \epsilon^{ijk}A^j x^k\right)^2 + \frac{1}{2\sigma}ds^2_{S^4}\right)\,, \tag{3.2}$$

where x^i are such that $\sum_{i=1}^3 (x^i)^2 = 1$, A^i are the components of an SU(2) connection on S^4 (with $p_1 = 1$), and $ds^2_{S^4}$ is the round metric on S^4 (with radius one). R is an overall radius, related to the AdS radius by

$$R_{\mathrm{AdS}} \equiv L = \frac{R}{2}\sqrt{\frac{5}{(2\sigma+1)}}\,. \tag{3.3}$$

The parameter σ in (3.2) is in the interval $[2/5, 2]$; this implies, in particular, that L/R is of order 1 for these $\mathcal{N} = 1$ solutions. For $\sigma = 2$, (3.2) is the usual Fubini-Study metric, whose isometry group is $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$. For $\sigma \neq 2$, the isometry group is simply the SO(5) that rotates the base S^4.

The metric (3.1) depends on the two parameters L and σ. A third parameter in the supergravity solution is the string coupling g_s. Yet another parameter comes from the B field. For $2/5 < \sigma < 2$, supersymmetry requires the NS-NS 3-form H to be non-zero (see [12, eq. (2.2)]). One can solve that constraint by writing

$$B = -\frac{\sqrt{(2-\sigma)(\sigma-2/5)}}{\sigma+2}\,J + \beta \tag{3.4}$$

where $J = \frac{i}{2}(e^i \wedge \overline{e^i})$ is the Hermitian form (an analogue of the Kähler form: e^i, $i = 1, 2, 3$ are $(1, 0)$ vielbeine), and β is a closed two-form [12, eq. (4.5)]. Because of gauge invariance $B \cong B + d\lambda_1$, the space of such β is nothing but the second de Rham cohomology of the internal space, $H^2(\mathbb{C}P^3) = \mathbb{R}$. So we have one such parameter, which we can take to be the integral of β over the generating two-cycle in H_2,

$$b \equiv \frac{1}{(2\pi l_s)^2} \int_{\mathbb{CP}^1} \beta \,, \tag{3.5}$$

where we normalized b so that large gauge transformations shift it by an integer.

To summarize, the $\mathcal{N} = 1$ supergravity solutions depend on the four parameters (L, σ, g_s, b).

3.2 Inverting the flux quantization equations

We now apply the flux quantization conditions (2.6). It is convenient to separate the contribution from the zero-mode β:

$$\tilde{F}_k = e^{-\beta} \tilde{F}_k|_{\beta=0} \;; \tag{3.6}$$

we then define $\int \tilde{F}_k|_{\beta=0} \equiv n_k^b (2\pi l_s)^{k-1}$, which can be computed explicitly [12]. We have

$$\begin{pmatrix} \frac{1}{l g_s} f_0(\sigma) \\ \frac{l}{g_s} f_2(\sigma) \\ \frac{l^3}{g_s} f_4(\sigma) \\ \frac{l^5}{g_s} f_6(\sigma) \end{pmatrix} = \begin{pmatrix} n_0^b \\ n_2^b \\ n_4^b \\ n_6^b \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ \frac{1}{2}b^2 & b & 1 & 0 \\ \frac{1}{6}b^3 & \frac{1}{2}b^2 & b & 1 \end{pmatrix} \begin{pmatrix} n_0 \\ n_2 \\ n_4 \\ n_6 \end{pmatrix} , \tag{3.7}$$

where

$$l = L/(2\pi l_s) \,, \tag{3.8}$$

and

$$f_0(\sigma) = \frac{5}{4}\sqrt{\frac{(2-\sigma)(5\sigma-2)}{(2\sigma+1)}} \,, \qquad f_4(\sigma) = -\frac{2^5\pi^2}{3\cdot 5^2}\frac{(\sigma-1)(2\sigma+1)^{5/2}}{\sigma^2(\sigma+2)^2}\sqrt{(2-\sigma)(5\sigma-2)} \,,$$
$$f_2(\sigma) = \frac{8\pi}{\sqrt{5}}\frac{(\sigma-1)}{(\sigma+2)}\sqrt{2\sigma+1} \,, \qquad f_6(\sigma) = -\frac{2^7\pi^3}{3\cdot 5^{7/2}}\frac{(\sigma^2-12\sigma-4)(2\sigma+1)^{7/2}}{\sigma^2(\sigma+2)^2} \,. \tag{3.9}$$

Equation (3.7) is [14, eq. (4.26)], which in this paper we chose to reexpress in terms of l (the AdS radius in string units) rather than r (the internal size in string units), to harmonize notation with section 5.

We want to invert these formulas and get expressions for the parameters (l, g_s, σ, b) in terms of the flux integers n_i, as explicitly as possible. If one assumes $b = 0$, this is easy [12];

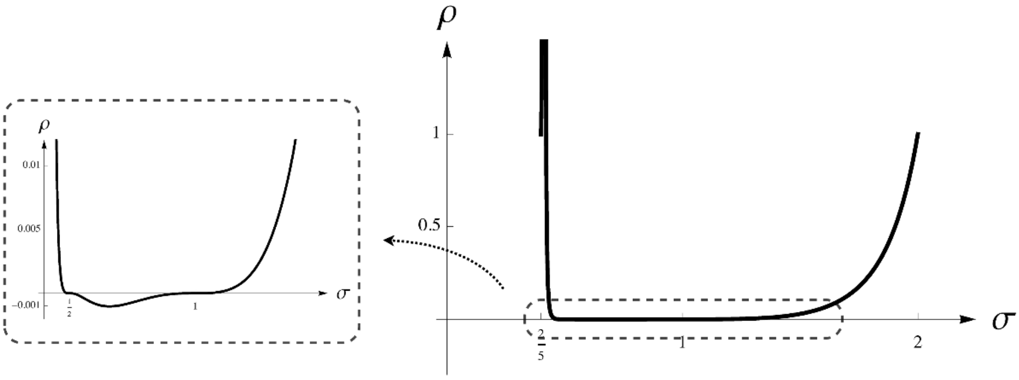

Figure 1. A plot of the function $\rho(\sigma)$ in (3.13).

with $b \neq 0$, it is a bit more complicated. A good strategy is to consider combinations of the flux integers that do not change under changes of the b field: in addition to n_0, two other combinations are

$$(n_2^b)^2 - 2n_0 n_4^b = n_2^2 - 2n_0 n_4 \,, \qquad (n_2^b)^3 + 3n_0^2 n_6^b - 3n_0 n_2^b n_4^b = n_2^3 + 3n_0^2 n_6 - 3n_0 n_2 n_4 \,. \tag{3.10}$$

We then find

$$n_2^2 - 2n_0 n_4 = (f_2^2 - 2f_0 f_4)\left(\frac{l}{g_s}\right)^2 = \frac{16}{15}\pi^2 \left(\frac{l}{g_s}\right)^2 \frac{(\sigma-1)(4\sigma^2-1)}{\sigma^2} \,, \tag{3.11}$$

$$\begin{aligned} n_2^3 + 3n_0^2 n_6 - 3n_0 n_2 n_4 =& (f_2^3 + 3f_0^2 f_6 - 3f_0 f_2 f_4)\left(\frac{l}{g_s}\right)^3 \\ =& \frac{8\pi^3}{5^{3/2}}\left(\frac{l}{g_s}\right)^3 \frac{(-6+17\sigma-6\sigma^2)(2\sigma+1)^{3/2}}{\sigma^2} \,. \end{aligned} \tag{3.12}$$

We see that (3.11) and (3.12) give two independent expressions for l/g_s; this implies

$$\frac{(n_2^2 - 2n_0 n_4)^3}{(n_2^3 + 3n_0^2 n_6 - 3n_0 n_2 n_4)^2} = \frac{64(\sigma-1)^3(2\sigma-1)^3}{27\sigma^2(-6+17\sigma-6\sigma^2)^2} \equiv \rho(\sigma) \,. \tag{3.13}$$

This determines σ implicitly in terms of the fluxes. The function $\rho(\sigma)$ (which we plot in figure 1) diverges at $\sigma = \frac{17-\sqrt{145}}{12} \sim .41$, and has zeros at $\sigma = \frac{1}{2}$ and $\sigma = 1$. These zeros have multiplicity three, and hence they are also extrema and inflection points. Moreover, it has a minimum at $\sigma \sim .65$; and it goes to 1 for both $\sigma = 2$ and $\sigma = \frac{2}{5}$.

We can now combine the equation for n_0 in (3.7), which determines $g_s l$, with the expression for l/g_s in either (3.11) or (3.12). We prefer using the latter, since it turns out to contain functions of σ which are of order one on most of the parameter space:

$$l = \frac{5^{3/4}}{2^{3/2}\sqrt{\pi}} \frac{(2-\sigma)^{1/4}(5\sigma-2)^{1/4}\sigma^{1/3}}{(2\sigma+1)^{1/2}(-6+17\sigma-6\sigma^2)^{1/6}} \left(\frac{n_2^3}{n_0^3} + 3\frac{n_6}{n_0} - 3\frac{n_2 n_4}{n_0^2}\right)^{1/6} \,, \tag{3.14}$$

$$g_s = 5^{1/4}\sqrt{\frac{\pi}{2}} \frac{(2-\sigma)^{1/4}(5\sigma-2)^{1/4}(-6+17\sigma-6\sigma^2)^{1/6}}{\sigma^{1/3} n_0^{\frac{1}{2}} (n_2^3 + 3n_0^2 n_6 - 3n_0 n_2 n_4)^{1/6}} \,. \tag{3.15}$$

The function in the expression for l diverges at $\sigma = \frac{17-\sqrt{145}}{12} \sim .41$ and vanishes for $\sigma = \frac{2}{5}$ and 2, whereas the function in the expression for g_s vanishes for $\sigma = \frac{2}{5}, \frac{17-\sqrt{145}}{12}$ and 2.

Finally, the second row of equation (3.7) determines b in terms of n_2, n_0 and the remaining fields l, g_s and σ. One could eliminate l and g_s from that expression using (3.14) and (3.15), but we will not bother to do so.

3.3 A phase transition

We will start by taking for simplicity

$$n_4 = 0 \,, \tag{3.16}$$

and we will call

$$n_2 \equiv k \,, \qquad n_6 \equiv N \tag{3.17}$$

as in [11].

In this case, (3.13) reads

$$\rho(\sigma) = \left(1 + 3\frac{Nn_0^2}{k^3}\right)^{-2} \,. \tag{3.18}$$

From the graph in figure 1, we see that the behavior of the solution depends crucially on the ratio $\frac{Nn_0^2}{k^3}$. If for example

$$N \ll \frac{k^3}{n_0^2} \,, \tag{3.19}$$

we have $\rho(\sigma) \sim 1$. Looking at figure 1, we see that a possible solution is $\sigma = 2$. Around this point, ρ goes linearly; so, if we write $\sigma = 2 - \delta\sigma$, we have $\delta\sigma \sim \frac{Nn_0^2}{k^3}$. From (3.14) and (3.15) we then have

$$l \sim \delta\sigma^{1/4}\left(\frac{k}{n_0}\right)^{1/2} = \frac{N^{1/4}}{k^{1/4}} \,, \qquad g_s \sim \delta\sigma^{1/4}(kn_0)^{-1/2} = \frac{N^{1/4}}{k^{5/4}} \,. \tag{3.20}$$

This is the same behavior as in the $\mathcal{N} = 6$ solution [11].

If, on the other hand,

$$N \gg \frac{k^3}{n_0^2} \,, \tag{3.21}$$

we have $\rho(\sigma) \sim 0$. The possible solutions are $\sigma \simeq 1$ or $\sigma \simeq \frac{1}{2}$. The σ-dependent functions in the expressions for l and g_s in (3.14) and (3.15) are then both of order one. We have

$$l \sim \frac{N^{1/6}}{n_0^{1/6}} \,, \qquad g_s \sim \frac{1}{N^{1/6}n_0^{5/6}} \,. \tag{3.22}$$

Notice that this behavior occurs for example in the nearly Kähler solutions of [21]. For those vacua, we have $l^5/g_s = n_6$ and $1/(lg_s) = n_0$, which gives the same behavior as in (3.22). Notice also that $\sigma = 1$ corresponds indeed to a nearly Kähler metric.

JHEP11(2010)047

Figure 2. The behavior of g_s as a function of $N = n_6$, for $n_2 = k = 100$, $n_4 = 0$ and $n_0 = 1$. We see both the growth in the first phase (3.20), for $n_6 \ll n_2^3/n_0^2 = 10^6$, and the decay in the second phase (3.22), for $n_6 \gg n_2^3/n_0^2 = 10^6$.

