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Families from supergroups and predictions for leptonic CP violation

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Abstract: As was shown in 1984 by Caneschi, Farrar, and Schwimmer, decomposing representations of the supergroup SU$(M|N)$, can give interesting anomaly-free sets of fermion representations of SU(M) $\times$ SU(N) $\times$ U(1). It is shown here that such groups can be used to construct realistic grand unified models with non-abelian gauged family symmetries. A particularly simple three-family example based on SU(5) $\times$ SU(2) $\times$ U(1) is studied. The forms of the mass matrices, including that of the right-handed neutrinos, are determined in terms of SU(2) Clebsch coefficients; and the model is able to fit the lepton sector and predict the Dirac CP-violating phase of the neutrinos. Models of this type would have a rich phenomenology if part of the family symmetry is broken near the electroweak scale.

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Contents

1 Introduction

One way of finding chiral sets of fermions that are anomaly-free under product gauge groups is to decompose anomaly-free multiplets of larger groups. For example, by decomposing the $\mathbf{10} + \overline{\mathbf{5}}$ of SU(5) under its SU(3) × SU(2) × U(1) subgroup, one finds the anomaly-free set that comprises the fermions of the Standard Model. And by decomposing the $\mathbf{16}$ of SO(10) under its SU(5) × U(1) subgroup, one finds the anomaly-free set $\mathbf{10}^1 + \overline{\mathbf{5}}^{-3} + \mathbf{1}^5$.

It was shown in [1] that interesting sets of fermions that are anomaly-free under groups of the form SU(M) × SU(N) × U(1) can be found by decomposing multiplets of the supergroup SU($M|N$) [2]. The idea is based on the fact that the Casimirs of SU($M|N$) only depend on $(M - N)$. Thus the third-order Casimirs for the groups SU($M + P|P$) are the same for any P, and thus the same as for SU(M). If one considers, therefore, an irreducible fermion representation that is anomaly-free (i.e. has vanishing third-order Casimir) under SU(M), the corresponding Young tableaux representation of SU($M+P|P$) will yield anomaly-free sets when decomposed under the bosonic subgroup SU($M+P$)×SU(P)×U(1).

To take a simple example, consider the totally antisymmetric rank-four tensor multiplet of SU(4). This is a singlet, and thus trivially anomaly free. Consequently, the representation with the same Young tableau yields anomaly-free sets of fermions when decomposed under the SU($4 + P$) × SU(P) × U(1) subgroup of SU($4 + P|P$). This decomposition gives the multiplets

$$([4], (0))^P + (\overline{[3]}, \overline{(1)})^{-(P+1)} + ([2], (2))^{(P+2)} + (\overline{[1]}, \overline{(3)})^{-(P+3)} + ([0], (4))^{P+4}, \qquad (1.1)$$

where $[m]$ and (m) stand for the rank-m tensor multiplets that are totally antisymmetric and totally symmetric in the indices, respectively, and the overbar stands for the conjugate multiplets. The superscripts are the U(1) charges. If we take $P = 1$, this gives the anomaly-free set of SU(5) × U(1) multiplets $\overline{\mathbf{5}}^1 + \mathbf{10}^{-2} + \mathbf{10}^3 + \overline{\mathbf{5}}^{-4} + \mathbf{1}^5$. If we take $P = 2$, this gives the anomaly-free set of SU(6) × SU(2) × U(1) multiplets $(\overline{\mathbf{15}}, \mathbf{1})^2 + (\mathbf{20}, \mathbf{2})^{-3} + (\mathbf{15}, \mathbf{3})^4 + (\overline{\mathbf{6}}, \mathbf{4})^{-5} + (\mathbf{1}, \mathbf{5})^5$.

The anomaly-free sets constructed in [1] are interesting from the point of view of gauged family symmetry. In a theory having the gauge group $\mathrm{SU}(M) \times \mathrm{SU}(N) \times \mathrm{U}(1)$, the first factor could contain the Standard Model group if $M \geq 5$, while $\mathrm{SU}(N)$ could be a family group if it has three-dimensional representations, whether irreducible or reducible.

An anomaly-free set of fermions that contains exactly three families of the Standard Model can be obtained by looking at the rank-3 tensors of $\mathrm{SU}(3+P|P)$. This gives the anomaly free-set of $\mathrm{SU}(3+P) \times \mathrm{SU}(P) \times \mathrm{U}(1)$ fermion multiplets

$$(\overline{[3]}, \overline{(0)})^{-P} + ([2], (1))^{(P+1)} + (\overline{[1]}, \overline{(2)})^{-(P+2)} + ([0], (3))^{P+3}, \tag{1.2}$$

An interesting case, which gives a family group SU(3), is obtained by setting $P = 3$ in eq. (1.2), in which case the group is $\mathrm{SU}(6) \times \mathrm{SU}(3) \times \mathrm{U}(1)$ and the multiplets are $(\mathbf{20}, \mathbf{1})^{-3} + (\mathbf{15}^{-3}, \mathbf{3})^{4} + (\overline{\mathbf{6}}, \mathbf{6})^{-5} + (\mathbf{1}, \mathbf{10})^{6}$. This contains in addition to the Standard Model fermions many fermions that are vector-like under the Standard Model group. Under SU(5), it contains in addition to the three families of $\mathbf{10} + \overline{\mathbf{5}}$, three sets of $\mathbf{5} + \overline{\mathbf{5}}$, one of $\mathbf{10} + \overline{\mathbf{10}}$, and sixteen singlets.

A more economical case is obtained by setting $P = 2$ in eq. (1.2). Then one has the following group and fermion multiplets:

$$\mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{U}(1): \qquad (\mathbf{10}, \mathbf{1})^{-2} + (\mathbf{10}, \mathbf{2})^{3} + (\overline{\mathbf{5}}, \mathbf{3})^{-4} + (\mathbf{1}, \mathbf{4})^{5}. \tag{1.3}$$

The only fermions this contains besides those of the Standard Model (SM) are four SM-singlets, which can play the role of the right-handed neutrinos. This is the simplest and most economical case based on the constructions of [1]. We shall therefore study it in detail. As will be seen, models can be constructed for this case in which the family SU(2) gives non-trivial forms for the fermion mass matrices, fits the lepton sector well, and predicts the Dirac CP phase of the neutrinos.

2 The minimal SU(5) × SU(2) × U(1) model

All the fermion multiplets given in eq. (1.3) can be given non-zero masses by just four Higgs multiplets (which will be distinguished from fermion multiplets by a subscript H):

$$(\mathbf{5}, \mathbf{1})^{4}_{H}, \;\; (\mathbf{5}, \mathbf{2})^{-1}_{H}, \;\; (\mathbf{5}, \mathbf{3})^{-6}_{H}, \;\; (\mathbf{1}, \mathbf{3})^{-10}_{H}. \tag{2.1}$$

These Higgs multiplets have the following Yukawa couplings to the fermions:

$$\begin{aligned}
&u \text{ masses from } \mathbf{10}\,\mathbf{10}\,\mathbf{5}_H \text{ terms}: && a\,(\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{1})^{-2}\;\;(\mathbf{5},\mathbf{1})^{4}_{H}\\
& && +b\,(\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{2})^{3}\;\;(\mathbf{5},\mathbf{2})^{-1}_{H}\\
& && +c\,(\mathbf{10},\mathbf{2})^{3}(\mathbf{10},\mathbf{2})^{3}\;\;(\mathbf{5},\mathbf{3})^{-6}_{H},\\
&d,\ell^- \text{ masses from } \mathbf{10}\,\overline{\mathbf{5}}\,\overline{\mathbf{5}}_H \text{ terms}: && e\,(\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{5}},\mathbf{3})^{-4}\;\;\left[(\mathbf{5},\mathbf{3})^{-6}_{H}\right]^{*}\\
& && +f\,(\mathbf{10},\mathbf{2})^{3}(\overline{\mathbf{5}},\mathbf{3})^{-4}\;\;\left[(\mathbf{5},\mathbf{2})^{-1}_{H}\right]^{*}\\
&\nu \text{ Dirac masses from } \overline{\mathbf{5}}\,\mathbf{1}\,\mathbf{5}_H \text{ terms}: && g\,(\overline{\mathbf{5}},\mathbf{3})^{-4}(\mathbf{1},\mathbf{4})^{5}\;\;(\mathbf{5},\mathbf{2})^{-1}_{H}\\
&\nu^c \text{ masses from } \mathbf{1}\,\mathbf{1}\,\mathbf{1}_H \text{ terms}: && h\,(\mathbf{1},\mathbf{4})^{5}(\mathbf{1},\mathbf{4})^{5}\;\;(\mathbf{1},\mathbf{3})^{-10}_{H}.
\end{aligned} \tag{2.2}$$

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Note that an SU(2)-singlet mass for the right-handed neutrinos, i.e. $(\mathbf{1},\mathbf{4})^5\mathbf{1},\mathbf{4})^5(\mathbf{1},\mathbf{1})_H^{-10}$, is forbidden by Fermi statistics, since the symmetric product of two 4-plets of SU(2) does not contain a singlet. (Note that from the fact that the Higgs multiplets in eq. (2.1) can couple to the fermions as in eq. (2.2) it is evident that they also can arise from decomposing multiplets of the supergroup SU(5|2).)

