

Combinatoric topological string theories and group theory algorithms

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ABSTRACT: A number of finite algorithms for constructing representation theoretic data from group multiplications in a finite group G have recently been shown to be related to amplitudes for combinatoric topological strings (G -CTST) based on Dijkgraaf-Witten theory of flat G -bundles on surfaces. We extend this result to projective representations of G using twisted Dijkgraaf-Witten theory. New algorithms for characters are described, based on handle creation operators and minimal multiplicative generating subspaces for the centers of group algebras and twisted group algebras. Such minimal generating subspaces are of interest in connection with information theoretic aspects of the AdS/CFT correspondence. For the untwisted case, we describe the integrality properties of certain character sums and character power sums which follow from these constructive G -CTST algorithms. These integer sums appear as residues of singularities in G -CTST generating functions. S -duality of the combinatoric topological strings motivates the definition of an inverse handle creation operator in the centers of group algebras and twisted group algebras.

KEYWORDS: Discrete Symmetries, Topological Field Theories, AdS-CFT Correspondence, Topological Strings

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1 Introduction

Two-dimensional Dijkgraaf-Witten theories are simple examples of topological field theories associated to finite groups [1–4]. At a basic level, these theories describe orbifolds of points, $[\text{point}/G]$, possibly with discrete torsion (described in this context as a twisting). In the case when the group is a symmetric group S_n , these theories admit defects, which have applications in describing counting and correlators in $U(N)$ gauge theories [5] of interest in AdS/CFT [6–8]. Recent work on wormhole physics and baby universes [9–15], in the context of topology change in quantum gravity, considers sums over Riemann surfaces weighted by a string coupling g_{st} , where each surface supports a Dijkgraaf-Witten theory. We will refer to these theories, summing over worldsheets, as combinatoric topological string theories or G -CTST. Motivations and insights on the mathematical properties of these strings thus arise both from AdS/CFT and from models of topology change in quantum gravity.

Another place Dijkgraaf-Witten theories arise is in couplings to physical theories. For example, consider an orbifold $[X/\Gamma]$, where a subgroup $K \subset \Gamma$ acts trivially on X , as studied in e.g. [16–24]. This can be interpreted as a coupling of the orbifold $[X/G]$ (for $G = \Gamma/K$) to Dijkgraaf-Witten theory for the group K , as will be discussed in greater detail in [25]. The orbifold $[X/\Gamma]$ is in any event equivalent to a disjoint union of orbifolds, a result known as decomposition [21], which when viewed as a coupling of a topological field theory, reflects the fact that as a topological field theory, Dijkgraaf-Witten theory itself is a disjoint union of invertible field theories [26–29]. Applied to G -CTST, decomposition implies that the ‘string field theory’ of Dijkgraaf-Witten theory (in the same sense as [30]) is a theory on a disjoint union of points, which could be interpreted as a noninteracting statistical mechanical theory.

In the recent paper [11] it was observed that well-known formulae for amplitudes in G -CTST can be used to give a finite algorithm which starts from group multiplications in G and arrives at the integer ratios $|G|/(\dim R)$ (relating the order of a finite group G and the dimension of an irreducible representation R). The integrality of these ratios is an interesting old result at the intersection of finite group theory and number theory (see for example [31, 32]) and plays an important role in the algorithm. The form of the group multiplications in the input is understood geometrically in terms of the fundamental groups of two dimensional surfaces, which are interpreted in G -CTST as string worldsheets. The algorithm proceeds by finding the zeroes of a polynomial equation which has integer coefficients (which are G -CTST amplitudes) and has roots which are also known to be integers (i.e. the $|G|/(\dim R)$). The construction of representation theoretic quantities using combinatoric methods is an interesting general theme in representation theory [33], with

implications for computational complexity theory [34, 35]. G -CTST provides an interesting topological perspective on this theme. A quantum mechanics of bipartite ribbon graphs which constructs Kronecker coefficients as eigenvalue degeneracies of Hamiltonians [36] is another angle on the theme of exploiting stringy geometric/algebraic structures to address questions in combinatorial representation theory.

It is natural to consider twisted G -CTST (involving Dijkgraaf-Witten theories of orbifolds with discrete torsion) and its relation to the combinatorics of projective representations of G . In this paper we will show how the amplitudes in the vacuum sector of G -CTST can be used to obtain the integer ratios $|G|/(\dim R)$, where $\dim R$ is the dimension of a projective representation R . The algorithm takes as input group multiplications weighted by cocycle factors defining the twist, and proceeds by solving a polynomial equation as in [11]. (The fact that these ratios are always integers in the projective case is proven in [32], [37, theorem 3.5].)

Standard algorithms for the construction of characters were also shown in [11] to be related to amplitudes in G -CTST, for two dimensional surfaces with boundary circles. In this paper we show that the geometrical picture based on G -CTST, along with the study of generating subspaces of centers of group algebras [38], can be used to give new algorithms for characters. The handle creation operator of G -CTST plays a role in one class of such algorithms. An interesting corollary of this discussion is that string amplitudes with one boundary in G -CTST determine a distinguished subspace of the center of the twisted group algebra, $\mathcal{Z}(\mathbb{C}_\omega(G))$, of dimension equal to the number of distinct integers $\dim R$. This discussion will be presented for both the untwisted and the twisted case.

The study of generating subspaces of centers of symmetric group algebras in [38] was motivated by the consideration of a toy model for black hole information loss arising from the AdS/CFT correspondence [39]. A family of supergravity solutions [40] with $AdS_5 \times S^5$ asymptotics are dual to half-BPS states in the dual CFT labelled by Young diagrams [41]. As explained in [39] the asymptotic gravitational charges of the SUGRA solutions correspond to Casimirs of the $U(N)$ gauge symmetry in the CFT. The information loss model considers the information content in a finite number of Casimirs. For quantum states having energy n in the natural units, the Casimirs are related by Schur-Weyl duality to central elements in the group algebra of $\mathbb{C}(S_n)$. The information content in low order Casimirs translates into a question about how effectively low order cycle operators in the center of $\mathbb{C}(S_n)$ distinguish Young diagrams. This is in turn related to the dimensions of subspaces of the center generated by a finite set of central elements. In this paper we will be considering the generating subspaces for general finite groups G in connection with Dijkgraaf-Witten topological field theories. The embedding of this discussion into gauge-string dualities is an interesting problem for the future.

The paper is organised as follows. Section 2 explains the use of amplitudes in the vacuum sector of G -CTST to give finite algorithms starting from group multiplications in G weighted by appropriate cocycle factors and deriving the integer ratios $|G|/\dim R$ for projective representations R of finite groups G . The handle creation operator (C.39) for twisted group algebras plays an important role in this discussion. By considering one-point functions of twist field operators on higher genus surfaces, expressible combinatorially

using the handle creation operator, we give a combinatoric construction for the number of distinct dimensions $\dim R$ for irreducible representations of G , or irreducible projective representations of G . Section 3 extends the discussion to amplitudes in G -CTST for surfaces having boundaries to obtain algorithms for calculating characters. The constructions in sections 3.1, 3.2, 3.3 are used to obtain some integrality properties of certain sums of characters and sums of powers of characters in section 3.4, which in turn have implications for factorisation properties of certain polynomials which are used in character algorithms [45–47]. The integer sums and power sums of characters appear as residues for singularities in appropriate G -CTST partition functions. For simplicity this section focuses on the untwisted case. Section 4 collects a few remarks on G -CTST: we elaborate on the connection between determinants appearing in the algorithms of sections [2, 3] and plethystic exponentials of stringy amplitudes at low genus. We also comment on S -duality in G -CTST, which leads to the definition of an inverse handle creation operator. This is given as an expansion in terms of the projector basis of $\mathcal{Z}(\mathbb{C}_\omega(G))$, while its expansion in terms of the conjugacy class basis is an interesting question for the future.

2 Fourier transform and vacuum sector for G -CTST

The previous paper [11] studied computations of characters of ordinary representations of finite groups, as relevant to e.g. the AdS/CFT correspondence. In this section we generalize those computations to include discrete torsion, which twists the representations to projective representations. In broad brushstrokes, much of the analysis is formally similar to [11], so we will combine a review of the results of [11] while simultaneously describing novel features present in cases with discrete torsion.

To improve readability, we have banished a number of technical definitions and computations in cases with discrete torsion to appendix C, to which we refer as needed.

2.1 The twisted group algebra of a finite group $\mathbb{C}_\omega(G)$

Let G be a finite group and $[\omega] \in H^2(G, \mathbb{U}(1))$. In this section we will review properties of the twisted group algebra $\mathbb{C}_\omega(G)$ and its center $\mathcal{H} = \mathcal{Z}(\mathbb{C}_\omega(G))$, which will play an important role in our computations. Physically, the center \mathcal{H} is the state space of a two-dimensional (twisted) Dijkgraaf-Witten theory, which we will call G -CTST for short. Setting $\omega = 1$ in the formulae that follow recovers formulae for centers of ordinary group algebras $\mathcal{Z}(\mathbb{C}(G))$.

The twisted group algebra $\mathbb{C}_\omega(G)$ is a vector space, with basis elements we label τ_g corresponding to elements g of the group G , equipped with the product

$$\tau_g \tau_h = \omega(g, h) \tau_{gh}, \tag{2.1}$$

which is generically non-commutative. ω is a 2-cocycle representing the cohomology class $[\omega]$. Generic elements take the form

$$\sum_{g \in G} a_g \tau_g, \tag{2.2}$$

where $a_g \in \mathbb{C}$. $\mathbb{C}_\omega(G)$ has an inner product where the group elements are orthonormal:

$$\langle g_1 | g_2 \rangle = \delta(g_1 g_2^{-1}). \tag{2.3}$$

For general elements

$$\left\langle \sum_{g_1} a_{g_1} \tau_{g_1} \middle| \sum_{g_2} b_{g_2} \tau_{g_2} \right\rangle = \sum_{g_1, g_2} a_{g_1}^* b_{g_2} \delta(g_1 g_2^{-1}). \quad (2.4)$$

Now, we are interested in the center of $\mathbb{C}_\omega(G)$, denoted \mathcal{H} earlier, which is the subspace of $\mathbb{C}_\omega(G)$ which commutes with τ_g for any $g \in G$. It inherits an inner product from $\mathbb{C}_\omega(G)$ by restriction of (2.4). One basis for the center is given by twist fields, which are associated with ω -regular conjugacy classes. An element $g \in G$ is said to be ω -regular if for all h commuting with g ,

$$\omega(g, h) = \omega(h, g), \quad (2.5)$$

and an ω -regular conjugacy class is defined [42, section 3.6] to be a conjugacy class in which every element is ω -regular.

Given an ω -regular conjugacy class $[g]$ represented by $g \in G$, we define a twist field [42]

$$T_{[g]} = \frac{1}{|G|} \sum_{h \in G} \tau_h \tau_g \tau_h^{-1}, \quad (2.6)$$

$$= \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g) \omega(hg, h^{-1})}{\omega(h, h^{-1})} \tau_{hgh^{-1}}. \quad (2.7)$$

It can be shown (see for example [16, section 2.2.1]) that the twist fields commute with all elements of the twisted group algebra $\mathbb{C}_\omega(G)$, meaning

$$T_{[g]} \tau_h = \tau_h T_{[g]} \quad (2.8)$$

for all $h \in G$, and also the $\{T_{[g]}\}$ form a basis for the center.

Note that these operators $T_{[g]}$ depend upon the representative g of the conjugacy class: as shown in e.g. [42, section 3],

$$T_{[hgh^{-1}]} = \frac{\omega(gh^{-1}, h)}{\omega(h, gh^{-1})} T_{[g]}. \quad (2.9)$$

There is a second basis for the center, given by projectors associated to irreducible projective representations, which are in (noncanonical) one-to-one correspondence with ω -regular conjugacy classes. (Thus, there are as many projectors as twist fields.) Let us review some pertinent results on projective representations before defining those projectors.

Projectors will be constructed using characters of projective representations. Unlike characters of ordinary representations, characters of projective representations are not class functions, as they are not invariant under conjugation. If R is a projective representation of G , associated to some cocycle ω , and χ^R denotes the character, then [42, section 7.2, prop. 2.2]

$$\chi^R(g) = \frac{\omega(g, h^{-1})}{\omega(h^{-1}, hgh^{-1})} \chi^R(hgh^{-1}). \quad (2.10)$$

As a consistency check, it may be useful to note that

$$\chi^R(T_{[g]}) = \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g) \omega(hg, h^{-1})}{\omega(h, h^{-1})} \chi^R(hgh^{-1}), \quad (2.11)$$

$$= \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g) \omega(hg, h^{-1}) \omega(h^{-1}, hgh^{-1})}{\omega(h, h^{-1}) \omega(g, h^{-1})} \chi^R(g), \quad (2.12)$$

$$= \frac{1}{|G|} \sum_{h \in G} \chi^R(g) = \chi^R(g), \quad (2.13)$$

using the fact that

$$\begin{aligned} \frac{\omega(h, g) \omega(hg, h^{-1}) \omega(h^{-1}, hgh^{-1})}{\omega(h, h^{-1}) \omega(g, h^{-1})} &= (d\omega)(h^{-1}, hg, h^{-1}) (d\omega)(h^{-1}, h, g) \\ &\quad \cdot (d\omega)(h, h^{-1}, h), \\ &= 1. \end{aligned} \quad (2.14)$$

In fact, using this identity, one can show

$$\tau_h \tau_g \tau_h^{-1} = \frac{\omega(h, g) \omega(hg, h^{-1})}{\omega(h, h^{-1})} \tau_{hgh^{-1}} = \frac{\omega(g, h^{-1})}{\omega(h^{-1}, hgh^{-1})} \tau_{hgh^{-1}}, \quad (2.15)$$

so we can write (2.10) as

$$\chi^R(\tau_g) = \chi^R(\tau_h \tau_g \tau_h^{-1}). \quad (2.16)$$

As a result, although characters of projective representations are not invariant under conjugating group elements, they are invariant under conjugating τ 's.

Another important property of characters of projective representations is that they vanish on non- ω -regular group elements, see e.g. [42, section 7.2, prop. 2.2].

Now, we can define projectors, following [42, section 7.3], which are associated to irreducible projective representations, and which form another basis for the center of the twisted group algebra. These are given by

$$P_R = \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} \tau_g = \frac{\dim R}{|G|} \sum_{g \in G} \chi^R(\tau_g^{-1}) \tau_g, \quad (2.17)$$

where R is an irreducible projective representation. (Instead of summing over all group elements, one can equivalently sum only over ω -regular elements, as the character χ^R will vanish on non- ω -regular elements.) These form a complete, mutually orthogonal, basis for the center of the twisted group algebra, meaning that they obey

$$P_R P_S = \delta_{RS} P_S, \quad \sum_R P_R = 1. \quad (2.18)$$

They also obey the relation (B.18)

$$\delta(P_R) = \frac{(\dim R)^2}{|G|}. \quad (2.19)$$

These two bases (of twist fields, and of projectors) are related as follows:

$$P_R = \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} T_{[g]}, \tag{2.20}$$

(which formally matches the result of taking the definition (2.17) and replacing $\tau_g \in \mathbb{C}_\omega(G)$ with $T_{[g]}$, an element of the center), and

$$T_{[g]} = \sum_R \frac{\chi^R(g)}{\dim R} P_R. \tag{2.21}$$

These Fourier transforms are known, but for completeness, as they are perhaps somewhat obscure, next we will perform a consistency check and provide derivations.

As a consistency check, recall both $T_{[g]}$ and $\chi^R(g)$ transform under conjugation. However, using the identity

$$\frac{\omega(gh^{-1}, h)}{\omega(h, gh^{-1})} = \frac{\omega(h^{-1}, hgh^{-1})}{\omega(g, h^{-1})}, \tag{2.22}$$

a consequence of

$$(d\omega)(h^{-1}, h, gh^{-1}) (d\omega)(g, h^{-1}, h) = 1, \tag{2.23}$$

we see that both $T_{[g]}$ and $\chi^R(g)$ transform in the same way under $g \mapsto hgh^{-1}$, and so the identity (2.21) is consistent.

As a consequence, if C is any element of the center of the twisted group algebra, it can be expressed similarly. Write

$$C = \sum_{i=1}^n C_i T_{[h_i]}, \tag{2.24}$$

for $C_i \in \mathbb{C}$, so that

$$\chi^R(C) = \sum_{i=1}^n C_i \chi^R(h_i), \tag{2.25}$$

then from (2.21) we have

$$C = \sum_{i=1}^m C_i \left[\sum_R \frac{\chi^R(T_{[h_i]})}{\dim R} P_R \right], \tag{2.26}$$

$$= \sum_R \frac{\chi^R(C)}{\dim R} P_R. \tag{2.27}$$

We can establish (2.20) by direct computation, as follows.

$$\begin{aligned} & \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} T_{[g]} \\ &= \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} \frac{1}{|G|} \sum_h \frac{\omega(h, g) \omega(hg, h^{-1})}{\omega(h, h^{-1})} \tau_{hgh^{-1}}, \end{aligned} \tag{2.28}$$

$$= \frac{\dim R}{|G|^2} \sum_{h \in G} \frac{1}{\omega(h, h^{-1})} \tau_h \left[\sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} \tau_g \right] \tau_{h^{-1}}, \tag{2.29}$$

$$= \frac{1}{|G|} \sum_{h \in G} \frac{1}{\omega(h, h^{-1})} \tau_h P_R \tau_{h^{-1}}, \tag{2.30}$$

$$= \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, h^{-1})}{\omega(h, h^{-1})} P_R, \tag{2.31}$$

$$= P_R, \tag{2.32}$$

where we have used the fact that P_R is central in the group algebra.

