Published for SISSA by Springer

Received: June 10, 2019
Revised: August 24, 2019
Accepted: September 10, 2019
Published: October 7, 2019

Crossing, modular averages and $N \leftrightarrow k$ in WZW models

Ratul Mahanta and Anshuman Maharana

Harish-Chandra Research Institute, HBNI,
Chhatnag Road, Jhunsi, Allahabad, 211019, India

E-mail: ratulmahanta@hri.res.in, anshumanmaharana@hri.res.in

Abstract: We consider the construction of genus zero correlators of $\mathrm{SU}(N)_k$ WZW models involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation from modular averaging of the contribution of the vacuum conformal block. We perform the averaging by two prescriptions — averaging over the stabiliser group associated with the correlator and averaging over the entire modular group. For the first method, in cases where we find the orbit of the vacuum conformal block to be finite, modular averaging reproduces the exact result for the correlators. In other cases, we perform the modular averaging numerically, the results are in agreement with the exact answers. Construction of correlators from averaging over whole of the modular group is more involved. Here, we find some examples where modular averaging does not reproduce the correlator. We find a close relationship between the modular averaging sums of the theories related by level-rank duality. We establish a one to one correspondence between elements of the orbits of the vacuum conformal blocks of dual theories. The contributions of paired terms to their respective correlators are simply related. One consequence of this is that the ratio between the OPE coefficients associated with dual correlators can be obtained analytically without performing the sums involved in the modular averagings. The pairing of terms in the modular averaging sums for dual theories suggests an interesting connection between level-rank duality and semi-classical holographic computations of the correlators in the theories.

Keywords: Conformal and W Symmetry, Conformal Field Theory

ArXiv ePrint: 1905.02816

https://doi.org/10.1007/JHEP10(2019)061

JHEP10(2019)061

Contents

1 Introduction

The bootstrap [1, 2] serves as an extremely useful tool in the study of conformal field theories (see [3–6] for reviews). An interesting direction of study is its interplay with duality symmetries. For example, in [7] it was found that S-duality invariant points of N=4 supersymmetric Yang-Mill saturate the bootstrap bounds on the anomalous dimensions of low twist non-BPS operators, in [8] it was found that crossing has interesting implications for the structure of the S-matrix in Chern Simons theories with matter. Recently, a rather simple proposal has been put forward to generate crossing symmetric genus zero correlation functions in two dimensional conformal field theories [9]. In this paper, we construct correlation functions in $\mathrm{SU}(N)_k$ WZW models using the proposal and examine level-rank duality of the models in this context.

In two dimensions, crossing together with modular invariance has provided strong constraints from the early days [11–20]. For some recent developments in 2D bootstrap see [21]–[41], and in particular [42]–[48] for work on theories with currents. The basic idea in [9] is to make use of transformation properties of conformal blocks under crossing to arrive at crossing symmetric candidate correlation functions. Correlation functions are generated by starting from a seed contribution (as given by the contributions of conformal blocks of some primaries of low dimension running in the intermediate channel) and summing over the orbit of the seed under crossing transformations to obtain a crossing symmetric candidate correlation function. In two dimensions, crossing symmetry acts as the modular group on conformal blocks. Thus the sum over the orbit of the seed contribution corresponds to "modular averaging".[1] It was shown in [9] that modular averaging can be used to successfully compute genus zero four point functions of minimal models. Modular averaging has appeared in the physics literature in the context of three-dimensional quantum gravity and is often referred to as Farey tail sums (see e.g. [49–55]). It was argued in [9] that terms that arise from the orbit of the seed contribution would arise naturally in a semiclassical holographic AdS_3 dual computation of the CFT correlator.

Our focus will be on WZW correlators of [12], involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation. Here, we perform modular averaging by both the prescriptions given in [9] - averaging over the stabiliser subgroup of the correlator and over the entire modular group, mostly focussing on the first one (we review these prescription in section 2). For averaging performed using the stabilser group, we find that the correlators can be constructed from modular averaging of the contribution of the vacuum block in all the cases we examine. Primary examples of models where the sums can be done exactly are models with $N = k$ (the orbits for these models are finite). For models where we have not been able to show that the orbit is finite, we consider examples with specific values of N and k, and perform the averaging numerically. Construction of correlators from averaging over whole of the modular group is more involved. Here, we find some examples where modular averaging does not reproduce the correlator.

An interesting feature of WZW models is level-rank duality [56]. Dual primary fields under $N \leftrightarrow k$ are related by transposition of the Young tableaux of their representations. The correlators considered in this paper are the simplest related to each other by this duality. From the point of view of modular averaging, both N and k simply appear as parameters in the matrices associated with the action of the modular group on the conformal blocks. Thus modular averaging puts N and k in a more equal footing; one can hope that writing correlators as modular averages can reveal various aspects of level-rank duality. This expectation is borne out. We establish a one to one correspondence between elements of the orbits of the vacuum conformal blocks of dual theories. The contributions of paired terms to their respective correlators are simply related. This allows us to obtain the ratio between the OPE coefficients associated with dual correlators analytically without performing the sums involved in the modular averagings. The pairing of terms also indicates that holographic computations can make some properties of the level-rank duality manifest.

This paper is organised as follows. In section 2, we briefly review some basic ingredients that will be necessary for our analysis. In section 3 (and appendix A) we obtain the

[1] This is very similar in spirit to the proposal of [10] to compute partition functions from vacuum characters.

JHEP10(2019)061

transformation properties of the conformal blocks of the correlators under the action of the modular group. In section 4 (and appendices C, D) we compute correlators by modular averaging. In section 5, we examine level-rank duality.

2 Review

We start by recalling some basic facts about four point functions in two dimensional conformal field theories. We then go on to describe the proposal of [9] to construct crossing symmetric correlation functions from modular averaging.

The four-point correlator of operators O_1, O_2, O_3 and O_4 in 2D CFTs on the Riemann sphere can be written as the product of a factor that determines its transformation properties under global conformal transformations and a function of a conformally invariant cross-ratio. It will be our convention to take

$$\langle O_1(z_1,\bar{z}_1)O_2(z_2,\bar{z}_2)O_3(z_3,\bar{z}_3)O_4(z_4,\bar{z}_4)\rangle = G_0\big(z_a,\bar{z}_a\big)G_{1234}(x,\bar{x}) \tag{2.1}$$

with

$$G_0\big(z_a,\bar{z}_a\big) = \prod_{a<b}\big(z_{ab}^{\mu_{ab}}\cdot\bar{z}_{ab}^{\bar{\mu}_{ab}}\big), \tag{2.2}$$

where $z_{ab} = z_a - z_b$ $(a,b = 1..4)$, $\mu_{ab} = (\frac{1}{3}\sum_{c=1}^{4} h_c) - h_a - h_b$ (h_i being the dimensions of the operators O_i) and the cross-ratio

$$x = \frac{z_{12}z_{34}}{z_{14}z_{32}}. \tag{2.3}$$

Conformal transformations can be used to set z_2 to 0 and z_3 to 1 and set z_4 to infinity, the coordinate z_1 then corresponds to the cross-ratio. Thus the cross-ratio space is the Riemann sphere with three punctures.

Correlators in two dimensional CFTs can be constructed from holomorphic and anti-holomorophic conformal blocks. Although correlators need to be single valued functions of the cross-ratio space,[2] there is no such requirement on the conformal blocks. Conformal blocks have monodromies in the cross-ratio space. Thus it is natural to consider conformal blocks as functions in the universal covering space of the cross-ratio space. This is $\mathbb{H}_+ = \{u+iv \mid v>0 \text{ and } u,v\in\mathbb{R}\}$, the upper half plane.[3] The elliptic lambda function

$$\lambda(\tau) = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^4, \tag{2.4}$$

where $\tau = u+iv$ provides a surjective map ($x = \lambda(\tau)$) from $\mathbb{H}_+$ to the cross-ratio space [57]. $\mathrm{PSL}(2,\mathbb{Z})$ action on the upper half plane has a close connection to the map. Under the action of the generators of the modular group

$$T:\tau\to\tau+1 \quad\text{and}\quad S:\tau\to-\frac{1}{\tau}, \tag{2.5}$$

[2] We will be dealing with bosonic operators.

[3] The observation that conformal blocks should be single-valued on the upper half plane was made in [58], where an elliptic recursion representation was obtained for them.

images in the cross-ratio space have rather simple transformations

$$T \cdot x = \frac{x}{x-1} \quad \text{and} \quad S \cdot x = 1 - x. \tag{2.6}$$

Furthermore, the function $\lambda(\tau)$ is invariant under the normal subgroup $\Gamma(2)$ of $\mathrm{PSL}(2,\mathbb{Z})$:

$$\lambda(\gamma\tau) = \lambda(\tau),\ \forall\gamma \in \Gamma(2). \tag{2.7}$$

Thus, the condition that correlators have to be single valued in the cross-ratio space translates to invariance under $\Gamma(2)$ in $\mathbb{H}_+$.

At this stage, it is natural to seek for the interpretation of the action of the entire $\mathrm{PSL}(2,\mathbb{Z})$ on the correlators in the CFT. For this, one has to look at crossing symmetry. For a general ordering of the operators, we define

$$\langle O_p(z_p,\bar{z}_p)O_q(z_q,\bar{z}_q)O_r(z_r,\bar{z}_r)O_s(z_s,\bar{z}_s)\rangle = G_0(z_a,\bar{z}_a)G_{pqrs}(x_{pqrs},\bar{x}_{pqrs}), \tag{2.8}$$

with G_0 as defined in (2.2) and

$$x_{pqrs} = \frac{z_{pq}z_{rs}}{z_{ps}z_{rq}}. \tag{2.9}$$

Note that with this we have $x = x_{1234}$, where x is the cross-ratio introduced in (2.3). Our choice of G_0 is invariant under permutations of the operators $\{O_a(z_a)\}$ inside the correlator thus crossing symmetry reduces to the statement that $G_{abcd}(x_{abcd})$ is invariant under action of the same permutation on $\{a,b,c,d\}$ in both the subscripts. Permutations that leave the cross ratio x invariant yield:

$$G_{1234}(x,\bar{x}) = G_{2143}(x,\bar{x}) = G_{3412}(x,\bar{x}) = G_{4321}(x,\bar{x}). \tag{2.10}$$

On the other hand, permutations which act non-trivially on the cross-ratio[4] give

$$\begin{aligned} G_{1234}(x,\bar{x}) &= G_{1243}\left(\frac{x}{x-1},\frac{\bar{x}}{\bar{x}-1}\right) = G_{3241}\left(\frac{1}{1-x},\frac{1}{1-\bar{x}}\right) = G_{3214}\left(\frac{1}{x},\frac{1}{\bar{x}}\right) \\ &= G_{4231}(1-x,1-\bar{x}) = G_{4213}\left(\frac{x-1}{x},\frac{\bar{x}-1}{\bar{x}}\right). \end{aligned} \tag{2.11}$$

The arguments of the functions in (2.11) can be related by the actions of S and T as given in (2.6). The actions are isomorphic to the anharmonic group, S_3. This is precisely equal to $\mathrm{PSL}(2,\mathbb{Z})/\Gamma(2)$. Thus crossing symmetry and single valuedness[5] together specify the full $\mathrm{PSL}(2,\mathbb{Z})$ action on the correlators. Combining (2.6), (2.10) and (2.11) they can be written in a very compact form [9]:

$$\vec{G}(\gamma\tau,\gamma\bar{\tau}) = \sigma(\gamma)\cdot\vec{G}(\tau,\bar{\tau}), \qquad \gamma \in \mathrm{PSL}(2,\mathbb{Z}) \tag{2.12}$$

where

$$\vec{G} = (G_{1234}(\tau,\bar{\tau}),G_{2134}(\tau,\bar{\tau}),G_{4132}(\tau,\bar{\tau}),G_{1432}(\tau,\bar{\tau}),G_{2431}(\tau,\bar{\tau}),G_{4231}(\tau,\bar{\tau}))^t \tag{2.13}$$

[4]These relations differ from the ones in [9] since our choice for the cross-ratio x is different.

[5]Recall that correlators need to be invariant under $\Gamma(2)$ so that they are single valued.

JHEP10(2019)061

and $\sigma(\gamma)$ are the six dimensional matrices associated with the linear representation of $\mathrm{PSL}(2,\mathbb{Z})/\Gamma(2) = S_3$ with

$$\sigma(S) = \begin{pmatrix} 0&0&0&0&0&1\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \quad \text{and} \quad \sigma(T) = \begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0 \end{pmatrix}. \tag{2.14}$$

We note that there is further simplification when all or some of the operators O_a are identical. For instance, in the case that all the four operators are identical $\vec{G}$ has only one independent component. Equation (2.12) requires it to be a modular invariant scalar.

Modular averaging can be used to obtain solutions of equations of the form of (2.12). The general structure of four point functions in a CFT gives fiducial functions over which the averaging can be performed. Conformal invariance implies that the stripped correlators in (2.8) can be written as a sum over contributions associated with conformal primaries (ϕ_k):

$$G_{pqrs}(y,\bar{y}) = \sum_k C_{O_pO_q\phi_k} C_{O_rO_s\phi_k} \times y^{h_{\phi_k}-\frac{\mathfrak{H}}{3}} \bar{y}^{\bar{h}_{\phi_k}-\frac{\bar{\mathfrak{H}}}{3}} F^{\phi_k}_{pqrs}(y,\bar{y}), \tag{2.15}$$

where $C_{O_pO_q\phi_k}$, $C_{O_rO_s\phi_k}$ are three point structure constants, $\mathfrak{H} = (h_p + h_q + h_r + h_s)$ and $\bar{\mathfrak{H}} = (\bar{h}_p + \bar{h}_q + \bar{h}_r + \bar{h}_s)$. The functions $F^{\phi_k}_{pqrs}(y,\bar{y})$ are analytic at $y,\bar{y}=0$ and $F^{\phi_k}_{pqrs}(0,0) = 1$. It will be our convention to call $\{y^{h_{\phi_k}-\frac{\mathfrak{H}}{3}} \bar{y}^{\bar{h}_{\phi_k}-\frac{\bar{\mathfrak{H}}}{3}} F^{\phi_k}_{pqrs}(y,\bar{y})\}$ as the conformal block corresponding to primary ϕ_k. These can be further factorized into holomorphic and anti-holomorchic conformal blocks for each ϕ_k. Given the form of (2.15), in the limit of $y \to 0$ the stripped correlator is well approximated by including contributions from the low lying primaries that appear in the sum i.e.

$$\begin{aligned} G_{pqrs}(y,\bar{y}) &\approx G^{\text{light}}_{pqrs}(y,\bar{y}) \\ &= \sum_{k \le k_{\max}} C_{O_pO_q\phi_k} C_{O_rO_s\phi_k} \times y^{h_{\phi_k}-\frac{\mathfrak{H}}{3}} \bar{y}^{\bar{h}_{\phi_k}-\frac{\bar{\mathfrak{H}}}{3}} F^{\phi_k}_{pqrs}(y,\bar{y}) \quad \text{for} \quad y \to 0. \end{aligned} \tag{2.16}$$

where the sum now runs over primaries which have weights less than or equal to $(h_{k_{\max}}, \bar{h}_{k_{\max}})$. The simplest approximation is to keep only the primary with the lowest weight. Reference [9] proposed that modular averaging of $\vec{G}^{\text{light}}$ can be used to construct candidate CFT correlators which satisfy the requirements single-valuedness and crossing.

$$\vec{G}^{\text{candidate}}(\tau,\bar{\tau}) = \mathcal{N}^{-1} \cdot \sum_{\gamma \in \mathrm{PSL}(2,\mathbb{Z})} \sigma^{-1}(\gamma) \cdot \vec{G}^{\text{light}}(\gamma\tau, \gamma\bar{\tau}), \tag{2.17}$$

where $\mathcal{N}$ is a normalisation which can be determined from the $\tau \to i\infty$ $(y \to 0)$ behaviour of $\vec{G}(\tau,\bar{\tau})$. In general, the sum in (2.17) is difficult to perform and might even need regularisation. The complications associated with dealing with a sum involving vector valued modular objects can be ameliorated for correlators with identical operators. As described earlier, in the presence of identical operators, various components of $\vec{G}$ (as defined in (2.13)) become related - the vector space effectively collapses to a lower dimensional one. As a result, the subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ that leaves any particular component of the vector inert under action of $\sigma(\gamma)$ is enhanced.[6] If the subgroup associated with the component G_a in the collapsed vector space is Γ_a, a natural candidate G_a can be constructed by defining

$$G_a^{\text{candidate}}(\tau,\bar{\tau}) = \mathcal{N}^{-1}\cdot\sum_{\gamma\in\Gamma_a} G_a^{\text{light}}(\gamma\tau,\gamma\bar{\tau}). \tag{2.18}$$

The above program to obtain CFT correlators was implemented for minimal models in [9]. It was found that for a large number of them, the candidate correlators did match with the exact ones by taking only the contribution of the Virasoro vacuum block while constructing G_a^{light} - the lightest block served the purpose.