As we shall see in detail, the forms of mass matrices of the quarks and leptons that arise from the Yukawa terms in eq. (2.2) are determined by the SU(2) family symmetry, and the Clebsch coefficients of SU(2).

Let us denote the vacuum expectation values (VEVs) of the Higgs multiplets shown in eq. (2.1) as follows

$$\begin{aligned} \langle(\mathbf{5},\mathbf{1})_H^4\rangle &= S, \\ \langle(\mathbf{5},\mathbf{2})_H^{-1}\rangle &= (d_\downarrow, d_\uparrow), \\ \langle(\mathbf{5},\mathbf{3})_H^{-6}\rangle &= (v_1, v_2, v_3), \\ \langle(\mathbf{1},\mathbf{3})^{-10}\rangle &= (t_1, t_2, t_3). \end{aligned} \tag{2.3}$$

Here we have expressed the VEVs of the SU(2) triplets in a "Cartesian basis". But we can also denote them in a "spherical basis", with $v_\pm \equiv (v_1 \pm i v_2)/\sqrt{2}$, $v_0 \equiv v_3$, and $t_\pm \equiv (t_1 \pm i t_2)/\sqrt{2}$, $t_0 \equiv t_3$. The VEV (t_1, t_2, t_3) is a complex vector. If we assume that its real and imaginary parts are aligned, then we can choose the basis in SU(2) space so that $(t_1, t_2, t_3) = (0, 0, t)$. Such alignment happens if a certain quartic self-coupling of $(\mathbf{1},\mathbf{3})_H^{-10}$ has the right sign. The most general renormalizable potential for this field is of the form $V(\vec{t}) = -\mu^2(\vec{t}^* \cdot \vec{t}) + \lambda(\vec{t}^* \cdot \vec{t})^2 + \lambda'(\vec{t} \cdot \vec{t})(\vec{t}^* \cdot \vec{t}^*) + \lambda''(\vec{t}^* \times \vec{t})^2$. If we write $\vec{t} = \vec{a} + i\vec{b}$, where $\vec{a}$ and $\vec{b}$ are real vectors, then $V = -\mu^2(a^2+b^2) + (\lambda+\lambda')(a^2+b^2)^2 - 4(\lambda'+\lambda'')a^2b^2\sin^2\theta_{ab}$. There are two cases: case I with $\lambda' + \lambda'' < 0$, and Case II with $\lambda' + \lambda'' > 0$. In Case I, the angle θ_{ab} between $\vec{a}$ and $\vec{b}$ vanishes. Then $\vec{t} = \hat{a}(a + ib)$, where the phase of $a + ib$ can be gauged away. Choosing $\hat{a}$ to point in the 3 direction, and defining $t \equiv \sqrt{a^2 + b^2}$, one ends up with the form $(t_1, t_2, t_3) = (0, 0, t)$. In Case II, $\theta_{ab} = \pi/2$, so $\vec{a}$ and $\vec{b}$ are perpendicular to each other, and one can choose the basis in SU(2) space so that $(t_1, t_2, t_3) = (0, it', t)$. Moreover, in this case the term with $\sin^2\theta_{ab}$ becomes $-|\lambda' + \lambda''|a^2b^2$, meaning that a and b become of equal magnitude, and one has $(t_1, t_2, t_3) = (0, it, t)$. These two cases give different mass matrices for the right-handed neutrinos and will both be examined below.

In eq. (2.1) we have written several 5-plets of SU(5). These contain altogether six electroweak doublets of scalars. All of them would "naturally" be expected to have superheavy masses. In a non-SUSY SU(5) model, one fine-tuning is done to make the mass-squared matrix of the six electroweak doublets have one small (i.e. electroweak-scale) eigenvalue. The linear combination of electroweak doublets corresponding to this eigenvalue is the Standard Model Higgs doublet; the five orthogonal linear combinations are superheavy. If the Standard Model Higgs doublet is a linear combination of the six doublets in $(\mathbf{5},\mathbf{1})_H^4$, $(\mathbf{5},\mathbf{2})_H^{-1}$, $(\mathbf{5},\mathbf{3})_H^{-6}$ with all six of the coefficients being non-zero, then all six of the VEVs denoted S, $d_\uparrow$, $d_\downarrow$, v_1, v_2, and v_3 in eq. (2.3) will be non-zero. That

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this can be achieved will be shown in detail in section 5. In SUSY SU(5) models, the situation is similar. There one considers the 6×6 mass matrix of the electroweak doublet Higgsinos and arranges, either by fine-tuning or by technically natural mechanism (such as the missing partner mechanism), that one of its eigenvalues is of electorweak scale.

We will define the complex numbers

$$x \equiv d_\downarrow / d_\uparrow, \quad z_1 \equiv v_1/v_3, \quad z_2 \equiv v_2/v_3. \tag{2.4}$$

Let us similarly denote the fermion multiplet $(\overline{\mathbf{5}}, \mathbf{3})^{-4}$ by $(\overline{\mathbf{5}}_1, \overline{\mathbf{5}}_2, \overline{\mathbf{5}}_3)$ or $(\overline{\mathbf{5}}_-, \overline{\mathbf{5}}_0, \overline{\mathbf{5}}_+)$ and the fermion multiplet $(\mathbf{10}, \mathbf{2})^3$ by $(\mathbf{10}_\downarrow, \mathbf{10}_\uparrow)$. The $(\mathbf{10}, \mathbf{1})^{-2}$ we will denote simply by $\mathbf{10}$, without any subscript. Then the 3×3 mass matrix of the up quarks mass can be written

$$(\mathbf{10}_\downarrow, \mathbf{10}_\uparrow, \mathbf{10})_u \begin{pmatrix} cv_+ & cv_0/\sqrt{2} & bd_\uparrow/2 \\ cv_0/\sqrt{2} & cv_- & -bd_\downarrow/2 \\ bd_\uparrow/2 & -bd_\downarrow/2 & aS \end{pmatrix} \begin{pmatrix} \mathbf{10}_\downarrow \\ \mathbf{10}_\uparrow \\ \mathbf{10} \end{pmatrix}_{u^c}, \tag{2.5}$$

so that the up quark mass matrix can be written in the form

$$M_u = \mu_u \begin{pmatrix} \delta(z_1 + iz_2) & \delta & \epsilon \\ \delta & \delta(z_1 - iz_2) & -\epsilon x \\ \epsilon & -\epsilon x & 1 \end{pmatrix}, \tag{2.6}$$

where $\mu_u = aS$, $\epsilon = \frac{bd_\uparrow}{2aS}$, $\delta = \frac{cv_0}{\sqrt{2}aS}$.

Since the VEVs that give the fermions mass do not break $\mathrm{SU}(4)_c$, one obtains the unrealistic "minimal SU(5)" relation [7] between the down quark and charged lepton mass matrices: $M_d = M_\ell^T$. These come from the term

$$(\mathbf{10}_\downarrow, \mathbf{10}_\uparrow, \mathbf{10})_{d\ (\text{or}\ \ell^c)} \begin{pmatrix} fd_\uparrow^*/\sqrt{3} & ifd_\uparrow^*/\sqrt{3} & fd_\downarrow^*/\sqrt{3} \\ -fd_\downarrow^*/\sqrt{3} & ifd_\downarrow^*/\sqrt{3} & fd_\uparrow^*/\sqrt{3} \\ ev_1^* & ev_2^* & ev_3^* \end{pmatrix} \begin{pmatrix} \overline{\mathbf{5}}_1 \\ \overline{\mathbf{5}}_2 \\ \overline{\mathbf{5}}_3 \end{pmatrix}_{d^c\ (\text{or}\ \ell)}, \tag{2.7}$$

This gives

$$M_d = \mu_d \begin{pmatrix} \eta & i\eta & \eta x^* \\ -\eta x^* & i\eta x^* & \eta \\ z_1^* & z_2^* & 1 \end{pmatrix}, \quad M_\ell = \mu_d \begin{pmatrix} \eta & -\eta x^* & z_1^* \\ i\eta & i\eta x^* & z_2^* \\ \eta x^* & \eta & 1 \end{pmatrix}, \tag{2.8}$$

where $\mu_d = ev_0^*$. $\eta = \frac{fd_\uparrow^*}{\sqrt{3}ev_0^*}$, and we have used the complex parameters x, z_1, and z_2 parameters in eq. (2.4).

