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Conformal integrals in four dimensions

Aritra Pal and Koushik Ray

Indian Association for the Cultivation of Science,
Calcutta 700 032, India

E-mail: intap@iacs.res.in, koushik@iacs.res.in

Abstract: We obtain analytic expressions of four-dimensional Euclidean N-point conformal integrals for arbitrary N by solving a Lauricella-like system of differential equations derived earlier. We demonstrate their relation to the GKZ A-hypergeometric systems. The conformal integrals are solutions to these expressed in terms of leg factors and infinite series in the conformal invariant cross ratios.

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Conformal integrals [1] are the *sine qua non* of theories dealing with conformal symmetry. The integrals make an appearance in the evaluation of Feynman diagrams in quantum field theories [2, 3], as well as studying renormalization groups [4]. In particular, they furnish representations of the conformal group. Conformal blocks, which in turn determine the correlation functions of a conformal field theory, are expressed in terms of conformal integrals. The integrals for N points have been evaluated in certain cases, for relatively small values of N, looked at from different angles and various methods have been employed to this end [5–18]. We present a general method to obtain analytic expressions of four-dimensional N-point conformal integrals as infinite series in terms of conformal invariants, namely, the cross ratios, obtained as solutions to previously derived Lauricella-like equations [19]. We derive explicit expressions of the conformal integrals for arbitrary N, by showing that the Lauricella-like equations are solved by certain GKZ A-hypergeometric functions.

Let us outline the strategy before presenting the details of the computation. We restrict to the four-dimensional Euclidean space $\mathbf{R}^4$, indicating generalisation to higher dimensions at the end. As a normed vector space $\mathbf{R}^4$ can be identified with the space of quaternions $\mathbf{H}$, the norm-squared being equal to the determinant of a quaternion. The conformal group of $\mathbf{R}^4$ is the Möbius group of 2×2 block matrices, each block being a quaternion (1). The N-point conformal integral is defined in terms of quaternions in equation (6) to utilize this connection. Differentiating with respect to quaternions within the integral sign, a system of linear second-order differential equations (13) is then obtained of which the integral is a solution, analogous to its two-dimensional counterpart [20]. Next, the conformal integral is interpreted as the sheaf of germs of functions on the Fulton-MacPherson completion of the configuration space of ordered N-tuple of points on the Euclidean space, allowing it to be envisaged as a function of the determinant of pairwise differences of the N quaternions (22). Inserting it as an ansatz in (13) leads to a Lauricella-like system of differential equations (24) for the invariant part of the conformal integral written in terms of the cross ratios [19], generalising the Lauricella system for the two-dimensional case [20]. In the current article we observe that this system of equations when cast in the form (27) is satisfied by the solution of a GKZ A-hypergeometric system (47) and (48) corresponding to a matrix of exponents of the norm of pairwise differences of the quaternions under the Möbius transformation. The solutions are then explicitly obtained as infinite series (71). We discuss the examples of $N = 4, 5, 6$ at length. These are consistent with previously obtained results [6], but to the best of our knowledge the general solution has not appeared in literature before.

We now elaborate on the procedure, starting with a recount of the derivation of the differential equations [19]. The conformal or Möbius group of $\mathbf{R}^4 \cup \{\infty\}$ is isomorphic to a certain group of matrices written as 2×2 blocks of quaternions [21, 22], namely,

$$\mathrm{SL}(2, \mathbf{H}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \, |AC^{-1}DC - BC| = 1; A, B, C, D \in \mathbf{H} \right\}. \tag{1}$$

The Möbius group acts on a quaternion Q as

$$Q \longmapsto Q' = (AQ + B)(CQ + D)^{-1}. \tag{2}$$

A real Euclidean four-vector $\mathbf{q} = (q_0, q_1, q_2, q_3)$ in $\mathbf{R}^4$ is fashioned into a quaternion as

$$Q = \begin{pmatrix} q_0 + iq_3 & q_1 + iq_2 \\ -q_1 + iq_2 & q_0 - iq_3 \end{pmatrix}. \tag{3}$$

The determinant of the quaternion Q is the Euclidean norm-squared of the four-vector, written as

$$|Q| = q_0^2 + q_1^2 + q_2^2 + q_3^2. \tag{4}$$

The determinant of the difference of two quaternions, denoted $Q_{ij} = Q_i - Q_j$ from now on, transforms under the conformal transformation (2) as

$$|Q'_{ij}| = |CQ_i + D|^{-1}|CQ_j + D|^{-1}|Q_{ij}|. \tag{5}$$

A conformal integral is defined in terms of quaternions as

$$I_N^{\boldsymbol{\mu}}(\boldsymbol{Q}) = \int \frac{d^4Q}{|Q - Q_1|^{\mu_1}|Q - Q_2|^{\mu_2} \cdots |Q - Q_N|^{\mu_N}}, \tag{6}$$

where $\boldsymbol{Q}$ denotes an N-tuple of quaternions, $\boldsymbol{Q} = (Q_1, Q_2, \cdots, Q_N)$, $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_N)$ is an N-tuple of real numbers and

$$d^4Q = dq_0 \wedge dq_1 \wedge dq_2 \wedge dq_3 \tag{7}$$

denotes the volume form of $\mathbf{R}^4$, the integral being over the whole space. The integral transforms under the Möbius transformation (2) as

$$I_N^{\boldsymbol{\mu}}(\boldsymbol{Q}') = |CQ_1 + D|^{\mu_1}|CQ_2 + D|^{\mu_2} \cdots |CQ_N + D|^{\mu_N} I_N^{\boldsymbol{\mu}}(\boldsymbol{Q}), \tag{8}$$

provided $|\boldsymbol{\mu}| = \mu_1 + \mu_2 + \cdots + \mu_N = d$. Here, $d = 4$. Representations of the Möbius group $\mathrm{SL}(2, \mathbf{H})$ may be constructed out of $|Q_{ij}|$ and $I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})$. Our goal is to obtain the conformal integral as a solution to a system of differential equations. The system, which generalizes the Lauricella system appearing in two dimensions, is set up by differentiating the integral (6) with respect to the Q_i under the integration sign. Let us denote the integrand of (6) as

$$F_N^{\boldsymbol{\mu}}(Q, \boldsymbol{Q}) = \prod_{i=1}^{N} \frac{1}{|Q - Q_i|^{\mu_i}}. \tag{9}$$

Denoting the matrix component of a quaternion Q_i by $(Q_i)_{ab}$, with $a, b = 1, 2$, as defined in (3), we have

$$\frac{\partial F_N^{\boldsymbol{\mu}}(Q, \boldsymbol{Q})}{\partial (Q_i)_{ba}} = \mu_i (Q - Q_i)^{-1}_{ab} F_N^{\boldsymbol{\mu}}(Q, \boldsymbol{Q}). \tag{10}$$

Differentiating twice and using the identity

$$(Q - Q_i)^{-1} Q_{ij} (Q - Q_j)^{-1} = (Q - Q_i)^{-1} - (Q - Q_j)^{-1} \tag{11}$$

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we derive

$$\sum_{b,c=1}^{2}(Q_{ij})_{bc}\frac{\partial}{\partial(Q_i)_{ba}}\frac{\partial F_N^{\boldsymbol{\mu}}(Q,\boldsymbol{Q})}{\partial(Q_j)_{dc}} = \mu_i\mu_j\left[(Q-Q_i)^{-1}-(Q-Q_j)^{-1}\right]_{ad}F_N^{\boldsymbol{\mu}}(Q,\boldsymbol{Q}), \tag{12}$$

with $i \neq j$. Using (10) and (12) to perform the differentiations under the integral sign in (6) we arrive at the Lauricella-like equation [19],

$$\sum_{b,c=1}^{2}(Q_{ij})_{bc}\frac{\partial}{\partial(Q_i)_{ba}}\frac{\partial I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})}{\partial(Q_j)_{dc}} = \mu_j\frac{\partial I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})}{\partial(Q_i)_{da}} - \mu_i\frac{\partial I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})}{\partial(Q_j)_{da}}, \tag{13}$$

where $i, j = 1, 2, \cdots, N$ and $i \neq j$.