We can establish (2.21) by direct computation, as follows.

$$\sum_R \frac{\chi^R(g)}{\dim R} P_R = \sum_R \frac{\chi^R(g)}{\dim R} \frac{\dim R}{|G|} \sum_{h \in G} \frac{\chi^R(h^{-1})}{\omega(h, h^{-1})} T_{[h]}, \tag{2.33}$$

$$= \frac{1}{|G|} \sum_{h \in G} \left[\sum_R \frac{\chi^R(g) \chi^R(h^{-1})}{\omega(h, h^{-1})} \right] T_{[h]}, \tag{2.34}$$

$$= \frac{1}{|G|} \sum_{h=aga^{-1} \in [g]} \frac{|G|}{|[g]|} \frac{\omega(a, g)}{\omega(h, a)} T_{[aga^{-1}]}, \tag{2.35}$$

$$= \frac{1}{|[g]|} \sum_{h=aga^{-1} \in [g]} \frac{\omega(a, g)}{\omega(aga^{-1}, a)} \frac{\omega(ga^{-1}, a)}{\omega(a, ga^{-1})} T_{[g]}, \tag{2.36}$$

$$= T_{[g]}, \tag{2.37}$$

using the index formula (B.4) and the fact that

$$\frac{\omega(a, g)}{\omega(aga^{-1}, a)} \frac{\omega(ga^{-1}, a)}{\omega(a, ga^{-1})} = (d\omega)(a, ga^{-1}, a) = 1. \tag{2.38}$$

2.2 Vacuum string amplitudes and $\mathcal{H}_0 \leftrightarrow \mathcal{H}$

We observe that the vacuum amplitudes of G -CTST are constructed by applying the delta-function on the twisted group algebra $\mathbb{C}_\omega(G)$ to powers of a handle creation operator Π . We show in section 2.2.1 that these powers generate a subspace of $\mathcal{Z}(\mathbb{C}_\omega(G))$ with dimension equal to the number of distinct dimensions ($\dim R$) of irreducible representations of $\mathbb{C}_\omega(G)$. In section 2.2.2 we show that one point functions of twist fields on higher genus surfaces can be used to determine sums of irreducible characters over irreducible representations having the same dimension.

2.2.1 The handle creation operator and twist fields

One convenient way of expressing the partition function of (twisted) Dijkgraaf-Witten theory on a genus h Riemann surface is as

$$Z_h = \frac{1}{|G|} \delta(\Pi^h), \tag{2.39}$$

where Π is the handle creation operator (a map $\mathbb{C}_\omega(G) \rightarrow \mathbb{C}_\omega(G)$ which descends to $\mathcal{H} \rightarrow \mathcal{H}$) which, for twisted theories, is defined in section C.2.

We can express the partition function more explicitly as follows. Using the identity (C.39), namely

$$\Pi = \sum_R \left(\frac{|G|}{\dim R} \right)^2 P_R, \tag{2.40}$$

so that

$$\Pi^h = \sum_R \left(\frac{|G|}{\dim R} \right)^{2h} P_R \tag{2.41}$$

(since P_R is an idempotent), and the identity (B.18), namely

$$\delta(P_R) = \frac{(\dim R)^2}{|G|}, \tag{2.42}$$

we have that the partition function is

$$Z_h = \frac{1}{|G|} \sum_R \left(\frac{|G|}{\dim R} \right)^{2h} \delta(P_R), \tag{2.43}$$

$$= |G|^{2h-2} \sum_R (\dim R)^{2-2h}, \tag{2.44}$$

which matches the expression (C.28) obtained independently. We conclude that

$$Z_h = \frac{1}{|G|} \delta(\Pi^h) = \sum_R \left(\frac{|G|}{\dim R} \right)^{2h-2}. \tag{2.45}$$

Using the formula (C.38) for Π , the calculation of the delta function on the left-hand side can be done from the combinatorics of multiplying the elements τ_g and picking up the coefficient of the identity. The formula, in the untwisted case, is well known in the mathematical literature [43, 44]. The combinatoric input from the left-hand side serves to give the power sums of $\frac{|G|}{\dim R}$. As explained in [11], we can go from the powers sums to the integers in a finite number of steps by solving for the zeroes of a polynomial with integer coefficients. We further elaborate in section 4.1 on the stringy interpretation of the polynomial in the context of G -CTST.

Products of Z_h , with appropriate symmetry factors, give us the vacuum sector of G -CTST. The vacuum sector of G -CTST defines two distinguished subspaces of $\mathcal{H} = \mathcal{Z}(\mathbb{C}_\omega(G))$. Complex multiples of Π form a one-dimensional subspace of \mathcal{H} . Powers of Π span a (generically) higher-dimensional vector subspace of \mathcal{H} .

Proposition. The powers of the handle creation operator Π span a vector subspace $\mathcal{H}_0 \hookrightarrow \mathcal{H}$ which has dimension D_0 equal to the number of distinct integers $\dim R$ as R runs over the set of irreducible projective representations.

Lemma ([48, Lemma 2.1, Prop. 2.3]). If we have a complete set of L orthogonal projectors P_i acting on a vector space and take a linear combination with distinct coefficients a_i

$$P = \sum_{i=1}^L a_i P_i, \tag{2.46}$$

then the powers of P generate a space of dimension equal to L .

Proof of proposition: We can write

$$\Pi = \sum_R \frac{|G|^2}{(\dim R)^2} P_R = \sum_{R'} \frac{|G|^2}{(\dim R')^2} \tilde{P}_{R'} \tag{2.47}$$

where R runs over all the distinct irreducible projective representations, and R' runs over a maximal list of irreducible projective representations having distinct dimensions, while $\tilde{P}_{R'}$ is a sum of the projectors for irreducible projective representations with the same dimension as R' . The list of projectors $\tilde{P}_{R'}$ spans a subspace $\mathcal{H}_0 \hookrightarrow \mathcal{H}$ of dimension D_0 . In this subspace \mathcal{H}_0 , we can use the Lemma to show that the powers of Π span \mathcal{H}_0 .

The proposition has a physical interpretation in terms of the rank of a matrix of one-point functions in G -CTST. Consider the one-point functions $M_{l,[g]} \equiv \delta(\Pi^l T_{[g]})$, with l ranging from 1 to K and g ranging over representatives of all the ω -regular conjugacy classes. (In the untwisted case this reduces to the set of all the conjugacy classes.) This matrix has rank D_0 . In the case where $M_{l,[g]}$ is a matrix with rational entries (this is the case for all untwisted cases and when the twists $\omega(g, h)$ can all be chosen to be rational), an integer basis for the null space can be found using discrete integer matrix algorithms. One approach is to use algorithms for Hermite normal forms (such as algorithm 2.4.4 of [49]) and extract the null vectors as explained for example in [36, section 4.1]. Such discrete algorithms for null vectors are available in computational group theory software GAP [50]. This gives a combinatoric algorithm, starting from group multiplication combinatorics, which produces an interesting representation theoretic integer: the number of distinct $(\dim R)$ among the irreducible (projective) representations of a (twisted) group algebra.

2.2.2 Character algorithm from higher genus one-point functions

By considering the one-point functions $\delta(\Pi^l T_{[g]})$ on general genus, for fixed $[g]$, we can extract information about characters of $\chi^R(T_\mu)/\dim R$. Consider

$$\frac{1}{|G|} \delta(\Pi^h T_{[g]}) = \sum_R \left(\frac{|G|^2}{(\dim R)^2} \right)^{h-1} \frac{\chi^R(g)}{\dim R}, \tag{2.48}$$

$$= \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2} \right)^{h-1} \sum_{R:R'} \frac{\chi^R(g)}{\dim R}, \tag{2.49}$$

for the range $l \in \{1, 2, \dots, D_0\}$, where we have used the identity (C.67). The primed sum runs over a maximal set $\{R'\}$ of irreducible representations R' having distinct dimensions. The sum over $\{R : R'\}$ is a sum over the distinct irreducible representations R with the same dimension as R' . Let us define \tilde{R}' to be the direct sum of irreducible projective representations R with the same dimension as R' . Then we can write

$$\frac{1}{|G|} \delta(\Pi^h T_{[g]}) = \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2} \right)^{h-1} \frac{\chi^{\tilde{R}'}(g)}{\dim R'}. \tag{2.50}$$

As h runs over the set $\{1, \dots, D_0\}$, we have a linear system of equations of size $D_0 \times D_0$ for the normalized characters $\chi^{\tilde{R}'}(g)/\dim R'$. As R' and l range over the D_0 possibilities, we have a matrix

$$\mathcal{V}_{R',h} = \left(\frac{|G|^2}{(\dim R')^2} \right)^{h-1} \tag{2.51}$$

of size $D_0 \times D_0$. The equation (2.50) takes the form

$$Y = \mathcal{V} \cdot X \tag{2.52}$$

where

$$\begin{aligned} Y_h &= \frac{1}{|G|} \delta(\Pi^h T_{[g]}), \\ X_{R'} &= \frac{\chi^{\tilde{R}'}(g)}{\dim R'}, \end{aligned} \tag{2.53}$$

and we recognize \mathcal{V} as a Vandermonde matrix. Since the R' have been chosen to run over a set of irreducible (projective) representations with distinct dimensions, the integers $\left(\frac{|G|^2}{(\dim R')^2}\right)$ are distinct. This ensures that \mathcal{V} is invertible. The inverse matrix can thus be used to construct the normalized characters $X_{R'}$ from the combinatoric G -CTST data Y_h . As explained earlier, the construction of the ratios $\left(\frac{|G|^2}{(\dim R')^2}\right)$ from G -CTST data follows using the formulae in section 2 in the twisted case, using the same algorithm described for the untwisted case in [11].

3 Character algorithms and string amplitudes

In the AdS/CFT correspondence, one is led in connection with toy models of black hole information loss [39] to consider questions of when sequences of central elements suffice to distinguish representations and multiplicatively generate the center of the group algebra [38]. In the context of TQFTs such as Dijkgraaf-Witten theory, it is natural to supplement such lists by the handle creation operator. To this end, in this section we present some general statements about subsets that multiplicatively generate the center of a (twisted) group algebra. We also use these generating subspaces to give algorithms for the construction of characters from string amplitudes in G -CTST. In the last subsection, we use these constructions to derive some integrality properties of characters and factorisation properties of character polynomials.

3.1 Minimal generating subspaces of (twisted) group algebras

We will say that a set of elements¹ $\{C_1, C_2, \dots, C_k\}$, with $C_i \in \mathcal{Z}(\mathbb{C}_\omega(G))$, multiplicatively generate $\mathcal{Z}(\mathbb{C}_\omega(G))$ if every element $T \in \mathcal{Z}(\mathbb{C}_\omega(G))$ can be written as a linear combination of products of elements C_i :

$$T = \sum_{n_1, n_2, \dots, n_k \geq 0} t_{n_1, n_2, \dots, n_k} C_1^{n_1} C_2^{n_2} \dots C_k^{n_k}. \quad (3.1)$$

The coefficients t_{n_1, n_2, \dots, n_k} are in \mathbb{C} , and C^0 is defined as 1, the identity element of the group algebra.

Proposition. The following two statements are equivalent:

- (1) A set of central elements $\{C_1, C_2, \dots, C_k\}$ multiplicatively generate $\mathcal{Z}(\mathbb{C}_\omega(G))$
- (2) The ordered lists of normalized characters $\{\frac{\chi^R(C_1)}{\dim R}, \frac{\chi^R(C_2)}{\dim R}, \dots, \frac{\chi^R(C_k)}{\dim R}\}$, for irreducible representations R of $\mathbb{C}_\omega(G)$ distinguish the irreducible representations, i.e. no two irreducible representations have the same list.

The proof uses the fact that each element C has an expansion in projectors P_R given by (2.27), which we repeat here:

$$C = \sum_R \frac{\chi^R(C)}{\dim R} P_R, \quad (3.2)$$

where the P_R form a complete set of orthogonal projectors, as in equation (2.18). Consider first the case where $k = 1$, and a single element $C_1 \in \mathcal{Z}(\mathbb{C}_\omega(G))$ has the property that $\{\frac{\chi^R(C_1)}{\dim R}\}$ distinguishes the irreducible representations R . The following fact is useful.

Lemma. If $T = \sum_R a_R P_R$ with a_R all distinct, then

$$P_R = \prod_{S \neq R} \frac{(T - a_S)}{(a_R - a_S)}. \quad (3.3)$$

We know that $\mathcal{Z}(\mathbb{C}_\omega(G))$ is spanned by the projectors P_R . Since (in the case $k = 1$) C_1 can be written as a linear combination of P_R with distinct coefficients, the lemma above implies each P_R can be written as a linear combination of powers of C_1 , hence $\{C_1\}$ multiplicatively generates $\mathcal{Z}(\mathbb{C}_\omega(G))$. The powers of C_1 range from 0 up to a maximum of $K - 1$ where K is the dimension of $\mathcal{Z}(\mathbb{C}_\omega(G))$.

Suppose now that $k = 2$, i.e. $\{C_1, C_2\}$ have lists of normalized characters $\{\frac{\chi^R(C_1)}{\dim R}, \frac{\chi^R(C_2)}{\dim R}\}$ which distinguish the irreducible representations R . We have

$$C_1 = \sum_R \frac{\chi^R(C_1)}{\dim R} P_R = \sum_{R'} \frac{\chi^{R'}(C_1)}{\dim R'} \widetilde{P}_{R'}, \quad (3.4)$$

where R' runs over a set of irreducible representations with distinct normalized characters $\chi^{R'}(C_1)/\dim R'$ and $\widetilde{P}_{R'}$ is the sum of projectors P_R for all R such that

$$\frac{\chi^R(C_1)}{\dim R} = \frac{\chi^{R'}(C_1)}{\dim R'}. \quad (3.5)$$

¹This argument expands the one presented in [38]. It was described there for untwisted group algebras, but the extension to twisted group algebras which we develop here has the same form.

Let us define $[C_1]_{R'}$ to be this set of irreducible representations R with the same normalized characters as R' . Then we may write

$$\widetilde{P}_{R'} = \sum_{R \in [C_1]_{R'}} P_R. \tag{3.6}$$

Let us denote the number of distinct R' in the sum for C_1 in (3.4) by K_1 , where by assumption $K_1 \leq K - 1$. Using the Lemma, we can write each $\widetilde{P}_{R'}$ as a linear combination of powers of C_1 . The largest power in these expressions is $(K_1 - 1)$. Consider now, for each R' ,

$$\widetilde{P}_{R'} C_2 = \sum_{R \in [C_1]_{R'}} \frac{\chi^R(C_2)}{\dim R} P_R. \tag{3.7}$$

By assumption, $\{\frac{\chi^R(C_1)}{\dim R}, \frac{\chi^R(C_2)}{\dim R}\}$ distinguish the irreducible representations, so it follows that for each R' , the $\frac{\chi^R(C_2)}{\dim R}$ are distinct as R ranges over the set $[C_1]_{R'}$. This means that we can apply the Lemma to express P_R as a linear combination of powers of the form

$$(\widetilde{P}_{R'} C_2)^l = \widetilde{P}_{R'} C_2^l. \tag{3.8}$$

Let $K_{a_1;R'}$ be the number of elements R in the set $[C_1]_{R'}$. The powers l range up to $K_{a_1;R'} - 1$. Since the $\widetilde{P}_{R'}$ have already been expressed in terms of powers of C_1 , we conclude that each P_R can be expressed as a linear combination of powers of $\{C_1, C_2\}$.

We can express this more symmetrically by writing

$$C_1 = \sum_{R'_1 \in [C_1]} \frac{\chi^{R'_1}(C_1)}{\dim R'_1} \widetilde{P}_{R'_1}, \tag{3.9}$$

$$C_2 = \sum_{R'_2 \in [C_2]} \frac{\chi^{R'_2}(C_2)}{\dim R'_2} \widetilde{P}_{R'_2} \tag{3.10}$$

where the sums run over representations with distinct normalized characters, and the projectors \widetilde{P} are defined with respect to the various sets $[C_i]$.

It is easy to see that this argument can be iterated for the cases of multiplicative generating subsets with more elements ($k > 2$).

We now describe another way to see that any projector P_R is a linear combination of products of central elements $\{C_1, C_2, \dots, C_k\}$ with the property given in (2) of the proposition. For each C_i , we can write

$$C_i = \sum_R \frac{\chi^R(C_i)}{\dim R} P_R = \sum_{R'_i} \frac{\chi^{R'_i}(C_i)}{\dim R'_i} \widetilde{P}_{R'_i}, \tag{3.11}$$

where R'_i runs over a maximal set S_i of irreducible representations with distinct normalized characters $\chi^{R'_i}(C_i)/\dim R'_i$. Let the cardinality of the set S_i be K_i and

$$\widetilde{P}_{R'_i} = \sum_{R_i \in [R'_i: C_i]} P_{R_i}. \tag{3.12}$$

We have introduced the notation $[R'_i : C_i]$ for the set of irreducible representations R_i with the property that

$$\frac{\chi^{R_i}(C_i)}{\dim R_i} = \frac{\chi^{R'_i}(C_i)}{\dim R'_i}. \quad (3.13)$$

The set S_i is not unique because the sets $[R'_i : C_i]$ generically have more than one element, but we will make a choice of S_i . Using the Lemma, the projectors $\widetilde{P}_{R'_i}$ can be written as a linear combination of powers of C_i . Now we know, by assumption, that any irreducible representation R is uniquely characterised by its normalised characters

$$\left\{ \frac{\chi^R(C_1)}{\dim R}, \frac{\chi^R(C_2)}{\dim R}, \dots, \frac{\chi^R(C_k)}{\dim R} \right\}. \quad (3.14)$$

This means that there is a unique list $[R'_1(R), R'_2(R), \dots, R'_k(R)]$ with $R'_1(R) \in S_1, R'_2(R) \in S_2, \dots, R'_k(R) \in S_k$, with the property that

$$\{R\} = [R'_1(R); C_1] \cap [R'_2(R); C_2] \cap \dots \cap [R'_k(R); C_k]. \quad (3.15)$$

This list is defined by the property that

$$\frac{\chi^R(C_1)}{\dim R} = \frac{\chi^{R'_1(R)}(C_1)}{\dim R'_1}, \quad (3.16)$$

$$\frac{\chi^R(C_2)}{\dim R} = \frac{\chi^{R'_2(R)}(C_2)}{\dim R'_2}, \quad (3.17)$$

$$\dots \quad (3.18)$$

$$\frac{\chi^R(C_k)}{\dim R} = \frac{\chi^{R'_k(R)}(C_k)}{\dim R'_k}. \quad (3.19)$$

It follows that

$$P_R = \widetilde{P}_{R'_1(R)} \widetilde{P}_{R'_2(R)} \cdots \widetilde{P}_{R'_k(R)}. \quad (3.20)$$

In the next several subsections we will apply these ideas to examples of sets of twist operators motivated by AdS/CFT, sometimes combined with handle creation operators as also motivated by Dijkgraaf-Witten theory.

