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Superconformal index of higher derivative $\mathcal{N}=1$ multiplets in four dimensions

Matteo Beccaria[a] **and Arkady A. Tseytlin**[b,1]

[a] *Dipartimento di Matematica e Fisica Ennio De Giorgi, Università del Salento & INFN, Via Arnesano, 73100 Lecce, Italy*
[b] *The Blackett Laboratory, Imperial College, South Kensington Campus, London SW7 2AZ, U.K.*

E-mail: matteo.beccaria@le.infn.it, tseytlin@imperial.ac.uk

Abstract: Supersymmetric partition function of $\mathcal{N}=1$ superconformal theories on $S^1_\beta \times S^3$ is related to the superconformal index receiving contributions from short representations. The leading coefficients in the small β (high "temperature") expansion of the index were previously related to the conformal anomaly coefficients of the theory. Assumptions underlying universality of these relations were tested only for simplest low-spin unitary multiplets. Here we consider examples of higher derivative non-unitary $\mathcal{N}=1$ multiplets that naturally appear in the context of extended conformal supergravities and compute their superconformal index. We compare the coefficients in the small β expansion of the index with those proposed earlier for unitary multiplets and suggest some modifications that should apply universally to all types of theories. We also comment on the structure of subleading terms and the case of $\mathcal{N}=4$ conformal supergravity.

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[1] Also at Lebedev Institute, Moscow, Russia.

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Contents

1 Introduction

Unitary representations of $\mathcal{N}=1$ superconformal algebra play an important role in many aspects of supersymmetric quantum field theory. Applications of non-unitary representations are less studied. They appear, in particular, in extended conformal supergravities [1–4] written in terms of $\mathcal{N}=1$ multiplets. Recently, such multiplets were considered in the computation of conformal anomalies (in 4 and 6 dimensions) in the context of AdS/CFT [5–9]; similar massless and massive multiplets were also discussed in [10]. Non-unitary superconformal theories may have other interesting applications (see, e.g., [11]).

Given a CFT one may define the standard partition function on $S^1_\beta \times S^3$ (with fermions treated as antiperiodic on S^1 of length β to have usual thermodynamic interpretation). When CFT is also a superconformal theory one may formally define also another — "supersymmetric" — partition function $Z^{\rm susy}$ on $S^1_\beta \times S^3$ by (i) taking fermions to be periodic on S^1_β and (ii) introducing extra R-symmetry gauge-field couplings in the action to preserve global supersymmetry on S^3 [12–14]. While having no thermodynamic interpretation[1] $Z^{\rm susy}$ will instead be related to the superconformal index $\mathrm{I}(\beta)$ [15–17]. It will thus be protected by supersymmetry, receiving contributions only from short multiplets (thus being computable exactly, e.g., using localization, see [18] for a comprehensive review).

In this paper we shall study the properties of the superconformal index for higher-derivative (and higher spin) $\mathcal{N}=1$ non-unitary multiplets. We shall compute the coefficients in the small β expansion of the index $\mathrm{I}(\beta)$ and compare with their expected expressions in terms of conformal anomaly coefficients proposed earlier on the basis of studies of unitary low-spin examples [19, 20]. We shall find that some modifications of these expressions are required in the non-unitary cases.[2]

We shall start in section 2 with a short review of the definition of the superconformal index for a 4d $\mathcal{N}=1$ theory and its relation to the supersymmetric partition function on $S^1_\beta \times S^3$. We shall then discuss what is known about the leading coefficients in their small β expansion, emphasizing that the assumptions used to derive the general expressions for the coefficients were checked only in models with simplest unitary (chiral and vector) multiplets.

In section 3 we shall introduce four basic higher-derivative $\mathcal{N}=1$ superconformal multiplets for which we shall later compute the superconformal index. These non-unitary multiplets are the $\mathcal{N}=1$ building blocks of extended conformal supergravities. We shall discuss the superfield structure of the multiplets and check the relation between their chiral and conformal anomalies as predicted by the superconformal invariance.

In section 4 we shall compute the superconformal index $\mathrm{I}(\beta)$ of these free non-unitary multiplets by an explicit "letter"-counting algorithm. The multiplets that contain higher

[1] We may still formally refer to β as an inverse "temperature". We shall also assume that the radius of S^3 is fixed to be 1.

[2] Our discussion will be restricted to abelian free superconformal theories. In presence of a non-trivial semi-simple gauge group the asymptotic behaviour discussed in [19, 20] may require corrections when the theory has moduli spaces on the "thermal" cycle [21–23], as in the case of the ISS model [24, 25], i.e. SU(2) $\mathcal{N}=1$ SYM with a single chiral field in the spin 3/2 representation. The general reason for such corrections in models with simple gauge groups has been further elucidated in [26, 27] by taking into account contributions from all vacua in the 3d limit.

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spin gauge fields (conformal gravitino and graviton) require careful treatment of equation of motion constraints and Bianchi identities for the field strengths. In section 4.2 where we present an efficient method to extract the small β expansion of the index, including all possible subleading terms.

In section 5 we shall compare the expressions for the coefficients in the expansion of $\mathrm{I}(\beta)$ with those proposed earlier for unitary multiplets and propose some modifications that should apply universally to all types of theories. We also comment on subleading terms and the case of finite $\mathcal{N}=4$ conformal supergravity.

In appendix A we shall present the free action of the $[\frac{1}{2}]$ tensor multiplet. In appendix B we shall discuss the results for the chiral gravitational and gauge anomalies of the conformal gravitino and non-gauge antisymmetric tensor. In appendix C we shall review the expression for the conformal higher spin partition function on $S^1_\beta \times S^3_b$ and work out its small β expansion. Appendix D will contain a discussion of how to one may compute the constant and $\log\beta$ term in the partition function using the direct expansion and ζ-function regularization in terms of spectrum of dimensionally reduced theory on S^3. In appendix E we shall repeat the computation of the superconformal index and its small β expansion for non-unitary multiplets in case of unequal fugacities which is related to supersymmetric partition function on $S^1_\beta \times S^3_b$ (with S^3_b being a squashed 3-sphere). Finally, in appendix F we demonstrate how to perform similar analysis of the index of six-dimensional theories with $(1,0)$ supersymmetry considering the examples of scalar, tensor and non-unitary higher derivative vector multiplets.

One of the conclusions of this paper is that the relation between the leading term in the expansion of the superconformal index and the anomaly coefficients suggested in [19] requires a certain modification in non-unitary theories where the structure of the index appears to be more involved. The results of this paper may be useful for further studies of properties of higher derivative superconformal multiplets in 4 and 6 dimensions closely related to conformal supergravities. In particular, it is interesting to investigate further superconformal index or supersymmetric partition function for a finite model of interacting $\mathcal{N}=4$ conformal supergravity (mentioned in section 5) viewed as a special $\mathcal{N}=1$ superconformal theory. It would be interesting also to explore the index for higher spin superconformal theories which are "shadow" boundary counterparts of supersymmetric higher spin theories in AdS in the context of vectorial AdS/CFT correspondence. A (modified) relation between the superconformal index and the conformal anomaly coefficients may shed light on the values of the latter for superconformal higher spin theories.

2 Superconformal index and its small β expansion: a review

The superconformal index of an $\mathcal{N}=1$ theory on $\mathbb{R}^4$ is defined as [15–17]

$$\mathrm{I}(p,q) = \mathrm{Tr}\Big[(-1)^F\, e^{-\mu\,(\Delta-2\,j_2-\frac{3}{2}\,r)}\, p^{j_1+j_2+\frac{1}{2}\,r}\, q^{-j_1+j_2+\frac{1}{2}\,r}\Big]. \tag{2.1}$$

Here, quantum numbers j_1, j_2, Δ, r label representations of the bosonic compact subgroup $\mathrm{SU}(2)_{j_1}\times\mathrm{SU}(2)_{j_2}\times\mathrm{U}(1)_\Delta\times\mathrm{U}(1)_r$ of the $\mathrm{SU}(2,2|1)$ superconformal group. In particular, Δ

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is the conformal dimension and r is R-charge.[3] The chemical potentials p and q are free parameters. Due to supersymmetry, the trace receives contributions only from states with $\delta \equiv \Delta - 2\, j_2 - \frac{3}{2}\, r = 0$ and thus $\mathrm{I}(p, q)$ is independent of the third parameter μ. Examples of exact results for the index obtained by counting or localization can be found in [16, 17, 28–40].

Setting

$$p = q \equiv t = e^{-\beta}, \tag{2.2}$$

one finds the special case of the index that we shall consider below

$$\mathrm{I}(\beta) = \mathrm{Tr}\Big[(-1)^F\, e^{-\beta\,(\Delta - \frac{1}{2}\, r)}\Big], \qquad\qquad \mathrm{Tr} \equiv \mathrm{Tr}|_{\delta=0}\,. \tag{2.3}$$

This index that happens to be directly related to the supersymmetric partition function $Z^{\rm susy}(\beta)$ on $S^1_\beta \times S^3$ by [29, 34, 41]

$$Z^{\rm susy}(\beta) = \; e^{-\beta\, E_{\rm susy}}\; \mathrm{I}(\beta)\,. \tag{2.4}$$

Here $E_{\rm susy}$ is the "supersymmetric" Casimir energy [34, 41–43] which can be expressed in terms of the conformal anomaly a and c coefficients[4]

$$E_{\rm susy} = \frac{4}{27}(\mathrm{a} + 3\,\mathrm{c})\;. \tag{2.5}$$

The small β expansion of $Z^{\rm susy}(\beta)$ takes the following form

$$\log Z^{\rm susy}(\beta) \overset{\beta\to 0}{=} C_1\, \frac{\pi^2}{\beta} + C_2 + C_3\, \log\beta + 0\cdot\beta + \mathcal{O}(\beta^2), \tag{2.6}$$

where C_i are theory-dependent numerical coefficients and the absence of the linear in β term is due to supersymmetry. Then (2.4) implies the following expansion of the index[5]

$$\log \mathrm{I}(\beta) \overset{\beta\to 0}{=} C_1\, \frac{\pi^2}{\beta} + C_2 + C_3\, \log\beta + E_{\rm susy}\,\beta + \mathcal{O}(\beta^2). \tag{2.7}$$

A more general specialization than (2.2), depending on a 2-parameter family of unequal fugacities p and q in (2.1), is related [14, 48, 49] to supersymmetric partition function on $S^1_\beta \times S^3_b$ where S^3_b is squashed sphere and will be discussed in appendix E.

Let us now review what was claimed in the past about each coefficient in the expansion (2.6) or (2.7).

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[3] Here j_1, j_2 in (2.1) denote the third components of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ angular momenta.

[4] For a conformal theory on curved space, the coefficient of the logarithmic UV divergence in the standard partition function is

$$\log Z_\infty = \frac{1}{(4\pi)^2} \log\Lambda \int d^4x \sqrt{g}\, b_4\,, \qquad\qquad b_4 = -\mathrm{a}\, R^\star R^\star + \mathrm{c}\, C^2\,.$$

Here we ignored possible $\nabla^2 R$ term, C^2 is the square of the Weyl tensor and $R^\star R^\star = C^2 - 2R^2_{\mu\nu} + \frac{2}{3}R^2$ is $32\pi^2$ times the Euler number density. Note that in contrast to the standard Casimir energy [44–46] the "supersymmetric" one may be viewed as "scheme-independent" [43, 47] as supersymmetry should prohibit adding extra local counterterms that may modify the expression for $E_{\rm susy}$.

[5] The fact that the index encodes $E_{\rm susy}$ was first suggested in [41]. The relation between the index and $Z^{\rm susy}$ was later clarified in [29] who showed that the $e^{-\beta\, E_{\rm susy}}$ factor in (2.4) is a normal-ordering effect like for the standard Casimir-like contribution. Further discussion of the universality of the relation (2.4) appeared in [34].

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Leading term $\sim 1/\beta$. It was argued in [19] that the coefficient of the leading Cardy-type [50] term in (2.7) can be expressed in terms of the conformal anomaly coefficients as[6]

$$C_1 = \frac{16}{3}\,(\mathrm{c}-\mathrm{a}). \tag{2.8}$$

The proof in [19] was based on the expected form of the effective action of the dimensionally reduced 3d theory corresponding to the limit of small radius of S^1. It contains a Chern-Simons term $k\int_{S^3} a\wedge F$ where a is the Kaluza-Klein graviphoton (mixed component of the metric tensor in reduction to 3d) and F is the R-symmetry gauge field strength. The coefficient $k \sim C_1$ is then proportional to the R-current gravitational anomaly. It was computed by considering the example of a 4d Weyl fermion leading to

$$C_1 = -\frac{1}{3}\,\mathrm{Tr(R)}\,, \tag{2.9}$$

where Tr(R) is the sum of r-charges. In general, the $\mathcal{N}=1$ superconformal symmetry relates the gravitational R-current anomaly to the trace anomaly coefficients as [57]

$$\nabla_\mu R^\mu = -\frac{1}{384\,\pi^2}\mathrm{Tr(R)}\,RR^\star + \frac{1}{16\pi^2}\,\mathrm{Tr(R^3)}\,FF^\star = \frac{\mathrm{c}-\mathrm{a}}{24\,\pi^2}\,RR^\star + \frac{5\mathrm{a}-3\mathrm{c}}{9\pi^2}FF^\star\,, \tag{2.10}$$

i.e.

$$\mathrm{a} = \frac{3}{32}\Big[3\mathrm{Tr(R^3)}-\mathrm{Tr(R)}\Big], \qquad \mathrm{c} = \frac{1}{32}\Big[9\mathrm{Tr(R^3)}-5\mathrm{Tr(R)}\Big], \tag{2.11}$$

$$\mathrm{Tr(R)} = 16\,(\mathrm{a}-\mathrm{c})\,, \qquad \mathrm{Tr(R^3)} = \frac{16}{9}\,(5\,\mathrm{a}-3\,\mathrm{c})\,. \tag{2.12}$$

Here we use the somewhat loose notation Tr(R) and Tr(R^3) for the gravitational and gauge anomaly coefficients: they are literally the sum of r-charges and their cubes only in the case of the standard chiral fermions but in general contain also field-dependent coefficients, i.e.

$$\mathrm{Tr(R)} \equiv \sum_i \kappa_{1,i}\,r_i\,, \qquad \mathrm{Tr(R^3)} \equiv \sum_i \kappa_{3,i}\,r_i^3\,, \tag{2.13}$$

where $\kappa_1 = \kappa_3 = 1$ for a left Weyl spinor.

Constant term $\sim \beta^0$. Motivated by the study of explicit examples of standard chiral and vector multiplets, the constant C_2 in (2.6) was conjectured in [20] to be equal to the logarithm of the supersymmetric partition function of the dimensionally reduced theory on S^3[7]

$$C_2 = \mathrm{k} \equiv \log Z^{\mathrm{susy}}_{S^3}\,. \tag{2.14}$$

Logarithmic term $\sim \log\beta$. The coefficient C_3 of the logarithmic term in (2.6) was conjectured to be [20]

$$C_3 = -4\,(2\mathrm{a}-\mathrm{c}), \tag{2.15}$$

again motivated by the examples of chiral and vector multiplets.[8]

[6] Related observations appeared previously in [49, 51, 52] with further developments in [20, 53–56].

[7] To be precise, the Ansatz in [20] is slightly different by a multiple of $\log(2\pi)$ because the logarithmic term in (2.6) or (2.7) is written as $\log\frac{\beta}{2\pi}$. We claim that the identification (2.14) is correct if the logarithmic term is simply $\log\beta$, see appendix D.

[8] Let us note that the combination $2\,\mathrm{a}-\mathrm{c}$ plays a special role in the analysis of the $\mathcal{N}=2$ models in [58, 59] where it is essentially the sum of dimensions of operators parametrizing the Coulomb branch.

Linear term $\sim \beta$. As already mentioned, the coefficient of the linear term in (2.7) is the "supersymmetric" analog of Casimir energy.[9] The relation (2.4) may be studied in the low temperature $\beta \to \infty$ or high temperature $\beta \to 0$ limit. For $\beta \to \infty$ the leading contribution comes only from the energy exponential, i.e.

$$Z^{\rm susy} \overset{\beta\to\infty}{\sim} e^{-\beta\, E_{\rm susy}}, \qquad\qquad \mathrm{I}(\beta) \overset{\beta\to\infty}{\sim} 1\,, \tag{2.16}$$

where the asymptotics of the index corresponds to the vacuum contribution. Equivalently, $E_{\rm susy} = -\lim_{\beta\to\infty}\frac{d}{d\beta}\log Z^{\rm susy}$. On the other hand, in the $\beta \to 0$ limit the partition function (and the index) is governed by the high-energy part of the spectrum. This leads to the singular Cardy-type term in (2.6).[10] The relation (2.5) of $E_{\rm susy}$ to the conformal anomaly coefficients was demonstrated explicitly in the case of the chiral and vector multiplets [29, 34]. In addition, a general derivation was proposed in [61] based on a representation of $E_{\rm susy}$ in terms of the anomaly polynomial and assuming the standard relations (2.11) between the anomaly coefficients.

Higher order terms $\sim \beta^n$. It was claimed in [20] that the $\mathcal{O}(\beta^2)$ corrections are exponentially suppressed, i.e. schematically go as $e^{-1/\beta}$ for $\beta \to 0$. Let us recall, as an analogy, that absence of β^n ($n \geq 2$) corrections for in the case of the logarithm of the thermodynamic partition function $Z(\beta)$ was observed in the past for the standard conformal fields [62] where this follows from a simple modular transformation property of $Z(\beta)$. It is unclear if similar modular properties play a role in the supersymmetric context.

The above discussion leaves several open questions. One is whether the prediction (2.9) and thus (2.8) for the leading-order coefficient C_1 is completely universal, i.e. holds for general $\mathcal{N}=1$ superconformal theories, including also non-unitary and higher spin multiplets. The argument in [19] relying on Chern-Simons term in the reduced 3d effective theory appears to be specific to case of standard Weyl fermions. As we shall see below, the relation (2.9) indeed requires a generalization in the non-unitary case.

The second question is about the expression (2.15) for coefficient C_3 of the $\log\beta$ term which was checked only for standard multiplets. We shall find that (2.15) needs a modification in the case of higher spin superconformal multiplets. We shall propose an alternative universal expression for the $\log\beta$ term in terms of the integer numbers of conformal Killing tensors associated with each conformal gauge field in the multiplet.

Finally, it is not clear a priori if the non-unitary multiplets will also have exponentially decaying corrections in their index expansion, i.e. if all power $\beta^2, \beta^3, \ldots$ corrections in (2.7)

[9] The effective Hamiltonian appearing in (2.3) is $H_{\rm susy} = \Delta - \frac{1}{2}\mathrm{R}$. Normal ordering $:\!H_{\rm susy}\!: = H_{\rm susy} - \langle H_{\rm susy}\rangle = H_{\rm susy} - E_{\rm susy}$ is implicitly understood, and is essentially the reason why $Z^{\rm susy}$ is not equal to the index in (2.4) (see [29] for details). This is a general feature of the relation between the QFT partition function and the thermodynamic partition function $\mathrm{Tr}(e^{-\beta H})$ [60].

