

A note on inhomogeneous ground states at large global charge

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ABSTRACT: In this note we search for the ground state of the $D = 3$ Wilson-Fisher conformal $O(4)$ model, at nonzero values of the two independent charge densities $\rho_{1,2}$, on the torus spatial slice. Using an effective theory valid on scales longer than the scale defined by the charge density, we show that the ground-state configuration is inhomogeneous for generic ratios ρ_1/ρ_2 . This result confirms, within the context of a well-defined effective theory, a recent no-go result of [1]. We also show that any spatially periodic ground state solutions have an energetic preference towards longer periods, within some range of ρ_1/ρ_2 containing a neighborhood of zero. This suggests that the scale of variation of the ground state solution in finite volume will be the infrared scale, and that the use of the effective theory at large charge in finite volume is self-consistent. Note added: the statements in this paper are true for arbitrary ratio of ρ_1/ρ_2 , which we proved after we uploaded this paper. See [2].

KEYWORDS: Conformal Field Theory, Effective Field Theories, Global Symmetries

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1 Introduction

Conformal field theories with global symmetries display interesting and useful simplifications in the sector of large global charge. These simplifications make it possible to calculate asymptotic expansions of charged operator dimensions and OPE coefficients [1, 3–5] to any desired accuracy in terms of a small number of undetermined coefficients in an effective Lagrangian describing the local dynamics of the system at large charge density. Although these calculations use Lagrangian methods, the results are strikingly parallel to the large-spin expansion of operator dimensions obtained by the light-cone bootstrap¹ [8, 9] and along with those results, work best in a regime of large charge and large operator dimensions, complementing the regime of $O(1)$ charges and operator dimensions [10–14] that is accessed efficiently by numerical linear programming methods of solving the conformal bootstrap [15–17].

In [1, 3, 4], the properties of charged local operators are calculated in radial quantization by quantizing the large-charge effective Lagrangian on a spherical spatial slice. The hierarchy between the ultraviolet scale $E \ll \Lambda_{UV} \equiv \rho^{\frac{1}{D-1}}$ and the infrared scale, $E \ll \Lambda_{IR} = r^{-1} \ll \text{sphere}$ is $E \ll \Lambda_{IR}/E \ll \Lambda_{UV} \propto J^{-\frac{1}{D-1}}$, where J is the global charge of the local operator in the CFT. This large hierarchy renders the large-charge effective Lagrangian weakly coupled, and allows the perturbative computation of CFT data with quantum corrections and higher-derivative operators in the large-charge EFT, suppressed by inverse powers of J .

¹See also more recent work [6, 7].

In order to get started on such a calculation, one needs to know the structure of the large-charge effective Lagrangian, and the nature of the ground state carrying a given set of global charges. In the limit where the charge is taken to infinity, one can try to flatten out the sphere and consider the system in infinite flat space at fixed charge density ρ . Naively, then, it would seem that each large-charge limit in a CFT should correspond to a homogeneous ground state of a CFT with a chemical potential. And indeed, various interesting new phases of matter with spontaneously broken conformal and Lorentz symmetries have been derived through these considerations [1, 3, 4].

The expectation that the large-charge limit always defines a homogeneous phase of matter is a bit too naive however, as it assumes the classical solution describing the large-charge ground state on the sphere, is spherically symmetric. It is interesting to note that this expectation can be proven false in some very simple cases. In [1], the authors studied the conformal Wilson-Fisher $O(2N)$ model [18] in $D = 3$ at large Noether charge, and found that a homogeneous ground state in flat space exists only in the case where the element of the adjoint of $O(2N)$ defined by the total charge, has minimal rank, which is to say a single nonvanishing antisymmetric 2×2 block, and zeroes everywhere else. In the case where the charge matrix has minimal nonzero rank, the homogenous ground state in flat space was studied in detail and many interesting properties were extracted. Left unanswered is the question of the nature of the ground state when the charge matrix has nonminimal rank.