The neutrino mass matrix arises through a Type I see-saw mechanism [3–6]. There are three left-handed neutrinos in the $(\overline{\mathbf{5}}, \mathbf{3})^{-4}$ and four left-handed anti-neutrinos in the $(\mathbf{1}, \mathbf{4})^5$. The Dirac neutrino mass matrix comes from

$$(\overline{\mathbf{5}}_+, \overline{\mathbf{5}}_0, \overline{\mathbf{5}}_-)_\nu \begin{pmatrix} 0 & 0 & -\sqrt{\frac{1}{3}}gd_\downarrow & gd_\uparrow \\ 0 & \sqrt{\frac{2}{3}}gd_\downarrow & -\sqrt{\frac{2}{3}}d_\uparrow & 0 \\ -gd_\downarrow & \sqrt{\frac{1}{3}}gd_\uparrow & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{3/2} \\ \mathbf{1}_{1/2} \\ \mathbf{1}_{-1/2} \\ \mathbf{1}_{-3/2} \end{pmatrix}_{\nu^c}, \tag{2.9}$$

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where the form is entirely determined by SU(2) Clebsch coefficients. The 4×4 Majorana mass matrix of the ν^c is also determined by Clebsch coefficients. In Case I, where $(t_-, t_0, t_+) = (0, t, 0)$, one then finds

$$\left(\mathbf{1}_{3/2}, \mathbf{1}_{1/2}, \mathbf{1}_{-1/2}, \mathbf{1}_{-3/2}\right)_{\nu^c} \begin{pmatrix} 0 & 0 & 0 & \frac{3}{\sqrt{20}}ht \\ 0 & 0 & -\frac{1}{\sqrt{20}}ht & 0 \\ 0 & -\frac{1}{\sqrt{20}}ht & 0 & 0 \\ \frac{3}{\sqrt{20}}ht & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{3/2} \\ \mathbf{1}_{1/2} \\ \mathbf{1}_{-1/2} \\ \mathbf{1}_{-3/2} \end{pmatrix}_{\nu^c}, \tag{2.10}$$

From the see-saw formula $M_\nu = -M_{Dirac} M_R^{-1} M_{Dirac}^T$, one finds

$$\left[-2\sqrt{5}\frac{g^2 d_\uparrow^2}{3ht}\right] (\nu_+, \nu_0, \nu_-) \begin{pmatrix} 0 & \sqrt{2}x^2 & 0 \\ \sqrt{2}x^2 & 4x & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \nu_+ \\ \nu_0 \\ \nu_- \end{pmatrix}. \tag{2.11}$$

Writing this in the Cartesian basis (ν_1, ν_2, ν_3), and defining $\mu_\nu = -\frac{2\sqrt{5}g^2 d_\uparrow^2}{3ht}$, one has

$$\mu_\nu (\nu_1, \nu_2, \nu_3) \begin{pmatrix} 0 & 0 & x^2+1 \\ 0 & 0 & i\left(x^2-1\right) \\ x^2+1 & i\left(x^2-1\right) & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. \tag{2.12}$$

This is not, however, the most general form of the neutrino mass matrix, because another operator can contribute to it, namely the effective dim-5 operator $(\overline{\mathbf{5}}, \mathbf{3})^{-4} (\overline{\mathbf{5}}, \mathbf{3})^{-4} (\mathbf{5}, \mathbf{1})_H^4 (\mathbf{5}, \mathbf{1})_H^4$. In the Cartesian basis, this just gives the identity matrix. Defining the ratio of the coefficient of this term to μ_ν by the complex number y, we have

$$M_\nu = \mu_\nu \begin{pmatrix} y & 0 & x^2+1 \\ 0 & y & i\left(x^2-1\right) \\ x^2+1 & i\left(x^2-1\right) & 4x+y \end{pmatrix}. \tag{2.13}$$

Note that the complex parameter y actually makes a difference for the neutrino mixing angles and mass splittings, despite appearing as the coefficient of the identity matrix. This is so, because M_ν is complex and *symmetric* and thus diagonalized by $U_\nu M_\nu U_\nu^T$ rather than by $U_\nu M_\nu U_\nu^\dagger$.

For Case II, where $(t_1, t_2, t_3) = (0, it, t)$, one has $(t_-, t_0, t_+) = (-t/\sqrt{2}, t, +t/\sqrt{2})$, This gives the following mass matrix for the right-handed neutrinos:

$$\left(\mathbf{1}_{3/2}, \mathbf{1}_{1/2}, \mathbf{1}_{-1/2}, \mathbf{1}_{-3/2}\right)_{\nu^c} \begin{pmatrix} 0 & 0 & -\frac{\sqrt{3}}{\sqrt{20}}ht & \frac{3}{\sqrt{20}}ht \\ 0 & \frac{1}{\sqrt{5}}ht & -\frac{1}{\sqrt{20}}ht & \frac{\sqrt{3}}{\sqrt{20}}ht \\ -\frac{\sqrt{3}}{\sqrt{20}}ht & -\frac{1}{\sqrt{20}}ht & -\frac{1}{\sqrt{5}}ht & 0 \\ \frac{3}{\sqrt{20}}ht & \frac{\sqrt{3}}{\sqrt{20}}ht & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1}_{3/2} \\ \mathbf{1}_{1/2} \\ \mathbf{1}_{-1/2} \\ \mathbf{1}_{-3/2} \end{pmatrix}_{\nu^c}, \tag{2.14}$$

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After straightforward algebra, this gives the following mass matrix for the three light neutrinos in a Cartesian basis:

$$M_\nu = \frac{1}{2}\mu_\nu \begin{pmatrix} y & -i\left(x^2+1\right) & x^2+1 \\ -i\left(x^2+1\right) & y-4x & i\left(1-2x-x^2\right) \\ x^2+1 & i\left(1-2x-x^2\right) & y+2\left(x^2-1\right) \end{pmatrix}, \tag{2.15}$$

which is to be compared to eq. (2.13).

The model described above, while not fully realistic because of the "minimal SU(5) relation $M_d = M_\ell^T$ [7], can account for many of the qualitative features of the quark and lepton masses and mixing angles. The fact that the overall scales of the up quark masses, down quark and charged lepton masses, and neutrino masses are very different can be explained by the fact that they are determined by the three independent parameers μ_u, μ_d, and μ_ℓ (see eqs. (2.6), (2.8), (2.13), and (2.15)), which are in turn determined by the VEVs of different types of Higgs multiplets. Moreover, several features of the inter-family mass ratios can also be accounted for.

The most striking feature of the observed inter-family fermion mass ratios is that they are hierarchical. That can partly be explained in this model by the fact that the three families are distinguished from each other by how they transform under the SU(2) family symmetry. For instance, because of SU(2), three different types of Higgs multiplet contribute to the up quark masses, as one sees from eq. (2.2). If one assumes a hierarchy among the VEVs (or Yukawa coefficients, or both) of those three Higgs multiplets, one can have $\delta \ll \epsilon \ll 1$, which gives $m_u \ll m_c \ll m_t$, as is apparent from eq. (2.6). Two types of Higgs multiplets contribute to the down quark (and charged lepton) masses, as shown in eq. (2.2). If one assumes a hierarchy in their VEVs (or Yukawa couplings, or both), one can have $\eta \ll 1$. This would explain why the third family of down quarks and charged leptons is heavier than the first two families, as eq. (2.8) shows. However, it would not explain the lightness of the first family compared to the second for the down quarks and charged leptons. As one can see from eq. (2.8), that would require a certain relationship (which will be given later) to hold among the parameters x, z_1 and z_2.

Each of the parameters x, z_1 and z_2 is defined as a ratio of VEVs of different components of an SU(2) Higgs multiplet. One would therefore naturally expect that these (complex) parameters would have magnitudes of $O(1)$. Thus, the relationship among them that would make $m_e/m_\mu \ll 1$ and $m_d/m_s \ll 1$ would involve a fine-tuning of order 10^{-2}.

The hierarchies $\delta \ll \epsilon \ll 1$ and $\eta \ll 1$ would also partially explain the smallness of the CKM angles. An examination of eqs. (2.6) and (2.8) shows that V_{cb} and V_{ub} come out to be of order η, while the Cabibbo mixing V_{us} comes out to be $O(1)$ if x, z_1 and z_2 are arbitrary parameters of $O(1)$. The "fine-tuning? required to fit the Cabibbo angle is mild, but a tuning of order 10^{-1} is required to explain the smallness of $|V_{ub}|$. This tuning takes the form of a relation among x, z_1 and z_2 that must be approximately satisfied.