[10] For comparison, let us mention what happens in the case of the standard partition function of a non-supersymmetric CFT. For example, for a free conformal scalar the derivative $\mathcal{E}(\beta) \equiv -\partial_\beta \log\mathrm{Tr}(e^{-\beta\Delta})$ obeys $\mathcal{E}(\beta) = \frac{\pi^4}{15\beta^4} - \frac{1}{240} + \left(\frac{2\pi}{\beta}\right)^4 \mathcal{E}(\frac{2\pi}{\beta})$. For $\beta \to 0$ we get $\mathcal{E}(\beta) \overset{\beta\to 0}{=} \frac{\pi^4}{15\beta^4} - \frac{1}{240} + \ldots$ (up to exponentially suppressed terms). This implies $\log\mathrm{Tr}(e^{-\beta\Delta}) \overset{\beta\to 0}{=} \frac{C}{\beta^3} + E_{\rm cas}\,\beta + \ldots$., where $E_{\rm cas}$ is the standard Casimir energy $\langle H\rangle = \langle\Delta\rangle$.

will be absent. Indeed, we shall find that such power corrections will survive for multiplets containing higher spin ($s > 1$) conformal spins.

To address these questions, below we shall consider several non-unitary (higher derivative and higher spin) superconformal multiplets that appear in the context of extended conformal supergravities. For each of these free multiplets we will explicitly compute the index (2.3), obtain the coefficients in its small β expansion and compare them with their expected values based on relations found earlier in the studies of unitary multiplets.

3 Higher derivative $\mathcal{N} = 1$ superconformal multiplets

Here we will describe the content of the four basic higher-derivative $\mathcal{N} = 1$ superconformal multiplets for which we will later compute the index (2.3). They naturally appear in the decomposition of $\mathcal{N} \leq 4$ conformal supergravities in terms of $\mathcal{N} = 1$ multiplets.

3.1 $\mathcal{N} = 1$ multiplet content of extended conformal supergravities

Let us start with reviewing the field content of conformal supergravities (for details see [3, 4]). Expanded near flat-space vacuum they are given by a collection of the following free conformal fields:

ϕ: standard 2-derivative real scalar, $L \sim \phi\Box\phi$
$\phi^{(4)}$: 4-derivative real scalar, $L \sim \phi^{(4)}\Box^2\phi^{(4)}$
ψ: standard Weyl fermion, $L \sim \bar{\psi}\partial\!\!\!/\psi$
$\psi^{(3)}$: 3-derivative Weyl fermion, $L \sim \bar{\psi}^{(3)}\partial\!\!\!/^3\psi^{(3)}$
V_μ: standard gauge vector, $L \sim F_{\mu\nu}F^{\mu\nu}$
$T_{\mu\nu}$: non-gauge real antisymmetric tensor, $L \sim \partial^\mu T^+_{\mu\nu}\partial_\lambda T^{-\lambda\nu}$, $\quad T^\pm = T \pm T^*$
Ψ_μ: conformal gravitino, $L \sim \overline{\Psi}_\mu\partial\!\!\!/^3\Psi^\mu$
$h_{\mu\nu}$: conformal (Weyl) graviton, $L \sim h\Box^2 h$

The $\mathcal{N} = 1$ multiplet content of extended conformal supergravities (CSG) is [4]

$$\begin{aligned}
&\mathcal{N} = 1\ \mathrm{CSG} = [2],\\
&\mathcal{N} = 2\ \mathrm{CSG} = [2] + \left[\frac{3}{2}\right] + [1],\\
&\mathcal{N} = 3\ \mathrm{CSG} = [2] + 2\left[\frac{3}{2}\right] + 4\,[1] + \left[\frac{1}{2}\right] + 2\,[0],\\
&\mathcal{N} = 4\ \mathrm{CSG} = [2] + 3\left[\frac{3}{2}\right] + 8\,[1] + 3\left[\frac{1}{2}\right] + 6\,[0] + [0'] \,.
\end{aligned} \tag{3.1}$$

Here $[0] = (2\phi, \psi)$ and $[1] = (V_\mu, \psi)$ are the standard unitary scalar and vector multiplets while $[2]$, $[\frac{3}{2}]$, $[\frac{1}{2}]$ and $[0']$ are the four non-unitary multiplets containing higher derivative fields:

$$\begin{aligned}
&[0'] = (2\,\phi^{(4)}, \psi^{(3)}, 2\,\phi)\,, &&\left[\frac{1}{2}\right] = (\psi, 2\,\phi, T_{\mu\nu}, \psi^{(3)})\,,\\
&\left[\frac{3}{2}\right] = (\Psi_\mu, 2\,V_\mu, T_{\mu\nu}, \psi)\,, &&[2] = (h_{\mu\nu}, \Psi_\mu, V_\mu)\,.
\end{aligned} \tag{3.2}$$

	ϕ	$\phi^{(4)}$	ψ	$\psi^{(3)}$	$T_{\mu\nu}$	V_μ	Ψ_μ	$h_{\mu\nu}$
a	$\frac{1}{360}$	$-\frac{7}{90}$	$\frac{11}{720}$	$-\frac{3}{80}$	$-\frac{19}{60}$	$\frac{31}{180}$	$-\frac{137}{90}$	$\frac{87}{20}$
c	$\frac{1}{120}$	$-\frac{1}{15}$	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{1}{20}$	$\frac{1}{10}$	$-\frac{149}{60}$	$\frac{199}{30}$

Table 1. Conformal anomaly coefficients of fields appearing in $\mathcal{N}=1$ superconformal multiplets.

The maximal $\mathcal{N}=4$ supergravity has local SU(4) R-symmetry under which the fields transform in various representations. Decomposing SU(4) $\to$ SU(3) $\times$ U(1) allows one to identify the U(1) with the $\mathcal{N}=1$ R-symmetry corresponding to the $\mathcal{N}=1$ multiplets and thus fix the r-charges of the component fields. In particular, the conformal graviton is a singlet so has $r=0$ while for other fields one finds

$$\begin{aligned}
&\Psi_\mu: \quad \mathbf{4} = \underset{[2]}{\mathbf{1}_1} + \underset{3\,[\frac{3}{2}]}{\mathbf{3}_{-1/3}}, \qquad\qquad T_{\mu\nu}: \quad \mathbf{6} = \underset{3\,[\frac{3}{2}]}{\mathbf{3}_{\frac{2}{3}}} + \underset{3\,[\frac{1}{2}]}{\overline{\mathbf{3}}_{-\frac{2}{3}}},\\
&V_\mu: \quad \mathbf{15} = \underset{[2]}{\mathbf{1}_0} + \underbrace{\mathbf{3}_{-4/3} + \overline{\mathbf{3}}_{4/3}}_{3\,[\frac{3}{2}]} + \underset{[1]}{\mathbf{8}_0},\\
&\phi^{(4)}: \quad 2 \times \underset{[0']}{\mathbf{1}_0}, \qquad\qquad \psi^{(3)}: \quad \overline{\mathbf{4}} = \underset{[0']}{\mathbf{1}_{-1}} + \underset{3\,[\frac{1}{2}]}{\overline{\mathbf{3}}_{1/3}},\\
&\phi: \quad 2 \times \overline{\mathbf{10}} = 2 \times \underset{[0']}{\mathbf{1}_{-2}} + 2 \times \underset{3\,[\frac{1}{2}]}{\overline{\mathbf{3}}_{-2/3}} + 2 \times \underset{6\,[0]}{\overline{\mathbf{6}}_{2/3}},\\
&\psi: \quad \mathbf{20} = \underset{3\,[\frac{3}{2}]}{\mathbf{3}_{-1/3}} + \underset{3\,[\frac{1}{2}]}{\overline{\mathbf{3}}_{-5/3}} + \underset{6[0]}{\overline{\mathbf{6}}_{-1/3}} + \underset{8\,[1]}{\mathbf{8}_1}\,.
\end{aligned} \tag{3.3}$$

Here we indicated how the SU(4) representations of each field in $\mathcal{N}=4$ theory is decomposed, resulting $\mathcal{N}=1$ R-charges and the $\mathcal{N}=1$ multiplets each field belongs to. For example, the 4 gravitinos of the $\mathcal{N}=4$ supergravity split into the one belonging to [2] and having $r=1$, and three from the three $[\frac{3}{2}]$ multiplets with $r=-\frac{1}{3}$. The resulting R-charge values are in agreement with the representation theory of $\mathcal{N}=1$ superconformal algebra discussed below.

For future reference in table 1 we list the conformal anomaly coefficients of the individual fields (see [4, 63, 64]). In table 2 we present the resulting values of a and c for the basic unitary and non-unitary multiplets introduced above. We also give particular combinations of a and c corresponding to the expected values of (i) the coefficients Tr(R) and Tr(R^3) of the gravitational and R-symmetry chiral anomalies computed according to (2.12) (ii) $E_{\rm susy}$ as given by (2.5), and (iii) the coefficient C_3 as defined in (2.15). It is interesting to observe [4] that the values for the [1] and [0′] multiplets are exactly opposite to each other. Note also that the combinations 3 Tr(R) and C_3 are always integer.

3.2 Structure of the $\mathcal{N}=1$ multiplets

Let us now discuss the $\mathcal{N}=1$ superfield description of the above multiplets that allows one to independently fix the R-charges of the individual fields which will be in agreement the R-charge assignment in (3.3) following from the $\mathcal{N}=4$ conformal supergravity.

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		a	c	Tr(R)	Tr(R^3)	$E_{\rm susy}$	C_3
[0]	$(2\phi,\psi)$	$\frac{1}{48}$	$\frac{1}{24}$	$-\frac{1}{3}$	$-\frac{1}{27}$	$\frac{7}{324}$	0
[1]	(V_μ,ψ)	$\frac{3}{16}$	$\frac{1}{8}$	1	1	$\frac{1}{12}$	-1
[0′]	$(2\,\phi^{(4)},\psi^{(3)},2\,\phi)$	$-\frac{3}{16}$	$-\frac{1}{8}$	-1	-1	$-\frac{1}{12}$	1
$[\frac{1}{2}]$	$(\psi,2\phi,T_{\mu\nu},\psi^{(3)})$	$-\frac{1}{3}$	$\frac{1}{12}$	$-\frac{20}{3}$	$-\frac{92}{27}$	$-\frac{1}{81}$	3
$[\frac{3}{2}]$	$(\Psi_\mu,2\,V_\mu,T_{\mu\nu},\psi)$	$-\frac{71}{48}$	$-\frac{53}{24}$	$\frac{35}{3}$	$-\frac{37}{27}$	$-\frac{389}{324}$	3
[2]	$(h_{\mu\nu},\Psi_\mu,V_\mu)$	3	$\frac{17}{4}$	-20	4	$\frac{7}{3}$	-7

Table 2. The values of the conformal anomaly coefficients and their combinations $\mathrm{Tr(R)} = 16\,(\mathrm{a}-\mathrm{c})$, $\mathrm{Tr(R^3)} = \frac{16}{9}\,(5\,\mathrm{a}-3\,\mathrm{c})$, $E_{\rm SUSY} = \frac{4}{27}(\mathrm{a}+3\mathrm{c})$, and $C_3 = -4(2\mathrm{a}-\mathrm{c})$ for the 2 unitary and 4 non-unitary superconformal $\mathcal{N}=1$ multiplets.

As a consistency check, we will then be able to show that the resulting chiral anomaly coefficients Tr(R) and Tr(R^3) computed directly from (2.13) will be in agreement with the values in table 2 found assuming the supersymmetry-implied relations (2.12).

To this end we will need, in addition to the values of R-charges r_i, also the field-dependent chiral gravitational and gauge anomaly coefficients κ_1 and κ_3 in (2.13). Their values for the fields contributing to the chiral anomalies — Weyl fermions $\psi, \psi^{(3)} \sim (\frac{1}{2},0)$, Weyl conformal gravitino $\Psi_\mu \sim (1,\frac{1}{2})$ and self-dual tensor $T^+_{\mu\nu} \sim (1,0)$ are given by (see [65–68])

	ψ	$\psi^{(3)}$	$T^+_{\mu\nu}$	Ψ_μ
κ_1	1	1	8	-20
κ_3	1	1	-4	4

(3.4)

The chiral anomaly does not depend on extra derivatives in the kinetic term and thus is the same for ψ and $\psi^{(3)}$. The Lorentz index of the gravitino is inert under R-symmetry and thus its gauge anomaly is 4 times that of the Weyl spinor (cf. [69]). The non-trivial cases of the antisymmetric non-gauge tensor and conformal gravitino are further reviewed in appendix B.

3.2.1 Unitary scalar [0] and vector [1] multiplets

The [0] chiral multiplet containing one complex scalar and one Weyl fermion corresponds to a chiral superfield $\Phi = \phi + \theta\psi + \theta^2\varphi$. Omitting the auxiliary field φ and using that θ^α has R-charge 1 we then find the following dimensions Δ and R-charges

	Δ	(j_1,j_2)	r
ϕ	1	$(0,0)$	r
ψ_α	$\frac{3}{2}$	$(\frac{1}{2},0)$	$r-1$

	Δ	(j_1,j_2)	r
$\overline{\phi}$	1	$(0,0)$	$-r$
$\overline{\psi}_{\dot\alpha}$	$\frac{3}{2}$	$(0,\frac{1}{2})$	$-r+1$

(3.5)

The $\mathcal{N}=1$ superconformal algebra requires that for the superconformal primary or the lowest chiral superfield component (here the scalar field) one should have $r = \frac{2}{3}\Delta$ (see, e.g., [70]). This fixes $r = \frac{2}{3}$, consistently with the SU(4) decomposition in (3.3). The resulting chiral anomaly coefficients in (2.13), (3.4) are then

$$\mathrm{Tr}(\mathrm{R}) = \underbrace{r-1}_{\psi} = -\frac{1}{3}, \qquad \mathrm{Tr}(\mathrm{R}^3) = \underbrace{(r-1)^3}_{\psi} = -\frac{1}{27}\,, \tag{3.6}$$

in agreement with the values in table 2. Similar agreement will be found for all other multiplets discussed below.

The vector multiplet [1] is related to the chiral spinor field strength superfield $W_\alpha = \psi_\alpha + \theta^\beta F^+_{\alpha\beta} + \ldots$ so that we find

$$\begin{array}{|c|c|c|c|}\hline & \Delta & (j_1,j_2) & r \\ \hline \psi_\alpha & \frac{3}{2} & (\frac{1}{2},0) & r \\ F^+_{\alpha\beta} & 2 & (1,0) & r-1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline & \Delta & (j_1,j_2) & r \\ \hline \overline{\psi}_{\dot\alpha} & \frac{3}{2} & (0,\frac{1}{2}) & -r \\ F^-_{\dot\alpha\dot\beta} & 2 & (0,1) & -r+1 \\ \hline \end{array} \tag{3.7}$$

The symmetric tensors $F^+_{\alpha\beta}$ and $F^-_{\dot\alpha\dot\beta}$ are the (anti) self-dual parts of the Maxwell field strength in spinor notation. Here the lowest component is ψ_α so that $r = \frac{2}{3}\Delta_\psi = 1$. This gives, in agreement with table 2,

$$\mathrm{Tr}(\mathrm{R}) = \underbrace{r}_{\psi} = 1, \qquad \mathrm{Tr}(\mathrm{R}^3) = \underbrace{r^3}_{\psi} = 1. \tag{3.8}$$

3.2.2 Higher derivative scalar multiplet [0′]

A general discussion of $\mathcal{N} = 1$ non-unitary multiplets can be found in [71] (at the level of states), and in [72] (at the level of fields). We need to embed the [0′] multiplet in (3.2) into a chiral superfield with the superconformal primary being the 4-derivative scalar $\phi^{(4)}$ of dimension 0. Other fields are then obtained by applications of the supersymmetry generator $Q_\alpha \sim (\frac{1}{2}, 0)$ leading to

$$\begin{array}{|c|c|c|c|}\hline & \Delta & (j_1,j_2) & r \\ \hline \phi^{(4)} & 0 & (0,0) & r \\ \psi^{(3)}_\alpha & \frac{1}{2} & (\frac{1}{2},0) & r-1 \\ \phi & 1 & (0,0) & r-2 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|c|}\hline & \Delta & (j_1,j_2) & r \\ \hline \overline{\phi}^{(2)} & 0 & (0,0) & -r \\ \overline{\psi}^{(3)}_{\dot\alpha} & \frac{1}{2} & (0,\frac{1}{2}) & -r+1 \\ \bar\phi & 1 & (0,0) & -r+2 \\ \hline \end{array} \tag{3.9}$$

This multiplet may be viewed as resulting from the application of extra $\Box$ to the standard chiral multiplet, i.e. $(\phi, \psi, \varphi) \to (\phi^{(4)}, \psi^{(3)}, \phi)$, where, in particular, the auxiliary field φ becomes a (complex) dynamical scalar. As this multiplet is non-unitary, it is not a priori obvious how to fix the value of r. Nevertheless, from the analysis of [72] (see also [73]) the vanishing scaling dimension $\Delta = 0$ should imply $r = 0$, i.e. vanishing chiral weight. In practice, this is still consistent with the rule $r = \frac{2}{3}\Delta_{\phi^{(4)}} = 0$. From the point of view of $\mathcal{N} = 4$ supergravity (cf. (3.3)) this is also consistent with the higher derivative scalar being an SU(4) singlet. The direct evaluation of Tr(R) and $\mathrm{Tr}(\mathrm{R}^3)$ then gives

$$\mathrm{Tr}(\mathrm{R}) = \underbrace{r-1}_{\psi^{(3)}} = -1, \qquad \mathrm{Tr}(\mathrm{R}^3) = \underbrace{(r-1)^3}_{\psi^{(3)}} = -1\,, \tag{3.10}$$

in agreement with the values in table 2.