3 $\mathrm{SU}(N)_k$ WZW model: conformal blocks, actions of S and T

As mentioned in the introduction, our focus will be on WZW correlators involving two Kac-Moody primaries in the fundamental and two in the anti-fundamental representation. In this section, we will obtain the transformation properties of the conformal blocks associated with the correlators under the action of crossing.

We begin by recalling some basic facts about the correlators (our discussion follows that of [12, 13, 59, 60]) and in the process set up our notation. The $\mathrm{SU}(N)$ WZW model at level k on the two sphere is described by the action:

$$S_k^{\text{WZW}}[g] = \frac{k}{16\pi}\int d^2z\ \mathrm{Tr}(\partial^\mu g^{-1}\partial_\mu g) - \frac{ik}{24\pi}\int_B d^3\vec{X}\ \epsilon_{\alpha\beta\gamma}\mathrm{Tr}(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g),\qquad k=1,2,\ldots \tag{3.1}$$

where $g(z,\bar{z})$ is a matrix valued bosonic field which takes values in the group $\mathrm{SU}(N)$. The second term is an integral over the three ball B, whose boundary is the two sphere. The pre-factors of the two terms in the action are chosen so that theory is conformal at the quantum level. The action enjoys an $\mathrm{SU}(N)(z)\times\mathrm{SU}(N)(\bar{z})$ invariance. The associated currents are

$$j(z)\equiv -k(\partial_z g)g^{-1},\ \bar{j}(\bar{z})\equiv kg^{-1}(\partial_{\bar{z}}g) \tag{3.2}$$

which can be expanded in terms of the generators of $\mathrm{SU}(N)$ as

$$j(z)=\sum\nolimits_a j^a(z)t^a,\ \bar{j}(\bar{z})=\sum\nolimits_a \bar{j}^a(\bar{z})t^a. \tag{3.3}$$

The Laurent series expansion coefficients of the currents together with the Virasoro generators generate two copies of the Kac-Moody algebra at level k.

Kac-Moody primaries serve as the highest weight states in the theory. For the (N,k) theory the spectrum of Kac-Moody primaries consists operators transforming in all representations of $\mathrm{SU}(N)$ which have integrable Young tableaux i.e. those in which the number

[6] In the case that all the operators are distinct, this subgroup is $\Gamma(2)$ for all the components.

JHEP10(2019)061

of columns is at most k. The conformal dimension of a Kac-Moody primary transforming in a representation R is

$$h_R = \frac{C(R)}{2(k+N)}, \tag{3.4}$$

where $C(R)$ is the quadratic Casimir of the representation.

We will follow the notation of [12] and denote a fundamental Kac-Moody primary by ${g_\alpha}^\beta(z,\bar{z})$, where α is a fundamental index of the SU(N) left and β is a fundamental index of the SU(N) right. On the other hand, an anti-fundamental will be denoted by $g_\rho^{-1\sigma}$, where where ρ is an anti-fundamental index of the SU(N) right and σ is an anti-fundamental index of the SU(N) left. The conformal dimension of these fields can be easily obtained from (3.4)

$$h_g = h_{g^{-1}} = \frac{N^2-1}{2N(k+N)}. \tag{3.5}$$

For correlators involving two fundamentals and two anti-fundamentals, primaries that run in the intermediate channels will be as per the fusion rules

$$g \times g^{-1} = \mathbb{1} + \theta, \qquad g \times g = \xi + \chi, \qquad g^{-1} \times g^{-1} = \xi + \chi, \tag{3.6}$$

where $\mathbb{1}$ is the identity field, θ the adjoint, ξ the antisymmetric and χ the symmetric. The associated dimensions are

$$h_{\mathbb{1}} = 0, \quad h_\theta = \frac{N}{N+k}, \quad h_\xi = \frac{(N-2)(N+1)}{N(N+k)} \quad \text{and} \quad h_\chi = \frac{(N+2)(N-1)}{N(N+k)}. \tag{3.7}$$

Our main interest will be the correlator

$$\langle g g^{-1} g^{-1} g\rangle \equiv \langle {g_{\alpha_1}}^{\beta_1}(z_1,\bar{z}_1) \cdot g_{\beta_2}^{-1\alpha_2}(z_2,\bar{z}_2) \cdot g_{\beta_3}^{-1\alpha_3}(z_3,\bar{z}_3) \cdot {g_{\alpha_4}}^{\beta_4}(z_4,\bar{z}_4)\rangle \tag{3.8}$$

Recall that as per our conventions α_1, α_4 are SU(N) left fundamental indices, α_2, α_3 are SU(N) left anti-fundamental indices, β_1, β_4 are SU(N) right fundamental indices, β_2, β_3 are SU(N) right anti-fundamental indices. We will be eventually interested in making choices for the indices such that the correlator contains two pairs of identical operators so that we can carry out modular averaging as per the prescription in (2.18). For this we need the conformal blocks associated with the correlator and their transformations under the modular group.

The correlator has been studied in detail in [12]. We briefly describe their analysis adopting the discussion to our conventions. First, we define the stripped correlator $G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x})$ as in (2.1)

$$\langle g g^{-1} g^{-1} g\rangle = \Big(\prod_{a<b} z_{ab}^{\mu_{ab}} \bar{z}_{ab}^{\bar{\mu}_{ab}}\Big) G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x}), \tag{3.9}$$

where x is the cross-ratio defined in (2.3). Invariance of the correlator under SU(N) left and right implies

$$G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x}) = \sum_{A,B=1,2} (I_A)(\bar{I}_B) G_{AB}(x,\bar{x}), \tag{3.10}$$

where

$$I_1 = \delta^{\alpha_2}_{\alpha_1}\delta^{\alpha_3}_{\alpha_4}, \;\; \bar{I}_1 = \delta^{\beta_1}_{\beta_2}\delta^{\beta_4}_{\beta_3}, \;\; I_2 = \delta^{\alpha_3}_{\alpha_1}\delta^{\alpha_2}_{\alpha_4} \;\text{ and }\; \bar{I}_2 = \delta^{\beta_1}_{\beta_3}\delta^{\beta_4}_{\beta_2}. \tag{3.11}$$

One then imposes the Knizhnik-Zamolodchikov (KZ) equations on the correlator. The KZ equations are a consequence of the Kac-Moody symmetries. For a correlator involving Kac-Moody primaries ϕ_i, transforming in the representations R_i they are

$$\left[\partial_{z_i} - \frac{1}{k+N}\sum_{j\neq i}\frac{\sum_a t^a_{R_i}\otimes t^a_{R_j}}{z_i - z_j}\right]\langle\phi_1(z_1,\bar{z}_1)\cdots\phi_n(z_n,\bar{z}_n)\rangle = 0, \;\forall\, i, \tag{3.12}$$

where $t^a_{R_i}$ are SU(N) generators in the representation R_i. Similar set of equations hold in the anti-holomorphic coordinates. Imposing them on the correlator (3.8) yields the following equations for the matrix G_{AB} defined in (3.10).

$$\frac{\partial G}{\partial x} = \left[\frac{1}{x}P + \frac{1}{x-1}Q\right]G \;\text{ and }\; \frac{\partial G}{\partial \bar{x}} = G\left[\frac{1}{\bar{x}}P^t + \frac{1}{\bar{x}-1}Q^t\right], \tag{3.13}$$

where the matrices P and Q are given by

$$P = -\frac{1}{N(k+N)}\begin{pmatrix}\frac{2(N^2-1)}{3} & N \\ 0 & -\frac{N^2+2}{3}\end{pmatrix} \;\text{ and }\; Q = -\frac{1}{N(k+N)}\begin{pmatrix}-\frac{N^2+2}{3} & 0 \\ N & \frac{2(N^2-1)}{3}\end{pmatrix}. \tag{3.14}$$

The general solution to these equations takes the form

$$G_{AB}(x,\bar{x}) = X_{ij}F^i_A(x)F^j_B(\bar{x}), \tag{3.15}$$

where the indices i,j run over the primaries in the intermediate channel. These are the identity ($\mathbb{1}$) and the adjoint (θ) fields. $F^i_A(x)$ are the conformal blocks

$$\begin{aligned}
F^{\mathbb{1}}_1(x) &= x^{-\frac{4h_g}{3}}(1-x)^{h_\theta - \frac{4h_g}{3}}F\left(\frac{1}{\tilde{k}}, -\frac{1}{\tilde{k}}; 1-\frac{N}{\tilde{k}}; x\right), \\
F^{\mathbb{1}}_2(x) &= \frac{1}{k}x^{1-\frac{4h_g}{3}}(1-x)^{h_\theta - \frac{4h_g}{3}}F\left(1+\frac{1}{\tilde{k}}, 1-\frac{1}{\tilde{k}}; 2-\frac{N}{\tilde{k}}; x\right), \\
F^{\theta}_1(x) &= x^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta - \frac{4h_g}{3}}F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; 1+\frac{N}{\tilde{k}}; x\right), \\
F^{\theta}_2(x) &= -Nx^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta - \frac{4h_g}{3}}F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}}, \frac{N}{\tilde{k}}+\frac{1}{\tilde{k}}; \frac{N}{\tilde{k}}; x\right),
\end{aligned} \tag{3.16}$$

where $\tilde{k} = k+N$ and $F(a,b,c;x)$ is the Gauss hypergeometric function.[7] We define the holomorphic and the anti-holomorphic blocks:

$$\mathcal{F}^{\mathbb{1}}(x) = I_1F^{\mathbb{1}}_1(x) + I_2F^{\mathbb{1}}_2(x) \tag{3.17}$$
$$\bar{\mathcal{F}}^{\mathbb{1}}(\bar{x}) = \bar{I}_1F^{\mathbb{1}}_1(\bar{x}) + \bar{I}_2F^{\mathbb{1}}_2(\bar{x}) \tag{3.18}$$
$$\mathcal{F}^{\theta}(x) = I_1F^{\theta}_1(x) + I_2F^{\theta}_2(x) \tag{3.19}$$
$$\bar{\mathcal{F}}^{\theta}(\bar{x}) = \bar{I}_1F^{\theta}_1(\bar{x}) + \bar{I}_2F^{\theta}_2(\bar{x}). \tag{3.20}$$

[7] Our conventions for the definition of the Gauss hypergeometric function will be same as that of [61].

JHEP10(2019)061

With this, the correlator factorises into holomorphic and anti-holomorphic parts:

$$G^{\beta_1\alpha_2\alpha_3\beta_4}_{\alpha_1\beta_2\beta_3\alpha_4}(x,\bar{x}) = X_{ij}\mathcal{F}^i(x)\bar{\mathcal{F}}^j(\bar{x}). \tag{3.21}$$

As discussed in section 2, general correlators transform as a six dimensional modular vector under the action of the modular group. Just as in the correlator described above, there are two holomorphic and two anti-holomorphic blocks associated with each correlator. This implies that the vector valued modular form requires 24 coefficients for its specification. This number is large even if one wants to carry out modular averaging as per (2.17) numerically. Luckily, one can simplify the computation by exploiting the fact that (3.21) implies that the X_{ij} are independent of the $\mathrm{SU}(N)$ left and right tensor indices. We will make choices for these so that the correlator has two pairs of identical operators i.e. we will take $\alpha_1 = \alpha_4$, $\beta_1 = \beta_4$, $\alpha_2 = \alpha_3$, $\beta_2 = \beta_3$. With this we have

$$I_1 = I_2 \equiv I \qquad \text{and} \qquad \bar{I}_1 = \bar{I}_2 \equiv \bar{I}. \tag{3.22}$$

As a result, the six dimensional vector space collapses to a three dimensional one (after use of equation (2.10)):

$$\vec{G} = \left(G^{\beta_1\alpha_2\alpha_2\beta_1}_{\alpha_1\beta_2\beta_2\alpha_1}(\tau,\bar{\tau}), G^{\beta_1\alpha_2\alpha_1\beta_2}_{\alpha_1\beta_2\beta_1\alpha_2}(\tau,\bar{\tau}), G^{\beta_1\alpha_1\alpha_2\beta_2}_{\alpha_1\beta_1\beta_2\alpha_2}(\tau,\bar{\tau})\right), \tag{3.23}$$

its transformations under the modular group as given by (2.12) reduces to

$$\begin{aligned} \vec{G}(T\cdot\tau, T\cdot\bar{\tau}) &= \sigma(T)\cdot\vec{G}(\tau,\bar{\tau}),\\ \vec{G}(S\cdot\tau, S\cdot\bar{\tau}) &= \sigma(S)\cdot\vec{G}(\tau,\bar{\tau}), \end{aligned} \tag{3.24}$$

where

$$\sigma(T) = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix}, \ \sigma(S) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}. \tag{3.25}$$

We list the conformal blocks associated with the three correlators in (3.23) and their transformation properties under the modular group in appendix A.

We will primarily perform the modular averaging as per the algorithm in (2.18) (although also briefly consider averaging as per the prescription in (2.17) in appendix D). For the representation of $\mathrm{PSL}(2,\mathbb{Z})$ generated by the matrices in (3.25), it is easy to see that the vector $(1,0,0)$ is left invariant by the subgroup generated by the actions of S and T^2. This is called the theta group [62]. This subgroup is an index 3 subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ which contains $\Gamma(2)$ as an index 2 normal subgroup. In order to carry out the modular averaging as per (2.18), we require the actions of the elements of this subgroup on the conformal blocks associated with the stripped correlator $G^{\beta_1\alpha_2\alpha_2\beta_1}_{\alpha_1\beta_2\beta_2\alpha_1}(\tau,\bar{\tau})$. These blocks are

$$\begin{aligned} \mathcal{H}^{\mathbb{1}}(x) &= IF^{\mathbb{1}}_1(x) + IF^{\mathbb{1}}_2(x)\\ \mathcal{H}^{\theta}(x) &= IF^{\theta}_1(x) + IF^{\theta}_2(x), \end{aligned} \tag{3.26}$$

with I and $\bar{I}$ as defined in (3.22).