In order to obtain explicit expressions for the conformal integrals as a solution to (13) we first interpret the solutions as sheaf of germs of functions on the Fulton-MacPherson completion of the configuration space

$$C_N(M) = M^N \setminus \{q_i \in M,\ q_i \neq q_j;\ i, j = 1, 2, \cdots, N\} \tag{14}$$

of N non-coalescing points on the Euclidean space, $M = \mathbf{R}^4 \cup \{\infty\}$. The Fulton-MacPherson completion is furnished by the embedding [23, 24]

$$\begin{aligned} C_N(M) &\hookrightarrow M^N \times (S^3)^{\binom{N}{2}} \times [0,\infty]^{\binom{N}{3}}, \\ (q_1, q_2, \cdots, q_N) &\longmapsto (q_1, q_2, \cdots, q_N, v_{12}, \cdots, v_{(N-1)N}, a_{123}, \cdots, a_{(N-2)(N-1)N}), \end{aligned} \tag{15}$$

where every

$$v_{ij} = \frac{Q_{ij}}{|Q_{ij}|} \tag{16}$$

describes a three-sphere S^3 and the scalars

$$a_{ijk} = \frac{|Q_{ij}|}{|Q_{ik}|} \tag{17}$$

are non-negative real numbers. We used the correspondence (3) to express v_{ij} and a_{ijk} in terms of quaternions. With this interpretation the conformal integral can be expressed in terms of the variables a_{ijk}, while invariance under translation and rotation forbids a representation to depend on Q_i alone and v_{ij}, respectively. However, a product of powers of a_{ijk} can be uniquely written as a product of powers of $|Q_{ij}|$. Comparing (5) and (8) we conclude that a product of $|Q_{ij}|$ with appropriate exponents reproduces the transformation property of the conformal integral under the Möbius group. While this takes care of the equivariant part, the conformal integral, in general, is also a function of conformal invariants, for example,

$$|\chi_{ijkl}| = a_{ijk}a_{lkj}, \tag{18}$$

obtained as the determinant of the quaternion $\chi_{ijkl} = Q_{ij}Q_{ik}^{-1}Q_{kl}Q_{jl}^{-1}$. All the $|\chi|$'s can be expressed in terms of $N_0 = N(N-3)/2$ conveniently chosen invariants. We refer to these special invariants as the cross ratios from now on and denote by ξ. We often collect the cross ratios in a vector

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_{N_0}). \tag{19}$$

The number N_0 is actually an upper bound for $N \geqslant 7$ in four dimensions [13]. The cross ratios satisfy relations amongst themselves. We shall ignore this subtlety here as the conformal integrals may be obtained in those cases by restricting to the subspace of independent cross ratios using the relations. The cross ratios are written as products of ratios of $|Q_{ij}|$ as

$$\xi_A = \prod_{\substack{i,j \\ 1\leq i\neq j\leq N}} |Q_{ij}|^{\frac{1}{2}\alpha_{ij}^A}, \tag{20}$$

where $A = 1, 2, \cdots, N_0$, and each α_{ij}^A is an integer, satisfying

$$\alpha_{ji}^A = \alpha_{ij}^A, \quad \alpha_{ii}^A = 0, \forall i; \quad \sum_{j=1}^{N} \alpha_{ij}^A = 0, \forall i, \tag{21}$$

for each A. The factor of $\frac{1}{2}$ is accounted for by the symmetry of the cross ratios under the exchange of i and j. From this discussion it follows that the conformal integral (6) as a function on the Fulton-MacPherson completion of the configuration space of N points on M can be expressed as

$$I_N^{\boldsymbol{\mu}}(\boldsymbol{Q}) = \prod_{\substack{i,j \\ 1\leq i\neq j\leq N}} |Q_{ij}|^{\frac{1}{2}\beta_{ij}} I_0(\boldsymbol{\xi}), \tag{22}$$

where $I_0(\boldsymbol{\xi})$ is a function of the cross ratios $\boldsymbol{\xi}$, and

$$\beta_{ji} = \beta_{ij}, \quad \beta_{ii} = 0, \forall i; \quad \sum_{j=1}^{N} \beta_{ij} = -\mu_i; \quad \sum_{\substack{i,j \\ 1\leq i<j\leq N}} \beta_{ij} = -|\boldsymbol{\mu}|/2 = -d/2, \tag{23}$$

The conformal integral is, therefore, a function of the determinant $|Q_{ij}|$ of the quaternions.

Plugging in (22) as an ansatz and taking trace over the matrix indices a, b of quaternions, the differential equation (13) gives rise to a system of equations for the invariant part, namely [19],

$$\begin{aligned}
&\sum_{A,B} \sum_{\substack{k,l \\ 1\leq k,l\leq N \\ k\neq i, l\neq j}} \alpha_{ik}^A \alpha_{jl}^B \tau_{ijkl}\; \xi_A \xi_B \partial_A \partial_B I_0(\boldsymbol{\xi}) \\
&+ \sum_A \Bigg(4\alpha_{ij}^A + \sum_{\substack{k,l \\ 1\leq k,l\leq N \\ k\neq i, l\neq j}} \left(\alpha_{ik}^A \alpha_{jl}^A + \alpha_{ik}^A \beta_{jl} + \alpha_{jl}^A \beta_{ik}\right) \tau_{ijkl}\Bigg) \xi_A \partial_A I_0(\boldsymbol{\xi}) \\
&\qquad + \Bigg(4\beta_{ij} + \sum_{\substack{k,l \\ 1\leq k,l\leq N \\ k\neq i, l\neq j}} \beta_{ik}\beta_{jl}\tau_{ijkl}\Bigg) I_0(\boldsymbol{\xi}) = 0,
\end{aligned} \tag{24}$$

where $\tau_{ijkl} = \mathrm{Tr}\ \chi_{ijkl}$ and $\partial_A = \frac{\partial}{\partial \xi_A}$ for $A, B = 1, 2, \cdots, N_0$. In order to express this set of equations in terms of cross ratios alone we need to express the trace of χ_{ijkl} in terms of its determinant. From the identity [19]

$$\chi_{ijkl}\chi_{ijlk} = \chi_{ijkl} + \chi_{ijlk}, \tag{25}$$

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taking determinant and using $\det(1+M) = 1 + \mathrm{Tr}\ M + \det M$ for 2×2 matrices, we derive