3.2.3 Tensor multiplet $\left[\frac{1}{2}\right]$

The superfield embedding of this multiplet can be found, e.g., in [74].[11] The field content in (3.2) may be organized into a chiral superfield, see appendix A. Its lowest component is $\psi^{(3)}_\alpha$ and other components are built by acting with Q_α decreasing r by one (or, in conjugate case, with $\overline{Q}_{\dot\alpha}$ increasing r by one). As a result, we find

$$\begin{array}{|cccc|}\hline & \Delta & (j_1,j_2) & r \\ \hline \psi^{(3)}_\alpha & \frac{1}{2} & (\frac{1}{2},0) & r \\ \phi & 1 & (0,0) & r-1 \\ T^+_{\alpha\beta} & 1 & (1,0) & r-1 \\ \psi_\alpha & \frac{3}{2} & (\frac{1}{2},0) & r-2 \\ \hline \end{array} \qquad \begin{array}{|cccc|}\hline & \Delta & (j_1,j_2) & r \\ \hline \overline{\psi}^{(3)}_{\dot\alpha} & \frac{1}{2} & (0,\frac{1}{2}) & -r \\ \overline{\phi} & 1 & (0,0) & -r+1 \\ T^-_{\dot\alpha\dot\beta} & 1 & (0,1) & -r+1 \\ \overline{\psi}_{\dot\alpha} & \frac{3}{2} & (0,\frac{1}{2}) & -r+2 \\ \hline \end{array} \tag{3.11}$$

The R-charge is again determined by $r = \frac{2}{3}\Delta_{\psi^{(3)}} = \frac{1}{3}$. The direct computation of the chiral anomaly coefficients $\mathrm{Tr(R)}$ and $\mathrm{Tr(R^3)}$ based on (2.13), (3.4) involves summing up contributions from all the fields but the scalars

$$\mathrm{Tr(R)} = \underbrace{r}_{\psi^{(3)}} \underbrace{+8\,(r-1)}_{T^+} \underbrace{+r-2}_{\psi} = \frac{1}{3} + 8\times\left(-\frac{2}{3}\right) - \frac{5}{3} = -\frac{20}{3},$$

$$\mathrm{Tr(R^3)} = \underbrace{r^3}_{\psi^{(3)}} \underbrace{-4\,(r-1)^3}_{T^+} \underbrace{+(r-2)^3}_{\psi} = \left(\frac{1}{3}\right)^3 - 4\times\left(-\frac{2}{3}\right)^3 + \left(-\frac{5}{3}\right)^3 = -\frac{92}{27}. \tag{3.12}$$

3.2.4 Conformal gravitino multiplet $\left[\frac{3}{2}\right]$

A review of the conformal gravitino supermultiplet can be found in appendix C of [75]. It was also discussed recently in [76] in the context of higher spin generalizations. In general, one can consider a superconformal multiplet associated with an integer superspin s and described in terms of an unconstrained superfield

$$\Psi_{\alpha(s)\,\dot\alpha(s-1)} \equiv \Psi_{\alpha_1\ldots\alpha_s\dot\alpha_1\ldots\dot\alpha_{s-1}}, \qquad \overline{\Psi}_{\alpha(s-1)\,\dot\alpha(s)} \equiv \overline{\Psi}_{\alpha_1\ldots\alpha_{s-1}\dot\alpha_1\ldots\dot\alpha_s}, \tag{3.13}$$

where $\alpha(s)$ denotes a set of s symmetrized indices. The gauge freedom is

$$s>1: \qquad \delta\Psi_{\alpha_1\ldots\alpha_s\dot\alpha_1\ldots\dot\alpha_{s-1}} = D_{(\alpha_1}\overline{\Lambda}_{\alpha_2\ldots\alpha_s)\dot\alpha_1\ldots\dot\alpha_{s-1}} + \overline{D}_{(\dot\alpha_1}\zeta_{\alpha_1\ldots\alpha_s\dot\alpha_2\ldots\dot\alpha_{s-1})}, \tag{3.14}$$

$$s=1: \qquad \delta\Psi_\alpha = D_\alpha\overline{\Lambda} + \zeta_\alpha\,, \tag{3.15}$$

with unconstrained gauge parameters $\overline{\Lambda}_{\alpha(s-1)\dot\alpha(s-1)}$ and $\zeta_{\alpha(s)\dot\alpha(s-2)}$. The superfield $\Psi_{\alpha(s)\,\dot\alpha(s-1)}$ has superconformal weights $(q,\overline{q}) = (-\frac{s}{2},\frac{1-s}{2})$ [76], so that its dimension and R-charge are $\Delta = q+\overline{q} = \frac{1}{2}-s$ and $r = \frac{2}{3}(q-\overline{q}) = -\frac{1}{3}$. One may choose a Wess-Zumino gauge where

$$\begin{aligned}\Psi_{\alpha_1\ldots\alpha_s\dot\alpha_1\ldots\dot\alpha_{s-1}}(\theta,\overline{\theta}) = {} & \theta^\beta\overline{\theta}^{\dot\beta}\psi_{(\beta\alpha_1\ldots\alpha_s)(\dot\beta\dot\alpha_1\ldots\dot\alpha_{s-1})} + \overline{\theta}^2\theta^\beta T_{(\beta\alpha_1\ldots\alpha_s)\dot\alpha_1\ldots\dot\alpha_{s-1}} \\ & - \theta^2\overline{\theta}^{\dot\beta}\,\mathrm{V}_{\alpha_1\ldots\alpha_s(\dot\beta\dot\alpha_1\ldots\dot\alpha_{s-1})} + \theta^2\overline{\theta}^2\psi_{\alpha_1\ldots\alpha_s\dot\alpha_1\ldots\dot\alpha_{s-1}},\end{aligned} \tag{3.16}$$

[11] In [74] the antisymmetric tensor component is the standard gauge-invariant one but this is not relevant for the purpose of fixing R-charges we are concerned with here.

with complex bosonic fields $\mathrm{V}_{\alpha_1\ldots\alpha_s(\dot\beta\dot\alpha_1\ldots\dot\alpha_{s-1})} = (V + i\,V')_{\alpha_1\ldots\alpha_s(\dot\beta\dot\alpha_1\ldots\dot\alpha_{s-1})}$ and $T_{\alpha(s+1)\dot\alpha(s-1)}$. Specialization to our case of interest $s=1$ gives

$$\Psi_\alpha(\theta,\overline{\theta}) = \theta^\beta\overline{\theta}^{\dot\beta}\psi_{(\alpha\beta)\dot\beta} + \overline{\theta}^2\theta^\beta T_{(\alpha\beta)} - \theta^2\overline{\theta}^{\dot\beta}\,\mathrm{V}_{\alpha\dot\beta} + \theta^2\overline{\theta}^2\psi_\alpha\,, \tag{3.17}$$

with the same field content as in (3.2). For $s>1$ the fields $\mathrm{V}_{\alpha(s)\dot\alpha(s)}$ and $T_{\alpha(s+1)\dot\alpha(s-1)}$ have residual gauge invariances. In the special case of $s=1$ the field $T_{(\alpha\beta)} \equiv T^+_{\alpha\beta}$ is a non-gauge one [76]. From (3.17) we find (here $\Psi_{(\alpha\beta)\dot\beta}$ is the gravitino in spinor notation)

$$\begin{array}{c|ccc}
 & \Delta & (j_1,j_2) & r \\ \hline
\Psi_{(\alpha\beta)\dot\beta} & \frac12 & (1,\frac12) & r \\
\mathrm{V}_{\alpha\dot\alpha} & 1 & (\frac12,\frac12) & r-1 \\
T^+_{\alpha\beta} & 1 & (1,0) & r+1 \\
\psi_\alpha & \frac32 & (\frac12,0) & r
\end{array}
\qquad
\begin{array}{c|ccc}
 & \Delta & (j_1,j_2) & r \\ \hline
\overline{\Psi}_{(\dot\alpha\dot\beta)\beta} & \frac12 & (\frac12,1) & -r \\
\overline{\mathrm{V}}_{\alpha\dot\alpha} & 1 & (\frac12,\frac12) & -r+1 \\
T^-_{\dot\alpha\dot\beta} & 1 & (0,1) & -r-1 \\
\overline{\psi}_{\dot\alpha} & \frac32 & (0,\frac12) & -r
\end{array} \tag{3.18}$$

According to the above general discussion here we should have $r=-\frac13$.

It is useful to consider also an alternative and more transparent description of the $[\frac32]$ multiplet in terms of the gauge-invariant chiral superfield (see [77])

$$W_{\alpha\beta} = T^+_{\alpha\beta} + \theta^\gamma\,(\Psi_{\alpha\beta\gamma} + \varepsilon_{\gamma(\alpha}\psi_{\beta)}) + \theta^2\,\mathrm{F}_{\alpha\beta}\,, \tag{3.19}$$

where $\Psi_{\alpha\beta\gamma}$ is the gravitino field strength (i.e. the self-dual part of $\partial_{[\mu}\Psi_{\nu]}$) and F is the field strength of the complex vector. Here the dimensions of the components are $\Delta = 1, \frac32, 2$ and the R-charges are $r+1, r, r-1$ (here we set $r_{_T} \equiv r+1$ to match the notation in (3.18)). As this is a chiral superfield, its lowest component should have $\Delta = \frac32 r$. This implies $r+1 = \frac23$ and once again $r=-\frac13$. The chiral anomaly coefficients Tr(R) and Tr(R^3) receive contributions from all the fields except the vectors and thus we find from (2.13), (3.4)

$$\begin{aligned}
\mathrm{Tr}(\mathrm{R}) &= \underbrace{-20\,r}_{\Psi_\mu}\underbrace{+8\,(r+1)}_{T^+}\underbrace{+r}_{\psi} = -20\left(-\frac13\right) + 8\left(\frac23\right) - \frac13 = \frac{35}{3}, \\
\mathrm{Tr}(\mathrm{R}^3) &= \underbrace{4\,r^3}_{\Psi_\mu}\underbrace{-4\,(r+1)^3}_{T^+}\underbrace{+r^3}_{\psi} = 4\left(-\frac13\right)^3 - 4\left(\frac23\right)^3 + \left(-\frac13\right)^3 = -\frac{37}{27}.
\end{aligned} \tag{3.20}$$

3.2.5 Conformal graviton multiplet [2]

The superfield description of the linearized $\mathcal{N}=1$ conformal supergravity multiplet [2] was discussed in [2, 3]. The corresponding real superfield starting with graviton contains (in a Wess-Zumino gauge) the components corresponding to the fields in (3.2)

$$\begin{array}{c|ccc}
 & \Delta & (j_1,j_2) & r \\ \hline
h_{(\alpha\beta)(\dot\alpha\dot\beta)} & 0 & (1,1) & r \\
\Psi_{(\alpha\beta)\dot\beta} & \frac12 & (1,\frac12) & r+1 \\
\overline{\Psi}_{(\dot\alpha\dot\beta)\beta} & \frac12 & (\frac12,1) & r-1 \\
V_{\alpha\dot\alpha} & 1 & (\frac12,\frac12) & r
\end{array} \tag{3.21}$$

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The expected value of the graviton R-charge is $r = 0$. To confirm this one may consider the corresponding chiral field strength superfield [77, 78] (cf. (3.19))

$$W_{\alpha\beta\gamma} = \Psi_{\alpha\beta\gamma} + \theta^\delta (C_{\alpha\beta\gamma\delta} + \varepsilon_{\delta(\alpha} F_{\beta\gamma)}) + \theta^2 \Phi_{\alpha\beta\gamma}\,, \tag{3.22}$$

where $\Psi_{\alpha\beta\gamma}$ is the gravitino field strength and $\Phi_{\alpha\beta\gamma}$ is the "second" gravitino field strength (self-dual part of the strength of $\Phi_\mu \sim \gamma^\nu \partial_{[\mu}\Psi_{\nu]}$). Here $\Psi_{\alpha\beta\gamma}$ should have $\Delta_{\Psi} = \frac{3}{2} r_{\Psi} = \frac{3}{2}$ so that $r_{\Psi} = 1$ and thus in (3.21) we should have $r = 0$ (equivalently, this follows from the fact that Weyl tensor $C_{\alpha\beta\gamma\delta}$ has $r_C = r_{\Psi} - 1 = 0$).

The chiral anomaly coefficients $\mathrm{Tr}(\mathrm{R})$ and $\mathrm{Tr}(\mathrm{R}^3)$ here receive contributions only from the Weyl gravitino (see (2.13), (3.4))

$$\mathrm{Tr}(\mathrm{R}) = \underbrace{-20\,(r+1)}_{\Psi_\mu} = -20, \qquad \mathrm{Tr}(\mathrm{R}^3) = \underbrace{4\,(r+1)^3}_{\Psi_\mu} = 4\,. \tag{3.23}$$

These values, like those in (3.12) and (3.20), are once again in agreement with the corresponding values in table 2 demonstrating consistency with the supersymmetry which underlies the relations (2.12).

4 The superconformal index of $\mathcal{N} = 1$ multiplets

The explicit evaluation of the index (2.1) in a free superconformal theory can be done in terms of the plethystic exponential[12]

$$\log \mathrm{I}(p,q) = \sum_{n=1}^{\infty} \frac{1}{n}\, \mathrm{i}(p^n, q^n), \tag{4.1}$$

where the single-particle index $\mathrm{i}(p,q)$ can be computed by letter counting [15–17]. We shall consider the special case of $p = q = e^{-\beta} \equiv t$ corresponding to (2.3) and use the notation, cf. (2.3),

$$\mathrm{I}(e^{-\beta}, e^{-\beta}) \equiv \mathrm{I}(\beta), \qquad \mathrm{i}(e^{-\beta}, e^{-\beta}) \equiv \mathrm{i}(\beta)\,. \tag{4.2}$$

Below we will first compute the single-particle index $\mathrm{i}(\beta)$ for the multiplets introduced in the previous section and then discuss the $\beta \to 0$ expansion.

4.1 Computing the single-particle index

For the familiar unitary multiplets [0] and [1] one finds [70]

$$\mathrm{i}_{[0]}(\beta) = \frac{t^{\frac{2}{3}} - t^{\frac{4}{3}}}{(1-t)^2}, \qquad \mathrm{i}_{[1]}(\beta) = -\frac{2\,t}{1-t}\,, \qquad t \equiv e^{-\beta}\,. \tag{4.3}$$

[12] Eq. (4.1) is a standard way to build symmetric multi-particle states in terms of the single-particle states. This is made explicit by the illustrative relation

$$\exp \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} p^{n\gamma_m} = 1 + \sum_{m=1}^{\infty} p^{\gamma_m} + \sum_{m\le m'} p^{\gamma_m+\gamma_{m'}} + \sum_{m\le m'\le m''} p^{\gamma_m+\gamma_{m'}+\gamma_{m''}} + \dots\,.$$

	Δ	(j_1, j_2)	r	$(-1)^F\, t^{2j_2+r}$
$\phi^{(4)}$	0	(0,0)	0	1
$\overline{\phi}^{(2)}$	0	(0,0)	0	1
$\overline{\psi}^{(3)}_{\dot{-}}$	$\frac{1}{2}$	$(0, -\frac{1}{2})$	1	-1
$\partial_{\pm\dot{-}}\overline{\psi}^{(3)}_{\dot{+}}$	$\frac{3}{2}$	$(\pm\frac{1}{2}, 0)$	1	$-2t$
$\partial_{\pm\dot{-}}\overline{\phi}$	2	$(\pm\frac{1}{2}, -\frac{1}{2})$	2	$2t$

Table 3. Contributions to the index of the $[0']$ multiplet. j_1 and j_2 (which can take positive and negative values) stand for third components of the two SU(2) spins which label the states in the superconformal index. Each undotted (dotted) $\pm$ index contributes $\pm\frac{1}{2}$ to $j_1(j_2)$.

For the $[0']$ multiplet in (3.9) the analysis goes as follows. One has to consider the fields X ("letters") with $\delta \equiv \Delta - 2\, j_2 - \frac{3}{2}\, r = 0$ contributing

$$\delta(X) = 0: \qquad \mathrm{i}|_X = (-1)^F\, t^{2j_2+r} = (-1)^F\, t^{\Delta - \frac{r}{2}}\,. \tag{4.4}$$

Applying derivatives $\partial_{\alpha\dot{\alpha}}$ one builds new letters. In the following it will be convenient to denote the spinor indices $\alpha = 1, 2$ by $\pm$: $1 \to +$ and $2 \to -$ (and similar for dotted indices). The $+/-$ notation is convenient because each type of index (dotted or undotted) with such a value increases/decreases the third component of the associated Lorentz spin by $\frac{1}{2}$. The derivatives $\partial_{\pm\dot{+}}$ do not change δ. They can be applied repeatedly leading to a universal factor $1/(1-t)^2$ in the single-particle index i. Instead, $\partial_{\pm\dot{-}}$ increase δ by two units. Any derivative $\partial_{\alpha\dot{\alpha}}$ does not change r so, on the $\delta = 0$ states, it increases $2j_2 + r \overset{\delta=0}{=} \Delta - \frac{r}{2}$ by one unit. This gives the set of contributions in table 3. The descendants (obtained by the application of $\partial_{\pm\dot{+}}$ leaving δ invariant) that have the form of equations of motion are

$$\partial_{-\dot{+}}(\partial_{+\dot{-}}\overline{\phi}) \sim \Box\overline{\phi} = 0\,, \tag{4.5}$$

$$\partial_{\pm\dot{+}}\partial_{-\dot{+}}(\partial_{+\dot{-}}\overline{\psi}^{(3)}_{\dot{+}}) \sim \Box\partial_{\pm\dot{+}}\overline{\psi}^{(3)}_{\dot{+}} = 0\,. \tag{4.6}$$

This gives the index (cf. (4.3))

$$\mathrm{i}_{[0']}(\beta) = \frac{1 + 1 - 1 - 2\,t + (2\,t - t^2)}{(1-t)^2} = \frac{1-t^2}{(1-t)^2}\,. \tag{4.7}$$

A similar analysis can be carried out for the tensor multiplet $[\frac{1}{2}]$ in (3.11). In this case, the list of non-zero contributions is collected in table 4. The contribution of the last line of this table should not be included: due to the spinor equations of motion, this derivative may be replaced by another one with $\delta \neq 0$. This gives the contributions $-2\,t^{\frac{1}{3}}$ and $+2\,t^{\frac{5}{3}}$ from the "left" and "right" chiral fields. Next, we have to take into account the equations of motion for the $(j_1, j_2) = (1, 0)$ and $(j_1, j_2) = (0, 1)$ chiral components. One can check that there are no contributions with $\delta = 0$ and thus

$$\mathrm{i}_{[\frac{1}{2}]}(\beta) = \frac{-2\,t^{\frac{1}{3}} + 2\,t^{\frac{5}{3}}}{(1-t)^2}. \tag{4.8}$$

	Δ	(j_1,j_2)	r	$(-1)^F\,t^{2j_2+r}$
$\psi^{(3)}_{\pm}$	$\frac{1}{2}$	$(\pm\frac{1}{2},0)$	$\frac{1}{3}$	$-2\,t^{\frac{1}{3}}$
$\overline{\psi}^{(3)}_{\dot{+}}$	$\frac{1}{2}$	$(0,+\frac{1}{2})$	$-\frac{1}{3}$	$-t^{\frac{2}{3}}$
$\overline{\phi}$	1	$(0,0)$	$\frac{2}{3}$	$t^{\frac{2}{3}}$
$T^{-}_{\dot{+}\dot{-}}$	1	$(0,0)$	$\frac{2}{3}$	$t^{\frac{2}{3}}$
$\partial_{\pm\dot{-}}T^{-}_{\dot{+}\dot{+}}$	2	$(\pm\frac{1}{2},\frac{1}{2})$	$\frac{2}{3}$	$2\,t^{\frac{5}{3}}$
$\overline{\psi}_{\dot{-}}$	$\frac{3}{2}$	$(0,-\frac{1}{2})$	$\frac{5}{3}$	$-t^{\frac{2}{3}}$
$\partial_{\pm\dot{-}}\overline{\psi}_{\dot{+}}$	$\frac{5}{2}$	$(\pm\frac{1}{2},0)$	$\frac{5}{3}$	$-2\,t^{\frac{5}{3}}$

Table 4. Contributions to the index of the $[\frac{1}{2}]$ multiplet. The components of the (anti) self-dual tensor are symmetric in the (dotted) undotted indices.