In this note, we will address this question in the simplest nontrivial example, that of the Wilson-Fisher $O(4)$ fixed point in three dimensions, on the spatial torus T^2 . We will find that there are no exactly homogeneous ground states, but a family of inhomogeneous, spatially periodic solutions of spatial period matching the larger cycle of the spatial torus, in some range of ratios of the two independent large charges, $0 < J_1/J_2 \ll 1$. (*Note: the same holds for arbitrary ratio $0 < J_1/J_2$, which was proven in [2] after this paper was completed.*) In this range of charges, then, the system will be driven dynamically into a regime where the fields vary slowly compared with the scale of the charge density, and the large-charge effective Lagrangian is parametrically reliable.

2 The $O(4)$ model at finite charge density

We now analyse the $O(4)$ model on the torus spatial slice, for general global charges. That is, we examine what the ground state looks like when we let the charge be proportional to a general element of the adjoint of $SO(4)$. We first refine and make more rigorous the result of [1] by following the recipe of [3], integrating out the heavy mode and working strictly within a conformal sigma model that is singular in the vacuum but non-singular around a state with large charge density.

In this framework, we rigorously reproduce the no-go result of [1]: we find that all candidate ground state solutions are inhomogeneous, and break the translational symmetry in one direction down to at most a discrete subgroup with period ℓ , if the antisymmetric matrix defining the charge has non-vanishing determinant. In particular, for each value of the ratio $\hat{\rho}_1/\hat{\rho}_2$ of the two eigenvalues of the charge matrix, there is one spatially peri-

odic solution with period ℓ that also satisfies a helical symmetry, *i.e.*, a symmetry under combined time translation and global symmetry rotation.

This raises two closely related questions: first, which candidate periodic solution is the true ground state? That is, which value of the spatial period ℓ , if any, minimizes the energy for given global charge densities? Second, for what range of ℓ is the effective field theory (EFT) reliable?

2.1 Parametrizing the charge density

To answer these questions quantitatively, we must find a convenient way to express the charge density itself, as an element of the adjoint of $SO(4)$, that is, a general 4×4 imaginary antisymmetric matrix. Such a matrix has real eigenvalues that occur in pairs with equal magnitude and opposite sign. The two independent positive eigenvalues are $\hat{\rho} \ll 1, 2$

Rather than parametrizing the charge density directly by the two independent eigenvalues $\hat{\rho} \ll 1, 2$ of the charge matrix, we follow [1] in choosing a basis for the chemical potential, which is equivalent to diagonalizing the generator defining the symmetry of the helical solution. Choose a complex basis for the fundamental of $U(2) \subset SO(4)$, and parametrize the charge generator by the two matrix elements $\rho \ll 1, 2$ on the diagonal. This will turn out to be equivalent: for helical solutions, the charge matrix commutes with the chemical potential, its off-diagonal terms always vanish, and $\hat{\rho}_1/\hat{\rho}_2$ is simply equal to ρ_1/ρ_2 .

We will see that there is an unstable direction of the classical solution, such that minimizing the energy at fixed charge densities in infinite volume, leads to an instability towards an infinite spatial period, for sufficiently small values of the ratio ρ_1/ρ_2 . In other words, on the torus spatial slice, the spatial period of the inhomogeneity becomes the large cycle of the spatial torus.

For purposes of computing the operator spectrum in radial quantization, we would ultimately want to put the theory on $\mathbb{R} \times S^2$, but in the present note we will aim to understand some local aspects of the charged ground state by taking a limit of large charge and fixed average density, which amounts to quantizing the theory on $\mathbb{R} \times \mathbb{R}^2$. We will comment in the Discussion section on the relevance to the ground state in finite volume.

2.2 Conformal sigma model from linear sigma model

The $O(4)$ model is described by four real scalars $X^{1,2,3,4}$, which we organize into a complex $SU(2)$ doublet $Q \equiv \begin{pmatrix} X_1 + iX_2 \\ X_3 + iX_4 \end{pmatrix}$. The $O(4)$ critical point is obtained by starting in the ultraviolet, giving the scalars a quartic potential proportional to $(X^2)^2 = |q|^4$, and fine-tuning the mass term $m^2|Q|^2$ to the unique strength such that the system has infinite correlation length and flows to a nontrivial fixed point of the renormalization group.