3.1.1 Untwisted example: \mathbb{Z}_n

The group \mathbb{Z}_n has n irreducible representations, which we label ρ_r for $r \in \{0, \dots, n-1\}$. If g denotes the generator of \mathbb{Z}_n , and $\xi = \exp(2\pi i/n)$ the generator of n th roots of unity, then

$$\rho_r(g) = \xi^r = \chi_r(g). \quad (3.21)$$

From (2.7), the twist fields are

$$T_{[g^k]} = \frac{1}{|G|} \sum_{h \in G} \tau_{hg^k h^{-1}} = \tau_{g^k}, \quad (3.22)$$

and from the definition (2.17), we have that the projectors are

$$P_r = \frac{1}{n} \sum_{k=0}^{n-1} \chi_r(g^{-k}) \tau_{g^k} = \frac{1}{n} \sum_{k=0}^{n-1} \xi^{-rk} \tau_{g^k}. \quad (3.23)$$

Representation	1	g	g^2
1	1	1	1
R_1	1	ξ	ξ^2
R_2	1	ξ^2	ξ

Table 1. Character table for \mathbb{Z}_3 .

In particular, in this case the center of the group algebra $\mathbb{C}(\mathbb{Z}_n)$ coincides with the group algebra, and has dimension n .

From (C.39) we have that the handle creation operator is

$$\Pi = \sum_{r=0}^{n-1} n^2 P_r, \tag{3.24}$$

$$= \sum_{k=0}^{n-1} n^2 \left(\frac{1}{n} \sum_{r=0}^{n-1} \xi^{-rk} \right) \tau_{g^k}, \tag{3.25}$$

$$= \sum_{k=0}^{n-1} n^2 \delta_{k,0} \tau_{g^k}, \tag{3.26}$$

$$= n^2 \tau_1 = n^2. \tag{3.27}$$

Thus, we see that in this example the handle creation operator and its powers can only ever generate a one-dimensional subspace of the center of the group algebra. This is expected from section 2 since all the irreducible representations are one-dimensional, so the number of distinct values of $\dim R$ (D_0 in the discussion of section 2) is 1.

Now, let us turn to the question of constructing multiplicative generators. Consider for example the case of \mathbb{Z}_3 . Let g denote the generator of the group, and R_1, R_2 the two nontrivial representations, then the character table is given in table 1, where ξ generates cube roots of unity. In this case, we see that the irreducible representations are uniquely determined by the (normalized) characters of g , and it is also easy to check that $T_{[g]}$ generates all the twist fields multiplicatively:

$$T_{[g]}^2 = T_{[g^2]}, \quad T_{[g]}^3 = 1 = T_{[1]}. \tag{3.28}$$

3.1.2 Untwisted example: D_4

List the elements of the dihedral group D_4 as

$$\{1, z, a, b, az, bz, ab, ba = abz\}, \tag{3.29}$$

where z generates the \mathbb{Z}_2 center.

D_4 has five irreducible representations: four one-dimensional representations, and one two-dimensional representation. The character table of D_4 is given in table 2.

Since there are five conjugacy classes (also five irreducible representations), the center $\mathcal{Z}(\mathbb{C}(D_4))$ has dimension five. Note, however, that knowing the normalized characters of

	{1}	{z}	{a, az}	{b, bz}	{ab, ba}
1	1	1	1	1	1
1 _a	1	1	1	-1	-1
1 _b	1	1	-1	1	-1
1 _{ab}	1	1	-1	-1	1
2	2	-2	0	0	0

Table 2. Character table of D_4 (without a twist).

just two conjugacy classes suffices to distinguish characters. For example, from table 2, the characters of $T_{[a]}, T_{[b]}$ suffice to distinguish all the irreducible representations. (By contrast, for example, the normalized characters of $T_{[1]}$ and $T_{[z]}$ can only be used to distinguish the two-dimensional representation from the one-dimensional representation, but cannot distinguish between the one-dimensional representations.)

This tells us that although the center $\mathcal{Z}(\mathbb{C}(D_4))$ is a five-dimensional vector space, it is generated multiplicatively by $T_{[a]}$ and $T_{[b]}$, for example. Indeed, from (2.7) one finds

$$T_{[a]} = \frac{1}{2}(\tau_a + \tau_{az}), \quad T_{[b]} = \frac{1}{2}(\tau_b + \tau_{bz}), \quad (3.30)$$

and it is straightforward to check that

$$T_{[a]}^2 = \frac{1}{2}(1 + \tau_z) = T_{[b]}^2, \quad T_{[a]}T_{[b]} = T_{[ab]}, \quad (3.31)$$

$$T_{[a]}(1 + \tau_z) = 2T_{[a]}, \quad T_{[b]}(1 + \tau_z) = 2T_{[b]}. \quad (3.32)$$

Thus, the products of nonzero powers of $T_{[a]}$ and $T_{[b]}$ generate themselves, $T_{[ab]}$, and the combination $1 + T_z$, and when we include the zeroth power of $T_{[a]}, T_{[b]}$, we get all of the elements of the center.

3.1.3 Untwisted example: S_n

This question for the case of S_n is motivated by AdS/CFT and was recently studied [38] in untwisted cases. In that paper, central elements T_k correspond to conjugacy classes defined by permutations with a single non-trivial cycle of length k , and remaining cycles of length 1. For any $\mathcal{Z}(\mathbb{C}(S_n))$, the set $\{T_2, T_3, \dots, T_n\}$ generates the center [38] (Since there is no discrete torsion in this example, twist fields depend only upon conjugacy classes, not upon representatives, and so we only list the former.).

Typically a much smaller set $\{T_2, T_3, \dots, T_{k_*(n)}\}$ generates the center [38], where $k_*(n)$ is much smaller than n , which is equivalent to the statement that the normalized characters distinguish the irreducible representation R . For example, the single normalized character $\chi_R(T_2)/\dim R$ distinguishes R for n up to 5 and 7. The normalized characters of T_2, T_3 distinguish the Young diagrams up to $n = 14$. Using the formulae for normalized characters given in [51, 52] the lists $\{\frac{\chi_R(T_2)}{\dim R}, \frac{\chi_R(T_3)}{\dim R}\}$ were constructed for all the R at fixed n , and verified (in Mathematica) to be distinct for n up to 14. For tests at higher n (up to 80) it was convenient to convert the question (using formulae in [51, 52]) of comparing lists of

Irrep	(1 ⁴)	(21 ²)	(2 ²)	(31)	(4)
1	1	1	1	1	1
R ₂	1	-1	1	1	-1
R ₃	2	0	2	-1	0
R ₄	3	1	-1	0	-1
R ₅	3	-1	-1	0	1

Table 3. Character table of S_4 , from [53, table 4.5]. Conjugacy classes are indicated by the number of elements exchanged. For example, a “1” indicates that an element is mapped to itself, whereas a “4” indicates that all four elements are permuted, for example $1 \mapsto 2 \mapsto 3 \mapsto 4$. (The fact that this distinguishes conjugacy classes is discussed in e.g. [53, theorem 3.7].) In particular, (1⁴) is the conjugacy class of the identity.

normalized characters to a question of comparing lists of power sums of contents of Young diagrams (for the precise procedure see [38]).

The discussion in [38] is generalised here to consider central elements including the handle creation operator, alongside the cycle operators. Using computations in GAP, we verify that the pairs

$$[\dim R, \chi^R(T_2)] \tag{3.33}$$

uniquely determine all the Young diagrams of S_n for n up to 11, as well as 13. For example, the list of pairs at $n = 6$ is:

$$\{[1, -15], [5, -45], [9, -45], [5, -15], [10, -30], [16, 0], [5, 15], [10, 30], [9, 45], [5, 45], [1, 15]\}.$$

There is one such pair for every Young diagram. No two pairs are identical. Note that the list of ratios $\chi^R(T_2)/\dim R$ is

$$\{[-15], [-9], [-5], [-3], [-3], [0], [3], [3], [5], [9], [15]\}. \tag{3.34}$$

These numbers are not unique: -3 and 3 each appear twice. This means that T_2 does not generate the center of the group algebra of S_6 (as in [38]) but Π and T_2 together do.

Computations in GAP also show the lists $\{\dim R, \chi^R(T_2), \chi^R(T_3), \chi^R(T_4)\}$ distinguish all the irreducible representations for S_n at n up to at least 30. This means that the center is generated by $\{\Pi, T_2, T_3, T_4\}$ for $\mathbb{C}(S_n)$ with n up to at least 30.

For later comparisons, we give the character table of S_4 in table 3, from [53, table 4.5].

3.1.4 Untwisted example: \tilde{S}_n

The group \tilde{S}_n is a central extension of the symmetric group S_n by \mathbb{Z}_2 :

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \tilde{S}_n \longrightarrow S_n \longrightarrow 1. \tag{3.35}$$

It is described in [53, chapter 2] by generators $z, t_1, t_2, \dots, t_{n-1}$ and relations

$$z^2 = 1, \quad zt_j = t_jz, \quad t_j^2 = z, \tag{3.36}$$

$$(t_j t_{j+1})^2 = z \quad \text{for } 1 \leq j \leq n-2, \tag{3.37}$$

$$t_j t_k = z t_k t_j \quad \text{for } |j - k| > 1 \text{ and } 1 \leq j, k \leq n-1. \tag{3.38}$$

Irrep	$(1^4)'$	$(1^4)''$	(21^2)	(2^2)	$(31)'$	$(31)''$	$(4)'$	$(4)''$
1	1	1	1	1	1	1	1	1
R_2	1	1	-1	1	1	1	-1	-1
R_3	2	2	0	2	-1	-1	0	0
R_4	3	3	1	-1	0	0	-1	-1
R_5	3	3	-1	-1	0	0	1	1
R_6	2	-2	0	0	1	-1	$\sqrt{2}$	$-\sqrt{2}$
R_7	2	-2	0	0	1	-1	$-\sqrt{2}$	$\sqrt{2}$
R_8	4	-4	0	0	-1	1	0	0

Table 4. Character table for \tilde{S}_4 , from [53, table 4.7]. The notation for conjugacy classes references their images in S_4 , which are indicated with the same notation as in table 3. The primes refer to differences arising from including the central element z . For example, $(1^4)'$ is the conjugacy class of the identity, whereas $(1^4)''$ is the conjugacy class of z . See [53, theorem 3.8] for further details.

The character table of \tilde{S}_4 is given in table 4 (from [53, table 4.7]).

From table 4, we see for example that the normalized characters of the conjugacy classes $(31)'$ and $(4)'$ uniquely distinguish all the representations, hence, using the proposition in section 3.1, we expect that the center $\mathcal{Z}(\mathbb{C}(\tilde{S}_4))$ is multiplicatively generated by twist fields corresponding to those two elements.

3.1.5 Twisted example: $\mathbb{Z}_2 \times \mathbb{Z}_2$

Let us now turn to a simple twisted example, namely $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, with $[\omega] \in H^2(G, \mathbb{U}(1))$ the nontrivial element. A representative 2-cocycle ω is

$$\omega(a, b) = \omega(b, ab) = \omega(ab, a) = +i, \tag{3.39}$$

$$\omega(b, a) = \omega(ab, b) = \omega(a, ab) = -i, \tag{3.40}$$

where $G = \langle a, b \rangle$, and with $\omega(g, h) = +1$ for other g, h .

The only ω -regular conjugacy class in this case is $\{1\}$. From the definition (2.7), the twist fields are

$$T_{[1]} = \tau_1 = 1, \quad T_{[a]} = 0 = T_{[b]} = T_{[ab]}. \tag{3.41}$$

(Although there is only one ω -regular conjugacy class, we can certainly compute twist fields for other conjugacy classes, though as we see we do not get any further twist fields.)

There is only one irreducible projective representation [42, section 3.7], which we label ρ . It is two-dimensional, and for the 2-cocycle above can be represented by

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho(b) = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \rho(ab) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.42}$$

in the sense that

$$\rho(g)\rho(h) = \omega(g, h) \rho(gh). \tag{3.43}$$

From this and the definition (2.17), one quickly computes that the single projector is given by

$$P_\rho = \frac{\dim \rho}{|G|} \sum_{g \in G} \frac{\chi^\rho(g^{-1})}{\omega(g, g^{-1})} \tau_g = \tau_1 = 1, \tag{3.44}$$

essentially because only $\rho(1)$ has a nonzero trace. Then, using the identity (C.39), the handle creation operator is easily computed to be

$$\Pi = \sum_R \left(\frac{|G|}{\dim R} \right)^2 P_R = 4P_\rho = 4. \tag{3.45}$$

In this case, the center of the twisted group algebra is also one-dimensional, corresponding to complex multiples of the identity, and so Π generates the center, essentially trivially.

In passing, let us also compare to the character table of D_4 , table 2. Since D_4 is an extension of $\mathbb{Z}_2 \times \mathbb{Z}_2$, it includes information about the irreducible projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, which in this case is an honest representation of D_4 . Looking at table 2, we see the first four D_4 representations descend to representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$, because they take the same value on z as on the identity. The fifth representation, the two-dimensional one, takes a different value on z than on 1, and so does not arise from an ordinary representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$. This representation corresponds to the irreducible projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

3.1.6 Twisted example: D_4

Now, consider the $2n$ -element dihedral group $G = D_n$. This can be generated by a, b , such that

$$a^2 = 1, \quad b^n = 1, \quad aba = b^{-1}. \tag{3.46}$$

For simplicity, we assume n is even. This has a nontrivial element of $H^2(D_n, \mathbb{U}(1))$, given by

$$\omega(b^i, b^j a^k) = 1, \quad \omega(b^i a, b^j a^k) = \epsilon^j, \tag{3.47}$$

where ϵ generates the n th roots of unity. For n even, $b^{n/2}$ is central, and the dihedral group D_n has $n/2$ irreducible projective representations, each two-dimensional, described as follows² [42, section 3.7]. For $r \in \{1, \dots, n/2\}$, define

$$A_r = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} \epsilon^r & 0 \\ 0 & \epsilon^{1-r} \end{bmatrix}, \tag{3.48}$$

and then the r th representation is given by

$$\rho_r(b^i a^j) = B_r^i A_r^j, \tag{3.49}$$

for $i \in \{0, \dots, n-1\}$ and $j \in \{0, 1\}$.

²In our conventions we exchanged the roles of a and b relative to [42, section 3.7].

r	1	b^2	b	b^3	$\{a, ab^2\}$	$\{ab, ab^3\}$
$r = 1$	2	0	$+(1+i)$	$+(1-i)$	0	0
$r = 2$	2	0	$-(1+i)$	$-(1-i)$	0	0

Table 5. Character table for irreducible projective representations of D_4 . Note that although b and b^3 are in the same conjugacy class of D_4 , their characters are different, so we list them separately. Also note that only characters of representatives of ω -regular conjugacy classes are nonzero.

To make this more concrete, we specialize to D_4 , which has center \mathbb{Z}_2 , generated by b^2 . Here, $H^2(D_4, \mathbb{U}(1)) = \mathbb{Z}_2$, with a representative of the nontrivial cocycle given above. The conjugacy classes in D_4 are

$$\{1\}, \{b^2\}, \{b, b^3\}, \{a, ab^3\}, \{ab, ab^3\}, \tag{3.50}$$

of which only two are ω -regular, namely $\{1\}$ and $\{b, b^3\}$. From (2.7), twist fields are

$$T_{[1]} = \tau_1 = 1, \quad T_{[b]} = \frac{1}{2}(\tau_b + \epsilon\tau_{b^3}), \quad T_{[b^3]} = \frac{1}{2}(\epsilon^3\tau_b + \tau_{b^3}), \tag{3.51}$$

$$T_{[b^2]} = 0 = T_{[a]} = T_{[ab]}, \tag{3.52}$$

where ϵ generates fourth roots of unity, hence we can take $\epsilon = i$. (Only for the ω -regular conjugacy classes are the twist fields produced by (2.7) nonzero. Also, although b, bz are in the same equivalence class, $T_{[g]}$ is not invariant under conjugation, but instead are related by (2.9), as is easily checked to relate $T_{[b]}, T_{[b^3]}$ above.)

Since there are two ω -regular conjugacy classes, there are two (two-dimensional) irreducible projective representations, which are given by

$$\rho_r(1) = I, \quad \rho_r(a) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \rho_r(b) = \begin{bmatrix} \epsilon^r & 0 \\ 0 & \epsilon^{1-r} \end{bmatrix}, \tag{3.53}$$

$$\rho_r(b^2) = \begin{bmatrix} \epsilon^{2r} & 0 \\ 0 & \epsilon^{2-2r} \end{bmatrix}, \quad \rho_r(b^3) = \begin{bmatrix} \epsilon^{3r} & 0 \\ 0 & \epsilon^{3-3r} \end{bmatrix}, \quad \rho_r(ba) = \begin{bmatrix} 0 & \epsilon^4 \\ \epsilon^{1-r} & 0 \end{bmatrix}, \tag{3.54}$$

$$\rho_r(b^2a) = \begin{bmatrix} 0 & \epsilon^{2r} \\ \epsilon^{2-2r} & 0 \end{bmatrix}, \quad \rho_r(b^3a) = \begin{bmatrix} 0 & \epsilon^{3r} \\ \epsilon^{3-3r} & 0 \end{bmatrix}. \tag{3.55}$$

Since there are two irreducible projective representations, the twisted group algebra of D_4 has a two-dimensional center. We give the character table for projective representations of D_4 in table 5.

Plugging into (2.17), we have

$$P_r = \frac{1}{4} \left[2 + (\epsilon^{3r} + \epsilon^{3-3r})\tau_b + (\epsilon^r + \epsilon^{1-r})\tau_{b^3} \right], \tag{3.56}$$

using the fact that $\epsilon^{2r} + \epsilon^{2-2r} = 0$. (As a consistency check, it is straightforward to show that $P_r^2 = P_r$, $P_1P_2 = 0$, and $P_1 + P_2 = 1$.)

From (C.39), we have

$$\Pi = \sum_r \left(\frac{|D_4|}{\dim \rho_r} \right)^2 P_r = (16)(P_1 + P_2) = 16, \tag{3.57}$$

using the fact that $P_1 + P_2 = 1$. We see immediately that $\Pi^2 \propto \Pi$, and so the handle creation operator generates a one-dimensional subspace of the two-dimensional center of the twisted group algebra of D_4 . On the other hand, note that

$$T_{[b]}^2 = \epsilon/2 \propto 1, \tag{3.58}$$

hence the center can be multiplicatively generated by $T_{[b]}$ alone, which is consistent with table 5.

