The 4d superconformal index near roots of unity and 3d Chern-Simons theory

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Abstract: We consider the $S^3 \times S^1$ superconformal index $\mathcal{I}(\tau)$ of 4d $\mathcal{N} = 1$ gauge theories. The Hamiltonian index is defined in a standard manner as the Witten index with a chemical potential τ coupled to a combination of angular momenta on S^3 and the U(1) R-charge. We develop the all-order asymptotic expansion of the index as $q = e^{2\pi i\tau}$ approaches a root of unity, i.e. as $\widetilde{\tau} \equiv m\tau + n \to 0$, with m, n relatively prime integers. The asymptotic expansion of $\log \mathcal{I}(\tau)$ has terms of the form $\widetilde{\tau}^k$, $k = -2, -1, 0, 1$. We determine the coefficients of the $k = -2, -1, 1$ terms from the gauge theory data, and provide evidence that the $k = 0$ term is determined by the Chern-Simons partition function on $S^3/\mathbb{Z}_m$. We explain these findings from the point of view of the 3d theory obtained by reducing the 4d gauge theory on the S^1. The supersymmetric functional integral of the 3d theory takes the form of a matrix integral over the dynamical 3d fields, with an effective action given by supersymmetrized Chern-Simons couplings of background and dynamical gauge fields. The singular terms in the $\widetilde{\tau} \to 0$ expansion (dictating the growth of the 4d index) are governed by the background Chern-Simons couplings. The constant term has a background piece as well as a piece given by the localized functional integral over the dynamical 3d gauge multiplet. The linear term arises from the supersymmetric Casimir energy factor needed to go between the functional integral and the Hamiltonian index.

Contents

1 Introduction

In the last few years there has been a renewed interest in the study of the superconformal index of 4d $\mathcal{N} = 1$ superconformal field theories (SCFTs) and, in particular, $\mathcal{N} = 4$ super Yang-Mills (SYM). The index in question is the supersymmetric partition function of the SCFT on $S^3 \times S^1$ which receives contributions from BPS states that preserve two supercharges $(\mathcal{Q}, \overline{\mathcal{Q}})$. In the large-$N$ limit, the expectation from AdS/CFT is that the index should account for the entropy of the BPS black holes (BH) that preserve the same two supercharges in the dual supergravity on AdS$_5$. This question was introduced in [1–3], and the work of the last few years has shown that the index indeed captures the BH entropy in different asymptotic limits [4–24].

The focus of the present paper is the *Cardy-like limit* in which the BH entropy becomes very large. In the canonical ensemble, this translates to the study of the exponential growth of the index as $\tau \to 0$, where the parameter τ is the chemical potential dual to the charge. As pointed out in [13], the $\tau \to 0$ limit is in fact one of an infinite number of inequivalent Cardy-like limits in which the index is expected to grow exponentially. These limits correspond to τ approaching a rational number or, equivalently, $q = e^{2\pi i \tau}$

approaching a root of unity. In this paper we analyze the 4d superconformal index near a general root of unity, and find interesting relations to three-dimensional Chern-Simons (CS) theory. The main statement is that the asymptotics of the index near a rational point $-n/m$ is equal (to all orders in perturbation theory in deviations $\widetilde{\tau} = m\tau + n$ from the rational point) to the partition function of a certain 3d $\mathcal{N} = 2$ gauge theory with Chern-Simons couplings that involve background as well as dynamical fields on an $S^3/\mathbb{Z}_m$ orbifold. The background couplings give rise to singular terms at $\mathrm{O}(1/\widetilde{\tau}^2)$ and $\mathrm{O}(1/\widetilde{\tau})$ that govern the growth of the index, while the constant $\mathrm{O}(1)$ term receives contributions from both background fields and the dynamical Chern-Simons theory.

We demonstrate this statement from two points of view — by direct asymptotic analysis of the index near rational points, and from an analysis of the reduced three-dimensional theory and calculating the various couplings using high-temperature effective-field theory (EFT) techniques. The latter method, based on [25, 26], relates the high-temperature asymptotics of the index to a low-energy effective field theory, in the spirit of the Cardy formula.[1]

The four-dimensional superconformal index and its asymptotic growth. In this paper we study $\mathcal{N} = 1$ gauge theories with a Lagrangian description and a $\mathrm{U}(1)_R$ symmetry, with a focus on $\mathcal{N} = 4$ SYM which we use to illustrate some statements in detail. The symmetry algebra of $\mathcal{N} = 1$ SCFT on $S^1 \times S^3$ is $\mathrm{SU}(2,2|1)$, which includes the energy E which generates translations around S^1, the angular momenta J_1, J_2 on S^3, and the $\mathrm{U}(1)$ R-charge Q. One can pick a complex supercharge obeying the following algebra,

$$\left\{ \mathcal{Q}, \overline{\mathcal{Q}} \right\} = E - J_1 - J_2 - \tfrac{3}{2} Q \,. \tag{1.1}$$

The most general index built out of the $\mathcal{N} = 1$ superconformal algebra is an extension of the Witten index of $\mathcal{Q}$ and is defined as the following trace over the physical Hilbert space,

$$\mathcal{I}(\sigma, \tau) = \mathrm{Tr}_{\mathcal{H}} \, (-1)^F \, \mathrm{e}^{-\gamma \left\{ \mathcal{Q}, \overline{\mathcal{Q}} \right\} + 2\pi\mathrm{i}\sigma \left(J_1 + \frac{1}{2}Q \right) + 2\pi\mathrm{i}\tau \left(J_2 + \frac{1}{2}Q \right)} \,. \tag{1.2}$$

The trace (1.2) only receives contributions from states annihilated by the supercharges ($\tfrac{1}{4}$-BPS states) so that the right-hand side of (1.1) vanishes for these states. This index $\mathcal{I}(\sigma, \tau)$ can be calculated from either Hamiltonian or functional integral methods and reduces to a unitary matrix integral [3, 32–34], which can be written as an integral over the space of gauge holonomies around the S^1 of certain infinite products, as written in equation (2.1).

Our focus in this paper is the analog, in the present context, of the high-temperature Cardy limit of 2d CFT. This means fixing the rank and taking the charges (J_i, Q) to be larger than any other scale in the theory. In the canonical ensemble this translates to

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[1]In the high-temperature picture the (Euclidean) time direction is taken along the S^1, while in the low-energy picture time is a fiber inside the S^3. Relating the two pictures involves swapping time and space as in the derivation of the 2d Cardy formula [27]. Unlike in the two-dimensional context where one uses $\mathrm{SL}(2,\mathbb{Z})$ automorphy to relate the swapped problem to the original one, here we do not have an a priori understanding of the automorphic properties of the 4d index $\mathcal{I}(\tau)$. Aspects of this question are being addressed in [28]. See also [29, 30] for related work on modular-type transformation properties relating different indices, and [31] for a discussion of the automorphic behavior of a different index in $\mathcal{N} = 2$ SCFTs.

taking $\text{Im}\,\sigma$, $\text{Im}\,\tau \to 0$ at fixed rank. In order to calculate the asymptotic growth of states along a certain direction in the charge lattice, one needs to fix the relation between σ and τ. We study[2] the slice $\sigma = \tau - n_0$ with n_0 an integer, as in [12, 13, 19]. Setting $2J = J_1 + J_2$, the resulting canonical index $\mathcal{I}$ is given by

$$\mathcal{I}\left(\tau; n_0\right) = \text{Tr}_{\mathcal{H}} \, (-1)^F \, \mathrm{e}^{-\gamma\left\{\mathcal{Q}, \overline{\mathcal{Q}}\right\} - 2\pi \mathrm{i} n_0 \left(J_1 + \frac{1}{2} Q\right) + 2\pi \mathrm{i} \tau (2J + Q)} \,. \tag{1.3}$$

The large-charge asymptotics then implies $\text{Im}\,\tau \to 0$, while $\text{Re}\,\tau$ is not fixed a priori by the limit. We consider asymptotic limits as τ approaches a rational number $\tau \to -n/m$ with $\gcd(m, n) = 1$, introduced in the present context as new Cardy-like limits in [13]. The index $\mathcal{I}$ clearly depends on the gauge group G. We generally suppress it in our notation, but sometimes use the notation $\mathcal{I}_N$ to emphasize the dependence on N for $\text{U}(N)$ or $\text{SU}(N)$ $\mathcal{N} = 4$ SYM theory (which should be clear from the context).

Our motivation to consider these rational points comes from the study of the index $\mathcal{I}_N(\tau)$ of $\mathcal{N} = 4$ SYM in the large-N limit.[3] In this limit one considers charges scaling as N^2 as $N \to \infty$, which translates to $N \to \infty$ at fixed τ in the canonical ensemble [20]. In this large-N limit one expects the field theory index $\mathcal{I}_N(\tau)$ to be written as a sum over saddles. This picture has been partially realized in the last few years using two different approaches — the Bethe-ansatz-like approach developed in [7, 35, 36], and the direct study of large-N saddle points using an elliptic extension of the action [13, 19]. In particular, the large-N approach in [13] found a class of saddles labelled by rational numbers $-n/m$, where the perturbation expansion around each saddle is given by the asymptotic limit $\tau \to -n/m$.[4] Setting $n_0 = -1$, we have

$$\log \mathcal{I}_N(\tau) \sim -S_{\text{eff}}(m, n; \tau)\,, \qquad \tau \to -n/m\,, \tag{1.4}$$

where the effective action at each saddle is given by

$$S_{\text{eff}}\left(m, n; \tau\right) = \frac{N^2 \pi \mathrm{i}}{27\, m} \frac{\left(2\widetilde{\tau} + \chi_1\left(m + n\right)\right)^3}{\widetilde{\tau}^2}\,, \qquad \widetilde{\tau} := m\tau + n\,. \tag{1.5}$$

where $\chi_1(n)$ is the Dirichlet character equal to $0, \pm 1$ when $n \equiv 0, \pm 1 \pmod 3$, respectively. There was one caveat in the above result, which was stressed in [13, 19], namely that the pure-imaginary $\widetilde{\tau}$-independent term could not be fixed by the methods used in those papers. The constant term in the effective action (1.4), therefore, was a convenient choice made using inputs coming from outside the field-theory analysis.

Although we do not have a rigorous notion of the sum over saddles yet, it should be clear that if the effective action of the (m, n) saddle has negative real part it dominates

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[2]Our methods can be generalized to study the case where σ and τ are linearly dependent over the rationals, but we shall not develop this in the present paper.

[3]Another motivation comes from the mathematical literature on q-series, where it is also natural to consider expansions around roots of unity. We thank D. Zagier for emphasizing this point to us.

[4]These saddles map to residues of the Bethe-ansatz type approach — see [22] for a recent discussion of the connections between the two approaches. A larger set of saddles have been classified in [19], but the full set of important/contributing saddles is not understood in either approach. In particular, interesting continuum configurations of the Bethe-ansatz equations have been recently discovered in [37–39] whose role in the large-N limit is not fully understood.

over the others as $m\tau + n \to 0$. It is also clear from (1.5) that the fastest growth among these saddles comes from $(m, n) = (1, 0)$. The $(1, 0)$ saddle in the SYM theory is identified as a fully deconfined phase whose entropy scales as N^2, while the other (m, n) saddles have entropies that are suppressed by a factor of m. For this reason they can be called partially deconfined saddles (in the sense of asymptotic growth, but not in the sense of center symmetry breaking — cf. [37]). On the gravitational side, the action $S_{\text{eff}}(1, 0; \tau)$ agrees precisely with the canonical on-shell action of the black hole solution in the dual AdS$_5$ supergravity [5], which leads to the identification of the AdS$_5$ BH as the saddle $(1, 0)$. The (m, n) solutions have been identified with orbifolds of the Euclidean AdS$_5$ BH [40].

Because of the dominance of the $(1, 0)$ saddle near $\tau \to 0$, one can capture it directly in an asymptotic expansion — even for finite N. In this calculation, one writes the index (1.3) as an integral over gauge holonomies u_i (see (2.1) below), estimates the integrand in the Cardy-like limit $\tau \to 0$, and then performs the integrals. The initial studies [8–12] successfully reproduced the singular parts of the action as $\tau \to 0$, i.e. the $1/\tau^2$ and the $1/\tau$ terms with the correct coefficients. More recently, the complete action (1.5) for $(m, n) = (1, 0)$ was obtained in [41] by a direct method, involving a careful analysis of all perturbative terms in the Cardy-like limit. (See [42, 43] for more recent related work.)

Our first goal in this paper is to obtain the complete perturbative action at all the (m, n) saddles by a direct asymptotic analysis of the index as $\tau \to -n/m$. This analysis is described in section 2, the result of which is a perfect agreement with the action (1.5), up to the constant terms as mentioned above. The asymptotic analysis requires developing the asymptotics of the elliptic gamma function [44, 45] near rational points. The $\tau \to 0$ asymptotic estimates were available in previous literature [46]. Here we develop the analysis for τ approaching rational numbers. The analysis is presented in appendix A. (See also [47] for related work motivated by integrable-systems considerations.)

Furthermore, we note that for given m, n, depending on the sign of $\arg\widetilde{\tau} - \pi/2$ the action in (1.5) can have negative or positive real part, which yields, respectively, a growing or decaying contribution to the index. Therefore in essentially half of the parameter space the saddles in (1.5) do not capture any growth in the index. As demonstrated in section 2.3, when the (m, n) saddle in (1.5) gives a decaying contribution to the index as $\widetilde{\tau} \to 0$, a "2-center saddle" takes over which yields exponential growth again. In other words, in half of the parameter space the growth of the index $\mathcal{I}_N(\tau)$ is captured by 2-center saddles.[5]

Chern-Simons theory from the asymptotics of the 4d index. The second goal of the paper is partly inspired by an interesting pattern appearing in the asymptotic calculations. As emphasized in the context of SU(N) $\mathcal{N} = 4$ SYM in [41], in the part of the parameter space where the index is dominated by isolated, 1-center saddles, the complete asymptotic expansion in τ terminates at $O(\tau)$ — i.e. the perturbation theory only con-

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[5]We use the terminology of [37]: in a 1-center saddle all the gauge holonomies condense at a single point on the circle, while in a 2-center saddle half of the gauge holonomies condense at one point and the other half condense at the opposite point on the circle. The 2-center saddles turn out to be partially deconfined saddles both in the sense of asymptotic growth and in the sense of center symmetry breaking [37]. See also [48].

tains $1/\tau^2$, $1/\tau$, τ^0 and τ up to exponentially suppressed corrections. (This is, in fact, more generally true when the index is dominated by isolated saddles, and not true when there are flat directions; see [49].) Interestingly, it was found in [41] that the constant term in the expansion contains the partition function of SU(N) pure Chern-Simons theory on S^3 at level $\pm N$.

In this paper we find that the same structure persists at all rational points. We see that the constant term in the expansion as $\tau \to -n/m$ involves Chern-Simons theory whose action is $1/m$ times the action as $\tau \to 0$. We present evidence that this corresponds to CS theory on an orbifold space $S^3/\mathbb{Z}_m$ (with the action of $\mathbb{Z}_m$ depending on n such that the orbifold coincides with the lens space $L(m, -1)$ when $n = 1$) at level $\pm N$ [50, 51]. In other words, the 4d SYM index appears to play the role of a master index which governs the partition function of three-dimensional CS theory on an infinite family of S^3 orbifolds.

The appearance of 3d Chern-Simons theory from the 4d superconformal index is intriguing, and gives rise to two related questions:

(a) is there a direct three-dimensional physics explanation of the appearance of Chern-Simons theory?

(b) can we also understand the singular terms in the asymptotic expansions around rational points as being related to 3d Chern-Simons theory?

The answers to both these questions are positive, as we now explain.

The asymptotics of the 4d index from supersymmetric Chern-Simons theory. The natural idea is that the reduction of the four-dimensional theory on S^1 gives rise to a three-dimensional theory on S^3 in a "high-temperature" expansion in powers of the circumference β of the shrinking circle. If we calculate the functional integral of the three-dimensional theory, we should recover the four-dimensional functional integral as $\beta \to 0$. The three-dimensional effective field theory is known to have a derivative expansion, where the most relevant terms are Chern-Simons terms [52, 53]. This EFT approach was developed in the supersymmetric context in [25, 26] who presented *supersymmetrized* CS actions involving the dynamical as well as background fields, which are necessary for preserving supersymmetry on $S^3 \times S^1$. In particular, the $1/\beta^2$ and $1/\beta$ effective actions derived this way in [26] reproduced the asymptotics of the index as found in [49] for $n_0 = 0$ and $\arg(\tau) = \pi/2$. (Note that when the metric on $S^3 \times S^1$ has a direct product form with S^3 the unit round three-sphere, a real value of β determines a purely imaginary $\tau = \frac{i\beta}{2\pi}$.) The coefficient of the leading $1/\beta^2$ term in these works is pure imaginary, and also does not grow as N^2 (it is in fact zero for non-chiral theories), therefore the exponential growth of states corresponding to the BH is not captured there.

One of the motivations for the current paper is to explain the exponential growth associated to the bulk black holes from the three-dimensional point of view, which requires $\arg(\tau) \neq \pi/2$ and $n_0 \neq 0$.[6] (Note, in particular, that the $(m, n) = (1, 0)$ saddle in (1.5)

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[6]The leading order $1/\tau^2$ behavior was found from similar considerations in [6] for $\mathcal{N} = 4$ SYM with flavor chemical potentials, and in [11] for more general gauge theories in a setting similar to ours. In this paper we follow a systematic, manifestly supersymmetric approach developed in [25, 26], which allows us to obtain all-order results for general gauge theories around generic rational points.

given for $n_0 = -1$, would have its leading piece a pure phase if $\arg(\tau) = \pi/2$.) For this purpose we consider, as in [5], a background geometry of the form $S^3 \times_\Omega S^1$, with S^3 the unit round three-sphere, γ the circumference of the circle, and Ω a twist parameter[7] controlling the deviation of the metric from a direct product form (equation (3.1)). The imaginary part of the twist parameter determines a non-zero real part of τ via (equation (3.4))

$$\tau = \frac{\mathrm{i}\gamma}{2\pi}(1 - \Omega)\,. \tag{1.6}$$

As shown in [5], the integer n_0 in (1.3) controls the periodicity of the fermions in this background, and $n_0 = \pm 1$ (which is naturally dual to the BH) corresponds to anti-periodic fermions, i.e., as in a Scherck-Schwarz reduction. In the present context we insist on supersymmetry being preserved — and that necessitates the turning on of other background fields under which the fermions are charged. In the three-dimensional background supergravity, we have a non-zero graviphoton from the fibration as well as non-zero auxiliary background gauge and scalar fields. As we explain in section 3, the resulting configuration is effectively described by a circle of radius R, which in the limit $\gamma \to 0$, $\Omega \to \infty$ with τ fixed obeys $R \to \tau$.

Now, what is the actual calculation? There are two types of fields in the three-dimensional functional integral — background fields which take constant values, and dynamical modes which fluctuate in the integral. The latter is further made up of light modes (with zero momentum around S^1) and heavy (Kaluza-Klein) modes. The first step is to integrate out the heavy modes in order to obtain an effective action for the light modes. The integration over heavy modes also generates corrections to the coefficients of the supersymmetric Chern-Simons terms of the non-zero background fields, see e.g. [25, 55]. In these calculations, we need to include, in addition to the couplings discussed in [26], the supersymmetrized RR and gravitational CS actions which were discussed in [56]. The effective actions of the background gauge fields turn out to produce precisely the singular pieces $1/\tau^2$ and $1/\tau$ in the asymptotic expansion of the index, as well as a constant piece. The remaining functional integral is described by an $\mathcal{N} = 2$ SYM theory with a certain one-loop induced CS coupling on S^3, whose partition function is known to agree, up to a sign, with that of pure Chern-Simons theory [57]. This explains the appearance of the dynamical Chern-Simons theory in the constant term of the asymptotic expansion.

Two technical remarks are in order. Firstly, recall that supersymmetry implies that the 4d superconformal index should not depend on γ and Ω separately, but only on their combination τ as in (1.6). In [5] this was shown to be true in 5d gravitational variables, as well as through a localization computation in 4d field theory. In this paper we verify this also in 3d effective field theory. Secondly, the order of limits is important to have a smooth calculational set up. We first send $\gamma \to 0$ keeping Ω fixed, so that the three-dimensional geometry is smooth and finite. Then we take $\Omega \to \infty$ at fixed τ and express the result in terms of τ using (1.6). We find there are no singularities generated in the latter step and thus the limiting procedure is perfectly smooth.

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[7]Similar twists had been described in slightly different contexts in [33, 54].