We wish to parametrise Q as follows in terms of amplitudes and angles:

$$Q = A \times q, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{2.1}$$

where $|q_1|^2 + |q_2|^2 = 1$. We can expand the solution at large A . The leading action at large and approximately constant A is sextic potential which is generated along the RG flow as explained in [3].

The Lagrangian of the theory in the IR becomes

$$\mathcal{L}_{\text{IR}} = \frac{1}{2}(\partial A)^2 + \frac{\gamma}{2}A^2\partial q^\dagger\partial q - \frac{h^2}{6}A^6, \tag{2.2}$$

under a field-reparametrization condition that the kinetic term of A is canonical.

We have omitted other terms as well. In the present note we use only the leading large-density term, and so we omit the Ricci coupling and higher derivative terms. The justification for the omission of these terms, is important and we must consider it carefully. Higher-derivative terms are suppressed when the fields vary on scales L which are long compared to the ultraviolet scale $(\rho_1 + \rho_2)^{-\frac{1}{2}}$. For values of L smaller than $(\rho_1 + \rho_2)^{-\frac{1}{2}}$, the large-charge effective theory is not within its range of validity, because the conformal goldstone fields are varying rapidly on the scale of the charge density itself. Higher-derivative operators and quantum corrections are unsuppressed, and there is no obvious simplification of the dynamics.

For generic charge densities $\rho \ll 1, 2$, we will find that there is no homogeneous ground state classical solution, so the question of the scale of variation L of the classical solution is crucial. If the ground state of the system has L smaller than or comparable to $(\rho_1 + \rho_2)^{-\frac{1}{2}}$, then the use of the effective theory is not allowed. If the ground state of the system only has inhomogeneities on scales L longer than $(\rho_1 + \rho_2)^{-\frac{1}{2}}$, the use of the EFT will be justified. We will see that the latter situation holds for some range of ρ_1/ρ_2 that contains a neighborhood of zero. For now, simply assume the fields are slowly varying on the scale set by the density itself and the effective theory will be usable; we will then justify this assumption *a posteriori*.

With this assumption, the field A has a mass scale set by the density, and therefore should be integrated out in such a limit. The equilibrium value of A is given by

$$\frac{\delta\mathcal{L}_{\text{IR}}}{\delta A} = 0 \iff A^2 = \sqrt{\frac{\partial q^\dagger\partial q}{\gamma^{-1}h^2}}. \tag{2.3}$$

Plugging this into (2.2), we get the conformal sigma model on S^3 as follows:

$$\mathcal{L} = b_q\mathcal{L}_0^{3/2} = b_q(\partial q^\dagger\partial q)^{3/2}, \tag{2.4}$$

where $|q| = 1$ and $b_q = \sqrt{\gamma^3 h^{-2}}/3$ is an undetermined coefficient which should come from the complicated, original RG flow equation, as in [3].

2.3 Restriction to fixed average charge densities $\rho \ll 1, 2$

Because we are putting the theory on \mathbb{R}^2 , and the concept of total charge is ill-defined, we can only fix the average charge density instead of the total charge itself. We impose the

following conditions unto Noether currents:

$$-\frac{2ib_q}{3} \int d^2x \sqrt{\mathcal{L}_0} \left[q^\dagger \partial_t q - \text{c.c.} \right] / \mathcal{V} = \rho_1 + \rho_2, \quad (2.5)$$

$$-\frac{2ib_q}{3} \int d^2x \sqrt{\mathcal{L}_0} \left[q^\dagger \sigma^3 \partial_t q - \text{c.c.} \right] / \mathcal{V} = \rho_1 - \rho_2, \quad (2.6)$$

where \mathcal{V} indicates the total volume of the space. Under these constraints, we look for a field configuration that has the lowest energy, whose density is given by

$$\mathcal{H} = b_q \sqrt{\dot{q}^\dagger \dot{q} - \partial_i q^\dagger \partial^i q} \times \left(2\dot{q}^\dagger \dot{q} + \partial_i q^\dagger \partial^i q \right). \quad (2.7)$$

2.4 Equation of motion for the conformal sigma model

Now we are ready to derive the equation of motion for (2.4). We set an ansatz for the ground state solution that it is at least homogeneous in the one of the spatial directions, the y direction, and varies spatially only in the x direction.