	Δ	(j_1,j_2)	r	$(-1)^F\,t^{2j_2+r}$
$T^{+}_{\alpha\beta}$	1	$(j_1,0)_{j_1=0,\pm1}$	$\frac{2}{3}$	$3\,t^{\frac{2}{3}}$
$\mathrm{F}_{\alpha\beta}$	2	$(j_1,0)_{j_1=0,\pm1}$	$\frac{4}{3}$	$3\,t^{\frac{4}{3}}$
$T^{-}_{\dot{+}\dot{+}}$	1	$(0,+1)$	$-\frac{2}{3}$	$t^{\frac{4}{3}}$
$\overline{\Psi}_{\dot{+}\dot{+}\dot{-}}$	$\frac{3}{2}$	$(0,+\frac{1}{2})$	$\frac{1}{3}$	$-t^{\frac{4}{3}}$
$\partial_{\pm\dot{-}}\overline{\Psi}_{\dot{+}\dot{+}\dot{+}}$	$\frac{5}{2}$	$(\pm\frac{1}{2},1)$	$\frac{1}{3}$	$-2\,t^{\frac{7}{3}}$
$\overline{\Phi}_{\dot{+}\dot{+}\dot{+}}$	$\frac{5}{2}$	$(0,+\frac{3}{2})$	$-\frac{1}{3}$	$-t^{\frac{8}{3}}$
$\overline{\psi}_{\dot{+}}$	$\frac{3}{2}$	$(0,+\frac{1}{2})$	$\frac{1}{3}$	$-t^{\frac{4}{3}}$
$\overline{\mathrm{F}}_{\dot{+}\dot{-}}$	2	$(0,0)$	$\frac{4}{3}$	$t^{\frac{4}{3}}$
$\partial_{\pm\dot{-}}\overline{\mathrm{F}}_{\dot{+}\dot{+}}$	3	$(\pm\frac{1}{2},+\frac{1}{2})$	$\frac{4}{3}$	$2\,t^{\frac{7}{3}}$

Table 5. Contributions to the index of the $[\frac{3}{2}]$ multiplet.

To find the index for the gravitino multiplet $[\frac{3}{2}]$ in (3.18) we should take into account that relevant letters should be gauge invariant, i.e. use the gravitino field strength in (3.19) and also the "second" gravitino field strength $\Phi_{\alpha\beta\gamma}$ (cf. (3.22)). The latter obeys the Bianchi identity

$$\partial_{\alpha}{}^{\dot{\beta}}\partial_{\beta}{}^{\dot{\gamma}}\overline{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}} = \partial_{\dot{\alpha}}{}^{\gamma}\Phi_{\alpha\beta\gamma}, \tag{4.9}$$

and thus its dimension is $\frac{5}{2}$ while the R-charge is opposite to that of $\Psi_{\alpha\beta\gamma}$. The non-zero contributions are collected in table 5. The Bianchi identity for the (complex) Maxwell field strength

$$\partial_{\alpha}{}^{\dot{\beta}}\,\overline{\mathrm{F}}_{\dot{\alpha}\dot{\beta}} = \partial^{\beta}{}_{\dot{\alpha}}\,\mathrm{F}_{\alpha\beta} \tag{4.10}$$

lead to additional vector contribution $-2\,t^{\frac{4}{3}+1}-2\,t^{\frac{7}{3}} = -4\,t^{\frac{7}{3}}$. Finally, we should account for the equations of motion of $T^{+}_{\alpha\beta}$, i.e. $\partial_{\alpha\dot{\alpha}}\partial_{\beta\dot{\beta}}\,T^{+\alpha\beta} = 0$ getting another $-t^{\frac{2}{3}+2}$ contribution. As a result, the final expression for the index is

$$\mathrm{i}_{[\frac{3}{2}]}(\beta) = \frac{-2t^{\frac{8}{3}} - 4\,t^{\frac{7}{3}} + 3\,t^{\frac{4}{3}} + 3\,t^{\frac{2}{3}}}{(1-t)^2}. \tag{4.11}$$

For the graviton multiplet [2] in (3.21) the computation of the index should be again done in terms of the gauge invariant field strengths appearing in (3.22). The resulting non-zero

	Δ	(j_1, j_2)	r	$(-1)^F\, t^{2j_2+r}$
$\Psi_{\alpha\beta\gamma}$	$\frac{3}{2}$	$(j_1, 0)_{j_1=\pm\frac{1}{2},\pm\frac{3}{2}}$	1	$-4\,t$
$\overline{\Psi}_{\dot{+}\dot{+}\dot{+}}$	$\frac{3}{2}$	$(0, +\frac{3}{2})$	-1	$-t^2$
$\overline{\Phi}_{\dot{+}\dot{+}\dot{-}}$	$\frac{5}{2}$	$(0, +\frac{1}{2})$	1	$-t^2$
$\partial_{\pm\dot{-}}\overline{\Phi}_{\dot{+}\dot{+}\dot{+}}$	$\frac{7}{2}$	$(\pm\frac{1}{2}, +1)$	1	$-2\,t^3$
$\overline{C}_{\dot{+}\dot{+}\dot{+}\dot{-}}$	2	$(0, +1)$	0	t^2
$\partial_{\pm\dot{-}}\overline{C}_{\dot{+}\dot{+}\dot{+}\dot{+}}$	3	$(\pm\frac{1}{2}, +\frac{3}{2})$	0	$2\,t^3$
$\overline{F}_{\dot{+}\dot{+}}$	2	$(0, +1)$	0	t^2

Table 6. Contribution to the index of the [2] multiplet.

	[0]	[1]	[0′]	$[\frac{1}{2}]$	$[\frac{3}{2}]$	[2]
$P(t)$	$t^{\frac{2}{3}} - t^{\frac{4}{3}}$	$-2t + 2t^2$	$1 - t^2$	$-2\,t^{\frac{1}{3}} + 2\,t^{\frac{5}{3}}$	$-2t^{\frac{8}{3}} - 4\,t^{\frac{7}{3}} + 3\,t^{\frac{4}{3}} + 3\,t^{\frac{2}{3}}$	$-4t + 4t^3$

Table 7. Numerators $P(t)$ of the single particle indices $\mathrm{i}(\beta) = \frac{P(t)}{(1-t)^2}$ of the $\mathcal{N} = 1$ multiplets.

contributions are collected in table 6. Before taking into account Bianchi identities, the index is simply $\frac{-4t}{(1-t)^2}$. The Bianchi identities may only contribute a term proportional to t^3. The condition of vanishing of the index numerator for $t = 1$ then gives[13]

$$\mathrm{i}_{[2]}(t) = \frac{-4t + 4\,t^3}{(1-t)^2}. \tag{4.12}$$

The summary of the computed indices is presented in table 7.[14]

4.2 Small β expansion of the index $\mathrm{I}(\beta)$

Let us now use the above results for $\mathrm{i}(\beta)$ to compute the small β expansion of the superconformal index $\mathrm{I}(\beta)$ in order to compare with the expected expansion (2.7). A generalization to a 2-parameter family of unequal p and q in (2.1) will be discussed in appendix E.

The usual approach to derivation of the small β expansion of the index for models involving unitary multiplets starts from the summation in (4.1) in terms of elliptic Γ function. Modular properties of the resulting expressions [79] are then exploited to discuss the small β limit. Here we propose a simpler approach based on the techniques developed for studying similar limit of standard partition functions using that $\mathrm{I}(\beta)$ has a formal structure of a partition function. Let m be a label a particular multiplet and let us define the Mellin transform of the single-particle index $\mathrm{i}(\beta)$ as

$$\mathrm{z}_\mathrm{m}(u) \equiv \frac{1}{\Gamma(u)} \int_0^\infty d\beta\, \beta^{u-1}\, \mathrm{i}_\mathrm{m}(\beta)\,. \tag{4.13}$$

[13]This extra $+4\,t^3$ comes from the field in the 4th line of the table 6 (which enters a Bianchi identity contributing $+2t^3$) and from $\partial^{\alpha}{}_{\dot{\alpha}}\partial^{\beta}{}_{\dot{\beta}}\Psi_{\alpha\beta\gamma} + \cdots = 0$ (which is another Bianchi identity similar to (4.9) conserving $\delta = 0$ when $\dot{\alpha} = \dot{\beta} = -$ and $\gamma = \pm$ is arbitrary).

[14]The supersymmetric index is the $n = 1$ case of similar indices for theories on the lens space $S^1_\beta \times S^3/\mathbb{Z}_n$ that have been computed in [32] for unitary theories. In the $n > 1$ cases the index receives contributions from twisted sectors. It would be interesting to extend the analysis to the non-unitary multiplets considered here and explore the $n \to \infty$ limit when $S^3/\mathbb{Z}_n \to S^2$ and the index reduces to that of a 3d theory.

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Then for $\mathrm{I}(\beta)$ in (4.2) one finds[15]

$$\log \mathrm{I_m}(\beta) = \sum_{n=1}^{\infty} \frac{1}{n}\, \mathrm{i_m}(n\beta) = \frac{1}{2\pi\, i} \int_{v-i\infty}^{v+i\infty} du\; \beta^{-u}\, \Gamma(u)\, \zeta(u+1)\, \mathrm{z_m}(u)\,, \tag{4.14}$$

which is valid when v is sufficiently large. When the vertical contour in (4.14) is moved to the left, we pick up residues of poles at integer u and this has the form of a small β expansion with the coefficients involving the residues of $\mathrm{z_m}(u)$.[16] In general, given a term $\frac{t^q}{(1-t)^2}$ in the index, we may use that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{\Gamma(u)}(n+1) \int_0^{\infty} d\beta\, \beta^{u-1}\, e^{-n\,\beta} e^{-q\,\beta} &= \sum_{n=0}^{\infty}(n+1)(n+q)^{-u}\\ &= \zeta(u-1,q) + (1-q)\,\zeta(u,q),\end{aligned} \tag{4.15}$$

and taking residues in (4.14) we immediately obtain the expansion of the index.

Let us apply this method to the known cases of the chiral and vector multiplets. For the [0] chiral multiplet we have from (4.3)

$$\begin{aligned}\mathrm{z}_{[0]}(u) &= \frac{1}{\Gamma(u)} \int_0^{\infty} d\beta\, \beta^{u-1} \frac{e^{-\frac{2}{3}\beta} - e^{-\frac{4}{3}\beta}}{(1-e^{-\beta})^2} = \frac{1}{\Gamma(u)} \sum_{n=0}^{\infty}(n+1) \int_0^{\infty} d\beta\, \beta^{u-1} e^{-n\,\beta}\ (e^{-\frac{2}{3}\beta} - e^{-\frac{4}{3}\beta})\\ &= \sum_{n=0}^{\infty}(n+1)\left[\left(n+\frac{2}{3}\right)^{-u} - \left(n+\frac{4}{3}\right)^{-u}\right]\\ &= \zeta\left(u-1,\frac{2}{3}\right) - \zeta\left(u-1,\frac{4}{3}\right) + \frac{1}{3}\zeta\left(u,\frac{2}{3}\right) + \frac{1}{3}\zeta\left(u,\frac{4}{3}\right).\end{aligned} \tag{4.16}$$

Multiplying this by $\Gamma(u)\zeta(u+1)\beta^{-u}$ (to get the integrand in (4.14)) and taking the residues of the poles, we get contributions to (4.14) coming from $u = -1, 0, 1$ only. As a result,

$$\log \mathrm{I}_{[0]}(\beta) = \frac{\pi^2}{9\,\beta} + \mathrm{k}[0] + \frac{7}{324}\,\beta + \mathcal{O}(e^{-1/\beta}), \tag{4.17}$$

where

$$\mathrm{k}[0] = \frac{\pi}{9\sqrt{3}} - \frac{1}{6}\log 3 - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{6\sqrt{3}\pi}\,. \tag{4.18}$$

This is in full agreement with (2.7) with the proposed values of the coefficients C_i, see (2.5), (2.8), (2.14), (2.15) and table 2. The constant $\mathrm{k}[0]$ can be identified with the 3d

[15] Eq. (4.14) follows from the Mellin inversion formula and may be checked for a typical single-particle contribution to the single-particle index starting from the relation $e^{-\beta} = \frac{1}{2\pi i}\int_{v-i\infty}^{v+i\infty} du\, \beta^{-u}\,\Gamma(u)$ valid for $v > 0$ (see, e.g., [80]). Here $\zeta(u+1)$ is the Riemann zeta function. Note that the relations below apply to both fermions and bosons (as fermions are treated as periodic on the circle).

[16] In some cases symmetry properties of the integrand allow one to relate the contour associated with $-v$ to that at $+v$. Then the remainder of the small β (high temperature) pole expansion can be found explicitly and gives rise to a "temperature inversion" relation [80].

partition function $\log Z^{\rm susy}_{S^3}$.[17] Similarly, for a vector multiplet [1] we find using (4.3)

$$\begin{aligned} \mathrm{z}_{[1]}(u) &= -\frac{2}{\Gamma(u)}\int_0^\infty d\beta\,\beta^{u-1}\frac{e^{-\beta}}{1-e^{-\beta}} = \\ &= -\frac{2}{\Gamma(u)}\sum_{n=0}^\infty\int_0^\infty d\beta\,\beta^{u-1}e^{-(n+1)\,\beta} = -2\sum_{n=0}^\infty (n+1)^{-u} = -2\,\zeta(u)\,. \end{aligned} \tag{4.19}$$

Taking residues in (4.14), we obtain

$$\log \mathrm{I}_{[1]}(\beta) = -\frac{\pi^2}{3\,\beta} - \log\beta + \log(2\pi) + \frac{1}{12}\,\beta + \mathcal{O}(e^{-1/\beta})\,. \tag{4.20}$$

This is again in agreement with (2.7) and (2.5), (2.8), (2.15) and table 2. Eq. (4.20) implies that $\mathrm{k}[1] = \log Z^{\rm susy}_{S^3} = \log(2\pi)$. Consistency of this result is further discussed in appendix (D.2).

Applying the same method also to the four non-unitary multiplets with single-particle indices given in table 7 the final results can be summarized as follows[18]

$$\begin{aligned}
&[0] = (2\,\phi,\psi): && \log \mathrm{I}_{[0]}(\beta) = \frac{\pi^2}{9\,\beta} + \mathrm{k}[0] + 0\cdot\log\beta + \frac{7}{324}\,\beta + \mathcal{O}(e^{-1/\beta}),\\
&[1] = (V_\mu,\psi): && \log \mathrm{I}_{[1]}(\beta) = -\frac{\pi^2}{3\,\beta} + \mathrm{k}[1] - \log\beta + \frac{1}{12}\,\beta + \mathcal{O}(e^{-1/\beta}),\\
&[0'] = (2\,\phi^{(4)},\psi^{(3)},2\,\phi): && \log \mathrm{I}_{[0']}(\beta) = \frac{\pi^2}{3\,\beta} + \mathrm{k}[0'] + 0\cdot\log\beta - \frac{1}{12}\,\beta + \mathcal{O}(e^{-1/\beta}),\\
&\left[\frac{1}{2}\right] = (\psi, 2\varphi, T_{\mu\nu},\psi^{(3)}): && \log \mathrm{I}_{[\frac{1}{2}]}(\beta) = -\frac{4\,\pi^2}{9\,\beta} + \mathrm{k}\left[\frac{1}{2}\right] + 0\cdot\log\beta - \frac{1}{81}\,\beta + \mathcal{O}(e^{-1/\beta}),\\
&\left[\frac{3}{2}\right] = (\Psi_\mu, 2\,V_\mu, T_{\mu\nu},\psi): && \log \mathrm{I}_{[\frac{3}{2}]}(\beta) = \frac{13\,\pi^2}{9\,\beta} + \mathrm{k}\left[\frac{3}{2}\right] + 6\,\log\beta - \frac{389}{324}\,\beta + \mathcal{O}(\beta^2),\\
&[2] = (h_{\mu\nu},\Psi_\mu,V_\mu): && \log \mathrm{I}_{[2]}(\beta) = -\frac{4\,\pi^2}{3\,\beta} + \mathrm{k}[2] - 8\,\log\beta + \frac{7}{3}\,\beta + \mathcal{O}(\beta^2)\,,
\end{aligned} \tag{4.21}$$

where

$$\begin{aligned}
&\mathrm{k}[0] = \frac{\pi}{9\sqrt{3}} - \frac{1}{6}\log 3 - \frac{\psi^{(1)}\left(\frac{1}{3}\right)}{6\sqrt{3}\pi}\,, && \mathrm{k}[1] = \log(2\pi)\,,\\
&\mathrm{k}[0'] = 0\,, && \mathrm{k}\left[\frac{1}{2}\right] = \frac{2\pi}{9\sqrt{3}} + \frac{2}{3}\log 3 - \frac{\psi^{(1)}(\frac{1}{3})}{3\sqrt{3}\,\pi},\\
&\mathrm{k}\left[\frac{3}{2}\right] = \frac{\pi}{9\sqrt{3}} - \frac{49}{6}\log 3 - \frac{\psi^{(1)}(\frac{1}{3})}{6\sqrt{3}\,\pi}, && \mathrm{k}[2] = 4\,\log(2\pi).
\end{aligned} \tag{4.22}$$

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[17] According to [81] $\log Z^{\rm susy}_{S^3} = \ell(\frac{1}{3})$, where $\ell(R) = -R\,\log(1-e^{2\,\pi\,i\,R}) - \frac{1}{2\pi\,i}\mathrm{Li}_2(e^{2\,\pi\,i\,R}) + \frac{i\,\pi\,R^2}{2} - \frac{i\,\pi}{12}$. It is possible to prove that $\mathrm{k}[0] = \ell(\frac{1}{3})$. The relation of the index to 3d partition function in the $\beta\to 0$ limit after the removal of singular terms is a non-trivial fact depending on regularization, see [49, 82–86]. For a discussion of this relation in the case of non-supersymmetric conformal partition functions see appendix D.

[18] Let us note that for the $[0']$ multiplet one has to be careful with the contribution of the higher derivative scalar $\phi^{(4)}$. This field has canonical dimension 0 (like a scalar in 2d) and one finds terms of the form $\sum_{n=0}^\infty n^{-u}$. The $n=0$ term is ambiguous and we used the natural analytical continuation $0^{-u}\equiv 0$ for all (complex) u.