3.2 Character algorithms and generating subspaces

In [11, section 3.1] the first author and his collaborators interpreted the Burnside construction [45](see [46, 47] for subsequent improvements) in terms of (untwisted) combinatoric amplitudes on genus one surfaces. The key formula, which takes the same form in the twisted case, is (C.67), which implies

$$\frac{1}{|G|} \delta(\Pi T_{[g]}^l) = \sum_R \left(\frac{\chi^R(g)}{\dim R} \right)^l. \tag{3.59}$$

Using the power sums, we solve a polynomial equation to get the normalized characters for the twist fields $T_{[g]}$. The polynomial equation is actually the eigenvalue equation for the matrix of structure constants $(\mathfrak{C}_{[g]})_\alpha^\beta = \mathfrak{C}_{[g]\alpha}^\beta$ where $T_{[g]}T_\alpha = \mathfrak{C}_{[g]\alpha}^\beta T_\beta$. After the normalized characters have been found, the dimensions can be found using the orthogonality relation (B.4), which implies

$$\sum_R \frac{1}{\omega(g, g^{-1})} \chi^R(g) \chi^R(g^{-1}) = \frac{|G|}{|[g]|}. \tag{3.60}$$

It is interesting to consider the implications for the character algorithms of knowing a subset of (ω -regular) conjugacy classes whose normalized characters determine the irreducible representations. Suppose a set of central elements $\{C_1, C_2, \dots, C_k\}$ (possibly including Π) are known to multiplicatively generate the center $\mathcal{Z}(\mathbb{C}_\omega(G))$ of a (possibly twisted) group algebra. In the case of the untwisted group algebra of S_n (for $n < 80$) it has been shown [38] that there are interesting small (compared to n) subsets which have this property. In section 3.3 we explain how to find such minimal generating subsets.

Let us first consider the case where a single operator $C_1 \in \mathcal{Z}(\mathbb{C}_\omega(G))$ multiplicatively generates the center, as we have seen occurs in examples in sections 3.1.1, 3.1.3. In this case, following a construction similar to the use of the Vandermonde matrices in section 2, we can compute the characters of any (represented, ω -regular) conjugacy class \mathcal{C}_μ from the genus one amplitudes associated with C_1, T_μ and the normalized characters of C_1 . Specifically we start with the string amplitudes (C.67)

$$\frac{1}{|G|} \delta(\Pi C_1^k T_\mu) = \sum_R \left(\frac{\chi^R(C_1)}{\dim R} \right)^k \frac{\chi^R(T_\mu)}{\dim R}. \tag{3.61}$$

(It suffices to only consider $k \in \{0, 1, \dots, K - 1\}$, where K is the number of conjugacy classes.)

In terms of the Vandermonde matrix

$$\mathcal{V}_{k,R} = \left(\frac{\chi^R(C_1)}{\dim R} \right)^k \tag{3.62}$$

the expression (3.61) is an invertible linear system of equations relating string amplitudes to the normalized characters of T_μ . By using the inverse of the Vandermonde matrix, we can solve for the normalized characters $\chi^R(T_\mu)/\dim R$ in terms of the string amplitudes in (3.61) and the normalized characters of the generator C_1 , both assumed known.

Suppose now that $\{C_1, C_2\}$ are a minimal set that multiplicatively generate $\mathcal{Z}(\mathbb{C}_\omega(G))$, as we have seen in examples in sections 3.1.2, 3.1.3, 3.1.6. In such cases the lists $\{\chi^R(C_1)/\dim R, \chi^R(C_2)/\dim R\}$ uniquely determine the irreducible representations R . Now, we can again consider the problem of determining the normalized characters for a general conjugacy class (with specified representative) T_μ , from the string amplitudes. Start with the amplitudes

$$\frac{1}{|G|} \delta(\Pi C_1^k T_\mu) = \sum_R \left(\frac{\chi^R(C_1)}{\dim R} \right)^k \frac{\chi^R(T_\mu)}{\dim R}. \tag{3.63}$$

(As before, it suffices to restrict to $k \in \{0, 1, \dots, K' - 1\}$, where K' is the number of distinct normalized characters $\chi_R(C_1)/\dim R$.) Let R' run over a set of irreducible representations (of size K') with distinct normalized characters $\chi_{R'}(C_1)/\dim R'$, and $[R : R']$ over the irreducible representations with the same normalized characters as R' . We write

$$\frac{1}{|G|} \delta(\Pi C_1^k T_\mu) = \sum_{R'} \left(\frac{\chi^{R'}(C_1)}{\dim R'} \right)^k \sum_{[R:R']} \frac{\chi^R(T_\mu)}{\dim R}. \tag{3.64}$$

By inverting³ the $K' \times K'$ Vandermonde matrix with matrix elements

$$\mathcal{V}_{k,R'} = \left(\frac{\chi^{R'}(C_1)}{\dim R'} \right)^k \tag{3.65}$$

we now determine the sums

$$\sum_{[R:R']} \frac{\chi^R(T_\mu)}{\dim R} \tag{3.66}$$

ranging over the distinct irreducible representations R having the same normalized character $\chi_R(C_1)/\dim R$ as R' . We denote the number of such R (the number of elements of $[R : R']$) by $D_{1,R'}$.

Using the fact that

$$\left\{ \frac{\chi_R(C_1)}{\dim R}, \frac{\chi_R(C_2)}{\dim R} \right\} \tag{3.67}$$

³The reader should note that if for example $C_1 = T_{[1]} = 1$, then the Vandermonde matrix may not be invertible, and this procedure would not work. However, we exclude that case from consideration, by restricting to minimal generating sets.

distinguish all irreducible representations, we know that for any R' , as R ranges over the set $[R : R']$, the list $\{\chi_R(C_2)/\dim R\}$ has no repeated elements. Now for each R' , and each $l \in \{0, 1, \dots, D_{1,R'} - 1\}$ we can consider

$$\begin{aligned} \frac{1}{|G|} \delta \left(\Pi C_1^k (C_2^l T_\mu) \right) &= \sum_R \left(\frac{\chi^R(C_1)}{\dim R} \right)^k \left(\frac{\chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(T_\mu)}{\dim R}, \\ &= \sum_{R'} \left(\frac{\chi^{R'}(C_1)}{\dim R'} \right)^k \sum_{[R:R']} \left(\frac{\chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(T_\mu)}{\dim R}. \end{aligned} \quad (3.68)$$

As k ranges over $\{0, 1, \dots, K' - 1\}$, we have a linear system of equations for

$$\sum_{R:R'} \left(\frac{\chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(T_\mu)}{\dim R} \quad (3.69)$$

given by the invertible $K' \times K'$ Vandermonde matrix (3.65). By using the inverse of the Vandermonde matrix, we obtain

$$\sum_{R:R'} \left(\frac{\chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(T_\mu)}{\dim R}. \quad (3.70)$$

Collecting the results for all the $l \in \{0, 1, \dots, D_{1,R'} - 1\}$, we now have a linear system for $\chi^R(T_\mu)/\dim R$ for all the R in the set $[R; R']$, given by the invertible $D_{1,R'} \times D_{1,R'}$ Vandermonde matrix with matrix elements

$$\mathcal{V}_{\ell,R} = \left(\frac{\chi^R(C_2)}{\dim R} \right)^l. \quad (3.71)$$

By inverting the Vandermonde matrix, we obtain $\chi^R(T_\mu)/\dim R$ for all R with the property that

$$\frac{\chi^R(C_1)}{\dim R} = \frac{\chi^{R'}(C_1)}{\dim R'}. \quad (3.72)$$

It is clear that the above procedure can be iterated to give a procedure for constructing normalized characters T_μ in cases where a longer list

$$\left\{ \frac{\chi^R(C_1)}{\dim R}, \frac{\chi^R(C_2)}{\dim R}, \dots, \frac{\chi^R(C_k)}{\dim R} \right\} \quad (3.73)$$

distinguish the irreducible representations (equivalently $\{C_1, C_2, \dots, C_k\}$ generate the center). Note that the generating set of central elements can all be obtained by averaging over fixed conjugacy classes, and may also include central operators such as the handle operator Π as discussed in section 3.1.

3.2.1 Untwisted example: \mathbb{Z}_n

In this section we will illustrate the method in a case with well-known results, specifically, the case $G = \mathbb{Z}_3$.

As discussed in section 3.1.1, if g generates the group \mathbb{Z}_n , then T_g generates the center multiplicatively. Following the prescription given above, the Dijkgraaf-Witten amplitudes determine the normalized characters of any other conjugacy class. Specifically, write

$$\frac{1}{|G|} \delta \left(\prod T_{[g]}^k T_\mu \right) = \sum_R \mathcal{V}_{k,R} \frac{\chi^R(T_\mu)}{\dim R}, \quad (3.74)$$

where

$$\mathcal{V}_{k,R} = \left(\frac{\chi^R(T_{[g]})}{\dim R} \right)^k. \quad (3.75)$$

Using table 1, we have

$$\mathcal{V}_{k,R=1} = 1, \quad (3.76)$$

$$\mathcal{V}_{k,R=R_1} = \xi^k, \quad (3.77)$$

$$\mathcal{V}_{k,R=R_2} = \xi^{2k}, \quad (3.78)$$

for ξ a generator of cube roots of unity, hence

$$\mathcal{V} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \xi & \xi^2 \\ 1 & \xi^2 & \xi \end{bmatrix}, \quad \mathcal{V}^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \xi^2 & \xi \\ 1 & \xi & \xi^2 \end{bmatrix}. \quad (3.79)$$

Let us also take as given the string amplitudes

$$\frac{1}{|G|} \delta \left(\prod T_{[g]}^k T_1 \right) = 3\delta_{0,k \pmod 3}, \quad (3.80)$$

$$\frac{1}{|G|} \delta \left(\prod T_{[g]}^k T_{[g]} \right) = 3\delta_{0,k+1 \pmod 3}, \quad (3.81)$$

$$\frac{1}{|G|} \delta \left(\prod T_{[g]}^k T_{[g^2]} \right) = 3\delta_{0,k+2 \pmod 3}. \quad (3.82)$$

From these string amplitudes we then compute

$$\left(\frac{\chi^R(T_{[1]})}{\dim R} \right) = \mathcal{V}^{-1} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad (3.83)$$

matching the known result

$$\frac{\chi^R(T_{[1]})}{\dim R} = 1 \quad (3.84)$$

for each representation R . Similarly,

$$\left(\frac{\chi^R(T_{[g]})}{\dim R} \right) = \mathcal{V}^{-1} \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ \xi \\ \xi^2 \end{bmatrix}, \quad (3.85)$$

matching the result

$$\frac{\chi^1(T_{[g]})}{\dim \mathbf{1}} = 1, \quad \frac{\chi^{R_1}(T_{[g]})}{\dim R_1} = \xi, \quad \frac{\chi^{R_2}(T_{[g]})}{\dim R_2} = \xi^2. \quad (3.86)$$

Finally,

$$\left(\frac{\chi^R(T_{[g^2]})}{\dim R} \right) = \mathcal{V}^{-1} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \xi^2 \\ \xi \end{bmatrix}, \quad (3.87)$$

matching the result

$$\frac{\chi^1(T_{[g]})}{\dim \mathbf{1}} = 1, \quad \frac{\chi^{R_1}(T_{[g]})}{\dim R_1} = \xi^2, \quad \frac{\chi^{R_2}(T_{[g]})}{\dim R_2} = \xi. \quad (3.88)$$

Again, we emphasize that the point of this section is merely to illustrate the method in a simple well-known example.

3.2.2 Twisted example: $\mathbb{Z}_2 \times \mathbb{Z}_2$

Let us apply the algorithm above to the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with a twist, as discussed in section 3.1.5.

As discussed there, the center is one-dimensional, generated by Π .

Now, suppose we are given the string amplitudes

$$Y_k = \frac{1}{|G|} \delta \left(\Pi \Pi^k T_{[1]} \right), \quad (3.89)$$

and we want to compute the normalized characters of $T_{[1]}$. (Clearly, this will be trivial, but $[1]$ is the only ω -regular conjugacy class, so for purposes of illustrating the method, we will walk through this example.) From (C.67), we know that

$$\frac{1}{|G|} \delta \left(\Pi^{k+1} T_{[g]} \right) = \sum_R \left(\frac{|G|}{\dim R} \right)^{2(k+1)-2} \left(\frac{\chi^R(g)}{\dim R} \right), \quad (3.90)$$

which is a linear system of equations relating the normalized characters $\chi^R(1)/\dim R$ to the Y_k and the Vandermonde matrix

$$\mathcal{V}_{k,R} = \left(\frac{|G|}{\dim R} \right)^{2k}, \quad (3.91)$$

and can be written in the form

$$\vec{Y} = \mathcal{V} \vec{\chi}, \quad (3.92)$$

where $\vec{\chi}$ is the vector of normalized characters $\chi^R(1)/\dim R$ desired.

In the present case, $\mathbb{Z}_2 \times \mathbb{Z}_2$ with a twist, there is only one irreducible projective representation, of dimension 2, hence

$$\mathcal{V}_{k,R} = \left(\frac{|G|}{2} \right)^{2k} = 2^{2k}, \quad (3.93)$$

so our system of equations is simply

$$Y_k = (2^{2k}) \left(\frac{\chi^R(1)}{\dim R} \right). \tag{3.94}$$

(To be clear, this is many equations for one unknown, which is why in general we restrict to a finite number of values of k .)

In principle this allows one to compute the normalized characters in terms of the Y_k . In this particular case, it is a fact that $Y_k = 2^{2k}$, so we see that

$$\frac{\chi^R(1)}{\dim R} = 1, \tag{3.95}$$

or simply,

$$\chi^R(1) = \dim R, \tag{3.96}$$

a result which will not surprise the reader, but which will hopefully help to illuminate the idea of the method.

3.2.3 Twisted example: D_4

Now, let us apply these ideas to the case of D_4 with a twist, using the computations in section 3.1.6. Here, let us take the (two-dimensional) center of the twisted group algebra to be generated by $\{T_{[b]}\}$, and use the string amplitudes (Dijkgraaf-Witten correlation functions) to compute the normalized characters and reproduce the character table 5.

As before, suppose we are given the string amplitudes

$$Y(\mu)_k = \frac{1}{|G|} \delta \left(\prod T_{[b]}^k T_\mu \right), \tag{3.97}$$

which are related to the normalized characters of T_μ by

$$Y(\mu)_k = \sum_R \mathcal{V}_{k,R} \frac{\chi^R(T_\mu)}{\dim R}, \tag{3.98}$$

for

$$\mathcal{V}_{k,R} = \left(\frac{\chi^R(T_{[b]})}{\dim R} \right)^k. \tag{3.99}$$

As there are only two irreducible projective representations, it suffices to take $k \in \{0, 1\}$ and write $\mathcal{V}_{k,R}$ as the entries of a matrix

$$\mathcal{V} = \begin{bmatrix} 1 & 1 \\ +(1+i)/2 & -(1+i)/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ +\exp(i\pi/4)/\sqrt{2} & -\exp(i\pi/4)/\sqrt{2} \end{bmatrix}. \tag{3.100}$$

Using

$$\mathcal{V}^{-1} = \frac{1}{2} \begin{bmatrix} 1 + \sqrt{2} \exp(-i\pi/4) \\ 1 - \sqrt{2} \exp(-i\pi/4) \end{bmatrix}, \tag{3.101}$$

one can then compute normalized characters from string amplitudes, formally as

$$\left(\frac{\chi^R(T_\mu)}{\dim R}\right) = \mathcal{V}^{-1}\vec{Y}(\mu). \tag{3.102}$$

For example, for $\mu = [b^3]$, the string amplitudes are

$$\vec{Y}([b^3]) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \tag{3.103}$$

which implies

$$\left(\frac{\chi^R(T_{[b^3]})}{\dim R}\right) = \mathcal{V}^{-1}\vec{Y} = \begin{bmatrix} +(1-i) \\ -(1-i) \end{bmatrix}, \tag{3.104}$$

correctly matching table 5.

3.2.4 Twisted example: S_n

In this section we discuss the symmetric group S_n with discrete torsion.

First, let us describe the discrete torsion. We can do this implicitly using the extension \tilde{S}_n presented in section 3.1.4, and comparing to the presentation of S_n itself in section 3.1.3. Specifically, the extension is determined by an element of $H^2(S_n, \mathbb{Z}_2)$, which maps into $H^2(S_n, U(1))$ and so determines an element of discrete torsion.

We can compute the cocycle as follows, following [53, pp. 9–10]. Let $\theta : \tilde{S}_n \rightarrow S_n$ be the projection, with kernel $\{1, z\}$, and let r be a section, meaning a map $r : S_n \rightarrow \tilde{S}_n$, such that $\theta(r(a)) = a$ and $r(1) = 1$. A cocycle is given explicitly by

$$\alpha_r(a, b) = (-1)^{n_r(a,b)}, \tag{3.105}$$

where

$$r(a)r(b) = z^{n_r(a,b)}r(ab). \tag{3.106}$$

We can pick

$$\begin{aligned} \theta(t_i) &= x_i, \\ \theta(z) &= 1, \\ r(x_i) &= t_i \quad r(x_{i_1}x_{i_2}\cdots) = t_{i_1}t_{i_2}\cdots \end{aligned} \tag{3.107}$$

The section can also be used to construct projective representations of S_n from the ordinary representations of \tilde{S}_n . Given representation matrices $R(\tilde{g})$ for $\tilde{g} \in \tilde{S}_n$, one gets projective representation matrices $P(g)$ as

$$P(g) = R(r(g)) \tag{3.108}$$

as in [53, Theorem 1.4].