Finally, we repeat the same analysis as τ approaches rational points. The dimensional reduction in this case naturally leads us to considering orbifolds of $S^3 \times S^1$, which, as far as we understand, are related to the orbifolds discussed in [40]. The three-dimensional calculation then leads to the $1/\tilde{\tau}^2$ and $1/\tilde{\tau}$ terms as well as a constant piece from the background fields, and we provide evidence that the remaining dynamical piece is the partition function of $\mathcal{N} = 2$ SYM with a one-loop induced CS coupling on $S^3/\mathbb{Z}_m$.

Notation. We have $\sigma, \tau \in \mathbb{H}$ and $z, u \in \mathbb{C}$, and we set $p = \mathrm{e}^{2\pi i \sigma}$, $q = \mathrm{e}^{2\pi i \tau}$, $\zeta = \mathrm{e}^{2\pi i z}$.

We use $\simeq$ to mean an all-order asymptotic equality of the logarithms of the two sides.

2 The 4d superconformal index and its asymptotic expansion

We consider a four-dimensional $\mathcal{N} = 1$ gauge theory which flows to a superconformal fixed point. The theory has gauge group G (which we take to be semi-simple, and separately comment on the U(N) case), and a number of chiral multiplets labelled by I with R-charge r_I and in the representation $\mathcal{R}_I$ of the gauge group. We assume $0 < r_I < 2$ for all chiral multiplets. The superconformal index for these theories on $S^3 \times S^1$ has been calculated in the Hamiltonian as well as functional integral formalism [3, 32–34], and the answer is expressed as an integral over the Cartan torus which we parameterize by the vector of gauge holonomies $\underline{u} = (u_1, \ldots, u_{\mathrm{rk}(G)})$, with $u_i \in \mathbb{R}/\mathbb{Z}$. The index is given by the following expression [58–60]

$$\mathcal{I}(\sigma, \tau) = \int [D\underline{u}] \, \mathcal{Z}_{\mathrm{vec}}(\underline{u}; \sigma, \tau) \, \mathcal{Z}_{\mathrm{chi}}(\underline{u}; \sigma, \tau) \, . \tag{2.1}$$

Here we have used the measure $D\underline{u} = \frac{1}{|\mathcal{W}|} \prod_{i=1}^{\mathrm{rk}(G)} \mathrm{d}u_i$ with $|\mathcal{W}|$ the order of the Weyl group of G. For U(N) we have $D\underline{u} = \frac{1}{N!} \prod_{i=1}^{N} \mathrm{d}u_i$, while for SU($N$) one can work with $u_1, \ldots, u_N$ subject to $\sum_i u_i \in \mathbb{Z}$ and $D\underline{u} = \frac{1}{N!} \prod_{i=1}^{N-1} \mathrm{d}u_i$. The factors $\mathcal{Z}_{\mathrm{vec}}$, $\mathcal{Z}_{\mathrm{chi}}$ denote the vector multiplet and chiral multiplet contribution respectively given by

$$\mathcal{Z}_{\mathrm{vec}}(\underline{u}; \sigma, \tau) = (p; p)^{\mathrm{rk}(G)} (q; q)^{\mathrm{rk}(G)} \prod_{\alpha_+} \Gamma_{\mathrm{e}}(\alpha_+ \cdot u + \sigma + \tau; \sigma, \tau) \, \Gamma_{\mathrm{e}}(-\alpha_+ \cdot u + \sigma + \tau; \sigma, \tau) \, ,$$

$$\mathcal{Z}_{\mathrm{chi}}(\underline{u}; \sigma, \tau) = \prod_I \prod_{\rho \in \mathcal{R}_I} \Gamma_{\mathrm{e}}\left(\rho \cdot u + \frac{r_I}{2}(\sigma + \tau); \sigma, \tau\right) \, . \tag{2.2}$$

Here α_+ runs over the set of positive roots of the gauge group G, I runs over all the chiral multiplets of the theory, and ρ is the weight of the gauge representation $\mathcal{R}_I$. The *Pochhammer symbol* is defined by

$$(\zeta; q) = \prod_{k=0}^{\infty} \left(1 - \zeta q^k\right) \, , \tag{2.3}$$

and the *elliptic gamma function* [44, 61] is defined by the infinite product formula

$$\Gamma_{\mathrm{e}}(z; \sigma, \tau) = \prod_{j,k \geq 0} \frac{1 - p^{j+1} q^{k+1} \zeta^{-1}}{1 - p^j q^k \zeta} \, . \tag{2.4}$$

From now on in this section we set $\sigma = \tau - n_0$, and use the notation $\Gamma_e(z) = \Gamma_e(z; \tau, \tau)$. We have

$$\mathcal{I}(\tau; n_0) = \int [D\underline{u}] \exp(-S_{\text{ind}}(\underline{u}; \tau)), \tag{2.5}$$

where the action $S_{\text{ind}}(\underline{u}) = S_{\text{ind}}(\underline{u}; \tau)$ is given by

$$-S_{\text{ind}}(\underline{u}) = 2\operatorname{rk}(G)\log(q; q) + \sum_{\alpha_+} \log\left(\Gamma_e(\alpha_+ \cdot u + 2\tau)\Gamma_e(-\alpha_+ \cdot u + 2\tau)\right)$$
$$+ \sum_I \sum_{\rho \in R_I} \log\Gamma_e\left(\rho \cdot u + r_I(\tau - \tfrac{1}{2}n_0)\right). \tag{2.6}$$

Our goal now is to calculate the asymptotics of the function $\mathcal{I}(\tau, n_0)$ as τ approaches a rational number or, equivalently, $q = e^{2\pi i\tau}$ approaches a root of unity. For $\mathcal{N} = 4$ SYM we have

$$-S_{\text{ind}}^{\mathcal{N}=4}(\underline{u}) = 2N\log(q; q) + 3N\log\Gamma_e\left(\tfrac{1}{3}(2\tau - n_0)\right)$$
$$+ \sum_{i \neq j} \log\Gamma_e(u_{ij} + 2\tau) + 3\sum_{i \neq j} \log\Gamma_e\left(u_{ij} + \tfrac{1}{3}(2\tau - n_0)\right) \tag{2.7}$$

for $U(N)$ and a similar expression for $SU(N)$ with N replaced by $N-1$. Using the product expression (2.4) we see that for $\mathcal{N} = 4$ SYM the index $\mathcal{I}_N(\tau; n_0)$ has the symmetry $\tau \mapsto \tau + 3$ for fixed n_0, so that we can restrict our attention to, say, $\tau \in [0, 3]$. Relatedly, the independent values of n_0 are $0, \pm 1$. More generally, the periodicity of τ depends on the quantization of R-charge in the theory.

Before analyzing these asymptotic limits we briefly discuss a slightly independent motivation to study these new limits and, relatedly, the origin of the number n_0 in (2.5), (2.7). One of the motivations in the recent developments in this subject has been to "find the dual black hole" in the superconformal index. In terms of the microcanonical Fourier coefficients

$$\mathcal{I}_N(\tau; n_0) = \sum_n d_N(n; n_0) e^{2\pi in\tau}, \tag{2.8}$$

the problem in the context of the Cardy-like limit is to check if $|d_N(n; n_0)| \sim N^2 s(n/N^2)$ as $n \to \infty$ [20]. In the canonical setting this is reflected by a corresponding asymptotic growth of the function $\mathcal{I}(\tau)$ as τ approaches the real axis or, equivalently, as $q = e^{2\pi i\tau}$ approaches the unit circle. The leading asymptotics of the growth of microcanonical degeneracies is governed by the dominant singularity of $\mathcal{I}$. As it turns out, the index $\mathcal{I}_N(\tau; 0)$ of $\mathcal{N} = 4$ SYM does not have any exponential growth as $\tau \to 0$ (the growth is power-law [49]). It is the asymptotic growth of $\log\mathcal{I}_N(\tau; 0)$ as $\tau \to \pm 1$ instead that matches the on-shell action of the AdS$_5$ BH (the two points giving growth of equal magnitude and opposite phases). From a numerical study of the microcanonical degeneracies one can deduce that this is, in fact, the leading growth of the index [20]. In this case, noting that $\mathcal{I}_N(\tau \pm 1; n_0) = \mathcal{I}_N(\tau; n_0 \mp 1)$, we see that the leading growth can be stated as coming from the growth of the function $\mathcal{I}_N(\tau, \mp 1)$ as $\tau \to 0$. Actually, one finds that the growth of states at $n_0 = \pm 1$ matches the BH growth of states for very large classes of $\mathcal{N} = 1$ SCFTs [11, 12]. Once we understand that the growth can come from a region

with $\mathrm{Im}\,\tau \to 0$ but $\mathrm{Re}\,\tau \neq 0$, it is perhaps more natural to set $n_0 = 0$ (for which the two regions of leading growth have a symmetric placement around $\tau = 0$). We will, nevertheless, keep n_0 as an explicit parameter in the following to make contact with related literature.

It is clear from the above discussion that one should equally well explore other points on the unit circle in q.[8] As it turns out there is exponential growth near any root of unity consistent with (1.4), (1.5), i.e. partial deconfinement in the sense of asymptotic growth [37]. In the following subsections we proceed to analyze the asymptotic behavior of the index as $\tau \to 0$ and then as τ approaches any rational number.

2.1 Asymptotics of the index as $\tau \to 0$

In this subsection we perform an all-order asymptotic analysis of the integral (2.5) as $\tau \to 0$. This calculation was done for $\mathcal{N} = 4$ SYM recently in [41] using a saddle-point analysis. Here we find the asymptotics for the general class of theories discussed in the introduction, using the rigorous method of [46, 49] (see in particular section 3.1 of [49]). The application in [49] was restricted to real τ and $n_0 = 0$, but the method is more general and we apply it to the case of complex τ and general n_0.

We first calculate the all-order asymptotic expansion of the integrand. In order to do this we need the asymptotic behavior of the elements in (2.6), namely the Pocchammer symbol and the elliptic gamma function, which we review in equations (A.1), (A.2), (A.13). Using these estimates we find that in the range $\alpha_+ \cdot \underline{u} \in (-1+\delta, 1-\delta)$ (for fixed small δ) the integrand of (2.5) can be written, up to exponentially suppressed corrections, as

$$\exp\left(-S_{\mathrm{ind}}\left(\underline{u};\tau\right)\right) \simeq \frac{1}{(-\mathrm{i}\,\tau)^{\mathrm{rk}(G)}} \prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+ \cdot \underline{u}}{-\mathrm{i}\,\tau}\right) \exp\left(-2\pi\mathrm{i}\tau E_{\mathrm{susy}} - V\left(\underline{u}\right)\right). \quad (2.9)$$

The all-order effective potential as $\tau \to 0$ is given by

$$V(\underline{u}) = \frac{1}{\tau^2}V_2(\underline{u}) + \frac{1}{\tau}V_1(\underline{u}) + V_0(\underline{u}), \quad (2.10)$$

with

$$\begin{aligned}
V_2\left(\underline{u}\right) &= \sum_{I,\rho_I} \frac{\mathrm{i}\pi}{3}\,\overline{B}_3\left(\rho_I \cdot \underline{u} - \tfrac{1}{2}r_I n_0\right), \\
V_1\left(\underline{u}\right) &= \sum_{I,\rho_I} \mathrm{i}\pi\left(r_I - 1\right)\overline{B}_2\left(\rho_I \cdot \underline{u} - \tfrac{1}{2}r_I n_0\right) + \sum_{\alpha} \mathrm{i}\pi\left(\left(\alpha \cdot \underline{u}\right)^2 + \frac{1}{6}\right), \\
V_0\left(\underline{u}\right) &= \sum_{I,\rho_I} \mathrm{i}\pi\left(\left(r_I - 1\right)^2 - \frac{1}{6}\right)\overline{B}_1\left(\rho_I \cdot \underline{u} - \tfrac{1}{2}r_I n_0\right),
\end{aligned} \quad (2.11)$$

where α runs over all the roots of G including the $\mathrm{rk}(G)$ zero roots, I runs over all the chiral multiplets, and ρ_I runs over all the weights of the representation $\mathcal{R}_I$. Note that in (2.9) we have separated the supersymmetric Casimir energy given by [62]

$$E_{\mathrm{susy}} = \frac{1}{6}\,\mathrm{Tr}R^3 - \frac{1}{12}\,\mathrm{Tr}R. \quad (2.12)$$

---

[8]The superconformal index as a function of q is defined on a branched cover of the complex plane and one should explore the full covering space. For $\mathcal{N} = 4$ SYM one has a three-sheeted cover and the leading growth occurs on two of the three sheets [11, 12, 37].

We make a brief comparison to [12] in which the singular pieces were studied. The potential V_2 in (2.11) coincides (up a multiplicative $-\mathrm{i}\pi/6$ factor) with the V_2 studied in [12]. At finite $\underline{u}$, the $\sinh^2(\frac{\pi\alpha_+\cdot\underline{u}}{-\mathrm{i}\tau})$ factors in (2.9) also contribute to $\mathrm{O}(1/\tau)$. Including this piece in V_1 renormalizes it to V_1^r as[9]

$$V_1\left(\underline{u}\right) \to V_1^r\left(\underline{u}\right) = \sum_{I,\rho_I} \mathrm{i}\pi\left(r_I - 1\right)\overline{B}_2\left(\rho_I\cdot\underline{u} - \tfrac{1}{2}r_I n_0\right) + \sum_\alpha \mathrm{i}\pi\,\overline{B}_2\left(\alpha\cdot\underline{u}\right). \qquad (2.13)$$

The potential V_1^r coincides (up to a multiplicative $-\mathrm{i}\pi$ factor) with the V_1 studied in [12]. In our treatment below we keep the $\sinh^2$ factors separate and place them in the "dynamical measure"

$$\frac{D\underline{u}}{(-\mathrm{i}\tau)^{\mathrm{rk}(G)}}\prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+\cdot\underline{u}}{-\mathrm{i}\tau}\right). \qquad (2.14)$$

Compared to [12], here we also include the $\mathrm{O}(\tau^0)$ piece corresponding to $\exp(-V_0)$. Finally, the $\mathrm{O}(\tau)$ piece of the exponent is determined by the supersymmetric Casimir energy and, notably, there are no $\mathrm{O}(\tau^2)$ or higher corrections in the perturbative effective potential.

We now investigate the local behavior of the potential near $\underline{u} = 0$. The potentials V_2, V_1, V_0 are piecewise polynomials, and using $\overline{B}_j' = j\overline{B}_{j-1}$ we obtain their Taylor expansion near $\underline{u} = 0$ as

$$\begin{aligned}
V_2\left(\underline{u}\right) =& \sum_{I,\rho_I}\left(\frac{\mathrm{i}\pi}{3}\overline{B}_3\left(-\tfrac{1}{2}r_I\,n_0\right) + \mathrm{i}\pi\,\overline{B}_2\left(-\tfrac{1}{2}r_I\,n_0\right)\rho_I\cdot\underline{u} + 2\pi\mathrm{i}\,\overline{B}_1\left(-\tfrac{1}{2}r_I\,n_0\right)\frac{(\rho_I\cdot\underline{u})^2}{2}\right) \\
&+ \sum_{I,\rho_I} 2\pi\mathrm{i}\,\frac{(\rho_I\cdot\underline{u})^3}{3!}\,, \\
V_1\left(\underline{u}\right) =& \sum_{I,\rho_I}\mathrm{i}\pi\left(r_I - 1\right)\overline{B}_2\left(-\tfrac{1}{2}r_I\,n_0\right) + \frac{\mathrm{i}\pi}{6}\dim G + \sum_{\rho_I} 2\pi\mathrm{i}\left(r_I - 1\right)\overline{B}_1\left(-\tfrac{1}{2}r_I\,n_0\right)\rho_I\cdot\underline{u} \\
&+ \sum_{I,\rho_I} 2\pi\mathrm{i}\left(r_I - 1\right)\frac{(\rho_I\cdot\underline{u})^2}{2} + \sum_\alpha \mathrm{i}\pi\left(\alpha\cdot\underline{u}\right)^2, \\
V_0\left(\underline{u}\right) =& \sum_{I,\rho_I}\mathrm{i}\pi\left(\left(r_I - 1\right)^2 - \frac{1}{6}\right)\overline{B}_1\left(-\tfrac{1}{2}r_I\,n_0\right) \\
&+ \sum_{I,\rho_I}\mathrm{i}\pi\left(\left(r_I - 1\right)^2 - \frac{1}{6}\right)\rho_I\cdot\underline{u}.
\end{aligned}$$
$$(2.15)$$

Importantly, the second lines of V_2, V_1, V_0 above vanish due to gauge[3], $\mathrm{U}(1)_R$-gauge[2], and $\mathrm{U}(1)_R^2$-gauge and gravity[2]-gauge anomaly cancellations. Therefore V_2 is actually piecewise quadratic in $\underline{u}$, while V_1 is piecewise linear and V_0 is piecewise constant. This is similar to section 3.1 of [49].

The leading asymptotic behavior of V as $\tau \to 0$ is determined by V_2. In order to obtain a local minimum of $\mathrm{Re}(V)$ at $\underline{u} = 0$, we want (i) the linear term in V_2 to vanish,

---

[9]Instead of using (A.13) for vector multiplet gammas and then simplifying the sinh term for finite $\underline{u}$ as above, we could alternatively use (A.2) for vector multiplet gammas (assuming $\alpha_+\cdot\underline{u}\notin\mathbb{Z}$) and get V_1^r directly as in [12].

and (ii) the quadratic term to be on the negative (respectively positive) imaginary axis for $\arg(\tau) - \frac{\pi}{2} > 0$ (respectively $\arg(\tau) - \frac{\pi}{2} < 0$). As found in [12], we can achieve both of these requirements in any theory in which $0 < r_I < 2$ by specializing to $n_0 = \pm 1$. Explicitly, for $n_0 = \pm 1$ we can use the fact that for $|x| < 1$ we have $\overline{B}_2(x) = x^2 - |x| + \frac{1}{6}$ to deduce that the linear term in V_2 is equal to

$$\sum_{I, \rho_I} \frac{\mathrm{i}\pi}{4} \left((r_I - 1)^2 - \frac{1}{3} \right) \rho_I \cdot \underline{u}, \tag{2.16}$$

which vanishes thanks to the $\mathrm{U}(1)_R^2$-gauge and gravity^2-gauge anomaly cancellations. Similarly we can use the fact that for $0 < |x| < 1$ we have $\overline{B}_1(x) = x - \frac{\mathrm{sign}(x)}{2}$ to deduce that the quadratic term in V_2 is equal to

$$-\mathrm{i}\pi n_0 \sum_{I, \rho_I} (r_I - 1) \frac{(\rho_I \cdot \underline{u})^2}{2} = \mathrm{i}\pi n_0 \sum_{\alpha} \frac{(\alpha \cdot \underline{u})^2}{2}, \tag{2.17}$$

where the equality follows from $\mathrm{U}(1)_R$-gauge^2 anomaly cancellation, and we have used that $\mathrm{sign}(n_0) = n_0$ for $n_0 = \pm 1$. This quadratic piece is on the positive (respectively negative) imaginary axis for $n_0 = +1$ (respectively $n_0 = -1$). In this manner we see that $\underline{u} = 0$ is a local minimum of $\mathrm{Re}(V)$. Therefore in the rest of this subsection we focus on $n_0 = \pm 1$, and take $\arg(\tau) - \frac{\pi}{2}$ to have the opposite sign to n_0, i.e. $n_0 \left(\arg(\tau) - \frac{\pi}{2} \right) < 0$.

