Holographic entropy production

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ABSTRACT: The suspicion that gravity is holographic has been supported mainly by a variety of specific examples from string theory. In this paper, we propose that such a holography can actually be observed in the context of Einstein's gravity and at least a class of generalized gravitational theories, based on a definite holographic principle where neither is the bulk space-time required to be asymptotically AdS nor the boundary to be located at conformal infinity, echoing Wilson's formulation of quantum field theory. After showing the general equilibrium thermodynamics from the corresponding holographic dictionary, in particular, we provide a rather general proof of the equality between the entropy production on the boundary and the increase of black hole entropy in the bulk, which can be regarded as strong support to this holographic principle. The entropy production in the familiar holographic superconductors/superfluids is investigated as an important example, where the role played by the holographic renormalization is explained.

Contents

1 Introduction

Evidence has accumulated since the end of last century that quantum gravity is holographic [1, 2], i.e. quantum gravity in a $(d + 1)$-dimensional space-time region can be described by some sort of quantum field theory on the d-dimensional boundary of this region, especially since the discovery of AdS/CFT correspondence [3–5] in the framework of (super)string theory. On one hand, nowadays there have been many generalizations and/or applications of AdS/CFT correspondence, such as most of the phenomenological models in AdS/CMT (condensed matter theory), AdS/QCD and so on, which cannot be embedded in string theory. On the other hand, besides the black hole thermodynamics [6] that inspires the proposition of holography, there are already various hints from within the context of Einstein's gravity towards the speculation that gravity is essentially holographic, where neither string theory nor supersymmetry are involved. Here we would like list three of them as follows.

- Brown-Henneaux's asymptotic symmetry analysis for three dimensional gravity [7].

- Brown-York's surface tensor formulation of quasilocal energy and conserved charges [8].

- Bousso's covariant entropy bound [9].

In particular, Brown-York's surface tensor formulation bears a strong resemblance to the recipe in the dictionary for AdS/CFT correspondence, and has actually been incorporated into the latter (or its generalizations). Holography could have been explicitly implemented just in Einstein's gravity, in fact, if one was brave enough to declare that Brown-York's surface tensor is not only for the purpose of the bulk side but also for some sort of system living on the boundary.

In AdS/CFT, the radial direction of the (asymptotic AdS) bulk space-time corresponds to the energy scale of the dual field theory [10–13] and the change of radial coordinate r is regarded as equivalent to the corresponding renormalization group (RG) flow [14–20], where the conformal boundary $r \to \infty$ is its ultra-violet (UV) fixed point. Interestingly, from this point of view, the RG flows of many important transport coefficients of the boundary theory (at finite temperature) are trivial, which enables one to compute these coefficients by the so-called black-hole "membrane paradigm" [21]. Especially, it is proved that the ratio $\frac{\eta}{s} = \frac{1}{4\pi}$ of the shear viscosity η to the entropy density s does not run with the RG flow, so the universality of this ratio in both the black-hole membrane paradigm and the standard AdS/CFT follows.

In the above framework of the so-called holographic RG flow, physical quantities can be defined on any constant r surface (called the finite cutoff surface), while their RG flows are obtained by changing r. However, the finite cutoff surface itself is just a tool to relate the conformal boundary $r \to \infty$ (UV) and the bulk black-hole horizon $r \to r_h$ (IR), and no dual dynamical theory is directly defined on this surface, until later Strominger et al. [22, 23] establish hydrodynamics on the finite cutoff surface and then discuss the fluid/gravity correspondence from this point of view. The dual theory defined on the finite cutoff surface $r = r_c$ can be regarded as the effective field theory at the energy scale corresponding to r_c. In fact, the bulk space-time in this generalized framework of holography can be either asymptotic AdS [24–26] or not [23, 27], reflecting the fact that the dual theory does not need to have a UV completion.

Both in the standard AdS/CFT at the conformal boundary and in the generalized holography by Strominger et al, a holographic interpretation of the entropy production of the boundary system in non-equilibrium processes is an interesting problem. It has been well established in AdS/CFT that a static black hole in the bulk is dual to the boundary field theory at a thermal equilibrium state. Then what happens when the bulk black hole is perturbed? From the bulk point of view, the bulk perturbation will be eventually absorbed by the black hole, leading to an increase of the area of black hole, i.e., an increase of the black hole entropy [28]. On the other hand, such a bulk perturbation will induce the corresponding perturbation on the boundary, driving the boundary system away from the original equilibrium state. But the dissipation will bring the boundary system to a new

equilibrium state with the production of entropy. So a natural question is whether the entropy production by such a dissipative (transport) process on the boundary is equal to the increase of the black hole entropy in the bulk. Actually this problem has been raised by Strominger et al. in the generalized holography [22], but it still remains open until now.

So the main motivation of our paper is two-fold. On one hand, the UV fixed point of the dual field theory, which has a conformal dynamics, is not expected to be reached by experiments. Therefore, the generalized holography, which we call the general bulk/boundary correspondence, at a finite cutoff surface $r = r_c$ (corresponding to a finite energy scale) is important, where the dual (effective) theory is non-conformal in general. In order to study the general bulk/boundary correspondence systematically, we propose a general holographic principle, which leads to definite holographic dictionary on any cutoff surface. This dictionary should include the known cases [22, 23] as special examples, and should be consistent with the standard AdS/CFT when $r \to \infty$, if certain subtleties like the holographic renormalization[1] are taken into account.

On the other hand, as a support for our general bulk/boundary correspondence, we prove that the entropy production by the transport processes on the boundary is exactly equal to the increase of the black-hole entropy in the bulk an explicit construction of certain conserved currents, which is rather involved in the case of coupled transport processes. Since the holographic picture of general non-equilibrium processes has difficulties from both conceptual and technical aspects, we consider the near-equilibrium cases here, which corresponds to linear perturbations of the background bulk configuration. Then the discussion can be extended to the usual holographic models such as holographic superconductors/superfluids on the conformal boundary, after considering the holographic renormalization. Even without knowing holography, such an equality, together with the traditional black-hole membrane paradigm [29–31], can be viewed as generalization of the well-established black-hole thermodynamics to the black-hole "hydrodynamics" (see also ref. [32]).

The rest of our paper is structured as follows. In section II, we briefly review the basic idea of the general holographic principle, the corresponding dictionary and its implementation in the static case. In section III, we present our proof of the above equality by connecting the bulk with the boundary through the conserved current. We then analyze the entropy production in holographic superconductors/superfluids in section IV. The last section is dedicated to some discussions on our result.

2 Holographic dictionary and its implementation in the equilibrium thermodynamics

Our starting point is the following (Euclidean) holographic principle

$$Z_{\text{bulk}}[\bar{\phi}] = \int D\psi \exp(-I_{\text{FT}}[\bar{\phi}, \psi]) \tag{2.1}$$

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[1]For a nice review of the holographic renormalization, see, e.g., ref. [33].

for some quantum gravity theory with partition function $Z_{\text{bulk}}[\bar{\phi}]$ on some bulk space-time region and the corresponding quantum field theory with action $I_{\text{FT}}[\bar{\phi}, \psi]$ on its boundary, which is the refined and generalized version of the original AdS/CFT principle [4, 5]. Here the partition function $Z_{\text{bulk}}[\bar{\phi}]$ is evaluated by fixing the boundary value of the bulk field ϕ to be $\bar{\phi}$, which acts as some background field on the boundary, and ψ denotes all the dynamical fields in the boundary theory, which is integrated out to produce the partition function in the right hand side of (2.1). To be more precise, if ϕ is the metric or form fields, then the pull back of ϕ to the boundary is fixed to be $\bar{\phi}$. Infinitesimal variation of $\bar{\phi}$ in (2.1) gives

$$Z_{\text{bulk}}[\bar{\phi} + \delta\bar{\phi}] = Z_{\text{bulk}}[\bar{\phi}]\langle\exp\int_{\text{bdry}}\delta\bar{\phi}O_\phi\sqrt{\bar{g}}d^dx\rangle_{\text{FT}}$$

with $\sqrt{\bar{g}}d^dx$ the standard volume element on the boundary and

$$O_\phi(x) = -\frac{1}{\sqrt{\bar{g}}}\frac{\delta I_{\text{FT}}[\bar{\phi}, \psi]}{\delta\bar{\phi}(x)}$$

the "dual field", which should be understood as the corresponding quantum operator in the expression of expectation value.