As an example to illustrate the use of the above equations, consider the symmetric group S_4 . It has three generators $\{x_1, x_2, x_3\}$, which are the adjacent transpositions

$x_1 = (1, 2), x_2 = (2, 3), x_3 = (3, 4)$. The section r is defined by mapping words in the x_i to words in t_i . As an example of cocycle factors deduced from the above equations, note that

$$r(x_1).r(x_1) = t_1 t_1 = z = zr(x_1^2) \tag{3.109}$$

Hence

$$\alpha(x_1, x_1) = (-1). \tag{3.110}$$

Using the projection θ and the section r , the above equations specify a map from $\mathbb{C}(\tilde{S}_4)$ to $\mathbb{C}_\omega(S_4)$. Using the character table for \tilde{S}_4 (table 4), we note that the characters for elements in \tilde{S}_4 , in the last three rows associated with non-trivial twist, and corresponding to cycle structures $(2, 1^2), (3, 1)$ are zero. This means that the only non-zero ω -regular classes in $\mathbb{C}_\omega(S_4)$ correspond to cycle structures $(1^4), (3, 1), (4)$. The equality of the number of ω -regular conjugacy classes and irreducible projective reps illustrates our discussion of the center $\mathcal{Z}(\mathbb{C}_\omega(G))$: we observed that there is a basis for the center in terms of twist operators labelled by ω -regular conjugacy classes and another basis in terms of projectors, labelled by irreducible projective representations. The splitting of $(3, 1)$ and (4) into two columns illustrates the fact that characters are not class functions in the case of projective representations. Focusing on the column $(4)'$, and taking into account the dimensions of irreducible projective reps (given in the last three entries in the first column labelled by (1^4)), we find that the normalized characters $\{1/\sqrt{2}, -1/\sqrt{2}, 0\}$ distinguish the three irreducible projective reps. Following our discussion in section 3.1, this means that a central element labelled by conjugacy class (4) can be used to multiplicatively generate the center of $\mathcal{Z}(\mathbb{C}_\omega(S_4))$.

3.3 Algorithm for minimal generating subsets

In the above, we have assumed we are given central elements which distinguish irreducible representations, or equivalently, multiplicatively generate the center $\mathcal{Z}(\mathbb{C}_\omega(G))$. In this section, we outline an algorithm finding a minimal generating subset of the center of a twisted group algebra, using the topological field theory amplitudes. We start with a central element C_a . We can determine its normalized characters using the Burnside algorithm [45–47], equivalently as explained earlier, by considering genus one amplitudes with insertions of boundaries labelled by C_a . If the number of distinct eigenvalues, i.e. the number of distinct normalized characters $\frac{\chi_R(C_a)}{\dim R}$ is equal to the dimension of $\mathcal{Z}(\mathbb{C}_\omega(G))$, then we know that C_a generates the center. But suppose the number of distinct eigenvalues is smaller. Let us ask how to determine whether adding another central element C_b indeed generates the center. This can be done by considering the structure constants of the multiplication operator for C_a, C_b in the basis of central elements labelled by conjugacy class operators T_μ

$$C_a T_\mu = (\mathfrak{C}_a)_\mu^\nu T_\nu, \tag{3.111}$$

$$C_b T_\mu = (\mathfrak{C}_b)_\mu^\nu T_\nu. \tag{3.112}$$

These structure constants can be obtained from G -CTST amplitudes on the sphere:

$$\frac{1}{|T_\nu|} \delta(C_a T_\mu T_\nu) = \mathfrak{C}_{a\mu}^\nu, \tag{3.113}$$

$$\frac{1}{|T_\nu|} \delta(C_b T_\mu T_\nu) = \mathfrak{C}_{b\mu}^\nu. \tag{3.114}$$

We know from (2.27) that the projectors P_R obey

$$C_a P_R = \frac{\chi^R(C_a)}{\dim R} P_R, \tag{3.115}$$

$$C_b P_R = \frac{\chi^R(C_b)}{\dim R} P_R. \tag{3.116}$$

If C_a, C_b generate the center, then the simultaneous eigenspaces of the matrices $(\mathfrak{C}_a), (\mathfrak{C}_b)$ are one-dimensional with eigenvalues

$$\left(\frac{\chi^R(C_a)}{\dim R}, \frac{\chi^R(C_b)}{\dim R} \right). \tag{3.117}$$

Motivated by AdS/CFT applications of minimal generating subspaces, we can start with the twist field associated to (a representative of) the smallest conjugacy class T_{a_1} (excluding the conjugacy class of the identity) and the associated structure constant matrix \mathfrak{C}_{a_1} obtained from G -CTST amplitudes involving T_{a_1} , then alongside consider \mathfrak{C}_{a_2} for the next smallest conjugacy class. If the simultaneous eigenspaces are one-dimensional, we have a generating subspace spanned by (T_{a_1}, T_{a_2}) . If the simultaneous eigenspaces are more than one-dimensional, we add another central element T_{a_3} and simultaneously diagonalize $\mathfrak{C}_{a_1}, \mathfrak{C}_{a_2}, \mathfrak{C}_{a_3}$. If the eigenspaces are one-dimensional, then the ordered lists of eigenvalues of $\{\mathfrak{C}_{a_1}, \mathfrak{C}_{a_2}, \mathfrak{C}_{a_3}\}$ which give

$$\left\{ \frac{\chi^R(T_{a_1})}{\dim R}, \frac{\chi^R(T_{a_2})}{\dim R}, \frac{\chi^R(T_{a_3})}{\dim R} \right\}, \tag{3.118}$$

can be used to label the irreducible representations.

To find the eigenvalues for these basis elements in a minimal generating subspace, we have to solve the eigenvalue equations for the $K \times K$ matrices, where K is the dimension of the center. For the characters of the remaining conjugacy classes, we use the inversion of Vandermonde matrices of smaller size as explained above.

3.4 G -CTST and properties of characters of finite groups

In this section we will use the properties of handle-creation operators in G -CTST from section 2, and the AdS/CFT-inspired construction of characters using minimal generating subspaces from the previous subsections 3.1, 3.2, 3.3, to derive certain integrality properties of residues of poles of partition functions appearing in G -CTST.

Along the road to those physics results, we will derive some mathematical properties of characters of finite groups. We expect that these properties are already known in the

mathematical literature; we are not claiming any fundamental mathematical novelty. We include them because they follow from the framework of G -CTST and are related to the properties of singularities in generating functions arising therein. The methods in the proof are based on the combinatorics of group multiplications along with linear algebra. Similar methods have been used to obtain integrality properties of characters in, for example, [54]. A comprehensive textbook discussion of these properties is in chapter 3 of [55].

In section 3.4.1 we begin by deriving integrality properties for sums of characters of a given conjugacy class \mathcal{C}_μ , where the characters are being summed over certain restricted classes of irreducible representations. The restrictions depend on the dimension of the irreducible representation or the character of certain additional specified conjugacy classes, where these conjugacy classes have the property that all their characters are integers. In section 3.4.2 we extend the discussion to obtain integrality properties of sums of powers of characters, where the sums are constrained by similar restrictions as in 3.4.1. We show that the integrality of these power sums is equivalent to factorisation properties of polynomials arising in the Burnside algorithm [45–47] for the computation of characters, which we will refer to as Burnside character polynomials. In section 3.4.3 we show that the integer sums of normalized characters considered in 3.4.1 and 3.4.2 arise as residues of singularities in generating functions of G -CTST.

For simplicity we will restrict to Dijkgraaf-Witten theories without discrete torsion (twisting) in this section.

3.4.1 Integrality properties of some character sums

In this subsection we will derive some properties of characters that we will use in the analysis of poles of G -CTST generating functions.

First, it is useful to rewrite (3.59) with an adjusted normalization

$$\frac{1}{|G|} \delta \left(\Pi \left(|[g]| T_{[g]} \right)^l \right) = \sum_R \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^l. \tag{3.119}$$

The ratios $|[g]| \chi^R(g) / \dim R$ in the right-hand side are known to be algebraic integers. This follows from the fact that eigenvalues of integer matrices (in this case, the matrix of structure constants of multiplication by the central elements $|[g]| T_{[g]}$ in $\mathcal{Z}(\mathbb{C}(G))$) are algebraic integers (see e.g. [31, chapter 3]). It is also known that algebraic integers form a ring. Hence a sum of algebraic integers is an algebraic integer. Thus, the sum

$$\frac{|[g]| \chi^{\tilde{R}'}(g)}{\dim R'} = \sum_{R:R'} \frac{|[g]| \chi^R(g)}{\dim R}, \tag{3.120}$$

(where the sum is over all the irreducible representations R with a fixed $\dim R' = \dim R$ as in section 2.2.2) is an algebraic integer. It is useful to rewrite (2.50) with the normalization

$$\frac{1}{|G|} \delta \left(\Pi^h |[g]| T_{[g]} \right) = \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2} \right)^{h-1} \frac{|[g]| \chi^{\tilde{R}'}(g)}{\dim R'}. \tag{3.121}$$

For the untwisted case $\mathbb{C}(G)$ the left-hand side gives a sequence of rational numbers for different values of h . In section 2.2.2 we inverted the Vandermonde matrix of integers, applied it to a finite vector with the rational numbers on the left-hand side above, to give the characters $\chi^{\tilde{R}'}(g)/\dim R'$. Applying the same procedure here, we see that the normalized characters $||g||\chi^{\tilde{R}'}(g)/\dim R'$ are rational numbers. Now, any algebraic integer which is rational is also integer (see e.g. [31, chapter III]). This means that the sums of normalized characters in (3.120) are always integers, for any $\mathbb{C}(G)$ (even though the individual terms in the sum may not be integers).

To summarize, these arguments suggest the following

Proposition 3.4.1-I. The sum of normalized characters

$$\sum_{R:R'} \frac{||g||\chi^R(g)}{\dim R} = \frac{||g||}{\dim R'} \sum_{R:R'} \chi^R(g) \tag{3.122}$$

over all the irreducible representations R of a fixed dimension $\dim R'$ is an integer for any finite group G .

This is easy to verify in examples by inspection of finite group character tables. In addition, we expect that the statement above, as well as the other propositions in this section, likely already exist in the literature, though we are not able to give precise references. We include them here because we will use these results in the analysis of poles of G -CTST generating functions. We are not claiming any fundamental mathematical novelty.

It is also known that the characters $\chi^R(g)$ are algebraic integers (e.g. [31, chapter III]), hence the sum

$$\chi^{\tilde{R}'}(g) \equiv \sum_{R:R'} \chi^R(g) \tag{3.123}$$

is an algebraic integer. The rationality of $||g||\chi^{\tilde{R}'}(g)/\dim R'$ explained above also implies that $\chi^{\tilde{R}'}(g)$ is rational. Using again the fact that rational algebraic integers are integers, we conclude that $\chi^{\tilde{R}'}(g)$ are integers. We state this as

Proposition 3.4.1-II. The sum of the characters

$$\sum_{R:R'} \chi^R(g) \tag{3.124}$$

over all irreducible representations R with a fixed dimension $\dim R'$, is an integer, for any finite group G .

A corollary of the discussion on integrality of character sums above, is that if for every irreducible representation R of a finite group G which has a unique value of the dimension, i.e. a value not shared by any other irreducible representation, the characters $\frac{||g||\chi^R(g)}{\dim R}$ and $\chi^R(g)$ are integers for g in any conjugacy class.

Following the discussion in section 3.2 where we consider linear systems for a given $\frac{\chi^R(T_\mu)}{\dim R}$ using a pair of central elements, we can generalize the above argument. We start again with the untwisted case $\mathbb{C}(G)$. Consider central elements $\{C_1, C_2\}$, chosen to have the property that $\frac{\chi^R(C_1)}{\dim R}$ and $\frac{\chi^R(C_2)}{\dim R}$ are both integers for all R . We do not require here that

C_1, C_2 generate the center $\mathcal{Z}(\mathbb{C}(G))$ in the present discussion. The key equation is (3.68), part of which we repeat for convenience, is

$$\frac{1}{|G|} \delta \left(\Pi C_1^k (C_2^l | C_\mu | T_\mu) \right) = \sum_R \left(\frac{\chi^R(C_1)}{\dim R} \right)^k \left(\frac{\chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(|C_\mu| T_\mu)}{\dim R}. \quad (3.125)$$

It is worth noting that the product in the sum above, namely

$$\left(\frac{\chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(|C_\mu| T_\mu)}{\dim R}, \quad (3.126)$$

is a product of algebraic integers and hence itself an algebraic integer. Using the discussion in section 3.2, we can construct the character sums

$$\sum_{R:[C_1, C_2]} \frac{\chi^R(|C_\mu| T_\mu)}{\dim R} \quad (3.127)$$

where R is being summed over all the irreducible representations having a fixed pair of eigenvalues for $[C_1, C_2]$, using inverses of integer Vandermonde matrices multiplying the combinatoric data on the left-hand side of (3.125) consisting of rational numbers. Thus we conclude that these sums, which are known to be algebraic integers, are in fact integers. This also means that the character of a $\chi^R(g)$ of a group element $g \in C_\mu$ is rational, and since it is known to be an algebraic integer, also in fact integer. By taking C_1 to be the handle creation operator with eigenvalues $\frac{|G|^2}{(\dim R)^2}$ and C_2 the sum of elements in a conjugacy class \mathcal{C} with the property that $\chi^R(g)$ for $g \in \mathcal{C}$ is an integer for all irreducible representations R , we conclude

Proposition 3.4.1-III. The character sums

$$\sum_{R:[\dim R', \chi^{R''}(\mathcal{C})]} \frac{\chi^R(|C_\mu| T_\mu)}{\dim R} \quad (3.128)$$

and

$$\sum_{R:[\dim R', \chi^{R''}(\mathcal{C})]} \chi^R(g) \text{ for } g \in C_\mu \quad (3.129)$$

for any conjugacy class \mathcal{C}_μ , over irreducible representations with a fixed specified dimension denoted $\dim R'$ and a fixed value of the character for the conjugacy class \mathcal{C} , are integers.

If we take $[C_1, C_2]$ to be two conjugacy classes having integer characters, then we have

Proposition 3.4.1-IV. The character sums

$$\sum_{R:[\chi^{R_1}(C_1), \chi^{R_2}(C_2)]} \frac{\chi^R(|C_\mu| T_\mu)}{\dim R} \quad (3.130)$$

and

$$\sum_{R:[\chi^{R_1}(C_1), \chi^{R_2}(C_2)]} \chi^R(g) \text{ for } g \in C_\mu \quad (3.131)$$

for any conjugacy class \mathcal{C}_μ are integers, where the sum is over all irreducible representations which have fixed characters $[\chi^{R_1}(\mathcal{C}_1), \chi^{R_2}(\mathcal{C}_2)]$ for two conjugacy classes $\mathcal{C}_1, \mathcal{C}_2$, and where these latter are conjugacy classes known to have integer characters for all irreducible representations R . This property for \mathcal{C}_μ generalizes to the case where we fix the characters for any number of conjugacy classes $\{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m\}$ having the property that all their irreducible characters are integers. We also have this integrality property for \mathcal{C}_μ when we fix $\{\dim R, \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m\}$.

Integrality properties of fusion matrices and quantum dimensions have recently been studied using Galois theory methods [56] in the context of 3D topological quantum field theories. The combination of Galois theory methods with the constructive methods used here in general classes of topological field theories would be an interesting area for future investigation.

3.4.2 Integrality of power sums and factorisation properties of character polynomials

In the previous subsection, as part of our physical analysis of G -CTST, we derived some intermediate mathematical integrality properties involving single characters. In this subsection we similarly derive integrality properties for power sums of characters which have implications for the factorization properties of the Burnside character polynomials. In the next subsection we will apply these properties to the analysis of generating functions in G -CTST.

For a conjugacy class \mathcal{C}_μ consider a diagonal matrix X_μ of size K , with entries $\frac{|\mathcal{C}_\mu| \chi^{R(g)}}{\dim R}$ for $g \in \mathcal{C}_\mu$ where K is the number of conjugacy classes in G . The determinant $\det(x - X_\mu)$ is a polynomial in x

$$\begin{aligned} \det(x - X_\mu) &= x^K - x^{K-1} \text{tr} X + \dots + (-1)^K \det X_\mu, \\ &= \sum_{i=1}^K (-1)^i x^{K-i} e_i(X) \end{aligned} \tag{3.132}$$

where $e_i(X)$ are elementary symmetric polynomials. They can be expressed in terms of traces of X and in terms of the eigenvalues of x_i of X as

$$\begin{aligned} e_k(X) &= \sum_{p \vdash k} \frac{(-1)^{k - \sum_i p_i}}{\prod_i i^{p_i} p_i!} \prod_i (\text{tr} X^i)^{p_i}, \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq n} x_{i_1} x_{i_2} \dots x_{i_p} \end{aligned} \tag{3.133}$$

Here p is a partition of k , with p_i parts of length i , so that $\sum_i i p_i = k$. As reviewed in [11] the quantity $\det(x - X_\mu)$, viewed as a polynomial in x , is also the characteristic polynomial for the integer matrix $\frac{|\mathcal{C}_\mu| |\mathcal{C}_\nu|}{|\mathcal{C}_\lambda|} (\mathfrak{C}_\mu)_\nu^\lambda$ of structure constants of $\mathcal{Z}(\mathbb{C}(G))$. Solving for the eigenvalues of the matrix of structure constants for conjugacy classes \mathcal{C}_μ is a step in determining the character table in the Burnside algorithm [45]. A useful piece of terminology is that $\det(x - X_\mu)$ is an integer monic polynomial: a monic polynomial has the coefficient of the highest power of x to be equal to 1 while all the other coefficients are also integers.

The above arguments for integrality of sums of characters apply equally well for the power sums. In this case we consider, for fixed k and for $h \in \{0, 1, \dots, K - 1\}$

$$\begin{aligned} & \frac{1}{|G|} \delta \left(\Pi^h (|[g]| T_{[g]})^k \right) \\ &= \sum_R \left(\frac{|G|^2}{(\dim R)^2} \right)^{h-1} \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^k, \\ &= \sum_{R'} \left(\frac{|G|^2}{(\dim R')^2} \right)^{h-1} \sum_{R:R'} \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^k. \end{aligned} \tag{3.134}$$

The last line includes a sum over irreps R having a fixed dimension $\dim R'$. This allows us to write, in terms of an inverse Vandermonde matrix, the power sums over irreducible representations R of G with a fixed dimension $\dim R = \dim R'$

$$\sum_{R:R'} \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^k. \tag{3.135}$$

These are known, on general grounds, to be algebraic integers. Applying the reasoning in section 3.4.1 above to these power sums, they can be expressed as a matrix product of a rational matrix (inverse of a Vandermonde matrix) times a vector of rational numbers (obtained from the evidently rational numbers on the l.h.s. of (3.134)). This means that these sums of powers, restricted to all irreducible representations R having the same dimension as R' , are actually integers.