$$\tau_{ijkl} = 1 - |\chi_{lijk}| + |\chi_{ijkl}|. \tag{26}$$

Using this in (24) we obtain, after rearrangement of terms, the system of differential equations in the concise form

$$L_{ij} I_0(\boldsymbol{\xi}) = 0, \tag{27}$$

where for $i \neq j$, the indices $i, j = 1, 2, \cdots N$, and we define the differential operator

$$L_{ij} = \sum_{k,l} (|\chi_{ijkl}| - |\chi_{lijk}|) \Big(\sum_A \alpha_{ik}^A \theta_A + \beta_{ik} \Big) \Big(\sum_B \alpha_{jl}^B \theta_B + \beta_{jl} \Big) + 4 \Big(\sum_A \alpha_{ij}^A \theta_A + \beta_{ij} \Big) + \mu_i \mu_j, \tag{28}$$

with $A, B = 1, 2, \cdots, N_0$, in terms of the logarithmic derivatives

$$\theta_A = \xi_A \frac{\partial}{\partial \xi_A}. \tag{29}$$

Singular values of $|\chi|$'s are kept from appearing in the equations by choosing to cancel them in the sums at the level of symbols and using (21) and (23) prior to expressing $|\chi|$'s in terms of the invariants.

Let us also note that thanks to the relations

$$|\chi_{jilk}| = |\chi_{ijkl}|, \quad |\chi_{kjil}| = |\chi_{lijk}|, \tag{30}$$

the operators L_{ij} and L_{ji} give rise to identical equations, leaving $N(N-1)/2$ equations in (27). Moreover, since the ratios $|\chi_{ijkl}|$ and $|\chi_{lijk}|$ are interchanged under the exchange of the indices j and l, we have

$$\sum_{j=1}^{N} L_{ij} = 0, \tag{31}$$

for each value of i. This takes away another N equations, so that (27) is a system of $N_0 = N(N-3)/2$ independent ones. Hence, we have N_0 linear second order partial differential equations to solve in order to obtain I_0 as a function of the same number of variables, $\boldsymbol{\xi}$. We choose the ones from the L_{ij} by discarding the $N-1$ equations coming from L_{1i} for $i = 2, 3, \cdots, N$ and also L_{23}.

We now describe the method of solving (27). Let us introduce another notation for later use. Expressing the determinant $|\chi|$ defined in (18) in terms of the cross ratios as

$$|\chi_{ijkl}| = \prod_{A=1}^{N_0} \xi_A^{\gamma_{ijkl}^A}, \tag{32}$$

and using (20), the consistency of the definition of $|\chi|$ requires

$$\sum_{A=1}^{N_0} \alpha_{ab}^A \gamma_{ijkl}^A = (\delta_{ai}\delta_{bj} + \delta_{aj}\delta_{bi} + \delta_{ak}\delta_{bl} + \delta_{al}\delta_{bk}) - (\delta_{ai}\delta_{bk} + \delta_{ak}\delta_{bi} + \delta_{aj}\delta_{bl} + \delta_{al}\delta_{bj}), \tag{33}$$

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where a δ denotes a Kronecker delta. This relation can be inverted using a Gram matrix to express γ's in terms of the α's. Explicitly,

$$\sum_{A=1}^{N_0}\sum_{a,b=1}^{N}\alpha_{ab}^{B}\alpha_{ab}^{A}\gamma_{ijkl}^{A}=2(\alpha_{ij}^{B}+\alpha_{kl}^{B}-\alpha_{ik}^{B}-\alpha_{jl}^{B}). \tag{34}$$

Clearly, (32) fails to hold in the instances wherein χ is null or singular. As mentioned above, those will not appear in the formulæ below.

We now proceed to obtain solutions to the system of N_0 equations (27). First let us define the differential operators,

$$\partial_{ij}=\frac{\partial}{\partial|Q_{ij}|},\qquad \theta_{ij}=|Q_{ij}|\partial_{ij}, \tag{35}$$

$$\begin{aligned}\widehat{L}_{ijkl}&=\partial_{ij}\partial_{kl}-\partial_{ik}\partial_{jl}\\&=\frac{1}{|Q_{ij}||Q_{kl}|}(\theta_{ij}\theta_{kl}-|\chi_{ijkl}|\theta_{ik}\theta_{jl}).\end{aligned} \tag{36}$$

From (20), (22) and (29) we obtain

$$\theta_{ij}I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})=\prod_{\substack{m,n\\1\leq m\neq n\leq N}}|Q_{mn}|^{\frac{1}{2}\beta_{mn}}\Big(\sum_A\alpha_{ij}^A\theta_A+\beta_{ij}\Big)I_0(\boldsymbol{\xi}), \tag{37}$$

so that

$$\widehat{L}_{ijkl}I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})=\left(\prod_{\substack{m,n\\1\leq m\neq n\leq N}}|Q_{mn}|^{\frac{1}{2}\beta_{mn}}\right)\frac{1}{|Q_{ij}||Q_{kl}|}L_{ijkl}I_0(\boldsymbol{\xi}), \tag{38}$$

where

$$L_{ijkl}=\Big(\sum_A\alpha_{ij}^A\theta_A+\beta_{ij}\Big)\Big(\sum_B\alpha_{kl}^A\theta_B+\beta_{kl}\Big)-|\chi_{ijkl}|\Big(\sum_A\alpha_{ik}^A\theta_A+\beta_{ik}\Big)\Big(\sum_B\alpha_{jl}^A\theta_B+\beta_{jl}\Big). \tag{39}$$

Requiring the conformal integral (22), which is but a function of $|Q_{ij}|$ treated as independent variables, to satisfy

$$L_{ijkl}I_0(\boldsymbol{\xi})=0, \tag{40}$$

or, equivalently,

$$\widehat{L}_{ijkl}I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})=0, \tag{41}$$

we obtain the equation for the invariant part.

The crucial observation in the present article is that the equation (27) is obtained from this by summing over the k and l indices as

$$L_{ij}I_0(\boldsymbol{\xi})=\sum_{k,l=1}^{N}L_{lijk}I_0(\boldsymbol{\xi})-\sum_{k,l=1}^{N}L_{ijkl}I_0(\boldsymbol{\xi}), \tag{42}$$

where the symmetry of Q, α and β with respect to the indices has been used. We have indicated the sum in the two terms separately, since it is easier to derive (42) by performing the sums on the r.h.s. before subtracting. It can be verified by explicit computation that only the operators L_{ijkl} with all the four indices distinct appear in the final expression L_{ij}. Many of these are, in turn, related through the inter-relations among the $|\chi|$'s. We need to consider only a few of these operators in order to obtain I_0.