If the parameters x, z_1 and z_2 have magnitudes of $O(1)$, as one would naturally expect, then the forms of the lepton mass matrices given in eq. (2.8) and eqs. (2.13) and (2.15) show

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that the PMNS angles should typically be of $O(1)$ as well, and that the ratio of neutrino masses should not be small. Thus, this model can account in a natural way for most of the qualitative features of the quark and lepton mass ratios and mixing angles. The two exceptions are the smallness of m_e/m_μ (and m_d/m_s) and the smallness of $|V_{ub}|$, each of which requires a somewhat tuned condition to hold among the complex parameters x, z_1, and z_2.

The minimal model described above is very simple, and, as we shall see, the SU(2) family symmetry yields non-trivial predictions for the lepton sector, in particular for the Dirac CP phase of the neutrinos. As noted, however, this minimal model's predictions for the quark sector are not realistic, because the model gives the "minimal SU(5)" relation $M_d = M_\ell^T$ at the GUT scale. This defect can be repaired if some of the quarks and leptons obtain mass from effective higher-dimension Yukawa terms that contain the adjoint Higgs of SU(5) (or whatever Higgs breaks SU(5) down to the Standard Model group). This can be done in such a way that the quark sector is made realistic without changing the minimal model's predictions for the lepton sector. We shall therefore defer to section 4 a discussion of how this can be done, and first derive the lepton-sector predictions of the minimal model in section 3.

3 Predictions of neutrino properties

Let us now see whether the simple model we have presented can fit the lepton sector, i.e. the masses of the charged leptons and neutrinos, and the PMNS angles.

As noted before, the fact that $m_e \ll m_\mu$ requires a tuning of parameters. As can be seen from an inspection of eq. (2.8), for $|z_1|$, $|z_2|$ and $|x|$ of $O(1)$, and $|\eta|$ small, the three eigenvalues of M_ℓ are of order $|\mu_d|$, $|\eta\mu_d|$, and $|\eta\mu_d|$. To have $m_e \sim 10^{-2} m_\mu$ requires that $|\det M_\ell| \sim 10^{-2}|\eta^2\mu_d^3|$. This yields the condition that

$$\left|1 - \frac{x^2-1}{2x} z_1 - i\frac{x^2+1}{2x} z_2\right| \sim 10^{-2}. \tag{3.1}$$

It will make no significant difference, and will simply calculations, if in fitting the neutrino properties we simply set this small quantity to zero. In that case, solving a quadratic equation allows one to solve for x in terms of z_1 and z_2:

$$x \cong \frac{1 \pm \sqrt{1 + {z_1}^2 + {z_2}^2}}{z_1 + iz_2}. \tag{3.2}$$

Suppose the mass matrices M_ℓ and M_ν are diagonalized by the following unitary transformations: $U_\ell M_\ell V_\ell^\dagger = M_\ell^{diagonal}$ and $U_\nu M_\nu U_\nu^T = M_\nu^{diagonal}$. Then, with our conventions, the PMNS matrix is given by $U_{PMNS} = U_\ell^* U_\nu^T$. If we ignore effects that are subleading by order $|\eta|^2$, the unitary matrix U_ℓ depends only on the complex parameters z_1, z_2, and x, as can be seen by inspection of the form of M_ℓ given in eq. (2.8). (The matrix V_ℓ depends on η in leading order, but does not contribute to U_{PMNS}.) In fact it is easy to write an explicit form of U_ℓ:

$$U_\ell = \begin{pmatrix} \cos\theta_{\ell 12} & -\sin\theta_{\ell 12} & 0 \\ (\sin\theta_{\ell 12})^* & (\cos\theta_{\ell 12})^* & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{z_2^*}{N_{12}} & -\frac{z_1^*}{N_{12}} & 0 \\ \frac{z_1}{NN_{12}} & \frac{z_2}{NN_{12}} & -\frac{N_{12}}{N} \\ \frac{z_1}{N} & \frac{z_2}{N} & \frac{1}{N} \end{pmatrix}, \tag{3.3}$$

where $N_{12} \equiv \sqrt{|z_1|^2 + |z_2|^2}$, $N \equiv \sqrt{1 + N_{12}^2} = \sqrt{1 + |z_1|^2 + |z_2|^2}$, and

$$\frac{\sin\theta_{\ell 12}}{\cos\theta_{\ell 12}} \equiv \frac{i\left(z_1^{*2} + z_2^{*2}\right)\sqrt{1 + |z_1|^2 + |z_2|^2}}{|z_1 - iz_2|^2 + (|z_1|^2 + |z_2|^2)\left(-1 \pm \sqrt{1 + z_1^{*2} + z_2^{*2}}\right)}, \tag{3.4}$$

where we have used eq. (3.2) to eliminate the parameter x and write U_ℓ entirely in terms of z_1 and z_2.

The diagonalization of M_ν, given in eq. (2.13) for Case I and eq. (2.15) for Case II, must be done numerically. This requires searching over three complex parameters of $O(1)$, namely z_1, z_2, and y. For each choice of these parameters, one can compute the PMNS angles and the ratio of neutrino mass splittings $\Delta m_{12}^2/\Delta m_{23}^2$. (The overall scale of the neutrino masses is set by the parameter μ_ν.) One might think that one should be able to fit these four experimental numbers with the three complex model parameters z_1, z_2, and y. A good fit is not guaranteed to exist, however, as the equations are nonlinear.

For Case I, we have done a numerical search of parameter space and found that there are values of the parameters that give excellent fits to the three PMNS angles, but none of them also gives a small enough value for the ratio of mass splittings $\Delta m_{12}^2/\Delta m_{23}^2$.

For Case II, we have two found satisfactory solutions for the leptons, one corresponding the minus sign in eq. (3.2), and the other corresponding to the plus sign. We will call these Solutions 1 and 2, respectively. These two solutions give a good fit all three neutrino mixing angles and the ratio of neutrino mass splittings $\Delta m_{12}^2/\Delta m_{23}^2$, but give different predictions for the Dirac CP phase of the neutrinos δ_{CP}.

In table 1, we present the fits to the neutrino mixing angles and the predictions of δ_{CP} for the two solutions. These were found in the following way. We searched over the three complex parameters z_1, z_2, y and kept only those points which yielded values for the three PMNS angles and for the ratio of neutrino mass splittings that were each within one-sigma of the experimental value. The error bars in the second and third columns of table 1 represent the standard deviation of the values obtained in this way. One notes that the prediction for the Dirac CP phase of the neutrinos δ_{CP} is fairly sharp for each of the two solutions. The fourth column in table 1 gives the 1σ best fit values from the 2014 particle data group [8], and the fifth column gives the best fit values from the 2016 particle data group [9]. In table 2, we give the values of the complex model parameters z_1, z_2 and y for the two solutions.

This model illustrates the predictive potential of models with non-abelian family groups. The SU(2) family symmetry strongly constrains the forms of the mass matrices. The patterns arising from the SU(2) family symmetry allow the model to account for many qualitative features of the quark and lepton spectrum, as well as yielding very precise predictions for the Dirac CP phase of the neutrinos. We now show that the model

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Quantity	Solution 1	Solution 2	1σ best fit[5]	best fit[6]
$\sin^2\theta_{12}$	0.321 ± 0.004	0.307 ± 0.011	0.308 ± 0.017	0.297
$\sin^2\theta_{23}$	0.467 ± 0.0026	0.457 ± 0.0065	$0.437^{+0.033}_{-0.023}$	0.437
$\sin^2\theta_{13}$	0.0231 ± 0.001	0.0234 ± 0.0015	$0.0234^{+0.002}_{-0.0019}$	0.0214
δ_{CP} (rad)	0.829 ± 0.0035	-0.617 ± 0.0047		
δ_{CP}/π	0.264 ± 0.0011	-0.196 ± 0.0015		

Table 1. The values of the PMNS parameters for the two solutions of Case II.

Quantity	Solution 1	Solution 2
$\mathrm{Re}(z_1)$	1.51 ± 0.004	-0.098 ± 0.006
$\mathrm{Im}(z_1)$	-0.064 ± 0.009	-1.19 ± 0.0035
$\mathrm{Re}(z_2)$	0.13 ± 0.008	-0.056 ± 0.0028
$\mathrm{Im}(z_2)$	0.78 ± 0.016	0.755 ± 0.018
$\mathrm{Re}(y)$	0.488 ± 0.018	0.473 ± 0.033
$\mathrm{Im}(y)$	0.268 ± 0.004	-0.391 ± 0.0057

Table 2. The values of the complex parameters z_1, z_2, and y for the two solutions for Case II.

can be modified to allow a realistic quark sector, without affecting the predictions for the lepton sector.

4 Making the quark sector realistic

The quark sector of the minimal model described in previous sections is unrealistic in two ways. First, the down quark masses come out wrong, because of the relation $M_d = M_\ell^T$. Second, there are not enough free parameters to ensure that the CKM mixing parameters are fit.