It is now useful to consider a diagonal matrix $X_\mu^{(R')}$ of size equal to the number $K^{(R')}$ of distinct irreducible representations with the same dimension as R' , and with entries equal to $\frac{|[g]| \chi^R(g)}{\dim R}$ as R ranges over the distinct R with the specified dimension. We can construct a polynomial $\det(x - X_\mu^{(R)})$ of degree $K^{(R')}$. The coefficients of the powers of x are elementary symmetric polynomials $e_i(X^{(R)})$, expressible as polynomials in these normalized characters $\frac{|[g]| \chi^R(g)}{\dim R}$ for R having fixed dimension $\dim R'$. Since these normalized characters are known to be algebraic integers, the elementary symmetric polynomial functions of these (which are sums of products of these according to the second line in (3.133)) are algebraic integers. These elementary symmetric polynomials are also expressible in terms of linear combinations with rational coefficients of power sums (first line of (3.133)). These power sums are integers as explained above. Combining these facts, and since numbers which are rational and algebraic integer are also integers, we conclude that these coefficients of powers of x in $\det(x - X_\mu^{(R)})$ are actually integers. Thus $\det(x - X_\mu^{(R)})$ is an integer monic polynomial in the variable x . Since the diagonal entries of the diagonal matrix $X_\mu^{(R')}$ form a subset of the entries of the diagonal matrix X_μ defined above, we see that $\det(x - X_\mu^{(R)})$ is an integer monic polynomial which is a factor of the Burnside character polynomial $\det(x - X_\mu)$. We summarise this conclusion as

Proposition 3.4.2-I. The Burnside character polynomial for any conjugacy class \mathcal{C}_μ , which is an integer monic polynomial, factorises into lower degree integer monic polynomials

parametrised by the list of distinct dimensions $\dim R'$

$$\det(x - X_\mu) = \prod_{R'} \det(x - X_\mu^{(R')}). \quad (3.136)$$

Following the discussion in section 3.3.1, we can also consider further integrality properties for powers of normalised characters summed over sets of irreps restricted by dimension as well as characters of conjugacy classes. By following the argument above, this integrality of power sums leads to more refined factorisation properties of the Burnside character polynomials. Suppose \mathcal{C}_1 is a conjugacy class with integer characters. i.e. for all irreducible representations R of G , the characters $\chi^R(g)$ for $g \in \mathcal{C}_1$ are integers. Let $\chi^{\mathcal{C}_1; R'_1}$ be the list of the distinct values of these characters, and $K^{\mathcal{C}_1; R'_1}$ be the multiplicity of the eigenvalue. We have $\sum_{R'_1} K_{\mathcal{C}_1; R'_1} = K$. Let $X_\mu^{(\mathcal{C}_1; R'_1)}$ be the diagonal matrix with entries $\frac{\chi^R(\mathcal{C}_\mu)}{\dim R}$ for irreducible representations R having

$$\chi^R(g) = \chi^{R'_1}(g) \text{ for } g \in \mathcal{C}_1, \quad \chi^R(g) = \chi^{\mathcal{C}_1; R'_1}. \quad (3.137)$$

The polynomial $\det(x - X_\mu^{\mathcal{C}_1; R'_1})$ is an integer monic polynomial.

Proposition 3.4.2-II. The Burnside character polynomial for any conjugacy class \mathcal{C}_μ , which is an integer monic polynomial, factorises into lower degree integer monic polynomials parametrised by the list of distinct characters $\chi^{\mathcal{C}_1; R'_1}$

$$\det(x - X_\mu) = \prod_{R'_1} \det(x - X_\mu^{\mathcal{C}_1; R'_1}). \quad (3.138)$$

Let the pair $[R', R'_1]$ be labels for pairs of irreducible representations which run over the distinct possible values of $[\dim R, \chi^R(g)]$ for $g \in \mathcal{C}_1$. Let $K^{\Pi, \mathcal{C}_1; R', R'_1}$ be the multiplicity of the pair of values associated with $[R', R'_1]$. For any other conjugacy class $\mathcal{C}_\mu \neq \mathcal{C}_1$, we can construct the integer monic polynomial $\det(x - X_\mu^{\Pi, \mathcal{C}_1; R', R'_1})$ of degree $K^{\Pi, \mathcal{C}_1; R', R'_1}$. We have the factorisation property

Proposition 3.4.2-III. The Burnside character polynomial for any conjugacy class \mathcal{C}_μ , which is an integer monic polynomial, factorises into lower degree integer monic polynomials parametrised by the list of distinct ordered pairs $[\dim R', \chi^R(g)]$ for $g \in \mathcal{C}_1$

$$\det(x - X_\mu) = \prod_{R, R'_1} \det(x - X_\mu^{\Pi, \mathcal{C}_1; R', R'_1}). \quad (3.139)$$

These factorisation properties can further be generalised to run over lists $[\dim R', \chi^R(g_1), \dots, \chi^R(g_m)]$ for $g_1 \in \mathcal{C}_1, g_2 \in \mathcal{C}_2, \dots, g_m \in \mathcal{C}_m$ where $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m$ have integer characters. We can also drop $\dim R'$ from the lists to have factorisation over distinct lists $[\chi^R(g_1), \dots, \chi^R(g_m)]$.

3.4.3 Integral power sums as residues of singularities in G -CTST generating functions

In this subsection we now apply the properties we have derived to the analysis of G -CTST generating functions.

We observe that the integer sums of normalized characters and sums of powers of normalized characters derived in sections 3.4.1 and 3.4.2 arise as residues at singularities of G -CTST generating functions. The argument is an extension of the one in section 5 of [11]. Let us define a sum over arbitrary numbers of handles of the string amplitude with one boundary labelled by conjugacy class \mathcal{C}_μ (3.121) weighted by the appropriate power of the string coupling. Taking $g \in \mathcal{C}_\mu$, i.e. $[g] = \mathcal{C}_\mu$ we write

$$g_{st}^{-1} Z(g_{st}; \mathcal{C}_\mu) = \sum_{h=0}^{\infty} \frac{g_{st}^{2h-2}}{|G|} \delta\left(\Pi^h |[g]| T_{[g]}\right) = \sum_h \sum_{R'} \left(\frac{g_{st}^2 |G|^2}{(\dim R')^2} \right)^{h-1} \frac{|[g]| \chi^{\tilde{R}'}(g)}{\dim R'}, \quad (3.140)$$

$$= \sum_{R'} \frac{1}{(1 - g_{st}^2 |G|^2 / (\dim R')^2)} \frac{|[g]| \chi^{\tilde{R}'}(g)}{\dim R'}. \quad (3.141)$$

The poles of this generating function are at

$$g_{st} = \frac{(\dim R')}{|G|}, \quad (3.142)$$

and the residues are

$$\frac{|[g]| \chi^{\tilde{R}'}(g)}{\dim R'} = \frac{|[g]|}{\dim R'} \sum_{R:R'} \chi^R(g), \quad (3.143)$$

which we showed to be integers (proposition 3.4.1-I). Similarly we can define a stringy generating function for the k 'th power sums

$$g_{st}^{k-2} Z(g_{st}; \mathcal{C}_\mu, k) = \sum_{h=0}^{\infty} \frac{g_{st}^{2h-2}}{|G|} \delta\left(\Pi^h (|[g]| T_{[g]})^k\right), \quad (3.144)$$

$$= \sum_{h=0}^{\infty} \sum_R \left(\frac{g_{st}^2 |G|^2}{(\dim R)^2} \right)^{h-1} \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^k, \quad (3.145)$$

$$= \sum_{h=0}^{\infty} \sum_{R'} \left(\frac{g_{st}^2 |G|^2}{(\dim R')^2} \right)^{h-1} \sum_{R:R'} \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^k, \quad (3.146)$$

$$= \sum_{R'} \frac{1}{(1 - g_{st}^2 |G|^2 / (\dim R')^2)} \sum_{R:R'} \left(\frac{|[g]| \chi^R(g)}{\dim R} \right)^k. \quad (3.147)$$

The singularities are at

$$g_{st} = \frac{(\dim R')}{|G|}, \quad (3.148)$$

while the respective residues are

$$\sum_{R:R'} \left(\frac{|[g]|\chi^R(g)}{\dim R} \right)^k. \tag{3.149}$$

As shown in proposition 3.4.1-II these residues of the G -CTST generating function defined are integers.

The connection between integer character sums and residues of G -CTST partition functions extends to the more refined sums considered in sections 3.4.1 and 3.4.2. As an example consider (3.125) involving powers of two conjugacy class sums C_1, C_2 and a single power of C_μ and let us introduce a partition function depending on two chemical potentials μ_1, μ_2

$$\begin{aligned} Z(\mu_1, \mu_2; C_1, C_2) &= \sum_{k,l=0}^{\infty} \mu_1^k \mu_2^l \frac{1}{|G|} \delta \left(\Pi C_1^k (C_2^l | C_\mu | T_\mu) \right), \\ &= \sum_{k,l=0}^{\infty} \sum_R \left(\frac{\mu_1 \chi^R(C_1)}{\dim R} \right)^k \left(\frac{\mu_2 \chi^R(C_2)}{\dim R} \right)^l \frac{\chi^R(|C_\mu|T_\mu)}{\dim R}, \\ &= \sum_R \frac{1}{\left(1 - \frac{\mu_1 \chi^R(C_1)}{\dim R}\right) \left(1 - \frac{\mu_2 \chi^R(C_2)}{\dim R}\right)} \frac{\chi^R(|C_\mu|T_\mu)}{\dim R}, \\ &= \sum_{R_1, R_2} \frac{1}{\left(1 - \frac{\mu_1 \chi^{R_1}(C_1)}{\dim R_1}\right) \left(1 - \frac{\mu_2 \chi^{R_2}(C_2)}{\dim R_2}\right)} \sum_{R: [\chi^{R_1}(C_1), \chi^{R_2}(C_2)]} \frac{\chi^R(|C_\mu|T_\mu)}{\dim R}. \end{aligned} \tag{3.150}$$

In the last line, we have introduced sums over a complete set of pairs of irreducible representations R_1, R_2 which have distinct character values $[\chi^{R_1}(C_1), \chi^{R_2}(C_2)]$. For each pair of values, we have a sum over R running over the distinct irreducible representations having these characters. It follows that the singularities of these generating functions are at

$$\mu_1 = \frac{\dim R_1}{\chi^{R_1}(C_1)}, \quad \mu_2 = \frac{\dim R_2}{\chi^{R_2}(C_2)}. \tag{3.151}$$

The residues at these singularities are

$$\sum_{R: [\chi^{R_1}(C_1), \chi^{R_2}(C_2)]} \frac{\chi^R(|C_\mu|T_\mu)}{\dim R}. \tag{3.152}$$

These residues are integers as explained in Proposition 3.3.2-IV.

4 Further remarks on G -CTST and future directions

We collect a few comments here on the stringy interpretation of the determinants that have played a central role in the algorithms earlier in the paper. We find a link to plethystic exponentials of low genus amplitudes. The plethystic exponential function has well known applications in AdS/CFT relating the counting of single trace gauge invariants in CFT

to multi-trace counting [58]. It also has a related application in tensor model holography, relating the counting of connected and disconnected surfaces which are related to tensor model invariants [59]. Careful quantum gravitational discussions of the normalizations of partition functions relevant to combinatoric topological strings are in [9, 10, 13].

The second point we develop is S -duality for G -CTST. While S -duality was discussed in [11] in terms of entangled disconnected surfaces, we observe that there is also an interpretation of the S -dual amplitudes in terms of the inversion of the handle-creation operator in the group algebra of G . We observe that for both the untwisted and twisted case this inverse operator is well-defined. We give an expression for the inverse handle creation operator as an expansion in the projector basis $\mathcal{Z}(\mathbb{C}_\omega(G))$. A combinatoric description of the expansion in terms of the conjugacy class basis for $\mathcal{Z}(\mathbb{C}_\omega(G))$ is an interesting question. The third point concerns the implications of finiteness of G for relations between G -CTST amplitudes.

4.1 Construction of integer ratios $|G|^2/(\dim R)^2$ and stringy interpretation

4.1.1 Background

The construction of the integer ratios $|G|/\dim R$ from group multiplications in [11] used the determinant $\det(x - X)$, and its expansion in terms of products of traces

$$\det(x - X) = x^K - e_1(X)x^{K-1} + e_2(X)x^{K-2} + \dots + (-1)^K e_K(X), \quad (4.1)$$

where the e_i denote the elementary symmetric functions, given by

$$e_0(X) = 1, \quad (4.2)$$

$$e_1(X) = \sum_i X_i, \quad (4.3)$$

$$e_2(X) = \sum_{1 \leq i < j \leq K} X_i X_j, \quad (4.4)$$

$$e_l(X) = \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq K} X_{i_1} X_{i_2} \dots X_{i_l}. \quad (4.5)$$

The elementary symmetric functions can be expressed in terms of traces of X as in (3.133).

As was argued in [11], the algorithm presented there was a stringy construction more than a field theoretic construction, since it involved combining amplitudes of different genera, but there was not a crisp simple connection between the algorithm and a stringy observable.

As a first step in this direction, note that $e_1(X)$ is $\text{tr}X = Z_{h=2}$. In a stringy partition function this is naturally weighted with g_{st}^2 . The next elementary symmetric polynomial, $e_2(X)$, is a linear combination of $\text{tr}X^2 = Z_{h=3}$ and $(\text{tr}X)^2 = Z_{h=2}^2$. Both of these are weighted with g_{st}^4 . The next elementary symmetric polynomial, $e_3(X)$, is a linear combination of $Z_{h=4}$, $Z_{h=3}Z_{h=2}$, and $Z_{h=2}^3$, all of which are naturally weighted by g_{st}^6 . In general, $e_k(X)$ is associated with g_{st}^{2k} .

The determinant above can be written as

$$\det(X - x) = x^K \left(1 - e_1(X)x^{-1} + e_2(X)x^{-2} + \dots + (-1)^K x^{-K} e_K(X) \right). \quad (4.6)$$

By substituting $x \rightarrow g_{st}^{-2}$, we can write

$$x^{-K} \det(X - x)|_{x=g_{st}^{-2}} = 1 - e_1(X)g_{st}^2 + e_2(X)g_{st}^4 + \dots + (-1)^k g_{st}^{2K} e_K(X). \quad (4.7)$$

This looks like a stringy observable. We develop a link with disconnected string diagrams below.

4.1.2 Determinant from generating function of disconnected worldsheets

Start with the observation that a generating function of disconnected diagrams of genus 2 or higher can be obtained by expanding the exponential of a sum

$$\mathcal{Z}_{\text{disconn}}(g_{st}^2) = \exp \sum_{k=1}^{\infty} g_{st}^{2k} \frac{\mathcal{Z}_{k+1}}{k} = \exp \sum_{k=1}^{\infty} g_{st}^{2k} \frac{\text{tr} X^k}{k}, \quad (4.8)$$

$$= \prod_{k=1}^{\infty} \sum_{p_k=0}^{\infty} \frac{g_{st}^{2kp_k}}{p_k!} \left(\frac{\text{tr} X^k}{k} \right)^{p_k}, \quad (4.9)$$

$$= \sum_{p_k=0}^{\infty} \prod_{k=1}^{\infty} \frac{g_{st}^{\sum_k 2kp_k}}{k^{p_k} p_k!} (\text{tr} X^k)^{p_k}. \quad (4.10)$$

The argument of the exponential is motivated by the plethystic exponential function as studied in [58, 59]. Now observe that the first line is a determinant:

$$\exp \sum_{k=1}^{\infty} g_{st}^{2k} \frac{\text{tr} X^k}{k} = \exp \left(-\text{tr} \log \left(1 - g_{st}^2 X \right) \right) = \frac{1}{\det \left(1 - g_{st}^2 X \right)}, \quad (4.11)$$

where in the last equality, we used

$$\det(A) = \exp \text{tr} \log(A). \quad (4.12)$$

We conclude

$$\det(1 - g_{st}^2 X) = \frac{1}{\mathcal{Z}_{\text{disconn}}(g_{st}^2)}. \quad (4.13)$$

So the determinant used in the algorithm for $|G|^2/(\dim R)^2$ is nothing but the inverse of the generating function for the disconnected diagrams. The zeroes of this inverse generating function are at $g_{st}^2 = (\dim R)^2/|G|^2$, or $g_{st}^{-2} = |G|^2/(\dim R)^2$. A remarkable fact is that this inverse generating function truncates at a finite power of g_{st}^2 . This is due to the finiteness properties of the theory. Another way to express the remarkable fact is that the generating function of disconnected string diagrams is a rational function.

In [11], it was observed that finding the zeroes of $\det(x - X)$ in (4.1), viewed as a function of x , gives a finite algorithm (which uses as input the products of traces of X available from G -CTST partition functions) to arrive at the integer ratios $|G|/\dim R$. The identification of the formal variable x with g_{st}^{-2} above and the equation (4.13) shows that the integer ratios have the physical interpretation of being the locations in the g_{st}^{-2} plane of the poles of the generating function of disconnected amplitudes. It was also observed in [11] that the poles of the connected generating function as a function of g_{st} are given in terms of the integer ratios $|G|/\dim R$. Connected and disconnected generating functions are related through the plethystic exponential function (see [58] for applications of the plethystic exponential in the combinatorics of moduli spaces of supersymmetric gauge theories).

4.2 S -duality in G -CTST and the inverse handle-creation operator

Following the discussion in [11], the generating function of connected closed string amplitudes is

$$Z(g_{st}) = \sum_{h=0}^{\infty} g_{st}^{2h-2} \sum_R \frac{(|G|)^{2h-2}}{(\dim R)^{2h-2}}, \quad (4.14)$$

$$= \sum_R \frac{(\dim R)^4}{g_{st}^2 |G|^2 ((\dim R)^2 - |G|^2 g_{st}^2)}. \quad (4.15)$$

An S -dual generating function is defined as

$$\tilde{Z}(\tilde{g}_{st}) = -\tilde{g}_{st}^{-4} Z(g_{st} \rightarrow \tilde{g}_{st}^{-1}). \quad (4.16)$$

It is calculated to be

$$\tilde{Z}(\tilde{g}_{st}) = \sum_R \frac{(\dim R)^4}{|G|^4} \left(1 - \tilde{g}_{st}^2 \frac{(\dim R)^2}{|G|^2} \right)^{-1}, \quad (4.17)$$

$$= \sum_R \frac{(\dim R)^4}{|G|^4} \left(1 + \tilde{g}_{st}^2 \frac{(\dim R)^2}{|G|^2} + \tilde{g}_{st}^4 \frac{(\dim R)^4}{|G|^4} + \dots \right), \quad (4.18)$$

$$= \sum_R \sum_{k=1}^{\infty} \tilde{g}_{st}^{2+2k} \frac{(\dim R)^{2+2k}}{|G|^{2+2k}}. \quad (4.19)$$

In [11] a geometrical interpretation for the positive power sums of dimensions was given in terms of disconnected entangled surfaces. Here we develop an alternative interpretation of this S -dual expansion.