If one were to find a Chern-Simons dual to a vacuum whose only relevant fluxes are n_6 and n_0, such as the nearly Kähler solutions, it would be natural to identify n_6 with a rank N and n_0 with a Chern-Simons coupling $\tilde{k}$ (because F_0 induces a Chern-Simons coupling on D2-branes). In such a dual, $\frac{n_6}{n_0} = \frac{N}{\tilde{k}} \equiv \tilde{\lambda}$ would then be the new 't Hooft coupling. We see then that l and $g_s N$ in (3.22) are both functions of this $\tilde{\lambda}$, as expected.

From (3.22) one can calculate the finite temperature free energy to be $\beta F \sim V_2 T^2 \frac{N^2}{\tilde{\lambda}^{1/3}} \sim V_2 T^2 N^{5/3} n_0^{1/3}$, which grows with a higher power of N than in the massless case, for which at strong coupling $\beta F \sim V_2 T^2 N^{3/2} k^{1/2}$.

In figure 2 we show a graph of g_s as a function of N; we see both behaviors (3.20) and (3.22).

Our analysis above was limited for simplicity to the case $n_4 = 0$, but it is easy to argue that also for other values of n_4, g_s cannot become large. Equation (3.15) tells us that $g_s = f(\sigma)/n_0^{1/2} m^{1/6}$, where $f(\sigma)$ is bounded from above in the relevant range of values, and $m \equiv n_2^3 + 3n_0^2 n_6 - 3n_0 n_2 n_4$ is an integer. Thus, if $m \neq 0$, then g_s is clearly bounded from above in the massive theory by the maximal value of $|f(\sigma)|$. If $m = 0$, then (3.12) implies that $(-6 + 17\sigma - 6\sigma^2)$ also vanishes, and we can then use (3.13) to rewrite (3.15) in the form $g_s = \tilde{f}(\sigma)/n_0^{1/2} \tilde{m}^{1/4}$, where $\tilde{f}(\sigma)$ is again bounded in the relevant range and $\tilde{m} \equiv n_2^2 - 2n_0 n_4$ is another integer. Thus, if $\tilde{m} \neq 0$ then g_s is bounded from above by the maximal value of $|\tilde{f}(\sigma)|$, but this must be true since m and $\tilde{m}$ cannot vanish at the same time (as is clear from (3.11) and (3.12)). Thus, for any integer fluxes with $n_0 \neq 0$, g_s is bounded from above by a number of order one.

3.4 Probes

We will now see that the "phase transition" between (3.20) and (3.22) has a sharp consequence on the behavior of the probe branes in the geometry. We will consider branes which are particles in AdS$_4$ and that wrap different cycles in the internal space $\mathbb{CP}^3$.

JHEP11(2010)047

Not all such wrapped branes are consistent. In the $\mathcal{N}=6$ case, where $F_0=0$ and $\int_{\mathbb{CP}^1} F_2 = n_2 \neq 0$, the action for a D2-brane particle wrapping the internal $\mathbb{CP}^1$ has a tadpole for the world-sheet gauge field $\mathcal{A}$, because of the coupling

$$\frac{1}{2\pi l_s}\int_{\mathbb{R}\times\mathbb{CP}^1} \mathcal{A}\wedge F_2 = n_2 \int_{\mathbb{R}} \mathcal{A} \tag{3.23}$$

(the $\mathbb{R}$ factor in the D2-brane worldvolume being time). D0-branes, in contrast, have no such problem. In the field theory, they correspond [11] to gauge-invariant operators made of monopole operators and bifundamentals.

For the solutions with both $F_0 \simeq n_0 \neq 0$ and $\int_{\mathbb{CP}^1} F_2 \simeq n_2 \neq 0$, both D2's and D0's have a tadpole. If one considers a bound state of $n_{\rm D2}$ D2 branes and $n_{\rm D0}$ D0 branes, the tadpole for $\mathcal{A}$ is

$$(n_{\rm D2} n_2 + n_{\rm D0} n_0)\int_{\mathbb{R}} \mathcal{A}\ . \tag{3.24}$$

For relatively prime n_0 and n_2, the minimal choice that makes this vanish is $n_{\rm D2}=n_0$ and $n_{\rm D0}=-n_2$. These branes also correspond to a mix of monopole operators and bifundamentals; we will discuss analogous configurations in more detail in section 5.4.

Consider now the case $n_0=1$, and $n_2=k\gg 1$. Here we should consider a bound state of one D2 brane and k D0 branes. In the context of AdS/CFT, all masses are naturally measured in units of the AdS mass scale $m_{\rm AdS}\equiv \frac{1}{R_{\rm AdS}}=\frac{1}{L}$; recall also from (3.3) that R is of order L. The masses of a D2 and of a D0 particle would then be (setting the string scale to one)

$$m_{\rm D2}L \sim \frac{L^2/g_s}{1/L}=\frac{L^3}{g_s}\,, \qquad m_{\rm D0}L\sim \frac{1/g_s}{1/L}=\frac{L}{g_s}\ . \tag{3.25}$$

Thus, the bound states we are considering here (the particles that have no world-sheet tadpole) have a mass of order

$$m_{{\rm D2}-k{\rm D0}} = \frac{L}{g_s}\sqrt{k^2+L^4}\ . \tag{3.26}$$

Which of the two terms dominates? it turns out that the answer depends on which of the two phases, (3.20) or (3.22), we are considering. In both phases the ratio of the two masses is a function of $\frac{N}{k^3}$.

A simple computation gives that, in the first phase (3.20), the D2's mass is $\sim \sqrt{Nk}$, whereas the k D0 branes have mass $k\times k$. The D0's dominate the mass, which then goes like $k^2\sqrt{1+\frac{N}{k^3}}$.

In the second phase, the D2's mass is $\sim N^{2/3}$, whereas the k D0's mass goes like $k\times N^{1/3}$. Hence the D2 dominates. The mass then goes like $N^{2/3}\sqrt{1+\left(\frac{N}{k^3}\right)^{-2/3}}$.

Another type of branes that have no tadpole problems are D4 branes. In the field theory, these correspond [11] to baryon operators. In AdS units, these have a mass of order $\frac{L^4/g_s}{1/L}=\frac{L^5}{g_s}$. Interestingly, this turns out to be of order N in both phases (3.20) and (3.22), which looks reasonable for a baryon.

3.5 Field theory interpretation

The field theories dual to the vacua analyzed in this section were proposed in [14]. Because of the low amount of supersymmetry, we do not expect to be able to make here any useful check of this duality. However, we can use our gravity results to make some predictions about those field theories, under some assumptions.

First of all, let us recall briefly the $\mathcal{N}=1$ field theories defined in [14]. They are similar to the $\mathcal{N}=6$ theory of [11, 22], in that they also have a gauge group $\mathrm{U}(N_1)\times\mathrm{U}(N_2)$. The matter content can be organized in (complexified) $\mathcal{N}=1$ superfields X^I, $I=1,\ldots,4$; they transform in the $(\bar{N}_1,N_2)$ representation of the gauge group. The biggest difference is that the Chern-Simons couplings for the two gauge groups are now unrelated: we will call them k_1 and $-k_2$. For $k_1\neq k_2$, it is no longer possible to achieve $\mathcal{N}=6$ supersymmetry, and there are several choices as to the amount of flavor symmetry and supersymmetry that one can preserve. In this section, we consider a choice that leads to $\mathcal{N}=1$ supersymmetry and SO(5) flavor symmetry; in the following sections we will consider a different choice, that leads to $\mathcal{N}=2$ and SO(4) flavor symmetry.

This theory can be written in terms of $\mathcal{N}=1$ superfields; the superpotential then reads $W_{\mathcal{N}=1}=\mathrm{Tr}[c_1X^\dagger_IX^IX^\dagger_JX^J+c_2X^\dagger_IX^JX^\dagger_JX^I+c_3\omega^{IK}\omega_{JL}X^\dagger_IX^JX^\dagger_KX_L]$. Notice that all the terms are manifestly invariant under Sp(2)=SO(5), as promised. When $k_1=k_2\equiv k$, the theory has $\mathcal{N}=6$ supersymmetry when the parameters are $c_1=-c_2=2\pi/k$, $c_3=-4\pi/k$. For $k_1\neq k_2$, this choice is no longer possible, as we already mentioned. In spite of there being only $\mathcal{N}=1$ supersymmetry, however, it was argued in [14] that there still exists a choice of c_i that makes the theory superconformal, as long as k_1-k_2 is small enough with respect to the individual k_i.

If we define the 't Hooft couplings

$$\lambda_1=\frac{N}{k_1}\,,\qquad \lambda_2=\frac{N}{-k_2}\,,\qquad \lambda_\pm=\lambda_1\pm\lambda_2\,, \tag{3.27}$$

the $\mathcal{N}=6$ theory would correspond to $\lambda_+=0$. The argument in [14] then says that there is a CFT in this space of theories if $\lambda_+\ll\lambda_-$, although at strong coupling it is difficult to quantify just how much smaller it has to be.

Let us now try to translate in terms of these field theories the "phase transition" we saw in section 3.3. To do so, we can use the dictionary (5.35) between the field theory ranks and levels on one side, and flux integers on the other. This dictionary is also valid for $\mathcal{N}=1$ theories [14]. The phase transition in section 3.3 happens for $N\sim n_2^3/n_0^2$. Since

$$\frac{k_1\mp k_2}{N}=\frac{1}{\lambda_1}\pm\frac{1}{\lambda_2}=\frac{\pm4\lambda_\pm}{\lambda_+^2-\lambda_-^2}\,, \tag{3.28}$$

when $\lambda_+\ll\lambda_-$ we have $n_0/N\sim\lambda_+/\lambda_-^2$, $n_2/N\sim1/\lambda_-$. So the phase transition happens at

$$\lambda_-\sim\lambda_+^2\,. \tag{3.29}$$

In particular, the "ABJM phase" (3.20) corresponds to $\lambda_+ \ll \sqrt{\lambda_-}$; the "nearly-Kähler" phase (3.22) corresponds to $\lambda_+ \gg \sqrt{\lambda_-}$. At strong coupling, then, there is an intermediate regime where $\sqrt{\lambda_-} \ll \lambda_+ \ll \lambda_-$, where it is possible that the second phase (3.22) is also described by the field theories described in [14] and reviewed in this section. However, given the low amount of supersymmetry, this can only be a conjecture at this point.

Rather than trying to test further this correspondence, we will now turn our attention to $\mathcal{N} = 2$ theories, on which there is much better control.

4 Monopoles in $\mathcal{N} = 2$ Chern-Simons-matter theories

In this section, we will recall some general facts about monopole operators in Chern-Simons-matter theories, and we will apply them to a particular quiver theory, similar to the ABJM theory; its gravity dual will be examined in section 5.

4.1 Construction of monopole operators in general

Consider a $d = 3$ gauge theory with gauge group $\prod_{i=1}^m \mathrm{U}(N_i)$. Then there are m currents, $j_i = * \,\mathrm{Tr}\,(F_i)$, which are conserved by the Bianchi identity. If the theory flows to a CFT in the IR, then these must be dimension 2 operators in the IR. There may or may not be operators charged under the corresponding $\mathrm{U}(1)^m$ flavor symmetry; if they exist we will call them monopole operators.

In a conformal field theory, it is convenient to use radial quantization and consider the theory on $\mathbb{R} \times S^2$. Let us apply the state-operator correspondence to a monopole operator, with charge vector n_i. This results in a state in the theory on an S^2, such that $\int_{S^2} \mathrm{Tr}\,(F_i) = 2\pi n_i$. We will denote the diagonal values of F_i by w_i^a:

$$F_i = \mathrm{diag}(w_i^1, \dots, w_i^{N_i})\mathrm{vol}_{S^2} \ ; \tag{4.1}$$

taking the trace over the gauge group $\mathrm{U}(N_i)$, we have that the magnetic charges are

$$n_i = \sum_a w_i^a \ . \tag{4.2}$$

We are interested in $d = 3$ $\mathcal{N} = 2$ Chern-Simons-matter theories. We can take them to be weakly interacting at short distances by adding Yang-Mills terms as regulators [23–26]. In such a regulated theory, there are BPS classical configurations with the gauge field as in (4.1) and non-trivial values for the scalar fields. The BPS equations on $\mathbb{R}^{1,2}$ include the Bogomolnyi equations $F_i = *D\sigma_i$, where σ_i is the adjoint scalar in the vector multiplet. In $\mathbb{R}\times S^2$, this equation is different because the metric needs to be rescaled, and the fields need to be transformed accordingly; the equations then read $F_i = \sigma_i \mathrm{vol}_{S^2}$. Notice in particular that the σ_i are constant. There are also other BPS equations, which involve the other scalars in the theory (for explicit computations for $\mathcal{N} = 3$ theories, see [24, §3.2], and in $\mathcal{N} = 2$ language, [27]).