We also use the fact that the time dependence of the ground state solution must be helical, and also that it is invariant under the combination of $t \rightarrow -t$ and complex conjugation. Then the ground state solution for q can be parametrised as follows:

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} e^{i\omega_1 t} \sin(p(x)) \\ e^{i\omega_2 t} \cos(p(x)) \end{pmatrix}, \quad (2.8)$$

where we are free to set $\omega_1 > \omega_2$. The equation of motion for the p field is then

$$\mathcal{L} - p'(x) \frac{\delta \mathcal{L}}{\delta p'} = (\text{const.}). \quad (2.9)$$

Using a constant κ that is of the same mass dimension as $\omega_{1,2}$, we rewrite the above equation as

$$-\frac{\kappa^6}{4} = -\frac{b_q^{-2} T_{xx}^2}{4} = (p'(x)^2 - V(p(x))) \left(p'(x)^2 + \frac{V(p(x))}{2} \right)^2, \quad (2.10)$$

where

$$V(p) = \omega_2^2 + (\omega_1^2 - \omega_2^2) \sin^2(p). \quad (2.11)$$

The constraints imposed by (2.5) and (2.6) become

$$\rho_1 = \frac{8b_q}{3\mathcal{V}} \int d^2x \omega_1 \sqrt{-p'(x)^2 + V(p(x))} \sin^2(p(x)), \quad (2.12)$$

$$\rho_2 = \frac{8b_q}{3\mathcal{V}} \int d^2x \omega_2 \sqrt{-p'(x)^2 + V(p(x))} \cos^2(p(x)). \quad (2.13)$$

Notice from the equation of motion that the solution for p is inevitably inhomogeneous unless $\omega_1 = \omega_2$, which will never be the case if both ρ_1 and ρ_2 are nonvanishing.² This

²At first glance there might seem to be solutions of the type $q = \begin{pmatrix} e^{i\omega t} \sin(p_0) \\ e^{i\omega t} \cos(p_0) \end{pmatrix}$, but the (spatially averaged) charge density matrices come out non-diagonal (Remember our definitions of $\rho_{1,2}$ are the diagonal elements of the averaged charge density matrix.). Once you take a basis in which they are diagonal, those solutions reduce to $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the diagonalized charge density matrix becomes $\begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$.

means that the charged ground state configuration for the $O(4)$ theory is generically inhomogeneous, as promised in the introduction and demonstrated in the context of the model of [1]. Also, the energy density of this parametrised solution for p is, because of (2.7),

$$\begin{aligned} \mathcal{H} &= b_q \sqrt{-p'(x)^2 + V(p(x))} (p'(x)^2 + 2V(p(x))) \\ &= \frac{T_{xx}}{2} + \frac{3}{2} b_q \sqrt{-p'(x)^2 + V(p(x))} V(p) \\ &= 2T_{xx} - 3b_q \sqrt{-p'(x)^2 + V(p(x))} p'(x)^2, \end{aligned} \tag{2.14}$$

and the average energy density becomes, by using (2.12) and (2.13),

$$\mathcal{E} = \frac{1}{\mathcal{V}} \int d^2x \mathcal{H} = \frac{b_q \kappa^3}{2} + \frac{9}{16} (\rho_1 \omega_1 + \rho_2 \omega_2) \tag{2.15}$$

$$= 2b_q \kappa^3 - \frac{3b_q}{\mathcal{V}} \int d^2x \sqrt{-p'(x)^2 + V(p(x))} p'(x)^2. \tag{2.16}$$

2.5 Solving the equation of motion

We restrict our attention to solutions for the p -field that have a point where $p'(x) = 0$. This is because we are ultimately interested in putting the theory on S^2 , where we would have to impose the Neumann boundary condition (*Note: the analysis in this direction was later completed in [19]*). Even on the torus spatial slice, it seems reasonable to expect such solutions with $p(L) = p(0) + 2\pi n$ ($n \in \mathbb{Z}$), where L is the size of one cycle of the torus, has higher energy when $n \neq 0$, where $p(x)$ has a point at which $p'(x) = 0$. Without loss of generality, we set the location of such a point at the origin, *i.e.*, $p'(0) = 0$.