Using the explicit expressions of $\overline{B}_{1,2,3}(x)$ in the range $0 < |x| < 1$, and using the anomaly cancellation conditions, the potentials V_2, V_1, V_0 simplify, for $n_0 = \pm 1$, to

$$\begin{aligned}
V_2 \left(\underline{u} \right) &= -\frac{\mathrm{i}\pi n_0}{24} \left(\mathrm{Tr} R^3 - \mathrm{Tr} R \right) + \mathrm{i}\pi n_0 \sum_{\alpha} \frac{(\alpha \cdot \underline{u})^2}{2}, \\
V_1 \left(\underline{u} \right) &= \frac{\mathrm{i}\pi}{12} \left(3\mathrm{Tr} R^3 - \mathrm{Tr} R \right), \\
V_0 \left(\underline{u} \right) &= \sum_{I, \rho_I} \mathrm{i}\pi \left(\frac{r_I - 1}{6} - (r_I - 1)^3 \right) \frac{n_0}{2}.
\end{aligned} \tag{2.18}$$

Note that, as a bonus, V_1 also becomes independent of $\underline{u}$ for $n_0 = \pm 1$ and small enough $\underline{u}$.[10]

We now consider a small neighborhood $\mathfrak{h}_{cl}^{\epsilon}$ around $\underline{u} = 0$, defined by the cutoff $|u_j| < \epsilon$, whose contribution to the index is

$$\mathcal{I} \left(\tau; n_0 \right)_{|u_j| < \epsilon} \simeq \mathrm{e}^{-2\pi \mathrm{i}\tau E_{\mathrm{susy}}} \int_{\mathfrak{h}_{cl}^{\epsilon}} \frac{D\underline{u}}{(-\mathrm{i}\tau)^{\mathrm{rk}(G)}} \prod_{\alpha_+} 4 \sinh^2 \left(\frac{\pi \alpha_+ \cdot \underline{u}}{-\mathrm{i}\tau} \right) \exp \left(-V \left(\underline{u} \right) \right). \tag{2.19}$$

From the above discussion we have that

$$\mathcal{I} \left(\tau; n_0 = \pm 1 \right)_{|u_j| < \epsilon} \simeq \mathrm{e}^{-2\pi \mathrm{i}\tau E_{\mathrm{susy}}} Z^{\mathrm{bgnd}} \left(\tau; n_0 \right) \, Z_{\epsilon}^{\mathrm{dyn}} \left(\tau; n_0 \right), \tag{2.20}$$

---

[10]We will shortly interpret the quadratic term in V_2 as inducing a Chern-Simons type coupling in the integrand. If the linear term in V_1 were present, it would similarly induce an FI parameter in the integrand. While this is impossible for semi-simple gauge theories near $\underline{u} = 0$ which we are focussing on in this section, there are cases of semi-simple gauge theories in which one must expand around $\underline{u} \neq 0$ and as a result one finds the measure of an abelian gauge theory in the integrand, where such induced FI parameters do arise. See section 3.3.1 of [49] where the ISS model displaying an $\mathrm{SU}(2) \to \mathrm{U}(1)$ breaking pattern with an induced FI parameter in the $\tau \to 0$ limit is discussed, and see appendix A of [63] where that induced FI parameter is given an effective field theory explanation.

where the *background piece* is

$$Z^{\text{bgnd}}(\tau; n_0)$$
$$= \exp\left(\frac{\mathrm{i}\pi\, n_0}{24\tau^2}\left(\mathrm{Tr}R^3 - \mathrm{Tr}R\right) - \frac{\mathrm{i}\pi}{12\tau}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right) + \sum_{\rho_I}\frac{\mathrm{i}\pi\, n_0}{2}\left((r_I-1)^3 - \tfrac{1}{6}(r_I-1)\right)\right)$$
(2.21)

and the *dynamical piece* is

$$Z_\epsilon^{\text{dyn}}(\tau; n_0) = \int_{\mathfrak{h}_{cl}^\epsilon} \frac{D\underline{u}}{(-\mathrm{i}\,\tau)^{\mathrm{rk}(G)}} \prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+ \cdot \underline{u}}{-\mathrm{i}\,\tau}\right) \exp\left(\frac{\mathrm{i}\pi n_0}{2}\sum_\alpha\left(\frac{\alpha \cdot \underline{u}}{-\mathrm{i}\,\tau}\right)^2\right). \quad (2.22)$$

Here we suppress the dependence of these functions on the gauge group and the matter content.

To simplify Z_ϵ^{dyn} further, we first define $x_j = \frac{u_j}{-\mathrm{i}\tau}$, so that the integral becomes along straight contours from $x_j = -\frac{\epsilon}{-\mathrm{i}\tau}$ to $x_j = +\frac{\epsilon}{-\mathrm{i}\tau}$. With our choice of n_0 and $\arg(\tau)$, the integrand is locally exponentially suppressed away from $\underline{u} = 0$, so we can complete the tails of the contours along straight lines to infinity (i.e. send $\epsilon \to +\infty$) introducing only exponentially small error. The contours make an angle $\frac{\pi}{2} - \arg(\tau)$ with the positive real axis. However, observing that (i) the integrand is exponentially suppressed as $|x_j| \to \infty$, and (ii) there are no poles between the contour of x_j and the real axis, we can deform the contours back to the real axis. We thus obtain, with $\underline{x} = (x_1, \ldots, x_n)$

$$Z_\epsilon^{\text{dyn}}(\tau; n_0) \simeq \int_{-\infty}^\infty D\underline{x} \prod_{\alpha_+} 4\sinh^2(\pi\alpha_+ \cdot \underline{x}) \exp\left(\frac{\mathrm{i}\pi n_0}{2}\sum_\alpha (\alpha \cdot \underline{x})^2\right) =: Z^{\text{dyn}}(n_0). \quad (2.23)$$

As noted in [41] for $\mathcal{N} = 4$ SYM, and in [42, 43] for more general groups, Z^{dyn} is related to the partition function of pure Chern-Simons theory [64] on S^3 as

$$Z^{\text{dyn}}(n_0) = (-1)^{(\dim G - \mathrm{rk}G)/2}\, Z^{\text{CS}}(k_{ij}), \quad (2.24)$$

with the gauge group implicit and the same on both sides, and with Chern-Simons coupling given by

$$k_{ij} = -\frac{n_0}{2}\sum_\alpha \alpha_i\, \alpha_j. \quad (2.25)$$

Notice that $Z^{\text{dyn}}(n_0)$ is independent of τ. The tails completion (i.e. sending $\epsilon \to \infty$) and contour deformation mentioned above removed the τ-dependence of Z_ϵ^{dyn} at the cost of an exponentially small error.

The considerations of three-dimensional effective field theory in the next section show that Z^{dyn} arises naturally in fact as the partition function of three-dimensional $\mathcal{N} = 2$ gauge theory on S^3 with the same gauge group and the same CS coupling. (The latter is well-known to be related to Z^{CS} exactly as in (2.24).)

The above analysis was local around $\underline{u} = 0$. We now focus on SU(N) $\mathcal{N} = 4$ SYM for which we know that $\underline{u} = 0$ is a global minimum of the leading potential V_2/τ^2 for $n_0 = \pm 1$

and $n_0(\arg(\tau) - \frac{\pi}{2}) < 0.$[11] However, this is not the only global minimum — there are N isolated global mimima labelled by $k = 1, \ldots, N$ which are related to $u_j = 0$ by center symmetry, namely the points $u_j = (k-1)/N$, $k = 1, \ldots, N$ [10]. Upon summing over these minima we obtain

$$\mathcal{I}_N(\tau; n_0) \simeq N e^{-2\pi i \tau E_{\mathrm{susy}}} Z^{\mathrm{bgnd}}(\tau; n_0) Z^{\mathrm{dyn}}(n_0). \tag{2.26}$$

The factor of N arises from the sum over N minima as explained above. The other three factors can be calculated by specializing our general discussion to this case:

$$E_{\mathrm{susy}} = \frac{4}{27}\left(N^2 - 1\right), \tag{2.27}$$

$$Z^{\mathrm{bgnd}}(\tau; n_0) = \exp\left(-\frac{i\pi}{\tau^2}\left(N^2 - 1\right)\left(\left(\frac{-n_0 + 2\tau}{3}\right)^3 + \frac{5n_0\tau^2}{12}\right) + 2\pi i \tau \cdot \frac{4}{27}\left(N^2 - 1\right)\right), \tag{2.28}$$

$$Z^{\mathrm{dyn}}(n_0) = \int_{-\infty}^{\infty} D\underline{x} \prod_{i<j} 4\sinh^2(\pi x_{ij}) \exp\left(i\pi n_0 N \sum_{j=1}^N x_j^2\right). \tag{2.29}$$

In this case the matrix of Chern-Simons couplings reduces to a single level ($k_{ij} = k\delta_{ij}$), and we have

$$Z^{\mathrm{dyn}}(n_0) = (-1)^{N(N-1)/2} Z^{\mathrm{CS}}(k), \qquad k = -n_0 N. \tag{2.30}$$

For $n_0 = \pm 1$ the SU(N) Chern-Simons partition was found in [41] to simplify to

$$Z^{\mathrm{CS}}(-n_0 N) = (-1)^{N(N-1)/2} \exp\left(5 i\pi n_0 (N^2 - 1)/12\right). \tag{2.31}$$

Upon combining this equation with (2.30) and (2.26), we obtain[12]

$$\mathcal{I}_N(\tau; n_0) \simeq N \exp\left(-\frac{i\pi}{\tau^2}(N^2 - 1)\left(\frac{-n_0 + 2\tau}{3}\right)^3\right). \tag{2.32}$$

The analogous result for the case with $U(N)$ gauge group is obtained by adding the contribution of a decoupled $U(1)$ $\mathcal{N} = 4$ multiplet to that of the SU(N) theory:

$$\mathcal{I}^{\mathrm{U}(N)}(\tau; n_0) \simeq N \frac{1}{-i\tau} \exp\left(-\frac{i\pi}{\tau^2} N^2 \left(\frac{-n_0 + 2\tau}{3}\right)^3 - \frac{5\pi i n_0}{12}\right). \tag{2.33}$$

This finishes the discussion of our methods illustrated in the special case $\tau \to 0$. Before moving on to the more general case of rational points, we make a few technical remarks.

Firstly, since we are analyzing the index by estimating its integrand, we need uniform estimates. For $n_0 = \pm 1$, the estimate (A.2) when applied to the chiral multiplet gamma functions gives uniformly valid asymptotics near $\underline{u} = 0$, because the $-n_0 r_I/2$ shift in the argument takes us safely into the domain of validity of (A.2). For the vector multiplet

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[11]This was shown in [11, 12] where experimental evidence that this is true for a large class of theories was also discussed.

[12]For comparison with [16], we set $\xi_a^{\mathrm{there}} = -n_0^{\mathrm{here}}/3$. The result in that paper contains the number $\eta \in \{-1, +1\}$. This is related to our n_0 as $\eta = 6\overline{B}_1(-\frac{n_0}{3})$. For $n_0 = \pm 1$, a simple calculation shows that $\eta = n_0$.

gamma functions, however, there is no finite shift in the argument, so (A.2) does not apply uniformly around $\underline{u} = 0$. We had to use instead (A.13) to obtain uniform asymptotics near $\underline{u} = 0$ for the vector multiplet gammas.

Secondly, we emphasize that our asymptotic analysis is essentially real-analytic (as in [46, 49]). We only appeal to complex-analytic tools (specifically, contour deformation), after having done the asymptotic analysis, to simplify the final answer for $Z_\epsilon^{\mathrm{dyn}}$ in (2.22) to the more familiar form (2.23).

Thirdly, we note that when actually doing the saddle-point analysis, one finds that the dominant holonomy configurations spread into the complex plane, as in the analysis of [7, 13, 41]. Upon taking the $\tau \to 0$ limit the spreading shrinks, and the answers from those approaches indeed agree with our results.

2.2 Asymptotics of the index as $\tau \to \mathbb{Q}$

We now study the index (2.5) in the limit

$$\widetilde{\tau} \equiv m\tau + n \to 0 \,, \tag{2.34}$$

with m, n relatively prime, keeping $\arg(\widetilde{\tau})$ away from integer multiples of $\pi/2$.

As in the previous subsection we first calculate the all-order asymptotics of the integrand. The required small-$\widetilde{\tau}$ estimates for the Pochhammer symbol and the elliptic gamma function are given in equations (A.17), (A.23), (A.26). Using these estimates we find that in the range $\alpha_+ \cdot \underline{u} \in (-\frac{1}{m} + \delta, \frac{1}{m} - \delta)$, for some fixed small δ, the integrand of (2.5) can be written up to exponentially suppressed corrections as

$$\exp\left(-S_{\mathrm{ind}}\left(\underline{u};\tau\right)\right) \simeq \frac{1}{(-\mathrm{i}\widetilde{\tau})^{\mathrm{rk}(G)}} \prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+ \cdot \underline{u}}{-\mathrm{i}\widetilde{\tau}}\right) \exp\left(-2\pi\mathrm{i}\widetilde{\tau}\frac{E_{\mathrm{susy}}}{m} - \frac{\widetilde{V}\left(\underline{u}\right)}{m}\right). \tag{2.35}$$

The all-order effective potential as $\widetilde{\tau} \to 0$ is given by

$$\widetilde{V}(\underline{u}) = \frac{1}{\widetilde{\tau}^2}\widetilde{V}_2(\underline{u}) + \frac{1}{\widetilde{\tau}}\widetilde{V}_1(\underline{u}) + \widetilde{V}_0(\underline{u}) \,, \tag{2.36}$$

with

$$\begin{aligned}
\widetilde{V}_2\left(\underline{u}\right) &= \sum_{I, \rho_I} \frac{\mathrm{i}\pi}{3}\,\overline{B}_3\left(m\rho_I \cdot \underline{u} + m\xi_I\right) \,, \\
\widetilde{V}_1\left(\underline{u}\right) &= \sum_{I, \rho_I} \mathrm{i}\pi\left(r_I - 1\right)\overline{B}_2\left(m\rho_I \cdot \underline{u} + m\xi_I\right) + \sum_\alpha \mathrm{i}\pi\left((m\alpha \cdot \underline{u})^2 + \frac{1}{6}\right) \,, \\
\widetilde{V}_0\left(\underline{u}\right) &= -2\pi\mathrm{i} \cdot \dim\left(G\right)s\left(n, m\right) + \sum_{I, \rho_I} 2\pi\mathrm{i}\,C\left(m, n, \rho_I \cdot \underline{u} - \frac{n_0}{2}r_I, r_I\right) \,,
\end{aligned} \tag{2.37}$$

where α runs over all the roots of G including the $\mathrm{rk}(G)$ zero roots, I runs over all the chiral multiplets, and ρ_I runs over all the weights of the representation $\mathcal{R}_I$. Here $s(n,m)$ is the Dedekind sum defined in (A.18) and the function $C(m,n,r,z)$ is defined in (A.24). We have defined

$$\xi_I := -\frac{r_I}{2}\left(n_0 + \frac{2n}{m}\right), \tag{2.38}$$

– 14 –

to emphasize an analogy with the analysis in [10, 37] of the index with flavor chemical potential ξ, although we do not have flavor fugacities in our problem. Note also that we have separated the supersymmetric Casimir energy in (2.35) as in the $\tau \to 0$ case.

Next, as in the previous subsection we expand the potentials near $\underline{u} = 0$. Anomaly cancellations again lead to simplifications, but here we further assume the theory is non-chiral (i.e. that ρ_I come in pairs of opposite sign) so that the answer takes a particularly simple form. Analogously to (2.18) we obtain

$$
\begin{aligned}
\widetilde{V}_2\left(\underline{u}\right) &= \sum_{I,\rho_I} \left(\frac{\mathrm{i}\pi}{3}\,\overline{B}_3\left(m\xi_I\right) + 2\pi\mathrm{i}\,\overline{B}_1\left(m\xi_I\right)\frac{\left(m\rho_I\cdot\underline{u}\right)^2}{2} \right), \\
\widetilde{V}_1\left(\underline{u}\right) &= \sum_{I,\rho_I} \mathrm{i}\pi\left(r_I-1\right)\overline{B}_2\left(m\xi_I\right) + \frac{\mathrm{i}\pi}{6}\dim\left(G\right), \\
\widetilde{V}_0\left(\underline{u}\right) &= \widetilde{V}_0\left(0\right) = -2\pi\mathrm{i}\dim\left(G\right)s\left(n,m\right) + \sum_{I,\rho_I} 2\pi\mathrm{i}\,C\left(m,n,-\frac{n_0}{2}r_I,r_I\right).
\end{aligned}
\tag{2.39}
$$

Next we focus on a small neighborhood $\mathfrak{h}_{cl}^\epsilon$ around $\underline{u} = 0$, defined by the cutoff $|u_j| < \epsilon$, whose contribution to the index as $\widetilde{\tau} \to 0$ is

$$
\mathcal{I}(\tau;n_0)_{|u_j|<\epsilon} \simeq \mathrm{e}^{-2\pi\mathrm{i}\widetilde{\tau}\frac{E_{\mathrm{susy}}}{m}} \int_{\mathfrak{h}_{cl}^\epsilon} \frac{D\underline{u}}{\left(-\mathrm{i}\widetilde{\tau}\right)^{\mathrm{rk}(G)}} \prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+\cdot\underline{u}}{-\mathrm{i}\widetilde{\tau}}\right) \exp\left(-\frac{\widetilde{V}(\underline{u})}{m}\right).
\tag{2.40}
$$

Upon putting the above discussion together we obtain

$$
\mathcal{I}(\tau;n_0)_{|u_j|<\epsilon} \simeq \mathrm{e}^{-2\pi\mathrm{i}\widetilde{\tau}\frac{E_{\mathrm{susy}}}{m}} Z^{\mathrm{bgnd}}(\tau;m,n,n_0)\,Z_\epsilon^{\mathrm{dyn}}(\tau;m,n,n_0),
\tag{2.41}
$$

where the *background piece* is

$$
\begin{aligned}
&Z^{\mathrm{bgnd}}\left(\tau;m,n,n_0\right) \\
&= \exp\left(-\frac{1}{m\widetilde{\tau}^2}\sum_{I,\rho_I}\frac{\mathrm{i}\pi}{3}\,\overline{B}_3\left(m\xi_I\right) - \frac{1}{m\widetilde{\tau}}\sum_{I,\rho_I}\mathrm{i}\pi\left(r_I-1\right)\overline{B}_2\left(m\xi_I\right) + \frac{\mathrm{i}\pi}{6}\dim\left(G\right) + \widetilde{V}_0\left(0\right) \right),
\end{aligned}
\tag{2.42}
$$

and the *dynamical piece* is

$$
Z_\epsilon^{\mathrm{dyn}}\left(\tau;m,n,n_0\right) = \int_{\mathfrak{h}_{cl}^\epsilon} \frac{D\underline{u}}{\left(-\mathrm{i}\widetilde{\tau}\right)^{\mathrm{rk}(G)}} \prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+\cdot\underline{u}}{-\mathrm{i}\widetilde{\tau}}\right) \exp\left(+\frac{\mathrm{i}\pi}{m}\sum_{I,\rho_I}\overline{B}_1\left(m\xi_I\right)\left(\frac{m\rho_I\cdot\underline{u}}{-\mathrm{i}\tau}\right)^2 \right).
\tag{2.43}
$$

Upon defining the rescaled variable $x_j = \frac{u_j}{-\mathrm{i}\widetilde{\tau}}$, we recognize $Z_\epsilon^{\mathrm{dyn}}(\tau;m,n,n_0)$ as the CS partition function on S^3 with gauge group G and level $k^{ij} = -\frac{1}{m}\sum_{I,\rho_I}\overline{B}_1(m\xi_I)\rho_I^i\rho_I^j$. We will see momentarily that it is more natural to define the rescaled variable as $x_j = \frac{mu_j}{-\mathrm{i}\widetilde{\tau}}$. Upon tails completion and deforming the integration contour we obtain

$$
\begin{aligned}
Z_\epsilon^{\mathrm{dyn}}\left(\tau;m,n,n_0\right) &\simeq m^{-\mathrm{rk}(G)} \int_{-\infty}^{\infty} D\underline{x} \prod_{\alpha_+} 4\sinh^2\left(\frac{\pi\alpha_+\cdot\underline{x}}{m}\right) \exp\left(+\frac{\mathrm{i}\pi}{m}\sum_{I,\rho_I}\overline{B}_1\left(m\xi_I\right)\left(\rho_I\cdot\underline{x}\right)^2 \right) \\
&=: m^{-\mathrm{rk}(G)}\,Z_{\underline{0}}^{\mathrm{dyn}}\left(m,n,n_0\right).
\end{aligned}
\tag{2.44}
$$

Up to the overall $m^{-\mathrm{rk}(G)}$ factor, this coincides [51] with the topologically trivial sector of the $S^3/\mathbb{Z}_m$ partition function of $\mathcal{N} = 2$ SYM with Chern-Simons coupling

$$k^{ij} = -\sum_{I, \rho_I} \overline{B}_1(m\xi_I)\, \rho_I^i\, \rho_I^j \,. \tag{2.45}$$

While the explicit expression for the dominant potential $\widetilde{V}_2$ in (2.37) was derived in a neighborhood $(-\frac{1}{m} + \delta, \frac{1}{m} - \delta)$ of $\underline{u} = 0$, it is actually correct more generally, because it follows from (A.23) which we can use as long as $m\rho_I \cdot \underline{u} + m\xi_I \notin \mathbb{Z}$. Moreover, since ρ_I^j are integers and u_j appears in $\widetilde{V}_2$ in the combination $m\, \rho_I \cdot \underline{u}$, the 1-periodicity of $\widetilde{V}_2$ implies that any holonomy configuration with u_j a multiple of $1/m$ gives the same leading asymptotics as the $\underline{u} = 0$ configuration. In the SU(N) case these non-trivial holonomy configurations correspond to

$$\underline{u} = \frac{1}{m}\, \underline{m} = \left(\frac{m_1}{m}, \ldots, \frac{m_N}{m} \right), \tag{2.46}$$

with $m_j \in \mathbb{Z}/m\mathbb{Z}$, and $\sum_{j=1}^N m_j = 0 \ (\mathrm{mod}\ m)$.