After adding the regulating Yang-Mills term, g_{YM}^2 becomes small in the UV, so the $\mathcal{N}=2$ vector multiplet should be treated classically, while the chiral matter fields should not. This justifies not solving Gauss's law in describing the "pure" monopole operator, which then behaves as if it were a local, non-gauge invariant chiral field [24].

The scalar σ_i is set by the BPS equations to be equal to the inverse Chern-Simons level times the moment maps of the matter fields,

$$\sigma_i = 2\pi \frac{D_i}{k_i} , \tag{4.3}$$

and to have a spin 0 operator, one should satisfy the constraint [27]

$$\sigma_i X_{ij} = X_{ij} \sigma_j \tag{4.4}$$

for any bi-fundamental field X_{ij} connecting the i-th and j-th gauge group [28–30]. In addition, the fields X_{ij} need to satisfy the F-term equations of the $\mathcal{N}=2$ theory. We see from (4.3) that some matter fields, which should be neutral under the background U(1), are necessarily non vanishing in the background; hence, the possibility of satisfying all equations gives a nontrivial constraint on the possible BPS monopole operators.

In general, monopole operators T creating such configurations at a point will not be gauge invariant. However, they will behave exactly like local fields. Hence, they can be combined with other local operators $\mathcal{O}$, to write gauge-invariant expressions of the schematic form

$$\mathrm{Tr}\,(T\mathcal{O}) , \tag{4.5}$$

where the indices are contracted as appropriate for the representations in which the operators transform.

Let us determine how the monopole transforms under the gauge group. This is easy to find for the Abelian factors of the theory. There is an obvious contribution to the electric charges of a monopole operator, from the Chern-Simons term $\sum \frac{k_i}{4\pi} \int \mathrm{Tr}\,(A_i \wedge F_i)$. This says that a monopole with magnetic charges n_i will behave like a particle with charges $n_i k_i$ under the i^{th} electric U(1) Abelian factor.

This result gives a constraint on the possible gauge invariant operators $\mathrm{Tr}\,(T\mathcal{O})$ we can obtain. If all matter is in bifundamental and adjoint representations, no gauge invariant operators can be formed from monopoles that are charged under the overall U(1), since no matter field transforms under it. Since we just computed the electric charge under the i^{th} U(1) to be $k_i n_i$, the charge under the overall U(1) is $\sum k_i n_i$. Thus, if we are to form any gauge invariant operators of the form (4.5), we need to require

$$\sum_{i=1}^{m} k_i n_i = 0 . \tag{4.6}$$

This result will be useful in the theories with gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$, which we will discuss in section 4.2.

Let us now ask how the monopole will transform under the full non-Abelian group. One method to determine this is the following. One fixes a particular configuration on the sphere (breaking the gauge symmetry), one computes the charge under the whole Cartan subalgebra of the gauge group, and then one integrates over the gauge orbits. Thus monopole operators can be labelled by U(1) subgroups of the gauge groups. For the $\mathrm{U}(N_i)$ factor of the gauge group, the charges under this Cartan subalgebra are the w_i^a, with $a = 1, ..., N_i$, that we saw in (4.1).

Therefore, a monopole associated with magnetic flux w_i^a is in the representation with weight vector [31, section 4.2]

$$(k_i w_i^1, \dots, k_i w_i^{N_i}) \tag{4.7}$$

of the i^{th} gauge group. (Our notation here is that the weight vector denotes the number of boxes in each row of a Young diagram of the representation; thus $(k, 0, \dots, 0)$ is, for example, the completely symmetric representation.)

Quantization in a background of the type discussed in this section can result in anomalous contributions to the charges and energy of the state. For the non-chiral theories we we will consider in section 4.2, there is no such a correction to the gauge charges.[3] However, as we will discuss in section 4.2, the dimension of the monopole operator is given by the energy of the state on the sphere, which often includes a non-zero Casimir energy.

4.2 Dimensions and charges of the monopoles

We will apply in this section the results of section 4.1 to the Chern-Simons-matter theories with $\mathcal{N} = 2$ and $\mathcal{N} = 3$ given in [14]. In particular, we will compute the dimensions of particular monopoles, which will be useful later, when comparing to the gravity solutions of section 5.

Let us first recall some details about the field theories of interest. They are similar to the $\mathcal{N} = 6$ theory of [11, 22], in that they also have a gauge group $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$, and $\mathcal{N} = 2$ "chiral" superfields A_i, B_i, $i = 1, 2$; the A_i transform in the $(\bar{N}_1, N_2)$ representation, whereas the B_i transform in the $(N_1, \bar{N}_2)$. Just as in section 3.5, the crucial difference between the $\mathcal{N} = 6$ theory and the $\mathcal{N} = 2$ theory is that the Chern-Simons couplings for the two gauge groups are now unrelated; we again call them k_1 and $-k_2$. The theories we want to consider in this section are defined by the superpotential

$$W = \mathrm{Tr}(c_1 (A_i B_i)^2 + c_2 (B_i A_i)^2) \ . \tag{4.8}$$

For generic c_i, the theory has $\mathcal{N} = 2$ supersymmetry and SU(2) flavor symmetry. For $c_i = \frac{1}{k_i}$, supersymmetry turns out to be enhanced to $\mathcal{N} = 3$, while the flavor symmetry is

[3] If the matter content is chiral, as for the theories in [15], there will be an additional one-loop correction to the gauge charges. One way to understand this effect is that the state on the sphere has a constant value for the scalar in the vector multiplet, which gives a mass to any matter fields charged under that U(1) subgroup in which the magnetic flux lives. Integrating them out at one-loop can shift the effective Chern-Simons level in that background if the theory has chiral matter.

still SU(2). For $c_1 = -c_2 = c$, the supersymmetry stays $\mathcal{N} = 2$, but W can be rewritten as

$$W = c\mathrm{Tr}(\epsilon^{ij}\epsilon^{kl} A_i B_k A_j B_l)\,, \tag{4.9}$$

which shows that the flavor symmetry is enhanced to SU(2) × SU(2). This $\mathcal{N} = 2$ theory is dual to the gravity solution discussed in the next section.

We will now apply to these theories the discussion of section 4.1 about monopole operators. The following computation is a straightforward generalization of that done in [23, 24] for the $\mathcal{N} = 3$ theory. The results for the $\mathcal{N} = 3$ and $\mathcal{N} = 2$ theories appear to be identical, since the flavor symmetry guarantees that the matter fields have the same dimensions as in the more supersymmetric theory.

As we saw in section 4.1, there are non-trivial conditions on the scalars for the monopole to be BPS. For the theories we are considering, the conditions read

$$\begin{aligned} A\,A^\dagger - B^\dagger B =& \frac{k_1}{2\pi}\sigma_1\,, \\ B\,B^\dagger - A^\dagger A =& -\frac{k_2}{2\pi}\sigma_2\,, \\ \sigma_1 A_i =& A_i\sigma_2\,, \\ \sigma_2 B_i =& B_i\sigma_1\,, \end{aligned} \tag{4.10}$$

together with the F-term constraints coming from (4.8). In view of (4.1) and $F_i = \sigma_i \mathrm{vol}_2$, such equations relate the magnetic fluxes w_1^a with the w_2^a. The simplest monopoles we can consider are defined by magnetic charges w_1^a which are all either 0 or 1: namely, $w_1 = (1, \ldots, 1, 0, \ldots)$ (with n_1 1's) and $w_2 = (1, \ldots, 1, 0, \ldots)$ (with n_2 1's). The non-zero elements of the fields A_i and B_i are $n_1 \times n_2$ and $n_2 \times n_1$ rectangular matrices, respectively, which are required to satisfy the first two lines in (4.10) and the F-term constraints. The problem of finding appropriate vacuum expectation values for the matter fields is equivalent to finding the BPS moduli space of the generalized $\mathrm{U}(n_1)\times\mathrm{U}(n_2)$ Klebanov-Witten theory with superpotential (4.8) and Fayet-Iliopoulos (FI) parameters turned on. Many of these moduli spaces are non-empty.

Using now (4.6), we see that this monopole operator can be coupled to elementary fields in a gauge invariant way only if

$$n_1 k_1 = n_2 k_2\;. \tag{4.11}$$

The matter content is non-chiral, so there are no anomalous contributions to the gauge charges of the monopole. There is, however, a one-loop correction to the dimension of the operator, which is given by [24–26]

$$\begin{aligned} \Delta &= -\frac{1}{2}\sum_{\text{fermions}} |q| R \\ &= -\frac{1}{2}\left[2\times 1\left(n_1(N_1-n_1) + n_2(N_2-n_2)\right) - 4\times\frac{1}{2}\left(n_1(N_2-n_2) + n_2(N_1-n_1)\right)\right] \\ &= (n_1-n_2)^2 - (N_1-N_2)(n_1-n_2)\,, \end{aligned} \tag{4.12}$$

where R is the R-charge of the fermion and q the charge under the U(1) subgroup specified by the vectors $w_1 = (1, ..{}_{n_1}.., 1, 0, \ldots)$ and $w_2 = (1, ..{}_{n_2}.., 1, 0, \ldots)$. The various contributions arise as follows. The four bi-fundamental fermions have R-charge $-1/2$.[4] Each bi-fundamental fermion is a matrix with N_1N_2 entries; the $n_1(N_2 - n_2) + n_2(N_1 - n_1)$ off-diagonal entries have charge ± 1 under the magnetic U(1) subgroup, while the remaining entries are neutral. The two adjoint gauginos have R-charge $+1$. They are square matrices with N_i^2 entries; the $2n_i(N_i - n_i)$ off-diagonal entries of the i-th fermion have charge ± 1, while the other are neutral. We used the fact that in this theory, both for $\mathcal{N} = 3$ and $\mathcal{N} = 2$, the R-charges in the UV and IR are identical.

In the $\mathcal{N} = 6$ theory, $k_1 = k_2$ and it follows from (4.11) that $n_1 = n_2$. The simplest monopole has just $w_1 = w_2 = (1, 0, \ldots, 0)$. We need to turn on A_i and B_i fields that solve the U(1) Klebanov-Witten theory with a FI term. According to (4.7), such a monopole transforms in the k-fold symmetric representation of $\mathrm{U}(N_1)$ and in the conjugate k-fold symmetric representation of $\mathrm{U}(N_2)$. The monopole can combine with k fields A_i to form a gauge-invariant operator (we can analogously form a gauge-invariant operator with the conjugate monopole and k fields B_i).

In the $\mathcal{N} = 3$ and $\mathcal{N} = 2$ theories, we cannot have $n_1 = n_2$, but we can now take $n_1 = k_2$ and $n_2 = k_1$ and rectangular matrices A_i and B_i that solve (4.10). In general, the matrices A_iB_j will not be diagonalizable. Recalling equation (4.7), the monopole operator is in a representation of the gauge group with weight vectors $(k_1, ..{}_{k_2}.., k_1, 0, \ldots)$ and $(k_2, ..{}_{k_1}.., k_2, 0, \ldots)$. A gauge invariant combination must include k_1k_2 matter bifundamentals (if k_1 and k_2 are not relatively prime, some operator with smaller dimension could exist). The total dimension of the gauge-invariant operator, dressed with k_1k_2 elementary fields, is then

$$\Delta = \frac{k_1k_2}{2} + (k_2 - k_1)^2 - (k_2 - k_1)(N_1 - N_2)\ . \tag{4.13}$$

Note that we have determined the vacuum expectation values of the matter fields needed to "support" the flux to form a BPS state on the sphere using the classical moduli space. This is justified since the Higgs branch does not receive quantum corrections. More precisely, the ring of chiral operators is the ring of algebraic functions on the moduli space. There is a natural map [28–30] from the moduli space of Chern-Simons-matter theories to the moduli space of the four-dimensional Yang-Mills theory with the same field content. That moduli space cannot receive quantum corrections (aside from wavefunction renormalization which fixes the coefficients of the superpotential), and only the S^1 bundle over that space, associated to the dual gauge fields, is quantum corrected. This precisely corresponds to 1-loop corrections to the charges and dimensions of monopole operators, which are, however, constructed in the UV weakly coupled Yang-Mills-Chern-Simons-matter theory.

[4] Since the superpotential (4.8) must have R-charge $+2$ and there is a discrete symmetry between A_i and B_i we have that $R(A_i) = R(B_i) = 1/2$; the R-charge of the fermionic partners is $R(A) - 1$ by supersymmetry.

JHEP11(2010)047

Let us summarize the results of this section. The monopole operators that create k_2 units of flux for the first gauge group and k_1 for the second have k_1k_2 bifundamental indices, and hence we can contract them with k_1k_2 bifundamental fields to construct a gauge-invariant operator. Such operators have dimension given by (4.13). In particular, they stay light when $N_1 = N_2 \equiv N \to \infty$. Since in general monopole operators correspond to D-branes, this seems to indicate a limit where D-branes become light, which usually signals some sort of breakdown of the perturbative description. We will see in section 5.4 precisely how this happens.