JHEP10(2019)061

The transformation properties of these blocks under S and T^2 can be obtained from appendix A. The action of T^2 is given by

$$\mathcal{H}_i\left(T^2.x\right) = \mathcal{H}_j\left(x\right) M_{ji}(T^2), \tag{3.27}$$

where

$$M(T^2) = e^{-i4\pi(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi N/\tilde{k}} \end{pmatrix}. \tag{3.28}$$

The action of S is given by

$$\mathcal{H}_i\left(S.x\right) = \mathcal{H}_j\left(x\right) M_{ji}(S), \tag{3.29}$$

where

$$M(S) = \begin{pmatrix} -\frac{\tilde{k}\Gamma\left(N/\tilde{k}\right)\Gamma\left(k/\tilde{k}\right)}{\Gamma\left(1/\tilde{k}\right)\Gamma\left(-1/\tilde{k}\right)} & -\frac{N\Gamma^2\left(N/\tilde{k}\right)}{\Gamma\left(N/\tilde{k}-1/\tilde{k}\right)\Gamma\left(N/\tilde{k}+1/\tilde{k}\right)} \\ -\frac{\Gamma^2\left(k/\tilde{k}\right)}{N\Gamma\left(k/\tilde{k}-1/\tilde{k}\right)\Gamma\left(k/\tilde{k}+1/\tilde{k}\right)} & \frac{\tilde{k}\Gamma\left(N/\tilde{k}\right)\Gamma\left(k/\tilde{k}\right)}{\Gamma\left(1/\tilde{k}\right)\Gamma\left(-1/\tilde{k}\right)} \end{pmatrix}. \tag{3.30}$$

Successive actions of $M(T^2)$ and $M(S)$ can be used to obtain the action of any element γ of the theta subgroup of the modular group on $\mathcal{H}_i(x)$, we shall denote the associated matrix by $M(\gamma)$. With the definitions in (3.26), the most general form of solutions to the KZ equations with two identical operators can be written as

$$G^{\beta_1\alpha_2\alpha_2\beta_1}_{\alpha_1\beta_2\beta_2\alpha_1}(x,\bar{x}) = X_{ij}\mathcal{H}^i(x)\bar{\mathcal{H}}^j(\bar{x}). \tag{3.31}$$

Under the action of an element γ of the theta subgroup, the matrix X transforms as

$$X \to M(\gamma)XM^\dagger(\gamma). \tag{3.32}$$

We note that under composition

$$M(\gamma_2.\gamma_1) = M(\gamma_1).M(\gamma_2). \tag{3.33}$$

4 Correlators from modular averaging

Having obtained the transformation properties of the conformal blocks we now turn to constructing correlators from modular averaging. In this section, we will carry out the modular averaging as per the prescription in (2.18). As described in the previous section, we will focus on the correlator (3.8) after making choices for SU(N) left and right indices so that two pairs of operators are identical. G^{light} will be taken to be the contribution of the vacuum conformal block, as in [9] we will refer to this as the seed contribution. The transformation (3.32) of the matrix X implies that one can write the result of modular averaging as

$$X^{\text{av}} = \mathcal{N}^{-1}\cdot\sum_{\gamma\in\,\Gamma} M(\gamma)\cdot C_{\text{seed}}\cdot M(\gamma)^\dagger, \tag{4.1}$$

where we have used Γ to denote the theta subgroup and

$$C_{\text{seed}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

The normalization constant $\mathcal{N}$ is determined by demanding $[X]_{11} = 1$, so that the $x \to 0$ behaviour of the correlator is correct. For comparison we record the (exact)result of [12]:

$$X^{\mathrm{KZ}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})\Gamma^2(1-N/\tilde{k})}{N^2\Gamma(1-N/\tilde{k}+1/\tilde{k})\Gamma(1-N/\tilde{k}-1/\tilde{k})\Gamma^2(N/\tilde{k})} \end{pmatrix}. \tag{4.3}$$

Before carrying out the sum in explicit examples, let us discuss some generalities. Any element of Γ can be expressed as

$$\gamma = T^{2n_1} S T^{2n_2} S \cdots S T^{2n_k}, \tag{4.4}$$

for some choice of integers n_i (see e.g. [59]). Since we are dealing with a normalised sum, the sum can be reduced to be over the orbit of C_{seed}. Given this, our interest shall be in γ whose action will generate distinct elements. In this context, note that for all (N, k) the action of $M(T^2)$ on C_{seed} is trivial. Also, in the representations under consideration (which are given in (3.28)), T^2 has finite order. Thus, all distinct $M(\gamma)$ can be generated by considering non-negative values of n_i upto the order of T^2. Furthermore, for $M(\gamma)$ of the form $e^{i\alpha}\mathbb{1}$, its action (3.32) on any X is trivial. We define $m(N, k)$ as the smallest positive integer such that

$$M(T^{2m(N,k)}) \propto \mathbb{1}. \tag{4.5}$$

With this, given the trivial actions described above, a list of γs whose actions contain the orbit of C_{seed} can be constructed by considering $\mathbb{1}$ and all elements of the form

$$\gamma = S T^{2r_1} S \cdots S T^{2r_\ell}, \tag{4.6}$$

with ℓ taking values over natural numbers, $r_i = 1 \cdots (m-1)$ for $i = 1 \cdots (\ell - 1)$ and $r_\ell = 0 \cdots (m-1)$. We define the length of an element in the list to be the value of ℓ associated with it (and denote it as $\ell(\gamma)$). $\mathbb{1}$ is defined to be the element of zero length. The composition rule (3.33) implies

$$M(\gamma) = M(T^{2r_\ell})M(S)\cdots M(S)M(T^{2r_1})M(S). \tag{4.7}$$

If the stabiliser of C_{seed} under the action $C_{\mathrm{seed}} \to M(\gamma) \cdot C_{\mathrm{seed}} \cdot M(\gamma)^\dagger$ has finite index, then the sum reduces to a finite number of terms. Otherwise, one has to deal with an infinite sum. We begin by discussing some models in which the stabiliser is of finite index.

Models with $N = k$ are particularly simple. For $N = k$, the actions of S and T as given by (3.30) and (3.28) can be written as

$$M(S) = \begin{pmatrix} \sin\frac{\pi}{2k} & -k\cos\frac{\pi}{2k} \\ -\frac{1}{k}\cos\frac{\pi}{2k} & -\sin\frac{\pi}{2k} \end{pmatrix}, \qquad M(T^2) = e^{-\frac{2\pi i}{3}\cdot\frac{(N^2-1)}{N^2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.8}$$

Note that $M(T^4) \propto \mathbb{1}$, thus the highest power of T that needs to be included while generating the matrices $M(\gamma)$ in the list in (4.6) is T^2. Let us start by discussing a particular example.

JHEP10(2019)061

γ	$M(\gamma)\cdot C_{\text{seed}}\cdot M(\gamma)^\dagger$
$\mathbb{1}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$
S	$\begin{pmatrix}\frac{1}{4}&-\frac{1}{4\sqrt{3}}\\-\frac{1}{4\sqrt{3}}&\frac{1}{12}\end{pmatrix}$
ST^2	$\begin{pmatrix}\frac{1}{4}&\frac{1}{4\sqrt{3}}\\\frac{1}{4\sqrt{3}}&\frac{1}{12}\end{pmatrix}$
X^{av}	$\begin{pmatrix}1&0\\0&\frac{1}{9}\end{pmatrix}$

Table 1. Orbit of the vacuum block for $N=3, k=3$

$\boldsymbol{N=3,\ k=3:}$ for $N=3, k=3$, the matrices $M(S)$ and $M(T^2)$ are

$$M(S)=\begin{pmatrix}\frac{1}{2}&-\frac{3\sqrt{3}}{2}\\-\frac{1}{2\sqrt{3}}&-\frac{1}{2}\end{pmatrix},\qquad M(T^2)=e^{-\frac{16\pi i}{27}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\tag{4.9}$$

The orbit of C_{seed} consists of three matrices. It is generated by the action of $\mathbb{1}, S$ and ST^2. We tabulate the results of these actions in table 1. The normalised sum over the orbit (4.1) reproduces the KZ result.

For general values $N\ (=k)$, one can show that the orbit of C_{seed} is finite by taking repeated products of the matrices $M(S)$ and $M(T^2)$. The orbit is the set

$$\left\{\begin{pmatrix}\sin^2\alpha&-\frac{1}{k}\sin\alpha\cos\alpha\\-\frac{1}{k}\sin\alpha\cos\alpha&\frac{1}{k^2}\cos^2\alpha\end{pmatrix}\right\}\tag{4.10}$$

where $\alpha=\frac{\pi(2s+1)}{2k}$ with $s=0\cdots(k-1)$ for k odd, and $\alpha=\frac{\pi s}{2k}$ with $s=0\cdots(2k-1)$ for k even (we derive this in appendix B).

The sums over the orbits can be performed using the identities

$$\sum_{s=0}^{k-1}\sin^2\frac{\pi(2s+1)}{2k}=\frac{k}{2}=\sum_{s=0}^{k-1}\cos^2\frac{\pi(2s+1)}{2k},\qquad\sum_{s=0}^{k-1}\sin\frac{\pi(2s+1)}{k}=0$$

for k odd and

$$\sum_{s=0}^{2k-1}\sin^2\frac{\pi s}{2k}=k=\sum_{s=0}^{2k-1}\cos^2\frac{\pi s}{2k},\qquad\sum_{s=0}^{2k-1}\sin\frac{\pi s}{k}=0$$

for k even. Normalising the sum, one finds

$$X^{\text{av}}=\begin{pmatrix}1&0\\0&1/k^2\end{pmatrix},\tag{4.11}$$

which is in agreement with (4.3).

We now turn to models with $N\neq k$ models with finite orbits. For $k=1$ and any finite N the actions of S and T^2 as given by (3.30) and (3.28) take the identity block to a multiple of itself. Thus the adjoint block decouples and upon modular averaging the correlator is given by $|\mathcal{F}^1_{\mathbb{1}}(\tau)|^2$, in keeping with [12]. Next, we discuss two models: $N=4, k=2$ and $N=2, k=4$. These examples will reappear in our discussion of the properties of modular averaging under interchange of N and k in section 5.

$\boldsymbol{N=4,\ k=2}$**:** for $N=4, k=2$ we note that $M(T^6)\propto \mathbb{1}$. The orbit of $C_{\rm seed}$ consists of four matrices. It is generated by the action of $\mathbb{1}$, S, ST^2 and ST^4. The normalised sum over the orbit (4.1) reproduces the KZ result which is $\frac{1}{16\sqrt[3]{2}}$.

$\boldsymbol{N=2,\ k=4}$**:** for $N=2, k=4$ we note that $M(T^6)\propto \mathbb{1}$. The orbit of $C_{\rm seed}$ consists of four matrices. It is generated by the action of $\mathbb{1}$, S, ST^2 and ST^4. The normalised sum over the orbit (4.1) reproduces the KZ result which is $\frac{1}{2\sqrt[3]{4}}$.

Finally, we present some models whose orbits do not seem to be finite. We will analyse the models numerically. As described in our general discussion in the beginning of the section, a list of γs whose actions contain the orbit of $C_{\rm seed}$ can be obtained by considering elements of the form (4.6). To implement the numerics, we will organise the sum over the actions of the elements of the list in terms of the length of the elements. We define[8]

$$X^{\rm av}(\ell_{\rm max}) = \mathcal{N}(\ell_{\rm max})^{-1}\cdot \sum_{\ell(\gamma)\le \ell_{\rm max}}{}' M(\gamma)\cdot C_{\rm seed}\cdot M(\gamma)^\dagger, \tag{4.12}$$

where the primed sum indicates that we include distinct elements of the orbit of $C_{\rm seed}$ in the sum. The normalisation constant $\mathcal{N}(\ell_{\rm max})$ is determined by requiring $X^{\rm av}_{11}(\ell_{\rm max})=1$, so that the $x\to 0$ behaviour of the correlator is correctly reproduced at every value of $\ell_{\rm max}$.

$\boldsymbol{N=2,\ k=3}$**:** for $N=2, k=3$, we have performed sum in (4.12) upto $\ell_{\rm max}=9$. This involves 429226 distinct contributions to the sum. We find $X^{\rm av}_{22}(9)=0.29863$, which is in good agreement with the exact result (4.3), $X^{\rm KZ}_{22}\approx 0.29831$. The off diagonal entries of $X^{\rm av}(9)$ are of the order of 10^{-13}. Figure 1 shows our results for $X^{\rm av}_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$. Note that $X^{\rm av}_{22}(\ell_{\rm max})$ approaches the exact result in an oscillatory manner. Prior to normalisation of the sum, both the $(1,1)$-element as well as the $(2,2)$-element of the matrix have approximately linear growths (all terms in the sum make positive definite contributions to these elements). However, as exhibited by the plot, the ratio of the two quantities (which is $X^{\rm av}_{22}(\ell_{\rm max})$) tends to a constant. Off-diagonal entries are small as a result of phase cancellations.

$\boldsymbol{N=3,\ k=2}$**:** for $N=3, k=2$, we have performed sum in (4.12) upto $\ell_{\rm max}=9$. This involves 429226 distinct contributions to the sum. We find $X^{\rm av}_{22}(9)=0.0932166$, which is in good agreement with the exact result (4.3), $X^{\rm KZ}_{22}\approx 0.0931172$. The off diagonal entries of $X^{\rm av}(9)$ are of the order of 10^{-14}. Figure 2 shows our results for $X^{\rm av}_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$. As in the previous example, $X^{\rm av}_{22}(\ell_{\rm max})$ approaches the exact result in an oscillatory manner. Other features of the numerics are also similar.[9]

[8] Our implementation of the numerics is similar to [9].
[9] This is also true for all models that we study numerically.

JHEP10(2019)061

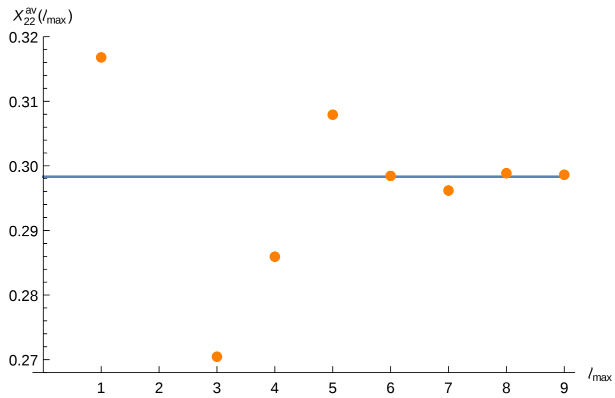

Figure 1. Orange dots show $X_{22}^{\rm av}(\ell_{\rm max})$ in the range $[0.268, 0.320]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.29831 represents $X_{22}^{\rm KZ}$.

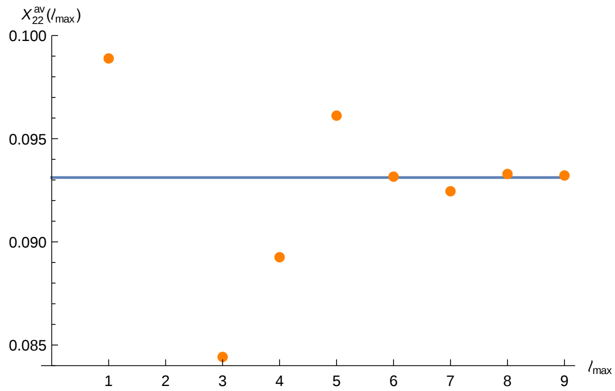

Figure 2. Orange dots show $X_{22}^{\rm av}(\ell_{\rm max})$ in the range $[0.084, 0.100]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0931172 represents $X_{22}^{\rm KZ}$.

$N = 4$, $k = 3$: for $N = 4, k = 3$, we have performed sum in (4.12) upto $\ell_{\rm max} = 8$. This involves 2338785 distinct contributions to the sum. We find $X_{22}^{\rm av}(8) = 0.0592407$, which is in good agreement with the exact result (4.3), $X_{22}^{\rm KZ} \approx 0.0591147$. The off diagonal entries of $X^{\rm av}(8)$ are of the order of 10^{-14}. Figure 3 shows our results for $X_{22}^{\rm av}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

$N = 3$, $k = 4$: for $N = 3, k = 4$, we have performed sum in (4.12) upto $\ell_{\rm max} = 8$. This involves 2338785 distinct contributions to the sum. We find $X_{22}^{\rm av}(8) = 0.117725$, which is in good agreement with the exact result (4.3), $X_{22}^{\rm KZ} \approx 0.117474$. The off diagonal entries of $X^{\rm av}(8)$ are of the order of 10^{-14}. Figure 4 shows our results for $X_{22}^{\rm av}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

It is interesting to ask whether it is possible to develop an understanding of the nature of the orbit associated with the (N, k) model and at what value of ℓ it terminates (if at all). We have developed a systematic algorithm for this purpose, we discuss this in appendix F.