These constant terms will be further discussed in appendix D.2.[19]

5 Structure of small β expansion of the index of non-unitary multiplets

Let us now compare the explicit values of the coefficients appearing in the small β expansion (2.6) of the indices in (4.21) with their expected values discussed in section 2, i.e. with the previously suggested relations (2.9), (2.14), (2.5).[20] We shall denote the true values of the coefficients as $\widehat{C}_i$ with C_i being the expected values:

$$\log \mathrm{I}(\beta) \overset{\beta\to 0}{=} \widehat{C}_1 \frac{\pi^2}{\beta} + \widehat{C}_2 + \widehat{C}_3 \log\beta + \dots . \tag{5.1}$$

Leading term $\sim 1/\beta$. Comparing the values of the coefficient of the π^2/β term in (4.21) with their expected (2.9) values $C_1 = -\frac{1}{3}\,\mathrm{Tr(R)}$ in table 2, we find agreement for the $[0]$, $[1]$, $[0']$ multiplets but discrepancies for the non-unitary multiplets $[\frac{1}{2}]$, $[\frac{3}{2}]$, $[2]$ containing the antisymmetric tensor or conformal gravitino:

$$\widehat{C}_1 = C_1 - \frac{1}{3}\nu = -\frac{1}{3}\left[\mathrm{Tr(R)} + \nu\right], \tag{5.2}$$

$$\nu_{[0]} = \nu_{[1]} = \nu_{[0']} = 0\,, \qquad \nu_{[\frac{1}{2}]} = 8, \qquad \nu_{[\frac{3}{2}]} = -16, \qquad \nu_{[2]} = 24. \tag{5.3}$$

Remarkably, the correction terms $\nu_{\rm m}$ are all integer multiples of 8. For the collection of multiplets appearing (3.1) in the $\mathcal{N}$-extended conformal supergravities we then get $\nu_{\mathcal{N}=1} = 24$, $\nu_{\mathcal{N}=2} = 8$, $\nu_{\mathcal{N}=3} = \nu_{\mathcal{N}=4} = 0$. The $\mathcal{N}=3$ and $\mathcal{N}=4$ conformal supergravities also have $\mathrm{a} = \mathrm{c}$ [4] or $\mathrm{tr}\,\mathrm{R} = 0$ and thus $\widehat{C}_1 = C_1 = 0$. The same result is found also for $\mathcal{N}=4$ vector multiplet or $[1] + 3[0]$, i.e.

$$\widehat{C}_{1\,\mathcal{N}=1\,\mathrm{CSG}} = -\frac{4}{3}, \quad \widehat{C}_{1\,\mathcal{N}=2\,\mathrm{CSG}} = -\frac{2}{9}, \quad \widehat{C}_{1\,\mathcal{N}=3\,\mathrm{CSG}} = \widehat{C}_{1\,\mathcal{N}=4\,\mathrm{CSG}} = 0, \quad \widehat{C}_{1\,\mathcal{N}=4\,\mathrm{SYM}} = 0. \tag{5.4}$$

The reason why the relation (2.9) between the $1/\beta$ term and Tr(R) suggested in [19] fails to be universal may be due to the fact that the argument in [19] may not directly apply to theories containing more complicated chiral fields (self-dual tensors, conformal gravitions, etc.) rather than just the standard Weyl fermions.[21] One possibility is that in reconstructing 3d effective action by matching anomalies there is an integer-shift ambiguity

[19] Let us note that part of the expansions in (4.21) can be found by a naive procedure of first expanding the single-particle index and then applying (4.1) term by term. In general, $\mathrm{i}(\beta) = \frac{A_{-1}}{\beta} + A_0 + A_1\,\beta + A_2\,\beta^2 + \dots$. From (4.1), we then formally obtain: $\log \mathrm{I}(\beta) = \zeta(2)\,\frac{A_{-1}}{\beta} + \zeta(1)\,A_0 + \zeta(0)\,A_1\,\beta + \zeta(-1)\,A_2\,\beta^2 + \dots = \frac{A_{-1}}{6}\,\frac{\pi^2}{\beta} + \zeta(1)\,A_0 - \frac{1}{2}\,A_1\,\beta - \frac{1}{12}\,\beta^2 + \dots$. One can check that $\frac{A_{-1}}{6}$ is indeed the coefficient of the leading term in (4.21) in all cases. The same agreement is found for the linear in β term. The term proportional to A_0 is ill-defined but a heuristic replacement rule $\zeta(1) \to -\log\beta$ reproduces indeed the $\log\beta$ term in (4.21). However, all other subleading corrections are not captured correctly by this procedure.

[20] Here we will not attempt to compare the expressions in (4.22) with their expected (2.14) values $\log Z^{\rm susy}_{S^3}$ since to compute the latter requires first the construction of the explicit supersymmetric Lagrangians for the non-unitary multiplets on $S^1 \times S^3$ that should contain extra couplings to the R-symmetry gauge field background. Nevertheless, we remark that $\mathrm{k}[\frac{1}{2}] = \frac{2}{3}\,\ell(\frac{2}{3}) + \frac{8}{3}\ell(\frac{1}{3})$, where $\ell(R)$ was defined in footnote 17.

[21] Indications that there are subtleties in reconstruction of 3d effective actions in the case of (non-conformal) gravitinos appeared in [87, 88].

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in the coefficient of the 3d Chern-Simons term used in [19] leading in general to the presence of the correction term ν in (5.2). Note that a shift of $\widehat{C}_1$ from its value C_1 in (2.8) was also discussed for non-abelian gauge theories in [21].

Let us note also that the presence of the correction ν is similar to what happens in non-unitary 2d CFT. In a generic CFT the partition function $Z(\beta)$ is related to the density of states $\rho(E)\,dE$. Writing the energies in terms of the conformal dimensions and the 2d central charge, $E = 2\pi\,(\Delta + \overline{\Delta} - \frac{1}{12}\mathrm{c})$, the modular invariance $Z(\beta) = Z(\beta^{-1})$ implies that the limit $\beta \to 0$ is related to the $\beta \to \infty$ one in which Z is dominated by the lowest-energy states $Z(\beta \to \infty) \sim A\,\beta^{-\lambda} e^{-\beta\,E_{\min}}$. Here λ is non-zero for gapless systems so that [89, 90]

$$Z(\beta) = \int dE\,\rho(E)\,e^{-\beta\,E} \overset{\beta\to 0}{\sim} A\,\beta^{\lambda}\,e^{\frac{\pi}{6\,\beta}\,\mathrm{c_{eff}}}, \qquad \mathrm{c_{eff}} = -\frac{6}{\pi}E_{\min} = \mathrm{c} - 24\,\Delta_{\min}, \tag{5.5}$$

where $\Delta_{\min} \equiv \frac{1}{2}\min(\Delta + \overline{\Delta})$. In unitary theories one has $\Delta_{\min} = 0$ and $\mathrm{c_{eff}} = \mathrm{c}$. Instead, in non-unitary theories, $\Delta_{\min}$ is typically negative and $\mathrm{c_{eff}} > \mathrm{c}$. This inequality may be violated in the case of supersymmetric partition function where fermions are taken to be periodic and contribute with a negative sign. In non-unitary case one has generically [90] $\mathrm{c_{eff}(B)} - \mathrm{c_{eff}(F)} \neq \mathrm{c(B)} - \mathrm{c(F)}$. The correction $\Delta_{\min}$ in $\mathrm{c_{eff}} = \mathrm{c} - 24\,\Delta_{\min}$ is analogous to the parameter ν in (5.2). There are, however, important differences: modular invariance is not available in general and the role of the 2d central charge is played by $\mathrm{c} - \mathrm{a}$ (cf. (2.8)). Nevertheless, it is tempting to relate the presence of the non-zero ν in (5.2) with the existence of negative norm states in the case of non-unitary multiplets.

Logarithmic term $\sim \log\beta$. The comparison between the coefficients of $\log\beta$ in (4.21) and the suggested values $C_3 = -4\,(2\mathrm{a} - \mathrm{c})$ in (2.15) [20] given in table 2 again implies the presence of an integer correction:

$$\widehat{C}_3 = C_3 + \gamma = -4\,(2\mathrm{a} - \mathrm{c}) + \gamma\,, \tag{5.6}$$

$$\gamma_{[0]} = \gamma_{[1]} = 0\,, \qquad \gamma_{[0']} = -1, \qquad \gamma_{[\frac{1}{2}]} = -3, \qquad \gamma_{[\frac{3}{2}]} = 3, \qquad \gamma_{[2]} = -1\,, \tag{5.7}$$

$$\widehat{C}_{3\,[0]} = \widehat{C}_{3\,[0']} = \widehat{C}_{3\,[\frac{1}{2}]} = 0\,, \qquad \widehat{C}_{3\,[1]} = -1\,, \quad \widehat{C}_{3\,[\frac{3}{2}]} = 6\,, \quad \widehat{C}_{3\,[2]} = -8\,. \tag{5.8}$$

Considering the collections of multiplets appearing in $\mathcal{N}$-extended conformal supergravities (3.1) one finds

$$\widehat{C}_{3_{\mathcal{N}=1\,\mathrm{CSG}}} = 24, \quad \widehat{C}_{3_{\mathcal{N}=2\,\mathrm{CSG}}} = 8, \quad \widehat{C}_{3_{\mathcal{N}=3\,\mathrm{CSG}}} = 0, \quad \widehat{C}_{3_{\mathcal{N}=4\,\mathrm{CSG}}} = 2, \quad \widehat{C}_{3_{\mathcal{N}=4\,\mathrm{SYM}}} = -1. \tag{5.9}$$

Instead of trying to understand why $\widehat{C}_3$ is, in general, different from $C_3 = -4\,(2\mathrm{a} - \mathrm{c})$ proposed in [20] let us suggest an alternative general expression for it. Let us start by noting that the logarithmic term in the expansion of the index is the same as in the expansion of the supersymmetric partition function (2.6) and thus may have a universal origin. One may attempt to interpret the singular $\log\beta$ term appearing in the $\beta \to 0$ limit as associated with the KK modes that become "massless" in the limit of shrinking S^1. In practice, this relation is not straightforward and depends on regularization, see also appendix D.[22] One

[22]Let us note also that the $\log\beta$ term in the case of unitary non-abelian gauge theories was discussed in [21] where its coefficient was related to the dimension of the space of flat directions (with no curvature coupling) in the 3d theory.

may then expect that the coefficient of the $\log\beta$ term should be the same as in the standard partition function for the conformal gauge fields on $S^1_\beta \times S^3$ (with both bosons and fermions taken to be periodic on the circle). Then the $\log\beta$ term should receive contributions only from the conformal gauge fields in each multiplet. These can be found from the conformal higher spin partition functions derived in [5] and reviewed in appendix C below.

The analysis in appendix C shows that the $\log\beta$ contribution comes from a specific SO(4, 2) conformal character and is determined by a particular integer equal up to sign to the number $n_{\rm CKT}$ of conformal Killing tensors for the bosons and the number $n_{\rm CKS}$ of conformal Killing spinor-tensors for the fermions.[23]

We propose that the coefficients $\widehat{C}_3$ of the $\log\beta$ term in the expansion of the supersymmetric partition function on $S^1_\beta \times S^3$ for a generic superconformal multiplet should be given by the sum of the contributions from the conformal higher spin gauge fields in this multiplet, i.e.

$$\widehat{C}_3 \equiv -\mathrm{n}\,, \qquad \mathrm{n} = \sum_i n_{\rm CKT}(i) - \sum_i n_{\rm CKS}(i)\,. \tag{5.10}$$

In the case of the multiplets discussed in this paper the relevant conformal gauge fields are the standard vector V_μ $(s=1)$, the conformal graviton $h_{\mu\nu}$ $(s=2)$ and the conformal gravitino Ψ_μ (s = 1) for which we find from (C.4), (C.6):

$$n_{\rm CKT}(V) = 1, \qquad n_{\rm CKT}(h) = 15, \qquad n_{\rm CKS}(\Psi) = 8\,. \tag{5.11}$$

As a result, from (3.2) we get

$$\widehat{C}_{3\,[0]} = 0\,, \qquad \widehat{C}_{3\,[1]} = -1\,, \qquad \widehat{C}_{3\,[0']} = 0, \qquad \widehat{C}_{3\,[\frac{1}{2}]} = 0\,, \tag{5.12}$$

$$\widehat{C}_{3\,[\frac{3}{2}]} = 2\times(-1)+8 = 6\,, \qquad \widehat{C}_{3\,[2]} = -1-15+8 = -8\,, \tag{5.13}$$

exactly in agreement with (4.21) and (5.8).

Linear term $\sim \beta$. For this term there is full agreement between the expected values of the supersymmetric energy (2.5) in table 2 and the coefficients in (4.21).

Higher order corrections. Higher order corrections in the small β expansion of $\log \mathrm{I}(\beta)$ can be found by taking residues in (4.14) at the points $u = -2, -3, \ldots$. For the unitary multiplets one can check that (4.16) as well as (4.19) have no poles at these points. This means that the corrections to the expansions in (4.21) are exponentially suppressed as $\beta \to 0$ (see (4.17), (4.19)). The same conclusion can be drawn by repeating the analysis for the "non-gauge" multiplets $[0']$ and $[\frac{1}{2}]$.

[23] Note that the UV finite partition function on $S^1_\beta \times S^3$ where S^3 has radius R (set to 1 in the above discussion) contains the $\log\beta$ term as part of the dimensionless $\log(\beta/R)$ term. This suggests (see also appendix D) that may be the dependence on R coming from some zero modes is determined by $n_{\rm CKT}$. The total power of R depends on a regularization of the contribution of all other modes (for a related discussion on S^3 see [91]).

Instead, for the non-unitary multiplets $[\frac{3}{2}]$ and $[2]$ containing gauge fields we find, in addition to the leading terms given in (4.21), an infinite series of power corrections

$$\begin{aligned}\log \mathrm{I}_{[\frac{3}{2}]}(\beta) &= \ldots - \frac{389}{324}\,\beta + \frac{\beta^2}{18} - \frac{\beta^4}{6480} + \frac{11\beta^6}{11022480} - \frac{43\beta^8}{5290790400} + \frac{19\beta^{10}}{261894124800} + \ldots, \\ \log \mathrm{I}_{[2]}(\beta) &= \ldots + \frac{7}{3}\,\beta - \frac{\beta^2}{6} + \frac{\beta^4}{720} - \frac{\beta^6}{45360} + \frac{\beta^8}{2419200} - \frac{\beta^{10}}{119750400} + \ldots . \end{aligned} \tag{5.14}$$

A similar pattern is found for the non-supersymmetric partition function of the conformal higher spin fields. For instance, in the case of a spin s bosonic conformal field the correction to (C.3) can be written as

$$\log Z_s = \text{terms in (C.3)} + R_s(\beta)\,, \tag{5.15}$$

where $R_s(\beta)$ is a infinite series that can be found in a closed form

$$R_s(\beta) = \frac{1}{3}\sum_{k=1}^{s-1} k(k+1)\big[(2s+1)\,k - 3s^2 - 2s - 1\big]\,\log\frac{2\,\sinh\left[\frac{(s-k)}{2}\beta\right]}{(s-k)\,\beta}\,. \tag{5.16}$$

This vanishes for the Maxwell field ($s = 1$), while for $s > 1$ one finds

$$\begin{aligned} R_s(\beta) = &-\frac{(s-1)s^2(s+1)^2(s+2)(2s+1)}{2160}\,\beta^2 \\ &+ \frac{(s-1)s^2(s+1)^2(s+2)(2s+1)\left(9s^2+9s-26\right)}{7257600}\beta^4 + \ldots . \end{aligned} \tag{5.17}$$

To summarize, the above analysis of non-unitary $\mathcal{N} = 1$ multiplets shows that in general (cf. (2.7))

$$\log \mathrm{I}(\beta) \overset{\beta\to 0}{=} -\big[16\,(\mathrm{a}-\mathrm{c}) + \nu\big]\,\frac{\pi^2}{3\beta} - \mathrm{n}\,\log\beta + \mathrm{k} + \frac{4}{27}(\mathrm{a}+3\mathrm{c})\,\beta + R(\beta)\,. \tag{5.18}$$

Here ν is a integer multiple of 8 which is non-zero for non-unitary multiplets with higher spin fields and $\mathrm{n} = \widehat{C}_3$ given by (5.10) is another integer which is non-zero only for multiplets with gauge fields. $\mathrm{k} \equiv C_2$ is a constant that should be related (2.14) to the partition function on S^3 and $R(\beta)$ contains power-like corrections for multiplets with conformal higher spin gauge fields but is $\mathcal{O}(\beta^k e^{-1/\beta})$ otherwise.

It is of interest to consider special combinations of the basic $\mathcal{N} = 1$ multiplets $[0], [1]$ and $[2], [\frac{3}{2}], [\frac{1}{2}], [0']$ that have vanishing leading β^{-1} and β coefficients in the expansion of the index (5.18), i.e. have vanishing total a, c and ν coefficients

$$\begin{aligned} &\mathrm{a}_{\rm tot} = \mathrm{c}_{\rm tot} = 0\,, \qquad \nu_{\rm tot} = 0: \\ &[2] + (k+3)\left[\frac{3}{2}\right] + k'[1] + (2k+3)\left[\frac{1}{2}\right] + (22k+18)[0] + (k'-9k-11)[0']\,. \end{aligned} \tag{5.19}$$

Here k and k' are integers and we assumed that there is just one graviton multiplet. As $k > -1$ (for the number of [0] not to be negative) the simplest solutions are $k = 0$ and

$k' = 11, 12, \ldots$ for which there are 4 conformal gravitini as in $\mathcal{N} = 4$ conformal supergravity. Using (5.13) we then find that the coefficient n of the $\log\beta$ term in (5.18) is n=$10 - k'$.

The case with $k = 0$, $k' = 12$, i.e. $[2]+3[\frac{3}{2}]+12[1]+3[\frac{1}{2}]+18[0]+[0']$, corresponds to the familiar case of the $\mathcal{N} = 4$ conformal supergravity coupled to four copies of $\mathcal{N} = 4$ SYM multiplets ($[1]_4 = [1]_1 + 3[0]_1$) which is a superconformal theory not only at the quadratic but also the interacting level [4, 92]. The small β expansion of the superconformal index of this theory (which does not depend on the conformal supergravity and the SYM coupling constants) is given by

$$\log \mathrm{I}(\beta) \overset{\beta\to 0}{=} -2\, \log\beta + \mathrm{k} + \frac{\beta^4}{1080} + \mathcal{O}(\beta^6)\,, \tag{5.20}$$

where the infinite series of power corrections come only from the $\mathcal{N} = 4$ conformal supergravity contribution (cf. (5.14)).

The minimal solution for the superconformal theory (5.19) having $k = 0$, $k' = 11$ has a smaller field content (it corresponds to removing the pair $[0'] + [1]$ that has zero anomalies, see table 2): $[2]+3[\frac{3}{2}]+11[1]+3[\frac{1}{2}]+18[0]$. It has similar expansion of the index: $\log\mathrm{I}(\beta) = -\log\beta + \mathrm{k} + \ldots$. This combination cannot be written as a collection of $\mathcal{N} = 3$ or $\mathcal{N} = 2$ multiplets; it was not included in classification of finite theories in section 6.3 of [4] as having separate $[\frac{3}{2}]$ multiplets that are not part of an extended conformal supergravity theory is not expected to lead to a classically consistent theory at a non-linear level.

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A Free action for the $\left[\frac{1}{2}\right]$ multiplet

Let us first recall the action for the standard chiral multiplet described by a chiral superfield $\Phi = \phi(y) + \theta\psi(y) + \theta^2\,\varphi(y)$ (with $y_{\alpha\dot\alpha} = x_{\alpha\dot\alpha} + i\,\theta_\alpha\bar\theta_{\dot\alpha}$)

$$S = \int d^4x\, d^4\theta\ \ \Phi^\dagger\Phi \to \int d^4x\,\Big[\phi^*\Box\phi + \psi^\alpha\partial_{\alpha\dot\alpha}\overline{\psi}^{\dot\alpha} + \varphi^*\varphi\Big]. \tag{A.1}$$

If instead one starts with a chiral spinor superfield

$$\Phi_\alpha = \chi_\alpha + \theta^\beta\, Q_{\alpha\beta} + \theta^2\,\psi_\alpha, \qquad\qquad Q_{\alpha\beta} = T_{\alpha\beta} + \varepsilon_{\alpha\beta}\phi\,, \tag{A.2}$$

where χ_α is a spinor with dimension $\frac{1}{2}$, ψ_α is a standard dimension $\frac{3}{2}$ spinor and the boson $Q_{\alpha\beta}$ is a combination of symmetric tensor $T_{\alpha\beta}$ (corresponding to self-dual part of the antisymmetric tensor $T_{\mu\nu}$) and a complex scalar ϕ. The corresponding conformally invariant action is then

$$S = \int d^4x\, d^4\theta\, \Phi^\alpha\partial_{\alpha\dot\alpha}\overline{\Phi}^{\dot\alpha}\,. \tag{A.3}$$

In components this gives

$$S = \int d^4x \left[\chi^\alpha \Box \partial_{\alpha\dot\alpha} \overline{\chi}^{\dot\alpha} + Q^{\alpha\beta} \partial_{\alpha\dot\alpha} \partial_{\beta\dot\beta} \overline{Q}^{\dot\alpha\dot\beta} + \psi^\alpha \partial_{\alpha\dot\alpha} \overline{\psi}^{\dot\alpha} \right], \tag{A.4}$$

where the bosonic term may be written explicitly as

$$\int d^4x \, Q^{\alpha\beta} \partial_{\alpha\dot\alpha} \partial_{\beta\dot\beta} \overline{Q}^{\dot\alpha\dot\beta} = \int d^4x \left[\phi^* \Box \phi + T^{\alpha\beta} \partial_{\alpha\dot\alpha} \partial_{\beta\dot\beta} \overline{T}^{\dot\alpha\dot\beta} \right]. \tag{A.5}$$

Eqs. (A.4), (A.5) give the action for the superconformal $[\frac{1}{2}]$ multiplet in (3.2) (after the renaming $\chi \to \psi^{(3)}$). The action for the antisymmetric tensor in (A.5) in spinor notation is equivalent to $\partial^\mu T^+_{\mu\nu} \partial_\lambda T^{-\lambda\nu}$ in vector notation with $T^+ \to T_{\alpha\beta} \sim (1,0)$, $T^- \to \overline{T}_{\dot\alpha\dot\beta} \sim (0,1)$.