5 The $\mathcal{N} = 2$ solution

We now turn to writing and studying the $\mathcal{N} = 2$ solution predicted to exist in [14], and found in [32] at first order in F_0. This solution will be the gravity dual of the field theory defined by the superpotential (4.9), and it will serve as another illustration of the general result of section 2.

We will start in section 5.1 by reducing the equations of motion and the supersymmetry conditions to a system of three equations for three functions of one variable. This procedure closely parallels [15], where an analogous solution for the gravity dual of the Chern-Simons theory based on the $\mathbb{C}^3/\mathbb{Z}_3$ quiver was found. In section 5.2, we will impose flux quantization, and derive expressions for the supergravity parameters in terms of the flux integers; in section 5.3 we find, just like in section 3.3, a "phase transition" that prevents the dilaton from growing arbitrarily large. Finally, in section 5.4 we find light D-brane states dual to the monopole operators discussed in section 4.2.

5.1 The $\mathcal{N} = 2$ solutions

The ten dimensional metric we will consider is a warped product of AdS_4 with a compact six-dimensional internal metric with the topology of $\mathbb{CP}^3$:

$$\mathrm{d}s^2_{10} = e^{2A}\mathrm{d}s^2_{AdS_4} + \mathrm{d}s^2_6 \ . \tag{5.1}$$

As discussed in [32, 33], there is a foliation of $\mathbb{CP}^3$ in copies of $T^{1,1}$, which is in turn a S^1 fibration over $S^2 \times S^2$. The usual Fubini-Study metric can be written as

$$\mathrm{d}s^2_6 = \frac{\cos^2(t)}{4}\mathrm{d}s^2_{S^2_1} + \frac{\sin^2(t)}{4}\mathrm{d}s^2_{S^2_2} + \mathrm{d}t^2 + \frac{1}{16}\sin^2(2t)(\mathrm{d}a + A_2 - A_1)^2 \,, \tag{5.2}$$

where A_i, $i = 1, 2$, are one-form connections, with curvatures

$$dA_i = J_i \,, \tag{5.3}$$

where J_i are the volume forms of the two spheres S^2_i. The coordinate t parametrizes the interval $[0, \pi/2]$; all the functions in our solution (including A in (5.1)) will depend on this coordinate alone. At one end of the interval $[0, \pi/2]$, one S^2 shrinks; at the other end, the other S^2 shrinks. To make this metric regular, we take the periodicity of a to be 4π.

The Fubini-Study metric is appropriate for the $\mathcal{N} = 6$ solution, which has $F_0 = 0$. Once we switch F_0 on, as we saw in section 4.2, AdS/CFT predicts the existence of an $\mathcal{N} = 2$ solution with isometry group $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ (the first two factors being the flavor symmetry which is manifest in (4.9), and the third being the R-symmetry). The internal metric for such a deformed solution is then given by[5]

$$ds_6^2 = \frac{e^{2B_1(t)}}{4} ds^2_{S_1^2} + \frac{e^{2B_2(t)}}{4} ds^2_{S_2^2} + \frac{1}{8}\,\epsilon^2(t) dt^2 + \frac{1}{64}\Gamma^2(t)(da + A_2 - A_1)^2 . \tag{5.4}$$

Were the functions e^{2B_i} non-vanishing, we would have a metric on the total space of an S^2 bundle over $S^2 \times S^2$. To maintain the topology of $\mathbb{CP}^3$, we require that e^{2B_2} vanishes at $t = 0$ and e^{2B_1} vanishes at $t = \pi/2$. With an abuse of language, we will refer to $t = 0$ as the North pole and $t = \pi/2$ as the South pole, although there is no real S^2 fiber. To have a regular metric, $\epsilon(t)$ and $\Gamma(t)$ must behave appropriately at the poles.

It is convenient to define the combinations

$$w_i = 4e^{2B_i - 2A} \tag{5.5}$$

which control the relative sizes of the two S^2's. As discussed in appendix A, the supersymmetry equations reduce to three coupled first order ordinary differential equations for w_1, w_2 and a third function ψ which enters in the spinors:

$$\begin{aligned}
\psi' &= \frac{\sin(4\psi)}{\sin(4t)} \frac{C_{t,\psi}(w_1 + w_2) + 2\cos^2(2t) w_1 w_2}{C_{t,\psi}(w_1 + w_2)\cos^2(2\psi) + 2w_1 w_2} ,\\
w_1' &= \frac{4w_1}{\sin(4t)} \frac{C_{t,\psi}(w_1 w_2 - 2\,w_2 - 2\sin^2(2\psi) w_1)}{C_{t,\psi}(w_1 + w_2)\cos^2(2\psi) + 2w_1 w_2} ,\\
w_2' &= \frac{4w_2}{\sin(4t)} \frac{C_{t,\psi}(w_1 w_2 - 2\,w_1 - 2\sin^2(2\psi) w_2)}{C_{t,\psi}(w_1 + w_2)\cos^2(2\psi) + 2w_1 w_2} ,
\end{aligned} \tag{5.6}$$

where

$$C_{t,\psi} \equiv \cos^2(2t)\cos^2(2\psi) - 1. \tag{5.7}$$

All other functions in the metric and the dilaton are algebraically determined in terms of w_1, w_2, ψ:

$$\epsilon = \sqrt{2} e^A (\cot(\psi) - \tan(\psi)) \frac{\csc^2(2t)\, \sin(2\psi) - \cos(2\psi)\, \cot(2t)\, \psi'}{2\sqrt{1 + \cot^2(2t)\, \sin^2(2\psi)}} \tag{5.8}$$

$$\Gamma = 4e^A \frac{\sin(2t) + \cos(2t)\cot(2t)\sin^2(2\psi)}{2\sqrt{1 + \cot^2(2t)\, \sin^2(2\psi)}} \tag{5.9}$$

[5]One could have reparameterized the coordinate t so as to set one of the functions in (5.4) to a constant, for example ϵ, as in (5.2). We have chosen, however, to fix this reparameterization freedom in another way: by choosing the pure spinors (A.9) to be as similar as possible to those for the $\mathcal{N} = 6$ solution, see in particular (A.12).

$$e^{4A} = -\frac{4c}{F_0}\csc(4t)\,\sec(2\psi)\,\tan(2\psi) \tag{5.10}$$

$$e^{3A-\Phi} = c\sec(2\psi)\sqrt{1+\cot^2(2t)\,\sin^2(2\psi)}\;. \tag{5.11}$$

Here, c is an integration constant, that so far is arbitrary. The fluxes are determined as well, and have the general form

$$\begin{aligned}
F_2 &= k_2(t)e^{2B_1}J_1 + g_2(t)e^{2B_2}J_2 + \tilde{k}_2(t)\frac{i}{2}z\wedge\bar{z}\,,\\
F_4 &= k_4(t)e^{2B_1+2B_2}J_1\wedge J_2 + \tilde{k}_4(t)e^{2B_1}\frac{i}{2}z\wedge\bar{z}\wedge J_1 + \tilde{g}_4(t)e^{2B_2}\frac{i}{2}z\wedge\bar{z}\wedge J_2\,,\\
F_6 &= k_6(t)\frac{e^{2B_1+2B_2}}{16}\frac{i}{2}z\wedge\bar{z}\wedge J_1\wedge J_2\,,
\end{aligned} \tag{5.12}$$

where $\frac{i}{2}z\wedge\bar{z} = \frac{\epsilon\Gamma}{16\sqrt{2}}dt\wedge(da+A_2-A_1)$. The full expressions for the coefficients k_i, $\tilde{k}_i$, g_i, $\tilde{g}_i$ can be found in (A.13). The fluxes satisfy the Bianchi identities, which require that

$$\tilde{F} \equiv e^{-B}(F_0+F_2+F_4+F_6) \tag{5.13}$$

is closed. This dictates in particular that F_0 is constant.

We can now study the regularity of the differential equation near its special points, $t=0$ and $t=\pi/2$, by finding a power series solution of the equations. The general solution will depend on three arbitrary constants. However, we are after solutions with particular topology, where w_2 vanishes at $t=0$ (the "North pole") and w_1 vanishes at $t=\pi/2$ (the "South pole"). Near $t=0$, we obtain

$$\begin{aligned}
\psi &= \psi_1 t - \frac{2}{3}(4\psi_1+5\psi_1^3)t^3 + O(t^5),\\
w_1 &= w_0 + (4+4\psi_1^2-2w_0+2w_0\psi_1^2)t^2 + O(t^4),\\
w_2 &= (4+4\psi_1^2)t^2 + O(t^4)\;.
\end{aligned} \tag{5.14}$$

In our solution, w_0 and ψ_1 are not independent: imposing that w_1 vanishes at $t=\pi/2$ determines w_0 in terms of ψ_1. The power series expansion in $\tilde{t}\equiv\pi/2-t$ near $t=\pi/2$ is identical, with the role of w_1 and w_2 exchanged:

$$\begin{aligned}
\psi &= \tilde{\psi}_1\tilde{t} - \frac{2}{3}(4\tilde{\psi}_1+5\tilde{\psi}_1^3)\tilde{t}^3 + O(\tilde{t}^5),\\
w_1 &= (4+4\tilde{\psi}_1^2)\tilde{t}^2 + O(\tilde{t}^4),\\
w_2 &= \tilde{w}_0 + (4+4\tilde{\psi}_1^2-2\tilde{w}_0+2\tilde{w}_0\tilde{\psi}_1^2)\tilde{t}^2 + O(\tilde{t}^4)\;.
\end{aligned} \tag{5.15}$$

The constants $\tilde{w}_0$ and $\tilde{\psi}_1$ should also be determined by ψ_1; we can then think of ψ_1 as the only parameter in the internal metric. To find a solution with the required topology, we note that the equations (5.6) are symmetric under the operation $t\to\frac{\pi}{2}-t, \psi\to-\psi$, $w_1\leftrightarrow w_2$, and we look for solutions which are left invariant by this symmetry. This determines $\tilde{\psi}_1=\psi_1$ and $\tilde{w}_0=w_0$, and it allows us to restrict the study of the equations to

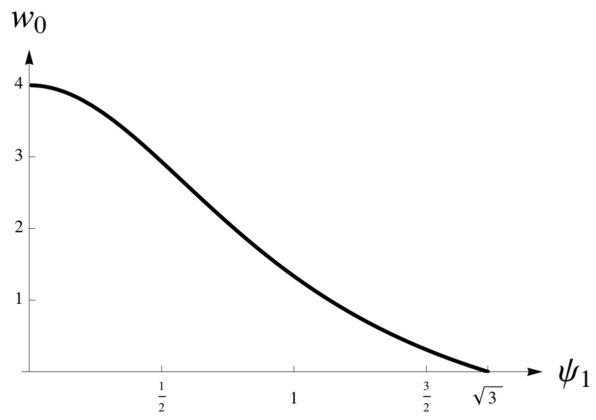

Figure 3. A plot of w_0 as a function of ψ_1. It vanishes linearly around the point $\psi_1 = \sqrt{3}$.

the "north hemisphere" $t \in [0, \pi/4]$. The only thing left to impose is that the solution is differentiable at $t = \pi/4$. This is what determines w_0 as a function of ψ_1, which we plot in figure 3. $w_0(\psi_1)$ is monotonicaly decreasing; our numerical analysis shows that it vanishes at a point very well approximated by $\psi_1 = \sqrt{3}$.

The perturbative expansion of the solutions near the "poles" $t = 0$ and $t = \pi/2$ allows to check the regularity of the six-dimensional metric. In fact, the only special points in the metric are the poles, where a copy of S^2 degenerates. Using the previous expansion, it is straightforward to check that, at both poles, the shrinking S^2 combines with (t, a) to give a piece of the metric proportional to

$$dt^2 + \frac{1}{4}t^2 \left(ds^2_{S^i} + (da \pm A_i)^2\right) \ . \tag{5.16}$$

Thanks to the fact that the periodicity of a is 4π, this is the flat metric of $\mathbb{R}^4$. For all $\psi_1 \in [0, \sqrt{3})$ the metric is then regular. For $\psi_1 = \sqrt{3}$, both spheres degenerate at each pole and the metric develops two conifold singularities. $\psi_1 = \sqrt{3}$ is thus the natural limiting point in our family of solutions.

We can examine now the number of parameters in the solution. As discussed above, the differential equations provide just one parameter, ψ_1, the value of the derivative of ψ at the North pole $t = 0$. It is convenient to define two more parameters by

$$g_s \equiv e^{\phi}|_{t=0} \, , \qquad 2L = e^{A}|_{t=0} \ . \tag{5.17}$$

Both ϕ and A vary over the internal manifold, but numerical study reveals that they only do so by order one functions. So g_s and L can be thought of as the order of magnitude of the dilaton and AdS radius in our solutions.[6] We can now reexpress the integration

[6]The normalization has been chosen so that in the metric, at $t = 0$, L^2 multiplies an Anti-de Sitter space of unit radius, and the relation between the mass of a particle at $t = 0$ and the conformal dimension of the dual operator is $mL = \Delta(\Delta - 4)$. This normalization is related to the fact that, in our conventions, $ds^2_{AdS_4}$ has cosmological constant $\Lambda = -3|\mu|^2$ and, as discussed in appendix A, we chose $\mu = 2$.