For $n = 1$, we can use the estimate (A.31) for the vector multiplet gamma functions to compute the contribution of the saddles (2.46). The result is similar to (2.40), with the same $\widetilde{V}_{2,1}$ and the same SUSY Casimir piece, but with the dynamical piece modified to (modulo an overall constant factor)

$$
\begin{aligned}
Z_{\epsilon_{\underline{m}}}^{\mathrm{dyn}}(\tau; m, n, n_0) &\simeq \int_{-\infty}^{\infty} D\underline{x}' \prod_{\alpha_+} 4\sinh^2 \left(\frac{\pi\alpha_+ \cdot (\underline{x}' + \mathrm{i}\underline{m})}{m} \right) \exp \left(+ \frac{\mathrm{i}\pi}{m} \sum_{I, \rho_I} \overline{B}_1(m\xi_I) (\rho_I \cdot \underline{x}')^2 \right) \\
&=: Z_{\underline{m}}^{\mathrm{dyn}}(m, n, n_0) \,,
\end{aligned}
\tag{2.47}
$$

where $\epsilon_{\underline{m}}$ indicates that we are considering the contribution from a neighborhood $|u_j - m_j/m| < \epsilon$, and the re-scaled integration variable arises as $x_j' = \frac{m(u_j - m_j/m)}{-\mathrm{i}\widetilde{\tau}}$. This coincides (again up to an overall constant factor) with the topologically non-trivial sector of the partition function of SU(N) Chern-Simons theory with coupling (2.45) on the lens space $L(m, -1)$ [51].

We expect that similarly for general n, including the contribution of the non-trivial saddles (2.46) to the index would complete $Z_0^{\mathrm{dyn}}(m, n, n_0)$ to the full $S^3/\mathbb{Z}_m$ partition function, including all the topologically non-trivial sectors. We motivate this expectation further from an EFT perspective in the next section where we also present the (n-dependent) action of $\mathbb{Z}_m$ on the S^3. The explicit demonstration, which we leave to future work, requires generalizing the estimate (A.31) to arbitrary n, and improving it to include the overall constant.

The above analysis was local in nature: we considered the contribution to the index from only a small neighborhood of $\underline{u} = 0$. We now study the specific case of SU(N) $\mathcal{N} = 4$ SYM for which we present a global picture of the dominant holonomy configurations. Note that in the previous subsection rather than performing the global analysis from scratch we had borrowed the result of [11, 12] that in a certain domain of parameters $(n_0(\arg(\tau) - \pi/2) < 0)$ the $\underline{u} = 0$ configuration is globally dominant (see section 2.3 for the complementary domain).
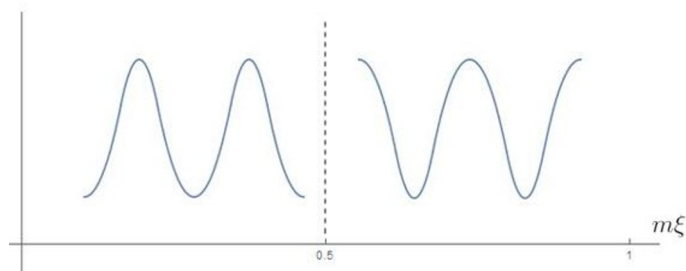
Figure 1. The catastrophic behavior of $V^Q(u_{ij})$, drawn over the range $mu_{ij} \in (-1, 1)$, for $\arg\widetilde{\tau} > \frac{\pi}{2}$. The control parameter $m\xi$ determines the M or W type behavior.

The global structure of the leading potential for $\mathcal{N} = 4$ SYM. For SU(N) $\mathcal{N} = 4$ theory the potential $\widetilde{V}_2$ reads

$$
\widetilde{V}_2\left(\underline{u}; \xi\right) = \frac{\mathrm{i}\pi}{3} \times 3 \left((N-1)\, \overline{B}_3\left(m\xi\right) + \sum_{i<j} \left(\overline{B}_3\left(m\xi + mu_{ij}\right) + \overline{B}_3\left(m\xi + mu_{ji}\right) \right) \right),
$$
(2.48)

where the factor of 3 comes from the sum over three chiral multiplets, and with

$$
\xi = -\tfrac{1}{3}\left(n_0 + \tfrac{2n}{m}\right).
$$
(2.49)

As mentioned below (2.45) the expression (2.48) applies as long as $u_{ij} + \xi_I$ avoid $\frac{\mathbb{Z}}{m}$.

We now have to minimize the real part of $\widetilde{V}_2(\underline{u})/\widetilde{\tau}^2$ as $|\widetilde{\tau}| \to 0$. Since the u_{ij}-independent piece and the real positive overall multiplicative constants are irrelevant in finding the dominant holonomy configurations, our problem boils down to minimizing the potential

$$
V^Q\left(u_{ij}; \arg\widetilde{\tau}, \xi\right) = -\mathrm{sign}\left(\arg\widetilde{\tau} - \frac{\pi}{2}\right) \left(\overline{B}_3\left(m\xi + mu_{ij}\right) + \overline{B}_3\left(m\xi - mu_{ij}\right) \right),
$$
(2.50)

which is analogous to the *pairwise holonomy potential* in [10]. As in that work, we first consider the *qualitative* behavior of V^Q. We assume $\arg(\widetilde{\tau}) - \frac{\pi}{2} > 0$, and comment below on what happens for the opposite sign. We find that the potential is (see figure 1)

$$
\begin{aligned}
&\text{M-shaped for} && 0 < \{m\xi\} < 1/2\,, \\
&\text{W-shaped for} && 1/2 < \{m\xi\} < 1\,.
\end{aligned}
$$

We also see from equation (2.49) that we have $\{m\xi\} \in \{0, \frac{1}{3}, \frac{2}{3}\}$.

The O($1/\widetilde{\tau}^2$) exponent. Let us now assume m, n are chosen such that $\{m\xi\} = \{\frac{-mn_0 - 2n}{3}\} = \frac{1}{3}$, so we are in the M-region with the dominant holonomy configurations corresponding to $\{mu_{ij}\} = 0$. Although this is analogous to the *1-center phase* in [37], as mentioned around (2.46) here in fact u_{ij} can be any integer multiple of $\frac{1}{m}$. All these

saddles contribute equally to the $O(1/\widetilde{\tau}^2)$ exponent though, and hence the preceding analysis around $\underline{u} = 0$ gives the correct leading asymptotics of the index, which up to $O(1/\widetilde{\tau})$ corrections in the exponent reads

$$\exp\left(-\frac{\pi i}{m\,\widetilde{\tau}^2}\left(N^2 - 1\right)\overline{B}_3\left(m\xi\right)\right) = \exp\left(-\frac{i\pi}{27m\,\widetilde{\tau}^2}\left(N^2 - 1\right)\right),$$
$$\text{for} \quad \arg(\widetilde{\tau}) > \frac{\pi}{2}, \qquad \{m\xi\} = \left\{\frac{-mn_0 - 2n}{3}\right\} = \frac{1}{3}. \tag{2.51}$$

For $\arg(\widetilde{\tau}) - \frac{\pi}{2} < 0$, the M- and W-regions switch places. So in order to have $u_{ij} = 0$ as the dominant saddle we must assume m, n are such that $\{m\xi\} = \{\frac{-mn_0 - 2n}{3}\} = \frac{2}{3}$. In this case we have $\overline{B}_3(2/3) = -\overline{B}_3(1/3) = 1/27$, which leads to

$$\exp\left(-\frac{\pi i}{m\,\widetilde{\tau}^2}\left(N^2 - 1\right)\overline{B}_3\left(m\xi\right)\right) = \exp\left(\frac{i\pi}{27m\,\widetilde{\tau}^2}\left(N^2 - 1\right)\right),$$
$$\text{for} \quad \arg\left(\widetilde{\tau}\right) < \frac{\pi}{2}, \qquad \{m\xi\} = \left\{\frac{-mn_0 - 2n}{3}\right\} = \frac{2}{3}. \tag{2.52}$$

In the remaining case where $\{m\xi\} = \{\frac{-mn_0 - 2n}{3}\} = 0$, we have $\widetilde{V}_2(\underline{u}; \xi) = 0$ and hence no $O(\frac{1}{\widetilde{\tau}^2})$ exponent. As we discuss momentarily there is no $O(\frac{1}{\widetilde{\tau}})$ exponent in this case either. There are thus $\text{rk}(G)$ flat directions in the moduli space, leading to a $(1/\widetilde{\tau})^{\text{rk}(G)}$ growth for the index, as in the $n_0 = 0$ and τ pure imaginary case studied in [49].

The $O(1/\widetilde{\tau})$ exponent. The $O(1/\widetilde{\tau})$ exponent comes from $\widetilde{V}_1/m\widetilde{\tau}$. Although the expression for $\widetilde{V}_1$ in (2.39) was obtained near $\underline{u} = 0$, the $O(1/\widetilde{\tau})$ exponent is correctly captured by (A.23), which implies that (2.39) remains correct near the nontrivial saddles with $u_{ij} \in \frac{1}{m}\mathbb{Z}$ as well. So we can specialize $\widetilde{V}_1$ in (2.39) to the SU(N) $\mathcal{N} = 4$ theory and obtain

$$\exp\left(-\frac{\pi i}{m\widetilde{\tau}}\left(N^2 - 1\right)\left(-\overline{B}_2(m\xi) + \tfrac{1}{6}\right)\right). \tag{2.53}$$

In this case we have that $\overline{B}_2(2/3) = +\overline{B}_2(1/3) = -1/18$, which leads to

$$\exp\left(-\frac{2\pi i}{9}\frac{(N^2 - 1)}{m\widetilde{\tau}}\right), \tag{2.54}$$

for $\arg(\widetilde{\tau}) > \frac{\pi}{2}$ as well as $\arg(\widetilde{\tau}) < \frac{\pi}{2}$. Note that since $\overline{B}_2(0) = 1/6$, we see from (2.53) that there is no $O(1/\widetilde{\tau})$ exponent for $\{m\xi\} = 0$, as alluded to above.

The Chern-Simons coupling. Specializing the Chern-Simons coupling (2.45) to SU(N) $\mathcal{N} = 4$ theory we find

$$k_{ij} = -\widetilde{\eta}\,N\,\delta_{ij}, \tag{2.55}$$

with

$$\widetilde{\eta} := 6\overline{B}_1\left(m\xi\right) = 6\overline{B}_1\left(\frac{-mn_0 - 2n}{3}\right). \tag{2.56}$$

We emphasize that all the topologically nontrivial sectors necessary for agreement with an $S^3/\mathbb{Z}_m$ partition function are present in our analysis, but we leave the investigation of their explicit contributions to future work.

The O($\widetilde{\tau}$) exponent. The linear (in $\widetilde{\tau}$) exponent can be read from (2.40) to be $-2\pi i \widetilde{\tau} E_{\text{susy}}/m$. Note again that while (2.40) was derived near $\underline{u} = 0$, as the estimate (A.31) shows the O($\widetilde{\tau}$) exponent remains valid near $\underline{u} \in \frac{\mathbb{Z}}{m}$ as well (at least for $n = 1$, and we expect more generally as well). Since for SU(N) $\mathcal{N} = 4$ theory $E_{\text{susy}} = \frac{4}{27}(N^2 - 1)$, we have the $\mathcal{O}(\widetilde{\tau})$ exponent as in

$$\exp\left(-\frac{8\pi \mathrm{i}}{27m}(N^2 - 1)\widetilde{\tau}\right). \tag{2.57}$$

Summary: the small-$\widetilde{\tau}$ asymptotics for $\mathcal{N} = 4$ SYM. We can summarize the asymptotics of the SU(N) $\mathcal{N} = 4$ SYM index analyzed above as follows

$$\mathcal{I}_N\left(\tau; n_0\right) \simeq N\, \widetilde{C}_N\left(n_0, m, n\right) \exp\left(-\frac{\mathrm{i}\pi}{m\,\widetilde{\tau}^2}\left(N^2 - 1\right)\left(\frac{-\widetilde{\eta} + 2\widetilde{\tau}}{3}\right)^3\right) Z_{S^3/\mathbb{Z}_m}^{\text{CS}}\left(k\right), \tag{2.58}$$

for τ near any rational point $-n/m$, with

$$\widetilde{\tau} = m\tau + n, \qquad \widetilde{\eta} = 6\overline{B}_1\left(\frac{-mn_0 - 2n}{3}\right) = -\text{sign}\left(\arg\left(\widetilde{\tau}\right) - \tfrac{\pi}{2}\right), \qquad k = -\widetilde{\eta}N, \tag{2.59}$$

and with $\widetilde{C}_N(n_0, m, n)$ an overall constant. Note that we have used $\widetilde{\eta}^3 = \widetilde{\eta} = \pm 1$ to simplify the final expression. Also, by completing the cube inside the exponent we have introduced an $\mathcal{O}(\widetilde{\tau}^0)$ factor at the cost of redefining $\widetilde{C}_N(n_0, m, n)$.

We have only demonstrated that there is a contribution to $Z_{S^3/\mathbb{Z}_m}^{\text{CS}}(k)$ from near $\underline{u} = 0$ that coincides with the topologically trivial sector of the $S^3/\mathbb{Z}_m$ partition function of Chern-Simons theory with coupling k. As mentioned below (2.46) we expect that summing over the contributions from neighborhoods of the non-trivial configurations $u_j = m_j/m$ would lead to the complete orbifold partition function.

We can include the contribution of a decoupled $U(1)$ $\mathcal{N} = 4$ multiplet in a straightforward manner. This effectively changes the dimension of the group in the exponent to N^2, introduces a prefactor $1/\widetilde{\tau}$, and change the constant from $\widetilde{C}_N(n_0, m, n)$ to a new constant $\widetilde{C'}_N(n_0, m, n)$, so that we have

$$\mathcal{I}^{\text{U}(N)}(\tau; n_0) \simeq \frac{N}{\mathrm{i}\widetilde{\tau}} \widetilde{C'}_N(n_0, m, n) \exp\left(-\frac{\mathrm{i}\pi}{m\,\widetilde{\tau}^2} N^2\left(\frac{-\widetilde{\eta} + 2\widetilde{\tau}}{3}\right)^3\right) Z_{S^3/\mathbb{Z}_m}^{\text{CS}}(k). \tag{2.60}$$

We see that the background (and the SUSY Casimir) piece in (2.60) matches the effective action (1.5) and, in addition, we have a dynamical Chern-Simons term. In the following section we explain both these pieces from the point of view of 3d $\mathcal{N} = 2$ field theory.

2.3 C-center phases

Focussing on SU(N) $\mathcal{N} = 4$ theory, we now move on to studying the $\widetilde{\tau} \to 0$ limit of the index in the *W region*, which as shown in figure 1 for $\arg\widetilde{\tau} > \pi/2$ corresponds to $1/2 < \{m\xi\} < 1$. As before we assume $\arg\widetilde{\tau}$ is in compact domains avoiding integer multiples of $\pi/2$ as $|\widetilde{\tau}| \to 0$.

Recall from (2.49) that only the values $\{m\xi\} = 0, 1/3, 2/3$ are realized in our problem. But to highlight the parallels with the analysis of partially-deconfined phases in the W

regions of the (flavored) 4d $\mathcal{N} = 4$ index in [37], we will study the phase structure for arbitrary $\{m\xi\} \in (\frac{1}{2}, 1)$ below, and only at the end specialize our result to the single "physical" point $\{m\xi\} = 2/3$ in that interval.

Asymptotic analysis of the index for arbitrary $\{m\xi\} \in (\frac{1}{2}, 1)$ is difficult for general N, because finding the dominant holonomy configurations is not possible analytically in the W regions. Analogously to [37] we consider now the large-N limit (on top of the $\widetilde{\tau} \to 0$ limit), and conjecture that the C-center phases suffice for extremizing the potential in the W region. Also, similarly to [37] we consider only the leading (here $\mathrm{O}(1/\widetilde{\tau}^2)$) exponent of the index in the W region.

A C-center holonomy configuration consists of C packs of N/C holonomies uniformly distributed on the circle such that the $\mathrm{SU}(N)$ constraint is satisfied. While at finite N it is possible to have such configurations only for C a divisor of N, in the large-N limit any integer $C \geq 1$ provides an acceptable C-center configuration [37]. For such a distribution the "on-shell" value of the potential $\widetilde{V}_2$ in (2.48) becomes

$$\widetilde{V}_2^{(C)} = \mathrm{i}\pi\left((N-1)\,\overline{B}_3\,(m\xi) + \frac{N}{d}\frac{d(d-1)}{2}\,2\overline{B}_3\,(m\xi) + d^2\sum_{J=1}^{C-1}J\left(\overline{B}_3\left(m\xi + m\frac{J}{C}\right) + \overline{B}_3\left(m\xi - m\frac{J}{C}\right)\right)\right),$$
(2.61)

where $d := N/C$. The second term above is the contribution from pairs in the same pack, and the third term is from pairs with each end on a different pack. To simplify the above expression further, we use the following identity which can be proven from (A.6) and (A.21):

$$\sum_{J=1}^{C-1}J\left(\overline{B}_3\left(\Delta + m\frac{J}{C}\right) + \overline{B}_3\left(\Delta - m\frac{J}{C}\right)\right) = \frac{g^2\overline{B}_3\,(C'\Delta)}{C'} - C\overline{B}_3\,(\Delta),$$
(2.62)

where $g := \gcd(m, C)$ and $C' := C/g$. Keeping only the $O(N^2)$ terms we hence end up with

$$\widetilde{V}_2^{(C)} = \mathrm{i}\pi N^2\,\frac{\overline{B}_3(C'm\xi)}{C'^3}.$$
(2.63)

Since the leading asymptotics of the index is given as $\exp(-\widetilde{V}_2/\widetilde{\tau}^2)$, we then find the analog of the main result of [37] (equation (3.19) of that work) for our case to be

$$\mathcal{I}_{N\to\infty} \xrightarrow{\widetilde{\tau}\to 0} \sum_{C=1}^{\infty}\exp\left(-\frac{i\pi N^2}{m\widetilde{\tau}^2}\frac{\overline{B}_3(C'm\xi)}{C'^3}\right),$$
(2.64)

with $m\xi = -\frac{mn_0 + 2n}{3}$ as before.

The competition between various terms in (2.64) can be visualized by comparing the exponents as in figure 2, which shows the range of $\Delta := \{m\xi\}$ for which a given phase dominates when $\arg\widetilde{\tau} - \pi/2 > 0$. The figure implies that for the "physical" values $\{m\xi\} = 1/3, 2/3$, the index is respectively in the 1-center, and 2-center phase when $\arg\widetilde{\tau} - \pi/2 > 0$, and vice versa for $\arg\widetilde{\tau} - \pi/2 < 0$. As mentioned above, for $\{m\xi\} = 0$ the index is in a confined phase and does not yield exponential $O(N^2)$ growth. Therefore up to an
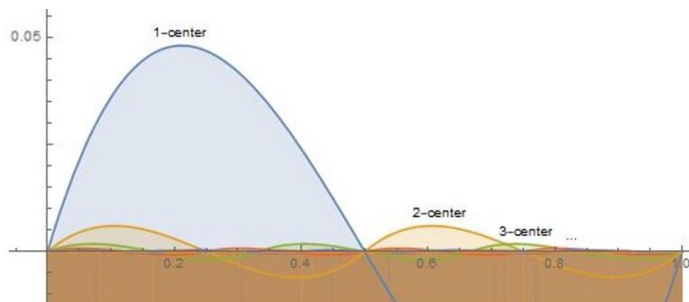
Figure 2. The functions $C'^{-3}\overline{B}_3(C'\Delta)$ with $C' = 1, \cdots, 13$, for $0 \leq \Delta \leq 1$. For $0 < \Delta < 1/2$ the blue curve corresponding to the fully-deconfined phase takes over. The take-over of the orange curve signifies the partially-deconfined 2-center phase in the corresponding region ($1/2 < \Delta \lesssim .72$), and so on.

$o(N^2/\widetilde{\tau}^2)$ error in the exponents we have the following simplification of (2.64) by restricting to $C' = 1, 2$:

$$\mathcal{I}_{N \to \infty} \xrightarrow{\widetilde{\tau} \to 0} e^{-\frac{i\pi N^2}{m\widetilde{\tau}^2}\overline{B}_3(m\xi)} + e^{-\frac{i\pi N^2}{m\widetilde{\tau}^2}\frac{\overline{B}_3(2m\xi)}{8}}. \tag{2.65}$$

This is the analog of Conjecture 1 in [37].