In the classical limit (or sometimes called the saddle point approximation), the bulk partition function is given by

$$Z_{\text{bulk}}[\bar{\phi}] = \exp(-I_{\text{bulk}}[\bar{\phi}])$$

with $I_{\text{bulk}}[\bar{\phi}]$ the on-shell action (Hamilton's principal functional). So the above holographic principle leads to

$$-\frac{1}{\sqrt{\bar{g}}}\frac{\delta I_{\text{bulk}}[\bar{\phi}]}{\delta\bar{\phi}(x)} = \langle O_\phi(x)\rangle_{\text{FT}}, \tag{2.2}$$

where the left hand side is just the canonical momentum conjugate to ϕ by virtue of the Hamilton-Jacobi equation regarding the boundary as the "time" slice. Now turn to the Minkowskian signature. The discussion in this case is similar to the above, but subtleties arise when one further considers correlation functions [34], which does not concern us in the present paper. For the bulk being (asymptotic) AdS space-time and the boundary tending to its conformal boundary, it is well known that the dual field theory is a (local) CFT.[2] But in more general cases, e.g. asymptotically flat bulk and/or boundary at finite distance [22, 23], the dual theory should be some effective field theory that is both non-local and non-conformal [35], inspired by the well-known AdS/CFT interpretation that the radial direction is related to renormalization group flow of the dual theory. Although the details of the general dual theory is unclear so far, macroscopic aspects of the general bulk/boundary correspondence turn out to be universal and can be clearly understood, which is part of the main motivation of this paper. In the macroscopic point of view, the boundary theory is described by thermodynamics and hydrodynamics, where we identify the expectation value in (2.2) with the macroscopic (classical) mechanical quantity $O_\phi(x)$.

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[2]In this case, some related discussions can be seen in section 4.3.

Two examples are of special interests. One is the case that ϕ is taken to be the metric $g_{\mu\nu}$, where $\bar{\phi}$ is just the induced metric $\bar{g}_{ab}$ on the boundary. Then the Minkowskian version of (2.2) tells us that the stress-energy tensor of the boundary system is given by the Brown-York tensor (see (2.7) for the explicit form)

$$t_{ab}(x) = \frac{2}{\sqrt{-g}} \frac{\delta I_{\text{bulk}}[\bar{g}]}{\delta \bar{g}_{ab}(x)},$$

where the bulk action is taken to be the standard Einstein-Hilbert action plus the Gibbons-Hawking term. The other is the case that ϕ is taken to be the electromagnetic potential A_μ. Similarly, the dictionary is that the electric current of the boundary system is given by

$$j^a(x) = \frac{1}{\sqrt{-g}} \frac{\delta I_{\text{bulk}}[\bar{A}]}{\delta \bar{A}_a(x)} = -n_\mu F^{\mu a}, \tag{2.3}$$

where the bulk action is just the Maxwell one in addition to the gravitational part. In this section, we first explore the macroscopic aspects of the general bulk/boundary correspondence in the equilibrium case, based on the above holographic dictionary.

2.1 Thermodynamics dual to the RN bulk space-time

We consider the RN black hole

$$ds^2_{d+1} = \frac{dr^2}{f(r)} - f(r)dt^2 + r^2 d\Omega^{(k)2}_{d-1},$$

$$f(r) = k + \frac{r^2}{\ell^2} - \frac{2M}{r^{d-2}} + \frac{Q^2}{r^{2d-4}},$$

$$d\Omega^{(k)2}_{d-1} = \hat{g}^{(k)}_{ij}(x)dx^i dx^j,$$

$$A = \sqrt{\frac{d-1}{8\pi(d-2)G}} \frac{Q}{r^{d-2}} dt, \tag{2.4}$$

with negative cosmological constant[3] in the Einstein-Maxwell theory as our bulk space-time (in equilibrium). Here M is the mass parameter, Q the charge parameter of the black hole, and $\hat{g}^{(k)}_{ij}(x)$ the metric on the "unit" sphere, plane or hyperbola for k equal to 1, 0 or -1, respectively, where in the planar or hyperbolic case some standard compactification is assumed. The boundary is the hypersurface $r = r_c$ outside the horizon, with an induced metric

$$ds^2_d = -f_c dt^2 + r_c^2 d\Omega^{(k)2}_{d-1}, \qquad f_c := f(r_c). \tag{2.5}$$

Due to static nature (with time-like Killing vector ∂_t) of both the bulk space-time and the boundary, and maximum symmetry on a time slice of the boundary, the boundary system is obviously in equilibrium. From the identification (2.1) of the Euclidean partition function (see the next subsection for detailed discussions), an argument of conical singularity leads to the conclusion that the entropy and temperature of the boundary system are equal to the Bekestein-Hawking entropy

$$S = \frac{\Omega^{(k)}_{d-1} r_h^{d-1}}{4G}$$

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[3]The case with positive cosmological constant can also be included by formally allowing $\ell^2 < 0$.

and local Hawking temperature

$$T = \frac{T_H}{\sqrt{f_c}} = \frac{f'_h}{4\pi\sqrt{f_c}}, \qquad f'_h := f'(r_h) \tag{2.6}$$

of the bulk black hole. Here $\Omega^{(k)}_{d-1}$ is the volume of the "unit" sphere, plane or hyperbola, and r_h the radius of the outer horizon satisfying $f(r_h) = 0$. Due to the bulk/boundary dictionary, the stress-energy tensor of the boundary system is given by the Brown-York tensor

$$t_{ab} = \frac{1}{8\pi G}(Kg_{ab} - K_{ab} - Cg_{ab}), \qquad K := K_{ab}g^{ab} \tag{2.7}$$

on the boundary with K_{ab} its extrinsic curvature and C some constant, which can be easily shown to have a form of ideal fluid

$$t_{ab} = \epsilon u_a u_b + p(u_a u_b + g_{ab})$$

with the velocity $u_a = (-\sqrt{f_c}, 0, \cdots, 0)$, the energy density

$$\epsilon = -\frac{d-1}{8\pi G}\frac{\sqrt{f_c}}{r_c} + C, \tag{2.8}$$

and the pressure

$$p = \frac{d-2}{8\pi G}\frac{\sqrt{f_c}}{r_c} + \frac{1}{16\pi G}\frac{f'_c}{\sqrt{f_c}} - C.$$

As well, the electric current (2.3) of the boundary system is

$$j^a = -n_\mu F^{\mu a}(r_c) = \left(-\sqrt{\frac{(d-1)(d-2)}{8\pi G f_c}}\frac{Q}{r_c^{d-1}}, 0, \cdots, 0\right). \tag{2.9}$$

Since the volume of the boundary system is

$$V = \Omega^{(k)}_{d-1}r_c^{d-1},$$

the energy density (2.8) gives the total energy

$$E = \Omega^{(k)}_{d-1}\left(-\frac{d-1}{8\pi G}\sqrt{f_c}r_c^{d-2} + Cr_c^{d-1}\right),$$

while the electric current (2.9) gives the total charge

$$\Omega^{(k)}_{d-1}\sqrt{\frac{(d-1)(d-2)}{8\pi G}}Q$$

that coincides with the physical charge of the black hole. The proportion coefficient here is not essential, so we will take Q as the total charge in the following discussion.

As a consistency check, if expressing E as a function of (S, V, Q), one can verify

$$\frac{\partial E}{\partial S} = T, \qquad \frac{\partial E}{\partial V} = -p.$$

Furthermore, one can obtain

$$\mu = \frac{\partial E}{\partial Q} = -\frac{d-1}{8\pi G} \frac{\Omega_{d-1}^{(k)} Q}{\sqrt{f_c}} \left(\frac{1}{r_c^{d-2}} - \frac{1}{r_h^{d-2}} \right), \tag{2.10}$$

which is proportional to the difference of electric potential between the horizon and the holographic screen, and is the appropriate generalization of the familiar chemical potential in AdS/CFT ($r_c \to \infty$). As a consistency check, we will show shortly that the chemical potential (2.10) gives the correct Einstein relation on the holographic screen $r = r_c$. Thus, we see that the first law

$$dE + pdV = TdS + \mu dQ \tag{2.11}$$

of thermodynamics holds for the boundary system. In the plane symmetric case ($k = 0$), a further relation

$$E + pV = TS + \mu Q$$

holds as the Gibbs-Duhem relation, as one may expect from extensibility arguments (see the next subsection). In this case, it is convenient to express the thermodynamic relations in terms of densities of the extensive quantities as [25]

$$\begin{aligned} \epsilon + p &= Ts + \mu\rho, \\ d\epsilon &= Tds + \mu d\rho, \end{aligned} \tag{2.12}$$

where $s = \frac{S}{V}$ is the entropy density and $\rho = \frac{Q}{V}$ the charge density.