JHEP11(2010)047

constant c by evaluating (5.11) at $t = 0$:

$$c = \frac{8L^3}{g_s} \frac{1}{\sqrt{1+\psi_1^2}} \ . \tag{5.18}$$

The F_0 flux is then determined by evaluating (5.10) at $t = 0$:

$$F_0 = -\frac{1}{Lg_s} \frac{\psi_1}{\sqrt{1+\psi_1^2}} \ . \tag{5.19}$$

Finally, a fourth parameter comes from the B field. As in section 3, there is a zero-mode ambiguity coming from the presence of a non-trivial cohomology in our internal manifold. To see this, let us call B_0 a choice of B-field such that $H = dB_0$ solves the equations of motion. For example, we can choose B_0 such that

$$\tilde{F}_2 = F_2 - B_0 F_0 = 0 \ . \tag{5.20}$$

$H = dB_0$ is guaranteed to solve the equations of motion, since equation (5.20) implies that $dF_2 = HF_0$, which we have already solved. However, this will also be true for any B of the form

$$B = B_0 + \beta \,, \tag{5.21}$$

for any β which is closed. We can apply to this β the same considerations as in section 3.1: because of gauge invariance $B \cong B + d\lambda_1$, the space of such β is nothing but the second de Rham cohomology of the internal space, $H^2(\mathbb{C}P^3) = \mathbb{R}$, so we have one such parameter. And, just as in (3.5), we define the integral of β over the generating two-cycle in H_2: $b \equiv \frac{1}{(2\pi l_s)^2} \int_{\mathbb{CP}^1} \beta$. The fact that we use the same notation as in section 3 should not generate confusion, as the contexts are different.

Summarizing, our solutions are parameterized by the four numbers (L, ψ_1, g_s, b). The situation is very similar to the $\mathcal{N} = 1$ solutions we studied in section 3, with σ replaced by ψ_1.

5.2 Inverting the flux quantization equations

This section will follow closely the corresponding treatment for the $\mathcal{N} = 1$ solutions in section 3.2. The equations are formally very similar:

$$\begin{pmatrix} \frac{1}{lg_s} f_0(\psi_1) \\ 0 \\ \frac{l^3}{g_s} f_4(\psi_1) \\ \frac{l^5}{g_s} f_6(\psi_1) \end{pmatrix} = \begin{pmatrix} n_0^b \\ n_2^b \\ n_4^b \\ n_6^b \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ \frac{1}{2}b^2 & b & 1 & 0 \\ \frac{1}{6}b^3 & \frac{1}{2}b^2 & b & 1 \end{pmatrix} \begin{pmatrix} n_0 \\ n_2 \\ n_4 \\ n_6 \end{pmatrix} , \tag{5.22}$$

where $l = L/(2\pi l_s)$, as in (3.8). The vector on the left hand side is given by the integrals $\frac{1}{(2\pi l_s)^{k-1}} \int_{C_k} \tilde{F}_k$, where C_k is the single k-cycle in $\mathbb{CP}^3$ $(k = 0, 2, 4, 6)$, and $\tilde{F}_k$ is defined

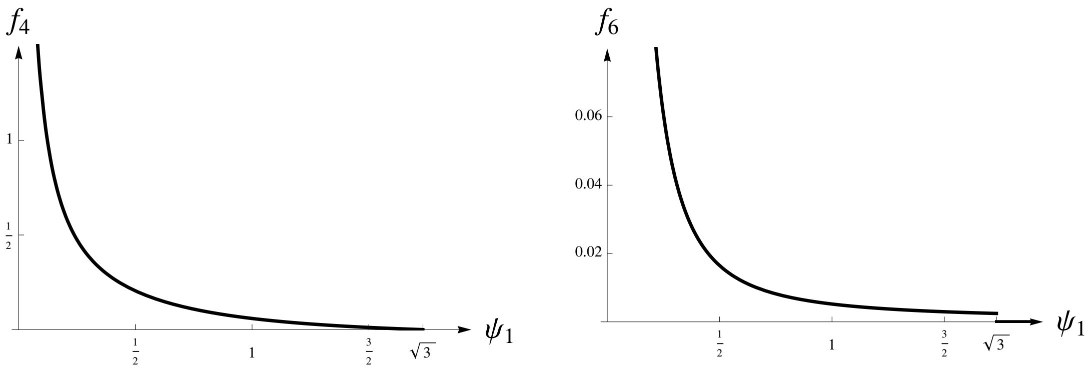

Figure 4. Plots of $f_4(\psi_1)$ and $f_6(\psi_1)$. Their asymptotic behavior near 0 and $\sqrt{3}$ is given in (5.24), (5.25). Notice in particular that $f_6(\sqrt{3})$ is small but non-zero.

using the particular B_0 in (5.20); this also explains why the second entry of the vector is zero (this is simply our choice for the definition of b). We could have made such a choice for the $\mathcal{N} = 1$ solution as well; we did not do so because for SU(3) structure solutions there is a different and particularly natural choice of B-field.

Using equation (5.19) we can write

$$f_0(\psi_1) = -\frac{\psi_1}{\sqrt{1+\psi_1^2}} \,. \tag{5.23}$$

We know the other functions $f_k(\psi_1)$ only numerically. We obtain $2f_4(\psi_1)$ by integrating $\tilde{F}_4$ over the diagonal S^2 times the "fiber" (t, a), which is a representative of twice the fundamental four-cycle. The plots of $f_k(\psi_1)$ are given in figure 4. Our numerical analysis indicates the following asymptotics at the two extrema $\psi_1 = 0$, $\psi_1 = \sqrt{3}$:

$$f_4 \sim \psi_1^{-1}, \qquad f_6 \sim \psi_1^{-2} \qquad \text{for } \psi_1 \to 0\,; \tag{5.24}$$

$$f_4 \sim (\sqrt{3} - \psi_1), \qquad f_6 \to \text{const} \qquad \text{for } \psi_1 \to \sqrt{3}\,. \tag{5.25}$$

We can now proceed in the same fashion as in the $\mathcal{N} = 1$ case to determine ψ_1 from the flux parameters. Namely, we write the combination

$$\frac{(n_2^2 - 2n_0n_4)^3}{(n_2^3 + 3n_0^2n_6 - 3n_0n_2n_4)^2} = -\frac{8f_4(\psi_1)^3}{9f_6(\psi_1)^2 f_0(\psi_1)} \equiv \rho(\psi_1)\,, \tag{5.26}$$

which allows us to determine ψ_1 in terms of fluxes. l and g_s are then given by

$$l^6 = \frac{f_0(\psi_1)}{3f_6(\psi_1)} \left(\frac{n_2}{n_0}\right)^3 \left(1 + 3\frac{n_0^2n_6}{n_2^3} - 3\frac{n_0n_4}{n_2^2}\right), \tag{5.27}$$

$$g_s^6 = 3f_0^5(\psi_1)f_6(\psi_1)\,(n_2n_0)^{-3}\left(1 + 3\frac{n_0^2n_6}{n_2^3} - 3\frac{n_0n_4}{n_2^2}\right)^{-1}. \tag{5.28}$$

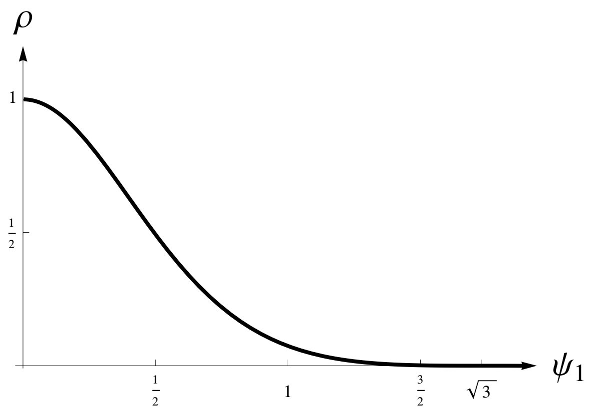

Figure 5. A plot of the function $\rho(\psi_1)$ in (5.26).

A crucial role is played by the function $\rho(\psi_1)$ which we plot in figure 5. It decreases monotonically from 1 at $\psi_1 = 0$ to zero at $\psi_1 = \sqrt{3}$. Its asymptotic behavior at $\psi_1 = 0$ and $\psi_1 = \sqrt{3}$ is:

$$\rho \sim 1 - \tilde{c}\,\psi_1^2 \quad \text{for } \psi_1 \to 0\,, \qquad \rho \sim (\sqrt{3} - \psi_1)^3 \quad \text{for } \psi_1 \to \sqrt{3}\,, \tag{5.29}$$

for some constant $\tilde{c}$. This is in agreement with (5.24), (5.25). The fact that ρ vanishes at the same point, $\psi_1 = \sqrt{3}$, where the solution develops a singularity, is strongly supported by our numerical analysis, and will be crucial in reproducing the field theory results.

5.3 A phase transition

As in the $\mathcal{N} = 1$ case, consider for simplicity the case $n_4 = 0$ and call, as usual, $n_6 = N$ and $n_2 = k$. Equation (5.26) becomes

$$\rho(\psi_1) = \left(1 + 3\frac{n_0^2 N}{k^3}\right)^{-2} . \tag{5.30}$$

As in the $\mathcal{N} = 1$ case, there are two interesting regimes. For $N \ll k^3/n_0^2$, $\psi_1 \to 0$ and we are near the undeformed solution. From (5.29), we see that $1 - \rho(\psi_1) \sim \psi_1^2$; hence we can identify in this regime

$$\psi_1 \sim \left(\frac{n_0^2 N}{k^3}\right)^{1/2} . \tag{5.31}$$

Moreover, we see from (5.23) that $f_0(\psi_1) \sim \psi_1$; using also (5.24), we easily compute from (5.27)

$$l \sim \frac{N^{1/4}}{k^{1/4}}\,, \qquad g_s \sim \frac{N^{1/4}}{k^{5/4}}\,, \tag{5.32}$$

which is indeed the behavior of the $\mathcal{N} = 6$ solution [11].

For $N \gg k^3/n_0^2$, the function $\rho(\psi_1)$ should approach zero, and this happens for $\psi_1 \to \sqrt{3}$. From (5.29) and (5.30), we see that

$$\delta\psi_1 \equiv \left(\sqrt{3} - \psi_1\right) \sim \left(\frac{k^3}{N n_0^2}\right)^{2/3} . \tag{5.33}$$

From (5.25) we then conclude

$$l \sim \frac{N^{1/6}}{|n_0|^{1/6}}\,, \qquad g_s \sim \frac{1}{N^{1/6}|n_0|^{5/6}}\,, \tag{5.34}$$

which is the same behavior as in (3.22). Again, as in the $\mathcal{N}=1$ case, we can also argue generally that g_s remains bounded for any integer values of the fluxes, with $n_0 \neq 0$.

At first sight, this seems puzzling. At the end of section 4.2, we noticed that this gravity solution is expected to develop light D-branes in the limit $N_1 = N_2 = N \to \infty$. As argued in [22], $N_1 = N_2$ precisely when $n_4 = 0$ (see also (5.35)). But there do not seem to be any light D-branes in a limit where g_s is small and the internal manifold is large, since a D-brane mass scales as L^k/g_s, with $k \geq 0$.

As we remarked after equation (5.16), however, in the limit $\psi \to \sqrt{3}$ (which is relevant for large N) the internal manifold develops two conifold-like singularities, since the two-cycle is now shrinking to zero at the "poles". As we will now see, the new light states are obtained from D-branes wrapping the vanishing cycle for that singularity.

5.4 Probes

We now want to compare this gravity solution with the field theory we saw in section 4.2; specifically, the one defined by the superpotential in (4.9), which has the right symmetries to be the dual of the gravity solution we found in section 5.1.

We can first of all try to predict what sort of bulk field corresponds to the monopole operators discussed in section 4.2. Let us recall how the duality works in the ABJM case, when $F_0 = 0$. Consider first a monopole operator that creates one unit of field strength for both gauge groups at a particular point. This operator has k indices under both gauge groups, and we can make it gauge-invariant by contracting it with k bifundamentals. The resulting bound state corresponds to a D0 brane in the gravity dual; notice that such a brane has no tadpole on its worldsheet for the worldsheet vector potential $\mathcal{A}$, as we already saw in section 3.4. Another monopole operator that can be considered is the one that creates one unit of flux for, say, the second gauge group. In this case, we cannot make this operator gauge-invariant: it will have k "dangling" indices. This corresponds to a D2 brane wrapping an internal two-cycle. As we also already saw in section 3.4, such a brane has a tadpole on its worldsheet, coming from the term $\int A_1 \mathcal{F} = \int F_2 \mathcal{A}$; so one needs to have k strings ending on it, and these k strings correspond to the k indices of the monopole operator.