As the values of N and k are increased the numerics can become quite involved. Getting accurate results might require large values of $\ell_{\rm max}$. Models with (N, k) equals to $(5, 6)$ and $(6, 5)$ provide examples of this. We discuss them in appendix C.

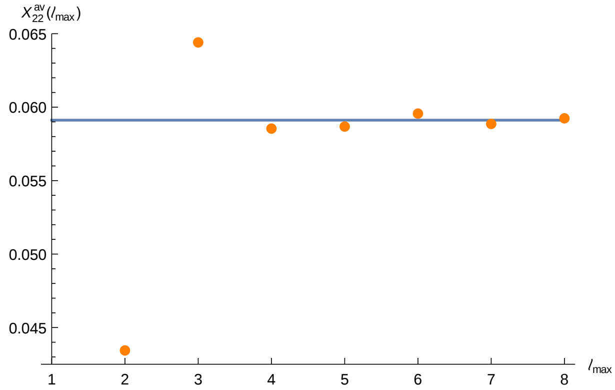

Figure 3. Orange dots show $X_{22}^{\rm av}(\ell_{\rm max})$ in the range $[0.0425, 0.0650]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0591147 represents $X_{22}^{\rm KZ}$.

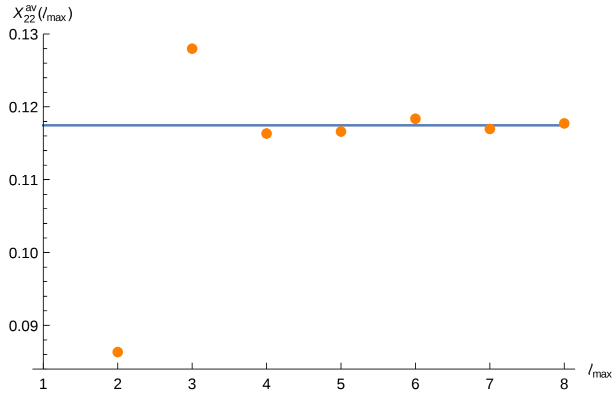

Figure 4. Orange dots show $X_{22}^{\rm av}(\ell_{\rm max})$ in the range $[0.084, 0.130]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.117474 represents $X_{22}^{\rm KZ}$.

Large N: it is interesting to consider the large N limit of the system, this can be interesting from the point of view of holography. For finite k, the matrices $M(S)$ and $M(T^2)$ have following $\frac{1}{N}$ expansions upto order $\frac{1}{N^2}$.

$$M(S) = \begin{pmatrix} \frac{1}{k} + \frac{\pi^2(k^2-1)}{6kN^2} & -N + \frac{\pi^2}{6N} - \frac{k}{3N^2}\left(\pi^2 k + 3k\psi^{(2)}(1)\right) \\ (-1+\frac{1}{k^2})\frac{1}{N} & -\frac{1}{k} - \frac{\pi^2(k^2-1)}{6kN^2} \end{pmatrix}, \tag{4.13}$$

$$M(T^2) = \begin{pmatrix} e^{\frac{2\pi i}{3}} - \frac{4\pi k}{3N} e^{\frac{\pi i}{6}} - \frac{4\pi}{9N^2}\{(2(-1)^{2/3}\pi - 3(-1)^{1/6})k^2 + 3(-1)^{1/6}\} & 0 \\ 0 & e^{\frac{2\pi i}{3}} + \frac{2\pi k}{3N} e^{\frac{\pi i}{6}} - \frac{2\pi}{9N^2}\{((-1)^{2/3}\pi + 3(-1)^{1/6})k^2 + 6(-1)^{1/6}\} \end{pmatrix}, \tag{4.14}$$

where $\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}} \ln\Gamma(z)$ is the Polygamma function. We have performed modular averaging using above matrices and obtained the associated correlators (it is not possible to carry out the sums analytically, we have performed them making specific choices of N

JHEP10(2019)061

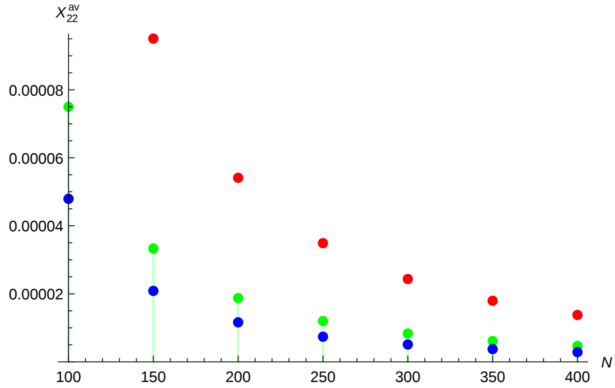

Figure 5. Plot for $k = 2$. Red dots show $X_{22}^{\rm av}(1)$ while blue dots show $X_{22}^{\rm av}(2)$ in the range $[0, 0.0000965]$ plotted against N. Green dots represent $X_{22}^{\rm KZ}$ against N.

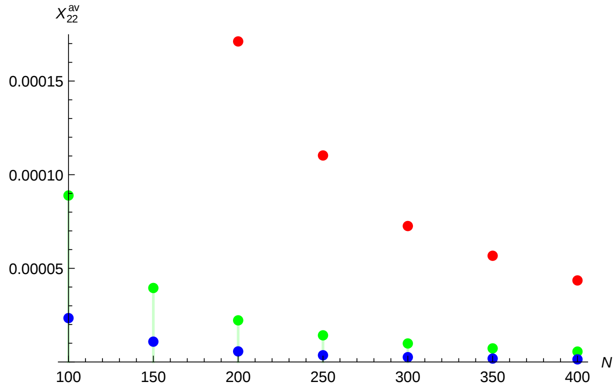

Figure 6. Plot for $k = 3$. Red dots show $X_{22}^{\rm av}(1)$ while blue dots show $X_{22}^{\rm av}(2)$ in the range $[0, 0.0001750]$ plotted against N. Green dots represent $X_{22}^{\rm KZ}$ against N.

and k with ($N \gg k$) using the numerical recipe described in the first part of this section). The agreement with the results of KZ is good even for low values of ℓ.

The results the for $k = 2,\ 3$ at $\ell_{\rm max} = 1,\ 2$ are summarised in figures 5, 6. The results indicate that one can obtain correlators by taking the large N limit of the matrices $M(S)$ and $M(T^2)$ (even working at low ℓ). This hints that low ℓ terms should be the most relevant in the context of semi-classical holography.

Finally, we have also considered the prescription for constructing correlators by averaging over the whole ${\rm PSL}(2, \mathbb{Z})$ (2.17). This involves averaging over a vector and hence is more complicated. We briefly present our results on this in appendix D and leave more detailed explorations for the future.

In summary, in all the cases that we have examined, modular averaging over the theta subgroup successfully reproduces the result of [12]. The correlators can be considered as extremal in the sense of [9]. For extremal correlators, modular averaging sums can be thought of as providing an alternate prescription for their computation. Next, we will examine the properties of these sums involved under interchange of N and k.

5 $N \leftrightarrow k$ in modular averages

As described in the introduction, an interesting property of WZW models is level-rank duality. In this section, we will show that there is a simple one to one correspondence between individual terms in the modular averaging sums for correlators in the (N,k) and (k,N) theories.

We will be simultaneously dealing with the (N,k) and (k,N) theories in this section, let us begin by introducing notation adapted for the purpose. We will include labels in the matrices (3.28) and (3.30) which generate the actions of S and T^2, to indicate the theory they belong to.

$$M_{N,k}(T^2) = e^{-i4\pi(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi N/\tilde{k}} \end{pmatrix} \equiv e^{i\alpha(N,k)} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi(N,k)} \end{pmatrix} \tag{5.1}$$

and

$$M_{N,k}(S) = \begin{pmatrix} -\frac{\tilde{k}\Gamma\left(N/\tilde{k}\right)\Gamma\left(k/\tilde{k}\right)}{\Gamma\left(1/\tilde{k}\right)\Gamma\left(-1/\tilde{k}\right)} & -\frac{N\Gamma^2\left(N/\tilde{k}\right)}{\Gamma\left(N/\tilde{k}-1/\tilde{k}\right)\Gamma\left(N/\tilde{k}+1/\tilde{k}\right)} \\ -\frac{\Gamma^2\left(k/\tilde{k}\right)}{N\Gamma\left(k/\tilde{k}-1/\tilde{k}\right)\Gamma\left(k/\tilde{k}+1/\tilde{k}\right)} & \frac{\tilde{k}\Gamma\left(N/\tilde{k}\right)\Gamma\left(k/\tilde{k}\right)}{\Gamma\left(1/\tilde{k}\right)\Gamma\left(-1/\tilde{k}\right)} \end{pmatrix} \equiv \begin{pmatrix} a_s(N,k) & b_s(N,k) \\ c_s(N,k) & d_s(N,k) \end{pmatrix}. \tag{5.2}$$

We note that $d_s(N,k) = -a_s(N,k)$ and $b_s(N,k).c_s(N,k) = 1 + a_s(N,k).d_s(N,k)$. Also, $a_s(N,k)$ and the product $b_s(N,k).c_s(N,k)$ are symmetric under the interchange of N and k, i.e.

$$a_s(N,k) = a_s(k,N), \qquad d_s(N,k) = d_s(k,N), \qquad b_s(N,k).c_s(N,k) = b_s(k,N).c_s(k,N). \tag{5.3}$$

Recall that the matrices given in (4.7) provide a list whose actions contain the orbit of C_{seed}. We will denote the matrices in the list by

$$M^{\ell}_{N,k}(r_1, r_2 \cdots, r_\ell) \equiv M^{\ell}_{N,k}(r_i) \equiv M_{N,k}(T^{2r_\ell})M_{N,k}(S)\cdots M_{N,k}(S)M_{N,k}(T^{2r_1})M_{N,k}(S). \tag{5.4}$$

Note that with this $M^{\ell}_{N,k}(r_i)$ is a function of $r_1, r_2 \cdots r_l$; with $r_i = 1 \cdots (m(N,k)-1)$ for $i = 1 \cdots (\ell - 1)$ and $r_\ell = 0 \cdots (m(N,k)-1)$ with $m(N,k)$ as defined in (4.5). We define $M^0_{N,k}$ to be the identity matrix. We now introduce another set of matrices

$$\begin{aligned} \tilde{M}^{\ell}_{N,k}(p_1, p_2 \cdots, p_\ell) &\equiv \tilde{M}^{\ell}_{N,k}(p_i) \\ &\equiv M_{N,k}(T^{-2p_\ell})M_{N,k}(S)\cdots M_{N,k}(S)M_{N,k}(T^{-2p_1})M_{N,k}(S). \end{aligned} \tag{5.5}$$

$\tilde{M}^{\ell}_{N,k}(p_i)$ is a function of $p_1, p_2 \cdots p_l$; with $p_i = 1 \cdots (m(N,k)-1)$ for $i = 1 \cdots (\ell-1)$ and $p_\ell = 0 \cdots (m(N,k)-1)$. We will define $\tilde{M}^0_{N,k}$ to be the identity matrix.

At any given length ℓ, the set of matrices generated from the action of $M^{\ell}_{N,k}(r_i)$ on C_{seed} is exactly same as the set generated from the action of $\tilde{M}^{\ell}_{N,k}(p_i)$ on C_{seed} i.e.

$$\begin{aligned} &\Big\{ M^{\ell}_{N,k}(r_i) C_{\text{seed}} M^{\dagger\ell}_{N,k}(r_i); r_i = 1\cdots(m(N,k)-1) \\ &\qquad\qquad \text{for } i = 1\cdots(\ell-1), r_\ell = 0\cdots(m(N,k)-1) \Big\} \\ &= \Big\{ \tilde{M}^{\ell}_{N,k}(p_i) C_{\text{seed}} \tilde{M}^{\dagger\ell}_{N,k}(p_i); p_i = 1\cdots(m(N,k)-1) \\ &\qquad\qquad \text{for } i = 1\cdots(\ell-1), p_\ell = 0\cdots(m(N,k)-1)) \Big\}. \end{aligned} \tag{5.6}$$

This is a consequence of the fact that for any X following equality (between sets) holds

$$\Big\{ M_{N,k}(T^{2r}) X M^{\dagger}_{N,k}(T^{2r}); r = 0 \cdots (m(N,k)-1) \Big\} \\ = \Big\{ M_{N,k}(T^{-2p}) X M^{\dagger}_{N,k}(T^{-2p}); p = 0 \cdots (m(N,k)-1) \Big\}. \quad (5.7)$$

Given the equivalence in (5.6), while carrying out modular averaging, either set can be used to generate the sum over the orbit of C_{seed}. While establishing the relationship between the modular averages in the (N,k) and (k,N) theories, it will be useful to generate the orbit for the (N,k) theory using the $M^{\ell}_{N,k}$ matrices and for the (k,N) theory using $\tilde{M}^{\ell}_{k,N}$ matrices. The essential point will be to establish that the actions of the two matrices[10]

$$M^{\ell}_{N,k}(r_1, r_2 \cdots r_\ell) \quad \text{and} \quad \tilde{M}^{\ell}_{k,N}(r_1, r_2 \cdots r_\ell) \qquad (5.8)$$

on C_{seed} are closely related. Let us begin by looking at the general from of the matrices $M^{\ell}_{N,k}(r_1, r_2 \cdots r_\ell)$ and $\tilde{M}^{\ell}_{N,k}(r_1, r_2 \cdots r_\ell)$. As shown in appendix E, they can be written as

$$\begin{aligned} M^{\ell}_{N,k}(r_1, \cdots r_\ell) &= \exp\Big(i\alpha(N,k)\Big(\sum r_i\Big)\Big) \\ &\quad \times \begin{pmatrix} a^{\ell}_{N,k}(r_1, \cdots r_\ell) & b_s(N,k) b^{\ell}_{N,k}(r_1, \cdots r_\ell) \\ c_s(N,k) c^{\ell}_{N,k}(r_1, \cdots r_\ell) & d^{\ell}_{N,k}(r_1, \cdots r_\ell) \end{pmatrix} \qquad (5.9) \\ \tilde{M}^{\ell}_{N,k}(r_1, \cdots r_\ell) &= \exp\Big(-i\alpha(N,k)\Big(\sum r_i\Big)\Big) \\ &\quad \times \begin{pmatrix} \tilde{a}^{\ell}_{N,k}(r_1, \cdots r_\ell) & b_s(N,k) \tilde{b}^{\ell}_{N,k}(r_1, \cdots r_\ell) \\ c_s(N,k) \tilde{c}^{\ell}_{N,k}(r_1, \cdots r_\ell) & \tilde{d}^{\ell}_{N,k}(r_1, \cdots r_\ell), \end{pmatrix} \qquad (5.10) \end{aligned}$$

with the functions appearing above obeying the relationships

$$\begin{aligned} \tilde{a}^{\ell}_{k,N}(r_1, \cdots r_\ell) &= a^{\ell}_{N,k}(r_1, \cdots r_\ell), & \tilde{b}^{\ell}_{k,N}(r_1, \cdots r_\ell) &= b^{\ell}_{N,k}(r_1, \cdots r_\ell), \\ \tilde{c}^{\ell}_{k,N}(r_1, \cdots r_\ell) &= c^{\ell}_{N,k}(r_1, \cdots r_\ell), & \tilde{d}^{\ell}_{k,N}(r_1, \cdots r_\ell) &= d^{\ell}_{N,k}(r_1, \cdots r_\ell). \end{aligned} \qquad (5.11)$$

Now, let us discuss the implications of these relations for modular averages. As mentioned before, we will generate the orbit of the (N,k) theory using the matrices $M^{\ell}_{N,k}$ and the (k,N) theory using the $\tilde{M}^{\ell}_{k,N}$ matrices. Firstly, note that (5.9) and (5.10) imply that any duplications in the action of $M^{\ell}_{N,k}$ on C_{seed} implies a duplication in the action of $\tilde{M}^{\ell}_{k,N}$ on C_{seed} and vice versa[11] i.e.

$$\begin{aligned} & M^{\ell}_{N,k}(r_i) C_{\text{seed}} M^{\dagger\ell}_{N,k}(r_i) = M^{\ell}_{N,k}(s_i) C_{\text{seed}} M^{\dagger\ell}_{N,k}(s_i) \\ \iff \quad & \tilde{M}^{\ell}_{k,N}(r_i) C_{\text{seed}} \tilde{M}^{\dagger\ell}_{k,N}(r_i) = \tilde{M}^{\ell}_{k,N}(s_i) C_{\text{seed}} \tilde{M}^{\dagger\ell}_{k,N}(s_i) \end{aligned} \qquad (5.12)$$

Furthermore, we have

$$M^{\ell}_{N,k}(r_i) C_{\text{seed}} M^{\dagger\ell}_{N,k}(r_i)\big|_{11} = \tilde{M}^{\ell}_{k,N}(r_i) C_{\text{seed}} \tilde{M}^{\dagger\ell}_{k,N}(r_i)\big|_{11} \qquad (5.13)$$

[10]Note since $\gcd(k+N,N) = \gcd(k,N) = \gcd(k+N,k)$, $m(N,k) = m(k,N)$. This implies that the arguments of $M^{\ell}_{N,k}$ and $\tilde{M}^{\ell}_{k,N}$ take the same values.