Now as we look for the lowest energy solution, in order to access the solution in the perturbative regime, we would like to set κ and ω_1 to be very close to ω_2 , *i.e.*, we have two perturbative parameter ϵ and η , which are defined by

$$\epsilon = \frac{\omega_1}{\omega_2} - 1, \quad \eta = \frac{\kappa}{\omega_2} - 1. \tag{2.17}$$

We take both of these parameters to be much less than 1.

We will also take $\eta \ll \epsilon$, which is equivalent to the condition $\rho_1 \ll \rho_2$. This is *not* a necessary consistency condition for the solution to be in the regime of validity of the EFT; it is merely a condition to simplify the classical equation of motion sufficiently that we can verify easily that the ground state lies in the regime accessible to the EFT. Indeed, the EFT may be applicable for a larger range of ρ_1/ρ_2 , and we shall comment later on this possibility.

As the ground state solution for p is periodic, we only have to evaluate the amplitude of the derivative of p , hereafter called $v_0 = p'(0)$, and p itself, hereafter called p_0 , in spite of the difficulty of solving the full equation of motion analytically. We assume that p_0 and v_0 are small, so that we can treat them as perturbative deviations from the homogeneous solution, an assumption we will verify later.

Let us evaluate p_0 and v_0 . When $p(x) = p_0$, the derivative of p must be vanishing, and we have the algebraic equation for p_0 ,

$$\kappa^2 = V(p_0) = \omega_2^2 + (\omega_1^2 - \omega_2^2) \sin^2(p_0) \iff \sin(p_0) = \sqrt{\frac{(1 + \eta)^2 - 1}{(1 + \epsilon)^2 - 1}}. \tag{2.18}$$

For small p_0 , we have

$$p_0 = \sqrt{\frac{(1+\eta)^2 - 1}{(1+\epsilon)^2 - 1}} (1 + O(\eta/\epsilon)) \sim \sqrt{\frac{\eta}{\epsilon}} \quad (2.19)$$

As for v_0 , the maximal value for p' is achieved when $p = 0$, so we have

$$-\frac{\kappa^6}{4} = (v_0^2 - \omega_2^2) \left(v_0^2 + \frac{\omega_2^2}{2} \right)^2. \quad (2.20)$$

Solving the equation for v_0 which is small, we have

$$v_0 = \omega_2 \sqrt{(1+\eta)^6 - 1} (1 + O(\eta^2)) \sim \sqrt{6\eta} \omega_2. \quad (2.21)$$

The spatial period of the solution, which is approximately $\ell = p_0/v_0$ modulo multiplicative constants, becomes

$$\ell \sim \frac{1}{\omega_2 \sqrt{\epsilon}}, \quad (2.22)$$

which becomes infinite as ϵ goes to zero, *i.e.*, we recover the homogeneous solution, as we must.

2.6 Resolving the equation of motion at leading order

We can also solve the equation of motion by noting (2.19) and (2.21) and expanding all quantities to first order in η . The equation of motion then becomes

$$2\eta = 2\epsilon p(x)^2 + \left(\frac{p'(x)}{\omega_2} \right)^2, \quad (2.23)$$

whose solution for p is then

$$p(x) = \sqrt{\frac{\eta}{\epsilon}} \sin(\sqrt{2\epsilon} \omega_2 x). \quad (2.24)$$