When we switch on F_0, even a D0 brane will have a tadpole on its worldsheet, coming from the coupling $\int F_0 \mathcal{A}$. On the field theory side, this corresponds to the fact that the monopole operator that creates one unit of field strength for both gauge groups has now k_1 fundamental indices for the first gauge group and k_2 antifundamental ones for the second. This cannot be made gauge-invariant; we are always left with at least $|k_1 - k_2|$ "dangling" indices. This fact was used in [14] to establish that the Romans mass integer is the sum of the Chern-Simons couplings, so that, in the present language, $n_0 = k_1 - k_2$; see also [34, 35]. In [22] it was similarly shown that n_4 is the difference between the two gauge group ranks $N_2 - N_1$.[7] Putting this together, we obtain a dictionary between the flux integers and the ranks and levels of the field theory:

$$n_0 = k_1 - k_2\,, \qquad n_2 = k_2\,, \qquad n_4 = N_2 - N_1\,, \qquad n_6 = N_2\,. \tag{5.35}$$

In section 4.2, we considered monopole operators which create k_2 units of field strength for the first gauge group, and k_1 units of field strength for the second. We noticed that these have $k_1 k_2$ bifundamental indices, and thus can be made gauge-invariant. Following the identifications of D2 branes and D0 branes above, if we assume for example that $k_1 > k_2$, we can say that these new gauge-invariant monopoles correspond to a bound state k_2 D0 branes and $k_1 - k_2$ D2 branes. We have already noticed in section 3.4 that such a bound state can cancel the tadpole on the worldsheet, because it makes the prefactor in (3.24) vanish.

Let us make this expectation more precise. Consider a D2 brane wrapped on a two-cycle $\mathcal{B}_2$ in the $\mathcal{N} = 2$ solution. As we will see in appendix B.2, supersymmetry requires that the D2 brane lives at the North pole $t = 0$ or at the South pole $t = \pi/2$, and that it wraps the S^2 that does not shrink there. We also need to cancel the tadpole for the world-volume field $\mathcal{A}$ which arises from the Wess-Zumino coupling,

$$\mathcal{A} \wedge (F_2 + F_0(\mathcal{F} - B))\ . \tag{5.36}$$

We can split B into a fiducial choice plus a zero mode, as in (5.21). The tadpole cancellation requires

$$\mathcal{F} - B = \mathcal{F} - \beta - B_0 = -F_2/F_0. \tag{5.37}$$

Since B_0 was chosen to satisfy (5.20), we need to turn on a world-volume flux

$$\mathcal{F} = \beta\ . \tag{5.38}$$

There is a possible obstruction to doing this, coming from the quantization of the world-volume flux, that says that $\frac{1}{(2\pi l_s)^2}\int_{S^2}\mathcal{F} \in \mathbb{Z}$. The value of $b = \frac{1}{(2\pi l_s)^2}\int_{S^2}\beta$ from (5.22) is given by $b = -n_2/n_0$; hence in general it is rational and not an integer. So we see that a single D2 brane is generally not consistent. We can get around this, however, by considering n_0 D2-branes. In that case, the equation we want to satisfy actually reads

$$\mathcal{F} = \beta\, 1_{D0}\ . \tag{5.39}$$

The integral of the trace of the left hand side is the first Chern class on the world-volume, which is the induced D0-brane charge n_0. The integral of the trace of the right hand side

[7] The relative sign between the expressions for n_4 and n_0 had not been determined so far. We made here a choice consistent with our final result in formula (5.43).

now gives $bn_0 = -n_2$. We conclude that we can cancel the tadpole by considering a bound state of n_0 D2 branes and n_2 D0 branes, just as in section 3.4.

Naively, one might think that the mass of a D2-D0 bound state should be at least as heavy as a D0-brane, which in units of AdS mass is $m_{D0}L \sim L/g_s$. Since this is heavy in the limit (5.34), one might think that such a bound state can never reproduce the light mass predicted in section 4.2.

Fortunately, such pessimism proves to be unfounded. The mass of the state is given by

$$m_{D2}L = n_0 L \frac{1}{(2\pi)^2 g_s l_s^3} \int_{\mathcal{B}_2} \sqrt{\det\left(g + \mathcal{F} - B\right)} = n_0 L \frac{1}{(2\pi)^2 g_s l_s^3} \int_{\mathcal{B}_2} \sqrt{\det\left(g - \frac{F_2}{F_0}\right)} \,. \tag{5.40}$$

where we used the tadpole cancellation condition. We will take the cycle $\mathcal{B}_2$ to be a representative of the non-trivial cycle, which is the diagonal of the two S^2's. Using the explicit form for the metric in (5.4), as well as (5.12), (5.18) and (A.13), we get:

$$m_{D2}L = 4\pi \frac{n_0 L}{(2\pi)^2 g_s l_s^3} \sqrt{\left(\frac{e^{2B_1}}{4}\right)^2 + \left(\frac{k_2 e^{2B_1}}{F_0}\right)^2} = 2\pi^2 \frac{n_0 l^3}{g_s} w_0(\psi_1) \sqrt{\frac{1+\psi_1^2}{\psi_1^2}} \,. \tag{5.41}$$

The fact that the two expressions under the square root are proportional is related to the BPS condition, as discussed in appendix B.2.

Inserting the values of l and g_s from (5.27), (5.28) for generic fluxes we obtain

$$m_{D2}L = \left(\frac{n_2^2}{2} - n_0 n_4\right) \frac{2\pi^2 w_0(\psi_1)}{f_4(\psi_1)} \sqrt{\frac{1+\psi_1^2}{\psi_1^2}} \,. \tag{5.42}$$

Quite remarkably, the function of ψ_1 in the previous formula, which can be computed numerically, turns out to be constant with value 1. The final result for the mass formula is then

$$m_{D2}L = \left(\frac{n_2^2}{2} - n_0 n_4\right) \,. \tag{5.43}$$

Upon using the dictionary (5.35), this formula is identical to the field theory prediction (4.13) in the limit where $n_0 \ll n_2$. This is exactly the limit where we can trust the supergravity solution, since, as shown in (5.27), for a generic value of ψ_1, L is large only if $n_2 \gg n_0$. In this limit, it is also true that the dimension of the corresponding operator is given by $\Delta \sim m_{D2}L$.

In contrast with the $\mathcal{N} = 1$ results in section 3.4, and with the naive expectation expressed earlier, we see from (5.43) that the mass of the bound state remains finite also in the limit $N \gg k^3/n_0^2$. A contribution from the B field cancels the large mass $\sim L/g_s$ of the constituent D0-brane, leaving a smaller piece that is proportional to the volume of the shrinking S^2. These are precisely the new light states that we had predicted to exist from the field theory analysis in section 4.2.

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A Supersymmetry equations and pure spinors for the $\mathcal{N} = 2$ solution

We will give in this section more details about the $\mathcal{N} = 2$ solution we found in section 5.

The supersymmetry parameters for compactifications of the form $\mathrm{AdS}_4 \times M_6$ (or $\mathrm{Minkowski}_4 \times M_6$) decompose as

$$\epsilon_1 = \sum_{a=1}^{\mathcal{N}} \zeta_+^a \otimes \eta_+^{1a} + \zeta_-^a \otimes \eta_-^{1a} \,, \tag{A.1}$$

$$\epsilon_2 = \sum_{a=1}^{\mathcal{N}} \zeta_-^a \otimes \eta_+^{2a} + \zeta_+^a \otimes \eta_-^{2a} \,. \tag{A.2}$$

Here, $\mathcal{N}$ is the number of supersymmetries. The subscripts $\pm$ denote positive and negative chirality spinors, in four and six dimensions; the negative chirality spinors are conjugate to the positive chirality ones,

$$\zeta_-^a = (\zeta_+^a)^* \,, \qquad \eta_-^{ia} = (\eta_+^{ia})^* \,. \tag{A.3}$$

For each a, ζ_+^a can vary among a basis of four-dimensional Weyl spinors; we will take the elements of this basis to be "Killing spinors", which means that $D_\mu \zeta_+ = \frac{\mu}{2}\gamma_\mu \zeta_-$. The η_+^{ia}, with $i = 1, 2$, are a priori independent six-dimensional Weyl spinors. In this section, we will consider $\mathcal{N} = 2$.

A priori, one could have taken the ζ^a in ϵ_1 and ϵ_2 to be different. This can indeed be done for compactifications with vanishing RR flux; for example, for the usual $\mathcal{N} = 2$ Calabi-Yau compactifications. To recover that case in (A.1), one can take for example $\eta^{21} = \eta^{12} = 0$, and keep a non-vanishing η^{11} and η^{22}. However, in compactifications where RR fluxes are present, the ζ^a in ϵ_1 and ϵ_2 are required to be equal, up to a constant that can be reabsorbed in the η^{ia}. Hence (A.1) describes all possible $\mathcal{N} = 2$ compactifications, and is particularly appropriate for vacua with RR fluxes.

Using (A.1) in the supersymmetry equations yields equations for the internal spinors η^{ia}. In fact, these equations do not mix the η^{i1} with the η^{i2}. In what follows, we will first analyze the equations of the $\eta^{i1} \equiv \eta^i$; we will come back to the second pair later.

JHEP11(2010)047

We will construct a pair of pure spinors as tensor products of the supersymmetry parameters[8] η^1 and η^2

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\,\dagger}_\pm \ . \tag{A.4}$$

The type IIA supersymmetry conditions can be expressed as [36]:

$$(\mathrm{d} - H\wedge)(e^{A-\varphi}\mathrm{Re}\ (\Phi_-)) = 0\,, \tag{A.5a}$$
$$(\mathrm{d} - H\wedge)(e^{3A-\varphi}\mathrm{Im}\ (\Phi_-)) = -3e^{2A-\varphi}\mu\mathrm{Im}\ (\Phi_+) + \frac{e^{4A}}{8} * \lambda(F)\,, \tag{A.5b}$$
$$(\mathrm{d} - H\wedge)(e^{2A-\varphi}\Phi_+) = -2\mu e^{A-\varphi}\mathrm{Re}\ (\Phi_-)\ ; \tag{A.5c}$$
$$||\Phi_+|| = ||\Phi_-|| = e^A\ . \tag{A.5d}$$

Here, F are the internal fluxes (which determine also the external fluxes, by self-duality). A is the warping function, defined as $ds^2_{10} = e^{2A}ds^2_{\mathrm{AdS}_4} + ds^2_6$. The cosmological constant in $ds^2_{\mathrm{AdS}_4}$ is given by $\Lambda = -3|\mu|^2$. Since A is non-constant in the solution, however, this Λ has no independent meaning, since one can reabsorb it in A. We have normalized $\mu = 2$ in this paper. The symbol λ acts on a k-form by multiplying it by the sign $(-)^{\mathrm{Int}(k/2)}$. Finally, the norm in (A.5d) is defined as $||A||^2 = i(A \wedge \lambda(\bar{A}))_6$.