B Chiral anomalies of conformal gravitino and non-gauge tensor field

To compute anomalies of a higher spin field one needs to couple it to a gravitational and gauge fields (assuming certain chiral transformation properties), take into account contribution of ghosts, etc. One may use, e.g., a perturbative approach, computing triangle diagrams corresponding to the matrix element of the chiral current between the vacuum and a two-graviton state, or two chiral symmetry gauge fields.

An alternative topological approach is based on relating the anomaly of the chiral current to the Atiyah-Singer index of a certain elliptic operator mapping fields to fields of opposite chirality. This approach is somewhat heuristic and is practically useful only if the starting (higher-spin) field theory is consistent. For a detailed comparison of the perturbative and topological methods for standard gravitino see [69].

Here we shall discuss the chiral gravitational and gauge anomalies of the conformal gravitino and the antisymmetric non-gauge tensor field $T_{\mu\nu}$ justifying the values of their coefficients κ_1 and κ_2 given in (3.4).[24] Their embedding into conformal supergravity means that it is possible to consistently couple them to gravity and the chiral gauge field. Using topological approach we shall assume the existence of a suitable elliptic operator whose index computes the chiral anomaly. Earlier results for the chiral anomaly coefficients in (3.4) can be found in [66, 67, 69, 94].

Following [95], let us consider a compact 4-manifold M^4 and a field belonging to the general spinor bundle $X^{mn} \equiv X^{m,n}$, i.e. a tensor $X_{(\alpha_1 \ldots \alpha_m)(\dot\beta_1 \ldots \dot\beta_n)}$ with m symmetric spinor indices and n symmetric dotted spinor indices. In particular, the gravitino Ψ_μ corresponds to $X^{2,1} + X^{1,2}$ (modulo gauge symmetry) while the tensor $T_{\mu\nu}$ to $X^{2,0} + X^{0,2}$ (i.e. to the sum of the self-dual and antiselfdual parts in spinor notation). The index theorem computes the analytical index of the universal chirality-swapping elliptic operator $D_{mn} : X^{mn} \to X^{nm}$ in terms of topological quantities. In 4 dimensions, it reads

$$\mathrm{ind}\, D_{mn} = \frac{\mathrm{ch} X^{mn} - \mathrm{ch} X^{nm}}{\mathrm{e}(TM)} \mathrm{td}(TM \otimes \mathbb{C}), \tag{B.1}$$

[24] The standard antisymmetric tensor gauge field of rank $2n$ with self-dual strength $H = dB$ has gravitation anomaly in $d = 4n + 2$ dimensions [93], i.e. not in 4 dimensions where $B_{\mu\nu}$ is dual to a scalar.

where TM is the tangent bundle, and td, e, ch are the Todd class, the Euler class and the Chern character. It is understood that one has to extract the part of degree 4 and evaluate it on M^4. If the fields transform in a non-trivial U(1) gauge bundle V, we have to multiply the index by the Chern character $\mathrm{ch}(V) = 1 - c_2 + \dots$. The degree 4 terms give the gravitational and pure gauge contributions to the divergence of the corresponding chiral current (see, e.g., [95–98])

$$\begin{aligned}\mathcal{A}_{mn} = \mathrm{ind}\big[D_{mn}\,\mathrm{ch}(V)\big]\Big|_{\mathrm{deg}\ 4} = \\ &-\frac{(m+1)(n+1)}{720}\Big[n(n+2)(3n^2+6n-14) - m(m+2)(3m^2+6m-14)\Big]\,p_1 \\ &-\frac{1}{6}(m-n)(m+1)(n+1)(m+n+2)\,c_2\,,\end{aligned} \tag{B.2}$$

where $c_2 = -\frac{1}{8\pi^2}\mathrm{Tr}(F\wedge F)$ is the second Chern class and $p_1 = \frac{1}{8\pi^2}\mathrm{tr}(R\wedge R)$ is the Pontryagin class. For a Weyl fermion we get (cf. (2.10), (2.13), (3.4))

$$\mathcal{A}_{1,0} = -\frac{1}{24}\,p_1 - c_2. \tag{B.3}$$

For the conformal gravitino (taking into account the ghost subtraction [67]) and the self-dual 2-tensor or symmetric bispinor, we get

$$\mathcal{A}_{2,1} - \mathcal{A}_{1,0} = \frac{5}{6}\,p_1 - 4\,c_2, \qquad\qquad \mathcal{A}_{2,0} = \frac{1}{3}\,p_1 - 4\,c_2\,. \tag{B.4}$$

Thus in units of the Weyl fermion anomaly the chiral gravitational anomaly of the conformal gravitino is $\frac{5}{6} : (-\frac{1}{24}) = -20$ while its gauge anomaly is 4.[25]

For the self-dual antisymmetric tensor $T_{\alpha\beta}$ the chiral gravitation anomaly is $(-1)\frac{1}{3} : (-\frac{1}{24}) = 8$ while its gauge anomaly is -4 (where we have included a -1 factor due to the different statistics of the tensor with respect to the Weyl fermion). These values are in agreement with those given in (3.4).

The reason why the real $T_{\mu\nu}$ tensor contributes [66] to the chiral anomaly can be understood from the analogy of its kinetic operator in spinor basis in (A.5) with the (square of) Dirac operator for a Weyl spinor. Its chiral gravitational anomaly can be also obtained by adapting the analysis of [99], i.e. by observing that the antisymmetric tensor anomaly related to a chiral rotation between the self-dual and anti-selfdual parts is the same as the electromagnetic duality anomaly of a Maxwell field (cf. also [68, 100]).

C Conformal higher spin partition function on $S^1_\beta \times S^3$

Here we shall review the expression for the conformal higher spin partition function $Z(\beta)$ on $S^1_\beta \times S^3$ and consider the small β expansion of $\log Z(\beta)$ focussing on the interpretation of the coefficient of the $\log\beta$ term.

[25] Let us mention for completeness that in the case of the standard gravitino the total ghost contribution (taking into account chiralities) leads to an extra subtraction $-\mathcal{A}_{1,0}$ compared to the conformal gravitino case, i.e. $\mathcal{A}_{2,1} - 2\,\mathcal{A}_{1,0} = \frac{7}{8}\,p_1 - 3\,c_2$, so that the chiral gravitational anomaly coefficient is -21, while the gauge anomaly is +3 [67, 69, 94].

Let us start with the bosonic fields. The Maxwell vector and the conformal graviton are simplest cases of the 4d conformal higher spin fields with $\Box^s$ kinetic term for a spin s field [4]. The standard partition function (or character of the corresponding representation of the conformal group) for the conformal higher spin s field on $S^1_\beta \times S^3$ is given by [5]

$$\log Z_s(\beta) = \sum_{n=1}^{\infty} \frac{1}{n}\, \mathcal{Z}_s(n\beta)\,, \tag{C.1}$$

$$\mathcal{Z}_s(\beta) = \frac{2\,(2s+1)\,t^2 - 2\,(s+1)^2\,t^{s+2} + 2\,s^2\,t^{s+3}}{(1-t)^4}, \qquad t = e^{-\beta}\,. \tag{C.2}$$

Here $\mathcal{Z}_s$ is the single-particle partition function playing the role as the single-particle index in (4.1).

Using the method sketched in (4.14)–(4.17) one finds that the small β expansion of $\log Z_s$ has the form[26]

$$\log Z_s = \frac{\pi^4\, s\,(s+1)}{45\,\beta^3} - \frac{\pi^2\, s(s+1)(s^2+s+1)}{18\,\beta} + \mathrm{k}_s - n_{\rm CKT}\log\beta + E_{\rm cas}\,\beta + \mathcal{O}(\beta^2)\,, \tag{C.3}$$

$$n_{\rm CKT} = \frac{s^2(s+1)^2(2s+1)}{12}\,, \qquad E_{\rm cas} = \frac{s(s+1)(18s^4+36s^3+4s^2-14s-11)}{720}\,\beta\,. \tag{C.4}$$

Here the coefficient of the $\log\beta$ is simply minus the number $n_{\rm CKT}$ of conformal Killing tensors in 4 dimensions.[27] $E_{\rm cas}$ is the standard Casimir energy. The constant k_s in (C.3) has a non-polynomial dependence on s and may be expressed as a linear combination of transcendental constants.[28]

Similarly, for the fermionic conformal fields with spin $s = \mathrm{s} + \frac{1}{2}$ (with $\mathrm{s} = 0$ corresponding to Weyl spinor, $\mathrm{s} = 1$ to conformal gravitino, etc.) one finds [101]

$$\mathcal{Z}_{\rm s}(\beta) = 4\,\frac{(\mathrm{s}+1)\,t^{\frac{3}{2}} + (\mathrm{s}+1)\,t^{\frac{5}{2}} - (\mathrm{s}+1)(\mathrm{s}+2)\,t^{\frac{5}{2}+\mathrm{s}} + \mathrm{s}(\mathrm{s}+1)\,t^{\frac{7}{2}+\mathrm{s}}}{(1-t)^4}\,. \tag{C.5}$$

[26]Note that on general grounds the leading term in (C.3) should scale as $\sim \beta^{-d}$ where d is the space-time dimension. The reason why the first term in the corresponding expansion of the supersymmetric partition function in (2.6) has "softer" β^{-1} behaviour is due to supersymmetric cancellations.

[27]For a spin s field it is the dimension of the SO(4, 2) representation $(s-1, s-1, 0)$ labelled by the Young tableau that has two rows of length $s-1$.

[28]It can be written as

$$\begin{aligned}
\mathrm{k}_s =& -s\,(s+1)\,\frac{\zeta(3)}{4\,\pi^2} + \frac{1}{6}\,s\,(s+1)\,(s^2+s+1)\,\log(2\pi) - \frac{1}{3}(s+1)(s+2)s^3\log\Gamma_0(s+3)\\
&+ \frac{1}{3}\left(3s^2+6s+2\right)s^2\log\Gamma_1(s+3) - (s+1)s^2\log\Gamma_2(s+3) + \frac{1}{3}s^2\log\Gamma_3(s+3)\\
&- \frac{1}{3}(s+1)^2\left(3s^2-1\right)\log\Gamma_1(s+2) + \frac{1}{3}(s-1)(s+1)^3 s\log\Gamma_0(s+2) + (s+1)^2 s\log\Gamma_2(s+2)\\
&- \frac{1}{3}(s+1)^2\log\Gamma_3(s+2),
\end{aligned}$$

where Bendersky's generalised gamma function $\log\Gamma_k(n) = \sum_{j=1}^{n-1} j^k \log j$ comes from simplification of derivatives of Hurwitz zeta function by $\log\Gamma_k(n) = \zeta'_{\rm H}(-k,n) - \zeta'(-k)$.

The $\beta \to 0$ expansion of the (periodic) fermionic partition function is given by

$$\log Z_{\rm s} = -\frac{2\pi^4({\rm s}+1)^2}{45\beta^3} + \frac{\pi^2({\rm s}+1)^2(2{\rm s}+1)(2{\rm s}+3)}{36\beta} + {\rm k}_{\rm s} + n_{\rm CKS}\log\beta + E_{\rm cas}\,\beta + \mathcal{O}(\beta^2)\,,$$
$$n_{\rm CKS} = \frac{{\rm s}({\rm s}+1)^3({\rm s}+2)}{3}, \qquad E_{\rm cas} = -\frac{({\rm s}+1)^2(144{\rm s}^4+576{\rm s}^3+584{\rm s}^2+16{\rm s}-51)}{2880}. \quad \text{(C.6)}$$

Here $n_{\rm CKS}$ is the number of the conformal Killing spinor-tensors. As in the bosonic case, the constant ${\rm k_s}$ has a non-polynomial dependence on s.

To further investigate why the coefficient of $\log\beta$ is related to the integers $n_{\rm CKT}$ and $n_{\rm CKS}$ let us write a general single particle partition functions associated with a conformal field in 4d as

$$\mathcal{Z}(\beta) = \frac{P(t)}{(1-t)^4}, \qquad P(t) = \sum_q c_q\, t^q, \qquad t = e^{-\beta}, \qquad \text{(C.7)}$$

where q runs over some finite set of integers or half-integers. This covers the cases of the bosonic and fermionic conformal higher spin fields in (C.2), (C.5) as well as other "matter" conformal fields in section 3.1 above [5]

$$P_\phi(t) = t(1-t^2), \qquad P_{\phi^{(4)}}(t) = 1-t^4, \qquad P_T(t) = 6\,t(1-t^2),$$
$$P_\psi(t) = 4\,t^{\frac{3}{2}}\,(1-t), \qquad P_{\psi^{(3)}}(t) = 4t^{\frac{1}{2}}\,(1-t^3). \qquad \text{(C.8)}$$

For a general $P(t)$ one finds using (C.1)[29]

$$\log Z(\beta) = \frac{\zeta(5)\,p_0}{\beta^4} + \frac{\pi^4\,(2p_0-p_1)}{90\,\beta^3} + \frac{\zeta(3)\,(3p_2-9p_1+11p_0)}{6\,\beta^2} \qquad \text{(C.9)}$$
$$-\frac{\pi^2\,(p_3-3p_2+6p_1-6p_0)}{36\,\beta} - \frac{30p_4-60p_3+150p_2-270p_1+251p_0}{720}\log\beta$$
$$+\,{\rm k}_P + \frac{6p_5+10p_3-30p_2+57p_1-54p_0}{1440}\,\beta + \mathcal{O}(\beta^2)\,, \qquad p_n \equiv \frac{d^n}{dt^n}P(t)\Big|_{t=1}.$$

The transcendental constant ${\rm k}_P$ is determined by the form of $P(t)$, but cannot be expressed as a linear combination of its derivatives at $t=1$. For a theory on $S^1_\beta \times S^3$, the $\beta \to 0$ or $t \to 1$ singularity of $\mathcal{Z}(\beta)$ is $\sim 1/(1-t)^3$, i.e. $P(t)$ should have an explicit $1-t$ factor.[30] This implies the constraint $p_0 = P(1) = 0$. Besides, for all conformal fields we get one extra constraint on $P(t)$[31]

$$P''(1) - 3\,P'(1) = 0\,, \qquad \text{(C.10)}$$

[29]This expression is formally valid for all fields as we assume that the fermions are also taken to be periodic on the "thermal" cycle S^1_β.

[30]This follows both from the conformal group representation theory and from the simple remark that $1/(1-t)^3$ factor takes into account the contributions of states on S^3 corresponding to all spatial derivatives of the field.

[31]This relation was discussed in [102] where it was related to the absence of suitable counterterms in the heat kernel calculation of the "energy" $E(\beta) = -\partial_\beta \log Z(\beta)$.

that implies the vanishing of the coefficient of the $1/\beta^2$ term in (C.9). Assuming these constraints, the expansion (C.9) simplifies to

$$\log Z(\beta) = -\frac{\pi^4\, p_1}{90\,\beta^3} - \frac{\pi^2\,(p_3-3p_1)}{36\,\beta} + \mathrm{k}_P - \frac{p_4-2p_3+6p_1}{24}\log\beta \\ + \frac{6p_5+10p_3-33p_1}{1440}\,\beta + \mathcal{O}(\beta^2). \tag{C.11}$$

This has similar structure as the expansion of the superconformal index in (2.7) (apart from the leading $1/\beta^3$ term that cancels in a supersymmetric combinations of fields).

The coefficient of the $\log\beta$ term is thus related to a particular combination $p_4-2p_3+6p_1$ of derivatives of $P(t)$ at $t=1$. To understand the meaning of this combination we may use the relation between the 4d conformal partition function and its AdS_5 counterpart.[32] As discussed in [5] there exists a close relation between the single-particle partition function $\mathcal{Z}$ of a conformal field on $S^1_\beta\times S^3$ and the partition function $\mathcal{Z}_{\rm HS}$ of the associated higher spin field in AdS_5 with quantum numbers determined by the Lorentz spins and conformal dimension of the 4d conformal field:[33]

$$\mathcal{Z}(t) = \mathcal{Z}_{\rm HS}(t^{-1}) - \mathcal{Z}_{\rm HS}(t) + \sigma(t)\,. \tag{C.12}$$

Here the function $\sigma(t)$ is a finite polynomial in $t+t^{-1}$ and is generically present in the case of 4d conformal higher spin fields related to massless higher spin fields in AdS_5. It is given by the character of the finite dimensional irreducible representation of $\mathrm{SO}(4,2)$ corresponding, in bosonic case, to the conformal Killing tensors in 4 dimensions. Its value at $t=1$ gives the dimension of this representation, i.e. the total number of conformal Killing tensors $\sigma(1) = n_{\rm CKT}$.[34] Using the conformal group representation theory and (C.12) one can show that

$$\mathcal{Z}_{\rm HS} = \frac{Q(t)}{(1-t)^4}\,, \quad \text{i.e.} \quad P(t) \equiv (1-t)^4\,\mathcal{Z}(t) = t^4\,Q(t^{-1}) - Q(t) + (1-t)^4\,\sigma(t), \tag{C.13}$$

where $Q(t)$ is a smooth function. The conditions $P(1)=0$ and (C.10) can be checked to hold automatically and the coefficient of the $\log\beta$ term in (C.11) then reads as

$$-\frac{p_4-2p_3+6p_1}{24} = -\sigma(1) = -n_{\rm CKT}\,. \tag{C.14}$$

Thus the coefficient of the logarithmic term is simply minus the number of the conformal Killing tensors of a rank related to the spin of the conformal gauge field. Similar result is found for the fermionic conformal higher spin fields.

For non-gauge conformal fields (like the scalars ϕ, $\phi^{(4)}$, spinors ψ, $\psi^{(3)}$, and the tensor $T_{\mu\nu}$) one finds that $\sigma(t)=0$ and thus there is no $\log\beta$ term in the small β expansion of the corresponding $\log Z(\beta)$.