Thus, a simultaneous solution of (40) for the operators that appear in L_{ij} is a solution to (27). For a given set of α, the equation (40) is solved using the Frobenius' method with

$$\prod_{A=1}^{N_0} \xi_A^{\nu_A} \sum_{n_1,n_2,\cdots,n_{N_0}=0}^{\infty} C_{n_1,n_2,\cdots,n_{N_0}} \xi_1^{n_1}\xi_2^{n_2}\cdots\xi_{N_0}^{n_{N_0}}, \tag{43}$$

where the solutions ν to the indicial equations can be chosen in terms of the parameters β and the coefficients are given by the recursion relation

$$\frac{C_{n_1-\gamma^1_{ijkl},n_2-\gamma^2_{ijkl},\cdots,n_{N_0}-\gamma^{N_0}_{ijkl}}}{C_{n_1,n_2,\cdots,n_{N_0}}} = \frac{\left(\sum_A \alpha^A_{ij}(n_A+\nu_A)+\beta_{ij}\right)\left(\sum_B \alpha^B_{kl}(n_B+\nu_B)+\beta_{kl}\right)}{\left(\sum_A \alpha^A_{ik}(n_A+\nu_A)+\beta_{ik}+1\right)\left(\sum_B \alpha^B_{jl}(n_B+\nu_B)+\beta_{jl}+1\right)}, \tag{44}$$

with the γ's obtained from (34). The coefficients $C_{n_1,n_2,\cdots,n_{N_0}}$ can now be written in terms of Gamma functions involving the combinations appearing within the braces.

Solving the system (27) thus reduces to the combinatorial problem of obtaining the exponents of $|Q|$'s in (20), that is, the α's, and expressing ν's in terms of β's. In order to obtain the α's we consider each $|Q_{ij}|$ in turn, which transforms according to (5) with a factor for each of the indices i and j. Let us form a matrix from the Möbius transformation of $|Q_{ij}|$. From (5) we note that it transforms by two factors, $(CQ_i + D)$ and $(CQ_j + D)$, with exponents -1 for each. We define an $N \times N(N-1)/2$ matrix $\mathcal{A}$ from this data. Its columns correspond to $|Q_{ij}|$ and rows correspond to Q_i. The entry of $\mathcal{A}$ in the column of Q_{ij} in both the rows i and j is unity. All other entries are taken to be zero. The indices of the invariants under the Möbius transformation constitute the kernel of $\mathcal{A}$. A choice of the basis of the kernel is taken to define the α's which in turn define the cross ratios $\boldsymbol{\xi}$ from (20). The N_0 cross rations are determined by the transpose of the matrix of these basis vectors, denoted v, which is an $N_0 \times N(N-1)/2$ matrix. Let us exemplify this construction with the example of $N = 4$. In this cases, the matrix $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{array}{c} \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{array} \begin{array}{c} \begin{array}{cccccc} Q_{12} & Q_{13} & Q_{14} & Q_{23} & Q_{24} & Q_{34} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}. \tag{45}$$

This encodes, for example, the fact that $|Q_{12}|$ transforms by factors involving Q_1 and Q_2, both having exponent -1, but does not contain factors involving Q_3 or Q_4, as can be read off from (5). Let us denote the entries of $\mathcal{A}$ by $a_{i,jk}$, with $j < k$. The kernel is two-dimensional. Its transpose with a certain choice of basis vectors is

$$v = \begin{array}{c} \\ \xi_1 \\ \xi_2 \end{array} \begin{array}{c} \begin{array}{cccccc} Q_{12} & Q_{13} & Q_{14} & Q_{23} & Q_{24} & Q_{34} \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix} \end{array}. \tag{46}$$

The A-th row of v gives α^A, for example, $\alpha^1_{12} = 1$, $\alpha^2_{34} = 0$ etc, so that $v = (\alpha^A_{ij})$. Let us recall that in our notation the GKZ A-hypergeometric system corresponding to the matrix $\mathcal{A}$ is given by [25]

$$\left(\sum_{\substack{j,k=1\\ j<k}}^{N} a_{i,jk}\frac{\partial}{\partial|Q_{jk}|} - \mu_i\right) f = 0, \quad \forall i, \tag{47}$$

$$\prod_{\alpha^A_{ij}>0}\left(\frac{\partial}{\partial|Q_{ij}|}\right)^{\alpha^A_{ij}} f - \prod_{\alpha^A_{ij}<0}\left(\frac{\partial}{\partial|Q_{ij}|}\right)^{-\alpha^A_{ij}} f = 0, \quad \forall A. \tag{48}$$

The operators acting on f form an ideal in the Weyl algebra corresponding to the matrix $\mathcal{A}$. It can be checked that these imply (41). Hence, L_{ijkl} belong to the GKZ ideal. Inserting (22) for f the first set (47) is satisfied using (23). In order to obtain $I_0(\boldsymbol{\xi})$ it thus suffices to solve the second set of equations (48). Expressing the GKZ operators in terms of logarithmic variables, a series solution to these equations are obtained with its coefficients satisfying (44). Hence the invariant $I_0(\boldsymbol{\xi})$ is given by the GKZ A-hypergeometric function corresponding to the matrix $\mathcal{A}$

We now present examples for $N = 4, 5, 6$. The general expression can be similarly written.

Example 1. For four points, $N = 4$, six operators L_{ij}, $i, , j = 1, 2, 3, 4$, $i < j$, are to be considered in (27). Two of such operators determine the rest through the relations

$$\begin{gathered} L_{12} = L_{34}, L_{13} = L_{24}, L_{14} = L_{23}, \\ L_{12} + L_{13} + L_{14} = L_{23} + L_{24} + L_{34} = 0. \end{gathered} \tag{49}$$

Choosing L_{24} and L_{34} as the independent ones leads to the equations

$$\begin{aligned} &(\xi_1 + \xi_2 - 1)\theta_1^2 I_0 + 2\xi_1\theta_1\theta_2 I_0 + \xi_1(2 + \beta_{13})\theta_2 I_0 \\ &-[\xi_1(\beta_{14} + \beta_{23}) + (1 - \xi_2)\beta_{13}]\theta_1 I_0 + \xi_1\beta_{14}\beta_{23} I_0 = 0, \\ &(\xi_1 + \xi_2 - 1)\theta_2^2 I_0 + 2\xi_2\theta_1\theta_2 I_0 + \xi_2(2 + \beta_{13})\theta_1 I_0 \\ &-\left[\xi_2(\beta_{14} + \beta_{23}) + (1 - \xi_1)\beta_{13}\right]\theta_2 I_0 + \xi_2\beta_{14}\beta_{23} I_0 = 0, \end{aligned} \tag{50}$$

respectively, where ξ_1 and ξ_2 are defined from v in (46) as

$$\xi_1 = \frac{|Q_{12}||Q_{34}|}{|Q_{14}||Q_{23}|}, \qquad \xi_2 = \frac{|Q_{13}||Q_{24}|}{|Q_{14}||Q_{23}|}. \tag{51}$$