The metric (5.4) can be written in terms of the vielbein

$$\begin{aligned}
E_1 &= \frac{1}{2\sqrt{2}}\epsilon dt + \frac{i}{8}\Gamma(da + A_2 - A_1),\\
E_2 &= \frac{1}{2}\left(e^{B_1}\sin(t)e^{-ia/2}e_1 - e^{B_2}\cos(t)e^{ia/2}e_2\right),\\
E_3 &= \frac{1}{2}\left(e^{B_1}\cos(t)e^{-ia/2}e_1 + e^{B_2}\sin(t)e^{ia/2}e_2\right),
\end{aligned} \tag{A.6}$$

where $e_i = d\theta_i + i\sin\theta_i d\phi_i$ are the natural one-forms on the spheres S^2_i. For $e^{B_1} = \cos(t)$ and $e^{B_2} = \sin(t)$ we recover the Fubini-Study metric of $\mathbb{CP}^3$ with natural Kähler form $J = \frac{i}{2}\sum_{i=1}^3 E_i \wedge \bar{E}_i$ and natural three form section $\Omega = E_1 \wedge E_2 \wedge E_3$ (see for example [32, (5.31)]). It is also convenient to use the forms

$$J_i = dA_i = \frac{i}{2}e_i \wedge \bar{e}_i \ \ (\text{not summed})\,, \qquad o \equiv \frac{i}{2}e^{ia}e_2 \wedge \bar{e}_1\ ; \tag{A.7}$$

the J_i were already defined in (5.3). These forms satisfy

$$dJ_i = 0\,, \qquad do = i(da + A_2 - A_1)\wedge o\,, \qquad o \wedge \bar{o} = -J_1 \wedge J_2\ . \tag{A.8}$$

The generic pure spinors corresponding to an SU(3) $\times$ SU(3) structure can be written in terms of the "dielectric Ansatz"

$$\begin{aligned}
\Phi_+ &= \frac{i}{8}\cos(2\psi)\, e^{A+i\theta}\exp\left(-i\left(\frac{j}{\cos(2\psi)} + \frac{i}{2}z\wedge\bar{z}\right) + \tan(2\psi)\,\mathrm{Re}(\omega)\right)\,,\\
\Phi_- &= -\frac{i}{8}\sin(2\psi)e^{A+i\theta}\, z\wedge\exp\left(-\cot(2\psi)\,\mathrm{Re}(\omega) - \frac{i}{\sin(2\psi)}\mathrm{Im}(\omega)\right)\,,
\end{aligned} \tag{A.9}$$

[8] As usual, we left implicit a Clifford map on the left hand side, that sends $dx^m \to \gamma^m$.

where θ and ψ are two new angular variables; one can see easily that the supersymmetry equations (A.5) relate them by

$$\tan(\theta) = -\cot(2t)\,\sin(2\psi)\ . \tag{A.10}$$

The one-form z and the two-forms j and ω can also be used to describe an SU(2) structure on M_6. For our solution, these forms are given by

$$\begin{aligned} z &= -ie^{-i\theta}E_1\,,\\ j &= \frac{i}{2}\left(E_2\wedge\bar{E}_2+\bar{E}_3\wedge E_3\right)\,,\\ \omega &= iE_2\wedge\bar{E}_3\ . \end{aligned} \tag{A.11}$$

We can also characterize j and ω in terms of the forms in (A.7):

$$\begin{pmatrix} j \\ -\mathrm{Re}(\omega)\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\cos(2t) & -\sin(2t)\\ \sin(2t) & \cos(2t)\end{pmatrix}\begin{pmatrix}-e^{2B_1}J_1+e^{2B_2}J_2\\ 2e^{B_1+B_2}\mathrm{Re}(o)\end{pmatrix},\quad \mathrm{Im}(\omega)=\frac{1}{2}e^{B_1+B_2}\mathrm{Im}(o). \tag{A.12}$$

The RR fluxes are determined to be as in equation (5.12) with

$$\begin{aligned} k_2 &= \frac{c\,e^{-4A}}{2\,w_1}\frac{\sec(2\psi)}{\cos(2t)}\left(2\,C_{t,\psi}+w_1\right),\\ g_2 &= -\frac{c\,e^{-4A}}{2\,w_2}\frac{\sec(2\psi)}{\cos(2t)}\left(2\,C_{t,\psi}+w_2\right),\\ \tilde{k}_2 &= 2\,\frac{c\,e^{-4A}}{w_1\,w_2}\left(2\,C_{t,\psi}(w_1+w_2)+3\,w_1\,w_2\right),\\ k_4 &= -\frac{c\,e^{-4A}}{4\,w_1\,w_2}\frac{\sin(2\psi)}{\sin(4t)\cos^2(2\psi)}\left(2\,C_{t,\psi}(w_1+w_2)+w_1\,w_2\right),\\ \tilde{k}_4 &= \frac{c\,e^{-4A}}{2\,w_2}\frac{\tan(2\psi)}{\sin(2t)}\left(2\,C_{t,\psi}+3\,w_2\right),\\ \tilde{g}_4 &= -\frac{c\,e^{-4A}}{2\,w_1}\frac{\tan(2\psi)}{\sin(2t)}\left(2\,C_{t,\psi}+3\,w_1\right),\\ k_6 &= 6\,c\,e^{-4A}, \end{aligned} \tag{A.13}$$

where $C_{t,\psi}$ was defined in (5.7). Recall also that one possible choice of NS-NS field that satisfies the equations of motion is $B_0 = F_2/F_0$, as in (5.20).

So far we have described the solution as if it were an $\mathcal{N}=1$ solution: we have only paid attention to the $a=1$ part of (A.1). To show that the solution actually has $\mathcal{N}=2$ supersymmetry, we have to provide a second pair of spinors, η^{i2}, that satisfies the equations of motion for supersymmetry with the same expectation values for all the fields. In terms of pure spinors, we can now form the bilinears

$$\tilde{\Phi}_\pm = \eta^{12}_+\otimes\eta^{22\,\dagger}_\pm \tag{A.14}$$

and require that they solve again the equations (A.5), with the same values of the fluxes and the same metric.

In fact, one expects the two solutions Φ and $\tilde{\Phi}$ to be rotated by R-symmetry, so that there is actually a U(1)'s worth of solutions to (A.5). To see this U(1), rotate the two-form o in (A.7) by a phase:[9]

$$o \to e^{-i\alpha} o \equiv o_\alpha \ . \tag{A.15}$$

We can correspondingly define a pair of pure spinors $\Phi^\alpha_\pm$, by changing $o \to o_\alpha$ wherever it appears. The crucial fact about the rotation of o in (A.15) is that it keeps its differential properties (A.8) unchanged: namely, $do_\alpha = i(da + A_2 - A_1) \wedge o_\alpha$. Because of this fact, the computations to check (A.5) do not depend on α; and, since we checked already that $\alpha = 0$ gives a solution, it follows that any $\Phi^\alpha_\pm$ is a solution. A priori, this could be a solution with different fluxes; but we can see from (5.12) that o never appears in F_k. We conclude, then, that the solution we have found is an $\mathcal{N} = 2$ solution.

B BPS particles

In this section, we will give a general analysis of BPS particles in flux compactifications (subsection B.1), and we will then apply those general results to the $\mathcal{N} = 2$ background described in section 5 and appendix A.

B.1 General considerations

We will start with some general considerations about BPS states in $\mathcal{N} = 2$ backgrounds with fluxes. These will in general be states that are left invariant by a certain subalgebra of the supersymmetry algebra. This subalgebra is in general defined by the fact that the two supersymmetry parameters ϵ_i are related:

$$\Gamma_{||}\epsilon_2 = \epsilon_1 \ . \tag{B.1}$$

In first approximation, $\Gamma_{||}$ is the product of the gamma matrices parallel to the brane. When B fields or worldsheet fluxes $\mathcal{F}$ are present, $\Gamma_{||}$ receives additional contributions of $e^{\mathcal{F}-B}$. We will give a definition later on, in the context needed for this paper; for the general and explicit expression, see for example [37, eq. (3.3)]. For an $\mathrm{AdS}_4 \times M_6$ compactification, we would like to use the decomposition (A.1). For particles, this will lead to an equation involving the four-dimensional spinors $\zeta^i_\pm$ and $\gamma_0\zeta^a_\pm$; here and in what follows, the index $_0$ is meant to be a frame index. To have a chance to solve the resulting equations, we need to postulate a relation between these spinors. One can write for example

$$\gamma_0\zeta^a_+ = A^{ab}\zeta^b_- \ , \tag{B.2}$$

[9] Alternatively, one can change the vielbeine (A.6) by translating $a \to a + \alpha$.

for some matrix A. (Recall that in general the index a runs from 1 to $\mathcal{N}$; for us, $\mathcal{N} = 2$, and so $a = 1, 2$.) In fact, (B.2) is almost the most general choice one can make, compatibly with the symmetries of the problem. The only generalization one could make would be to multiply the left-hand side by another matrix B^{ab}. Whenever this matrix is invertible, one can reabsorb it by a redefinition of A^{ab}. In this sense, we can say that (B.2) is the "generic" Ansatz for a BPS particle.

The matrix A^{ab} in (B.2) needs to satisfy certain conditions. Let us work for simplicity in a basis where all the space-time gamma matrices γ_μ, $\mu = 0, \ldots, 3$ are real, and the internal γ^m, $m = 1, \ldots, 6$, are purely imaginary; the ten-dimensional gamma matrices are then given as usual by

$$\Gamma_\mu = e^A \gamma_\mu \otimes 1 \, , \qquad \Gamma_{m+3} = \gamma_5 \otimes \gamma_m \; . \tag{B.3}$$

It follows from these definitions that Γ_μ are real. Let us now conjugate (B.2); using (A.3), the fact that γ_0 is real, and that $\gamma_0^2 = -1$, we get

$$A^{ab}\overline{A^{bc}} = -\delta^{ac} \; . \tag{B.4}$$

If we were considering an $\mathcal{N} = 1$ background, A^{ab} would be a one-by-one matrix, and (B.4) would have no solution. This is just what one would expect: there are no BPS particles in a $\mathcal{N} = 1$ background. For $\mathcal{N} = 2$, one choice that satisfies (B.4) is

$$A = e^{-i\lambda} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; . \tag{B.5}$$

We can now use (B.3) to write

$$\Gamma_{\|} = \gamma_0 \otimes \gamma_{\|} \, , \tag{B.6}$$

where $\gamma_{\|}$ is now an element of the internal Clifford algebra; it contains the product of all the internal gamma matrices parallel to the brane, plus additional contributions from the worldsheet flux and B-field. Let $\mathcal{B}_p \subset M_6$ be the p-cycle wrapped by the brane, of dimension p and with coordinates σ^α, $\alpha = 1, \ldots, p$. Then we define the natural volume form on $\mathcal{B}$ to be

$$\mathrm{vol}_{\mathcal{B}} \equiv \sqrt{\det(g + \mathcal{F} - B)}\, d\sigma^1 \wedge \ldots \wedge d\sigma^p \; . \tag{B.7}$$

One can also define similarly an "inverse volume form" as the multivector

$$\mathrm{vol}_{\mathcal{B}}^{-1} = \frac{\partial_1 \wedge \ldots \wedge \partial_p}{\sqrt{\det(g + \mathcal{F} - B)}} \, , \tag{B.8}$$

which is a section of $\Lambda^p(T\mathcal{B})$. This multivector can be used to give an intrinsic definition of $\gamma_{\|}$: here is how. We can define $e^{\mathcal{F}}\mathrm{vol}^{-1}$ to be the multivector of mixed degree that one obtains by contracting the indices of the form $e^{\mathcal{F}}$ with the multi-vector vol^{-1}. Recall now that multivectors can be "pushed forward": if we call $x : \mathcal{B} \hookrightarrow M_6$ the embedding map, with components $x^m(\sigma)$, then $x_*(e^{\mathcal{F}}\mathrm{vol}^{-1})$ is a multivector in M_6, obtained by contracting all indices α on $\mathcal{B}$ with the tensor $\partial_\alpha x^m$. In fact:

$$\gamma_{\|} = x_*(e^{\mathcal{F}-B}\mathrm{vol}_{\mathcal{B}}^{-1}) \ . \tag{B.9}$$

Here, we left implicit on the right hand side a Clifford map that sends a vector ∂_m into a gamma matrix γ_m. We already used this map on forms (see footnote 8). One can show that $\gamma_{\|}$ is unitary:

$$\gamma_{\|}^\dagger \gamma_{\|} = 1 \ . \tag{B.10}$$

For a more explicit expression of $\gamma_{\|}$, see [37, eq. (3.5)].

If we now use (B.2), (B.6) and (A.1) in (B.1), we get

$$\gamma_{\|}\, \eta_+^{2a} = (A^{-1})^{ba}\eta_+^{1b} \ . \tag{B.11}$$

For our choice (B.5), this reads

$$\gamma_{\|}\, \eta_+^{22} = -e^{i\lambda}\eta_+^{11} \, , \tag{B.12a}$$
$$\gamma_{\|}\, \eta_+^{21} = e^{i\lambda}\eta_+^{12} \ . \tag{B.12b}$$

We are now left with solving (B.12), which are two purely internal equations. Each of the two equations is formally identical to others that have already appeared [37] in the context of BPS objects which do exist in $\mathcal{N} = 1$ flux compactifications: branes which extend along the time direction, plus one, two or three space directions. Hence we can simply follow the same steps; we will now summarize that procedure for (B.12a), and then apply the result to (B.12b).

Let us first define the new pure spinors

$$\Psi_\pm \equiv \eta_+^{11} \otimes \eta_\pm^{22\,\dagger} \ ; \tag{B.13}$$

notice that these are different from the pure spinors $\Phi_\pm$, defined in (A.4), which entered the supersymmetry equations (A.5). In (A.4), η^1 and η^2 were to be understood as η^{1a} and η^{2a}, for a either 1 or 2. In (B.13), we are mixing $a = 1$ with $a = 2$.