[32] Let us note that use of the AdS connection is useful but is not really necessary for the final conclusion as the form of $\mathcal{Z}(\beta)$ used below can be justified purely on the basis of the conformal group representation theory discussed in appendix F in [5].

[33] Changing notation slightly here instead of β we use $t=e^{-\beta}$ as the argument of the partition functions.

[34] The general expression for $\sigma(t)$ for a bosonic conformal spin s field is
$\sigma_s(t) = \frac{1}{6}\,s\,(s+1)\,(s^2+s+1) - \frac{1}{6}\sum_{p=1}^{s-1} p\,(p+1)\,\big[(2s+1)p - 3s^2 - 2s - 1\big]\,(t^{s-p}+t^{-s+p})$.
Some special cases are $\sigma_1(t)=1$, $\sigma_2(t) = 7+4\,(t+t^{-1})$, $\sigma_3(t) = 26+20\,(t+t^{-1})+9\,(t^2+t^{-2})$.

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D Expansion of partition function in terms of regularized theory on S^3

It is possible to compute the constant and logarithmic contributions to the small β expansion of the partition function directly in terms of the spectrum of the dimensionally reduced 3d theory. Below we shall explain this starting with the example of the standard (non-supersymmetric) partition function on $S^1_\beta \times S^3$.

D.1 Standard bosonic partition function

In general, for a free conformal field on $S^1_\beta \times S^3$, we can write the single particle partition function and the full partition function in terms of the (square roots of) eigenvalues λ_n and their multiplicities d_n of the corresponding Laplacian on S^3 (equal to energies of states or dimensions of CFT operators)

$$\mathcal{Z}(\beta) = \sum_n \mathrm{d}_n\, e^{-\beta\lambda_n}\,, \qquad \log Z(\beta) = -\sum_n \mathrm{d}_n\, \log(1-e^{-\beta\lambda_n}). \tag{D.1}$$

Instead of following the systematic derivation of the small β expansion of $\log Z$ following the approach of section 4.2, one may attempt the direct $\beta \to 0$ expansion of the expression for $\log Z$ in (D.1):[35]

$$\log Z(\beta) = -\sum_n \mathrm{d}_n\, \log\beta - \sum_n \mathrm{d}_n \log\lambda_n + \ldots \to -\mathrm{n}\, \log\beta + \mathrm{k} + \ldots\,, \tag{D.2}$$

$$\mathrm{n} = \sum_n \mathrm{d}_n\Big|_{\mathrm{reg}}\,, \qquad \mathrm{k} = -\sum_n \mathrm{d}_n \log\lambda_n\Big|_{\mathrm{reg}}\,. \tag{D.3}$$

Thus the coefficient of $\log\beta$ should be directly related to the (regularized) sum of the multiplicities, while the constant term k should be the partition function of the reduced 3d theory on S^3. Note that as in a conformal theory Z depends only on dimensionless ratio β/R where R is the radius of S^3 (the square roots of eigenvalues λ_n scale as R^{-1}) the dependence on R is also controlled by the regularized total number of the eigenvalues or n.[36]

The natural regularization is the spectral ζ-function one: if $\zeta_\Delta(z) = \sum_n \mathrm{d}_n\lambda_n^{-z}$ then $\mathrm{n} = \zeta_\Delta(0)$ and $\mathrm{k} = \zeta'_\Delta(0)$. This analytic regularization can be implemented simply by adding the factors $e^{-\varepsilon\lambda_n}$, doing the sums and then dropping all terms which are singular in the limit $\varepsilon \to 0$.

As an example, let us consider the partition function of a conformally coupled scalar on $S^1_\beta \times S^3$ where ($t \equiv e^{-\beta}$)

$$\mathcal{Z}_0(\beta) = \frac{t-t^3}{(1-t)^4} = \sum_{n=1}^\infty n^2\, t^n, \qquad \log Z_0(\beta) = -\sum_{n=1}^\infty n^2\, \log(1-e^{-\beta\, n}). \tag{D.4}$$

Using the method of section 4.2 one can show that the exact small β expansion of $\log Z_0$ is

$$\log Z_0(\beta) = \frac{\pi^4}{45\,\beta^3} + \frac{0}{\beta} + 0\cdot\log\beta - \frac{\zeta(3)}{4\pi^2} + \frac{1}{240}\beta + \mathcal{O}(e^{-1/\beta}). \tag{D.5}$$

[35] The leading singular $1/\beta^n$ terms (cf. (C.9)) are not explicit in this naive expansion approach.

[36] Notice that n is also the coefficient of the β^0 term in the small β expansion of the single particle partition function $\mathcal{Z}(\beta)$, cf. (D.1).

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The same results of the coefficient $\mathrm{n} = 0$ of $\log \beta$ term and the constant term $\mathrm{k} = -\frac{\zeta(3)}{4\pi^2}$ are indeed found by using directly the corresponding regularized expressions in (D.3) (here $\lambda_n = n$ and $\mathrm{d}_n = n^2$)

$$\mathrm{n} = \sum_{n=1}^{\infty} e^{-\varepsilon\, n}\, n^2 \Big|_{\varepsilon^0} = \frac{2}{\varepsilon^3} + \mathcal{O}(\varepsilon)\Big|_{\varepsilon^0} \;=\; 0\,, \tag{D.6}$$

$$\mathrm{k} = -\sum_{n=1}^{\infty} e^{-\varepsilon\, n}\, n^2\, \log n\Big|_{\varepsilon^0} = \frac{2\log\varepsilon}{\varepsilon^3} + \frac{-3+2\,\gamma_{\mathrm{E}}}{\varepsilon^3} - \frac{\zeta(3)}{4\pi^2} + \mathcal{O}(\varepsilon)\,\Big|_{\varepsilon^0} = -\frac{\zeta(3)}{4\pi^2}\,. \tag{D.7}$$

Thus the constant in (D.5) may be identified with the partition function of the dimensionally reduced scalar 3d theory on S^3 computed using natural analytic regularization.[37]

Similar computation can be done for the Maxwell vector field where

$$\mathcal{Z}_1(\beta) = \sum_{n=1}^{\infty} 2n(n+2)\, t^{n+1}, \qquad \log Z_1(\beta) = -\sum_{n=1}^{\infty} 2\, n(n+2)\, \log(1-e^{-\beta\,(n+1)}). \tag{D.8}$$

Here $\lambda_n = n+1$ $(n = 1, 2, \ldots)$ is the square root of the eigenvalue of the transverse 3-vector Laplacian on S^3 and $2n(n+2)$ is its degeneracy [5]. The exact small β expansion of Z_1 computed as in section 4.2 reads

$$\log Z_1(\beta) = \frac{2\pi^4}{45\,\beta^3} - \frac{\pi^2}{3\beta} - \log\beta + \log(2\pi) - \frac{\zeta(3)}{2\pi^2} + \frac{11}{120}\beta + \mathcal{O}(e^{-1/\beta}). \tag{D.9}$$

Using instead the direct expansion and (D.3) we find gives

$$\mathrm{n} = \sum_{n=1}^{\infty} e^{-\varepsilon\,(n+1)}\, 2\, n(n+2)\Big|_{\varepsilon^0} = \frac{2(3e^{\epsilon}-1)}{(e^{\epsilon}-1)^3}\Big|_{\varepsilon^0} = \left[\frac{4}{\varepsilon^3} - \frac{2}{\varepsilon} + 1 + \mathcal{O}(\varepsilon)\right]_{\varepsilon^0} = 1,$$

$$\begin{aligned}\mathrm{k} = -\sum_{n=1}^{\infty} e^{-\varepsilon\,(n+1)}\, 2\, n(n+2)\, \log(n+1)\Big|_{\varepsilon^0} &= \frac{4\log\varepsilon + 4\gamma_{\mathrm{E}} - 6}{\epsilon^3} + \frac{-2\log\varepsilon - 2\gamma_{\mathrm{E}}}{\epsilon} \\ &+ \log(2\pi) - \frac{\zeta(3)}{2\pi^2} + \mathcal{O}(\varepsilon)\Big|_{\varepsilon^0} = \log(2\pi) - \frac{\zeta(3)}{2\pi^2}\,,\end{aligned} \tag{D.10}$$

in agreement with the coefficients of the $\log\beta$ and the constant term in (D.9).

Another non-trivial example is that of the conformal graviton for which [5][38]

$$\mathcal{Z}_2(\beta) = \sum_{n=0}^{\infty} 2\,(3\,n^2 + 12n + 5)\, t^{n+2}\,, \tag{D.11}$$

$$\begin{aligned}\mathrm{n} &= \sum_{n=0}^{\infty} e^{-\varepsilon(n+2)} 2\,(3\,n^2+12n+5)\Big|_{\varepsilon^0} = \frac{2(9\sinh\varepsilon + \cosh\varepsilon + 5)}{(e^{\varepsilon}-1)^3}\Big|_{\varepsilon^0} \\ &= \frac{12}{\varepsilon^3} - \frac{14}{\varepsilon} + 15 + \mathcal{O}(\varepsilon)\Big|_{\varepsilon^0} = 15\,,\end{aligned} \tag{D.12}$$

[37] Note that the dimensionally reduced 3d theory does not, of course, correspond to a conformal scalar on S^3: the 4d conformal scalar operator $-\nabla^2 + \frac{R}{6}$ reduces to the same one on S^3 (with R here being the curvature of S^3) while the conformally coupled scalar on S^3 would have the kinetic operator $-\nabla^2 + \frac{R}{8}$.

[38] Here to determine the effective $\lambda_n = n+2$, $\mathrm{d}_n = 2\,(3\,n^2+12n+5)$ we re-expanded the final expression for the single-particle partition function of the conformal graviton on $S^1 \times S^3$ in eq. (3.22) of [5] that was obtained by combining the contributions of the transverse graviton and vector Laplacians on S^3. In the notation of [5], these are respectively $\mathcal{Z}_{2,0} = \sum_{n=0}^{\infty} 2\,(n+1)(n+5)\,(t^{n+2}+t^{n+4})$ and $\mathcal{Z}_{1,1} = \sum_{n=1}^{\infty} 2\,(n+1)(n+3)\, t^{n+2}$ with $\mathcal{Z}_2 = \mathcal{Z}_{2,0} + \mathcal{Z}_{1,1}$.

in agreement with (5.11).

Similar computations can be done also for other conformal fields appearing in the $\mathcal{N}=1$ multiplets discussed in the text, confirming that $\mathrm{n}=0$ for non-gauge fields and is always an integer (see (5.10), (C.4), (C.6), (C.14)) for the gauge fields. This fact suggests that it should have some "zero-mode" interpretation which remains to be clarified (cf. [91]).

D.2 Supersymmetric case

The supersymmetric partition function is the same as the superconformal index up to the normal ordering supersymmetric Casimir energy factor in (2.4). This means that we may use the expansion of the index to extract the analogs of d_n and λ_n in (D.1). These will have again the meaning of multiplicities and eigenvalues of the single particle (free) supersymmetric spectrum.

Application of (D.3) is expected to give the constant and logarithmic terms in the expansion of the index. Let us check this claim for few examples of $\mathcal{N}=1$ multiplets using the expressions in table 7. For the chiral multiplet, we have

$$\mathrm{i}_{[0]}(\beta)=\frac{t^{\frac{2}{3}}-t^{\frac{4}{3}}}{(1-t)^2}=\sum_{n=0}^{\infty}(n+1)\,(t^{n+\frac{2}{3}}-t^{n+\frac{4}{3}}). \tag{D.13}$$

Hence, from (D.3),

$$\mathrm{n}_{[0]}=\sum_{n=0}^{\infty}\left[e^{-\varepsilon\,(n+\frac{2}{3})}-e^{-\varepsilon\,(n+\frac{4}{3})}\right](n+1)\Big|_{\varepsilon^0}=\frac{2}{3\,\varepsilon}+\mathcal{O}(\varepsilon)\Big|_{\varepsilon^0}\;=\;0\,, \tag{D.14}$$

$$\begin{aligned}\mathrm{k}_{[0]}&=-\sum_{n=0}^{\infty}\left[e^{-\varepsilon\,(n+\frac{2}{3})}\log\left(n+\frac{2}{3}\right)-e^{-\varepsilon\,(n+\frac{4}{3})}\log\left(n+\frac{4}{3}\right)\right](n+1)\Big|_{\varepsilon^0}\\&=\lim_{a\to0}\partial_a\left\{\frac{1}{3}e^{-\frac{4}{3}\epsilon}\left[3\Phi\left(e^{-\epsilon},-a-1,\frac{4}{3}\right)-\Phi\left(e^{-\epsilon},-a,\frac{4}{3}\right)\right]\right.\\&\quad\left.-\frac{1}{3}e^{-\frac{2}{3}\epsilon}\left[3\Phi\left(e^{-\epsilon},-a-1,\frac{2}{3}\right)+\Phi\left(e^{-\epsilon},-a,\frac{2}{3}\right)\right]\right\}\Bigg|_{\varepsilon^0}\;=\;\frac{\pi}{9\sqrt{3}}-\frac{1}{6}\log3-\frac{\psi^{(1)}\left(\frac{1}{3}\right)}{6\sqrt{3}\pi}\,,\end{aligned} \tag{D.15}$$

where $\Phi(z,s,\alpha)=\sum_{k=0}^{\infty}\frac{z^k}{(k+\alpha)^s}$ is the Lerch function and we expanded around $\varepsilon=0$ using

$$\Phi(z,s,\alpha)=\zeta(s,\alpha)+(z-1)\left[\zeta(s-1,\alpha+1)-\alpha\zeta(s,\alpha+1)\right]+\dots. \tag{D.16}$$

As a result, these values of $\mathrm{n}_{[0]}=0$ and $\mathrm{k}_{[0]}$ are in agreement with (4.21).

For the vector multiplet

$$\mathrm{i}_{[1]}(\beta)=\frac{-2t+2t^2}{(1-t)^2}=-2\sum_{n=1}^{\infty}t^n, \tag{D.17}$$

and applying again (D.3) we find

$$\mathrm{n}_{[1]}=-2\sum_{n=1}^{\infty}e^{-\varepsilon\,n}\Big|_{\varepsilon^0}=-\frac{2}{\varepsilon}+1+\mathcal{O}(\varepsilon)\Big|_{\varepsilon^0}\;=\;1\,, \tag{D.18}$$

$$\mathrm{k}_{[1]}=2\sum_{n=1}^{\infty}e^{-\varepsilon\,n}\,\log n\Big|_{\varepsilon^0}=\frac{-2\,\log\varepsilon-2\,\gamma_{\mathrm{E}}}{\varepsilon}+\log(2\pi)+\mathcal{O}(\varepsilon)\Big|_{\varepsilon^0}\;=\;\log(2\pi), \tag{D.19}$$

in agreement with (4.21).

In the case of the graviton multiplet [2] we have

$$\mathrm{i}_{[2]}(\beta) = \frac{-4t+4t^3}{(1-t)^2} = -4t - 8\sum_{n=2}^{\infty} t^n, \tag{D.20}$$

and then the values of $\mathrm{n}_{[2]}$, $\mathrm{k}_{[2]}$ are, again, in agreement with (4.21)

$$\mathrm{n}_{[2]} = -4 - 8\sum_{n=2}^{\infty} e^{-\varepsilon\, n}\Big|_{\varepsilon^0} = -\frac{8}{\varepsilon} + 8 + \mathcal{O}(\varepsilon)\Big|_{\varepsilon^0} \;=\; 8\,, \tag{D.21}$$

$$\mathrm{k}_{[2]} = 8\sum_{n=2}^{\infty} e^{-\varepsilon\, n}\,\log n\Big|_{\varepsilon^0} = 4\,\mathrm{k}_{[1]} = 4\,\log(2\pi)\,. \tag{D.22}$$

Similar agreement with values in (4.21) is found also for other multiplets.

E Superconformal index corresponding to $\mathcal{N}=1$ multiplets on squashed S^3

In this appendix we shall consider the generalized 2-parameter superconformal index (2.1), (4.1) which happens to be related [14, 48, 49] to the supersymmetric partition on $S^1_\beta \times S^3_b$ where b is the squashing parameter of the 3-sphere. The corresponding choice of the fugacities generalizing $p=q=e^{-\beta}$ in (2.2), (2.3) is

$$p = e^{-\beta/b}\,, \qquad\qquad q = e^{-\beta\, b}\,. \tag{E.1}$$

The small β expansion of the corresponding index for the scalar chiral multiplet [0] was found in [20]

$$\log \mathrm{I}_{[0]}(\beta, b) = \frac{b+b^{-1}}{18}\,\frac{\pi^2}{\beta} + \left(\frac{b+b^{-1}}{216} + \frac{b^3+b^{-3}}{162}\right)\beta + \dots. \tag{E.2}$$

The supersymmetric Casimir energy, entering the general relation (2.4), is expected to have the general form [29, 34]

$$E_{\rm susy} = \frac{2}{9}\,(b+b^{-1})\,\mathrm{a} - \frac{2}{27}\,(b^3+b^{-3})\,(2\,\mathrm{a}-3\,\mathrm{c}), \tag{E.3}$$

that reduces to (2.5) for $b\to 1$. We can easily obtain the expansion (E.2) by the ζ-function methods described in section 4.2. Let us first recall that for general p,q the single-particle superconformal index in (4.1) is [70] (reducing to (4.3) for $b=1$ or $p=q=t$))

$$\mathrm{i}_{[0]}(p,q) = \frac{(pq)^{\frac{1}{3}} - (pq)^{\frac{2}{3}}}{(1-p)(1-q)}. \tag{E.4}$$

The Mellin transform (4.13) of the index gets the following contribution from a term of the form $\frac{e^{-a\beta}}{(1-e^{-\beta\, b})(1-e^{-\beta\, b^{-1}})}$

$$\zeta_2(u; b, b^{-1}, a) = \sum_{n,m=0}^{\infty} (a + b\, n + b^{-1}\, m)^{-u}\,, \tag{E.5}$$

where we adopted the standard notation for the Barnes double zeta function [103]. Expanding around $u = 1, 0, -1$ we have (cf. (4.14))

$$\beta^{-u}\Gamma(u)\zeta(u+1) = \begin{cases} \frac{\pi^2}{6\,\beta} + \dots\,, & u \to 1, \\ \frac{1}{u^2} - \frac{1}{u}\log\beta + \dots\,, & u \to 0, \\ \frac{\beta}{2\,(u+1)} + \dots\,, & u \to -1. \end{cases} \tag{E.6}$$

Using [103, 104]

$$\operatorname*{Res}_{u=1}\zeta_2(u; b, b^{-1}, a) = \frac{b+b^{-1}}{2} - a, \qquad \zeta_2(0; b, b^{-1}, a) = \frac{1}{4} + \frac{b^2+b^{-2}}{12} - \frac{a}{2}(b+b^{-1}) + \frac{a^2}{2},$$
$$\zeta_2(-1; b, b^{-1}, a) = -\frac{b+b^{-1}}{24} + \left(\frac{1}{4} + \frac{b^2+b^{-2}}{12}\right) a - \frac{b+b^{-1}}{4}a^2 + \frac{a^3}{6}\,, \tag{E.7}$$

we obtain from (E.4) the following expression for $\mathrm{z_m}(u)$ defined in (4.13)

$$\mathrm{z}_{[0]}(u) = \zeta_2\left(u; b, b^{-1}, \frac{1}{3}(b+b^{-1})\right) - \zeta_2\left(u; b, b^{-1}, \frac{2}{3}(b+b^{-1})\right). \tag{E.8}$$

Combining the results in (E.7) with (E.6) we reproduce (E.2).