Since $\overline{B}_3(2/3) = -\overline{B}_3(1/3)$, we see from (2.65) that the action of the 2-center saddle has the opposite sign and is smaller in absolute value by a factor of 8 compared to that of the 1-center saddle.

3 Asymptotics of the 4d index from 3d field theory

In this section we consider the dimensional reduction of the four-dimensional $\mathcal{N} = 1$ gauge theory on a Hopf surface. This surface is topologically $S^3 \times S^1$ and we reduce along the S^1 fiber. The dimensionally reduced theory describes a three-dimensional dynamical gauge supermultiplet coupled to background three-dimensional supergravity on S^3. The Wilsonian effective action of the gauge multiplet can be calculated by integrating out the tower of massive Kaluza-Klein modes, and the resulting theory is described by a functional integral over the gauge multiplet fields with this effective action. We find that the functional integral of the three-dimensional theory can be written as a perturbative expansion in τ. The singular terms in the expansion behave as $\mathrm{O}(1/\tau^2)$ and $\mathrm{O}(1/\tau)$, and are captured by three-dimensional effective field theory. In particular, these terms are independent of the dynamical fields, and are completely accounted by the (supersymmetrized) Chern-Simons couplings of the background supergravity. The result agrees with the corresponding singular terms in the microscopic expansion (2.26), (2.28).

The all-order asymptotic formula from the microscopic index includes, in addition to these singular terms, constant and linear terms in τ. Using a localization argument we show that the constant term in τ, besides a background part, has a dynamical piece captured by the integral over the fluctuations of the dynamical fields in three-dimensional path integral, which is essentially the partition function of $\mathcal{N} = 2$ supersymmetric CS theory at

level $\pm N$. Finally, the linear term in the microscopic formula is precisely the supersymmetric Casimir energy which is needed to translate between the microscopic Hamiltonian index and the macroscopic functional integral.[13] In this manner the full asymptotic formula for the four-dimensional index is explained by three-dimensional physics. The fact that the asymptotic formula does not contain any higher order terms in τ implies a non-renormalization theorem, namely that there are no corrections to the three-dimensional effective action at any polynomial order in τ. We leave the explanation of this interesting point to future work. Finally, we show that corresponding statements also hold near rational points when $\tau \to -n/m$. Here we present evidence that the relevant three-dimensional manifold is a $\mathbb{Z}_m$ orbifold of S^3 and the results agree with the microscopic asymptotic expansion given in (2.58).

We begin by recalling the functional integral definition of the $\mathcal{N} = 1$ superconformal index on $S^3 \times S^1$. In the Hamiltonian trace definition (1.2) we have two chemical potentials that couple to linear combinations of the two angular momenta J_1, J_2 on S^3 and the U(1) R-charge Q. This is equal to the supersymmetric functional integral of the theory on $S^3 \times S^1$ with twisted boundary conditions on the fields as we go around the S^1. Equivalently, one can explicitly introduce a background gauge field (for the R charge) and background off-diagonal terms in the metric (for the angular momenta) in a manner, so as to preserve supersymmetry. As explained in [66], such background configurations can be obtained as solutions to the condition of vanishing gravitino variations of off-shell supergravity (and then taking a rigid limit so as to decouple the fluctuations of gravity).

The relevant background configuration for the calculation of the 4d superconformal index for complex τ and nonzero n_0 was studied in [5] in the context of 4d new minimal supergravity [67, 68]. Recall that the bosonic fields of new minimal supergravity are the metric, a gauge field A^{nm}, and another vector field V^{nm} which is covariantly conserved. The background configuration [5] preserving the supercharges $(\mathcal{Q}, \overline{\mathcal{Q}})$ is[14]

$$
\begin{aligned}
\mathrm{d}s_4^2 &= \mathrm{d}t_E^2 + \mathrm{d}\theta^2 + \sin^2\theta \left(\mathrm{d}\phi_1 - \mathrm{i}\,\Omega_1\,\mathrm{d}t_E\right)^2 + \cos^2\theta \left(\mathrm{d}\phi_2 - \mathrm{i}\,\Omega_2\,\mathrm{d}t_E\right)^2, \\
A^{\mathrm{nm}} &= \mathrm{i}\left(\Phi - \frac{3}{2}\right)\mathrm{d}t_E, \qquad V^{\mathrm{nm}} = -\mathrm{i}\,\mathrm{d}t_E.
\end{aligned}
\tag{3.1}
$$

Here $\theta \in [0, \pi/2]$, the angles ϕ_1, ϕ_2 are 2π-periodic, and the Euclidean time coordinate has the independent periodicity condition[15]

$$
t_E \sim t_E + \gamma.
\tag{3.2}
$$

---

[13]The supersymmetric Casimir energy that appears in our asymptotic formulas is the one given in [62]. Note in particular that (unlike in [5]) this is independent of n_0. We can understand this in the path-integral picture by appealing to the result in section 4 of [10] (based on the regularization method of [65]) which demonstrated that the supersymmetric Casimir energy is independent of flavor fugacities when they are on the unit circle, and by noting that $\mathrm{e}^{2\pi\mathrm{i}(-n_0 r_I/2)}$ is effectively a flavor fugacity in our problem.

[14]A real metric corresponds to pure imaginary Ω_i. General complex Ω_i correspond to analytic continuation in the background metric.

[15]In [5] the parameter γ was called β.

This configuration admits the following Killing spinor which is identified with $\mathcal{Q}$,

$$\varepsilon = \begin{pmatrix} e^{i\,z\,t_E} \\ 0 \\ 0 \\ e^{-i\,z\,t_E} \end{pmatrix}, \qquad z = \frac{\pi n_0}{\gamma}. \tag{3.3}$$

The twist parameters Ω_i, Φ are related to the chemical potentials σ, τ in the index as follows[16]

$$\Omega_i = 1 + \frac{\omega_i}{\gamma}, \qquad \Phi = \frac{3}{2} + \frac{1}{\gamma}\left(\frac{\omega_1 + \omega_2}{2} - \pi i\, n_0\right), \tag{3.4}$$

with

$$\omega_1 = 2\pi i\, \sigma, \qquad \omega_2 = 2\pi i\, \tau. \tag{3.5}$$

In this section for ease of presentation we focus on the case with $\Omega_1 = \Omega_2 = \Omega$, which implies $\sigma = \tau = \frac{i\gamma}{2\pi}(1 - \Omega)$. The partition function on the above background is related to the index $\mathcal{I}(\sigma - n_0, \tau)$, which for $\sigma = \tau$ coincides with the index $\mathcal{I}(\tau; n_0)$ in (1.3). In appendix D we comment on the more general case with $\Omega_1 \neq \Omega_2$ and hence $\sigma \neq \tau$.

The four-dimensional supersymmetric partition function of the theory corresponding to the Hamiltonian index (1.2) can then be expressed as a functional integral of the gauge theory with 4d $\mathcal{N} = 1$ chiral and vector multiplets on the background (3.1).[17] As discussed in [5], this functional integral localizes to an integral over flat connections of the gauge field on the KK circle,

$$\oint A^i = 2\pi\, u_i. \tag{3.6}$$

The Wilson loop (3.6) maps to the scalar in the three-dimensional vector multiplet in the KK reduction. We now proceed to derive an expression for the supersymmetric partition function of the three-dimensional gauge theory.

3.1 Dimensional reduction to three dimensions

We first consider the reduction of the above four-dimensional background as a configuration in three-dimensional supergravity. In three dimensions we use the off-shell supergravity formalism [69–72], and follow the treatment [34, 73–75] for the reduction from four to three dimensions. The bosonic fields in the off-shell three-dimensional supergravity are the metric, the KK gauge field (the graviphoton) written as a one-form c, a two-form B, and the R-symmetry gauge field one-form $\mathcal{A}^R$. The equations are often presented in terms of the dual one-form $v = -i * dc$ and the dual scalar $H = i * dB$.

We begin by writing the background in (3.1) as a Kaluza-Klein (KK) compactification to three dimensions, i.e. a circle fibration on a 3-manifold $\mathcal{M}_3$. We define the rescaled S^1 coordinate

$$Y = \sqrt{1 - \Omega^2}\, t_E, \tag{3.7}$$

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[16]Here $(\Omega_1^*, \Omega_2^*, \Phi^*) = (1, 1, \frac{3}{2})$ are the values of the potentials on the supersymmetric BH solution.

[17]More precisely the Hamiltonian index equals the functional integral for the supersymmetric partition function up to the supersymmetric Casimir energy factor [62, 65].

which obeys the periodicity condition

$$Y \sim Y + 2\pi R, \qquad R = \frac{\gamma}{2\pi}\sqrt{1-\Omega^2}\,. \qquad (3.8)$$

Writing the metric (3.1) in the KK form,

$$\mathrm{d}s_4^2 = \mathrm{d}s_3^2 + (\mathrm{d}Y + c)^2\,, \qquad (3.9)$$

we find that the graviphoton field is

$$c = c_\mu \, \mathrm{d}x^\mu = -\mathrm{i}\frac{\Omega}{\sqrt{1-\Omega^2}}\left(\sin^2\theta\,\mathrm{d}\phi_1 + \cos^2\theta\,\mathrm{d}\phi_2\right)\,, \qquad (3.10)$$

and the metric on the 3-manifold $\mathcal{M}_3$ is

$$\mathrm{d}s_3^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi_1^2 + \cos^2\theta\,\mathrm{d}\phi_2^2 - c^2\,. \qquad (3.11)$$

The three-dimensional metric obeys

$$\sqrt{g} = \frac{\sin 2\theta}{2\sqrt{1-\Omega^2}}\,. \qquad (3.12)$$

We see that we effectively have a KK reduction on a circle of radius R.

In order to study the effective theory in three dimensions, we consider the limit $R \to 0$. From the relation (3.8) we see that this is implemented by taking the original circle size $\gamma \to 0$. Our eventual interest is in the limit $\tau \to 0$. The question is how to correlate these two limits of γ and τ. If we take $\gamma \to 0$ first, then we see from the relation (3.4) that $\Omega \to \infty$ and from (3.12) that $\mathcal{M}_3$ shrinks to zero size. Although the local Lagrangian involves background fields and terms such as the Ricci scalar which diverge in this limit, the three-dimensional effective action turns out to be finite. We can understand this in a cleaner manner as follows. We first scale τ and γ to zero at the same rate keeping Ω finite and fixed, i.e. take $\gamma = \varepsilon\tau$ with fixed $\varepsilon = 2\pi\mathrm{i}/(\Omega-1)$, and only take $\varepsilon \to 0$ at the end of all calculations. In particular, the three-dimensional calculations are all performed at finite ε, i.e. on smooth backgrounds. The action turns out to have two pieces, one of which stays finite and the other vanishing in the limit $\varepsilon \to 0$, and, in particular, there are no diverging terms in this limit. Thus we can safely take the limit $\Omega \to \infty$ at the end of calculations. In this limit we have that $R \to \tau$, so that the effective field theory answers are effectively written as a perturbative series in τ.

In the treatment of three-dimensional background supergravity we need the Hodge dual of the graviphoton,

$$v = v_\mu \, \mathrm{d}x^\mu = -\mathrm{i}*\mathrm{d}c\,, \qquad (3.13)$$

whose value in the above background is

$$v = \frac{2\,\Omega}{1-\Omega^2}\left(\sin^2\theta\,\mathrm{d}\phi_1 + \cos^2\theta\,\mathrm{d}\phi_2\right)\,, \qquad (3.14)$$

so that $v^\mu = 2\,\Omega(1,1,0)$. The associated Chern-Simons action is

$$S^{\mathrm{CS}}(c) = \int_{\mathcal{M}_3} c \wedge \mathrm{d}c = \mathrm{i}\int_{\mathcal{M}_3} \mathrm{d}^3x\,\sqrt{g}\,v^\mu\,c_\mu\,. \qquad (3.15)$$

The identification between the four-dimensional and the three-dimensional gauge fields is made by comparing the respective Killing spinor equations. As shown in [34, 75], one has[18]

$$\frac{1}{2} v_\mu = V_\mu^{\mathrm{nm}} - V_Y^{\mathrm{nm}} c_\mu \,, \qquad H = V_Y^{\mathrm{nm}} \,, \qquad \mathcal{A}_\mu^R = A_\mu^{\mathrm{nm}} - A_Y^{\mathrm{nm}} c_\mu + \frac{1}{2} v_\mu \,. \tag{3.16}$$

The background gauge fields in (3.1) are given by

$$A^{\mathrm{nm}} = \left(-\frac{\tau}{R} + \frac{n_0}{2R} \right) \mathrm{d}Y \,, \qquad V^{\mathrm{nm}} = -\frac{\mathrm{i}}{\sqrt{1 - \Omega^2}} \, \mathrm{d}Y \,, \tag{3.17}$$

so that the auxiliary fields in the background supergravity multiplet are

$$v_\mu = -2 \, V_Y^{\mathrm{nm}} \, c_\mu \,, \qquad H = V_Y^{\mathrm{nm}} \,, \qquad \mathcal{A}_\mu^R = -(A_Y^{\mathrm{nm}} + V_Y^{\mathrm{nm}}) \, c_\mu \,. \tag{3.18}$$

(The above equation for v_μ is consistent with equations (3.10), (3.13).)

We now discuss the Kaluza-Klein reduction of the dynamical gauge multiplet. The $\mathcal{N} = 1$ gauge multiplet in four dimensions reduces to an $\mathcal{N} = 2$ gauge multiplet in three dimensions, whose bosonic field content is a vector $\mathcal{A}_\mu$, a scalar σ, and the auxiliary $\mathcal{D}$ field. These are related to the four-dimensional fields as follows,

$$\sigma^i = A_Y^i \,, \qquad \mathcal{A}_\mu^i = A_\mu^i - A_Y^i \, c_\mu \,, \qquad \mathcal{D}^i = D^i - A_Y^i \, H \,, \tag{3.19}$$

and the three-dimensional fermions are the reduction of the corresponding four-dimensional fermions. As discussed above, the theory localizes on the BPS configurations given by

$$A^i = \frac{u_i}{R} \, dY \,, \qquad D^i = 0 \,, \tag{3.20}$$

with vanishing values of all other fields in the off-shell gauge and chiral multiplets. In the three-dimensional theory the non-zero fields on the BPS locus are

$$\sigma^i = \frac{u_i}{R} \,, \qquad \mathcal{A}_\mu^i = -\frac{u_i}{R} \, c_\mu \,, \qquad \mathcal{D}^i = -\frac{u_i}{R} \, H \,. \tag{3.21}$$

3.2 Effective action and functional integral of the three-dimensional theory

We now turn to the calculation of the partition function of the three-dimensional supersymmetric theory that we just discussed. Our strategy is to first calculate the three-dimensional Wilsonian effective action of u_i, and then use this to calculate the three-dimensional partition function. The tree-level action (coming from a mode expansion of the four-dimensional theory) consists of matter-coupled super Yang-Mills theory. The full quantum effective action of the three-dimensional theory is obtained by integrating out the tower of massive KK modes on the circle. In order to calculate this action, we draw from known results in the effective field theory in three dimensions.

The effective field theory on backgrounds of the type $\mathcal{M}_3 \times S_R^1$ was studied in a general context in [52, 53], and in the special context of supersymmetry in [25, 26]. The resulting

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[18]In [75] it is assumed that $V_Y^{\mathrm{nm}} = A_Y^{\mathrm{nm}}$, which is not satisfied in our background. Therefore we follow more closely the treatment of [34].

three-dimensional action begins with a term proportional to $1/R^2$, and continues as a perturbation expansion as the radius $R \to 0$. At each order in R one has a combination of three-dimensional actions of the background and the dynamical fields, which are all related by supersymmetry to a certain Chern-Simons term. The Chern-Simons terms are of the form $\int_{\mathcal{M}_3} A_x \wedge \mathrm{d}A_y$, where A_x and A_y represent the various gauge fields. As discussed in the previous subsection, these are the dynamical gauge field, the background graviphoton, the background R-gauge field, and the spin connection. We follow, and review in appendix B, the treatment of [26] for the supersymmetrized Chern-Simons action of all the background and the dynamical gauge fields up to $O(R^0)$. The full effective action also includes RR and gravitational supersymmetrized CS terms discussed in [56], which turn out to be crucial for our purposes.

It follows from the above discussion that the overall coefficient at each order in R can be fixed by calculating the coefficient of the Chern-Simons terms themselves. These coefficients, in turn, can be obtained by integrating out all the fermions coupling to the corresponding gauge fields. The resulting induced Chern-Simons coefficient is one-loop exact. Thus the strategy is to integrate out the fermions in each KK mode, write the resulting Chern-Simons action, and sum over all the fermions in the theory. The KK momenta of the fermions take the values $p_Y = k_Y/R$, with $k_Y = n + \frac{n_0}{2}$, $n \in \mathbb{Z}$. The shift $n_0/2$ appears because of the gauge fields in the background (3.1). (Recall, for example, that the four-dimensional Killing spinor (3.3) has momentum $n_0/2$.)

The result for the complete action obtained by integrating out a fermion f of R-charge r_f and transforming in a representation of weight ρ_f under the gauge group is given in appendix B and take the following form,

$$\delta S^\mathrm{f}_{\text{1-loop}} = \widetilde{S}^\mathrm{f}_{\text{g-g}} + 2\,\widetilde{S}^\mathrm{f}_{\text{g-R}} + S^\mathrm{f}_{\text{R-R}} + S^\mathrm{f}_{\text{grav}} \,. \tag{3.22}$$

The terms in (3.22) depend on the real mass m_f (related to the central charge appearing in the three-dimensional algebra). The first two terms depend on the dynamical gauge field. On the configuration (3.21) they take the following values,

$$\begin{aligned}
\widetilde{S}^\mathrm{f}_{\text{g-g}} &= -\mathrm{i}\pi\, \frac{\operatorname{sgn}(m_\mathrm{f})}{8R^2}\, (\rho_\mathrm{f} \cdot \underline{u} - k_Y)^2\, A_{\mathcal{M}_3} \,, \\
2\,\widetilde{S}^\mathrm{f}_{\text{g-R}} &= -\mathrm{i}\pi\, \frac{\operatorname{sgn}(m_\mathrm{f})}{8R}\, 2\, r_\mathrm{f}\, (\rho_\mathrm{f} \cdot \underline{u} - k_Y)\, L_{\mathcal{M}_3} \,,
\end{aligned} \tag{3.23}$$

where $A_{\mathcal{M}_3}$ and $L_{\mathcal{M}_3}$ are functions of the three-dimensional background given in (B.11). The last two terms in (3.22) do not depend on the dynamical gauge field, and given by

$$\begin{aligned}
S^\mathrm{f}_{\text{R-R}} &= -\mathrm{i}\pi\, \frac{\operatorname{sgn}(m_\mathrm{f})}{8}\, \left(r_\mathrm{f}^2 - \frac{1}{6}\right) R_{\mathcal{M}_3} \,, \\
S^\mathrm{f}_{\text{grav}} &= -\mathrm{i}\pi\, \frac{\operatorname{sgn}(m_\mathrm{f})}{192}\, G_{\mathcal{M}_3} \,,
\end{aligned} \tag{3.24}$$

where $R_{\mathcal{M}_3}$ and $G_{\mathcal{M}_3}$ are functions of the three-dimensional background given in (B.3).

In appendix C we calculate the values of these background actions.[19] As explained above, we perform the calculations keeping R, Ω finite so that the three-dimensional physics is manifestly smooth. The result is that there is a smooth limit as $\gamma \to 0$ keeping fixed τ. The limiting values of the actions are as follows,

$$A_{\mathcal{M}_3} = -4, \qquad\qquad L_{\mathcal{M}_3} = -4 \left(1 - \frac{n_0}{2R}\right),$$
$$R_{\mathcal{M}_3} = -4 \left(1 - \frac{n_0}{2R}\right)^2, \qquad G_{\mathcal{M}_3} = -16 + 4\,R_{\mathcal{M}_3}. \tag{3.25}$$

Using these values, we obtain the total effective action of the fermion f to be

$$\delta S^{\mathrm{f}}_{\text{1-loop}} = \mathrm{i}\pi\,\frac{\mathrm{sgn}\,(m_{\mathrm{f}})}{2R^2}\left(\rho_{\mathrm{f}} \cdot \underline{u} - k_Y - \tfrac{1}{2}n_0\,r_{\mathrm{f}}\right)^2 + \mathrm{i}\pi\,\frac{\mathrm{sgn}\,(m_{\mathrm{f}})}{R}\,r_{\mathrm{f}}\left(\rho_{\mathrm{f}} \cdot \underline{u} - k_Y - \tfrac{1}{2}n_0\,r_{\mathrm{f}}\right)$$
$$+ \mathrm{i}\pi\,\frac{\mathrm{sgn}\,(m_{\mathrm{f}})}{2}\,r_{\mathrm{f}}^2 - \mathrm{i}\pi\,\frac{\mathrm{sgn}\,(m_{\mathrm{f}})}{12}. \tag{3.26}$$

Now we turn to the sum over all the fermions in the theory. The value of the real mass is given in (B.7) to be, as $R \to 0$,[20]

$$m_{\mathrm{f},n} = -\frac{1}{R}\left(\rho_{\mathrm{f}} \cdot \underline{u} - n - \tfrac{1}{2}n_0\,(r_{\mathrm{f}} + 1)\right), \tag{3.27}$$

In order to obtain the full effective action we now have to sum over all the fermions. For the chiral multiplets, this implies summing over all the weights in representations $\rho_{\mathrm{f}} \in \mathcal{R}_{\mathrm{f}}$, as well as over all momenta labelled by $n \in \mathbb{Z}$. The summation over KK modes can be evaluated using

$$\sum_{n \in \mathbb{Z}} \mathrm{sgn}(n+x)(n+x)^{j-1} = -\frac{2}{j}\,\overline{B}_j(x), \tag{3.28}$$

with $x = \rho_{\mathrm{f}} \cdot \underline{u} - \tfrac{1}{2}n_0\,r_I$, for $j = 1, 2, 3$ (cf. section 4 of [10]).[21] Here we have used the relation $r_{\mathrm{f}} = r_I - 1$ between the R-charge of the fermion and that of the bottom component of the multiplet I to which the fermion belongs.