Now we show that the chemical potential (2.10) is consistent with the Einstein relation

$$\sigma = \Xi D, \tag{2.13}$$

where σ is the electric conductivity, Ξ the susceptibility, and D the diffusion constant. Following the corresponding discussion [29–31] in the standard AdS/CFT, we work at the linear order of ρ in μ, which is actually the limit of small Q. In this case, we have $\rho = \Xi\mu$, so the susceptibility

$$\Xi = \frac{\rho}{\mu} = \frac{\frac{(d-1)(d-2)}{8\pi G} \frac{Q}{r_c^{d-1}}}{-\frac{d-1}{8\pi G} \frac{Q}{\sqrt{f_c}} \left(\frac{1}{r_c^{d-2}} - \frac{1}{r_h^{d-2}} \right)} = (d-2) \frac{\sqrt{f_c}}{r_c^{d-1}} \left(\frac{1}{r_h^{d-2}} - \frac{1}{r_c^{d-2}} \right)^{-1}.$$

The conductivity σ and the diffusion constant D have been computed at the finite holographic screen [22, 29–31, 36].[4] In our notation and convention, the results are

$$\sigma = \frac{r_h^{d-3}}{r_c^{d-1}},$$

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[4] Note that in ref. [29–31, 36], the electric current at a finite cutoff surface differs from our definition (2.3) by a $\sqrt{-\bar{g}}$ factor, so does other quantities related to conjugate momenta. Our formalism is close to ref. [22], which treats the holographic screen as an effective physical system, so such quantities should be more suitably defined as intrinsic tensor (vector, scalar) fields on the screen. Correspondingly, although the famous ratio $\frac{\eta}{s}$ with η the shear viscosity does not run with r_c in both formalism, η and s independently do not run with r_c in the formalism of ref. [29–31, 36], while they do run with r_c in our formalism. See also the related discussion in section 4.3.

$$D = \frac{r_h^{d-3}}{(d-2)\sqrt{f_c}} \left(\frac{1}{r_h^{d-2}} - \frac{1}{r_c^{d-2}} \right),$$

so it is obvious that the Einstein relation (2.13) holds.

2.2 The general thermodynamics by Hamilton-Jacobi-like analysis

For more general gravitational theories with various matter content, the dual thermodynamic relation similar to (2.11) can be obtained through a Hamilton-Jacobi-like analysis, which we present here. There are several types of ensembles that we can choose. They are related to one another by Legendre transformations, in the thermodynamic limit. For the system with charges, such as that dual to the RN black hole (2.4), the most often used ensemble in AdS/CMT or fluid/gravity correspondence is the grand-canonical ensemble, so we will take this ensemble as our starting point. Other ensembles can be discussed similarly. First of all, at finite temperature the holographic principle (2.1) is naturally extended to

$$\begin{array}{cc} \text{black-hole Euclidean partition function} = \text{(grand-)canonical partition function,} \\ \text{(bulk)} \qquad\qquad\qquad\qquad\qquad \text{(boundary)} \end{array} \qquad (2.14)$$

where for the grand-canonical case the black-hole Euclidean partition function is evaluated under the boundary condition of fixed chemical potential (2.10), instead of fixed charge Q for the canonical case. The black-hole Euclidean partition functions (or more precisely the logarithm of them) under different sets of boundary conditions are also related to one another by Legendre transformations, in the classical limit of gravity. Here we see the classical limit/thermodynamic limit correspondence in the general (Euclidean) bulk/boundary holography, as already indicated in the standard AdS/CFT case.[5] In fact, using the Hamilton-Jacobi-like analysis and insisting on the micro-canonical ensemble, Brown and York obtain the first law-like relation from the purely gravitational point of view [37]. But we will present a simpler argument in the context of holography, which in the same time clearly shows the relation of different ensembles.

A natural requirement for the spatial section of our holographic setup is homogeneity and isotropy, since it is hard to define equilibrium otherwise. Recall that the central quantity of the grand-canonical ensemble is the grand potential Ω, as a function of the temperature T (or the inverse temperature β), the chemical potential μ and the volume V. Under the classical limit on the bulk side and the thermodynamic limit on the boundary side, the holographic principle (2.14) becomes

$$\exp[-I_{\text{bulk}}(\beta, \mu, V)] = \exp(\beta\Omega), \qquad (2.15)$$

where in the homogeneous and isotropic case the on-shell (Euclidean) action I_{bulk} is evaluated under the boundary condition of fixed β, μ and V. Here we have suppressed any other possible fields in the theory, such as the scalar field in the holographic superconductor/superfluid models, which are easily included in this discussion. Note that from the holographic point of view, β is just the periodicity of the Euclidean time, as a Killing vector field

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[5]In AdS/CFT, the "thermodynamic limit" here manifests itself more familiarly as a large N limit or a large central charge limit, which is apparently different from its usual meaning.

of the bulk space-time, measured by the proper time on the boundary, which is determined by the condition of regularity, i.e. having no conical defect, at the (Euclidean) horizon.

Now we give two equivalent ways to vary I_{bulk} and then compare the results. The first way is to vary I_{bulk} with respect to β, μ and $\bar{g}_{ii}$ (the diagonal spatial component of $\bar{g}_{ab}$). Since I_{bulk} is on shell, similar to the standard Hamilton-Jacobi equation, variation of I_{bulk} with respect to $\mu = \bar{A}_\tau$ and $\bar{g}_{ii}$ while keeping β and the other quantities fixed gives

$$\delta I_{\text{bulk}} = \rho\beta V \delta\mu + p\beta\delta V. \tag{2.16}$$

Note that the triviality of the Wilson loop of A_μ contracted at the Euclidean horizon requires $A_\tau = 0$ there automatically. On the other hand, variation of I_{bulk} with respect to β gives [38]

$$\delta I_{\text{bulk}} = \frac{I_{\text{bulk}}}{\beta}\delta\beta - \frac{S}{\beta}\delta\beta, \tag{2.17}$$

where the last term can be viewed as coming from the contribution of the conical singularity when β is perturbed away from the original periodicity of the Euclidean time. (See [39] for more detailed and rigorous discussions about subtleties that may arise here.) Combining (2.15), (2.16) and (2.17), we obtain

$$d(\beta\Omega) = \Omega d\beta + \frac{S}{\beta}d\beta - \rho\beta V d\mu - p\beta dV, \tag{2.18}$$

which is just the thermodynamic relation

$$\beta d\Omega = \frac{S}{\beta}d\beta - \rho\beta V d\mu - p\beta dV,$$

or more conveniently

$$d\Omega = -SdT - Qd\mu - pdV. \tag{2.19}$$

The second way is to vary I_{bulk} with respect to $\bar{g}_{\tau\tau}$, μ and $\bar{g}_{ii}$. Similar to (2.18) in the first way, we obtain

$$\begin{aligned} d(\beta\Omega) &= \epsilon V d\beta - \rho V d(\beta\mu) - p\beta dV \\ &= (E - \mu Q)d\beta - \rho\beta V d\mu - p\beta dV. \end{aligned}$$

Comparing the above equation with (2.18), we see

$$\Omega = E - TS - \mu Q,$$

which together with (2.19) shows that Ω is indeed related to the energy E by Legendre transformations. Then, the standard first law

$$dE + pdV = TdS + \mu dQ$$

of thermodynamics follows immediately.

In the case with planar symmetry, there is another important relation. Similar to the extensibility arguments in the ordinary thermodynamics in textbooks, we have

$$\Omega(T, \mu, \lambda V) = \lambda\Omega(T, \mu, V)$$

with an arbitrary scaling parameter λ, since V is the only extensive quantity in the arguments of Ω, and there is no extra independent scale in the system. That gives

$$\Omega = \left(\frac{\partial \Omega}{\partial V} \right)_{T,\mu} V = -pV,$$

and then the Gibbs-Duhem relation

$$E + pV = TS + \mu Q, \tag{2.20}$$

which we have seen in the $\varepsilon = 0$ case in the last subsection. Sometimes this relation is also expressed as the differential form

$$V dp - S dT - Q d\mu = 0.$$

In the other cases (i.e. with spherical or hyperbolic symmetry), it is expected that the usual Gibbs-Duhem relation (2.20) will no longer hold. Actually, one can easily show from the expressions in the preceding subsection that $E + pV \neq TS + \mu Q$ if $k \neq 0$ for the RN black hole (2.4). However, it turns out that a peculiar Gibbs-Duhem-like relation still holds, at least formally. In fact, let $k = \varepsilon^{\frac{2}{d-1}}$ in (2.4), one will obtain the conjugate quantity ς of ε as

$$\varsigma = \left(\frac{\partial E}{\partial \varepsilon} \right)_{S,Q,V} = -\frac{\Omega_{d-1}}{8\pi G} \frac{r_c^{d-2} - r_h^{d-2}}{\sqrt{f_c}} \varepsilon^{\frac{3-d}{d-1}}. \tag{2.21}$$