Recall the handle creation operator

$$\Pi = \sum_R \frac{|G|^2}{(\dim R)^2} P_R \quad (4.20)$$

with the property $\delta(P_R) = \frac{(\dim R)^2}{|G|}$ so that the genus h partition function is obtained by taking the trace of h powers of Π .

$$Z_h = \frac{1}{|G|} \delta(\Pi^h) = \sum_R \frac{(\dim R)^{2h-2}}{|G|^{2h-2}}. \quad (4.21)$$

We observe that the handle creation operator has an inverse element in the center of the group algebra, which is given by

$$\Pi^{-1} = \sum_R \frac{(\dim R)^2}{|G|^2} P_R. \quad (4.22)$$

We have

$$\Pi \Pi^{-1} = \sum_{R,S} \frac{(\dim R)^2}{|G|^2} \frac{|G|^2}{(\dim S)^2} P_R P_S, \quad (4.23)$$

$$= \sum_R P_R = 1. \quad (4.24)$$

We propose to interpret the inverse handle creation operator Π^{-1} as the handle creation operator of the S -dual theory and denote it as $\Pi^{-1} = \tilde{\Pi}$.

Note that the leading order term in the S -dual generating function (4.17) is

$$\frac{1}{|G|} \delta(\tilde{\Pi}) = \sum_R \frac{1}{|G|} \frac{(\dim R)^2}{|G|^2} \delta(P_R), \quad (4.25)$$

$$= \sum_R \frac{(\dim R)^4}{|G|^4}. \quad (4.26)$$

Since there is a single power of $\tilde{\Pi}$, it is natural to interpret this as the partition function at genus one of the S -dual theory. The higher powers are

$$\frac{1}{|G|} \delta(\tilde{\Pi}^k) = \sum_R \frac{(\dim R)^{2+2k}}{|G|^{2+2k}} \quad (4.27)$$

which can therefore be interpreted as genus k partition function of the dual theory.

Remark. It would be interesting to understand if there is a string field theory that generates the S -dual perturbation expansion above. One may be able to get some hints by examining the coefficients of the expansion of Π^{-1} in a basis of twist fields. Such an expression could be obtained using the character expansion of P_R to obtain a formula for Π^{-1} as an expansion in terms of the twist operator basis of $\mathcal{Z}(\mathbb{C}_\omega(G))$. The expansion coefficients involve the calculation of the sums

$$\sum_R (\dim R)^3 \chi^R(g). \quad (4.28)$$

These sums are some functions of g . It would be interesting to find out how these depend on the conjugacy class of g .

For example in $\mathbb{C}(S_3)$ it is easy to calculate

$$\begin{aligned} \Pi &= 18 + 9((1, 2, 3) + (1, 3, 2)), \\ \Pi^{-1} &= \frac{1}{12} - \frac{1}{36}((1, 2, 3) + (1, 2, 3)). \end{aligned} \quad (4.29)$$

It would be interesting to explore this for general $\mathbb{C}(S_n)$ and other group algebras.

Dualities in the context of discrete gauge theories have been discussed in [56, 57]. It will be interesting to investigate potential relations between these dualities and the S -duality considered here.

4.3 Finiteness relations

Systematic studies of the consequences of finiteness of G on the string amplitudes of G -CTST, both in untwisted and twisted case, are interesting future directions. For any group G , with K conjugacy classes, there are universal K -dependent finiteness relations which were described explicitly in [11]. Requiring that these finite K relations appear as null states of an inner product led to a discussion of the factorization puzzle in 2D/3D holography [60].

The inner product discussed in [11] was not uniquely determined. It would be interesting to investigate if there is a natural inner product, determined by the finiteness relations, possibly with additional data naturally related to G -CTST. As we have seen in this paper, the degeneracies of representation theoretic data (e.g. of values of dimensions of irreps) have important implications for integrality. They can be expected to play a role in G -dependent refinements of the finite K relations.

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A Basics of group cohomology

Briefly, group cohomology $H^n(G, U(1))$ can be represented by cochains $C^n(G, U(1))$, meaning maps $G^n \rightarrow U(1)$, which are closed in the sense $d\omega = 0$ for

$$d : C^n(G, U(1)) \longrightarrow C^{n+1}(G, U(1)) \tag{A.1}$$

defined by

$$(d\omega)(g_1, \dots, g_{n+1}) = \omega(g_2, \dots, g_{n+1}) \left[\prod_{i=1}^n (\omega(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots))^{(-)^i} \right] \cdot (\omega(g_1, \dots, g_n))^{(-)^{n+1}}, \tag{A.2}$$

modulo coboundaries, meaning the image of $d : C^{n-1}(G, U(1)) \rightarrow C^n(G, U(1))$.

For example, $[\omega] \in H^2(G, U(1))$ are maps $\omega : G^2 \rightarrow U(1)$ such that

$$\frac{\omega(g_2, g_3)}{\omega(g_1 g_2, g_3)} \frac{\omega(g_1, g_2 g_3)}{\omega(g_1, g_2)} = 1, \tag{A.3}$$

modulo equivalences

$$\omega(g_1, g_2) \sim \omega(g_1, g_2) \frac{b(g_1) b(g_2)}{b(g_1 g_2)} \tag{A.4}$$

for $b : G \rightarrow U(1)$.

One can always pick cocycles so that, for example for 2-cocycles,

$$\omega(1, g) = 1 = \omega(g, 1) \tag{A.5}$$

for any group element g . We work with such normalized cocycles in this paper.

B Characters of projective representations

In this appendix we review some basics facts and results on characters of projective representations of finite groups that are used elsewhere in this paper.

Perhaps the first result to recall is that, unlike characters of ordinary group representations, characters of projective representations are not class functions (not invariant under conjugation), but instead obey [42, section 7.2, prop. 2.2]

$$\chi^R(g) = \frac{\omega(g, h^{-1})}{\omega(h^{-1}, hgh^{-1})} \chi^R(hgh^{-1}), \quad (\text{B.1})$$

as was previously mentioned in (2.10).

Second, these characters vanish on non- ω -regular group elements, see e.g. [42, section 7.2, prop. 2.2], where an element $g \in G$ is said to be ω -regular if for all h commuting with g ,

$$\omega(g, h) = \omega(h, g). \quad (\text{B.2})$$

Irreducible projective representations are in one-to-one correspondence with ω -regular conjugacy classes.

Next, we know (see e.g. [42, section 7.3], [61, section 31.1], [16, equ'ns (B.4), (B.20)])

$$\frac{1}{|G|} \sum_{g \in G} \frac{D^R(g)_{ju} D^S(g^{-1})_{ik}}{\omega(g, g^{-1})} = \frac{\delta_{R,S}}{\dim R} \delta_{jk} \delta_{ui}, \quad (\text{B.3})$$

and, for $[g], [h]$ both⁴ ω -regular conjugacy classes,

$$\sum_R \frac{\chi^R(g) \chi^R(h^{-1})}{\omega(h, h^{-1})} = \begin{cases} 0 & g, h \text{ not conjugate,} \\ \frac{|G|}{|[g]|} & g = h, \\ \frac{\omega(a, g)}{\omega(h, a)} \frac{|G|}{|[g]|} & g = a^{-1}ha, \end{cases}, \quad (\text{B.4})$$

where R, S are irreducible projective representations (with respect to ω), $D^R(g)$ is a matrix representing $g \in G$ in R , meaning

$$D^R(g)D^R(h) = \omega(g, h)D^R(gh), \quad (\text{B.5})$$

the sum in the second identity is over irreducible projective representations, and $[g]$ denotes the number of elements in a conjugacy class containing g .

⁴If either is not an ω -regular conjugacy class, then the corresponding characters vanish, and the sum equals zero.

For use in other sections, from the expressions above one can show (see e.g. [16, appendix B])

$$\frac{1}{|G|} \sum_{g \in G} \frac{\omega(a, g) \omega(g^{-1}, b)}{\omega(g, g^{-1})} \chi^R(ag) \chi^S(g^{-1}b) = \frac{\delta_{R,S}}{\dim R} \omega(a, b) \chi^R(ab). \quad (\text{B.6})$$

$$\frac{1}{|G|} \sum_{g \in G} \frac{\omega(g, a) \omega(b, g^{-1})}{\omega(g, g^{-1})} \chi^R(ga) \chi^S(bg^{-1}) = \frac{\delta_{R,S}}{\dim R} \omega(a, b) \chi^R(ab). \quad (\text{B.7})$$

$$\frac{1}{|G|} \sum_{g \in G} \frac{\omega(g, a) \omega(g^{-1}, b) \omega(ga, g^{-1}b)}{\omega(g, g^{-1})} \chi^R(gag^{-1}b) = \frac{1}{\dim R} \chi^R(a) \chi^R(b). \quad (\text{B.8})$$

$$\frac{1}{|G|} \sum_{g \in G} \frac{\omega(a, g) \omega(b, g^{-1}) \omega(ag, bg^{-1})}{\omega(g, g^{-1})} \chi^R(agbg^{-1}) = \frac{1}{\dim R} \chi^R(a) \chi^R(b). \quad (\text{B.9})$$

(Alternatively, by writing in terms of characters of products of τ 's, one can produce equivalent expressions without factors of ω .)

Furthermore, from (B.4), it is straightforward to show that

$$\delta(g) = \sum_R \frac{\dim R}{|G|} \chi^R(g). \quad (\text{B.10})$$

Let us check that this identity is well-defined under conjugation. Using (B.1),

$$\delta(hgh^{-1}) = \sum_R \frac{\dim R}{|G|} \chi^R(hgh^{-1}), \quad (\text{B.11})$$

$$= \frac{\omega(h^{-1}, hgh^{-1})}{\omega(g, h^{-1})} \delta(g). \quad (\text{B.12})$$

If $hgh^{-1} \neq 1$, then both sides vanish, so there is no ambiguity. Similarly, if $hgh^{-1} = 1$, then $g = 1$, and

$$\frac{\omega(h^{-1}, hgh^{-1})}{\omega(g, h^{-1})} = 1, \quad (\text{B.13})$$

so again the identity is unambiguous.

In passing, for the projector P_R given in equation (2.17), note that this implies

$$\delta(P_R) = \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} \delta(g), \quad (\text{B.14})$$

$$= \frac{\dim R}{|G|^2} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} \sum_S (\dim S) \chi^S(g), \quad (\text{B.15})$$

but from (B.6), one has

$$\sum_{g \in G} \frac{\chi^R(g) \chi^S(g^{-1})}{\omega(g, g^{-1})} = |G| \delta_{RS}, \quad (\text{B.16})$$

hence

$$\delta(P_R) = \sum_S \frac{(\dim R)(\dim S)}{|G|^2} |G| \delta_{RS}, \quad (\text{B.17})$$

$$= \frac{(\dim R)^2}{|G|}. \quad (\text{B.18})$$

Another identity that will be useful involves the handle creation operator Π given in (C.39). Using (B.6), first note that

$$\chi^S(P_R) = \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})\chi^S(g)}{\omega(g, g^{-1})} = (\dim R)\delta_{R,S}. \quad (\text{B.19})$$

Then,

$$\chi^S(\Pi) = \sum_R \left(\frac{|G|}{\dim R} \right)^2 \chi^S(P_R), \quad (\text{B.20})$$

$$= \frac{|G|^2}{\dim R}. \quad (\text{B.21})$$

One of the consequences of the fact that characters of projective representations are not invariant under conjugation is that, unlike characters of ordinary representations for which

$$\chi^R(gh) = \chi^R(hg) \quad (\text{B.22})$$

characters of projective representations instead have the property

$$\chi^R(gh) \neq \chi^R(hg) \quad (\text{B.23})$$

in general. For example, for representations of D_4 with nontrivial discrete torsion, then from [16, section 4.5] and references therein,

$$\chi_r(b = baa) \neq \chi_r(aba = bz). \quad (\text{B.24})$$

For example, in the notation of that reference,

$$\chi_1(b) = 1 + i, \quad \chi_1(bz) = 1 - i. \quad (\text{B.25})$$

In fact, we can derive a general relation between $\chi^R(gh)$ and $\chi^R(hg)$ as follows. In principle,

$$\chi^R(g) = \text{Tr } \rho_R(g), \quad (\text{B.26})$$

where $\rho_R(g)$ is a matrix representing g . Now,

$$\chi^R(gh) = \text{Tr } \rho_R(gh), \quad (\text{B.27})$$

$$= \omega(g, h)^{-1} \text{Tr } \rho_R(g)\rho_R(h), \quad (\text{B.28})$$

$$= \omega(g, h)^{-1} \text{Tr } \rho_R(h)\rho_R(g), \quad (\text{B.29})$$

$$= \frac{\omega(h, g)}{\omega(g, h)} \text{Tr } \rho_R(hg), \quad (\text{B.30})$$

$$= \frac{\omega(h, g)}{\omega(g, h)} \chi^R(hg). \quad (\text{B.31})$$

As a consistency check, we claim that $\delta(gh) = \delta(hg)$. Now, from (B.10),

$$\delta(gh) = \sum_R \frac{\dim R}{|G|} \chi^R(gh), \tag{B.32}$$

$$= \frac{\omega(h, g)}{\omega(g, h)} \sum_R \frac{\dim R}{|G|} \chi^R(hg), \tag{B.33}$$

$$= \frac{\omega(h, g)}{\omega(g, h)} \delta(hg). \tag{B.34}$$

Now, if $gh \neq 1$, then $hg \neq 1$, so both sides of the relation above vanish, and in particular, $\delta(gh) = \delta(hg) = 0$. Suppose instead that $gh = 1$, so that $\delta(gh) = 1$. In this case, $h = g^{-1}$, and from

$$(d\omega)(g, g^{-1}, g) = 1, \tag{B.35}$$

we have

$$\omega(g, g^{-1}) = \omega(g^{-1}, g). \tag{B.36}$$

Thus, if $gh = 1$, then $\delta(gh) = \delta(hg) = 1$, so for all g and h , $\delta(gh) = \delta(hg)$.

C Two-dimensional Dijkgraaf-Witten theory

In this appendix we collect some technical results on two-dimensional twisted Dijkgraaf-Witten theory that are used in the main text. Although we have not located a complete set of prior references, we believe these results were known previously; we include them and their derivations here for completeness and to make the detailed arguments of the main text convincing.

C.1 Partition functions

In this section we will compute genus g partition functions of two-dimensional Dijkgraaf-Witten theory with discrete torsion, in the same style as the analysis of [11, section 2] to include discrete torsion. Now, to be clear, these partition functions have been computed previously in the literature, see for example [10] in the physics literature for a recent computation in two-dimensional Dijkgraaf-Witten theory specifically, [27, appendix C.1] for a recent review of results on partition functions of 2d TQFTs, and in the math literature, see for example [62–66] for partition functions and one-point functions in cases without⁵ discrete torsion, where these are given as the orbifold Euler characteristics of the moduli

⁵Partition functions including discrete torsion have certainly been computed previously in the physics literature, see e.g. [1]. We include such computations here for completeness. Our expectation is that partition functions including discrete torsion were also computed, albeit in different language, in the mathematics literature in the same era as [64, 65], though we have not been able to find a specific mathematics reference.

space of flat G bundles,

$$\chi_{\text{orb}}(\mathcal{M}_G(\Sigma)) = \chi_{\text{orb}}(\text{Hom}(\pi_1(\Sigma), G)/G), \quad (\text{C.1})$$

$$= \frac{|\text{Hom}(\pi_1(\Sigma), G)|}{|G|}, \quad (\text{C.2})$$

$$= \sum_{\rho \in \text{Hom}(\pi_1(\Sigma), G)/G} \frac{1}{|\text{Aut}(\rho)|}, \quad (\text{C.3})$$

$$= \sum_R \left(\frac{\dim R}{|G|} \right)^{\chi(\Sigma)}. \quad (\text{C.4})$$

(We do not claim to give a complete list of references, but merely list a few representative examples; additional references are given in e.g. [63, 66, 67].) Also, in passing, an alternative computation of the same result is given in section 2.2.1.

First, we consider the genus-one partition function. Using (B.3) it is straightforward to check

$$\sum_{g_1, g_2 \in G} \frac{D^R(g_1)_{ab} D^R(g_2)_{bc} D^R(g_1^{-1})_{cd} D^R(g_2^{-1})_{de}}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} = \left(\frac{|G|}{\dim R} \right)^2 \delta_{ae}. \quad (\text{C.5})$$

(Compare [11, equ'n (2.4)].) From this one immediately derives

$$\sum_{g_1, g_2} D_{ae}^R([g_1, g_2]) \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} = \left(\frac{|G|}{\dim R} \right)^2 \delta_{ae}, \quad (\text{C.6})$$

where

$$[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}. \quad (\text{C.7})$$

In particular,

$$\sum_{g_1, g_2} \chi^R([g_1, g_2]) \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} = \left(\frac{|G|}{\dim R} \right)^2 \dim R. \quad (\text{C.8})$$

By multiplying in

$$\frac{(d\omega)(g_2, g_1, g_1^{-1})}{(d\omega)(g_1 g_2, g_1^{-1}, g_2^{-1})} = 1, \quad (\text{C.9})$$

one finds that in the special case $g_1 g_2 = g_2 g_1$,

$$\frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} = \frac{\omega(g_1, g_2)}{\omega(g_2, g_1)}. \quad (\text{C.10})$$

Using (B.10),

$$\delta([g_1, g_2]) = \sum_R \frac{\dim R}{|G|} \chi^R([g_1, g_2]). \quad (\text{C.11})$$

Assembling these pieces, we have that the genus-one partition function (with discrete torsion) is given by

$$Z_{g=1} = \frac{1}{|G|} \sum_{g_1, g_2} \delta([g_1, g_2]) \frac{\omega(g_1, g_2)}{\omega(g_2, g_1)}, \quad (\text{C.12})$$

$$= \frac{1}{|G|} \sum_{g_1, g_2} \delta([g_1, g_2]) \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})}, \quad (\text{C.13})$$

$$= \frac{1}{|G|} \sum_{g_1, g_2} \left[\sum_R \frac{\dim R}{|G|} \chi^R([g_1, g_2]) \right] \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})},$$

$$= \sum_R \frac{\dim R}{|G|^2} \left[\sum_{g_1, g_2} \chi^R([g_1, g_2]) \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} \right], \quad (\text{C.14})$$

$$= \sum_R \frac{\dim R}{|G|^2} \left(\frac{|G|}{\dim R} \right)^2 \dim R, \quad (\text{C.15})$$

$$= \sum_R (1), \quad (\text{C.16})$$

where we used (C.8). This recovers the result in [1, equ'n (6.40)].