[11]This together with (5.6) explains why the number of duplicates for theories related under $N \leftrightarrow k$ were same in our numerical analysis in section 4.

JHEP10(2019)061

and

$$c_s^2(k,N) M^{\ell}_{N,k}(r_i) C_{\text{seed}} M^{\dagger\ell}_{N,k}(r_i)\big|_{22} = c_s^2(N,k) \tilde{M}^{\ell}_{k,N}(r_i) C_{\text{seed}} \tilde{M}^{\dagger\ell}_{k,N}(r_i)\big|_{22}. \tag{5.14}$$

With this,[12] it is natural to pair the matrix

$$M^{\ell}_{N,k}(r_i) C_{\text{seed}} M^{\dagger\ell}_{N,k}(r_i)$$

in the orbit of C_{seed} of the (N,k) theory with the matrix

$$\tilde{M}^{\ell}_{k,N}(r_i) C_{\text{seed}} \tilde{M}^{\dagger\ell}_{k,N}(r_i)$$

in the orbit of C_{seed} of the (k,N) theory. This establishes our one to one correspondence between the terms that appear in the modular averaging sums of the two theories. Note that (5.13) implies that the normalisations of both the sums are equal. With this, (5.14) implies that the all paired terms in the sums contribute to the sums with the ratio

$$\frac{c_s^2(N,k)}{c_s^2(k,N)}. \tag{5.15}$$

Of course, since the ratio is same for all the pairs, from the point of view of modular averaging one can trivially write the relation (even without performing the sums)

$$\frac{X_{\text{av}}(N,k)\big|_{22}}{X_{\text{av}}(k,N)\big|_{22}} = \frac{c_s^2(N,k)}{c_s^2(k,N)} = \frac{k^2\Gamma^4\left(k/\tilde{k}\right)\Gamma^2\left(N/\tilde{k}-1/\tilde{k}\right)\Gamma^2\left(N/\tilde{k}+1/\tilde{k}\right)}{N^2\Gamma^2\left(k/\tilde{k}-1/\tilde{k}\right)\Gamma^2\left(k/\tilde{k}+1/\tilde{k}\right)\Gamma^4\left(N/\tilde{k}\right)}. \tag{5.16}$$

One can check by making use of gamma function identities that this is indeed consistent with the KZ result (4.3). Thus, the one to one correspondence between the terms in the two sums has given us relations between OPE coefficients in the theories (as OPE coefficients can be obtained by taking the small cross-ratio limit of the expressions of the correlators in terms of conformal blocks).

It is natural to ask if the one to one correspondence between the terms in the modular averaging sums in the two theories has any physical interpretation. In this context, we note that it was argued in [9] that for "heavy operators" the modular averaging for genus zero correlators can be interpreted as a semiclassical AdS_3 dual computation. More specifically, if the operator dimensions are of the order of the central charge (c) of the theory but less than $c/12$ then the bulk path integral has saddles corresponding to geodesic propagation of heavy particles between the operator insertion points in the boundary [65–74]. Performing the sum over the saddles incorporating the back reaction of the heavy particle geodesics on the geometry and exchange of light primaries, yields the sum over modular channels. But, the operators considered in this article cannot be made heavy in the semiclassical limit, since $h_g/c \sim 1/Nk$. One possibility is that the situation is similar to [10] where the topological sectors for the saddle point sum was as given in the semi classical limit even in the quantum regime. In any case, a computation similar to ours for operators satisfying the heavy operator criterion should help reveal how level-rank duality works from a holographic point of view.

[12]It is easy to check that these relationships hold for the (4,2) and (2,4) models (which have finite orbits). For other models we have checked them numerically.

6 Conclusions

In this article, we have analysed correlators involving two fundamentals and two anti-fundamentals in $\mathrm{SU}(N)_k$ WZW theories using modular averaging. After determining the transformations of the conformal blocks under S and T transformations, correlators were expressed as sum of the action of the elements of the theta subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ on the vacuum block. We found that for all models with $N = k$ the orbit of the vacuum block is finite and modular averaging reproduces the correlators correctly. In models where we were unable to characterise the orbit we performed the sums numerically; modular averaging successfully reproduced the correlators, providing strong evidence that the correlators examined in this paper are extremal in the sense of [9]. We also considered construction of correlators from averaging of the entire modular group. This is more involved. Here we have found examples where the averaging does not reproduce the correlator (see appendix D). Interestingly, [9] argues that it is the modular averaging over the theta subgroup that has a direct interpretation in the holographic context.

We have found a close relationship between modular averaging for correlators involving fundamentals and anti-fundamentals in the (N,k) and (k,N) theories. In section 5, we established a one to one correspondence between the orbits of the vacuum conformal blocks of the two theories. The contributions of the paired terms to their respective sums was given by a ratio of elements of braids matrices in the theories. This allowed us to obtain a simple relationship between OPE coefficients. A prescription relating general correlators of WZW models under level-rank duality has been given in [56]. The braid matrices of the theories for general correlators have been related in [63, 64]. It will be interesting to study the implications of these relations for modular averaging in more general correlators.

As discussed in the later part of the previous section, we believe that our results give a strong hint that holographic computations can make various aspects of level-rank duality in WZW models manifest. A first step in this direction can be to consider correlators of heavy operators in the theories and analyse their conformal blocks in the semi-classical limit.

Acknowledgments

We would like to thank Anirban Basu, Dileep Jatkar, Henry Maxfield, Shiraz Minwalla, Satchi Naik, Gim Seng Ng, Mahasweta Pandit and Ashoke Sen for discussions.

A Conformal blocks and their transformations

In this appendix, we list the conformal blocks associate with the following three correlators[13]

$$\langle {g_{\alpha_1}}^{\beta_1}(z_1,\bar z_1)\cdot g^{-1\,\alpha_2}_{\beta_2}(z_2,\bar z_2)\cdot g^{-1\,\alpha_3}_{\beta_3}(z_3,\bar z_3)\cdot {g_{\alpha_4}}^{\beta_4}(z_4,\bar z_4)\rangle \tag{A.1}$$

$$\langle {g_{\alpha_1}}^{\beta_1}(z_1,\bar z_1)\cdot g^{-1\,\alpha_2}_{\beta_2}(z_2,\bar z_2)\cdot {g_{\alpha_4}}^{\beta_4}(z_3,\bar z_3)\cdot g^{-1\,\alpha_3}_{\beta_3}(z_4,\bar z_4)\rangle \tag{A.2}$$

$$\langle {g_{\alpha_1}}^{\beta_1}(z_1,\bar z_1)\cdot {g_{\alpha_4}}^{\beta_4}(z_2,\bar z_2)\cdot g^{-1\,\alpha_2}_{\beta_2}(z_3,\bar z_3)\cdot g^{-1\,\alpha_3}_{\beta_3}(z_4,\bar z_4)\rangle \tag{A.3}$$

[13]The other three independent correlators in (2.13) are related to these by the interchange $I_1 \leftrightarrow I_2$. Thus they can be easily obtained from the data in this appendix.

and their transformation properties under the modular tranformations (after the identification (3.22) described in section 3). We will refer to the correlators listed above as the first, second and third correlators. Blocks and their transformation matrices will be given subscripts to indicate the correlator they belong to.

For the first correlator

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g^{-1\,\alpha_2}_{\beta_2}(z_2,\bar{z}_2)\cdot g^{-1\,\alpha_3}_{\beta_3}(z_3,\bar{z}_3)\cdot g_{\alpha_4}{}^{\beta_4}(z_4,\bar{z}_4)\rangle$$

the holomorphic conformal blocks[14] are

$$\begin{aligned}\mathcal{F}^{\mathbb{1}}_{(1)}(x) &= I_1 F^{\mathbb{1}}_{(1)1}(x) + I_2 F^{\mathbb{1}}_{(1)2}(x),\\ \mathcal{F}^{\theta}_{(1)}(x) &= I_1 F^{\theta}_{(1)1}(x) + I_2 F^{\theta}_{(1)2}(x),\end{aligned}\tag{A.4}$$

where

$$\begin{aligned}F^{\mathbb{1}}_{(1)1}(x) &= x^{-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(\frac{1}{\tilde{k}},-\frac{1}{\tilde{k}};1-\frac{N}{\tilde{k}};x\right),\\ F^{\mathbb{1}}_{(1)2}(x) &= \frac{1}{k}x^{1-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(1+\frac{1}{\tilde{k}},1-\frac{1}{\tilde{k}};2-\frac{N}{\tilde{k}};x\right),\\ F^{\theta}_{(1)1}(x) &= x^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};1+\frac{N}{\tilde{k}};x\right),\\ F^{\theta}_{(1)2}(x) &= -Nx^{h_\theta-\frac{4h_g}{3}}(1-x)^{h_\theta-\frac{4h_g}{3}}F\left(\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};\frac{N}{\tilde{k}};x\right).\end{aligned}\tag{A.5}$$

The holomorphic blocks for the correlator

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g^{-1\,\alpha_2}_{\beta_2}(z_2,\bar{z}_2)\cdot g_{\alpha_4}{}^{\beta_4}(z_3,\bar{z}_3)\cdot g^{-1\,\alpha_3}_{\beta_3}(z_4,\bar{z}_4)\rangle$$

are

$$\begin{aligned}\mathcal{F}^{\mathbb{1}}_{(2)}(x) &= I_1 F^{\mathbb{1}}_{(2)1}(x) + I_2 F^{\mathbb{1}}_{(2)2}(x),\\ \mathcal{F}^{\theta}_{(2)}(x) &= I_1 F^{\theta}_{(2)1}(x) + I_2 F^{\theta}_{(2)2}(x),\end{aligned}\tag{A.6}$$

where

$$\begin{aligned}F^{\mathbb{1}}_{(2)1}(x) &= x^{-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}}F\left(\frac{1}{\tilde{k}},1-\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};1-\frac{N}{\tilde{k}};x\right),\\ F^{\mathbb{1}}_{(2)2}(x) &= -\frac{1}{k}x^{1-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}}F\left(1+\frac{1}{\tilde{k}},1-\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};2-\frac{N}{\tilde{k}};x\right),\\ F^{\theta}_{(2)1}(x) &= x^{h_{\hat{\theta}}-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}}F\left(1+\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};1+\frac{N}{\tilde{k}};x\right),\\ F^{\theta}_{(2)2}(x) &= -Nx^{h_{\hat{\theta}}-\frac{4h_g}{3}}(1-x)^{h_\chi-\frac{4h_g}{3}}F\left(\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};\frac{N}{\tilde{k}};x\right).\end{aligned}\tag{A.7}$$

[14]The blocks for this correlator have already been discussed in the main text. We rewrite them here with the subscript convention discussed above, so as to have a consistent notation for this appendix.

The holomorphic blocks for the correlator

$$\langle g_{\alpha_1}{}^{\beta_1}(z_1,\bar{z}_1)\cdot g_{\alpha_4}{}^{\beta_4}(z_2,\bar{z}_2)\cdot g^{-1\alpha_2}_{\beta_2}(z_3,\bar{z}_3)\cdot g^{-1\alpha_3}_{\beta_3}(z_4,\bar{z}_4)\rangle$$

are

$$\begin{aligned}\mathcal{F}^{\xi}_{(3)}(x) &= I_1 F^{\xi}_{(3)1}(x) + I_2 F^{\xi}_{(3)2}(x),\\ \mathcal{F}^{\chi}_{(3)}(x) &= I_1 F^{\chi}_{(3)1}(x) + I_2 F^{\chi}_{(3)2}(x),\end{aligned}\tag{A.8}$$

where

$$\begin{aligned}F^{\xi}_{(3)1}(x) &= x^{h_\xi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(1-\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}};1-\frac{2}{\tilde{k}};x\right),\\ F^{\xi}_{(3)2}(x) &= -x^{h_\xi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(-\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}-\frac{1}{\tilde{k}};1-\frac{2}{\tilde{k}};x\right),\\ F^{\chi}_{(3)1}(x) &= x^{h_\chi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(1+\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};1+\frac{2}{\tilde{k}};x\right),\\ F^{\chi}_{(3)2}(x) &= x^{h_\chi-\frac{4h_g}{3}}(1-x)^{h_{\hat\theta}-\frac{4h_g}{3}}F\left(\frac{1}{\tilde{k}},\frac{N}{\tilde{k}}+\frac{1}{\tilde{k}};1+\frac{2}{\tilde{k}};x\right).\end{aligned}\tag{A.9}$$

With the choices for tensor indices as in (3.22), we will denote the holomorphic blocks of the three correlators by $\mathcal{H}^{i}_{(q)}(x)$ with $q=1,2,3$ i.e.

$$\begin{aligned}\mathcal{H}^{\mathbb{1}}_{(1)}(x) &= IF^{\mathbb{1}}_{(1)1}(x)+IF^{\mathbb{1}}_{(1)2}(x),\\ \mathcal{H}^{\theta}_{(1)}(x) &= IF^{\theta}_{(1)1}(x)+IF^{\theta}_{(1)2}(x),\\ \mathcal{H}^{\mathbb{1}}_{(2)}(x) &= IF^{\mathbb{1}}_{(2)1}(x)+IF^{\mathbb{1}}_{(2)2}(x),\\ \mathcal{H}^{\theta}_{(2)}(x) &= IF^{\theta}_{(2)1}(x)+IF^{\theta}_{(2)2}(x),\\ \mathcal{H}^{\xi}_{(3)}(x) &= IF^{\xi}_{(3)1}(x)+IF^{\xi}_{(3)2}(x),\\ \mathcal{H}^{\chi}_{(3)}(x) &= IF^{\chi}_{(3)1}(x)+IF^{\chi}_{(3)2}(x).\end{aligned}\tag{A.10}$$

We note that with $I_1=I_2$ the three correlators are equal to those in (3.23).