---

[19]We note that there is a subtlety with the gravitational CS term in (B.3), concerning the dependence of the term on the frame [64]. There should be a choice of frame which is consistent with the supersymmetry and the 4d to 3d reduction. We do not work out the details of this issue in this paper, and instead rely on consistency with [56] where this term is obtained indirectly by considering integrating out chiral multiplets. We thank Cyril Closset for a discussion on this point.

[20]In fact the first three terms in (3.26) sum up to

$$\mathrm{i}\frac{\pi}{2}\,\mathrm{sgn}\,(m_{\mathrm{f}})\left(\frac{\rho_{\mathrm{f}} \cdot \underline{u} - n - \tfrac{1}{2}n_0\,r_I + (r_I - 1)R}{R}\right)^2,$$

and, using (3.28) with $x = \rho_{\mathrm{f}} \cdot \underline{u} - n - \tfrac{1}{2}n_0\,r_I + (r_I - 1)R$ to perform the sum over the KK modes, we obtain an effective potential which reproduces the chiral multiplet contributions in (3.32). Essentially the same comment can be made in the microscopic analysis of section 2.

[21]Here, the sgn function is interpreted as applying to m_{f} with R a real positive number (which therefore scales out of the formula so as to give $\mathrm{sgn}(\rho_{\mathrm{f}} \cdot \underline{u} - \tfrac{1}{2}n_0\,r_I)$). Note that in the subsequent formulas R is taken to be complex.

For the vector multiplet contribution the analysis is quite similar: there is a tower of massive KK gaugino modes that are integrated out. These generate CS actions whose supersymmetrization yields the vector multiplet contribution to $\delta S_{\text{1-loop}}$. In the present context there is an important difference with the chiral multiplet analysis however. Near $\underline{u} = 0$ there is a single gaugino mode in the tower that has real mass of order $\alpha \cdot \underline{u}/R$, and is therefore considered a "light" mode for small enough $|\alpha \cdot \underline{u}|$. Therefore we do not integrate out this mode and, instead, keep it as a dynamical mode in the path integral of the three-dimensional theory.

More precisely, recall that the nth KK gaugino mode associated to a root α of the gauge group has $p_Y = (n + n_0)/R$ and hence a real mass $(\alpha \cdot \underline{u} - n - n_0)/R$. Therefore the mode corresponding to $n = -n_0$ is light near $\alpha \cdot \underline{u} = 0$. We now describe how removing this term from the sum over the KK tower modifies the result compared to the chiral multiplet computation. The vector multiplet contributions is a sum over roots α that come in pairs $\pm \alpha_+$, as a result of which they give vanishing contributions to the quadratic and constant terms in $\underline{u}$ in the action of a single KK mode. We therefore focus on the contribution to the linear term in $\underline{u}$, which is proportional to $1/R$. The calculation is similar to the corresponding chiral multiplet calculation. Upon summing over all the KK modes, we obtain the vector multiplet contribution from a root α to be

$$-\frac{\pi\mathrm{i}}{R} \sum_{n\in\mathbb{Z}}{}' \mathrm{sgn}\left(\alpha \cdot \underline{u} - n - n_0\right)\left(\alpha \cdot \underline{u} - n - n_0\right), \tag{3.29}$$

where the prime indicates that we are not including the light mode corresponding to $n = -n_0$. Upon adding and subtracting the $n = -n_0$ contribution, we obtain, using (3.28),

$$\frac{\mathrm{i}\pi}{R}\left(\overline{B}_2(\alpha \cdot \underline{u}) + |\alpha \cdot \underline{u}|\right). \tag{3.30}$$

Now, since we are interested in the proximity of $\underline{u} = 0$, we use the fact that for $|x| < 1$ we have $\overline{B}_2(x) = x^2 - |x| + \frac{1}{6}$, to simplify the result to

$$\frac{\mathrm{i}\pi}{R}\left((\alpha \cdot \underline{u})^2 + \frac{1}{6}\right). \tag{3.31}$$

Upon putting all the pieces together, we obtain the total one-loop correction to the Wilsonian action of the three-dimensional theory, which we call $V_{\text{eff}}(\underline{u})$ (we justify this name below). We have

$$
\begin{aligned}
V_{\text{eff}}(\underline{u}) &= \sum_{\text{f}} \sum_{\rho_{\text{f}} \in \mathcal{R}_{\text{f}}} \delta S_{\text{1-loop}}^{\text{f}} \\
&= \mathrm{i}\pi \sum_{I, \rho_I} \Bigg(\frac{1}{3R^2} \overline{B}_3\left(\rho_{\text{f}} \cdot \underline{u} - \tfrac{1}{2} n_0 r_I\right) + \frac{r_I - 1}{R} \overline{B}_2\left(\rho_{\text{f}} \cdot \underline{u} - \tfrac{1}{2} n_0 r_I\right) \\
&\quad + \frac{1}{R} \sum_{\alpha} \left((\alpha \cdot \underline{u})^2 + \tfrac{1}{6}\right) + \left((r_I - 1)^2 - \tfrac{1}{6}\right) \overline{B}_1\left(\rho_{\text{f}} \cdot \underline{u} - \tfrac{1}{2} n_0 r_I\right) \Bigg).
\end{aligned} \tag{3.32}
$$

We now localize the path integral of the light gauge multiplet mode that was excluded from the sum (3.29), using its Wilsonian effective action, which consists of the tree-level

action coming from the light $n = -n_0$ mode in 4d, as well as the one-loop action $\delta S_{\text{1-loop}}$ derived above (in the bosonic sector, which is relevant for the localization calculation) from integrating out the heavy modes. It is useful to keep in mind the different but related problem of calculating the partition function of superconformal CS theory coupled to matter on $\mathcal{M}_3$ [57, 76]. In that case the theory localizes onto arbitrary constant values of the scalar σ and is supported by the auxiliary scalar H. The measure including the one-loop determinant of the localizing action in the non-BPS directions is[22]

$$\int \frac{D\underline{\sigma}}{\sqrt{-\omega_1^{\text{thf}} \omega_2^{\text{thf}}}} \prod_{\alpha_+} 4 \sinh\left(\frac{\pi\alpha_+ \cdot \underline{\sigma}}{-\mathrm{i}\,\omega_1^{\text{thf}}}\right) \sinh\left(\frac{\pi\alpha_+ \cdot \underline{\sigma}}{-\mathrm{i}\,\omega_2^{\text{thf}}}\right), \tag{3.33}$$

with $\omega_{1,2}^{\text{thf}}$ the moduli of the transversely holomorphic foliation (THF) [78] of $\mathcal{M}_3$, which we expect to be

$$\omega_1^{\text{thf}} = \omega_2^{\text{thf}} = \mathrm{i}\sqrt{\frac{1-\Omega}{1+\Omega}}. \tag{3.34}$$

Recalling from (3.21) that $\sigma^i = u_i/R$, and adding the contribution from $\delta S_{\text{1-loop}}$ in (3.32) (which although arises at one-loop in high-temperature EFT, contributes as a "classical" piece in the localization computation), we obtain the final result for the three-dimensional partition function

$$Z\left(\tau\right) = \int \frac{D\underline{u}}{(-\mathrm{i}\,\tau)^{\text{rk}(G)}} \prod_{\alpha_+} 4 \sinh^2\left(\frac{\pi\alpha_+ \cdot \underline{u}}{-\mathrm{i}\,\tau}\right) \exp\left(-V_{\text{eff}}(\underline{u})\right). \tag{3.35}$$

Noting that the supersymmetric partition function and the Hamiltonian index are related as [62, 65]

$$Z(\tau) = e^{2\pi\mathrm{i}\tau E_{\text{susy}}} \mathcal{I}(\tau), \tag{3.36}$$

we see that the result (3.35) agrees precisely with the microscopic result (2.9)–(2.12).

We emphasize that while the above derivation of V_{eff} in (3.32) applies to $\underline{u}$ near 0, it can be easily extended to generic finite $\underline{u}$ by modifying the vector multiplet discussion. For generic $\underline{u}$, the non-Cartan components of the $n = -n_0$ mode of the vector multiplet are also heavy, and ought to be integrated out. Consequently the sum in (3.29) would no longer have a prime, and we end up with V_1^r as in (2.13) rather than V_1 in (3.32). This is the EFT derivation of the finite-$\underline{u}$ potentials $V_{1,2}$ found microscopically in [12].

On the other hand, when $n_0 = 0$, the small-$\underline{u}$ discussion leading up to (3.32) needs to be modified because now the chiral multiplets have light modes (corresponding to $n = 0$). As in the discussion around (3.29) the light mode should be removed from the KK sum and instead be included in the dynamical part (to be localized). Indeed, it is well-known that for $n_0 = 0$ the constant piece of the small-τ expansion coming from the $\underline{u} = 0$ saddle contains the (localized) S^3 partition function of the dimensionally reduced chiral as well as vector multiplets [49] (see [79–83] for earlier work on the connection between 4d indices and S^3 partition functions).

---

[22]Compare with section 5 of [77], noting that for squashed S^3 with squashing parameter b one has $\omega_1^{\text{thf}} = \mathrm{i}b$, $\omega_2^{\text{thf}} = \mathrm{i}b^{-1}$. We leave the derivation of (3.34) from the metric (3.11) to future work.

A technical remark is in order regarding our EFT derivation of (3.35). To reproduce the desired asymptotics, we have sent $\varepsilon(= \frac{\gamma}{\tau} = \frac{2\pi i}{\Omega - 1}) \to 0$, and hence $\Omega \to \infty$, when evaluating the CS actions in appendix C. It would be interesting to have a formula of the type (3.35) for $\gamma \to 0$ at finite ε, which would imply that Ω and the resulting 3d geometry would be finite-sized.[23]

Finally, as discussed in appendix D, we find that the effective potential for τ and σ not necessarily equal is given by making the replacement

$$\frac{1}{R^2} \to \frac{1}{\tau\,\sigma}\,, \qquad \frac{1}{R} \to \frac{\tau + \sigma}{2\tau\,\sigma} \tag{3.37}$$

in the effective potential (3.32). The singular pieces are indeed in agreement with the microscopic calculations reported in [11, 12].

3.3 Rational points

We now turn our attention to the limit of τ approaching a rational point. In the discussion of the previous subsection we used the fact that the radius of the circle R equals τ which becomes small in the limit, so that we could use an effective three-dimensional description. Now we are interested in $\widetilde{\tau} = m\tau + n \to 0$, with $n, m \in \mathbb{Z}$ (with no common factor) as in [13]. In terms of the variable $\widetilde{\tau}$ we have that $\omega = 2\pi i \tau = 2\pi i(\widetilde{\tau} - n)/m$ so that

$$\Omega = 1 + \frac{\omega}{\gamma} = 1 - \frac{2\pi i n}{m\gamma} + \frac{2\pi i \widetilde{\tau}}{m\gamma}\,, \tag{3.38}$$

and the four-dimensional metric background (3.1) is now

$$\begin{aligned}
ds_4^2 = dt_E^2 + d\theta^2 &+ \sin^2\theta \left(d\phi_1 - \frac{2\pi n}{m\gamma}\,dt_E - i\left(1 + \frac{2\pi i\widetilde{\tau}}{m\gamma}\right)dt_E\right)^2 \\
&+ \cos^2\theta \left(d\phi_2 - \frac{2\pi n}{m\gamma}\,dt_E - i\left(1 + \frac{2\pi i\widetilde{\tau}}{m\gamma}\right)dt_E\right)^2.
\end{aligned} \tag{3.39}$$

In terms of the following new coordinates and new parameters,

$$\widetilde{\gamma} = m\gamma\,, \qquad \widetilde{\Omega} = 1 + \frac{2\pi i\widetilde{\tau}}{\widetilde{\gamma}}\,, \qquad \widetilde{\phi}_i = \phi_i - \frac{2\pi n}{\widetilde{\gamma}}t_E\,, \tag{3.40}$$

the above metric is

$$ds_4^2 = dt_E^2 + d\theta^2 + \sin^2\theta \left(d\widetilde{\phi}_1 - i\widetilde{\Omega}\,dt_E\right)^2 + \cos^2\theta \left(d\widetilde{\phi}_2 - i\widetilde{\Omega}\,dt_E\right)^2\,, \tag{3.41}$$

with $\widetilde{\phi}_1$, $\widetilde{\phi}_2$ being 2π-periodic as before, and the periodic identification going around the time circle is

$$\left(t_E\,,\ \widetilde{\phi}_1\,,\ \widetilde{\phi}_2\right) \sim \left(t_E + \frac{\widetilde{\gamma}}{m}\,,\ \widetilde{\phi}_1 - \frac{2\pi n}{m}\,,\ \widetilde{\phi}_2 - \frac{2\pi n}{m}\right)\,. \tag{3.42}$$

---

[23]The recent work [84] presents such a derivation, although using a background different from ours. The precise relation between the two backgrounds is not clear to us at the moment.

The metric configuration (3.41) with the identifications (3.42) is simply a global identification, or orbifold, of the configuration considered in the previous subsection with the new parameters $(\widetilde{\gamma}, \widetilde{\tau}, \widetilde{\Omega})$ replacing (γ, τ, Ω).[24]

On the covering space, going around the time circle shifts $\widetilde{t}_E \to \widetilde{t}_E + \widetilde{\gamma}$ and $\widetilde{\phi}_i \to \widetilde{\phi}_i + 2\pi n$. The latter identification can be trivialized by using the independent 2π-periodicity of $\widetilde{\phi}_i$, so that we have the identification $(\widetilde{t}_E, \widetilde{\phi}_1, \widetilde{\phi}_2) \sim (\widetilde{t}_E + \widetilde{\gamma}, \widetilde{\phi}_1, \widetilde{\phi}_2)$. On this configuration we can perform the dimensional reduction to three dimensions. The relevant considerations of the previous subsection go through exactly as before with the replacement $(\gamma, \tau, \Omega) \mapsto (\widetilde{\gamma}, \widetilde{\tau}, \widetilde{\Omega})$. Actually, because the gauge holonomies on the cover wrap a circle m times larger than the original S^1, we also get a replacement $u_j \to mu_j$. Moreover, since ξ_I (which equals $-n_0 r_I/2$ for $(m,n) = (1,0)$) effectively plays the role of a flavor chemical potential in our problem as mentioned around (2.38), we expect a similar replacement $-n_0 r_I/2 \to m\xi_I$. We can see this replacement arise more directly as follows.

We multiply the first term in (3.26) by $\frac{m^2}{m^2}$, and the second term by $\frac{m}{m}$. This amounts to $\gamma \to m\gamma$ and $u_j \to mu_j$ as mentioned above, but also $k_Y \to mk_Y$ (which corresponds to keeping only the singlet modes under the $\mathbb{Z}_m$ quotient) as well as $n_0 \to mn_0$. On the other hand, writing A_Y^{nm} in (3.17) in terms of $\widetilde{\tau}$ instead of τ amounts to yet another replacement $n_0 \to n_0 + \frac{2n}{m}$. Combining these two effects yields the desired $-n_0 r_I/2 \to m\xi_I$ replacement.

With the preceding substitutions in the results of the previous subsection, we thus arrive at the potentials $\widetilde{V}_{2,1}$ in (2.37). We then take the $\mathbb{Z}_m$ quotient which has two effects as usual. Firstly it reduces the volume of the three-dimensional space, and secondly it introduces new topologically non-trivial sectors in the path integral over the gauge-field configurations. The change in calculations involving local gauge-invariant Lagrangians will therefore be only a reduction in the action by a factor of m. This explains the reduction of the effective potential by a factor of m as in (2.35).

Finally we discuss the constant terms (in $\widetilde{\tau}$) arising from the functional integral over the dynamical gauge multiplet. There are a few subtleties. Firstly the actions like the gravitational CS action will depend on the global properties of the orbifold. Then we need to calculate the partition function of the orbifold space with a background graviphoton. Assuming as in the previous subsection that the expected THF moduli arise, and that by re-scaling and contour deformation (as discussed around (2.44)) the THF moduli can be replaced with those of round S^3, the calculation presumably reduces to an $S^3/\mathbb{Z}_m$ partition function as in [51, 76, 85, 86], with the $\mathbb{Z}_m$ action following from (3.42) to be

$$\left(\widetilde{\phi}_1, \widetilde{\phi}_2\right) \sim \left(\widetilde{\phi}_1 - \frac{2\pi n}{m}, \widetilde{\phi}_2 - \frac{2\pi n}{m}\right), \tag{3.43}$$

which for $n = 1$ coincides with that of the lens space $L(m, -1)$. Here one has to be careful about how the measure on the space of constant scalars σ_i is affected by the four-dimensional orbifold (3.42). We leave these interesting questions to future work, noting

---

[24]We learned about these orbifolds from a talk by O. Aharony at the Stony Brook seminar series in November 2020 [40].

that the result of these considerations indeed agrees with the microscopic answer (2.58), with the $\mathrm{O}(\widetilde{\tau})$ piece explained by the supersymmetric Casimir energy factor as before.

Note added. The paper [84], which appeared on the arXiv the same day as the first version of this paper, has some overlap with our section 3. The paper [87], which appeared on the arXiv soon after, has some overlap with our section 2.

Acknowledgments

A Asymptotic estimates of the special functions

A.1 $\tau \to 0$

We first consider the limit $\tau \to 0$. More precisely, in the rest of this subsection we assume $\arg(\tau)$ is in compact domains avoiding integer multiples of $\frac{\pi}{2}$ as $|\tau| \to 0$.

For the Pochhammer symbol $(q;q)$ the small-τ asymptotics is standard:

$$(q;q) \simeq \frac{1}{\sqrt{-\mathrm{i}\,\tau}} \, \exp\!\Big(-\frac{2\pi\mathrm{i}}{24\,\tau} - \frac{2\pi\mathrm{i}\,\tau}{24}\Big) \qquad (\text{as } |\tau| \to 0)\,. \tag{A.1}$$

Recall that the symbol $\simeq$ means that logarithms (on appropriate branches) of the two sides (assumed to be non-zero) are equal to all orders in the small parameter (here in $|\tau|$).

For the chiral multiplet elliptic gamma functions we have the following estimate, valid for any $r \in \mathbb{R}$, uniformly in z over compact subsets of $\mathbb{R} \setminus \mathbb{Z}$ (see Proposition 2.11 of [46] or equation (3.53) of [49]):

$$\Gamma_{\mathrm{e}}(r\tau + z) \simeq \exp\left(-2\pi\mathrm{i}\left(\frac{\overline{B}_3(z)}{6\tau^2} + (r-1)\,\frac{\overline{B}_2(z)}{2\tau} + \frac{(r-1)^2 - \frac{1}{6}}{2}\,\overline{B}_1(z) + \frac{(r-1)^3 - \frac{r-1}{2}}{6}\,\tau\right)\right), \tag{A.2}$$

as $|\tau| \to 0$. Here $\overline{B}_j(z)$ are the *periodic Bernoulli polynomials* defined, for $z \in \mathbb{R}$ through their Fourier series expansion,

$$-\frac{(2\pi\mathrm{i})^j}{j!}\,\overline{B}_j(z) = \sum_{k\in\mathbb{Z}}' \frac{\mathrm{e}^{2\pi\mathrm{i}kz}}{k^j} \qquad (z \in \mathbb{R}\,,\ j \geq 1)\,. \tag{A.3}$$

The prime in the above formula means that $k = 0$ has to be omitted, and that in the $j = 1$ case — where the series is not absolutely convergent — the sum is in the sense of Cauchy principal value.