A possible basis for the space of spinors of positive chirality is given by η_+^{11} and $\gamma^m\eta_-^{11}$. Three linear combinations of the γ^m make $\gamma^m\eta_-^{11}$ vanish: they are its three "annihilators" γ^i, where i is a holomorphic index with respect to an almost complex structure I. Explicitly we have $\eta_+^{11\,\dagger}\gamma_m\gamma^n\eta_+^{11} = (1+iI)_m{}^n \equiv 2\bar{\Pi}_m{}^n$. In terms of this basis a priori one can expand

$$\gamma_{\|}\eta_+^{22} = a\eta_+^{11} + b_m\gamma^m\eta_-^{11} \ . \tag{B.14}$$

The coefficients a and b_m have a geometrical interpretation. To compute a, we can multiply (B.14) from the left by $\eta_+^{11\,\dagger}$; we get $ae^A = \eta_+^{11\,\dagger}\gamma_{\|}\eta_+^{22} = \mathrm{Tr}(\gamma_{\|}\eta_+^{22}\eta_+^{11\,\dagger}) = \mathrm{Tr}(\gamma_{\|}\Psi_+^\dagger)$. From the formula

$$\mathrm{Tr}(\not{A}\not{B}^\dagger) = \frac{8}{k!}A_{m_1\ldots m_k}\bar{B}^{m_1\ldots m_k} \, , \tag{B.15}$$

we see that $\mathrm{Tr}(\gamma_{\|}\Psi_+^\dagger)$ consists of contracting the free indices of $\gamma_{\|}$ with those of $\bar{\Psi}_+$. From (B.9), we see that $\gamma_{\|}$ contains factors of $\partial_\alpha x^m$; when contracting with $\bar{\Psi}_+$, these factors reconstruct a pull-back of that form. In conclusion we get[10]

$$(e^{\mathcal{F}-B}\bar{\Psi}_+)|_{\mathcal{B}} = \frac{a}{8}e^A\,\mathrm{vol}_{\mathcal{B}}\,, \tag{B.16}$$

where $|_{\mathcal{B}}$ denotes the top-form part on $\mathcal{B}$ of the pull-back. By similarly multiplying (B.14) from the left by $\eta_-^{11\,\dagger}\gamma_n$, we get

$$(dx^m\cdot e^{\mathcal{F}-B}\Psi_-)|_B = -\frac{1}{4}b^n\Pi_n{}^m e^A\,\mathrm{vol}_B\,, \tag{B.17}$$

where $\cdot$ denotes the Clifford product: $v\cdot = v\wedge + v\llcorner$. Here, $\partial_m\llcorner(dx^{m_1}\wedge\ldots\wedge dx^{m_p}) \equiv p\,\delta_m{}^{[m_1}dx^{m_2}\wedge\ldots dx^{m_p]}$.

We can now go back to (B.12a). Comparing to the expansion (B.14), we get

$$a = -e^{i\lambda}\,, \qquad b_m = 0\ . \tag{B.18}$$

Using the geometrical interpretations (B.16) and (B.17), we get

$$\mathrm{Re}(-e^{-i\lambda}e^{\mathcal{F}-B}\Psi_+)|_B = \frac{1}{8}e^A\mathrm{vol}_B\,, \tag{B.19}$$

and

$$\mathrm{Im}(e^{-i\lambda}e^{\mathcal{F}-B}\Psi_+)|_B = 0, \tag{B.20a}$$
$$(v\cdot e^{\mathcal{F}-B}\Psi_-)|_B = 0\ . \tag{B.20b}$$

Actually, one can show that (B.19) is equivalent to the system (B.20). To see this, observe that $\gamma_{\|}$ is unitary, as we saw in (B.10). This implies that $\gamma_{\|}\eta_+^{22}$ should have the same norm as η_+^{22}. Since all the spinors have norm e^A (see footnote 10), it follows that

$$|a|^2 + 2b_m\bar{b}^m = 1\ . \tag{B.21}$$

This means that imposing $\mathrm{Re}(a) = 1$ is equivalent to imposing $\mathrm{Im}(a) = 0$ and $b_m = 0$. Recalling (B.16) and (B.17), we get our claim that (B.19) is equivalent to (B.20).

This completes our analysis of (B.12a) (along the lines of [37]). For (B.12b), similar considerations apply; we obtain

$$\mathrm{Re}(e^{-i\lambda}e^{\mathcal{F}-B}\tilde{\Psi}_+)|_B = \frac{1}{8}e^A\mathrm{vol}_B\,, \tag{B.22}$$

and

$$\mathrm{Im}(e^{-i\lambda}e^{\mathcal{F}-B}\tilde{\Psi}_+)|_B = 0, \tag{B.23a}$$
$$(v\cdot e^{\mathcal{F}-B}\tilde{\Psi}_-)|_B = 0\,, \tag{B.23b}$$

[10]The factor of e^A comes from the fact that $\forall a,i$, $||\eta^{ia}|| = e^{A/2}$, which follows from (A.5).

for the pure spinors

$$\tilde{\Psi}_\pm \equiv \eta_+^{12} \otimes \eta_\pm^{21\,\dagger}. \tag{B.24}$$

Let us now summarize this section: we have shown that a brane wrapping an internal cycle $\mathcal{B}$, and extended along the time direction, is BPS if and only if (B.19) (or equivalently (B.20)) is satisfied by Ψ and, analogously, (B.22) (or equivalently (B.23)) is satisfied by $\tilde{\Psi}$, where Ψ and $\tilde{\Psi}$ are defined respectively in (B.13) and (B.24). We will now compute these pure spinors for the solution described in section 5 and in appendix A.

B.2 D2/D0 bound states in the $\mathcal{N} = 2$ solution

As discussed in the previous section, in order to study the supersymmetry of BPS particles obtained from wrapped branes, we need to form bilinears in the supersymmetry spinors η_+^{ia}. We first need to write them explicitly. A convenient basis to expand our spinors is given by the pair of spinors defining the $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure in (A.11). Recall that a $\mathrm{SU}(3)$ structure is specified by two invariant tensors (J, Ω) or, equivalently, by a spinor η_+ (of norm 1) such that

$$\begin{aligned} \eta_+ \otimes \eta_+^\dagger &= \frac{1}{8} e^{-iJ}\,, \\ \eta_+ \otimes \eta_-^\dagger &= -\frac{i}{8}\Omega\ . \end{aligned} \tag{B.25}$$

The $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure in (A.11) can be seen as the intersection of two $\mathrm{SU}(3)$ structures given by $(J_1, \Omega_1) = (j + \frac{i}{2} z \wedge \bar{z}, \omega \wedge z)$ and $(J_2, \Omega_2) = (-j + \frac{i}{2} z \wedge \bar{z}, -\bar{\omega} \wedge z)$. We call the corresponding spinors η_+ and χ_+. They are related by $\chi_+ = \frac{1}{\sqrt{2}} z \cdot \eta_-$, where $z\cdot$ denotes the Clifford multiplication by the one-form $z_m \gamma^m$. We will need in the following an expression for the tensor products of a generic linear combination

$$\begin{aligned} \mu_+ &= a\,\eta_+ + b\,\chi_+\,, \\ \nu_+ &= x\,\eta_+ + y\,\chi_+\ . \end{aligned} \tag{B.26}$$

This is given by [38]

$$\begin{aligned} \mu_+ \otimes \nu_+^\dagger &= \frac{1}{8}\Big[a\bar{x} e^{-ij} + b\bar{y} e^{ij} - i(a\bar{y}\omega + \bar{x} b\bar{\omega})\Big] e^{1/2 z\bar{z}}\,, \\ \mu_+ \otimes \nu_-^\dagger &= \frac{1}{8}\Big[i(by\bar{\omega} - ax\omega) + (bxe^{ij} - aye^{-ij})\Big] z\ . \end{aligned} \tag{B.27}$$

We can choose the spinors for the first supersymmetry as follows

$$\begin{aligned} \eta_+^{11} &= ie^{A/2+i\theta}(e^{i\frac{\pi}{4}}\cos(\psi)\,\eta_+ - ie^{-i\frac{\pi}{4}}\sin(\psi)\,\chi_+)\,, \\ \eta_+^{21} &= ie^{A/2}(e^{i\frac{\pi}{4}}\cos(\psi)\,\eta_+ + ie^{-i\frac{\pi}{4}}\sin(\psi)\,\chi_+)\ . \end{aligned} \tag{B.28}$$

It is easy to reproduce, using formula (B.27), the dielectric ansatz (A.9) for the pure spinors.

As discussed in appendix A, there is a U(1) family of supersymmetries obtained by rotating $o \to o_\alpha = e^{-i\alpha} o$. We can conveniently choose as a second independent supersymmetry the one with $o_\pi = -o$. This is defined by

$$
\begin{aligned}
\eta_+^{12} &= i e^{A/2+i\theta} (e^{i\frac{\pi}{4}} \cos(\psi)\, \tilde{\eta}_+ - i e^{-i\frac{\pi}{4}} \sin(\psi)\, \tilde{\chi}_+)\,, \\
\eta_+^{22} &= i e^{A/2} (e^{i\frac{\pi}{4}} \cos(\psi)\, \tilde{\eta}_+ + i e^{-i\frac{\pi}{4}} \sin(\psi)\, \tilde{\chi}_+)\,,
\end{aligned} \tag{B.29}
$$

where

$$
\begin{aligned}
\tilde{\eta}_+ &= -i \cos(2t)\, \eta_+ + i \sin(2t)\, \chi_+\,, \\
\tilde{\chi}_+ &= i \sin(2t)\, \eta_+ + i \cos(2t)\, \chi_+\ .
\end{aligned} \tag{B.30}
$$

This reproduces the rotated pure spinors $\Phi_\pm^\pi$.

With these ingredients, we can compute the spinors $\Psi_\pm$ and $\tilde{\Psi}_\pm$ defined in (B.13) and (B.24) and check the BPS conditions for a D2-brane. It is easy to see that the D2-brane considered in section 5, which wraps the diagonal S^2 and sits at the North or South pole, is indeed supersymmetric. Let us consider, for definiteness, the North pole. At $t = 0$, $\psi = 0$ and we see that $\eta_+^{i2} = -i\eta_+^{i1}$. As a consequence, at $t = 0$,

$$
\Psi_\pm = \pm i \Phi_\pm\,, \qquad \tilde{\Psi}_\pm = -i \Phi_\pm\,, \tag{B.31}
$$

and we are reduced to check expressions for the pure spinors $\Phi_\pm$ at the North pole. Taking into account that $\psi = 0$ there, we have

$$
\begin{aligned}
\Phi_+|_{t=0} &= \frac{i}{8} e^{A+i\theta} e^{-iJ}\,, \\
\Phi_-|_{t=0} &= \frac{i}{8} e^{A+i\theta} z \wedge \omega\ .
\end{aligned} \tag{B.32}
$$

The condition (B.20b) for Ψ_- (and the analogous (B.23b) for $\tilde{\Psi}_-$) gets contributions only from the contraction with the vector z and it is automatically satisfied because ω vanishes at the North pole, $t = 0$. It is easily seen that the conditions for Ψ_+ and $\tilde{\Psi}_+$ are equivalent and it is enough to analyze those for Ψ_+. Equation (B.20a) reads

$$
\mathrm{Im}\left(e^{i(\theta-\lambda)} e^{-ij}\right) \wedge e^{\mathcal{F}-B}|_{\mathcal{B}_2} = 0, \tag{B.33}
$$

and determines the world-volume field

$$
\mathcal{F} = (B + \cot(\theta - \lambda) j)|_{\mathcal{B}_2}. \tag{B.34}
$$

We see that a wrapped D2 brane can be made supersymmetric by choosing an appropriate world-volume field. However, as discussed in section 5.4, to have a consistent BPS state we need to impose the quantization of the world-volume field and the cancellation of tadpoles. As discussed there, the quantization condition requires to take n_0 D2-branes. On the other hand, the tadpole condition requires $\mathcal{F} = \beta$ or, equivalently, $\mathcal{F} - B = -F_2/F_0$. At $t = 0$,

JHEP11(2010)047

using the explicit form for the metric in (5.4), as well as (5.12), (5.18), (A.11), (A.10) and (A.13), we evaluate $\tan(\theta) = -\psi_1$ and $j = -\frac{1}{4}e^{2B_1}J_1$ and $F_2/F_0 = -\frac{1}{4}e^{2B_1}J_1/\psi_1$. Recall that J_1 is the volume form of one of the two S^2's, as defined in (5.3). We thus see that the tadpole condition is satisfied by $\lambda = 0$. The mass of n_0 D2 branes is then obtained by integrating the volume form in (B.19)

$$n_0 \int_{\mathcal{B}_2} \sqrt{\det(g + \mathcal{F} - B)} = n_0 \frac{1}{\sin(\theta)} \int_{\mathcal{B}_2} J = n_0 \frac{1}{4} e^{2B_1} \sqrt{\frac{1+\psi_1^2}{\psi_1}} \int_{\mathcal{B}_2} J_1 \ . \tag{B.35}$$

Using this, one exactly reproduces the result (5.41) of section 5.4.

A more detailed analysis of equations (B.20) (and the analogous ones for $\tilde{\Psi}_\pm$) shows that a D2-brane sitting at $t \neq 0, \pi/2$ cannot be supersymmetric and simultaneously satisfy the tadpole condition.

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