Note that the volume Ω_{d-1} of the "unit" sphere, plane or hyperbola can be arbitrarily dependent on k in the discussion of the preceding subsection, but here we assume that this volume is a constant independent of k, so that the total volume V of the boundary system can be simply identified with r_c^{d-1} when varying k. Then one can easily check that the Gibbs-Duhem-like relation

$$E + pV = TS + \mu Q + \varsigma \varepsilon \tag{2.22}$$

holds. But how can one understand the physical or geometric meaning of ε (and ς)? The $d = 3$ case, where the bulk is of four dimensions and $k = \varepsilon$, is the simplest one to illustrate the meaning of ε. In this case $f(r) = \varepsilon + \frac{r^2}{\ell^2} - \frac{2M}{r} + \frac{Q^2}{r^2}$, it is easy to see that the transformation

$$\varepsilon \to \lambda^2 \varepsilon, \qquad\qquad r \to \lambda r, \qquad\qquad M \to \lambda M, \qquad\qquad Q \to \lambda Q$$
$$t \to \lambda^{-1} t, \qquad\qquad d\Omega_2^2 \to \lambda^{-2} d\Omega_2^2$$

leaves the configuration (2.4) invariant. Using the above transformation with $\lambda = (-\varepsilon)^{-1/2}$, one can transform an $\varepsilon \neq -1$ solution to an $\varepsilon = -1$ one. However, this transformation has an additional consequence $\Omega_2 \to -\varepsilon \Omega_2$. If ε is a negative integer, this multiplies the volume of the "unit hyperbola" by an integer. Recalling that every two dimensional compact surface with constant negative curvature is the original hyperbolic space (Poincaré upper-half plane) modulo some discrete group (see e.g. [40]), one recognize ε as (half of) the Euler number of the spatial section of the black hole horizon. This interpretation remains valid for $\varepsilon = 0, 1$, but becomes obscure for $\varepsilon > 1$ or ε not integer, in which case the interpretation

of ε as a Euler number is only formal. Thus, at least formally, one sees that including the *topological charge* ε, as well as the (gauge) charge Q, as thermodynamic quantities for the case without planar symmetry, the Gibbs-Duhem-like relation (2.22) can be obtained. Now we consider the $d \neq 3$ case. If d is even, then the spatial section of the horizon is of odd dimensions and has no well-defined Euler number. In the same time, $\varepsilon = k^{\frac{d-1}{2}}$ is complex for negative k, so (2.22) is very formal in this case. If d is odd, the Euler density of the section scales as $R^{\frac{d-1}{2}}$ with R the curvature tensor of the section, the similar discussion as in the $d = 3$ case above also leads to the conclusion that ε can be viewed as (half of) the Euler number of the section.[6] Strictly speaking, the topological charge ε should be an integer, as well as the charge Q should be quantized, which makes the corresponding first law-like relation

$$dE + pdV = TdS + \mu dQ + \varsigma d\varepsilon \qquad (2.23)$$

less interesting. However, when the above relation is expressed in terms of densities:

$$d\epsilon = Tds + \mu d\rho + \varsigma de$$

with the topological charge density $e = \frac{\varepsilon}{V}$, just as (2.12), the quantization of ε and Q can be smoothed out by a large volume V of the boundary system.[7] The expression (2.21) of ς also looks like that of the chemical potential (2.10), but it is not clear whether ς can be understood as the difference of some kind of potential. Furthermore, one may expect that the topological charge ε will also play some role in the ordinary thermodynamics of black holes. We refer the interested readers to appendix A.

3 Entropy production on the holographic screen and its equality with the increase of entropy in the bulk

If the boundary system is perturbed by some sort of external sources, various transport processes occur as the system relaxes back towards equilibrium, which causes entropy production. From the bulk point of view, the ingoing boundary condition at the future horizon implies that the (material or gravitational) perturbations at the boundary should propagate to the black hole and be absorbed, which causes increase of the area of the black hole horizon. Based on the equilibrium configuration we have discussed above, there are three kinds of transport processes that we can consider, i.e. heat conduction, viscosity of fluid and charge conduction. The heat conduction is energy transportation, caused by temperature inhomogeneity. The viscosity of fluid is momentum transportation, caused by velocity inhomogeneity (shear and expansion, more precisely). The charge conduction is caused by external electric field or inhomogeneity of chemical potential. Then a slight generalization of the textbook discussion (see, e.g. ref. [42]) to the case allowing a curved

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[6]Note that if we let $k = \varepsilon^\alpha$ with $\alpha \neq \frac{2}{d-1}$, then the relation (2.22) cannot hold, so the validity of the Gibbs-Duhem-like relation can be viewed as equivalent to the fact that ε is the topological charge.

[7]If considering the d-dimensional spatial section of the bulk black hole, the density e is just the local flux of a recently proposed "Euler current" [41] across the $(d-1)$-dimensional spatial section of the hypersurface $r = r_c$. However, further consequences of this fact still need to be investigated.

space gives the total entropy production rate[8]

$$\Sigma = \mathbf{j}_q \cdot \mathbf{D}\frac{1}{T} - \frac{1}{T}\Pi : \mathbf{Du} + \frac{1}{T}\mathbf{j} \cdot \mathbf{E} = j_q^i D_i \frac{1}{T} - \frac{1}{T}\Pi^{ij}\sigma_{ij} + \frac{1}{T}j^i E_i,$$

where $\mathbf{j}_q$ is the heat current, Π^{ij} the dissipative part of the stress-energy tensor, $\sigma_{ij} = D_{(i}u_{j)} \equiv \frac{1}{2}(D_i u_j + D_j u_i)$ the combination of shear tensor and expansion rate, $\mathbf{j}$ the electric current, $\mathbf{E}$ the electric field, and we have assumed a homogeneous chemical potential. With a slight abuse of terminology, we also call σ_{ij} the shear tensor for the sake of briefness, but keep in mind that its traceless part is the genuine shear tensor while its trace part is the expansion rate. In fact, the physical laws of transportation tell us that the transport current $(j_q^i, \Pi^{ij}, j^i, \cdots)$ is proportional (in the linear regime) to the driving force $(D_i\frac{1}{T}, -\frac{1}{T}\sigma_{ij}, \frac{1}{T}E_i, \cdots)$, while the entropy production rate is just their product. In the holographic context, this proportion factor (matrix), i.e. the transport coefficients, is determined by imposing the ingoing boundary condition at the horizon and then solving the bulk equations of motion (see e.g. ref. [43, 44] for the traditional AdS/CFT case and ref. [22, 36, 45] for the "finite cutoff" case). However, we do not need the precise values of them here. Note that on the bulk side we only consider classical gravitational theory with classical matter fields.

It is well known in AdS/CFT that the temperature inhomogeneity and shear can both be realized by gravitational perturbations, at least for some special configurations (see, e.g. [43, 44] and [14], respectively). Now we generalize the analyses to arbitrary (but small) temperature perturbation and shear field on the holographic screen $r = r_c$. For the sake of simplification, we do a constant rescaling of t such that the metric (2.5) on the holographic screen becomes

$$ds_d^2 = -dt^2 + \cdots . \tag{3.1}$$

The temperature perturbation can be introduced by the metric perturbation

$$ds_d^2 \rightarrow ds_d^2 + 2h_{ti}dtdx^i,$$

generalizing the discussion in [43, 44], as

$$D_i\frac{1}{T} = \frac{1}{T}\partial_t h_{ti},$$

which can be briefly argued as follows. First, we perturb the time-time component of the metric by h_{tt} before turning on an off-diagonal metric perturbation h_{ti}. Recall that the inverse temperature $\beta = 1/T$ is the period of the Euclidean time measured under the proper time units, so the metric perturbation h_{tt} induces the temperature perturbation

$$\frac{T_0^2}{T^2} = 1 - h_{tt}$$

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[8]Note that two kinds of independent viscous processes, i.e. shear viscosity and bulk viscosity, present here, whose contributions to the entropy production rate are written in a uniform way. The shear viscosity is the transport process for the tangent component of momentum, and the bulk viscosity is the transport process for the normal component of momentum.

with T_0 the constant equilibrium temperature, or in other words

$$\partial_i h_{tt} = 2\frac{\partial_i T}{T}$$

at leading (linear) order. Next, we make an infinitesimal coordinate transformation (diffeomorphism) to turn off h_{tt} and exhibit the temperature gradient by the off-diagonal metric perturbation h_{ti}. Using $\mathcal{L}_\xi g_{ab} = D_a \xi_b + D_b \xi_a$, it is easy to check that the diffeomorphism induced by the vector field ξ satisfying

$$\partial_t \partial_i \xi_t = -\frac{\partial_i T}{T}, \qquad \xi_i = 0$$

can do the task, which turns on an off-diagonal metric perturbation

$$\partial_t h_{ti} = -\frac{\partial_i T}{T}$$

and then completes our argument. Note that all the covariant derivatives D_a here have been replaced by the partial derivatives ∂_a, since for the induced metric (2.5) the Christoffel symbol vanishes when any of its indices is t.