The actions of T and S on these can be computed using the following identities of hypergeometric functions [61].

$$\begin{aligned}F(a,b;c;z) &= (1-z)^{c-a-b}F(c-a,c-b;c;z),\\ F(a,b;c;\frac{z}{z-1}) &= (1-z)^{a}F(a,c-b;c;z)=(1-z)^{b}F(c-a,b;c;z),\\ F(a,b;c;1-z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;z)\\ &\quad+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{c-a-b}F(c-a,c-b;c-a-b+1;z).\end{aligned}\tag{A.11}$$

$$\begin{aligned}F(a,b;c;1-z) &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}z^{c-a-b}(1-z)^{1-c}F(1-b,1-a;1+c-a-b,z)\\ &\quad+\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}(1-z)^{1-c}F(1+b-c,1+a-c;1+a+b-c;z)\end{aligned}\tag{A.12}$$

Action of T: the action of T on the blocks $\mathcal{H}^i_{(1)}(x)$ are given by

$$\mathcal{H}^i_{(1)}\,(T.x) = \mathcal{H}^j_{(2)}\,(x)\,M_{(1)ji}(T), \tag{A.13}$$

where

$$M_{(1)}(T) = (-1)^{-2(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{N/\tilde{k}} \end{pmatrix}. \tag{A.14}$$

The action of T on the blocks $\mathcal{H}^i_{(2)}(x)$ are given by

$$\mathcal{H}^i_{(2)}\,(T.x) = \mathcal{H}^j_{(1)}\,(x)\,M_{(2)ji}(T), \tag{A.15}$$

where

$$M_{(2)}(T) = (-1)^{-2(N^2-1)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{N/\tilde{k}} \end{pmatrix}. \tag{A.16}$$

The action of T on the blocks $\mathcal{H}^i_{(3)}(x)$ are given by

$$\mathcal{H}^i_{(3)}\,(T.x) = \mathcal{H}^j_{(3)}\,(x)\,M_{(3)ji}(T), \tag{A.17}$$

where

$$M_{(3)}(T) = -(-1)^{(N^2-3N-4)/3N\tilde{k}} \begin{pmatrix} 1 & 0 \\ 0 & -(-1)^{2/\tilde{k}} \end{pmatrix}. \tag{A.18}$$

Action of S: the action of S on the blocks $\mathcal{H}^i_{(1)}(x)$ are given by

$$\mathcal{H}^i_{(1)}\,(S.x) = \mathcal{H}^j_{(1)}\,(x)\,M_{(1)ji}(S), \tag{A.19}$$

where

$$M_{(1)}(S) = \begin{pmatrix} -\frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} & -\frac{N\Gamma^2(N/\tilde{k})}{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ -\frac{\Gamma^2(k/\tilde{k})}{N\Gamma(k/\tilde{k}-1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} & \frac{\tilde{k}\Gamma(N/\tilde{k})\Gamma(k/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(-1/\tilde{k})} \end{pmatrix}. \tag{A.20}$$

The action of S on the blocks $\mathcal{H}^i_{(2)}(x)$ are given by

$$\mathcal{H}^i_{(2)}\,(S.x) = \mathcal{H}^j_{(3)}\,(x)\,M_{(2)ji}(S), \tag{A.21}$$

where

$$M_{(2)}(S) = \begin{pmatrix} \frac{\Gamma(k/\tilde{k})\Gamma(2/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} & \frac{N\Gamma(N/\tilde{k})\Gamma(2/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ \frac{\Gamma(k/\tilde{k})\Gamma(-2/\tilde{k})}{\Gamma(k/\tilde{k}-1/\tilde{k})\Gamma(-1/\tilde{k})} & -\frac{N\Gamma(N/\tilde{k})\Gamma(-2/\tilde{k})}{\Gamma(N/\tilde{k}-1/\tilde{k})\Gamma(-1/\tilde{k})} \end{pmatrix}. \tag{A.22}$$

The action of S on the blocks $\mathcal{H}^i_{(3)}(x)$ are given by

$$\mathcal{H}^i_{(3)}\,(S.x) = \mathcal{H}^j_{(2)}\,(x)\,M_{(3)ji}(S), \tag{A.23}$$

where

$$M_{(3)}(S) = \begin{pmatrix} \frac{2\Gamma(-2/\tilde{k})\Gamma(N/\tilde{k})}{\Gamma(-1/\tilde{k})\Gamma(N/\tilde{k}-1/\tilde{k})} & \frac{2\Gamma(2/\tilde{k})\Gamma(N/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(N/\tilde{k}+1/\tilde{k})} \\ \frac{\Gamma(1-2/\tilde{k})\Gamma(-N/\tilde{k})}{\Gamma(-1/\tilde{k})\Gamma(k/\tilde{k}-1/\tilde{k})} & \frac{\Gamma(1+2/\tilde{k})\Gamma(-N/\tilde{k})}{\Gamma(1/\tilde{k})\Gamma(k/\tilde{k}+1/\tilde{k})} \end{pmatrix}. \tag{A.24}$$

B Generators of the orbit for $N = k$ theories

In this section, we show that for general values of $N(=k)$ the orbit of $C_{\rm seed}$ is as given in (4.10). We will do this by showing that the orbit can in effect be generated by considering the action of matrices of the form

$$\begin{pmatrix} \sin\alpha & -k\cos\alpha \\ -\frac{1}{k}\cos\alpha & -\sin\alpha \end{pmatrix}, \tag{B.1}$$

on $C_{\rm seed}$, where $\alpha = \frac{\pi(2s+1)}{2k}$ with $s = 0\cdots(k-1)$ for k odd, and $\alpha = \frac{\pi s}{2k}$ with $s = 0\cdots(2k-1)$ for k even. It is easy to check that the actions of these matrices on $C_{\rm seed}$ indeed generates the orbits described in (4.10). We begin by noting that for $M(\gamma)$ of the form

$$M(\gamma) \equiv \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix},$$

its action on $C_{\rm seed}$ yields

$$\begin{pmatrix} |a_\gamma|^2 & a_\gamma c_\gamma^* \\ a_\gamma^* c_\gamma & |c_\gamma|^2 \end{pmatrix}. \tag{B.2}$$

Thus, the result of the action only depends on a_γ and c_γ (and is independent of b_γ and d_γ). Furthermore, since (B.2) is quadratic in a_γ and c_γ, elements of the orbit are only sensitive to their relative sign. Thus deformations of $M(\gamma)s$ which modify b_γ, d_γ and the relative sign between a_γ, c_γ keep their actions on $C_{\rm seed}$ unchanged. We will use such deformations to show that the orbit is in effect generated by the matrices given in (B.1). Let us start by considering the first few matrices in the list (4.7) of $M(\gamma)$ (for theories with $N = k$). In what follows, we will use the symbol '$\sim$' to denote a deformation of a matrix $M(\gamma)$ which keeps its action on $C_{\rm seed}$ unchanged.

$$\begin{aligned}
M(\mathbb{1}) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \sin\frac{\pi k}{2k} & -k\cos\frac{\pi k}{2k} \\ -\frac{1}{k}\cos\frac{\pi k}{2k} & -\sin\frac{\pi k}{2k} \end{pmatrix};\\
M(S) &= \begin{pmatrix} \sin\frac{\pi}{2k} & -k\cos\frac{\pi}{2k} \\ -\frac{1}{k}\cos\frac{\pi}{2k} & -\sin\frac{\pi}{2k} \end{pmatrix};\\
M(ST^2) &= \begin{pmatrix} \sin\frac{\pi}{2k} & -k\cos\frac{\pi}{2k} \\ \frac{1}{k}\cos\frac{\pi}{2k} & \sin\frac{\pi}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{\pi(2k-1)}{2k} & -k\cos\frac{\pi(2k-1)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(2k-1)}{2k} & -\sin\frac{\pi(2k-1)}{2k} \end{pmatrix};\\
M(ST^2S) &= \begin{pmatrix} \sin\frac{\pi(2-k)}{2k} & -k\cos\frac{\pi(2-k)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(2-k)}{2k} & -\sin\frac{\pi(2-k)}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{\pi(2+k)}{2k} & -k\cos\frac{\pi(2+k)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(2+k)}{2k} & -\sin\frac{\pi(2+k)}{2k} \end{pmatrix};\\
M(ST^2ST^2) &= \begin{pmatrix} -\cos\frac{2\pi}{2k} & -k\sin\frac{2\pi}{2k} \\ \frac{1}{k}\sin\frac{2\pi}{2k} & -\cos\frac{2\pi}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{\pi(3k-2)}{2k} & -k\cos\frac{\pi(3k-2)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(3k-2)}{2k} & -\sin\frac{\pi(3k-2)}{2k} \end{pmatrix}\\
&\sim \begin{pmatrix} \sin\frac{\pi(k-2)}{2k} & -k\cos\frac{\pi(k-2)}{2k} \\ -\frac{1}{k}\cos\frac{\pi(k-2)}{2k} & -\sin\frac{\pi(k-2)}{2k} \end{pmatrix};\\
M(ST^2ST^2S) &= \begin{pmatrix} -\sin\frac{3\pi}{2k} & k\cos\frac{3\pi}{2k} \\ \frac{1}{k}\cos\frac{3\pi}{2k} & \sin\frac{3\pi}{2k} \end{pmatrix} \sim \begin{pmatrix} \sin\frac{3\pi}{2k} & -k\cos\frac{3\pi}{2k} \\ -\frac{1}{k}\cos\frac{3\pi}{2k} & -\sin\frac{3\pi}{2k} \end{pmatrix}.
\end{aligned}$$

γ	$M(\gamma)\cdot C_{\text{seed}}\cdot M(\gamma)^\dagger$
$\mathbb{1}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$
S	$\begin{pmatrix}\frac{1}{2}&-\frac{1}{4}\\-\frac{1}{4}&\frac{1}{8}\end{pmatrix}$
ST^2	$\begin{pmatrix}\frac{1}{2}&\frac{1}{4}\\\frac{1}{4}&\frac{1}{8}\end{pmatrix}$
ST^2S	$\begin{pmatrix}0&0\\0&\frac{1}{4}\end{pmatrix}$
X^{av}	$\begin{pmatrix}1&0\\0&\frac{1}{4}\end{pmatrix}$

Table 2. Orbit of the vacuum block for $N=2, k=2$.

Proceeding as above, all the $M(\gamma)$ can be brought to the form in (B.1) by making use of the identities

$$\begin{pmatrix}\sin\beta & -k\cos\beta\\ -\frac{1}{k}\cos\beta & -\sin\beta\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cdot\begin{pmatrix}\sin\alpha & -k\cos\alpha\\ -\frac{1}{k}\cos\alpha & -\sin\alpha\end{pmatrix}$$
$$=\begin{pmatrix}\sin(\alpha+\beta-\frac{\pi}{2}) & -k\cos(\alpha+\beta-\frac{\pi}{2})\\ -\frac{1}{k}\cos(\alpha+\beta-\frac{\pi}{2}) & -\sin(\alpha+\beta-\frac{\pi}{2})\end{pmatrix}$$

and

$$\begin{pmatrix}\sin\alpha & -k\cos\alpha\\ -\frac{1}{k}\cos\alpha & -\sin\alpha\end{pmatrix}\sim\begin{pmatrix}\sin(\alpha+\pi) & -k\cos(\alpha+\pi)\\ -\frac{1}{k}\cos(\alpha+\pi) & -\sin(\alpha+\pi)\end{pmatrix}$$

for any angle α and β.

For completeness, we provide the orbit the $N(=k)=2$ theory. It can easily be checked that this is same as that given by the matrices in (4.10). For $N=2, k=2$ the matrices $M(S)$ and $M(T^2)$ are

$$M(S)=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\sqrt{2}\\ -\frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}}\end{pmatrix},\qquad M(T^2)=e^{-\frac{i\pi}{2}}\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{B.3}$$

The orbit of C_{seed} consists of four matrices. It is generated by the action of $\mathbb{1}, S, ST^2$ and ST^2S. We tabulate the results of these actions in table 2. The normalised sum over the orbit (4.1) reproduces the KZ result.

C Further numerical examples

Here we provide a couple of examples where the numerics are quite involved as discussed at the end of section 4.

$N=5$, $k=6$: for $N=5, k=6$, the value of $m(5,6)$ as defined in (4.5) is 11. Thus with each increment in ℓ_{max} by 1, there is approximately a tenfold increase in the number of new terms added to the sum (4.12). With the available computing resources we have

JHEP10(2019)061

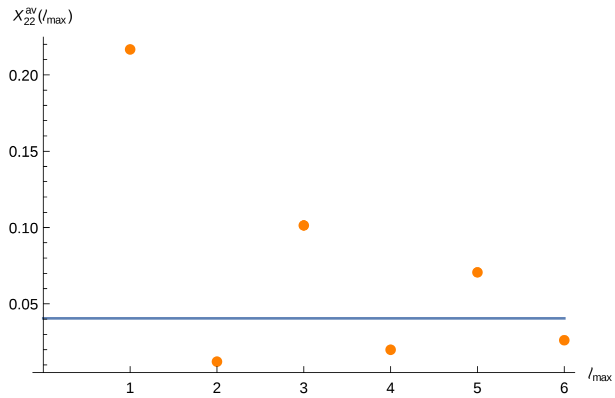

Figure 7. Orange dots show $X_{22}^{\text{av}}(\ell_{\text{max}})$ in the range $[0.005, 0.225]$ plotted against ℓ_{max}. Blue horizontal line at 0.0405346 represents X_{22}^{KZ}.

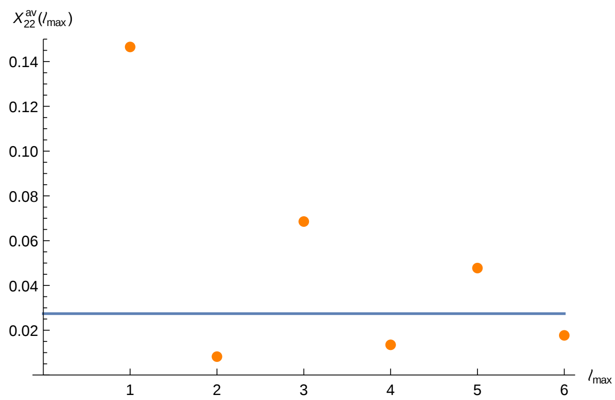

Figure 8. Orange dots show $X_{22}^{\text{av}}(\ell_{\text{max}})$ in the range $[0.000, 0.150]$ plotted against ℓ_{max}. Blue horizontal line at 0.0274114 represents X_{22}^{KZ}.

performed the sum upto $\ell_{\text{max}} = 6$. This involves 1193006 distinct contributions to the sum. We find $X_{22}^{\text{av}}(6) = 0.026177$, alongside we note the exact result (4.3), $X_{22}^{\text{KZ}} \approx 0.0405346$. The off diagonal entries of $X^{\text{av}}(6)$ are of the order of 10^{-14}. Figure 7 shows our results for $X_{22}^{\text{av}}(\ell_{\text{max}})$ as a function of ℓ_{max}, all qualitative features of the numerics are same as those in the examples discussed in section 4.

***N* = 6, *k* = 5:** for $N = 6, k = 5$, the value of $m(6,5)$ as defined in (4.5) is 11. Thus similarly, with each increment in ℓ_{max} by 1, there is approximately a tenfold increase in the number of new terms added to the sum (4.12). With the available computing resources we have performed the sum upto $\ell_{\text{max}} = 6$. This involves 1193006 distinct contributions to the sum. We find $X_{22}^{\text{av}}(6) = 0.0177022$, alongside we note the exact result (4.3), $X_{22}^{\text{KZ}} \approx 0.0274114$. The off diagonal entries of $X^{\text{av}}(6)$ are of the order of 10^{-14}. Figure 8 shows our results for $X_{22}^{\text{av}}(\ell_{\text{max}})$ as a function of ℓ_{max}. All the features of the numerics are similar to the previous example.