For $x \in \mathbb{R} \setminus \mathbb{Z}$, we have $\overline{B}_j(x) = B_j(\{x\})$ with $\{\cdot\} := \cdot - \lfloor \cdot \rfloor$ the fractional-part function. When $j > 1$ this also holds for $x \in \mathbb{Z}$ (and so $\overline{B}_j(\mathbb{Z}) = B_j(0)$). When $j = 1$ on the other hand $\overline{B}_1(\mathbb{Z}) = 0$, while $B_1(0) = -1/2$.

The Bernoulli polynomials are uniquely characterized by

$$B_0(u) = 1, \qquad B_j'(u) = jB_{j-1}(u), \qquad B_j(0) = B_j(1) \quad \text{for } j > 1, \qquad \text{(A.4)}$$

and the first three non-trivial ones are explicitly

$$
\begin{aligned}
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.
\end{aligned}
\qquad \text{(A.5)}
$$

The connection between $\overline{B}_j$ and the Bernoulli polynomials can be verified by first noting that for $j = 1$ the left-hand side of (A.3) is essentially the Taylor expansion of the logarithm function, and then observing that B_j are uniquely characterized by

$$B_0(u) = 1, \qquad B_j'(u) = jB_{j-1}(u), \qquad B_j(0) = B_j(1) \quad \text{for } j > 1.$$

With the aid of (A.3) one can easily prove relations such as

$$\sum_{\ell=1}^{C-1} \overline{B}_3\left(x + \frac{\ell}{C}\right) = \frac{\overline{B}_3(Cx)}{C^2} - \overline{B}_3(x), \qquad \text{(Raabe's formula)} \qquad \text{(A.6)}$$

and

$$
\begin{aligned}
\sum_{\ell=1}^{n-1} \left(\overline{B}_2\left(x + m\frac{\ell}{n}\right) - \overline{B}_2\left(x - m\frac{\ell}{n}\right) \right) &= 0, \\
\sum_{\ell=1}^{n-1} \left(\overline{B}_2\left(\frac{\ell}{n}\right)\overline{B}_2\left(x + m\frac{\ell}{n}\right) - \overline{B}_2\left(\frac{\ell}{n}\right)\overline{B}_2\left(x - m\frac{\ell}{n}\right) \right) &= 0,
\end{aligned}
\qquad \text{(A.7)}
$$

valid for $m, n \in \mathbb{Z}_{>0}$ relatively prime and $x \in \mathbb{R}$, by using the Fourier expansion of the Bernoulli functions, and swapping the sum over Fourier modes with the sum over ℓ.[25]

The estimate (A.2) is particularly useful for the chiral multiplet gamma functions in (2.6) when the integral is dominated by the 1-center holonomy configurations with $z_i - z_j = 0$. This is because the complex phase $2\pi n_0/3$ shifts the argument of the chiral multiplet gamma functions safely into the interior of the domain $z \in \mathbb{R} \setminus \mathbb{Z}$ where the estimate is uniformly valid. On the other hand, since the vector multiplet gamma functions in (2.6) lack such phase shifts in their arguments, the estimate (A.2) is not appropriate for them near the 1-center holonomy configurations when $\tau \to 0$.

---

[25]Note that similar operations with $\overline{B}_1$ are not allowed, because its Fourier expansion is not absolutely convergent. This is the source of sophistication of the Dedekind sum defined below in terms of $\overline{B}_1$ — or more specifically the source of the nontrivial dependence of (A.18) on n. (Readers familiar with Eisenstein series might recall similar "anomalous" behavior from E_2 and its associated elliptic functions.) A closely related fact is that $\overline{B}_{j>1}$ are continuous, but $\overline{B}_1$ has discontinuities on $\mathbb{Z}$.

The estimate (A.2) is not uniformly valid, with respect to z, over intervals containing $\mathbb{Z}$. There is a well-known improvement of it around $z = 0$ however, which is valid uniformly over compact subsets of $(-1, 1)$, and we will use for vector multiplet elliptic gamma functions in the index. It reads (see Proposition 2.10 of [46] or equation (2.16) of [49])

$$\Gamma_e(r\tau + z) \simeq e^{2\pi i R_0(r\tau + z;\tau)} \Gamma_h(r\tau + z; \tau, \tau), \tag{A.8}$$

where

$$R_0\left(z; \tau\right) = -\frac{z^3}{6\tau^2} + \frac{z^2}{2\tau} - \frac{\left(1 + 5\tau^2\right) z}{12\tau^2} + \frac{1}{12\tau} + \frac{\tau}{12}, \tag{A.9}$$

and $\Gamma_h(x; \omega_1, \omega_2)$ is the hyperbolic gamma function.

Using the estimate (A.8) and the "product formula"

$$\frac{1}{\Gamma_h\left(x; \omega_1, \omega_2\right) \Gamma_h\left(-x; \omega_1, \omega_2\right)} = -4 \sin\left(\frac{\pi x}{\omega_1}\right) \sin\left(\frac{\pi x}{\omega_2}\right), \tag{A.10}$$

the next estimate follows (cf. equation (2.18) of [49]):

$$\frac{1}{\Gamma_e\left(z\right) \Gamma_e\left(-z\right)} \simeq e^{-4\pi i\, R_0^+(z;\tau)} \, 4 \sin\left(\frac{\pi z}{\tau}\right) \sin\left(-\frac{\pi z}{\tau}\right), \tag{A.11}$$

valid uniformly in z over compact subsets of $(-1, 1)$, with

$$R_0^+(z; \tau) := \frac{R_0(z; \tau) + R_0(-z; \tau)}{2} = \frac{z^2}{2\tau} + \frac{1}{12\tau} + \frac{\tau}{12}. \tag{A.12}$$

Note that since $\Gamma_e(z + 2\tau) \Gamma_e(-z + 2\tau) = \frac{1}{\Gamma_e(z) \Gamma_e(-z)}$, and $\sin(i\, x) = i \sinh(x)$, we can write (A.13) alternatively as

$$\Gamma_e\left(z + 2\tau\right) \Gamma_e\left(-z + 2\tau\right) \simeq e^{-4\pi i\, R_0^+(z;\tau)} \, 4 \sinh^2\left(\frac{\pi z}{-i\tau}\right). \tag{A.13}$$

While we have presented two separate estimates (A.2) and (A.13) for the chiral and vector multiplet gamma functions, both of them can in fact be derived from the "central estimate" (A.8). Deriving (A.2) from the central estimate requires only an extra step to simplify the hyperbolic gamma functions arising from (A.8) using Corollary 2.3 of [46], as explained in Proposition 2.11 there.

A.2 $\quad \tau \to \mathbb{Q}$

We now consider

$$\widetilde{\tau} \equiv m\tau + n \to 0, \tag{A.14}$$

with m, n relatively prime. More precisely, in the rest of this subsection we assume that $\arg(\widetilde{\tau})$ is in compact domains avoiding integer multiples of $\pi/2$ as $|\widetilde{\tau}| \to 0$.

To obtain the asymptotics of the Pochhammer symbol we note that for integer a, b, c, d satisfying $ad - bc = 1$ with $c > 0$, we have

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \exp\left(2\pi i\left(\frac{a + d}{24c} - \frac{1}{8} - \frac{s\left(d, c\right)}{2}\right)\right) \left(c\tau + d\right)^{1/2} \eta\left(\tau\right), \tag{A.15}$$

with $s(d, c)$ the Dedekind sum

$$s(d, c) = \sum_{\ell=1}^{c-1} \frac{\ell}{c} \overline{B}_1 \left(d\frac{\ell}{c} \right) . \tag{A.16}$$

Since the $\gcd(m, n) = 1$, there exist integers a, b such that $an - bm = 1$. Now we use (A.15) with $(c, d) = (m, n)$. Noting that $a\tau + b = a(\widetilde{\tau} - n)/m + b = a\widetilde{\tau}/m - 1/m$, we obtain

$$(q; q) \simeq \frac{1}{\sqrt{-\mathrm{i}\widetilde{\tau}}} \exp\left(-\frac{2\pi\mathrm{i}}{24m\widetilde{\tau}} - \frac{2\pi\mathrm{i}\,\widetilde{\tau}}{24m} + \mathrm{i}\pi s(n, m) \right) , \tag{A.17}$$

in the limit of our interest. Our Dedekind sum is explicitly

$$s(n, m) = \sum_{\ell=1}^{m-1} \frac{\ell}{m} \overline{B}_1 \left(n\frac{\ell}{m} \right) . \tag{A.18}$$

To obtain an estimate for the elliptic gamma function we first note the identity [45]

$$\Gamma(\zeta; q, q) = \prod_{\ell=0}^{2(m-1)} \Gamma(\zeta q^\ell; q^m, q^m)^{m - |\ell - (m-1)|} = \prod_{\ell=0}^{2(m-1)} \Gamma(\zeta e^{-2\pi \mathrm{i} n\frac{\ell}{m}} \widetilde{q}^{\frac{\ell}{m}}; \widetilde{q}, \widetilde{q})^{m - |\ell - (m-1)|}, \tag{A.19}$$

with $\widetilde{q} = e^{2\pi\mathrm{i}\widetilde{\tau}}$.

Using (A.2) on the right-hand side of (A.19) we get

$$\frac{1}{2\pi\mathrm{i}} \log \Gamma_{\mathrm{e}}(z)$$

$$\sim -\frac{1}{6\widetilde{\tau}^2} \left(\sum_{\ell=1}^{m-1} \ell \left(\overline{B}_3 \left(z + \frac{n}{m} - n\frac{\ell}{m} \right) + \overline{B}_3 \left(z + \frac{n}{m} + n\frac{\ell}{m} \right) \right) + m \overline{B}_3 \left(z + \frac{n}{m} \right) \right)$$

$$- \frac{1}{2\widetilde{\tau}} \left(\sum_{\ell=1}^{n-1} \ell \left(\left(\frac{\ell-1}{m} - 1 \right) \overline{B}_2 \left(z + \frac{n}{m} - n\frac{\ell}{m} \right) + \left(\frac{2m-\ell-1}{m} - 1 \right) \overline{B}_2 \left(z + \frac{n}{m} + n\frac{\ell}{m} \right) \right) \right.$$

$$\left. + m \left(\frac{m-1}{m} - 1 \right) \overline{B}_2 \left(z + \frac{n}{m} \right) \right)$$

$$- \frac{1}{2} \left(\sum_{\ell=1}^{m-1} \ell \left(\left(\left(\frac{\ell-1}{m} - 1 \right)^2 - \frac{1}{6} \right) \overline{B}_1 \left(z + \frac{n}{m} - n\frac{\ell}{m} \right) + \left(\left(\frac{2m-\ell-1}{m} - 1 \right)^2 - \frac{1}{6} \right) \overline{B}_1 \left(z + \frac{n}{m} + n\frac{\ell}{m} \right) \right) \right.$$

$$\left. + m \left(\left(\frac{m-1}{m} - 1 \right)^2 - \frac{1}{6} \right) \overline{B}_1 \left(z + \frac{n}{m} \right) \right)$$

$$- \widetilde{\tau} \left(\sum_{\ell=1}^{m-1} \ell \left(\frac{1}{6} \left(\frac{\ell-1}{m} - 1 \right)^3 - \frac{1}{12} \left(\frac{\ell-1}{m} - 1 \right) + \frac{1}{6} \left(\frac{2m-\ell-1}{m} - 1 \right)^3 - \frac{1}{12} \left(\frac{2m-\ell-1}{m} - 1 \right) \right) \right.$$

$$\left. + m \left(\frac{1}{6} \left(\frac{m-1}{m} - 1 \right)^3 - \frac{1}{12} \left(\frac{m-1}{m} - 1 \right) \right) \right) . \tag{A.20}$$

Now using the identity,[26] for $\gcd(m, n) = 1$, $k > 1$,

$$\sum_{\ell=1}^{m-1} \ell \left(\overline{B}_k \left(x - n\frac{\ell}{m} \right) + \overline{B}_k \left(x + n\frac{\ell}{m} \right) \right) + m \overline{B}_k(x) = \frac{1}{m^{k-2}} \overline{B}_k(mx), \tag{A.21}$$

---

[26]See equation (4.54) in [13] or equation (3.12) in [37]. A simple proof is possible via (A.3).

and (A.3), we can simplify (A.20) to

$$\Gamma_e(z) \simeq \exp\left(-\frac{2\pi i}{m}\left(\frac{\overline{B}_3(mz)}{6\widetilde{\tau}^2} - \frac{\overline{B}_2(mz)}{2\widetilde{\tau}} + C(m,n,z) - \frac{1}{12}\widetilde{\tau}\right)\right),\qquad(A.22)$$

for $mz \in \mathbb{R}\setminus\mathbb{Z}$, as $\widetilde{\tau}\to 0$. Here $C(m,n,z)$ stands for ($-m$ times) the fourth and fifth lines of the right-hand side of (A.20).

Generalizing the above derivation in a straightforward manner leads to

$$\Gamma_e(z+r\tau) \simeq \exp\left(-\frac{2\pi i}{m}\left(\frac{\overline{B}_3(mz-nr)}{6\widetilde{\tau}^2} + (r-1)\frac{\overline{B}_2(mz-nr)}{2\widetilde{\tau}} + C(m,n,z,r) + \frac{(r-1)^3 - \frac{r-1}{2}}{6}\widetilde{\tau}\right)\right),$$
$$(A.23)$$

for $r \in \mathbb{R}$. This is the analog of (A.2) for $\widetilde{\tau}\to 0$.

The explicit expression for $C(m,n,z,r)$ is

$$\begin{aligned}C(m,n,z,r) = -\frac{m}{2}\Bigg[&\sum_{\ell=1}^{m-1}\ell\bigg(\bigg(\bigg(\frac{\ell+r-1}{m}-1\bigg)^2 - \frac{1}{6}\bigg)\overline{B}_1\bigg(z+\frac{n}{m}-n\frac{\ell+r}{m}\bigg)\\ &+\bigg(\bigg(\frac{2m-\ell+r-1}{m}-1\bigg)^2 - \frac{1}{6}\bigg)\overline{B}_1\bigg(z+\frac{n}{m}+n\frac{\ell-r}{m}\bigg)\bigg)\\ &+m\bigg(\bigg(\frac{m+r-1}{m}-1\bigg)^2 - \frac{1}{6}\bigg)\overline{B}_1\bigg(z+\frac{n}{m}-\frac{nr}{m}\bigg)\Bigg].\end{aligned}\qquad(A.24)$$

The estimate (A.23) is important to derive our results for the asymptotic expansion of the index near the roots of unity. It is valid uniformly over compact subsets of $z \in \mathbb{R}\setminus\frac{\mathbb{Z}}{m}$, because using (A.2) on the right-hand side of (A.19) is allowed only if $z - n\frac{l}{m}\notin\mathbb{Z}$ for $\ell = 0,\ldots,m-1$.

We can also use (A.19) for z near 0. More precisely, on the r.h.s. of (A.19), for fixed $z \in (-\frac{1}{m},\frac{1}{m})$, we can use (A.13) for the $\ell = 0, m$ terms, and use (A.2) for all other ℓ. With the aid of the "reflection formula"

$$\Gamma_h\left(x + \frac{\omega_1+\omega_2}{2};\omega_1,\omega_2\right)\Gamma_h\left(-x + \frac{\omega_1+\omega_2}{2};\omega_1,\omega_2\right) = 1.\qquad(A.25)$$

which gets rid of the hyperbolic gammas arising from $\ell = m$, and using the "product formula" (A.10) to trade the hyperbolic gammas arising from $\ell = 0$ for hyperbolic sines, we obtain

$$\frac{1}{\Gamma_e(z)\Gamma_e(-z)} \simeq \exp\left(-\frac{4\pi i}{m}\widetilde{R}_0^+(z;\widetilde{\tau}) + 4\pi i s(n,m)\right)4\sinh^2\left(\frac{\pi z}{-i\widetilde{\tau}}\right),\qquad(A.26)$$

where

$$\widetilde{R}_0^+(z;\widetilde{\tau}) := \frac{m^2 z^2}{2\widetilde{\tau}} + \frac{1}{12\widetilde{\tau}} + \frac{\widetilde{\tau}}{12}.\qquad(A.27)$$

This is the analog of (A.13)–(A.12) for $\widetilde{\tau}\to 0$, and is similarly useful (i.e. uniformly valid) in a neighborhood of $z = 0$.

An estimate similar to (A.26) for z near general nonzero $\frac{\mathbb{Z}}{m}$ can be obtained as well. We focus for simplicity on the $n = 1$ case (i.e. $\tau \to -\frac{1}{m}$). We write $z = \ell_0/m + z'$ and appeal

to (A.19). We have to use the estimate (A.8) for $\ell = \ell_0, \ell_0 + m$, and the estimate (A.2) for all other ℓ in the product (A.19) for $\Gamma_e(z)$. Similarly we have to use the estimate (A.8) for $\ell = -\ell_0 + m, -\ell_0 + 2m$, and the estimate (A.2) for all other ℓ in the product for $\Gamma_e(-z)$. The result is (up to a constant phase that we suppress)

$$
\frac{1}{\Gamma_e(z)\Gamma_e(-z)} \simeq \exp\left(-2\pi i \left[\frac{\frac{1}{6} + m^2 z'^2}{m\widetilde{\tau}} - \frac{1}{2} + \frac{\widetilde{\tau}}{6m}\right]\right) \times
$$
$$
\left[\Gamma_h\left(z' + \frac{\ell_0}{m}\widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau}\right)^{\ell_0+1} \Gamma_h\left(-z' + \frac{m-\ell_0}{m}\widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau}\right)^{m-\ell_0+1}
$$
$$
\Gamma_h\left(z' + \frac{\ell_0 + m}{m}\widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau}\right)^{m-\ell_0-1} \Gamma_h\left(-z' + \frac{2m-\ell_0}{m}\widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau}\right)^{\ell_0-1}\right]^{-1}.
$$

(A.28)

Using the reflection formula (A.25) we can simplify the above product of the hyperbolic gamma functions to find (up to the neglected constant phase)

$$
\frac{1}{\Gamma_e(z)\Gamma_e(-z)} \simeq \exp\left(-2\pi i \left[\frac{\frac{1}{6} + m^2 z'^2}{m\widetilde{\tau}} - \frac{1}{2} + \frac{\widetilde{\tau}}{6m}\right]\right) \times
$$
$$
\left[\Gamma_h\left(z' + \frac{\ell_0}{m}\widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau}\right) \Gamma_h\left(-z' + \frac{m-\ell_0}{m}\widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau}\right)\right]^{-2}.
$$

(A.29)

Now we use[27]

$$
[\Gamma_h(x, \widetilde{\tau}, \widetilde{\tau}) \Gamma_h(-x + \widetilde{\tau}, \widetilde{\tau}, \widetilde{\tau})]^{-2} = -4\sinh^2\left(\frac{\pi x}{-i\widetilde{\tau}}\right),
$$

(A.30)

to simplify (A.29) to (up to the neglected constant phase)

$$
\frac{1}{\Gamma_e(z)\Gamma_e(-z)} \simeq \exp\left(-2\pi i \left[\frac{\frac{1}{6} + m^2 z'^2}{m\widetilde{\tau}} + \frac{\widetilde{\tau}}{6m}\right]\right) 4\sinh^2\left(\frac{\pi\left(z' + \frac{\ell_0}{m}\widetilde{\tau}\right)}{-i\widetilde{\tau}}\right).
$$

(A.31)

B   Supersymmetric three-dimensional Chern-Simons actions

In this appendix we present the bosonic part of supersymmetrized three-dimensional Chern-Simons actions. We work in the context of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory coupled to off-shell three-dimensional supergravity. We first collect all allowed Chern-Simons terms including background and dynamical gauge fields, and then write the corresponding supersymmetrizations, following the presentation of appendix A of [26]. We then evaluate the actions for the field configurations that we consider in section 3.