There are two standard ways to avoid the unrealistic prediction $M_d = M_\ell^T$ in SU(5) GUT models. One is to introduce 45-plets of Higgs fields in addition to the 5-plets so that there are Yukawa couplings of the form $10 \cdot \overline{5} \cdot \overline{45}_H$ in addition to the $10 \cdot \overline{5} \cdot \overline{5}_H$. Because the VEV of the 45_H couples differently to the quarks and leptons, this would break the $d - \ell$ degeneracy. In our case, this would mean introducing at least $(45,2)^{-1}_H$ and $(45,3)^{-6}_H$ multiplets. This would create a problem, however, in that to maintain the same form of M_ℓ as in the minimal model (and thus the predictions for neutrino properties obtained in section 3) there would have to be near alignment in SU(2) family space of the VEVs of $(45,2)^{-1}_H$ with $(5,2)^{-1}_H$ and $(45,3)^{-6}_H$ with $(5,3)^{-6}_H$. While it may be possible to have a Higgs sector and Higgs potential that ensures such alignment, it appears to be quite difficult to achieve.

The second way to avoid the bad relation $M_d = M_\ell^T$ is to have some of the quark and lepton masses come from higher-dimension effective Yukawa terms that contain the SU(5) adjoint Higgs field (or whatever Higgs field breaks SU(5) down to the Standard Model group at the GUT scale). As we shall see, this altogether avoids the alignment problem mentioned in the previous paragraph if the adjoint is a singlet under the family SU(2), as then its insertion into the Yukawa terms does not affect the family structure of those terms. Using the adjoint Higgs to break the relation $M_d = M_\ell^T$ has the advantage that it does not require introducing any additional Higgs fields beyond those required in the minimal model. On the other hand, there need to be new fermion fields at the GUT scale that when integrated out yield the desired higher-dimension effective Yukawa terms. As we shall see, however, these new fermions can be vector-like pairs consisting of fermions in some of the same representations shown in eq. (1.3) together with their conjugates. So this is quite economical.

We shall now discuss one way to build a realistic extension of the minimal model described in the previous sections using higher-dimension effective Yukawa terms. Let us suppose there is a Z_2 symmetry under which the Higgs multiplets $(5,2)_H^{-1}$, $(5,3)_H^{-6}$. and the SU(5) adjoint Higgs field $(24,1)_H^0$ are odd and $(5,1)_H^4$ and $(1,3)_H^{-10}$ are even. The new vector-like fermion multiplets will all be odd under the Z_2, while the fermion multiplets that are in the minimal model are even. With these Z_2 assignments, several of the Yukawa terms in eq. (2.2) are forbidden, but can be made Z_2-invariant by insertion of the adjoint Higgs field, which converts them to dimension-5 operators. The allowed Yukawa terms (up to dimension 5) are then the following:

$$\begin{array}{ll}
u \text{ masses}: & a\ (\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{1})^{-2}\ \ (\mathbf{5},\mathbf{1})_H^4 \\
& +b\ (\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{2})^{3}\ \ (\mathbf{5},\mathbf{2})_H^{-1}(\mathbf{24},\mathbf{1})_H^0/M_{\rm GUT} \\
& +c\ (\mathbf{10},\mathbf{2})^{3}(\mathbf{10},\mathbf{2})^{3}\ \ (\mathbf{5},\mathbf{3})_H^{-6}(\mathbf{24},\mathbf{1})_H^0/M_{\rm GUT}, \\
 & \\
d,\ell^- \text{ masses}: & e\ (\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{5}},\mathbf{3})^{-4}\ \ [(\mathbf{5},\mathbf{3})_H^{-6}]^*(\mathbf{24},\mathbf{1})_H^0/M_{\rm GUT} \\
& +f\ (\mathbf{10},\mathbf{2})^{3}(\overline{\mathbf{5}},\mathbf{3})^{-4}\ \ [(\mathbf{5},\mathbf{2})_H^{-1}]^*(\mathbf{24},\mathbf{1})_H^0/M_{\rm GUT}, \\
 & \\
\nu \text{ Dirac masses}: & g\ (\overline{\mathbf{5}},\mathbf{3})^{-4}(\mathbf{1},\mathbf{4})^{5}\ \ (\mathbf{5},\mathbf{2})_H^{-1}(\mathbf{24},\mathbf{1})_H^0/M_{\rm GUT} \\
 & \\
\nu^c \text{ masses}: & h\ (\mathbf{1},\mathbf{4})^{5}(\mathbf{1},\mathbf{4})^{5}\ \ (\mathbf{1},\mathbf{3})_H^{-10}.
\end{array} \tag{4.1}$$

It should be noted that the terms in eq. (4.1) with coefficients denoted b, c, e, and f are each in reality two terms, since the indices in the SU(5) products of Higgs fields can be contracted in two distinct ways, corresponding to $\mathbf{5}\times\mathbf{24} = \mathbf{5}+\mathbf{45}$. So, b, c, e, and f really each represent two Yukawa coefficients, but we have not written this out explicitly in eq. (4.1). Note that the top quark mass still comes from a dimension-4 term, namely $(\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{1})^{-2}\ \ (\mathbf{5},\mathbf{1})_H^4$, so that it receives no suppression by a factor of $\langle(\mathbf{24},\mathbf{1})_H^0\rangle/M_{\rm GUT}$.

The terms with coefficients denoted e in eq. (4.1) contribute to the third row of M_d and the third column of M_ℓ in eq. (2.8), which are now no longer equal as in the minimal model, but are multiplied by different factors due to the VEV of $(\mathbf{24},\mathbf{1})_H^0$, which gives different contributions to quarks and leptons. We assumed that these are the largest row

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of M_d and column of M_ℓ, and therefore give the masses of the b quark and τ lepton. Thus the degeneracy of m_b and m_τ at the GUT scale is lifted.

Similarly the terms with coefficients denoted f in eq. (4.1) cause the first and second rows of M_d in eq. (2.8) to be multiplied by a different factor than the first and second columns of M_ℓ. As a result the degeneracy of m_s and m_μ at the GUT scale is lifted. However, as the first and second families get multiplied by the *same* factors, the terms in eq. (4.1) still give the bad prediction that $m_e/m_\mu = m_d/m_s$ at the GUT scale. We will return to this issue shortly.

While the effective Yukawa terms involving $(\mathbf{24},\mathbf{1})^0_H$ have made $M_d \neq M_\ell^T$, the *form* of M_ℓ given in eq. (1.3) is not changed. The group-theoretic factors introduced by the VEV of the adjoint Higgs field (which, of course, points in the weak hypercharge direction) can be absorbed by redefinitions of μ_d and η in eq. (1.3) (with different redefinitions for M_d and M_ℓ, so different μ_d and η parameters appear in the two matrices). It is obvious that the form of the neutrino mass matrices are also not changed from what they were in the minimal model. Thus the fitting of the lepton sector done in section 3 and the predictions for neutrino properties obtained there are also left unaffected.

Let us now return to the problem of the bad relation $m_e/m_\mu = m_d/m_s$. Because m_e and m_d are so small compared to their counterparts in the other families, they can be significantly affected by higher-order corrections to M_ℓ and M_d. There are, in fact, certain dimension-6 effective Yukawa terms that can violate this bad relation. They involve the VEV of Higgs fields that transform as $(\mathbf{1},\mathbf{2})^5_H$ and are odd under Z_2. As we will see in section 5, such Higgs fields must exist even in the minimal model in order to get realistic breaking of the family SU(2). The $\mathrm{SU}(5)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ representation of these fields is not chosen arbitrarily. It is one of the small representations that arise by decomposing multiplets of SU(5|2), as is evident from the fact that $(\mathbf{1},\mathbf{2})^5_H$ is in the products $(\mathbf{5},\mathbf{1})^4((\mathbf{5},\mathbf{2})^{-1})^*$ and $(\mathbf{5},\mathbf{2})^{-1}((\mathbf{5},\mathbf{3})^{-6})^*$. The existence of these fields allows effective Yukawa terms of the form $(\mathbf{10},\mathbf{2})^3(\overline{\mathbf{5}},\mathbf{3})^{-4}[(\mathbf{5},\mathbf{1})^4_H]^*(\mathbf{1},\mathbf{2})^5_H(\mathbf{24},\mathbf{1})^0/M^2_{\mathrm{GUT}}$. This term arises by integrating out fermions at the GUT scale, though it can also arise from Planck-scale effects. Because these operators are of dimension 6, it is not unreasonable to suppose that they contribute to M_d and M_ℓ at order $10^{-3}m_b \sim 10^{-3}m_\tau$, while the dimension-5 operators in eq. (4.1) that produce the second and third generation masses in M_d and M_ℓ give contributions of order $10^{-2}m_b$ to m_b. Because the dimension-6 operators slightly change the *forms* in eq. (2.3) and give different contributions to quarks and leptons, they break the relation $m_e/m_\mu = m_d/m_s$. The contributions to M_d of order $10^{-3}m_b$ coming from these operators will also have a significant effect on the real and imaginary parts of V_{ub}, since experimentally $|V_{ub}| \sim 3\times 10^{-3}$. They can also have a significant affect on the parameter V_{us}. A contribution to the 12 element of M_d of order $10^{-3}m_b \sim 0.05m_s$ would typically give a contribution to V_{us} of order 0.05. (It should be noted that there are also other dimension-6 operators, besides the one given above, that break the relation $m_e/m_\mu = m_d/m_s$.)