JHEP10(2019)061

D Averaging over all of PSL$(2,\mathbb{Z})$

In this appendix, we briefly discuss the construction of correlator from averaging over the full modular group. To implement the prescription (2.17), the six holomorphic blocks in (A.10) of the three correlators in (3.23) can be put in a six dimensional row:

$$\vec{\mathcal{H}}(\tau) = \left(\mathcal{H}^{\mathbb{1}}_{(1)}(\tau), \mathcal{H}^{\theta}_{(1)}(\tau), \mathcal{H}^{\mathbb{1}}_{(2)}(\tau), \mathcal{H}^{\theta}_{(2)}(\tau), \mathcal{H}^{\xi}_{(3)}(\tau), \mathcal{H}^{\chi}_{(3)}(\tau)\right) . \tag{D.1}$$

On this, T and S act as

$$\mathcal{H}^i(T.\tau) = \mathcal{H}^j(\tau)\mathcal{M}_{ji}(T) \ \text{ and } \ \mathcal{H}^i(S.\tau) = \mathcal{H}^j(\tau)\mathcal{M}_{ji}(S) \tag{D.2}$$

with

$$\mathcal{M}(T) = \begin{pmatrix} 0 & M_{(1)}(T) & 0 \\ M_{(2)}(T) & 0 & 0 \\ 0 & 0 & M_{(3)}(T) \end{pmatrix} \text{ and } \mathcal{M}(S) = \begin{pmatrix} M_{(1)}(S) & 0 & 0 \\ 0 & 0 & M_{(2)}(S) \\ 0 & M_{(3)}(S) & 0 \end{pmatrix} , \tag{D.3}$$

where the two dimensional matrices ($M_{(i)}(T)$ and $M_{(i)}(S)$) are as defined in appendix A. The light contribution as defined in (2.16) can be taken as

$$G^{\text{light}}_B(\tau,\bar\tau) = C^B_{i(B)j(B)}\mathcal{H}^{i(B)}(\tau)\bar{\mathcal{H}}^{j(B)}(\bar\tau), \ \ B = 1,2,3 , \tag{D.4}$$

where repeated indices are summed over with $i(1), j(1) \in \{1,2\}$, $i(2), j(2) \in \{3,4\}$ and $i(3), j(3) \in \{5,6\}$,

$$C^B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ \ B = 1,2,3 . \tag{D.5}$$

Under the action $\gamma \in \text{PSL}(2,\mathbb{Z})$,

$$C^B_{i(B)j(B)}\mathcal{H}^{i(B)}(\tau)\bar{\mathcal{H}}^{j(B)}(\bar\tau) \to \mathcal{M}(\gamma)_{ki(B)}C^B_{i(B)j(B)}\mathcal{M}(\gamma)^\dagger_{j(B)l}\mathcal{H}^k(\tau)\bar{\mathcal{H}}^l(\bar\tau) . \tag{D.6}$$

For each γ we arrange the three 6×6 matrices

$$\sigma^{-1}(\gamma)_{AB}\mathcal{M}(\gamma)_{ki(B)}C^B_{i(B)j(B)}\mathcal{M}(\gamma)^\dagger_{j(B)l} \ , \ \ A = 1,2,3 , \tag{D.7}$$

in a three dimensional column $\vec{X}(\gamma)$. The sum (2.17) then reads

$$\vec{X}^{\text{av}} = \mathcal{N}^{-1} \cdot \sum_{\gamma\in\text{PSL}(2,\mathbb{Z})} \vec{X}(\gamma) , \tag{D.8}$$

where the normalisation $\mathcal{N}$ is the $(1,1)$ element of $\left[\sum_\gamma \vec{X}(\gamma)\right]^1$. Hence the candidate for the vector-valued modular function (3.23) is given by

$$[\vec{X}^{\text{av}}]^A_{kl}\mathcal{H}^k(\tau)\bar{\mathcal{H}}^l(\bar\tau), \ \ A = 1,2,3 . \tag{D.9}$$

To incorporate the distinct contributions $\vec{X}(\gamma)$ to the sum (D.8), elements γ are arranged in a list similar to (4.6) where we replace all T^{2r_i} by T^{r_i}, and m denotes the smallest positive integer such that

$$\mathcal{M}(T^m) \propto \mathbb{1} .$$

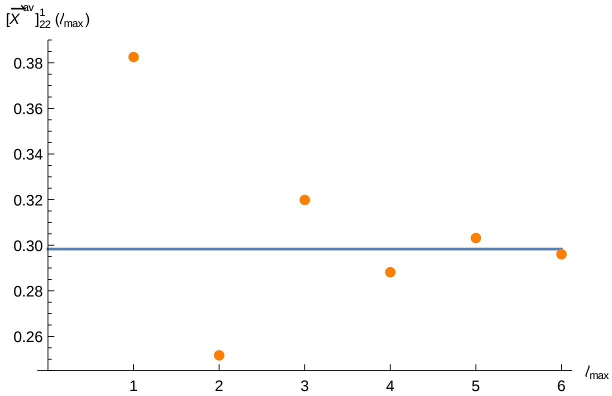

Figure 9. Orange dots show $\left[\vec{X}^{\rm av}\right]^1_{22}(\ell_{\rm max})$ in the range $[0.245, 0.390]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.29831 represents the KZ result.

We perform the sum (D.8) taking distinct contributions of elements γ of all lengths upto a maximum value $\ell_{\rm max}$:

$$\vec{X}^{\rm av}(\ell_{\rm max}) = \mathcal{N}(\ell_{\rm max})^{-1} \cdot \sum_{\ell(\gamma)\leq \ell_{\rm max}}{}' \vec{X}(\gamma)\,, \tag{D.10}$$

where the primed sum indicates that distinct elements are added. Our results are as follows

***N* = 2, *k* = 2:** for $N = 2, k = 2$, the sum (D.10) is finite and consists of six distinct contributions, reproducing the KZ result, $\left[\vec{X}^{\rm av}\right]^1_{22} = \frac{1}{4}$.

***N* = 2, *k* = 4:** for $N = 2, k = 4$, the sum (D.10) is finite and consists of four distinct contributions, reproducing the KZ result, $\left[\vec{X}^{\rm av}\right]^1_{22} = \frac{1}{2\sqrt[3]{4}}$.

***N* = 2, *k* = 3:** for $N = 2, k = 3$, the sum (D.10) seems to be infinite. We have performed the sum upto $\ell_{\rm max} = 6$. This invloves 83651 distinct contributions to the sum. We find $\left[\vec{X}^{\rm av}\right]^1_{22}(6) = 0.296026$, which is in good agreement with the KZ result. Figure 9 shows our results for $\left[\vec{X}^{\rm av}\right]^1_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

Finally, let us discuss some examples where modular averaging does not yield the correlator.

***N* = 3, *k* = 2:** for $N = 3, k = 2$, the sum (D.10) seems to be infinite. We have performed the sum upto $\ell_{\rm max} = 6$. This invloves 664111 distinct contributions to the sum. We find $\left[\vec{X}^{\rm av}\right]^1_{22}(6) = 0.151496$, which is not in agreement with the KZ result, although crossing symmetric. Figure 10 shows our results for $\left[\vec{X}^{\rm av}\right]^1_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$.

***N* = 4, *k* = 2:** for $N = 4, k = 2$, the sum (D.10) seems to be infinite. We have performed the sum upto $\ell_{\rm max} = 8$. This invloves 69219 distinct contributions to the sum. We find $\left[\vec{X}^{\rm av}\right]^1_{22}(8) = 0.111064$, which is not in agreement with the KZ result, although crossing symmetric. Figure 11 shows our results for $\left[\vec{X}^{\rm av}\right]^1_{22}(\ell_{\rm max})$ as a function of $\ell_{\rm max}$. Thus while summing over the entire modular group we have found examples where the

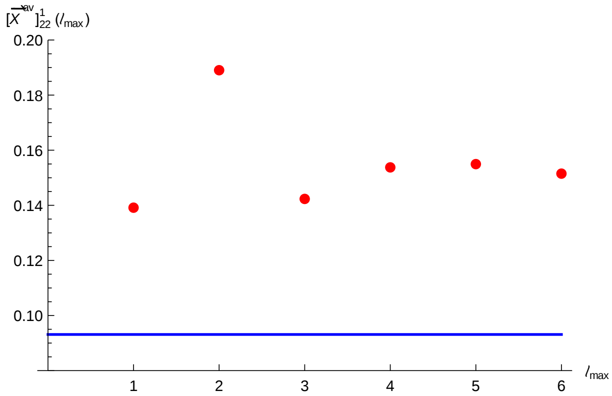

Figure 10. Red dots show $[\vec{X}^{\rm av}]^1_{22}(\ell_{\rm max})$ in the range $[0.08, 0.20]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0931172 represents the KZ result.

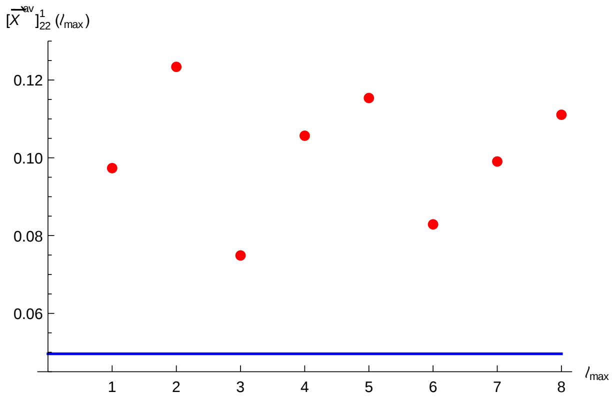

Figure 11. Red dots show $[\vec{X}^{\rm av}]^1_{22}(\ell_{\rm max})$ in the range $[0.045, 0.130]$ plotted against $\ell_{\rm max}$. Blue horizontal line at 0.0496063 represents the KZ result.

averaging does not reproduce the correlator (see appendix D). We note that, [9] argues that it is the modular averaging over the theta subgroup that has a direct interpretation in the holographic context.

Increasing N and k makes the numerics quite involved, we leave this for future work.

E The matrices $M^{\ell}_{N,k}$ and $\tilde{M}^{\ell}_{N,k}$

In this appendix, we obtain the general form of the matrices $M^{\ell}_{N,k}$ and $\tilde{M}^{\ell}_{N,k}$. We then use these to derive the relations given in (5.11). The elements of matrices $M^{\ell}_{N,k}$ can be computed recursively in ℓ using their defining equation in (5.4)

$$M^{\ell+1}_{N,k}(r_1, \cdots r_{\ell+1}) = M(T^{2r_{\ell+1}})M(S)M^{\ell}_{N,k}(r_1, \cdots r_\ell). \tag{E.1}$$

This gives the following relations for the functions that appear in (5.9)

$$\begin{aligned}
a^{\ell+1}_{N,k}(r_1, \cdots r_{\ell+1}) &= a_s(N,k)a^{\ell}_{N,k}(r_1 \cdots r_\ell) + b_s(N,k)c_s(N,k)c^{\ell}_{N,k}(r_1 \cdots r_\ell)\\
b^{\ell+1}_{N,k}(r_1, \cdots r_{\ell+1}) &= a_s(N,k)b^{\ell}_{N,k}(r_1 \cdots r_\ell) + d^{\ell}_{N,k}(r_1 \cdots r_\ell)
\end{aligned}$$

$$c^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = e^{ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)c^{\ell}_{N,k}(r_1\cdots r_\ell)+a^{\ell}_{N,k}(r_1\cdots r_\ell)\right)$$
$$d^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = e^{ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)d^{\ell}_{N,k}(r_1\cdots r_\ell)+b_s(N,k)c_s(N,k)b^{\ell}_{N,k}(r_1\cdots r_\ell)\right)$$

Similarly, the matrices $\tilde{M}^{\ell}_{N,k}$ can be computed recursively in ℓ using their defining equation n (5.5)

$$\tilde{M}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = M(T^{-2r_{\ell+1}})M(S)\tilde{M}^{\ell}_{N,k}(r_1,\cdots r_\ell). \tag{E.2}$$

This gives following relations for the functions that appear in (5.10)

$$\tilde{a}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = a_s(N,k)\tilde{a}^{\ell}_{N,k}(r_1\cdots r_\ell)+b_s(N,k)c_s(N,k)\tilde{c}^{\ell}_{N,k}(r_1\cdots r_\ell)$$
$$\tilde{b}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = a_s(N,k)\tilde{b}^{\ell}_{N,k}(r_1\cdots r_\ell)+\tilde{d}^{\ell}_{N,k}(r_1\cdots r_\ell)$$
$$\tilde{c}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = e^{-ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)\tilde{c}^{\ell}_{N,k}(r_1\cdots r_\ell)+\tilde{a}^{\ell}_{N,k}(r_1\cdots r_\ell)\right)$$
$$\tilde{d}^{\ell+1}_{N,k}(r_1,\cdots r_{\ell+1}) = e^{-ir_{\ell+1}\phi(N,k)}\left(d_s(N,k)\tilde{d}^{\ell}_{N,k}(r_1\cdots r_\ell)+b_s(N,k)c_s(N,k)\tilde{b}^{\ell}_{N,k}(r_1\cdots r_\ell)\right).$$

Now, making use of relations in (5.3) and the fact that[15]

$$e^{ir\phi(N,k)} = e^{-ir\phi(k,N)} \quad \text{for any integer r}, \tag{E.3}$$

it is easy to see that $\tilde{a}^{\ell}_{k,N}(r_i), \tilde{b}^{\ell}_{k,N}(r_i), \tilde{c}^{\ell}_{k,N}(r_i), \tilde{d}^{\ell}_{k,N}(r_i)$ have exactly the same recurrence relations as $a^{\ell}_{N,k}(r_i), b^{\ell}_{N,k}(r_i), c^{\ell}_{N,k}(r_i), d^{\ell}_{N,k}(r_i)$. Given that they have same initial values, hence the equalities in (5.11).

F Truncation of sums

The sum in (4.12) terminates after a (lowest) value ℓ^0_{max} if the actions of γs in the list (4.6) with $\ell(\gamma) > \ell^0_{\text{max}}$ do not generate new elements of the orbit of C_{seed}, i.e. the orbit is finite. Note that if there is an ℓ such that no new terms are generated, higher values of ℓ also do not generate new terms in the orbit (with this the value of the sum in (4.12) at higher ℓ_{max} does not change). Thus comparison of the terms generated at a certain ℓ with the ones at lower ℓ can be used to determine the cases with finite orbit. It is possible to implement this consideration at each point in the (N,k) lattice (of course the non-trivial cases are for $N,k \geq 2$). Before discussing the details, we summarise our results. Truncations start from $\ell^0_{\text{max}} = 1$. Here, it is possible to determine analytically the values of (N,k) for which the truncations occur — $(3,3),(2,4)$ and $(4,2)$ are the only points where the modular sum truncates at $\ell^0_{\text{max}} = 1$. For higher values of ℓ, except for cases with $N = k$ we have not been able to carry out a general analysis so as to determine the points in the (N,k) lattice for which truncations occur (the results for $N = k$ are given in section 4, recall that all these models exhibit truncation). For $N \neq k$ we have implemented the above algorithm numerically, and found that upto $\ell = 5$, for points in the (N,k) lattice with $N,k \leq 6$ (and $N \neq k$) there are no truncations.

[15] Recall that $\phi(N,k) = \frac{2\pi N}{k+N}$.