Then, we insist on the frame $u^a = (1, 0, \cdots, 0)$, while turn on the shear (and expansion, as always) by a metric perturbation

$$ds_d^2 \to ds_d^2 + h_{ij}dx^i dx^j.$$

In this case the shear tensor reads

$$\sigma_{ij} = \partial_{(i} u_{j)} - \gamma_{ij}^a u_a = \frac{1}{2}(\partial_i \tilde{g}_{tj} + \partial_j \tilde{g}_{ti}) - \gamma_{tij} = \frac{1}{2}\partial_t h_{ij}$$

with $\tilde{g}_{ab} = g_{ab} + h_{ab}$ the perturbed metric and γ_{bc}^a the corresponding (perturbed) Christoffel symbol. The heat current j_q^i is just the energy current $-t^{ti}$ (that vanishes in equilibrium),[9] while Π^{ij} is just the (first order) perturbation of t^{ij}. So we have the total entropy production rate

$$\Sigma = -\frac{1}{2T}t^{(1)ab}\partial_t h_{ab} + \frac{1}{T}j^i E_i, \tag{3.2}$$

where the contribution from the charge conduction is simply realized by electromagnetic perturbations. Our central task in this section is to check whether (3.2) matches the black hole side.

3.1 The case without cross-transportation

For clarity, we first assume that the equilibrium background is uncharged, i.e. $Q = 0$. Since in this case the gravitational perturbation and electromagnetic perturbation are decoupled from each other, it turns out that the first two kinds of transport processes and the charge

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[9]Note that in the case with cross-transportation, there is a mixing between the heat current and the charge current, as well as that between the temperature gradient and the potential gradient, just as shown in ref. [43, 44]. However, this complication does not ruin our discussion on the entropy production rate, since that rate is the very bilinear of conjugate quantities that is invariant under such mixing.

conduction are decoupled, which allows us to discuss them separately. First, we consider gravitational perturbations, realizing the heat conduction and viscosity of the dual fluid. For convenience, we rescale the r coordinate such that

$$ds^2_{d+1} = dr^2 + g(r)dt^2 + \cdots, \tag{3.3}$$

setting $g(r_c) = -1$ to guarantee (3.1). Now we introduce the gravitational perturbation with gauge

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}, \qquad h_{r\mu} = 0 \tag{3.4}$$

in the bulk, while in addition the time-time component h_{tt} vanishes on the holographic screen. The above metric implies the extrinsic curvature

$$K_{ab} = \frac{1}{2}\mathcal{L}_n g_{ab} = \frac{1}{2}\partial_r g_{ab} \to \frac{1}{2}\partial_r(g_{ab} + h_{ab})$$

for any hypersurface of constant r, with n its unit normal, even after perturbation (3.4). On the boundary side, since the background Brown-York tensor t_{ab} has no off-diagonal elements, the entropy production rate (3.2) is obviously of order $\mathcal{O}(h^2_{ab})$. To leading order, the entropy production rate (3.2) without the electromagnetic part is

$$\begin{aligned}
\Sigma &= -\frac{1}{2T}(t^{(1)}_{cd}g^{ca}g^{db} - t^{(0)}_{cd}h^{ca}g^{db} - t^{(0)}_{cd}g^{ca}h^{db})\partial_t h_{ab} \\
&= -\frac{1}{2T}(t^{(1)}_{ab}\partial_t h^{ab} - 2t^{(0)b}_c h^{ca}\partial_t h_{ab}) \\
&= -\frac{1}{16\pi GT}(K^{(1)}g_{ab} + K^{(0)}h_{ab} - K^{(1)}_{ab} - 2K^{(0)}h_{ab} + 2K^{(0)c}_b h_{ca})\partial_t h^{ab} \\
&= -\frac{1}{16\pi GT}\left(\frac{1}{2}\partial_r h\partial_t h - K^{(0)}h_{ab}\partial_t h^{ab} - \frac{1}{2}\partial_r h_{ab}\partial_t h^{ab} + 2K^{(0)c}_b h_{ca}\partial_t h^{ab}\right), \quad \tag{3.5}
\end{aligned}$$

where we have defined $h \equiv g^{ab}h_{ab}$, and the indices are lowered (or raised) with the background metric g_{ab}. Here the superscript (0) and (1) denote the background (equilibrium) quantities and the first order variations induced by the gravitational perturbation h_{ab}, respectively. Particularly, $K^{(1)}$ means the first order perturbation of $K = K_{ab}g^{ab}$, which can be written as

$$K^{(1)} = K^{(1)}_{ab}g^{ab} - K^{(0)}_{ab}h^{ab} = \frac{1}{2}g^{ab}\partial_r h_{ab} - \frac{1}{2}h^{ab}\partial_r g_{ab} = \frac{1}{2}\partial_r h.$$

On the bulk side, the physical picture is that the gravitational wave caused by the boundary perturbation propagates to the black hole, which will be absorbed and render the horizon area to increase. From the theory of gravitational waves (see, e.g. [46]), we know that the effective stress-energy tensor $T_{\mu\nu}$ of the wave is just $-\frac{1}{8\pi G}$ times $G^{(1,1)}_{\mu\nu}$, the second order contribution of $h_{\mu\nu}$ to the Einstein tensor, which satisfies

$$\nabla_\mu T^{\mu\nu} = 0$$

to order $\mathcal{O}(h^2_{ab})$ with respect to the background covariant derivative ∇_μ. Since $\xi = \partial_t$ is Killing, we have the conservation law

$$\nabla_\mu(T^\mu_\nu \xi^\nu) = 0 \tag{3.6}$$

of the current $T_\nu^\mu \xi^\nu = T_t^\mu$. Physically, our whole setup is in a static (equilibrium) state before the perturbation is turned on at some time, then the perturbation on the boundary system dissipates due to the transport processes, while the perturbation in the bulk (and on the horizon) gradually fades away due to the black hole absorption. Hence integrating the above equation over the perturbed bulk region with the perturbed horizon as the inner boundary and using the Gauss law, we end up with

$$\int_H T_t^\mu \lambda_\mu \tilde{\epsilon}_{[d]} = \int_{\text{bdry}} T_t^\mu n_\mu \hat{\epsilon}_{[d]}, \tag{3.7}$$

where H is the horizon and $\hat{\epsilon}_{[d]} = \sqrt{-\bar{g}} d^d x$. Here λ^μ is tangent to the affinely parametrized null geodesic generators of H, and $\tilde{\epsilon}_{[d]}$ satisfies

$$\lambda \wedge \tilde{\epsilon}_{[d]} = -\epsilon_{[d+1]}$$

with $\epsilon_{[d+1]}$ the standard volume element in the bulk. The left hand side of (3.7) is just the heat absorbed by the black hole [47, 48], which satisfies

$$\int_H T_t^\mu \lambda_\mu \tilde{\epsilon}_{[d]} = T_H \delta S. \tag{3.8}$$

This will be shown in appendix B.