Next, we compute the partition functions on Riemann surfaces of general genus. We will follow the notation of [68]. Consider a Riemann surface of genus g , with insertions defined by group elements $a_i, b_i, i \in \{1, \dots, g\}$. Define $\gamma_i = [a_i, b_i]$, and

$$X = \left[\prod_i \omega(a_i, a_i^{-1}) \prod_i \omega(b_i, b_i^{-1}) \right]^{-1}. \quad (\text{C.17})$$

Then, from (B.3), we have that

$$\begin{aligned} & \sum_{a_i, b_i} \frac{D^R(a_1) D^R(b_1) D^R(a_1^{-1}) D^R(b_1^{-1}) \dots D^R(b_g^{-1})}{\omega(a_1, a_1^{-1}) \omega(b_1, b_1^{-1}) \dots \omega(b_g, b_g^{-1})} = \\ & = \left(\frac{|G|}{\dim R} \right)^{2g} I, \end{aligned} \quad (\text{C.18})$$

$$\begin{aligned} & = \sum_{a_i, b_i} D^R(\gamma_1 \dots \gamma_g) X \omega(a_1, b_1) \omega(a_1 b_1, a_1^{-1}) \omega(a_1 b_1 a_1^{-1}, b_1^{-1}) \omega(\gamma_1, a_2) \omega(\gamma_1 a_2, b_2) \\ & \quad \cdot \omega(\gamma_1 a_2 b_2, a_2^{-1}) \omega(\gamma_1 a_2 b_2 a_2^{-1}, b_2^{-1}) \omega(\gamma_1 \gamma_2, a_3) \dots \omega(\gamma_1 \dots \gamma_{g-1} a_g b_g a_g^{-1}, b_g^{-1}). \end{aligned} \quad (\text{C.19})$$

Now, the phase factor assigned by discrete torsion to a genus g Riemann surface is [68, equ'n (15)] (see also [69])

$$\begin{aligned} \epsilon_g(a_i, b_i) \equiv & X \omega(a_1, b_1) \omega(a_1 b_1, a_1^{-1}) \omega(a_1 b_1 a_1^{-1}, b_1^{-1}) \omega(\gamma_1, a_2) \omega(\gamma_1 a_2, b_2) \omega(\gamma_1 a_2 b_2, a_2^{-1}) \\ & \cdot \omega(\gamma_1 a_2 b_2 a_2^{-1}, b_2^{-1}) \omega(\gamma_1 \gamma_2, a_3) \dots \omega(\gamma_1 \dots \gamma_{g-1} a_g b_g a_g^{-1}, b_g^{-1}). \end{aligned} \quad (\text{C.20})$$

Thus, we can write the expression above as

$$\sum_{a_i, b_i} D^R(\gamma_1 \cdots \gamma_g) \epsilon_g(a_i, b_i) = \left(\frac{|G|}{\dim R} \right)^{2g} I. \tag{C.21}$$

In particular,

$$\sum_{a_i, b_i} \chi^R(\gamma_1 \cdots \gamma_g) \epsilon_g(a_i, b_i) = \left(\frac{|G|}{\dim R} \right)^{2g} (\dim R). \tag{C.22}$$

Applying the identity (B.10)

$$\delta(\gamma_1 \cdots \gamma_g) = \sum_R \frac{\dim R}{|G|} \chi^R(\gamma_1 \cdots \gamma_g), \tag{C.23}$$

we then compute

$$Z_g = \frac{1}{|G|} \sum_{a_i, b_i} \delta \left(\prod_i \gamma_i \right) \epsilon_g(a_i, b_i), \tag{C.24}$$

$$= \frac{1}{|G|} \sum_{a_i, b_i} \left[\sum_R \frac{\dim R}{|G|} \chi^R(\gamma_1 \cdots \gamma_g) \right] \epsilon_g(a_i, b_i), \tag{C.25}$$

$$= \frac{1}{|G|} \sum_R \frac{\dim R}{|G|} \left[\sum_{a_i, b_i} \chi^R(\gamma_1 \cdots \gamma_g) \epsilon_g(a_i, b_i) \right], \tag{C.26}$$

$$= \frac{1}{|G|} \sum_R \frac{\dim R}{|G|} \left[\left(\frac{|G|}{\dim R} \right)^{2g} (\dim R) \right], \tag{C.27}$$

$$= \sum_R \left(\frac{|G|}{\dim R} \right)^{2g-2}, \tag{C.28}$$

where we have used equation (C.22).

C.2 Handle creation operator

In this section, we will describe the handle creation operator in the presence of discrete torsion, and its basic properties.

Without discrete torsion, the handle creation operator is [11, equ'n (6.23)]

$$\Pi = \sum_{g_1, g_2 \in G} \tau_{g_1} \tau_{g_2} \tau_{g_1}^{-1} \tau_{g_2}^{-1}, \tag{C.29}$$

and it is claimed that

$$\Pi = \sum_R \left(\frac{|G|}{\dim R} \right)^2 P_R, \tag{C.30}$$

for P_R the projection operator.

As a consistency test, note this implies

$$\sum_{g_1, g_2 \in G} D^S(g_1 g_2 g_1^{-1} g_2^{-1}) = \sum_R \left(\frac{|G|}{\dim R} \right)^2 D^S(P_R). \tag{C.31}$$

Let us check that this implication is correct for every irreducible representation S . First, from (C.6), in the absence of discrete torsion, we have

$$\sum_{g_1, g_2 \in G} D^S(g_1 g_2 g_1^{-1} g_2^{-1}) = \left(\frac{|G|}{\dim S} \right)^2 I. \quad (\text{C.32})$$

Now,

$$P_R = \frac{\dim R}{|G|} \sum_{g \in G} \chi^R(g^{-1}) \tau_g, \quad (\text{C.33})$$

hence

$$D^S(P_R) = \frac{\dim R}{|G|} \sum_{g \in G} \chi_R(g^{-1}) T^S(g), \quad (\text{C.34})$$

$$= \delta_{R,S} I \text{ using (B.3)}, \quad (\text{C.35})$$

hence

$$\sum_{g_1, g_2 \in G} D^S(g_1 g_2 g_1^{-1} g_2^{-1}) = \sum_R \left(\frac{|G|}{\dim R} \right)^2 \delta_{R,S} I, \quad (\text{C.36})$$

$$= \sum_R \left(\frac{|G|}{\dim R} \right)^2 D^S(P_R), \quad (\text{C.37})$$

confirming (C.31). Since this holds for any irreducible representation S , we take this as a confirmation of the handle creation operator identity (C.30).

Now, let us turn to the case with discrete torsion.

Here, we define the handle creation operator to be

$$\begin{aligned} \Pi &= \sum_{g_1, g_2 \in G} \tau_{g_1} \tau_{g_2} \tau_{g_1}^{-1} \tau_{g_2}^{-1} \\ &= \sum_{g_1, g_2 \in G} \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} \tau_{g_1 g_2 g_1^{-1} g_2^{-1}}, \end{aligned} \quad (\text{C.38})$$

and we claim that

$$\Pi = \sum_R \left(\frac{|G|}{\dim R} \right)^2 P_R. \quad (\text{C.39})$$

As a consistency check, this implies that

$$\sum_{g_1, g_2 \in G} \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} D^S(g_1 g_2 g_1^{-1} g_2^{-1}) = \sum_R \left(\frac{|G|}{\dim R} \right)^2 D^S(P_R). \quad (\text{C.40})$$

Let us check that this implication is correct for every irreducible representation S . First, from (C.6), we have

$$\sum_{g_1, g_2 \in G} \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} D^S(g_1 g_2 g_1^{-1} g_2^{-1}) = \left(\frac{|G|}{\dim R} \right)^2 I. \quad (\text{C.41})$$

Now, with discrete torsion,

$$P_R = \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} g, \quad (\text{C.42})$$

so

$$D^S(P_R) = \frac{\dim R}{|G|} \sum_{g \in G} \frac{\chi^R(g^{-1})}{\omega(g, g^{-1})} D^S(g), \quad (\text{C.43})$$

$$= \delta_{R,S} I \text{ using (B.3)}, \quad (\text{C.44})$$

hence

$$\begin{aligned} & \sum_{g_1, g_2 \in G} \frac{\omega(g_1, g_2) \omega(g_1^{-1}, g_2^{-1}) \omega(g_1 g_2, g_1^{-1} g_2^{-1})}{\omega(g_1, g_1^{-1}) \omega(g_2, g_2^{-1})} D^S(g_1 g_2 g_1^{-1} g_2^{-1}) \\ &= \sum_R \left(\frac{|G|}{\dim R} \right)^2 \delta_{R,S} I, \end{aligned} \quad (\text{C.45})$$

$$= \sum_R \left(\frac{|G|}{\dim R} \right)^2 D^S(P_R) \quad (\text{C.46})$$

confirming (C.40). Since this holds for any irreducible representation S , we take this as a confirmation of the handle creation operator identity (C.39).

C.3 Handle creation operator identities

In this section, we will describe some handle-operator creation identities.

First, we claim that

$$\delta \left(\Pi^n T_{[g]} \right) = \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \chi^R(g). \quad (\text{C.47})$$

To this end, recall the identity (C.39)

$$\Pi = \sum_R \left(\frac{|G|}{\dim R} \right)^2 P_R, \quad (\text{C.48})$$

where P_R is the projector given by [16, equ'n (2.43)]

$$P_R = \frac{\dim R}{|G|} \sum_{k \in G} \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \tau_k, \quad (\text{C.49})$$

hence

$$\Pi^n = \sum_R \left(\frac{|G|}{\dim R} \right)^{2n} P_R, \quad (\text{C.50})$$

and [16, equ'n (2.17)]

$$T_{[g]} = \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g) \omega(hg, h^{-1})}{\omega(h, h^{-1})} \tau_{hgh^{-1}}. \quad (\text{C.51})$$

Thus,

$$\begin{aligned} \delta \left(\Pi^n T_{[g]} \right) &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n} \frac{\dim R}{|G|} \sum_{k \in G} \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \frac{1}{|G|} \sum_{h \in G} \frac{\omega(h, g) \omega(hg, h^{-1})}{\omega(h, h^{-1})} \\ &\quad \cdot \omega(k, hgh^{-1}) \delta(khgh^{-1}), \end{aligned} \quad (\text{C.52})$$

where we have used the fact that

$$\tau_g \tau_h = \omega(g, h) \tau_{gh}. \quad (\text{C.53})$$

Using

$$\frac{(d\omega)(kh, g, h^{-1})}{(d\omega)(k, hg, h^{-1}) (d\omega)(k, h, g)} = 1 \quad (\text{C.54})$$

we have

$$\frac{\omega(h, g) \omega(hg, h^{-1}) \omega(k, hgh^{-1})}{\omega(h, h^{-1})} = \frac{\omega(k, h) \omega(g, h^{-1}) \omega(kh, gh^{-1})}{\omega(h, h^{-1})}, \quad (\text{C.55})$$

hence

$$\begin{aligned} \delta \left(\Pi^n T_{[g]} \right) &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_{k \in G} \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \frac{1}{|G|} \sum_{h \in G} \frac{\omega(k, h) \omega(g, h^{-1}) \omega(kh, gh^{-1})}{\omega(h, h^{-1})} \\ &\quad \cdot \sum_S \frac{\dim S}{|G|} \chi^R(khgh^{-1}). \end{aligned} \quad (\text{C.56})$$

Using (B.9),

$$\delta \left(\Pi^n T_{[g]} \right) = \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_{k \in G} \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \sum_S \frac{\dim S}{|G|} \frac{1}{\dim S} \chi^S(k) \chi^S(g), \quad (\text{C.57})$$

$$= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_S \delta_{R,S} \chi^S(g), \quad (\text{C.58})$$

$$= |G| \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-2} \frac{\chi^R(g)}{\dim R}, \quad (\text{C.59})$$

using (B.6).

Next, we compute

$$\begin{aligned} \delta \left(\Pi^n T_{[g_1]} T_{[g_2]} \right) &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n} \frac{\dim R}{|G|} \sum_k \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \frac{1}{|G|} \sum_{h_1} \frac{\omega(h_1, g_1) \omega(h_1 g_1, h_1^{-1})}{\omega(h_1, h_1^{-1})} \\ &\quad \cdot \frac{1}{|G|} \sum_{h_2} \frac{\omega(h_2, g_2) \omega(h_2 g_2, h_2^{-1})}{\omega(h_2, h_2^{-1})} \\ &\quad \cdot \omega(k, h_1 g_1 h_1^{-1}) \omega(kh_1 g_1 h_1^{-1}, h_2 g_2 h_2^{-1}) \\ &\quad \cdot \sum_S \frac{\dim S}{|G|} \chi^S(kh_1 g_1 h_1^{-1} h_2 g_2 h_2^{-1}). \end{aligned} \quad (\text{C.60})$$

Using (C.55) and (B.9), this reduces to

$$\begin{aligned} \delta \left(\Pi^n T_{[g_1]} T_{[g_2]} \right) &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_k \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \frac{1}{|G|} \sum_{h_1} \frac{\omega(h_1, g_1) \omega(h_1 g_1, h_1^{-1})}{\omega(h_1, h_1^{-1})} \\ &\quad \cdot \omega(k, h_1 g_1 h_1^{-1}) \sum_S \frac{\dim S \chi^R(k h_1 g_1 h_1^{-1}) \chi^S(g_2)}{|G| \dim S}. \end{aligned}$$

Modulo the factor of $\chi^S(g_2)$, this now essentially reduces to the previous computation. Using (C.55) and (B.9) again, we have

$$\delta \left(\Pi^n T_{[g_1]} T_{[g_2]} \right) = \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_k \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \sum_S \frac{\dim S \chi^S(k) \chi^S(g_1) \chi^S(g_2)}{|G| (\dim S)^2},$$

and using (B.6), we have

$$\delta \left(\Pi^n T_{[g_1]} T_{[g_2]} \right) = |G| \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-2} \left(\frac{\chi^R(g_1)}{\dim R} \right) \left(\frac{\chi^R(g_2)}{\dim R} \right). \quad (\text{C.61})$$

At this point, it is straightforward to derive an analogous expression for cases with more factors of $T_{[g]}$. Define

$$\gamma_i = h_i g_i h_i^{-1}, \quad A_i = \frac{1}{|G|} \sum_{h_i} \frac{\omega(h_i, g_i) \omega(h_i g_i, h_i^{-1})}{\omega(h_i, h_i^{-1})}, \quad (\text{C.62})$$

we have

$$\begin{aligned} \delta \left(\Pi^n T_{[g_1]} \cdots T_{[g_m]} \right) &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_k \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} A_1 \cdots A_m \\ &\quad \cdot \omega(k, \gamma_1) \omega(k \gamma_1, \gamma_2) \cdots \omega(k \gamma_1 \gamma_2 \cdots \gamma_{m-1}, \gamma_m) \\ &\quad \cdot \sum_S \frac{\dim S}{|G|} \chi^S(k \gamma_1 \gamma_2 \cdots \gamma_m). \end{aligned} \quad (\text{C.63})$$

Applying (C.55) and (B.9) to perform the sum over h_m , this becomes

$$\begin{aligned} \delta \left(\Pi^n T_{[g_1]} \cdots T_{[g_m]} \right) &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_k \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} A_1 \cdots A_{m-1} \\ &\quad \cdot \omega(k, \gamma_1) \omega(k \gamma_1, \gamma_2) \cdots \omega(k \gamma_1 \gamma_2 \cdots \gamma_{m-2}, \gamma_{m-1}) \\ &\quad \cdot \sum_S \frac{\dim S}{|G|} \chi^S(k \gamma_1 \gamma_2 \cdots \gamma_{m-1}) \frac{\chi^S(g_m)}{\dim S}. \end{aligned} \quad (\text{C.64})$$

Iterating that procedure, and then using (B.6), we find

$$\begin{aligned} & \delta \left(\Pi^n T_{[g_1]} \cdots T_{[g_m]} \right) \\ &= \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-1} \sum_k \frac{\chi^R(k^{-1})}{\omega(k, k^{-1})} \\ & \cdot \sum_S \frac{\dim S}{|G|} \chi^S(k) \frac{\chi^S(g_1)}{\dim S} \frac{\chi^S(g_2)}{\dim S} \cdots \frac{\chi^S(g_m)}{\dim S}, \end{aligned} \tag{C.65}$$

$$= |G| \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-2} \left(\frac{\chi^R(g_1)}{\dim R} \right) \left(\frac{\chi^R(g_2)}{\dim R} \right) \cdots \left(\frac{\chi^R(g_m)}{\dim R} \right). \tag{C.66}$$

Since this is linear in each factor, this immediately implies that for S_1, \dots, S_m any elements of the center of the group algebra,

$$\begin{aligned} & \delta \left(\Pi^n S_1 \cdots S_m \right) \\ &= |G| \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-2} \left(\frac{\chi^R(S_1)}{\dim R} \right) \left(\frac{\chi^R(S_2)}{\dim R} \right) \cdots \left(\frac{\chi^R(S_m)}{\dim R} \right). \end{aligned} \tag{C.67}$$

As a consistency check, note that if $S_1 = \Pi$, say, then from (B.21), we have

$$\begin{aligned} & \delta \left(\Pi^n S_1 \cdots S_m \right) \\ &= |G| \sum_R \left(\frac{|G|}{\dim R} \right)^{2n-2} \left(\frac{|G|}{\dim R} \right)^2 \left(\frac{\chi^R(S_2)}{\dim R} \right) \cdots \left(\frac{\chi^R(S_m)}{\dim R} \right), \end{aligned} \tag{C.68}$$

$$= |G| \sum_R \left(\frac{|G|}{\dim R} \right)^{2n} \left(\frac{\chi^R(S_2)}{\dim R} \right) \cdots \left(\frac{\chi^R(S_m)}{\dim R} \right), \tag{C.69}$$

$$= \delta \left(\Pi^{n+1} S_2 \cdots S_m \right). \tag{C.70}$$

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