A similar computation can be done for the vector multiplet index where

$$\mathrm{i}_{[1]}(p,q) = -\frac{p}{1-p} - \frac{q}{1-q} = \frac{-p-q+2pq}{(1-p)(1-q)}\,. \tag{E.9}$$

In this case

$$\mathrm{z}_{[1]}(u) = -\zeta_2\left(u; b, b^{-1}, b\right) - \zeta_2\left(u; b, b^{-1}, b^{-1}\right) + 2\,\zeta_2\left(u; b, b^{-1}, b+b^{-1}\right), \tag{E.10}$$

and using again (E.7) with (E.6) we find

$$\log \mathrm{I}_{[1]}(\beta, b) = -\frac{b+b^{-1}}{6}\frac{\pi^2}{\beta} + \mathrm{k}([1], b) - \log\beta + \frac{b+b^{-1}}{24}\beta + \dots, \tag{E.11}$$

which matches the expression in eq. (A.19) in [20]. Notice that the -1 coefficient of $\log\beta$ is independent of b, i.e. is the same as in (4.20), supporting its "topological" interpretation.

The same analysis can be repeated for the non-unitary multiplets. For the higher derivative multiplet $[0']$ using the data in table 3 to sum $(-1)^F p^{j_1+j_2+r/2}\, q^{-j_1+j_2+r/2}$ and dividing by $(1-p)(1-q)$ we find

$$\mathrm{i}_{[0']}(p,q) = \frac{1-pq}{(1-p)(1-q)}\,, \tag{E.12}$$

leading to the small β expansion

$$\log \mathrm{I}_{[0']}(\beta, b) = \frac{b+b^{-1}}{6}\frac{\pi^2}{\beta} + \mathrm{k}([0'], b) + 0\cdot\log\beta - \frac{b+b^{-1}}{24}\beta + \dots. \tag{E.13}$$

For the $[\frac{1}{2}]$ multiplet in table 4 we obtain

$$\mathrm{i}_{[\frac{1}{2}]}(p,q) = \frac{-p^{\frac{2}{3}}q^{-\frac{1}{3}} + p^{\frac{4}{3}}q^{\frac{1}{3}} - q^{\frac{2}{3}}p^{-\frac{1}{3}} + p^{\frac{1}{3}}q^{\frac{4}{3}}}{(1-p)(1-q)}, \tag{E.14}$$

$$\log \mathrm{I}_{[\frac{1}{2}]}(\beta,b) = -\frac{2}{9}(b+b^{-1})\,\frac{\pi^2}{\beta} + \mathrm{k}\left(\left[\frac{1}{2}\right],b\right) + 0\cdot\log\beta \tag{E.15}$$
$$+\left[-\frac{2}{27}(b+b^{-1}) + \frac{11}{162}(b^3+b^{-3})\right]\beta + \dots .$$

For the gravitino multiplet $[\frac{3}{2}]$ in table 5 we get

$$\mathrm{i}_{[\frac{3}{2}]}(p,q) = \frac{1}{(1-p)(1-q)}\Big[-2p^{\frac{5}{3}}q^{\frac{2}{3}} - 2p^{\frac{4}{3}}q^{\frac{4}{3}} + p^{\frac{4}{3}}q^{-\frac{2}{3}} \tag{E.16}$$
$$-2p^{\frac{2}{3}}q^{\frac{5}{3}} + p^{\frac{2}{3}}q^{\frac{2}{3}} + q^{\frac{4}{3}}p^{-\frac{2}{3}} + p^{\frac{5}{3}}q^{-\frac{1}{3}} + q^{\frac{5}{3}}p^{-\frac{1}{3}} + p^{\frac{1}{3}}q^{\frac{1}{3}}\Big],$$

$$\log \mathrm{I}_{[\frac{3}{2}]}(\beta,b) = \frac{13}{18}(b+b^{-1})\,\frac{\pi^2}{\beta} + \mathrm{k}\left(\left[\frac{3}{2}\right],b\right) + 6\,\log\beta \tag{E.17}$$
$$+\left[-\frac{71}{216}(b+b^{-1}) - \frac{22}{81}(b^3+b^{-3})\right]\beta + \dots .$$

A similar result is found for the graviton multiplet [2] in table 6

$$\mathrm{i}_{[2]}(p,q) = \frac{-p-q-p^2q^{-1}-q^2p^{-1}+2p^2q+2pq^2}{(1-p)(1-q)}, \tag{E.18}$$

$$\log \mathrm{I}_{[2]}(\beta,b) = -\frac{2}{3}(b+b^{-1})\,\frac{\pi^2}{\beta} + \mathrm{k}([2],b) + 8\,\log\beta \tag{E.19}$$
$$+\left[\frac{2}{3}(b+b^{-1}) + \frac{1}{2}(b^3+b^{-3})\right]\beta + \dots .$$

The pattern is thus the same as in the previous cases. In particular, the coefficient of $\log\beta$ does not depend on b and has the same (integer) value as we found in the undeformed case (see (4.21)).

F Small β expansion of superconformal index for (1,0) multiplets in six dimensions

The above discussion of the index in four dimensions can be readily extended to six dimensional (1,0) superconformal theories. Here we shall briefly outline what one finds for the unitary scalar and tensor multiplets, as well as for a non-unitary higher derivative vector multiplet.

The superconformal index for a (1,0) 6d theory is defined similarly to (2.1) [28]

$$\mathrm{I}(t,u,v) = \mathrm{Tr}\left[(-1)^F\, t^{\Delta-\frac{r}{2}}\, u^{j_1}\, v^{j_2}\right]_{\Delta=2r+\frac{1}{2}(j_1+2j_2+3j_3)} \tag{F.1}$$

Here $(\Delta, r, j_1, j_2, j_3)$ are associated to the subgroups of $\mathrm{OSp}(8^*|2) \subset \mathrm{SO}(2,6)\times\mathrm{SU}(2)_r \supset \mathrm{U}(1)_\Delta\times\mathrm{SU}(4)\times\mathrm{SU}(2)_r$ with (j_1,j_2,j_3) being the Dynkin labels of SU(4). We are interested in the specialization corresponding to a supersymmetric partition function on $S^1_\beta\times S^5$

$$\mathrm{I}(\beta) \equiv \mathrm{I}(e^{-\beta},1,1)\,. \tag{F.2}$$

	α	β	γ	δ
$\mathrm{S}^{(1,0)} = 4\,\phi + 2\,\psi^-$	0	0	$\frac{7}{240}$	$-\frac{1}{60}$
$\mathrm{T}^{(1,0)} = \phi + 2\,\psi^- + B^-$	1	$\frac{1}{2}$	$\frac{23}{240}$	$-\frac{29}{60}$
$\mathrm{V}^{(1,0)} = 3\phi + 2\,\psi^{(3)+} + V^{(4)}$	-1	$-\frac{1}{2}$	$-\frac{7}{240}$	$\frac{1}{60}$

Table 8. Anomaly coefficients of $(1,0)$ superconformal multiplets in six dimensions.

Leading terms in the $\beta \to 0$ expansion are expected to be related to the coefficients of the 8-form polynomial $\mathcal{A}_8$ encoding the chiral (R-symmetry and gravitational) anomaly (which in turn are related to the 6d conformal anomaly coefficients). These will play a role similar to that of the $\mathrm{Tr}\,\mathrm{R}$ and $\mathrm{Tr}\,\mathrm{R}^3$ in (2.11) or to the conformal anomaly coefficients a and c in four dimensions. The structure of $\mathcal{A}_8$ is [93, 105, 106]

$$\mathcal{A}_8 = \frac{1}{4!}\,(\alpha\, c_2^2 + \beta\, c_2\,\mathrm{p}_1 + \gamma\,\mathrm{p}_1^2 + \delta\,\mathrm{p}_2),$$
$$c_1 = \mathrm{tr}F, \qquad c_2 = \mathrm{tr}F^2, \qquad \mathrm{p}_1 = -\frac{1}{2}\,\mathrm{tr}R^2, \qquad \mathrm{p}_2 = -\frac{1}{4}\mathrm{tr}R^4 + \frac{1}{8}(\mathrm{tr}R^2)^2\,. \tag{F.3}$$

According to [19, 107], the small β expansion should have the following structure

$$\log \mathrm{I}(\beta) = \frac{8\pi^4}{9\,\beta^3}\left(\gamma + \frac{1}{4}\delta\right) + \frac{\pi^2}{6\beta}\left(\frac{9}{2}\beta - 8\gamma + \delta\right) - \mathrm{n}\,\log\beta + \mathrm{k} + E_{\rm susy}\,\beta + \dots, \tag{F.4}$$

where the six-dimensional supersymmetric Casimir energy is [61, 108]

$$E_{\rm susy} = -\frac{27}{128}\,\alpha + \frac{9}{32}\,\beta - \frac{3}{8}\,\gamma - \frac{1}{8}\delta\,. \tag{F.5}$$

The constant k and $\log\beta$ term were not, in fact, discussed in [19, 107] (the values of n and k given below will thus be new) but they should be expected from the general analysis we gave above in four dimension.

To test the validity of (F.4) let us consider the standard unitary (1,0) scalar $\mathrm{S}^{(1,0)}$ and tensor $\mathrm{T}^{(1,0)}$ 6d multiplets and also a non-unitary higher derivative multiplet $\mathrm{V}^{(1,0)}$; their field content and anomaly coefficients are summarized in table 8 (where we indicated chiralities of the fields). The scalar and tensor multiplets contain combinations of the 2-derivative scalar ϕ, the Majorana-Weyl (MW) spinor ψ and the standard gauge (anti)selfdual 2-tensor B^-. The higher-derivative (1,0) vector multiplet $\mathrm{V}^{(1,0)}$ contains a 4-derivative gauge vector $V^{(4)}$ and a 3-derivative spinor $\psi^{(3)}$ (in addition to scalars) [7, 109].[39]

For the 6d scalar and tensor multiplets the single-particle superconformal index is given by [111]

$$\mathrm{i}_{\mathrm{S}}(\beta) = 2\,\frac{t^{\frac{3}{2}} - t^{\frac{5}{2}}}{(1-t)^4}, \qquad\qquad \mathrm{i}_{\mathrm{T}}(\beta) = \frac{-3t^2 + 4t^3 - t^4}{(1-t)^4}. \tag{F.6}$$

[39]This multiplet may be identified with the $n = 2$ case of the $\mathcal{O}^*(n)$ multiplets recently discussed in [110].

Using the method of section 4.2 used above in 4d case we find

$$\log \mathrm{I_S}(\beta) = \frac{\pi^4}{45}\frac{1}{\beta^3} - \frac{\pi^2}{24}\frac{1}{\beta} + 0\cdot\log\beta + \mathrm{k_S} - \frac{17}{1920}\beta + \mathcal{O}(e^{-1/\beta}), \tag{F.7}$$

$$\log \mathrm{I_T}(\beta) = -\frac{\pi^4}{45}\frac{1}{\beta^3} + \frac{\pi^2}{6}\frac{1}{\beta} + \frac{1}{2}\log\beta + \mathrm{k_T} - \frac{11}{240}\beta + \mathcal{O}(e^{-1/\beta}). \tag{F.8}$$

The singular terms $\sim \beta^{-3}$ and $\sim \beta^{-1}$ as well as the Casimir term are in full agreement with (F.4), (F.5). In addition, we thus find that

$$\mathrm{n_S} = 0, \quad \mathrm{k_S} = \frac{1}{8}\log 2 + \frac{3\zeta(3)}{16\pi^2}, \qquad \mathrm{n_T} = -\frac{1}{2}, \quad \mathrm{k_T} = -\frac{1}{2}\log(2\pi) + \frac{\zeta(3)}{4\pi^2}. \tag{F.9}$$

For the non-unitary $\mathrm{V}^{(1,0)}$ multiplet, the following educated Ansatz for the single-particle index

$$\mathrm{i_V}(\beta) = \frac{-3t + 6t^2 - 5t^3 + 2t^4}{(1-t)^4}, \tag{F.10}$$

gives the following expansion

$$\log \mathrm{I_V}(\beta) = -\frac{\pi^4}{45}\frac{1}{\beta^3} - \frac{\pi^2}{3}\frac{1}{\beta} - \log\beta + \mathrm{k_V} + \frac{19}{240}\beta + \mathcal{O}(e^{-1/\beta}). \tag{F.11}$$

A check of the consistency of (F.10) is that the coefficients of the leading $\sim \beta^{-3}$, $\sim \beta^{-1}$ and β terms are again in agreement with (F.4), (F.5) and values in table 8. In addition, we find that

$$\mathrm{n_V} = 1, \qquad \mathrm{k_V} = \log(2\pi) + \frac{\zeta(3)}{4\pi^2}. \tag{F.12}$$

The constant term k has a natural interpretation of the logarithm of the partition function of the 5-dimensional reduced theory found in the limit $\beta \to 0$.

As for the $\log\beta$ term, it is present for the tensor and non-unitary vector multiplets. To understand its origin, let us briefly explain how to generalize the 4d analysis of appendix C to the present 6d case. As in the 4d case, we expect that the $\log\beta$ correction should come only from gauge fields in the multiplets. These are the gauge tensor B^- in the tensor multiplet and the 4-derivative gauge vector $V^{(4)}$ (with kinetic term $(\partial_\mu F^{\mu\nu})^2$) in the vector multiplet. We also expect it to be related to the 6d analog of the $\sigma(t)$ term in (C.12), (C.14), i.e. $\mathrm{n} = \sigma(1)$.

To check this claim, let us begin with the $V^{(4)}$ field. The associated dual higher spin field in AdS_7 (see discussion before (C.12)) has the $\mathrm{SO}(2,6)$ representation content [7]

$$(5;1,0,0) - (6;0,0,0), \tag{F.13}$$

where $(\Delta^+;\boldsymbol{h}) \equiv (\Delta^+;h_1,h_2,h_3)$ denote $\mathrm{SO}(2,6)$ quantum numbers (here $\Delta^+ = 6-\Delta$ where Δ is canonical dimension of 6d field). Each representation contributes to the analog of the partition function in (C.12) as $\mathcal{Z}^+_{\mathrm{HS}}(t) = \mathrm{d}(\boldsymbol{h})\,\frac{t^{\Delta^+}}{(1-t)^6}$ where

$$\mathrm{d}(\boldsymbol{h}) = \frac{1}{12}(1+h_1-h_2)(1+h_2-h_3)(1+h_2+h_3)(2+h_1-h_3)(2+h_1+h_3)(3+h_1+h_2) \tag{F.14}$$

is the dimension of the SO(6) representation with Dynkin labels $\boldsymbol{h}$. Thus, the higher spin partition function of the AdS$_7$ field transforming as (F.13) is given by

$$\mathcal{Z}^+_{\rm HS}(t) = \frac{{\rm d}(1,0,0)\, t^5 - {\rm d}(0,0,0)\, t^6}{(1-t)^6} = \frac{6t^5 - t^6}{(1-t)^6}\,. \tag{F.15}$$

The analog of the representation (C.12) then implies that the 6d partition function of $V^{(4)}$ is given by

$$\begin{aligned}\mathcal{Z}_{V^{(4)}}(t) &= \mathcal{Z}^+_{\rm HS}(t^{-1}) - \mathcal{Z}^+_{\rm HS}(t) + \sigma(t) = \frac{-1+6t-6t^5+t^6}{(1-t)^6} + \sigma(t)\\ &= (-1 + 15\, t^2 + 70\, t^3 + \dots) + \sigma(t).\end{aligned} \tag{F.16}$$

The $15\, t^2$ term is associated with the lowest dimension field which is the field strength $F_{\mu\nu}$ (which has indeed dimension 2 and $\binom{6}{2} = 15$ components). Hence, we need $\sigma(t) = +1$ in order to cancel the spurious -1 and get the correct 6d partition function for the conformal field $V^{(4)}$.[40] Thus, the expansion of $\log {\rm I_V}$ should contain the term $-\sigma(1) \log\beta = -\log\beta$ (same as for the 4d vector multiplet in (4.21)).

The gauge field B^- of the tensor multiplet is dual to the AdS$_7$ field with the conformal representation content [7]

$$\frac{1}{2}\Big[(4;1,1,0) - (5;1,0,0) + (6;0,0,0)\Big], \tag{F.17}$$

where the prefactor $\frac{1}{2}$ takes into account the anti-selfduality constraint, cf. table 1 of [7]. Repeating the steps leading to (F.16), here we find

$$\begin{aligned}\mathcal{Z}_{B^-}(t) &= \mathcal{Z}^+_{\rm HS}(t^{-1}) - \mathcal{Z}^+_{\rm HS}(t) + \sigma(t) = \frac{1-6t+15t^2-15t^4+6t^5-t^6}{2\,(1-t)^6} + \sigma(t)\\ &= \left(\frac{1}{2} + 10\, t^3 + 45\, t^4 + \dots\right) + \sigma(t).\end{aligned} \tag{F.18}$$

Again, the first t-dependent term $10\, t^3$ is associated with the lowest dimension field which in this case is the strength $H^-_{\mu\nu\rho}$ of B^- (which has indeed dimension 3 and $\frac{1}{2}\binom{6}{3} = 10$ components). Then cancellation of the constant term requires $\sigma(t) = -\frac{1}{2}$ and thus the expansion of $\log {\rm I_T}$ should contain the term $-\sigma(1) \log\beta = +\frac{1}{2}\log\beta$, in agreement with (F.8).

[40] The pattern is completely similar to the case of a Maxwell vector in 4d where the dual higher spin partition function in AdS$_5$ is the $s=1$ case of $\mathcal{Z}^+_{{\rm HS},s}(t) = \frac{(s+1)^2\, t^{s+2} - s^2\, t^{s+3}}{(1-t)^4}$ [5]. This gives $\mathcal{Z}_1(t) = \mathcal{Z}^+_{{\rm HS},1}(t^{-1}) - \mathcal{Z}^+_{{\rm HS},1}(t) + \sigma(t) = (-1 + 6\, t^2 + 16\, t^3 + \dots) + \sigma(t)$. The partition function here starts from the contribution of the field strength $F_{\mu\nu}$ which has dimension 2 and $\binom{4}{2} = 6$ components and thus contributes $6\, t^2$. The -1 term is spurious and is canceled by $\sigma_1(t) = 1$. This is, of course, the $s=1$ case of the general expression for $\sigma_s(t)$ in footnote 34. Canceling spurious terms with a polynomial in $t + t^{-1}$ is in general a convenient way to fix $\sigma(t)$. A non-trivial example is 4d conformal graviton where $\mathcal{Z}_2(t) = \mathcal{Z}^+_{{\rm HS},2}(t^{-1}) - \mathcal{Z}^+_{{\rm HS},2}(t) + \sigma(t) = (-4t^{-1} - 7 - 4\, t + 10\, t^2 + 40\, t^3 + \dots) + \sigma(t)$. Here the first contribution to $\mathcal{Z}_2(t)$ should be from the Weyl tensor $C_{\mu\nu\rho\sigma}$ that has dimension 2 and 10 components. and thus $\sigma_2(t) = 7 + 4\,(t + t^{-1})$ [5], again in agreement with footnote 34.

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