Let us now turn to the question of fitting the CKM matrix. The parameters that can be used to fit this must come from M_u and M_d. First let us consider the parameters in M_d. In the minimal model, the parameters in M_d are the same as those in M_ℓ and must be used to fit the lepton sector masses and mixing angles, so that none of them are available to fit the CKM matrix. In the extension of the minimal model, however, there are new parameters in M_d, as we just saw, coming from the dimension-6 operators. These are of order $10^{-3} m_b$ and should significantly affect both V_{ub} and V_{us}. This gives in effect four real adjustable parameters for fitting the CKM matrix.

Turning to M_u, it contains in the minimal model six complex parameters: x, z_1, z_2, μ_u, ϵ, and δ. The first three of these were fixed by fitting the lepton sector. The phase of μ_u is irrelevant, and three real parameters are needed to fit the masses of u, c, and t. That leaves two real adjustable parameters in M_u in the minimal model that are available to fit the CKM matrix. In the extended model, however, an additional complex parameter appears in M_u. The reason for this is that the form of M_u is changed when we go to the extended model. As noted before, the term $b\,(\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{2})^{3}\;\;(\mathbf{5},\mathbf{2})_H^{-1}(\mathbf{24},\mathbf{1})_H^{0}/M_{\rm GUT}$ in eq. (4.1) actually contains two terms, in which the SU(5) indices are contracted differently. The SU(5) tensor product of the Higgs fields is $\mathbf{5}_H \times \mathbf{24}_H = \mathbf{5} + \mathbf{45} + \mathbf{70}$. The SU(5) tensor product of the fermion fields is $\mathbf{10} \times \mathbf{10} = \overline{\mathbf{5}}_S + \overline{\mathbf{45}}_A + \overline{\mathbf{50}}_S$. That means there are effectively two Yukawa terms, where the Higgs are contracted into $\mathbf{5}$ and $\mathbf{45}$. Because of Fermi statistics, the former couples symmetrically in fermion flavor, whereas the latter couples anti-symmetrically. That means that these terms give (for both M_d and M_ℓ) $M_{13} \neq M_{31}$ and $M_{23} \neq M_{32}$. However, $M_{13}/M_{31} = M_{23}/M_{32}$. The result is that there is a new complex parameter in M_u in the extended model, namely $\zeta \equiv M_{13}/M_{31} = M_{23}/M_{32}$. (It should be noted that no such new parameter appears in the 12 block of M_u. If we look at the terms denoted $c\,(\mathbf{10},\mathbf{2})^{3}(\mathbf{10},\mathbf{2})^{3}\;\;(\mathbf{5},\mathbf{3})_H^{-6}(\mathbf{24},\mathbf{1})_H^{0}/M_{\rm GUT}$ in eq. (4.1), one sees that the flavor anti-symmetric part of $(\mathbf{10},\mathbf{2}) \times (\mathbf{10},\mathbf{2})$ is a singlet of SU(2), whereas the product of Higgs fields must be a triplet.)

Altogether then, we have four real parameters coming from M_u and four from M_d that are available to fit the CKM matrix, which is more than sufficient.

We now turn to the question of where the higher-dimension operators in eq. (4.1) come from. They can come from integrating out superheavy vectorlike fermion fields that we denote as follows (the subscript V standing for vector-like):

$$(\mathbf{10},\mathbf{1})_V^{-2} + (\overline{\mathbf{10}},\mathbf{1})_V^{2}, \quad (\mathbf{10},\mathbf{2})_V^{3} + (\overline{\mathbf{10}},\mathbf{2})_V^{-3}, \quad (\overline{\mathbf{5}},\mathbf{3})_V^{-4} + (\mathbf{5},\mathbf{3})_V^{4}. \tag{4.2}$$

These are all assumed to be odd under Z_2, while the fermion multiplets that appear in the minimal model are all assumed to be even. These vectorlike fermions have the explicit mass terms $M_1(\mathbf{10},\mathbf{1})_V^{-2}(\overline{\mathbf{10}},\mathbf{1})_V^{2} + M_2(\mathbf{10},\mathbf{2})_V^{3}(\overline{\mathbf{10}},\mathbf{2})_V^{-3} + M_3(\overline{\mathbf{5}},\mathbf{3})_V^{-4}(\mathbf{5},\mathbf{3})_V^{4}$. They also have the following couplings to the Z_2-even fermions: $y_1(\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{10}},\mathbf{1})_V^{2}(\mathbf{24},\mathbf{1})_H^{0} + y_2(\mathbf{10},\mathbf{2})^{3}(\overline{\mathbf{10}},\mathbf{2})_V^{-3}(\mathbf{24},\mathbf{1})_H^{0} + y_3(\overline{\mathbf{5}},\mathbf{3})^{-4}(\mathbf{5},\mathbf{3})_V^{4}(\mathbf{24},\mathbf{1})_H^{0}$, and

$$\begin{aligned}
&b'(\mathbf{10},\mathbf{1})^{-2}(\mathbf{10},\mathbf{2})_V^{3}(\mathbf{5},\mathbf{2})_H^{-1} + b''(\mathbf{10},\mathbf{1})_V^{-2}(\mathbf{10},\mathbf{2})^{3}(\mathbf{5},\mathbf{2})_H^{-1} \\
&+e'(\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{5}},\mathbf{3})_V^{-4}\left[(\mathbf{5},\mathbf{3})_H^{-6}\right]^* + e''(\mathbf{10},\mathbf{1})_V^{-2}(\overline{\mathbf{5}},\mathbf{3})^{-4}\left[(\mathbf{5},\mathbf{3})_H^{-6}\right]^* \\
&+f'(\mathbf{10},\mathbf{2})^{3}(\overline{\mathbf{5}},\mathbf{3})_V^{-4}\left[(\mathbf{5},\mathbf{2})_H^{-1}\right]^* + f''(\mathbf{10},\mathbf{2})_V^{3}(\overline{\mathbf{5}},\mathbf{3})^{-4}\left[(\mathbf{5},\mathbf{2})_H^{-1}\right]^*,
\end{aligned} \tag{4.3}$$

which just parallel the forms in eq. (4.1), but without the adjoint Higgs (which is not needed as each term here contains a Z_2-odd fermion multiplet).

Consider the two terms $y_1(\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{10}},\mathbf{1})_V^2(\mathbf{24},\mathbf{1})_H^0$ and $e''(\mathbf{10},\mathbf{1})_V^{-2}(\overline{\mathbf{5}},\mathbf{3})^{-4}[(\mathbf{5},\mathbf{3})_H^{-6}]^*$. It is evident that integrating out the pair $(\overline{\mathbf{10}},\mathbf{1})_V^2+(\mathbf{10},\mathbf{1})_V^{-2}$ gives the effective dimension-5 term $(e''y_1/M_1)(\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{5}},\mathbf{3})^{-4}\quad[(\mathbf{5},\mathbf{3})_H^{-6}]^*(\mathbf{24},\mathbf{1})_H^0$, which is one of the terms in eq. (4.1). Another contribution to this dimension-5 operator comes from the terms $e'(\mathbf{10},\mathbf{1})^{-2}(\overline{\mathbf{5}},\mathbf{3})_V^{-4}[(\mathbf{5},\mathbf{3})_H^{-6}]^*$ and $y_3(\overline{\mathbf{5}},\mathbf{3})^{-4}(\mathbf{5},\mathbf{3})_V^4(\mathbf{24},\mathbf{1})_H^0$ by integrating out the pair $(\overline{\mathbf{5}},\mathbf{3})_V^{-4}+(\mathbf{5},\mathbf{3})_V^4$.

It is obvious that all the other dimension-5 terms in eq. (4.1) arise in the same way. The dimension-6 operators can arise either by integrating out vectorlike fermions or from Planck-scale effects.

5 The Higgs sector

There are two issues that must be considered with respect to the Higgs sector: the breaking of the family group SU(2) at a large scale (which we take to be of order the GUT scale), and the electroweak breaking. We will consider them in turn.