Using the freedom of choice of β's from (23) to set β_{24} and β_{34} to zero these lead to the system of equations for the Appell function F_4 [19]. Here instead of solving (50), we solve the system of equations for the operators L_{ijkl}, as required from (42) without any *ad hoc* choice of β's. First, we write equation (42) for L_{24} and L_{34}. The r.h.s. of the two equations thus obtained contain the operators

$$L_{2431}, L_{3241}, L_{2413}, L_{1243}, L_{3412}, L_{2341}, L_{3421}, L_{1342} \tag{52}$$

in linear combinations. The operators L_{ijkl}, however, are also related among themselves through the relations

$$\begin{aligned} L_{1243} &= -\xi_1 L_{3241} = L_{3421} \\ L_{1342} &= -\xi_2 L_{2341} = L_{2431} \\ \xi_1 L_{2413} &= -\xi_2 L_{3412} = \xi_1 L_{2431} - \xi_2 L_{3421}, \end{aligned} \tag{53}$$

leaving only two of them independent. We choose these as,

$$\begin{aligned} L_{3421} &= (\theta_1+\beta_{12})(\theta_1+\beta_{34}) - \xi_1(\theta_1+\theta_2-\beta_{14})(\theta_1+\theta_2-\beta_{23}), \\ L_{2431} &= (\theta_2+\beta_{13})(\theta_2+\beta_{24}) - \xi_2(\theta_1+\theta_2-\beta_{14})(\theta_1+\theta_2-\beta_{23}). \end{aligned} \tag{54}$$

The solution $I_0(\xi_1,\xi_2)$ is annihilated by each of these operators. In this case there are four independent solutions corresponding to the solutions of the indicial equations ensuing from (54). For example, for the choice of indices, $\nu_1=-\beta_{34}$ and $\nu_2=-\beta_{24}$, the solution is

$$I_0(\xi_1,\xi_2) = \xi_1^{-\beta_{34}}\xi_2^{-\beta_{24}} P_2(\boldsymbol{\mu};\boldsymbol{\xi}), \tag{55}$$

where P_2 is an infinite series which can be identified with the Appell series F_4 up to an overall constant, namely,

$$P_2(\boldsymbol{\mu};\boldsymbol{\xi}) = \frac{\Gamma(1-\mu_1-\mu_2+d/2)\Gamma(1-\mu_1-\mu_3+d/2)}{\Gamma(\mu_4)\Gamma(-\mu_1+d/2)} \sum_{n_1,n_2=0}^{\infty} \frac{\xi_1^{n_1}}{n_1!}\frac{\xi_2^{n_2}}{n_2!} \frac{\Gamma(n_1+n_2+\mu_4)\Gamma(n_1+n_2-\mu_1+d/2)}{\Gamma(1+n_1-\mu_1-\mu_2+d/2)\Gamma(1+n_2-\mu_1-\mu_3+d/2)}, \tag{56}$$

where we have used $|\boldsymbol{\mu}|=d=4$ as well as (23) to replace linear combinations of β's with μ's in deriving this expression. With the prefactor chosen, the solution is the Appell function F_4. The four solutions to the indicial equations from (54) corresponds to the four solutions to the Appell equation for F_4, so that the general expression for the $N=4$ conformal integral becomes [19]

$$I_4^{(\boldsymbol{\mu})} = C_1(\boldsymbol{\mu})f_1 + C_2(\boldsymbol{\mu})f_2 + C_3(\boldsymbol{\mu})f_3 + C_3(\boldsymbol{\mu})f_4, \tag{57}$$

where C's are constants depending on the parameters $\boldsymbol{\mu}$ and

$$\begin{aligned} f_1 &= |Q_{34}|^{-\mu_3-\mu_4+d/2}|Q_{24}|^{-\mu_2-\mu_4+d/2}|Q_{14}|^{-\mu_1}|Q_{23}|^{\mu_4-d/2}F_4\Big(\mu_1,-\mu_4+d/2,1-\mu_3-\mu_4+d/2,1-\mu_2-\mu_4+d/2;\ \xi_1,\xi_2\Big), \\ f_2 &= |Q_{34}|^{-\mu_3-\mu_4+d/2}|Q_{13}|^{-\mu_1-\mu_3+d/2}|Q_{23}|^{-\mu_2}|Q_{14}|^{\mu_3-d/2}F_4\Big(\mu_2,-\mu_3+d/2,1-\mu_3-\mu_4+d/2,1-\mu_1-\mu_3+d/2;\ \xi_1,\xi_2\Big), \\ f_3 &= |Q_{12}|^{-\mu_1-\mu_2+d/2}|Q_{24}|^{-\mu_2-\mu_4+d/2}|Q_{23}|^{-\mu_3}|Q_{14}|^{\mu_2-d/2}F_4\Big(\mu_3,-\mu_2+d/2,1-\mu_1-\mu_2+d/2,1-\mu_2-\mu_4+d/2;\ \xi_1,\xi_2\Big), \\ f_4 &= |Q_{12}|^{-\mu_1-\mu_2+d/2}|Q_{13}|^{-\mu_1-\mu_3+d/2}|Q_{14}|^{-\mu_4}|Q_{23}|^{\mu_1-d/2}F_4\Big(\mu_4,-\mu_1+d/2,1-\mu_1-\mu_2+d/2,1-\mu_1-\mu_3+d/2;\ \xi_1,\xi_2\Big). \end{aligned} \tag{58}$$

The solution is independent of the choice of β's in the ansatz (22), as expected.

Our goal is to write the conformal integral in terms of a local system on the Fulton-MacPherson completion of the configuration space, which possesses a canonical action of the group of permutations of the points, to be reflected in the conformal integral. This action is lifted to the conformal integral as the permutation of Q_i and μ_i at once. While the integral (6) is invariant under these permutations, the solution (57) is not. The permutation symmetry has been broken by the choice of independent equations, namely (54). It can

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be restored by fixing the four constants such that they transform appropriately under permutation of μ_i. The details of the computations to fix the constants is presented in the appendix. The result is

$$
\begin{aligned}
C_1(\boldsymbol{\mu}) &= \Gamma(\mu_1)\Gamma(2-\mu_4)\Gamma(2-\mu_1-\mu_2)\Gamma(2-\mu_1-\mu_3),\\
C_2(\boldsymbol{\mu}) &= \Gamma(\mu_2)\Gamma(2-\mu_3)\Gamma(2-\mu_1-\mu_2)\Gamma(2-\mu_2-\mu_4),\\
C_3(\boldsymbol{\mu}) &= \Gamma(\mu_3)\Gamma(2-\mu_2)\Gamma(2-\mu_1-\mu_3)\Gamma(2-\mu_3-\mu_4),\\
C_4(\boldsymbol{\mu}) &= \Gamma(\mu_4)\Gamma(2-\mu_1)\Gamma(2-\mu_2-\mu_4)\Gamma(2-\mu_3-\mu_4),
\end{aligned}
\tag{59}
$$

up to an overall constant independent of $\boldsymbol{\xi}$ and $\boldsymbol{\mu}$, chosen to be unity here.