JHEP10(2019)061

The details of the analysis are as follows. We recall (B.2). The action of $\mathbb{1}$ on $C_{\rm seed}$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{F.1}$$

The actions of γs of $\ell(\gamma) = 1$ on $C_{\rm seed}$ are given by

$$\begin{pmatrix} a_S^2 & a_S c_S e^{-\frac{i2\pi N r_1}{\tilde{k}}} \\ a_S c_S e^{\frac{i2\pi N r_1}{\tilde{k}}} & c_S^2 \end{pmatrix}, \tag{F.2}$$

for $r_1 = 0, \cdots, (m(N,k)-1)$. Here a_S is the 1-1 entry of the matrix $M(S)$ (see for e.g (5.2)). Clearly the phases at the off-diagonal entries are the $m(N,k)$-th roots of unity, hence all distinct and add up to zero. The actions of γs of $\ell(\gamma) = 2$ on $C_{\rm seed}$ are given by[16]

$$\begin{pmatrix} 1-2a_S^2+2a_S^4-2a_S^2(a_S^2-1)\cos(\frac{2\pi N r_1}{\tilde{k}}) & a_S c_S e^{-\frac{i2\pi N r_2}{\tilde{k}}}(1-e^{-\frac{i2\pi N r_1}{\tilde{k}}})(a_S^2-(a_S^2-1)e^{\frac{i2\pi N r_1}{\tilde{k}}}) \\ a_S c_S e^{\frac{i2\pi N r_2}{\tilde{k}}}(1-e^{\frac{i2\pi N r_1}{\tilde{k}}})(a_S^2-(a_S^2-1)e^{-\frac{i2\pi N r_1}{\tilde{k}}}) & 2a_S^2 c_S^2(1-\cos(\frac{2\pi N r_1}{\tilde{k}})) \end{pmatrix}, \tag{F.3}$$

for $r_1 = 1, \cdots, (m(N,k)-1)$ and $r_2 = 0, \cdots, (m(N,k)-1)$. Comparing with the structure of the terms generated at length zero (F.1) and one (F.2), we see that truncation requires that the following equality necessarily holds for all $r_1 = 1 \cdots (m(N,k)-1)$.

$$\cos\left(\frac{2\pi N r_1}{\tilde{k}}\right) = \frac{2a_s^2 - 1}{a_S^2}\ . \tag{F.4}$$

Hence necessarily $a_S^2 \geq \frac{1}{4}$, which holds only when (N,k) lies on the line $N = 2$ or $k = 2$ or at the point $(3,3)$. Furthermore at any (N,k) the r.h.s. of (F.4) is fixed which restricts the number of values r_1 can take. This in turn gives a necessary condition for the possible values for $m(N,k)$: it must be 2 or 3. Hence $(2,2), (3,3), (2,4)$ and $(4,2)$ models are the only ones which satisfy this criterion. Going through each of these possibilities case by case one finds that truncation and $\ell^0_{\rm max} = 1$ occurs for $(3,3), (2,4)$ and $(4,2)$. Similar considerations necessary to determine truncations at higher ℓ are more involved (except for the cases with $N = k$); we have implemented them numerically and found for points in the (N,k) lattice with $N, k \leq 6$ (and $N \neq k$), there are no truncations upto $\ell = 5$.

References

[1] S. Ferrara, A.F. Grillo and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, *Annals Phys.* **76** (1973) 161 [INSPIRE].

[2] A.M. Polyakov, *Nonhamiltonian approach to conformal quantum field theory*, *Zh. Eksp. Teor. Fiz.* **66** (1974) 23 [*Sov. Phys. JETP* **39** (1974) 9] [INSPIRE].

[16] After using $b_S c_S = 1 + a_S d_S$ and $d_S = -a_S$.

JHEP10(2019)061

[3] D. Poland, S. Rychkov and A. Vichi, *The conformal bootstrap: theory, numerical techniques and applications*, Rev. Mod. Phys. **91** (2019) 015002 [arXiv:1805.04405] [INSPIRE].

[4] D. Simmons-Duffin, *The conformal bootstrap*, arXiv:1602.07982 [INSPIRE].

[5] S. Rychkov, *EPFL lectures on conformal field theory in $D \geq 3$ dimensions*, arXiv:1601.05000.

[6] M.F. Paulos, *JuliBootS: a hands-on guide to the conformal bootstrap*, arXiv:1412.4127 [INSPIRE].

[7] C. Beem, L. Rastelli, A. Sen and B.C. van Rees, *Resummation and S-duality in $N = 4$ SYM*, JHEP **04** (2014) 122 [arXiv:1306.3228] [INSPIRE].

[8] S. Jain et al., *Unitarity, crossing symmetry and duality of the S-matrix in large N Chern-Simons theories with fundamental matter*, JHEP **04** (2015) 129 [arXiv:1404.6373] [INSPIRE].

[9] A. Maloney, H. Maxfield and G.S. Ng, *A conformal block Farey tail*, JHEP **06** (2017) 117 [arXiv:1609.02165] [INSPIRE].

[10] A. Castro et al., *The gravity dual of the Ising model*, Phys. Rev. **D 85** (2012) 024032 [arXiv:1111.1987] [INSPIRE].

[11] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B 241** (1984) 333 [INSPIRE].

[12] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two-dimensions*, Nucl. Phys. **B 247** (1984) 83 [INSPIRE].

[13] D. Gepner and E. Witten, *String theory on group manifolds*, Nucl. Phys. **B 278** (1986) 493 [INSPIRE].

[14] A. Cappelli, C. Itzykson and J.B. Zuber, *Modular invariant partition functions in two-dimensions*, Nucl. Phys. **B 280** (1987) 445 [INSPIRE].

[15] J.L. Cardy, *Operator content of two-dimensional conformally invariant theories*, Nucl. Phys. **B 270** (1986) 186 [INSPIRE].

[16] E.P. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. **B 300** (1988) 360 [INSPIRE].

[17] R. Dijkgraaf and E.P. Verlinde, *Modular invariance and the fusion algebra*, Nucl. Phys. Proc. Suppl. **5** (1988) 87 [INSPIRE].

[18] G.W. Moore and N. Seiberg, *Polynomial equations for rational conformal field theories*, Phys. Lett. **B 212** (1988) 451 [INSPIRE].

[19] G.W. Moore and N. Seiberg, *Naturality in conformal field theory*, Nucl. Phys. **B 313** (1989) 16 [INSPIRE].

[20] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, Phys. Rept. **223** (1993) 183 [hep-th/9210010] [INSPIRE].

[21] S. Hellerman, *A universal inequality for CFT and quantum gravity*, JHEP **08** (2011) 130 [arXiv:0902.2790] [INSPIRE].

[22] S. Ribault and R. Santachiara, *Liouville theory with a central charge less than one*, JHEP **08** (2015) 109 [arXiv:1503.02067] [INSPIRE].

[23] M. Picco, S. Ribault and R. Santachiara, *A conformal bootstrap approach to critical percolation in two dimensions*, *SciPost Phys.* **1** (2016) 009.

[24] C.-M. Chang and Y.-H. Lin, *Bootstrap, universality and horizons*, *JHEP* **10** (2016) 068 [arXiv:1604.01774] [INSPIRE].

[25] T. Anous, R. Mahajan and E. Shaghoulian, *Parity and the modular bootstrap*, *SciPost Phys.* **5** (2018) 022 [arXiv:1803.04938] [INSPIRE].

[26] W. Li, *New method for the conformal bootstrap with OPE truncations*, arXiv:1711.09075 [INSPIRE].

[27] S. Collier, Y. Gobeil, H. Maxfield and E. Perlmutter, *Quantum Regge trajectories and the Virasoro analytic bootstrap*, *JHEP* **05** (2019) 212 [arXiv:1811.05710] [INSPIRE].

[28] H. Kim, P. Kravchuk and H. Ooguri, *Reflections on conformal spectra*, *JHEP* **04** (2016) 184 [arXiv:1510.08772] [INSPIRE].

[29] S. Collier, P. Kravchuk, Y.-H. Lin and X. Yin, *Bootstrapping the spectral function: on the uniqueness of Liouville and the universality of BTZ*, *JHEP* **09** (2018) 150 [arXiv:1702.00423] [INSPIRE].

[30] D. Friedan and C.A. Keller, *Constraints on 2d CFT partition functions*, *JHEP* **10** (2013) 180 [arXiv:1307.6562] [INSPIRE].

[31] M. Ashrafi and F. Loran, *Non-chiral 2d CFT with integer energy levels*, *JHEP* **09** (2016) 121 [arXiv:1607.08516] [INSPIRE].

[32] T. Hartman, C.A. Keller and B. Stoica, *Universal spectrum of 2d conformal field theory in the large c limit*, *JHEP* **09** (2014) 118 [arXiv:1405.5137] [INSPIRE].

[33] P. Kraus and A. Sivaramakrishnan, *Light-state dominance from the conformal bootstrap*, *JHEP* **08** (2019) 013 [arXiv:1812.02226] [INSPIRE].

[34] Y. Kusuki, *Light cone bootstrap in general 2D CFTs and entanglement from light cone singularity*, *JHEP* **01** (2019) 025 [arXiv:1810.01335] [INSPIRE].

[35] M. Cho, S. Collier and X. Yin, *Genus two modular bootstrap*, *JHEP* **04** (2019) 022 [arXiv:1705.05865] [INSPIRE].

[36] A. Belin, C.A. Keller and I.G. Zadeh, *Genus two partition functions and Rényi entropies of large c conformal field theories*, *J. Phys.* **A 50** (2017) 435401 [arXiv:1704.08250] [INSPIRE].

[37] N. Benjamin, E. Dyer, A.L. Fitzpatrick and Y. Xin, *The most irrational rational theories*, *JHEP* **04** (2019) 025 [arXiv:1812.07579] [INSPIRE].

[38] N. Afkhami-Jeddi, T. Hartman and A. Tajdini, *Fast conformal bootstrap and constraints on 3d gravity*, *JHEP* **05** (2019) 087 [arXiv:1903.06272] [INSPIRE].

[39] W. Song and J. Xu, *Structure constants from modularity in warped CFT*, arXiv:1903.01346 [INSPIRE].

[40] C.N. Gowdigere, J. Santara and Sumedha, *Conformal bootstrap signatures of the tricritical Ising universality class*, arXiv:1811.11442 [INSPIRE].

[41] A. de la Fuente, *Bootstrapping mixed correlators in the 2D Ising model*, arXiv:1904.09801 [INSPIRE].

[42] S. Collier, Y.-H. Lin and X. Yin, *Modular bootstrap revisited*, *JHEP* **09** (2018) 061 [arXiv:1608.06241] [INSPIRE].

JHEP10(2019)061

[43] N. Benjamin, E. Dyer, A.L. Fitzpatrick and S. Kachru, *Universal bounds on charged states in 2d CFT and 3d gravity*, *JHEP* **08** (2016) 041 [arXiv:1603.09745] [INSPIRE].

[44] E. Dyer, A.L. Fitzpatrick and Y. Xin, *Constraints on flavored 2d CFT partition functions*, *JHEP* **02** (2018) 148 [arXiv:1709.01533] [INSPIRE].

[45] J.-B. Bae, S. Lee and J. Song, *Modular constraints on conformal field theories with currents*, *JHEP* **12** (2017) 045 [arXiv:1708.08815] [INSPIRE].

[46] Y.-H. Lin, S.-H. Shao, Y. Wang and X. Yin, (2, 2) *superconformal bootstrap in two dimensions*, *JHEP* **05** (2017) 112 [arXiv:1610.05371] [INSPIRE].

[47] D. Das, S. Datta and S. Pal, *Charged structure constants from modularity*, *JHEP* **11** (2017) 183 [arXiv:1706.04612] [INSPIRE].

[48] J.-B. Bae, S. Lee and J. Song, *Modular constraints on superconformal field theories*, *JHEP* **01** (2019) 209 [arXiv:1811.00976] [INSPIRE].

[49] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole Farey tail*, hep-th/0005003 [INSPIRE].

[50] P. Kraus, *Lectures on black holes and the AdS_3/CFT_2 correspondence*, *Lect. Notes Phys.* **755** (2008) 193 [hep-th/0609074] [INSPIRE].

[51] A. Maloney and E. Witten, *Quantum gravity partition functions in three dimensions*, *JHEP* **02** (2010) 029 [arXiv:0712.0155] [INSPIRE].

[52] J.F. Duncan and I.B. Frenkel, *Rademacher sums, Moonshine and gravity*, *Commun. Num. Theor. Phys.* **5** (2011) 849 [arXiv:0907.4529] [INSPIRE].

[53] A. Castro, N. Lashkari and A. Maloney, *A de Sitter Farey tail*, *Phys. Rev.* **D 83** (2011) 124027 [arXiv:1103.4620] [INSPIRE].

[54] C.A. Keller and A. Maloney, *Poincaré series, 3D gravity and CFT spectroscopy*, *JHEP* **02** (2015) 080 [arXiv:1407.6008] [INSPIRE].

[55] S. Murthy and B. Pioline, *A Farey tale for $N = 4$ dyons*, *JHEP* **09** (2009) 022 [arXiv:0904.4253] [INSPIRE].

[56] S.G. Naculich and H.J. Schnitzer, *Duality between* $\mathrm{SU}(N)_k$ *and* $\mathrm{SU}(K)$-N *WZW Models*, *Nucl. Phys.* **B 347** (1990) 687 [INSPIRE].

[57] *MathWorld — A Wolfram Web Resource: Elliptic Lambda Function*, http://mathworld.wolfram.com/EllipticLambdaFunction.html.

[58] A.B. Zamolodchikov, *Conformal symmetry in two-dimensional space: Recursion representation of conformal block*, *Theor. Math. Phys.* **73** (1987) 1088.

[59] P.D. Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Graduate Texts in Contemporary Physics, Springer, Germany (1997).

[60] R. Blumenhagen and E. Plauschinn, *Introduction to conformal field theory: with applications to string theory*, Lecture Notes in Physics, Springer, Germany (2009), see chapter 3.

[61] *MathWorld — A Wolfram Web Resource: Hypergeometric Function*, http://mathworld.wolfram.com/HypergeometricFunction.html.

[62] R.W. Bruggeman, *Automorphic forms for the theta group*, The Modern Birkhäuser Classics book series , Springer, Germany (1994), see chapter 14.

[63] S.G. Naculich and H.J. Schnitzer, *Duality relations between* SU(N)-k *and* SU(K)-N *WZW models and their braid matrices*, *Phys. Lett.* **B 244** (1990) 235 [INSPIRE].

[64] S.G. Naculich, H.A. Riggs and H.J. Schnitzer, *Group level duality in WZW models and Chern-Simons theory*, *Phys. Lett.* **B 246** (1990) 417 [INSPIRE].

[65] S. Jackson, L. McGough and H. Verlinde, *Conformal bootstrap, universality and gravitational scattering*, *Nucl. Phys.* **B 901** (2015) 382 [arXiv:1412.5205] [INSPIRE].

[66] A.L. Fitzpatrick, J. Kaplan and M.T. Walters, *Virasoro conformal blocks and thermality from classical background fields*, *JHEP* **11** (2015) 200 [arXiv:1501.05315] [INSPIRE].

[67] A.L. Fitzpatrick and J. Kaplan, *Conformal blocks beyond the semi-classical limit*, *JHEP* **05** (2016) 075 [arXiv:1512.03052] [INSPIRE].

[68] E. Hijano, P. Kraus, E. Perlmutter and R. Snively, *Semiclassical Virasoro blocks from* AdS_3 *gravity*, *JHEP* **12** (2015) 077 [arXiv:1508.04987] [INSPIRE].

[69] E. Hijano, P. Kraus and R. Snively, *Worldline approach to semi-classical conformal blocks*, *JHEP* **07** (2015) 131 [arXiv:1501.02260] [INSPIRE].

[70] C.-M. Chang and Y.-H. Lin, *Bootstrapping* 2*D CFTs in the semiclassical limit*, *JHEP* **08** (2016) 056 [arXiv:1510.02464] [INSPIRE].

[71] A. Bhatta, P. Raman and N.V. Suryanarayana, *Holographic conformal partial waves as gravitational open Wilson networks*, *JHEP* **06** (2016) 119 [arXiv:1602.02962] [INSPIRE].

[72] J.C. Cresswell, I.T. Jardine and A.W. Peet, *Holographic relations for OPE blocks in excited states*, *JHEP* **03** (2019) 058 [arXiv:1809.09107] [INSPIRE].

[73] B. Carneiro da Cunha and M. Guica, *Exploring the BTZ bulk with boundary conformal blocks*, arXiv:1604.07383 [INSPIRE].

[74] V. Balasubramanian et al., *Heavy-heavy-light-light correlators in Liouville theory*, *JHEP* **08** (2017) 045 [arXiv:1705.08004] [INSPIRE].