One Higgs field that breaks the family group at the large scale is the $(\mathbf{1},\mathbf{3})_H^{-10}$ in eq. (2.3). It turns out, however, that to get a realistic model there must be additional Higgs fields that transform as $(\mathbf{1},\mathbf{2})^5$ and are odd under Z_2. It turns out that there need to be two of these in order to break the family SU(2) in a realistic way, so we will denote them by $(\mathbf{1},\mathbf{2})_{HK}^5$, $K=1,2$. We will sometimes use the notation $(\mathbf{1},\mathbf{3})^{-10}=\vec{t}=(t_1,t_2,t_3)$ and $(\mathbf{1},\mathbf{2})_{HK}^5=s_K=(s_\uparrow,s_\downarrow)_K$, $K=1,2$. In minimizing the Higgs potential for these fields, which will get GUT-scale VEVs, we can ignore the electroweak-breaking Higgs fields. Thus, we may consider the potential $V(\vec{t},s_K)=V_t(\vec{t})+V_s(s_K)+V_{ts}(\vec{t},s_K)$, where the most general renormalizable form is

$$\begin{aligned}
V_t\left(\vec{t}\right) &= -\mu^2\left(\vec{t}^*\cdot\vec{t}\right)+\lambda\left(\vec{t}^*\cdot\vec{t}\right)^2+\lambda'\left(\vec{t}\cdot\vec{t}\right)\left(\vec{t}^*\cdot\vec{t}^*\right)+\lambda''\left(\vec{t}^*\times\vec{t}\right)^2,\\
V_s(s_K) &= -\mu_{KL}^2\left(s_K^\dagger\cdot s_L\right)+\lambda_{KLMN}\left(s_K^\dagger\cdot s_L\right)\left(s_M^\dagger\cdot s_N\right),\\
V_{st}\left(\vec{t},s_K\right) &= \left[M_{KL}\left(s_K^Ti\sigma^2\sigma^a s_L\right)t^a+\text{h.c.}\right]+i\lambda_{KL}\epsilon^{abc}\left(s_K^\dagger\sigma^a s_L\right)t^{b*}t^c.
\end{aligned}\tag{5.1}$$

We already examined the minimization of $V_t(\vec{t})$ in section 2. For $(\lambda'+\lambda'')>0$, it was found that the VEV could be brought to the form $\vec{t}=\frac{t}{\sqrt{2}}(0,i,1)$ by a suitable choice of axes in SU(2) space. We will now assume that the VEVs of s_K are sufficiently small that the terms in V_{st} do not significantly affect the form of the VEV of $\vec{t}$. For example, if the quartic couplings in V are of order one, and $s_K\sim t/30$, then the VEVs of s_K would only affect the VEV of $\vec{t}$ at the 10^{-3} level. (Also, the dimension-6 operator $(\mathbf{1},\mathbf{4})^5(\mathbf{1},\mathbf{4})^5[(\mathbf{1},\mathbf{2})_H^5(\mathbf{1},\mathbf{2})_H^5]^*/M_{\rm GUT}$ would only affect the form of the right-handed neutrino mass matrix by order 10^{-3}.) These effects would be too small to be significant in the fits of lepton properties in section 3.

The directions of the VEVs of s_K are determined by V_s+V_{st}. It is straightforward to show that non-trivial values of $(s_\uparrow,s_\downarrow)_K$ can be obtained by choosing the coefficients in V_s+V_{st} appropriately. This is only the case because there are at least two s multiplets. If there were only one, the most general form of V_s+V_{st} would lead to $(s_\uparrow,s_\downarrow)$ being proportional to either $(1,1)$ or $(1,-1)$. (The reason for this is simple. The VEV $\vec{t}\propto(0,i,1)$

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picks out the 1 direction as special, and $(s_\uparrow, s_\downarrow)$ would end up being forced to be an eigenspinor of σ^1.) This in turn would end up forcing the VEVs of the electroweak-breaking Higgs fields to have unrealistic special forms.

Now we consider the electroweak symmetry breaking. Of the six electroweak doublets in $(\mathbf{5},\mathbf{1})^4_H$, $(\mathbf{5},\mathbf{2})^{-1}_H$, and $(\mathbf{5},\mathbf{3})^{-6}_H$, one linear combination has mass of electroweak scale, while the others have GUT-scale masses. In non-SUSY models, this must be achieved through fine-tuning of the 6×6 mass-squared matrix of these doublets to have one electroweak-scale eigenvalue (which can be done by fine-tuning its determinant). In SUSY models, it can be achieved either by fine-tuning or by a "technically natural" mechanism, such as the missing partner mechanism (though that is usually non-trivial in a realistic SUSY GUT model). We shall consider only the non-SUSY case, as the analysis is simpler.

To find which linear combination of Higgs doublets is the Standard Model Higgs doublet, i.e. the one whose mass is *not* of GUT scale, it is only necessary to look at the $O(M^2_{\mathrm{GUT}})$ contributions to the 6×6 mass-squared matrix of the electroweak scalar doublets. That is, we need only look at terms which are quadratic in the Higgs 5-plets and ignore the terms quartic in them. The most general set of terms in the Higgs potential that are invariant under $\mathrm{SU}(5)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times Z_2$ and quadratic in the 5-plets (up the relevant dimension) is

$$
\begin{aligned}
V_2 = {} & \mu_1^2 \left|(\mathbf{5},\mathbf{1})^4_H\right|^2 + \mu_2^2 \left|(\mathbf{5},\mathbf{2})^{-1}_H\right|^2 + \mu_3^2 \left|(\mathbf{5},\mathbf{3})^{-6}_H\right|^2 \\
& + \rho \left|\left[(\mathbf{1},\mathbf{3})^{-10}_H\right]^* \cdot (\mathbf{5},\mathbf{3})^{-6}_H\right|^2 + \overline{\rho}\left|(\mathbf{1},\mathbf{3})^{-10}_H \cdot (\mathbf{5},\mathbf{3})^{-6}_H\right|^2 \\
& + i\rho'_{KL}\left[\left[(\mathbf{1},\mathbf{2})^5_{HK}\right]^* (\mathbf{1},\mathbf{2})^5_{HL}\right] \cdot \left[(\mathbf{5},\mathbf{3})^{-6}_H\right]^* \times (\mathbf{5},\mathbf{3})^{-6}_H \\
& + \sigma_{KL}\left[(\mathbf{1},\mathbf{2})^5_{HK}\left[(\mathbf{5},\mathbf{2})^{-1}_H\right]^*\right]\left[\left[(\mathbf{1},\mathbf{2})^5_{HL}\right]^*(\mathbf{5},\mathbf{2})^{-1}_H\right] \\
& + \overline{\sigma}_{KL}\left[(\mathbf{1},\mathbf{2})^5_{HK}(\mathbf{5},\mathbf{2})^{-1}_H\right]^*\left[(\mathbf{1},\mathbf{2})^5_{HL}(\mathbf{5},\mathbf{2})^{-1}_H\right] \\
& + i\sigma'\left[\left[(\mathbf{5},\mathbf{2})^{-1}_H\right]^*(\mathbf{5},\mathbf{2})^{-1}_H\right]\cdot\left[(\mathbf{1},\mathbf{3})^{-10}_H\right]^*\times(\mathbf{1},\mathbf{3})^{-10}_H \\
& + \left[\tau^K_{12}\left[(\mathbf{5},\mathbf{1})^4_H\right]^*(\mathbf{5},\mathbf{2})^{-1}_H(\mathbf{1},\mathbf{2})^5_{HK} + \mathrm{h.c.}\right] \\
& + \left[\tau'^K_{12}\left[(\mathbf{5},\mathbf{1})^4_H\right]^*(\mathbf{5},\mathbf{2})^{-1}_H\left[(\mathbf{1},\mathbf{2})^5_{HK}\right]^*\left[(\mathbf{1},\mathbf{3})^{-10}_H\right]^* + \mathrm{h.c.}\right] \\
& + \left[\tau^K_{23}\left[(\mathbf{5},\mathbf{2})^{-1}_H\right]^*(\mathbf{1},\mathbf{2})^5_{HK}(\mathbf{5},\mathbf{3})^{-6}_H(\mathbf{24},\mathbf{1})^0_H + \mathrm{h.c.}\right] \\
& + \left[\tau_{13}\left[(\mathbf{5},\mathbf{1})^4_H\right]^*(\mathbf{5},\mathbf{3})^{-6}_H\left[(\mathbf{1},\mathbf{3})^{-10}_H\right]^*(\mathbf{24},\mathbf{1})^0_H + \mathrm{h.c.}\right] \\
& + \left[\tau'^{KL}_{13}\left[(\mathbf{5},\mathbf{1})^4_H\right]^*(\mathbf{5},\mathbf{3})^{-6}_H(\mathbf{1},\mathbf{2})^5_{HK}(\mathbf{1},\mathbf{2})^5_{HL}(\mathbf{24},\mathbf{1})^0_H/M_{\mathrm{GUT}} + \mathrm{h.c.}\right]
\end{aligned}
\tag{5.2}
$$