To evaluate the right hand side of (3.7), we should know the explicit form of $G_{\mu\nu}^{(1,1)}$. From

$$G_{\mu\nu}^{(1,1)} = R_{\mu\nu}^{(1,1)} - \frac{1}{2}(R^{(1,1)} g_{\mu\nu} + R^{(1)} h_{\mu\nu})$$

and the Einstein equations at the zeroth and first orders, it is not difficult to obtain

$$G_{\mu\nu}^{(1,1)} = R_{\mu\nu}^{(1,1)} - \frac{1}{2} R_{\alpha\beta}^{(1,1)} g^{\alpha\beta} g_{\mu\nu}.$$

The second order contribution of $h_{\mu\nu}$ to the Ricci tensor $R_{\mu\nu}^{(1,1)}$ is given by [49]

$$R_{\mu\nu}^{(1,1)} = \frac{1}{2}\left[\frac{1}{2} h_{\alpha\beta|\mu} h^{\alpha\beta}_{\ \ |\nu} + h^{\alpha\beta}(h_{\alpha\beta|\mu\nu} + h_{\mu\nu|\alpha\beta} - h_{\alpha\mu|\nu\beta} - h_{\alpha\nu|\mu\beta}) + h_\nu^{\alpha|\beta}(h_{\alpha\mu|\beta} - h_{\beta\mu|\alpha}) \right.$$
$$\left. - \left(h^{\alpha\beta}_{\ \ |\beta} - \frac{1}{2} h^{|\alpha} \right)(h_{\alpha\mu|\nu} + h_{\alpha\nu|\mu} - h_{\mu\nu|\alpha}) \right],$$

where the indices are lowered (or raised) with the background metric $g_{\mu\nu}$ and "|" denotes the background covariant derivative ∇. Some lengthy but straightforward calculation gives

$$G_{rt}^{(1,1)}(r_c) = \frac{1}{2}\left[-\frac{1}{2} \partial_r h_{ab} D_t h^{ab} + 2 K_a^{(0)c} h_{cb} D_t h^{ab} - K_t^{(0)a} h^{bc} D_a h_{bc} \right.$$
$$\left. + (K_t^{(1)a} - K_{tc}^{(0)} h^{ac}) D_a h + D_a J^a \right], \tag{3.9}$$

where D_a is the background covariant derivative on the screen and J^a an order $\mathcal{O}(h_{ab}^2)$ current. We do not need the explicit form of J^a, for the divergence term $D_a J^a$ on the

screen does not contribute to the right hand side of (3.7) in our case. Note that we also have the first two order "momentum constraints"

$$D_a t^{(0)ab} = \frac{1}{8\pi G} D_a(K^{(0)} g^{ab} - K^{(0)ab}) = 0, \tag{3.10}$$

$$D_a t_b^{(1)a} + D_a^{(1)} t_b^{(0)a} = D_a(K_b^{(1)a} - K_{bc}^{(0)} h^{ca} - K^{(1)} \delta_b^a) + \gamma_{ac}^{(1)a} K_b^{(0)c} - \gamma_{ab}^{(1)c} K_c^{(0)a} = 0. \tag{3.11}$$

where we have used the fact that the first order perturbation $D^{(1)}$ of the covariant derivative on the screen comes essentially from the first order perturbation $\gamma^{(1)}$ of the corresponding Christoffel symbol. For the latter, we have

$$\gamma_{ab}^{(1)c} = \frac{1}{2} g^{cd}(D_a h_{bd} + D_b h_{ad} - D_d h_{ab}), \qquad \gamma_{ac}^{(1)a} = \frac{1}{2} g^{ad} D_c h_{ad} = \frac{1}{2} D_c h, \tag{3.12}$$

which gives

$$D_a(K_t^{(1)a} - K_{tc}^{(0)} h^{ca} - K^{(1)} \delta_t^a) + \frac{1}{2} K_t^{(0)c} D_c h - \frac{1}{2} K^{(0)ca} D_t h_{ac} = 0 \tag{3.13}$$

from the $b = t$ component of (3.11). Since the background extrinsic curvature $K_{ab}^{(0)}$ has no $K_{tj}^{(0)}$ (or $K_{it}^{(0)}$) components and isotropy of the background space leads to $K_{ij}^{(0)} = \kappa g_{ij}$, we know

$$K^{(0)ca} D_t h_{ac} = \kappa D_t h,$$

so (3.13) multiplying by h gives

$$(K_t^{(1)a} - K_{tc}^{(0)} h^{ca}) D_a h \simeq K^{(1)} D_t h = \frac{1}{2} \partial_r h \partial_t h, \tag{3.14}$$

where "$\simeq$" stands for equality up to divergence terms on the screen. Substituting (3.9) into (3.7) and using (3.10), (3.14) when comparing with (3.5), we see from (3.8) that

$$T_H \delta S = T \int_{\text{bdry}} \Sigma \hat{\epsilon}_{[d]}.$$

Upon identification $T_H = T$ due to our setting (3.1),[10] we conclude that the entropy increase on the bulk side and the entropy production on the boundary side match exactly.

Next, we consider electromagnetic perturbations, which is much simpler. For the electromagnetic wave, the physical picture is similar to the gravitational case, except that the component of the stress-energy tensor appearing in (3.7) is

$$T_{rt}(r_c) = F_r{}^i(r_c) F_{ti}(r_c) = j^i E_i, \tag{3.15}$$

where in the second equality we have used the holographic dictionary. Then, combining (3.2) (with the metric perturbation h_{ab} turned off), (3.7), (3.8) and $T_H = T$, we obtain

$$\delta S = \int_{\text{bdry}} \Sigma \hat{\epsilon}_{[d]}. \tag{3.16}$$

To sum up, for the uncharged background, we see perfect matching between the entropy production from the above three kinds of transport processes on the boundary and the entropy increase of the black hole in the bulk.

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[10]It is easy to incorporate the corresponding red-shift factor in our discussion without setting (3.1), but we leave this generality to section 3.2.

3.2 For more general gravitational theories

We have shown by the above direct calculation the consistency of the bulk entropy increase and the boundary entropy production in the Einstein-Maxwell theory. For more general gravitational theories, similar calculations may be very difficult. In this subsection, we give a more formal derivation of the same result, which allows working in a more general class of gravitational theories. For illustration purpose, now we do not rescale g_{tt} on the holographic screen to -1, so u^a should be taken as $(1/\sqrt{f_c}, 0, \cdots, 0)$. As well, we do not transform r to obtain a bulk metric of the form (3.3) in this formal derivation. On the boundary side, similar discussion leads to a total entropy production rate

$$\Sigma = -\frac{1}{2\sqrt{f_c}T}t^{(1)ab}\partial_t h_{ab} + \frac{1}{T}j^i E_i$$

instead of (3.2). Noting

$$D_{(t}u_{i)} = \partial_{(t}u_{i)} - \gamma^a_{ti}u_a = \frac{1}{2\sqrt{f_c}}(\partial_t\tilde{g}_{ti} + \partial_i\tilde{g}_{tt}) - \gamma_{tti} = \frac{1}{2\sqrt{f_c}}\partial_t h_{ti},$$

$$D_t u_t = \partial_t u_t - \gamma^a_{tt}u_a = 0,$$

we can rewrite the above entropy production rate as

$$\Sigma = \frac{1}{T}j^i E_i - \frac{1}{T}t^{(1)ab}D_a u_b = \frac{1}{T}j^i E_i - \frac{1}{T}t^{(1)ab}(-\gamma^{(1)c}_{ab}u_c + D_a u_b^{(1)}), \qquad (3.17)$$

which is invariant under diffeomorphisms in the boundary system. In fact,

$$D_a u_b^{(1)} = D_a(h_{bc}u^c) = -\sqrt{f_c}D_a h_b^t, \qquad (3.18)$$

and $\gamma^{(1)c}_{ab}$ is given by (3.12).

We shall prove that the increase of black hole entropy in the bulk is precisely equal to the aforementioned entropy production on the holographic screen. As in the previous subsection, on the bulk side, the basic idea for such a proof is to relate the holographic screen to the bulk black hole horizon by the conserved current (3.6) with $\xi = \partial_t$. For the electromagnetic perturbation, the conserved current is just generated by the stress-energy tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta I_{\text{EM}}}{\delta g_{\mu\nu}} = F^{\mu\rho}F^\nu_{\ \rho} - \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \qquad (3.19)$$

of the electromagnetic field, with its rt component on the holographic screen given by (3.15). Then, integrating the conserved current (3.6) and using the Gauss law together with the first-law-like relation (2.11) as in the previous subsection, and noting

$$\frac{1}{T}j^i E_i = \frac{1}{\sqrt{f_c}T}n_\mu F^{\mu i}(r_c)F_{ti}(r_c)$$

by the holographic dictionary and the relation (2.6) between the temperature T on the screen and the black hole temperature T_H now, we see that (3.16) holds for the electromagnetic part of our correspondence, which is independent of whatever gravitational theories we work in.