Using this to rewrite (2.12) and (2.13), we have

$$\rho_1 = \frac{8b_q}{3\mathcal{V}} \omega_2^2 (1+\epsilon) \int d^2x \frac{\eta}{\epsilon} \left(1 + \eta - 2\eta \cos^2(\sqrt{2\epsilon} \omega_2 x) \right) \sin^2(\sqrt{2\epsilon} \omega_2 x) \quad (2.25)$$

$$= \frac{2b_q(1+\epsilon)}{3\epsilon} \eta (2-\eta) \omega_2^2 \sim \frac{4b_q \eta \omega_2^2}{3\epsilon} (1+\epsilon), \quad (2.26)$$

$$\rho_2 \sim \frac{8b_q \omega_2^2}{3} - \frac{4b_q \eta \omega_2^2}{3\epsilon}. \quad (2.27)$$

We have, as a consequence,

$$\omega_2 = \sqrt{\frac{3(\rho_1 + \rho_2)}{8b_q}} \left(1 - \frac{\eta}{4} \right) \quad (2.28)$$

and

$$\epsilon = \frac{\rho_1 + \rho_2}{2\rho_1} \eta \quad (2.29)$$

at first order in η .

2.7 Minimization of energy

We now check that the ground state configuration occurs near the homogeneous solution. By evaluating the average energy density for the solution of the equation of motion, we have

$$\mathcal{E} = b_q \kappa^3 + \frac{b_q}{\mathcal{V}} \int d^2x \sqrt{-p'(x)^2 + V(p(x))} (-p'(x)^2 + V(p(x))). \quad (2.30)$$

Now the second term, using the equation of motion, becomes

$$\frac{b_q}{\mathcal{V}} \int d^2x \sqrt{-p'(x)^2 + V(p(x))} (-p'(x)^2 + V(p(x))) = \frac{b_q \kappa^3}{\mathcal{V}} \int d^2x \frac{1 - p'(x)^2/V(p(x))}{1 + 2p'(x)^2/V(p(x))}. \quad (2.31)$$

By using the fact that $|p'(x)^2/V(p(x))| \sim \eta \ll 1$, we have, at first order in η ,

$$\mathcal{E} = b_q \kappa^3 + \frac{b_q}{\mathcal{V}} \int d^2x \kappa^3 \left[1 - \frac{3p'(x)^2}{V(p(x))} \right] = b_q \kappa^3 \left[2 - \frac{1}{\mathcal{V}} \int d^2x \frac{3p'(x)^2}{V(p(x))} \right] \quad (2.32)$$

$$\sim b_q \omega_2^3 (1 + \eta)^3 (2 - 3\eta) \sim \frac{3\sqrt{3}}{8\sqrt{2b_q}} (\rho_1 + \rho_2)^{3/2} \left(1 + \frac{3}{4}\eta \right). \quad (2.33)$$

This means that the minimal value of the total energy is achieved at $\eta = 0$. The configuration associated with this minimiser at fixed charge densities can be understood to have a constant amplitude $p_0 = \frac{2\rho_1}{\rho_1 + \rho_2}$ and an infinite spatial period, $\ell \sim \frac{4\sqrt{b_q\rho_1}}{3(\rho_1 + \rho_2)\sqrt{\eta}} \rightarrow \infty$.

The actual physical quantities which makes situation transparent is ρ_1 , ρ_2 , and ℓ , so let us write η and ϵ in terms of them and make the discussion above a bit clearer,

$$\begin{aligned} p_0 &= \sqrt{\frac{\eta}{\epsilon}} = \sqrt{\frac{2\rho_1}{\rho_1 + \rho_2}}, \\ \eta &= \frac{8b_q\rho_1}{3\ell^2(\rho_1 + \rho_2)^2}, \\ \epsilon &= \frac{4b_q}{3\ell^2(\rho_1 + \rho_2)}. \end{aligned} \quad (2.34)$$

You could also plug in these relations to the above argument for transparency. Most importantly, \mathcal{E} is given by

$$\mathcal{E} = \frac{3\sqrt{3}}{8\sqrt{2b_q}} (\rho_1 + \rho_2)^{3/2} \left(1 + \frac{3}{4} \frac{8b_q\rho_1}{3\ell^2(\rho_1 + \rho_2)^2} \right) + \dots, \quad (2.35)$$

so that it is apparent that ℓ as big as possible is the most favourable in terms of total energy.