Example 2. For $N = 5$, we have

$$
\mathcal{A} = \begin{array}{c} \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{23} & Q_{24} & Q_{25} & Q_{34} & Q_{35} & Q_{45} \end{array}\\
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\end{array}
\tag{60}
$$

and

$$
v = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{23} & Q_{24} & Q_{25} & Q_{34} & Q_{35} & Q_{45} \end{array}\\
\begin{pmatrix}
1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0\\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}.
\tag{61}
$$

In (27) we take the equations corresponding to L_{24}, L_{25}, L_{34}, L_{35}, L_{45} as the independent ones. The operators that contribute to these equations are

$$
L_{1243}, L_{1253}, L_{1342}, L_{1352}, L_{1435}, L_{1425}, L_{1524}, L_{1534}, L_{2435}, L_{2534}, L_{2415}, L_{2451}. \tag{62}
$$

The simultaneous solution of the equations ensuing from these is

$$
I_0(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = \xi_1^{-\beta_{45}}\xi_2^{-\beta_{35}}\xi_3^{-\beta_{34}}\xi_4^{-\beta_{25}}\xi_5^{-\beta_{24}}P_5(\boldsymbol{\mu};\boldsymbol{\xi}), \tag{63}
$$

with the series P_5 defined as

$$
\begin{aligned}
P_5(\boldsymbol{\mu};\boldsymbol{\xi}) = &\sum_{n_1,n_2,n_3,n_4,n_5=0}^{\infty} \frac{\xi_1^{n_1}}{n_1!}\frac{\xi_2^{n_2}}{n_2!}\frac{\xi_3^{n_3}}{n_3!}\frac{\xi_4^{n_4}}{n_4!}\frac{\xi_5^{n_5}}{n_5!} \\
&\times \frac{1}{\Gamma(1+n_1+n_2+n_3-\mu_1-\mu_2+d/2)\Gamma(1+n_1+n_4+n_5-\mu_1-\mu_3+d/2)}\\
&\times \frac{1}{\Gamma(1-n_1-n_3-n_5-\mu_4)\Gamma(1-n_1-n_2-n_4-\mu_5)\Gamma(1-n_1-n_2-n_3-n_4-n_5+\mu_1-d/2)}.
\end{aligned}
\tag{64}
$$

Plugging in (22), we obtain the conformal integral

$$
I_5^{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5)} = |Q_{12}|^{-\mu_1-\mu_2+d/2}|Q_{13}|^{-\mu_1-\mu_3+d/2}|Q_{14}|^{-\mu_4}|Q_{15}|^{-\mu_5}|Q_{23}|^{\mu_1-d/2}P_5(\boldsymbol{\mu};\boldsymbol{\xi}), \tag{65}
$$

independent of the choice of β's.

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Example 3. For $N = 6$ we have

$$
\mathcal{A} = \begin{array}{c} \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc} Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{34} & Q_{35} & Q_{36} & Q_{45} & Q_{46} & Q_{56} \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\end{array}
\tag{66}
$$

and

$$
v = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccc} Q_{12} & Q_{13} & Q_{14} & Q_{15} & Q_{16} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{34} & Q_{35} & Q_{36} & Q_{45} & Q_{46} & Q_{56} \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}.
\tag{67}
$$

Taking $L_{24}, L_{25}, \cdots, L_{56}$ as independent we derive

$$
I_0(\xi_1, \xi_2, \cdots, \xi_9) = \xi_1^{-\beta_{56}} \xi_2^{-\beta_{46}} \xi_3^{-\beta_{45}} \xi_4^{-\beta_{36}} \xi_5^{-\beta_{35}} \xi_6^{-\beta_{34}} \xi_7^{-\beta_{26}} \xi_8^{-\beta_{25}} \xi_9^{-\beta_{24}} P_9(\boldsymbol{\mu}; \boldsymbol{\xi}) \tag{68}
$$

as the simultaneous solution of the corresponding GKZ system. The series

$$
\begin{aligned}
P_9(\boldsymbol{\mu};\boldsymbol{\xi}) = & \sum_{n_1,n_2,\cdots,n_9=0}^{\infty} \frac{\xi_1^{n_1}}{n_1!} \frac{\xi_2^{n_2}}{n_2!} \frac{\xi_3^{n_3}}{n_3!} \frac{\xi_4^{n_4}}{n_4!} \frac{\xi_5^{n_5}}{n_5!} \frac{\xi_6^{n_6}}{n_6!} \frac{\xi_7^{n_7}}{n_7!} \frac{\xi_8^{n_8}}{n_8!} \frac{\xi_9^{n_9}}{n_9!} \\
& \times \frac{1}{\Gamma(1+n_1+n_2+n_3+n_4+n_5+n_6-\mu_1-\mu_2+d/2)} \\
& \times \frac{1}{\Gamma(1+n_1+n_2+n_3+n_7+n_8+n_9-\mu_1-\mu_3+d/2)} \\
& \times \frac{1}{\Gamma(1-n_2-n_3-n_6-n_9-\mu_4)\Gamma(1-n_1-n_3-n_5-n_8-\mu_5)\Gamma(1-n_1-n_2-n_4-n_7-\mu_6)} \\
& \times \frac{1}{\Gamma(1-n_1-n_2-n_3-n_4-n_5-n_6-n_7-n_8-n_9+\mu_1-d/2)}.
\end{aligned}
\tag{69}
$$

The conformal integral is

$$
I_6^{(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6)} = |Q_{12}|^{-\mu_1-\mu_2+d/2} |Q_{13}|^{-\mu_1-\mu_3+d/2} |Q_{14}|^{-\mu_4} |Q_{15}|^{-\mu_5} |Q_{16}|^{-\mu_6} |Q_{2,3}|^{\mu_1-d/2} P_9(\boldsymbol{\mu};\boldsymbol{\xi}). \tag{70}
$$

These examples can be generalized to any N, with the matrices v obtained using `Mathematica`. Let us point out the strategy to fix the ν's in general. The matrix v in the basis chosen has an exchange matrix, one with unity on the anti-diagonal entries as the only non-zero elements on the right. Then, if a ξ_A has a factor of $|Q_{ij}|$ coming from this part, we choose the corresponding index $\nu_A = -\beta_{ij}$. However, as we have shown, the choice of β's is obliterated in the final result. Generally, the N-point conformal integral is given by

$$I_N^{\boldsymbol{\mu}}(\boldsymbol{Q}) = |Q_{12}|^{-\mu_1-\mu_2+d/2}|Q_{13}|^{-\mu_1-\mu_3+d/2}|Q_{14}|^{-\mu_4}|Q_{15}|^{-\mu_5}\cdots|Q_{1N}|^{-\mu_N}|Q_{2,3}|^{\mu_1-d/2}P_{N_0}(\boldsymbol{\mu};\boldsymbol{\xi}), \tag{71}$$

where $P_{N_0}(\boldsymbol{\mu};\boldsymbol{\xi})$ is a power series in the cross ratios, with coefficients determined by the rows of the matrix v through the combinations appearing in (44).

The domain of definition of the series P_{N_0} and hence the choice of independent cross ratios vary in computing conformal correlation functions depending on the specific channel. The correct germ to be chosen is dictated by monodromy projection. Accordingly, the expressions presented here are to be analytically continued to other domains of convergence of P_{N_0} by Barnes' integrals. This can be performed since the coefficients are expressed in terms of Gamma functions. Unlike the case of four points wherein the series can be expressed in terms of an Appell series, however, the cases with higher number of points the series could not be identified with known functions. Also, let us point out that (57) has four terms, expressed in terms of the Appell series with different parameters. This stems from the special form of the Gale matrix (46), in which each column is repeated twice. For $N > 4$ the columns of the Gale matrices are all different, leading to a single series appearing in the expression for $I_N^{\boldsymbol{\mu}}(\boldsymbol{Q})$, as in (65) and (70).