The CS terms have the form

$$
\frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathcal{A}_x \wedge d\mathcal{A}_y,
$$

(B.1)

where x and y run over all possible gauge fields and with a coupling that we discuss below. Below we present the bosonic parts of the supersymmetric completions of the various cases

---

[27]This relation can be proven using the reflection formula (A.25) together with $\Gamma_h(x + \tau, \tau, \tau) = 2\sin\left(\frac{\pi x}{\tau}\right)\Gamma_h(x, \tau, \tau)$.

x-y, following [26, 56]. Firstly we have the gauge-gauge and gauge-R CS terms,

$$s_{\text{g-g}}^{ij} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\epsilon^{\mu\nu\rho} \mathcal{A}_\mu^i \partial_\nu \mathcal{A}_\rho^j + 2\mathrm{i}\, \mathcal{D}^i \sigma^j \right) ,$$

$$s_{\text{g-R}}^i = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\epsilon^{\mu\nu\rho} \mathcal{A}_\mu^i \partial_\nu \left(\mathcal{A}_\rho^{(R)} - \tfrac{1}{2} v_\rho \right) + \mathrm{i} \mathcal{D}^i H + \mathrm{i} \frac{\sigma^i}{4} \left(R^{(3)} + 2 v_\mu v^\mu + 2 H^2 \right) \right) .$$

(B.2)

Here i, j are Cartan labels for the gauge group, $\mathcal{A}$ is the three-dimensional gauge field, $\mathcal{D}$ is the D-term auxiliary scalar of the three-dimensional $\mathcal{N} = 2$ vector multiplet, σ is the (Coulomb branch) scalar of the three-dimensional $\mathcal{N} = 2$ vector multiplet, $\mathcal{A}^{(R)}$ is the three-dimensional background gauge field for the R current, $v^\mu = -\mathrm{i}\, \varepsilon^{\mu\nu\rho} \partial_\nu c_\rho$ is the Hodge dual of the graviphoton c_μ (the background $\mathrm{U}(1)_{\text{KK}}$ gauge field), H is the scalar in the supergravity multiplet, and $R^{(3)}$ is the Ricci scalar of $\mathcal{M}_3$.

Then we have the background R-R CS terms and the gravitational CS term for the spin connection ω,

$$R_{\mathcal{M}_3} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\epsilon^{\mu\nu\rho} \left(\mathcal{A}_\mu^{(R)} - \tfrac{1}{2} v_\mu \right) \partial_\nu \left(\mathcal{A}_\rho^{(R)} - \tfrac{1}{2} v_\rho \right) + \mathrm{i} \frac{H}{2} \left(R^{(3)} + 2 v_\mu v^\mu + 2 H^2 \right) \right) ,$$

$$G_{\mathcal{M}_3} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\epsilon^{\mu\nu\rho} \mathrm{Tr} \left(\omega_\mu \partial_\nu \omega_\rho - \tfrac{2}{3} \omega_\mu \omega_\nu \omega_\rho \right) + 4 \epsilon^{\mu\nu\rho} \left(\mathcal{A}_\mu^{(R)} - \tfrac{3}{2} v_\mu \right) \partial_\nu \left(\mathcal{A}_\rho^{(R)} - \tfrac{3}{2} v_\rho \right) \right) .$$

(B.3)

Finally we have the CS actions involving the graviphoton. These are gauge-KK, R-KK, and KK-KK, whose bosonic parts read [26]

$$s_{\text{g-KK}}^i = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\epsilon^{\mu\nu\rho} \mathcal{A}_\mu^i \partial_\nu c_\rho - \mathrm{i} \mathcal{D}^i + \mathrm{i} \sigma^i H \right) ,$$

$$s_{\text{R-KK}} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\mathrm{i}\, v^\mu \left(\mathcal{A}_\mu^{(R)} - \frac{1}{2} v_\mu \right) \right) - \frac{\mathrm{i}}{2} v^\mu v_\mu + \frac{\mathrm{i}}{2} H^2 - \frac{\mathrm{i}}{4} R^{(3)} ,$$

$$s_{\text{KK-KK}} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\mathrm{i}\, v^\mu c_\mu - 2\, \mathrm{i}\, H \right) = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \left(\epsilon^{\mu\nu\rho} c_\mu \partial_\nu c_\rho - 2\, \mathrm{i}\, H \right) .$$

(B.4)

The equations (B.2), (B.3), (B.4) together make up the complete list of CS terms. As we explain below, when we have a KK reduction these actions can be combined together into a succinct expression in a natural manner.

The coefficients of the above actions are obtained by calculating the coefficients of the CS pieces, which are obtained by integrating out all massive fermions that couple to the corresponding gauge fields. Integrating out a fermion f with real-mass m_{f} and charges $e_x^{\text{f}}, e_y^{\text{f}}$ under the gauge fields $\mathcal{A}_x, \mathcal{A}_y$ generates the term $\frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathcal{A}_x \wedge \mathrm{d}\mathcal{A}_y$ with coefficient given by the one-loop exact formula (we follow the conventions of [25])

$$- \frac{\mathrm{i}\pi}{8} \sum_{\text{f}} \mathrm{sgn} \left(m_{\text{f}} \right) e_x^{\text{f}} e_y^{\text{f}} .$$

(B.5)

The contribution of the fermion to the coefficient of the gravitational CS term is given by (see appendix A of [56])

$$- \frac{\mathrm{i}\, \pi}{192} \sum_{\text{f}} \mathrm{sgn}(m_{\text{f}}) .$$

(B.6)

The full effective action of the theory is the sum of the actions (B.2), (B.3) (B.4) with coefficients obtained by summing (B.5), (B.6) over all the massive fermions in the theory. (The actions with x and y different appear twice in the final action — as x-y and y-x — and therefore need to be multiplied by a factor of two.)

The situation of interest in section 3 is the Kaluza-Klein reduction of a four-dimensional theory on a circle of radius R. The bosonic fields in the three-dimensional vector multiplet are written in term of the 4d fields in (3.19), and the three-dimensional fermions are the reduction of the corresponding 4d fermions. Consider a fermion of R-charge $r_{\rm f}$ transforming in a representation of weight $\rho_{\rm f}$ under the gauge group. The tree-level real-mass of a KK mode of this fermion is given by (in the convention of [25])

$$m_{\rm f} = -\left(\rho_{\rm f} \cdot A_Y - p_Y - r_{\rm f} A_Y^{\rm nm} - \frac{1}{2} V_Y^{\rm nm} \right) , \tag{B.7}$$

where $A^{\rm nm}$, $V^{\rm nm}$ are the 4d background R-gauge fields given in (3.1). Note that p_Y also enters (B.5) as the charge of the fermion under $\mathrm{U}(1)_{\rm KK}$.[28] Since we take $R \to 0$ at the end of the calculations, it is enough to keep only the singular pieces in the formula (B.7).

Using these relations we proceed to write the three-dimensional effective action directly in terms of the dynamical 4d fields. The contribution of the actions coming from (B.2) and (B.4) to the full effective action can be written as the sum of the following two actions,

$$\widetilde{S}_{\text{g-g}}^{\rm f} = -\mathrm{i}\pi \frac{\text{sgn}\,(m_{\rm f})}{8} \left(\rho_{\rm f}^i \rho_{\rm f}^j s_{\text{g-g}}^{ij} + 2\, p_Y\, \rho_{\rm f}^i s_{\text{g-KK}}^i + p_Y^2\, s_{\text{KK-KK}} \right)$$

$$= -\mathrm{i}\pi \frac{\text{sgn}\,(m_{\rm f})}{8} \Big((\rho_{\rm f} \cdot A_Y - p_Y)^2 \int_{\mathcal{M}_3} \mathrm{d}^3 x\, \sqrt{g}\, (\mathrm{i}\, v^\mu c_\mu - 2\mathrm{i}\, H)$$

$$+ 2\,(\rho_{\rm f} \cdot A_Y - p_Y) \int_{\mathcal{M}_3} \mathrm{d}^3 x\, \sqrt{g}\, (-\mathrm{i}\, v^\mu\, (\rho_{\rm f} \cdot A_\mu) + \mathrm{i}\, (\rho_{\rm f} \cdot D))$$

$$+ \int_{\mathcal{M}_3} \mathrm{d}^3 x\, \sqrt{g}\, (\epsilon^{\mu\nu\rho}\, (\rho_{\rm f} \cdot A_\mu)\, \partial_\nu\, (\rho_{\rm f} \cdot A_\rho)) , \tag{B.8}$$

$$2\, \widetilde{S}_{\text{g-R}}^{\rm f} = -2\mathrm{i}\pi \frac{\text{sgn}\,(m_{\rm f})}{8}\, r_{\rm f} \left(\rho_{\rm f}^i s_{\text{g-R}}^i + p_Y\, s_{\text{R-KK}} \right)$$

$$= -2\mathrm{i}\pi \frac{\text{sgn}\,(m_{\rm f})}{8}\, r_{\rm f}$$

$$\left((\rho_{\rm f} \cdot A_Y - p_Y) \int_{\mathcal{M}_3} \mathrm{d}^3 x\, \sqrt{g}\, \left(-\mathrm{i}\, v^\mu \left(\mathcal{A}_\mu^{(R)} - \frac{1}{2} v_\mu \right) + \mathrm{i}\, \frac{1}{2} v^\mu\, v_\mu - \mathrm{i}\, \frac{1}{2} H^2 + \mathrm{i}\, \frac{1}{4} R^{(3)} \right) \right.$$

$$\left. + \int_{\mathcal{M}_3} \mathrm{d}^3 x\, \sqrt{g}\, \left(\epsilon^{\mu\nu\rho}\, (\rho_{\rm f} \cdot A_\mu)\, \partial_\nu \left(\mathcal{A}_\rho^{(R)} - \frac{1}{2} v_\rho \right) + \mathrm{i}\, (\rho_{\rm f} \cdot D)\, H \right) \right). \tag{B.9}$$

---

[28]In the Euclidean context that we discuss here, the background fields $A^{\rm nm}$, $V^{\rm nm}$, and the effective radius R are complex. The definition (B.7) is thought of as an analytic continuation, and is read off from the coupling of the fermion to the other fields and parameters in the component of the covariant derivative along the KK direction Y. The quantity $m_{\rm f}$ appears in the main text in formulas for the one-loop correction From the expression e.g. (3.26), we see that it appears only as the function $\text{sgn}(m_{\rm f})$. Then the sgn function needs to be appropriately defined. As discussed below equation (3.28), we can do that by defining it for real R, and then continuing the formulas to complex R.

Finally we specialize to the BPS configurations considered in the main text, given in (3.21). The above two terms take the following value

$$
\widetilde{S}_{\text{g-g}}^{\text{f}} = -\mathrm{i}\pi \, \frac{\text{sgn}(m_{\text{f}})}{8R^2} \left(\rho_{\text{f}} \cdot \underline{u} - k_Y \right)^2 A_{\mathcal{M}_3} \,,
$$
$$
2 \, \widetilde{S}_{\text{g-R}}^{\text{f}} = -\mathrm{i}\pi \, \frac{\text{sgn}(m_{\text{f}})}{8R} \, 2 \, r_{\text{f}} \left(\rho_{\text{f}} \cdot \underline{u} - k_Y \right) L_{\mathcal{M}_3} \,,
$$

$$(\text{B.10})$$

where $A_{\mathcal{M}_3}$ and $L_{\mathcal{M}_3}$ are functions of the three-dimensional background,[29]

$$
A_{\mathcal{M}_3} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \, \sqrt{g} \, (\mathrm{i} \, v^\mu \, c_\mu - 2 \, \mathrm{i} \, H) \,,
$$
$$
L_{\mathcal{M}_3} = \frac{1}{\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \, \sqrt{g} \, \left(-\mathrm{i} \, v^\mu \mathcal{A}_\mu^{(R)} + \mathrm{i} \, v^\mu \, v_\mu - \mathrm{i} \, \frac{1}{2} H^2 + \mathrm{i} \, \frac{1}{4} R^{(3)} \right) \,.
$$

$$(\text{B.11})$$

We now turn to the remaining terms in the full action, namely those coming from the terms in (B.3),

$$
S_{\text{R-R}}^{\text{f}} = -\mathrm{i}\pi \, \frac{\text{sgn}\,(m_{\text{f}})}{8} \left(r_{\text{f}}^2 - \frac{1}{6} \right) R_{\mathcal{M}_3} \,,
$$
$$
S_{\text{grav}}^{\text{f}} = -\mathrm{i}\pi \, \frac{\text{sgn}\,(m_{\text{f}})}{192} \, G_{\mathcal{M}_3} \,.
$$

$$(\text{B.12})$$

Note that both $R_{\mathcal{M}_3}$ and $G_{\mathcal{M}_3}$ contain $\mathcal{A}^{(R)} \wedge \mathrm{d}\mathcal{A}^{(R)}$ terms, and it is the sum of the corresponding coefficients that is fixed by (B.5). Since the coefficient in $S_{\text{grav}}^{\text{f}}$ is fixed by (B.6), the shift $-1/6$ in the coefficient of $S_{\text{R-R}}^{\text{f}}$ serves to cancel the $\mathcal{A}^{(R)} \wedge \mathrm{d}\mathcal{A}^{(R)}$ term coming from $S_{\text{grav}}^{\text{f}}$. The final result for the action of the BPS configurations up to $\mathrm{O}(R^0)$ obtained by integrating out a fermion f is given by the sum of the actions in (B.10), (B.12).

C Values of supersymmetrized Chern-Simons actions

In this appendix we record the values of various terms in the supersymmetrized actions of appendix B evaluated on the configurations discussed in section 3. We first recall from section 3 the values of the various fields entering the actions. The three-dimensional metric is

$$
\mathrm{d}s_3^2 = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi_1^2 + \cos^2\theta \, \mathrm{d}\phi_2^2 - c^2 \,,
$$

$$(\text{C.1})$$

the graviphoton and its Hodge dual are

$$
c = -\mathrm{i}\frac{\Omega}{\sqrt{1-\Omega^2}} \left(\sin^2\theta \, \mathrm{d}\phi_1 + \cos^2\theta \, \mathrm{d}\phi_2 \right) \,, \qquad v = \frac{2 \, \mathrm{i}}{\sqrt{1-\Omega^2}} \, c \,.
$$

$$(\text{C.2})$$

The auxiliary background supergravity multiplet fields are

$$
H = -\frac{\mathrm{i}}{\sqrt{1-\Omega^2}} \,, \qquad \mathcal{A}_\mu^R = \left(\frac{\tau}{R} - \frac{n_0}{2R} + \frac{\mathrm{i}}{\sqrt{1-\Omega^2}} \right) c_\mu \,.
$$

$$(\text{C.3})$$

The four-dimensional gauge fields are

$$
A_Y^i = \frac{u_i}{R} \,, \qquad D^i = 0 \,,
$$

$$(\text{C.4})$$

_____

[29]To compare with [26] note that $A_{\mathcal{M}_3} = -A_{\mathcal{M}_3}^{\text{there}}$ and $L_{\mathcal{M}_3} = \mathrm{i} \, L_{\mathcal{M}_3}^{\text{there}}$.

The Chern-Simons action for c

$$S^{\mathrm{CS}}(c) = \int_{\mathcal{M}_3} c \wedge \mathrm{d}c = \mathrm{i} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \, v^\mu c_\mu \,, \tag{C.5}$$

evaluates to

$$\frac{1}{4\pi^2} S^{\mathrm{CS}}(c) == \frac{\Omega^2}{1 - \Omega^2} = -1 + \mathrm{O}(\gamma) \,. \tag{C.6}$$

The other building blocks for the actions of the background fields in the three-dimensional theory are given below, including their limiting behavior as $\gamma \to 0$ with τ fixed (i.e. as $\Omega \to \infty$),

$$\frac{1}{4\pi^2} S^{(H)} = \frac{\mathrm{i}}{4\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \, H = \frac{1}{2(1 - \Omega^2)} = \mathrm{O}(\gamma) \,, \tag{C.7}$$

$$\frac{1}{4\pi^2} S^{(v)} = \frac{1}{4\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \, v^\mu v_\mu = \frac{\Omega^2}{(1 - \Omega^2)^{\frac{3}{2}}} = \mathrm{O}(\gamma) \,, \tag{C.8}$$

$$\frac{1}{4\pi^2} S^{(H^2)} = \frac{1}{4\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \, H^2 = -\frac{1}{2(1 - \Omega^2)^{\frac{3}{2}}} = \mathrm{O}(\gamma) \,, \tag{C.9}$$

$$\frac{1}{4\pi^2} S^{(R)} = \frac{1}{4\pi^2} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{g} \, R^{(3)} = -\frac{-6 + 8\Omega^2}{2(1 - \Omega^2)^{\frac{3}{2}}} = \mathrm{O}(\gamma) \,. \tag{C.10}$$

D   Dimensional reduction for the case $\Omega_1 \neq \Omega_2$

We begin by writing the background configuration in (3.1) as a KK compactification to three dimensions, i.e. a circle fibration on a 3-manifold M_3. We have

$$\mathrm{d}s_4^2 = \mathrm{d}s_3^2 + \mathrm{e}^{2\phi} \left(\mathrm{d}t_E + \widetilde{c} \right)^2 \,, \tag{D.1}$$

where the metric on M_3 is

$$\mathrm{d}s_3^2 = \widetilde{g}_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu = \mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi_1^2 + \cos^2 \theta \, \mathrm{d}\phi_2^2 - \mathrm{e}^{2\phi} \, \widetilde{c}^2 \,, \tag{D.2}$$

and the graviphoton and KK scalar are

$$\begin{aligned}
\mathrm{e}^{2\phi} &= 1 - \Omega_1^2 \sin^2 \theta - \Omega_2^2 \cos^2 \theta \,, \\
\widetilde{c} &= \widetilde{c}_\mu \, \mathrm{d}x^\mu = -\mathrm{i} \, \mathrm{e}^{-2\phi} \left(\Omega_1 \sin^2 \theta \, \mathrm{d}\phi_1 + \Omega_2 \cos^2 \theta \, \mathrm{d}\phi_2 \right) \,.
\end{aligned} \tag{D.3}$$

For the case $\Omega_1 = \Omega_2 = \Omega$, we have that $e^{2\phi} = 1 - \Omega^2$, so that the graviphoton c defined in (3.9) is related to $\widetilde{c}$ as $c = \mathrm{e}^\phi \, \widetilde{c}$. The magnitude of the volume form in three dimensions is

$$\sqrt{\widetilde{g}} = \frac{1}{2} \mathrm{e}^{-\phi} \sin 2\theta \,. \tag{D.4}$$

The associated Chern-Simons action

$$S^{\mathrm{CS}}(\widetilde{c}) = \int_{\mathcal{M}_3} \widetilde{c} \wedge \mathrm{d}\widetilde{c} = \mathrm{i} \int_{\mathcal{M}_3} \mathrm{d}^3 x \sqrt{\widetilde{g}} \, \widetilde{v}^\mu \widetilde{c}_\mu \tag{D.5}$$

(where $\widetilde{v} = -\mathrm{i} * \mathrm{d}\widetilde{c}$ is the Hodge dual) evaluates to

$$S^{\mathrm{CS}}(\widetilde{c}) = 4\pi^2 \,\Omega_1\,\Omega_2 \int_0^{\pi/2} \frac{\sin 2\theta}{\left(1 - \Omega_1^2 \sin^2\theta - \Omega_2^2 \cos^2\theta\right)^2} \,\mathrm{d}\theta = 4\pi^2 \,\frac{\Omega_1\Omega_2}{\left(1 - \Omega_1^2\right)\left(1 - \Omega_2^2\right)}\,. \qquad \text{(D.6)}$$

For the identification between the four-dimensional and the three-dimensional fields, we follow the treatment of [34] applied to the metric (D.1). The result is

$$\frac{1}{2}\,\mathrm{e}^\phi\,\widetilde{v}_\mu = V_\mu^{\mathrm{nm}} - V_{t_E}^{\mathrm{nm}}\,\widetilde{c}_\mu\,, \qquad \widetilde{H} = \mathrm{e}^{-\phi}\,V_{t_E}\,, \qquad \mathcal{A}_\mu^R = A_\mu^{\mathrm{nm}} - A_{t_E}^{\mathrm{nm}}\,\widetilde{c}_\mu + \frac{1}{2}\,\mathrm{e}^\phi\,\widetilde{v}_\mu\,. \quad \text{(D.7)}$$

The values of these fields are

$$\mathrm{e}^\phi\,\widetilde{v}_\mu = -2\,V_Y^{\mathrm{nm}}\,c_\mu = 2\,\mathrm{i}\,\widetilde{c}_\mu\,, \qquad \widetilde{H} = -\mathrm{i}\,e^{-\phi}\,, \qquad \text{(D.8)}$$

$$\mathcal{A}_\mu^R = \mathrm{i}\left(\frac{1}{2}\left(\Omega_1 + \Omega_2\right) - 1\right)\widetilde{c}_\mu + \frac{1}{2}\,\mathrm{e}^\phi\,\widetilde{v}_\mu = \frac{\mathrm{i}}{2}\left(\Omega_1 + \Omega_2\right)\widetilde{c}_\mu\,. \qquad \text{(D.9)}$$

We can now calculate the various actions as in appendix C, and we find that, in the $\gamma \to 0$, $\Omega \to \infty$ limit we have the effective replacement

$$\frac{1}{R^2} \to \frac{1}{\tau\,\sigma}\,, \qquad \frac{1}{R} \to \frac{\tau + \sigma}{2\tau\,\sigma} \qquad \text{(D.10)}$$

in the effective potential (3.32).

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