In eq. (2.35), the dots \dots signify omitted terms of order ℓ^{-3} and smaller, and also terms of order $\frac{\rho_1^2}{(\rho_1 + \rho_2)^3}$ and smaller in the limit where $\rho_1 \ll \rho_2$. Relaxing this latter approximation may not affect the result qualitatively. The only relevant consideration is the range of ρ_1/ρ_2 such that the ℓ^{-2} term in (2.35) has a positive coefficient. The restriction to small ρ_1/ρ_2 simply establishes that there is an open set of values of ρ_1/ρ_2 such that the true ground state occurs at large ℓ .

2.8 The case where ρ_1/ρ_2 is not small

We completed this proof a week after we published this paper, which is given in [2].

2.9 Summary and interpretation of the result

We have explicitly computed the ground state configuration of the conformal $O(4)$ model at finite chemical potential at large charges $J_{1,2} \equiv \mathcal{V}\rho_{1,2} \gg 1$, on the torus spatial slice with volume \mathcal{V} . Such a configuration turned out to be spatially inhomogeneous for large $J_{1,2}$ with fixed ratio of $0 < J_1/J_2 \ll 1$.

This signifies that the configuration we have computed breaks the translational symmetry spontaneously, and this inhomogeneity does not go away even when one takes the infinite volume limit (with fixed averaged charge densities). More explicitly in this case, the infinite volume limits for local quantities taken in this way do not exist. Especially the local charge density will not have a nice infinite volume limit, and varies by $O(1)$ of the averaged charge density, over the infrared scale.

3 Discussion

We have investigated the ground state of the three-dimensional critical $O(4)$ model in infinite volume, at general charge densities $\rho \ll 1, 2$. To do this, we have used a large-charge effective theory described by a conformal sigma model, which is weakly coupled when describing observables on distance scales large compared to $(\rho_1 + \rho_2)^{-\frac{1}{2}}$. To understand the true ground state, we have studied the helical solutions of the effective theory, *i.e.*, solutions preserving a combined time translation and global symmetry rotation. These solutions are equivalent to time-independent solutions with general chemical potentials, and the true charged ground state must always be among them. For $\rho_{1,2}$ both nonzero, we find the ground state is always inhomogeneous. In some range of ρ_1/ρ_2 , the inhomogeneity wishes to express itself on as large a distance scale as possible; that is, there is an energetic preference for arbitrarily large spatial period.

This outcome is a desirable one from the perspective of calculability. The effective Lagrangian (2.4) is only the first term in an infinite series of terms with higher derivatives in the numerator, and powers of $|\partial q^\dagger \partial q|$ in the denominator. For helical solutions with spatial period ℓ , these corrections are suppressed by negative powers of $(\rho_1 + \rho_2)\ell^2$. We see from eq. (2.35) that the energetically favored solutions are those where the leading action (2.4) is most reliable.

Ultimately, the main application of the large-charge effective theory is to compute observables, such as the ground state energy, in finite volume at large but finite total charge. The details will clearly depend on the topology and geometry of the spatial slice. For toroidal spatial slices, the ground state for sufficiently large global charges should be the homogeneous solution whose spatial period is the larger of the two cycles of the T^2 . For a spherical spatial slice, the situation is different, since there are no isometries on S^2 without fixed points, and so there may be rapid variation of the fields near the fixed point of the isometry. Nonetheless we expect the fields away from the poles to have gradients set

by the size of the sphere, rather than the ultraviolet scale, and the dominant contribution to the ground state energy on S^2 to be calculable using the leading order action (2.4).

Lastly, we emphasize again that we have only shown this situation holds for sufficiently small ρ_1/ρ_2 . It would be good to find the maximum possible range of ratios, for which long spatial periods are favoured.

Note added. This is true for arbitrary ratios and is proven in [2].

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