To conclude, we have presented a method for computing conformal integrals in the four-dimensional Euclidean space with explicit expressions in terms of infinite series of cross ratios. The method is very general and relates conformal integrals to the GKZ A-hypergeometric functions by defining them over the Fulton-MacPherson completion of the configuration space of N points on the real Euclidean space. In the case of $N = 4$ we have presented explicit expressions for the conformal integral invariant under permutation of points, with computational details given in the appendix. This is required for the integral to be a "good" function on the configuration space. Let us remark that in a conformal field theory the permutation symmetry is broken by the choice of radii of convergence of operator products. Thus, in using the integrals in such a theory, the constants need to be fixed anew, preserving only the required subgroup of S_4. For use in other contexts the constants are to be fixed according to physical requirements. The conformal integrals for higher points may be treated similarly with more cumbersome formulæ. From the scaling properties of the expressions it appears that the same formulæ will continue to hold in any dimension, d, as indicated in the expressions in anticipation. Finally, the appearance of the GKZ system seems to indicate an underlying real toric variety associated to the configuration space of points.

A Fixing the constants in example 1

We present the details of the computations to derive the constants (59). This is achieved by demanding invariance of the expression (57) under the action of the permutation group S_4 of $\{1, 2, 3, 4\}$ on Q_i and μ_i. It suffices to consider the generators $\sigma_{12}, \sigma_{23}, \sigma_{14}$ of S_4, where σ_{ij} denotes a cycle of the group exchanging i and j. A permutation of the Q_i transforms the cross ratios (51) according to

$$\sigma_{12} : (\xi_1, \xi_2) \longrightarrow (\frac{\xi_1}{\xi_2}, \frac{1}{\xi_2}) \tag{A.1}$$

$$\sigma_{23} : (\xi_1, \xi_2) \longrightarrow (\xi_2, \xi_1) \tag{A.2}$$

$$\sigma_{14} : (\xi_1, \xi_2) \longrightarrow (\xi_2, \xi_1), \tag{A.3}$$

thereby transforming the Appell functions appearing in (58). The first four arguments depending on μ_i also change. Using the transformation formulæ of F_4, namely,

$$F_4(a, b; c, d; x, y) = F_4(a, b; d, c; y, x), \tag{A.4}$$

which follows from the definition, and

$$\begin{aligned} F_4(a, b; c, d; x, y) = &\frac{\Gamma(d)\Gamma(b-a)}{\Gamma(d-a)\Gamma(b)}(-y)^{-a} F_4\left(a, a-d+1; c, a-b+1, \frac{x}{y}, \frac{1}{y}\right) \\ &+ \frac{\Gamma(d)\Gamma(a-b)}{\Gamma(d-b)\Gamma(a)}(-y)^{-b} F_4\left(b, b-d+1; c, b-a+1, \frac{x}{y}, \frac{1}{y}\right), \end{aligned} \tag{A.5}$$

the Appell functions can be expressed back in terms of F_4 with arguments (ξ_1, ξ_2). Since the functions (58) are the solutions to the four indicial equations associated to (54), these are the germs of the local system in a neighborhood of $\boldsymbol{\xi} = 0$ forming a basis. Hence, $I_4^{(\boldsymbol{\mu})}$ in (57) can be expressed in terms of the same functions (58) with new constants. In this manner the permutations induce an action on the C's. Let us denote the action of the generators σ_{ij} on the constants by

$$C_i'(\boldsymbol{\mu}) = \sigma_{12} C_i(\boldsymbol{\mu}), \quad C_i''(\boldsymbol{\mu}) = \sigma_{23} C_i(\boldsymbol{\mu}), \quad C_i'''(\boldsymbol{\mu}) = \sigma_{14} C_i(\boldsymbol{\mu}), \tag{A.6}$$

$i = 1, 2, 3, 4$. Writing the quadruple of C's as a vector we obtain a matrix representation of the permutations. For example,

$$\begin{pmatrix} C_1'(\boldsymbol{\mu}) \\ C_2'(\boldsymbol{\mu}) \\ C_3'(\boldsymbol{\mu}) \\ C_4'(\boldsymbol{\mu}) \end{pmatrix} = \Sigma_{12}(\boldsymbol{\mu}) \begin{pmatrix} C_1(\boldsymbol{\mu}) \\ C_2(\boldsymbol{\mu}) \\ C_3(\boldsymbol{\mu}) \\ C_4(\boldsymbol{\mu}) \end{pmatrix}, \tag{A.7}$$

where $\Sigma_{12}(\boldsymbol{\mu})$ denotes the transformation matrix under σ_{12}, and similarly for the other two generators. The three transformation matrices are

$$\Sigma_{12}(\boldsymbol{\mu}) =$$

$$\begin{pmatrix} \frac{(-1)^{\mu_4}\Gamma(3-\mu_1-\mu_4)\Gamma(2-\mu_1-\mu_3)}{\Gamma(\mu_2)\ \Gamma(1-\mu_1)} & \frac{(-1)^{-\mu_1}\Gamma(3-\mu_2-\mu_3)\Gamma(2-\mu_1-\mu_3)}{\Gamma(2-\mu_3)\Gamma(\mu_4-1)} & 0 & 0 \\ \frac{(-1)^{-\mu_2}\Gamma(3-\mu_1-\mu_4)\Gamma(2-\mu_2-\mu_4)}{\Gamma(\mu_3-1)\Gamma(2-\mu_4)} & \frac{(-1)^{\mu_3}\Gamma(3-\mu_2-\mu_3)\Gamma(2-\mu_2-\mu_4)}{\Gamma(1-\mu_2)\Gamma(\mu_1)} & 0 & 0 \\ 0 & 0 & \frac{(-1)^{-\mu_3}\Gamma(3-\mu_1-\mu_4)\Gamma(2-\mu_1-\mu_3)}{\Gamma(\mu_2-1)\Gamma(2-\mu_1)} & \frac{(-1)^{\mu_2}\Gamma(3-\mu_2-\mu_3)\Gamma(2-\mu_1-\mu_3)}{\Gamma(1-\mu_3)\Gamma(\mu_4)} \\ 0 & 0 & \frac{(-1)^{\mu_1}\Gamma(3-\mu_1-\mu_4)\Gamma(2-\mu_2-\mu_4)}{\Gamma(\mu_3)\Gamma(1-\mu_4)} & \frac{(-1)^{-\mu_4}\Gamma(3-\mu_2-\mu_3)\Gamma(2-\mu_2-\mu_4)}{\Gamma(2-\mu_2)\Gamma(\mu_1-1)} \end{pmatrix},$$

$$\Sigma_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \Sigma_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A.8}$$

The arguments of the latter two are suppressed since they do not depend on $\boldsymbol{\mu}$.