Next we consider the entropy production induced by the gravitational perturbation on the boundary. As promised, we can work in a more general class of gravitational theories, e.g. the general Lovelock gravity [50] as will be illustrated here. To proceed, let us firstly expand the bulk equations of motion on the black hole background to second order, i.e.,

$$E^{\mu\nu}[g] = 0, \tag{3.20}$$

$$E^{(1)\mu\nu}[g, h] = 0, \tag{3.21}$$

$$E^{(0,2)\mu\nu}[g, q] = -E^{(1,1)\mu\nu}[g, h] =: 8\pi G T^{\mathrm{G}\mu\nu}, \tag{3.22}$$

where the metric is expanded as $g_{\mu\nu} + h_{\mu\nu} + q_{\mu\nu}$ with $q_{\mu\nu}$ the second order perturbation and the indices are raised or lowered by the background metric $g_{\mu\nu}$. Furthermore, it follows from diffeomorphism invariance that the effective gravitational energy-momentum tensor $T^{\mathrm{G}\mu\nu}$ is conserved for the gravitational waves propagating on the background, i.e.

$$\nabla_\mu T^{\mathrm{G}\mu\nu} = 0,$$

which, as before, gives rise to

$$\delta S = \frac{1}{T_H} \int_H T^{\mathrm{G}\mu\nu} \lambda_\mu \xi_\nu = \frac{1}{T_H} \int_{\mathrm{bdry}} T^{\mathrm{G}\mu\nu} n_\mu \xi_\nu. \tag{3.23}$$

In Einstein's gravity, we know the Gauss-Codazzi equations

$$D_a t^{ab} = -\frac{1}{8\pi G} G^{\mu b} n_\mu$$

holding as geometric identities, which combines with the Einstein equations to give the momentum constraints. The counterparts of these constraints in the general Lovelock gravity can also be derived. The action functional of the Lovelock gravity is

$$I = \int_\Omega L + \int_{\partial\Omega} B, \tag{3.24}$$

where B is the surface term [51, 52] generalizing the usual Gibbons-Hawking term for Einstein's gravity. By construction of B, variation of the above action functional gives

$$\delta I = \int_\Omega \left(-\frac{1}{2} E^{MN} \delta g_{MN} \right) + 8\pi G \int_{\partial\Omega} \frac{1}{2} t^{ab} \delta \bar{g}_{ab}$$

with t_{ab} the generalization of the Brown-York surface tensor, which is holographically interpreted as the stress-energy tensor of the boundary system. Suppose that the above metric variation is a diffeomorphism induced by a vector field ξ tangent to $\partial\Omega$, the above equation becomes

$$\mathcal{L}_\xi I = \int_\Omega (-E^{MN} \nabla_M \xi_N) + 8\pi G \int_{\partial\Omega} t^{ab} D_a \xi_b$$

$$= \int_{\partial\Omega} (-E^{Mb} n_M - 8\pi G D_a t^{ab}) \xi_b,$$

where we have used $\nabla_M E^{MN} = 0$ (as a generalization of the Bianchi identity) coming from the diffeomorphism invariance of I without considering the boundary. The above expression vanishes due to the diffeomorphism invariance of I, then the arbitrariness of ξ_b gives

$$D_a t^{ab} = -\frac{1}{8\pi G} E^{\mu b} n_\mu.$$

Further, the right hand side of the above equation vanishes due to the equations of motion, which is just the momentum constraints.

So now the task boils down into whether one can express the above flux across the holographic screen in terms of entropy production on the screen, which can be achieved by a straightforward but lengthy calculation as in the previous subsection. But here we would like to present a shortcut towards the final result by taking advantage of the dual role played by the gravitational waves. Namely, as demonstrated in eqs. (3.21) and (3.22), the gravitational waves, albeit treated as sort of matter waves like light, are essentially ripples in the fabric of space-time. Thus we can relate the aforementioned flux to the quantities for the dual system on the holographic screen by the corresponding momentum constraints, which, expanded to second order, reads[11]

$$D_a t^{(0)ab} = -\frac{1}{8\pi G} E^{\mu b}[g]n_\mu = 0, \tag{3.25}$$

$$D_a t^{(1)ab} + D_a^{(1)} t^{(0)ab} = -\frac{1}{8\pi G} E^{(1)\mu b}[g,h]n_\mu = 0, \tag{3.26}$$

$$D_a t^{(2)ab} + D_a^{(1)} t^{(1)ab} + D_a^{(2)} t^{(0)ab} = -\frac{1}{8\pi G} E^{(1,1)\mu b}[g,h]n_\mu = T^{G\mu b}n_\mu, \tag{3.27}$$

where $D_a^{(2)}$ is determined by the second order Christoffel symbol, i.e.

$$D_a^{(2)} v^b = \gamma_{ac}^{(2)b} v^c = -\frac{1}{2} h^{bd}(D_a h_{cd} + D_c h_{ad} - D_d h_{ac}) v^c. \tag{3.28}$$

Then one can show

$$\begin{aligned}
\delta S &= \frac{\sqrt{f_c}}{T} \int_{\text{bdry}} (D_a t^{(2)at} + D_a^{(1)} t^{(1)at} + D_a^{(2)} t^{(0)at}) \\
&= \frac{\sqrt{f_c}}{T} \int_{\text{bdry}} \left(\frac{1}{2} D_d h t^{(1)dt} + \gamma_{cd}^{(1)t} t^{(1)cd} + \frac{1}{2} h^{ab} D_d h_{ab} t^{(0)dt} + \gamma_{cd}^{(2)t} t^{(0)cd} \right) \\
&= \frac{\sqrt{f_c}}{T_c} \int_{\text{bdry}} \left(\frac{1}{2} h D_d^{(1)} t^{(0)dt} + \gamma_{cd}^{(1)t} t^{(1)cd} + \gamma_{cd}^{(2)t} t^{(0)cd} \right) \\
&= \frac{\sqrt{f_c}}{T_c} \int_{\text{bdry}} \left(\frac{1}{2} h \left[\frac{1}{2} D_d h t^{(0)dt} + \gamma_{ad}^{(1)t} t^{(0)ad} \right] + \gamma_{cd}^{(1)t} t^{(1)cd} + \gamma_{cd}^{(2)t} t^{(0)cd} \right) \\
&= \frac{\sqrt{f_c}}{T_c} \int_{\text{bdry}} \left(\frac{1}{4} h g^{tb}[D_d h_{ba} + D_a h_{bd} - D_b h_{ad}] t^{(0)ad} + \gamma_{cd}^{(1)t} t^{(1)cd} + \gamma_{cd}^{(2)t} t^{(0)cd} \right)
\end{aligned}$$

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[11]Note that the perturbation of our boundary stress-energy tensor t_{ab} is only induced by $h_{\mu\nu}$, not by $q_{\mu\nu}$. Here our focus is on the tensor $T^{G\mu\nu}$, which is just $\frac{-1}{8\pi G} E^{(1,1)\mu\nu}[g,h]$ by definition (3.22). Note that $D_a t^{ab} = -\frac{1}{8\pi G} E^{\mu b} n_\mu$ is a geometric identity, which holds for any metric, so we take the metric in this identity to be $g_{\mu\nu} + h_{\mu\nu}$ (without $q_{\mu\nu}$) and do the following expansion. The relevance of the second order perturbation $q_{\mu\nu}$ is reflected in the first equality of (3.22).

$$= \frac{\sqrt{f_c}}{T_c} \int_{\text{bdry}} \left(-\frac{1}{2} h_a^t D_d h t^{(0)ad} - \frac{1}{4} h g^{tb} D_b h_{ad} t^{(0)ad} + \gamma_{cd}^{(1)t} t^{(1)cd} + \gamma_{cd}^{(2)t} t^{(0)cd} \right)$$

$$= \frac{\sqrt{f_c}}{T_c} \int_{\text{bdry}} \left(-\frac{1}{2} h_a^t D_d h t^{(0)ad} + \gamma_{cd}^{(1)t} t^{(1)cd} + \gamma_{cd}^{(2)t} t^{(0)cd} \right), \tag{3.29}$$

where we have thrown away all the total derivative terms at most of the steps, and employed (3.25), (3.26) as well as $h_{ad} t^{(0)ad} = ph$ (with p the pressure in equilibrium) for our isotropic background space in the last step. Noting

$$\begin{aligned} t^{(1)ab} D_a h_b^t &\simeq -h_b^t D_a t^{(1)ab} = h_b^t D_a^{(1)} t^{(0)ab} \\ &= h_b^t (\gamma_{ac}^{(1)a} t^{(0)cb} + \gamma_{ac}^{(1)b} t^{(0)ac}) \\ &= \frac{1}{2} h_b^t D_c h t^{(0)cb} - \gamma_{ac}^{(2)t} t^{(0)ac} \end{aligned}$$

by (3.26), (3.12) and (3.28), we see that (3.29) is exactly the integration of (the gravitational part of) (3.17) on the holographic screen. Recall that "$\simeq$" stands for equality up to divergence terms on the screen, which has been used in (3.14).