In eqs. (5.2), μ_1^2, μ_2^2, and μ_3^2 include both explicit mass-squared parameters and products of VEV that are invariant under $\mathrm{SU}(5)\times\mathrm{SU}(2)\times\mathrm{U}(1)\times Z_2$. Let us denote the electroweak Higgs doublets in $(\mathbf{5},\mathbf{1})^4_H$, $(\mathbf{5},\mathbf{2})^{-1}_H$, and $(\mathbf{5},\mathbf{3})^{-6}_H$ by ϕ, $\Phi = (\phi_\uparrow, \phi_\downarrow)$, $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$, respectively, and the SU(5) adjoint Higgs VEV by Ω. Then we may write the terms in

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eq. (5.2) by

$$
\begin{aligned}
V_2 = {} & \mu_1|\phi|^2 + \mu_2^2|\Phi|^2 + \mu_3^2|\vec{\phi}|^2 + \rho|\vec{t}^* \cdot \vec{\phi}|^2 + \overline{\rho}|\vec{t} \cdot \vec{\phi}|^2 + i\rho'_{KL}\left[s_K^\dagger \vec{\sigma} s_L\right] \cdot \vec{\phi}^* \times \vec{\phi} \\
& + \sigma_{KL}\left[s_K \Phi^*\right]\left[s_L^* \Phi\right] + \overline{\sigma}_{KL}\left[s_K \Phi\right]^*\left[s_L \Phi\right] + i\sigma'\left[\Phi^* \vec{\sigma} \Phi\right] \cdot \vec{t}^* \times \vec{t} \\
& + \left[\tau_{12}^K\, \phi^*(\Phi s_K) + \tau_{12}^{\prime K}\, \phi^*\left(\Phi \vec{\sigma} s_K^*\right) \cdot \vec{t}^* + \tau_{23}^K (\Phi^\dagger \vec{\sigma} s_K) \cdot \vec{\phi}\Omega + \text{h.c.}\right] \\
& + \left[\tau_{13}\, \phi^*\left(\vec{\phi} \cdot \vec{t}^*\right)\Omega + \tau_{13}^{\prime KL}\, \phi^* \vec{\phi} \cdot (s_K \vec{\sigma} s_L)\,\Omega/M_{\mathrm{GUT}} + \text{h.c.}\right]
\end{aligned}
\tag{5.3}
$$

One can write the above terms as a 6×6 mass-squared matrix

$$
(\phi\phi_\uparrow\phi_\downarrow\phi_1\phi_2\phi_3)^* \begin{pmatrix}
\mu_1^2 & \mu_+^2 + \delta^2 & \mu_-^2 - \delta^2 & \Delta_1^2 & \Delta_2^2 + i\Delta^2 & \Delta_3^2 + \Delta^2 \\
\mu_+^{2*} + \delta^{2*} & \mu_2^2 + \mu_{++}^2 & \mu_\pm^{\prime 2} + \mu_{+-}^2 & \mu_{+1}^2 & \mu_{+2}^2 & \mu_{+3}^2 \\
\mu_-^{2*} - \delta^{2*} & \mu_\pm^{\prime 2*} + \mu_{+-}^{2*} & \mu_2^2 + \mu_{--}^2 & \mu_{-1}^2 & \mu_{-2}^2 & \mu_{-3}^2 \\
\Delta_1^2 & \mu_{+1}^{2*} & \mu_{-1}^{2*} & \mu_3^2 & \mu_{12}^2 & \mu_{13}^2 \\
\Delta_2^{2*} - i\Delta^{2*} & \mu_{+2}^{2*} & \mu_{-2}^{2*} & \mu_{21}^2 & \mu_3^2 + \mu^2 & i\mu^{\prime 2} + \mu_{23}^2 \\
\Delta_3^{2*} + \Delta^{2*} & \mu_{+3}^{2*} & \mu_{-3}^{2*} & \mu_{31}^2 & -i\mu^{\prime 2} + \mu_{32}^2 & \mu_3^2 + \mu^2
\end{pmatrix}
\begin{pmatrix} \phi \\ \phi_\uparrow \\ \phi_\downarrow \\ \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},
\tag{5.4}
$$

where

$$
\begin{aligned}
\mu^2 &\equiv \frac{1}{2}(\rho + \overline{\rho})\, t^2, & \mu^{\prime 2} &\equiv \frac{1}{2}(\rho - \overline{\rho})\, t^2, \\
\mu_{ab}^2 &\equiv i\rho'_{KL}\left(s_K^\dagger \sigma^c s_L\right)\epsilon^{abc}, & & \\
\mu_{++}^2 &\equiv \sigma_{KL}\left(s_{\uparrow K} s_{\uparrow L}^*\right) + \overline{\sigma}_{KL}\left(s_{\downarrow K}^* s_{\downarrow L}\right), & \mu_{--}^2 &\equiv \sigma_{KL}\left(s_{\downarrow K} s_{\downarrow L}^*\right) + \overline{\sigma}_{KL}\left(s_{\uparrow K}^* s_{\uparrow L}\right), \\
\mu_{+-}^2 &\equiv \left(\sigma_{KL} + \overline{\sigma}_{LK}\right) s_{\uparrow K} s_{\downarrow L}^*, & \mu_\pm^2 &\equiv \sigma' t^2, \\
\mu_+^2 &\equiv -\tau_{12}^K s_{\downarrow K}, & \mu_-^2 &\equiv +\tau_{12}^K s_{\uparrow K}, \\
\delta^2 &\equiv \frac{1}{\sqrt{2}} \tau_{12}^{\prime K}\, t\,(s_\uparrow + s_\downarrow)_K^*, & & \\
\mu_{+a}^2 &\equiv \tau_{23}^K (\sigma^a s_K)_\uparrow \Omega, & \mu_{-a}^2 &\equiv \tau_{23}^K (\sigma^a s_K)_\downarrow \Omega, \\
\Delta^2 &\equiv \frac{1}{2\sqrt{2}} \tau_{13} t \Omega, & \Delta_a^2 &\equiv \tau_{13}^{\prime KL} (s_\uparrow, s_\downarrow)_K\, i\sigma^2 \sigma^a \begin{pmatrix} s_\uparrow \\ s_\downarrow \end{pmatrix}_L.
\end{aligned}
\tag{5.5}
$$

It is important that the eigenvector of the matrix in eq. (5.4) which has the electroweak-scale eigenvalue (i.e. the Standard Model Higgs doublet) be a linear combination of *all six* of the doublets, otherwise at least one of the six VEVs $S \equiv \langle\phi\rangle$, $(d_\uparrow, d_\downarrow) \equiv \langle(\phi_\uparrow, \phi_\downarrow)\rangle$, and $(v_1, v_2, v_3) \equiv \langle(\phi_1, \phi_2, \phi_3)\rangle$ would vanish, which would not allow the realistic fits obtained in section 3, as these involved all six of these VEVs being non-zero. For general values of the parameters in eq. (5.5), this condition is obviously satisfied.

6 Conclusions

By decomposition of multiplets of the supergroups SU$(M|N)$, anomaly-free sets of fermion multiplets of the bosonic groups SU$(M) \times$ SU$(N) \times$ U(1) can be found, as was shown in [1]. Models based on such groups and multiplets can give both grand unification of the Standard Model gauge interactions and gauged non-abelian family groups.

In this paper we have explored one potential of such models, namely that the family symmetry could constrain the form of the quark and lepton mass matrices in such a way as to explain the main qualitative features of the quark and lepton properties. We studied the smallest such model that contains three families, which has the group $\mathrm{SU}(5)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. In particular, we studied the minimal form of this model, and showed it can account in a simple way for many of the qualitative features of the spectrum of quark and lepton masses and mixing angles, as well as making definite predictions for the lepton sector, specifically the Dirac CP phase of the neutrinos. This predictiveness arises because very definite and non-trivial forms are obtained for the fermion mass matrices (including that of the right-handed neutrinos), determined in large part by the Clebsch coefficients of the SU(2) family group. We showed that the minimal form of the model can be modified to make it realistic for the quark sector without affecting the neutrino predictions.

In addition to their implications for quark and lepton masses and mixing angles, such models in general would have a rich phenomenology if a subgroup of the family gauge groups were broken near the electroweak scale. This phenomenology would include (a) extra Z' bosons, whose couplings to the quarks and leptons would be quite distinctive; (b) flavor-changing non-abelian gauge interactions, which would give rare flavor-violating decays of leptons, whose branching ratios would be constrained by family symmetry; and (c) extra vector-like quarks and leptons. Clearly, there are many possibilities that remain to be explored. Moreover, in such models, one would expect the flavor structure to be sufficiently constrained by the family symmetry to give predictions for proton-decay branching ratios.

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