To fix the constants, we first impose the condition that σ_{12} acting twice on $C_i(\boldsymbol{\mu})$ keeps them unchanged. From (A.7), then, we conclude that the vector of C's is an eigenvector of the product matrix $\Sigma'_{12}(\boldsymbol{\mu})\Sigma_{12}(\boldsymbol{\mu})$ with unit eigenvalue. Here the prime on Σ_{12} signifies that the arguments of Σ_{12} are permuted by σ_{12}. Solving the eigenvalue problem fixes the ratios

$$\frac{C_2(\boldsymbol{\mu})}{C_1(\boldsymbol{\mu})} = \frac{\sin\pi\mu_1 \sin\pi\mu_4}{\pi\sin\pi\,(\mu_2+\mu_4)} \frac{\Gamma(1-\mu_1)\Gamma(\mu_2)\Gamma(2-\mu_3)\Gamma(\mu_4-1)}{\Gamma(2-\mu_1-\mu_3)\Gamma(3-\mu_1-\mu_3)}, \tag{A.9}$$

$$\frac{C_4(\boldsymbol{\mu})}{C_3(\boldsymbol{\mu})} = \frac{\sin\pi\mu_2 \sin\pi\mu_3}{\pi\sin\pi\,(\mu_2+\mu_4)} \frac{\Gamma(2-\mu_1)\Gamma(\mu_2-1)\Gamma(1-\mu_3)\Gamma(\mu_4)}{\Gamma(2-\mu_1-\mu_3)\Gamma(3-\mu_1-\mu_3)}. \tag{A.10}$$

According to (A.8), $C''_1(\boldsymbol{\mu}) = C_1(\boldsymbol{\mu})$ and $C''_2(\boldsymbol{\mu}) = C_3(\boldsymbol{\mu})$ under the action of σ_{23}. Acting (A.9) with σ_{23} we obtain

$$\frac{C''_2(\boldsymbol{\mu})}{C''_1(\boldsymbol{\mu})} = \frac{C_3(\boldsymbol{\mu})}{C_1(\boldsymbol{\mu})} = \frac{\sin\pi\mu_1 \sin\pi\mu_4}{\pi\sin\pi\,(\mu_3+\mu_4)} \frac{\Gamma(1-\mu_1)\Gamma(2-\mu_2)\Gamma(\mu_3)\Gamma(\mu_4-1)}{\Gamma(2-\mu_1-\mu_2)\Gamma(3-\mu_1-\mu_2)}. \tag{A.11}$$

Using this in (A.10) we obtain

$$\frac{C_4(\boldsymbol{\mu})}{C_1(\boldsymbol{\mu})} = \frac{\sin\pi\mu_1 \sin\pi\mu_4}{\sin\pi(\mu_2+\mu_4)\sin\pi(\mu_3+\mu_4)} \frac{\Gamma(1-\mu_1)\Gamma(2-\mu_1)\Gamma(\mu_4)\Gamma(\mu_4-1)}{\Gamma(2-\mu_1-\mu_3)\Gamma(3-\mu_1-\mu_3)\Gamma(2-\mu_1-\mu_2)\Gamma(3-\mu_1-\mu_2)} \tag{A.12}$$

We have thus obtained $C_2(\boldsymbol{\mu})$, $C_3(\boldsymbol{\mu})$ and $C_4(\boldsymbol{\mu})$ in terms of $C_1(\boldsymbol{\mu})$ in (A.9), (A.11) and (A.12), respectively.

While we have used the invariance of the constants under σ_{12} acting twice up till now, we have not used the transformation (A.7) directly. Using (A.9), (A.11) and (A.12) in (A.7) we obtain the ratios

$$\begin{aligned}
\frac{C'_1(\boldsymbol{\mu})}{C_1(\boldsymbol{\mu})} &= \frac{\Gamma(\mu_2)\Gamma(2-\mu_2-\mu_3)}{\Gamma(\mu_1)\Gamma(2-\mu_1-\mu_3)},\\
\frac{C'_2(\boldsymbol{\mu})}{C_2(\boldsymbol{\mu})} &= \frac{\Gamma(\mu_1)\Gamma(2-\mu_1-\mu_4)}{\Gamma(\mu_2)\Gamma(2-\mu_2-\mu_4)},\\
\frac{C'_3(\boldsymbol{\mu})}{C_3(\boldsymbol{\mu})} &= \frac{\Gamma(2-\mu_1)\Gamma(2-\mu_2-\mu_3)}{\Gamma(2-\mu_2)\Gamma(2-\mu_1-\mu_3)},\\
\frac{C'_4(\boldsymbol{\mu})}{C_4(\boldsymbol{\mu})} &= \frac{\Gamma(2-\mu_2)\Gamma(2-\mu_1-\mu_4)}{\Gamma(2-\mu_1)\Gamma(2-\mu_2-\mu_4)}.
\end{aligned} \tag{A.13}$$

Since the primed constants are the transformed ones under σ_{12}, we deduce

$$\begin{aligned}
C_1(\boldsymbol{\mu}) &\propto \Gamma(\mu_1)\Gamma(2-\mu_1-\mu_3),\\
C_2(\boldsymbol{\mu}) &\propto \Gamma(\mu_2)\Gamma(2-\mu_2-\mu_4),\\
C_3(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_2)\Gamma(2-\mu_1-\mu_3),\\
C_4(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_1)\Gamma(2-\mu_2-\mu_4).
\end{aligned} \tag{A.14}$$

The constants transform under the other two generators as well. Under the action of σ_{23} these expressions transform to

$$\begin{aligned}
C''_1(\boldsymbol{\mu}) &\propto \Gamma(\mu_1)\Gamma(2-\mu_1-\mu_2),\\
C''_2(\boldsymbol{\mu}) &\propto \Gamma(\mu_3)\Gamma(2-\mu_3-\mu_4),\\
C''_3(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_3)\Gamma(2-\mu_1-\mu_2),\\
C''_4(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_1)\Gamma(2-\mu_3-\mu_4).
\end{aligned} \tag{A.15}$$

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The constants C_1 and C_4 remain invariant under Σ_{23}, while the other two are exchanged, as is seen from (A.8). Incorporating extra factors thus arising we obtain

$$\begin{aligned}
C_1(\boldsymbol{\mu}) &\propto \Gamma(\mu_1)\Gamma(2-\mu_1-\mu_2)\Gamma(2-\mu_1-\mu_3), \\
C_2(\boldsymbol{\mu}) &\propto \Gamma(\mu_2)\Gamma(2-\mu_1-\mu_2)\Gamma(2-\mu_3)\Gamma(2-\mu_2-\mu_4), \\
C_3(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_2)\Gamma(2-\mu_1-\mu_3)\Gamma(\mu_3)\Gamma(2-\mu_3-\mu_4), \\
C_4(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_1)\Gamma(2-\mu_2-\mu_4)\Gamma(2-\mu_3-\mu_4).
\end{aligned} \tag{A.16}$$

Furthermore, the constants C_2 and C_3 are invariant under σ_{14}, in accordance with Σ_{14} in (A.8), while C_1 and C_4 transform to

$$\begin{aligned}
C_1'''(\boldsymbol{\mu}) &\propto \Gamma(\mu_4)\Gamma(2-\mu_4-\mu_2)\Gamma(2-\mu_4-\mu_3) \\
C_4'''(\boldsymbol{\mu}) &\propto \Gamma(2-\mu_4)\Gamma(2-\mu_2-\mu_1)\Gamma(2-\mu_3-\mu_1).
\end{aligned} \tag{A.17}$$

The matrix Σ_{14} exchanges C_1 and C_4. Incorporating the additional factors in these two we derive the expressions (59) taking the constant of proportionality as unity.

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