As can be seen clearly, the above discussion applies to any gravitational theories, as long as there exists a surface term [like B in (3.24)] for the action functional to make the variational principle well defined.

3.3 The case with cross-transportation

When the bulk black hole background is charged ($Q \neq 0$), it is known (see e.g. [53]) that the gravitational and electromagnetic perturbations are coupled to each other. From the boundary point of view, it turns out that the three kinds of transport processes are coupled to one another, just as what happens in the thermoelectricity phenomena. In this case, the transport coefficients form a matrix with off-diagonal elements, which indicate the so-called cross-transport processes. Now the background metric of the bulk space-time has the general form of (2.4). Let us consider the perturbation of metric and Maxwell field[12]

$$\begin{aligned} \tilde{g}_{ab} &= g_{ab} + h_{ab} + q_{ab} + \cdots, \\ F_{ab} &= F_{ab}^{(0)} + F_{ab}^{(1)} + F_{ab}^{(2)} + \cdots, \end{aligned} \tag{3.30}$$

where q_{ab} is the second order perturbation of the metric. The perturbed Einstein tensor up to second order can be expanded as

$$\tilde{G}_{ab} = G_{ab}[g] + G_{ab}^{(1)}[g, h] + G_{ab}^{(1,1)}[g, h] + G_{ab}^{(0,2)}[g, q] + \cdots, \tag{3.31}$$

where $G_{ab}[g]$ is the Einstein tensor of the background metric g_{ab}; $G_{ab}^{(1)}[g, h]$ is the linearized Einstein tensor; $G_{ab}^{(1,1)}[g, h]$ is one part of the second order perturbed Einstein tensor which is only relevant to the first order metric perturbation h_{ab}; $G_{ab}^{(0,2)}[g, q]$ is the other part of the second order perturbed Einstein tensor which is contributed by the second order metric perturbation q_{ab}. The form of $G_{ab}^{(0,2)}[g, q]$ is the same as $G_{ab}^{(1)}[g, h]$ while replacing h_{ab} by q_{ab}.

---

[12]Due to numerous indices in this subsection, we use Latin letters to denote the bulk space-time indices within this subsection (and also section 4.2).

The perturbed inverse metric up to second order is

$$\tilde{g}^{ab} = g^{ab} - h^{ab} + h^{ac}g_{cd}h^{db} - q^{ab} + \cdots . \tag{3.32}$$

Explicitly, the first order perturbed Einstein tensor is

$$G_{ab}^{(1)} = R_{ab}^{(1)} - \frac{1}{2}R^{(0)}h_{ab} - \frac{1}{2}R_{cd}^{(0)}h^{cd}g_{ab} - \frac{1}{2}R^{(1)}g_{ab}. \tag{3.33}$$

For the second order perturbation $R_{ab}^{(2)}$ of the Ricci tensor, we have

$$R_{ab}^{(2)} = R_{ab}^{(1,1)} + R_{ab}^{(0,2)}, \tag{3.34}$$

similar to the expansion of the Einstein tensor. The second order Einstein equation $G_{ab}^{(2)} + \Lambda q_{ab} = 8\pi G T_{ab}^{F(2)}$ can be easily shown to have the following form:

$$
\begin{aligned}
G_{ab}^{(0,2)}&[g,q] + \Lambda q_{ab} \\
&= \left[\frac{1}{2}R^{(1,1)}g_{ab} - \frac{1}{2}R_{cd}^{(1)}h^{cd}g_{ab} + \frac{1}{2}R_{cd}^{(0)}h^{ce}g_{ef}h^{fd}g_{ab} \right. \\
&\qquad \left. -R_{ab}^{(1,1)} + \frac{1}{2}R^{(1)}h_{ab} - \frac{1}{2}R_{cd}^{(0)}h^{cd}h_{ab} \right] + 8\pi G T_{ab}^{F(2)} \\
&=: 8\pi G(T_{ab}^{\mathrm{G}} + T_{ab}^{F(2)}).
\end{aligned} \tag{3.35}
$$

In order to investigate the holographic entropy production in this case, we still want to construct a conserved current, but this time that turns out to be rather subtle. Let us consider the Bianchi identity of a (fictitious) metric $\hat{g}_{ab} = g_{ab} + q_{ab}$. It is easy to see that $\hat{\nabla} - \nabla = C_{ab}^d = \frac{1}{2}(\nabla_a q_b^d + \nabla_b q_a^d - \nabla^d q_{ab})$, so we have

$$
\begin{aligned}
0 &= \hat{\nabla}^a \hat{G}_{ab} \\
&= (g^{ac} - q^{ac} + \cdots)(\nabla_c \hat{G}_{ab} - C_{ca}^d \hat{G}_{db} - C_{cb}^d \hat{G}_{da}) \\
&= \nabla^a G_{ab} + \nabla^a G_{ab}^{(0,2)}[g,q] - q^{ac}\nabla_c G_{ab} - g^{ac}C_{ca}^d G_{db} - g^{ac}C_{cb}^d G_{da} + \cdots .
\end{aligned} \tag{3.36}
$$

The second order term of the above equation is

$$
\begin{aligned}
0 &= \nabla^a G_{ab}^{(0,2)}[g,q] - q^{ac}\nabla_c G_{ab} - g^{ac}C_{ca}^d G_{db} - g^{ac}C_{cb}^d G_{da} \\
&= \nabla^a G_{ab}^{(0,2)}[g,q] - q^{ac}\nabla_c G_{ab} - g^{ac}\frac{1}{2}(\nabla_a q_c^d + \nabla_c q_a^d - \nabla^d q_{ac})G_{db} \\
&\quad -g^{ac}\frac{1}{2}(\nabla_b q_c^d + \nabla_c q_b^d - \nabla^d q_{bc})G_{da} \\
&= \nabla^a G_{ab}^{(0,2)}[g,q] - \nabla_c(q^{ac}G_{ab}) - \frac{1}{2}G_{cd}\nabla_b q^{cd} + \frac{1}{2}\nabla^d(q_a^a G_{db}).
\end{aligned} \tag{3.37}
$$

Because the background is a stationary space-time with the time-like Killing vector ∂_t, we consider the t-component of the above equation, i.e.

$$0 = (\partial_t)^b \left[\nabla^c G_{bc}^{(0,2)}[g,q] - \nabla^c(q_c^a G_{ab}) - \frac{1}{2}G_{cd}\nabla_b q^{cd} + \frac{1}{2}\nabla^c(q_a^a G_{cb}) \right]$$

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$$= \nabla^c G_{ct}^{(0,2)}[g,q] - \nabla^c(q_c^b G_{bt}) + q_c^b G_{ab}\nabla^c(\partial_t)^a - \frac{1}{2}G_{cd}\nabla_t q^{cd} + \frac{1}{2}\nabla^c(q_a^a G_{ct})$$

$$= \nabla^a G_{ct}^{(0,2)}[g,q] - \nabla^c(q_c^b G_{bt}) + q_c^b G_{ab}\nabla^c(\partial_t)^a - \frac{1}{2}\nabla_t(G_{cd}q^{cd}) + \frac{1}{2}q_{cd}\nabla_t G^{cd} + \frac{1}{2}\nabla^c(q_a^a G_{ct})$$

$$= \nabla^a G_{ct}^{(0,2)}[g,q] - \nabla^c(q_c^b G_{bt}) + \frac{1}{2}q^{cd}\mathcal{L}_{\partial_t}G_{cd} - \frac{1}{2}\nabla_a[G_{cd}q^{cd}(\partial_t)^a] + \frac{1}{2}\nabla^c(q_a^a G_{ct})$$

$$= \nabla^c G_{ct}^{(0,2)}[g,q] - \nabla^c(q_c^b G_{bt}) - \nabla_a\left[\frac{1}{2}G_{cd}q^{cd}(\partial_t)^a\right] + \nabla^c\left(\frac{1}{2}q_a^a G_{ct}\right). \tag{3.38}$$

This means that $J_a := G_{at}^{(0,2)}[g,q] - q_a^b G_{bt} - \frac{1}{2}G_{cd}q^{cd}(\partial_t)_a + \frac{1}{2}q_b^b G_{at}$ is a conserved current.

Taking into account the second order perturbed Einstein equation (3.35), the above conserved current can also be written as

$$J_a = 8\pi G(T_{at}^{\mathrm{G}} + T_{at}^{F(2)}) - \frac{1}{2}G_{cd}q^{cd}(\partial_t)_a - q_a^b G_{bt} + \frac{1}{2}q_b^b G_{at} - \Lambda q_{at}. \tag{3.39}$$

Using the same method in the previous section, we consider the integral of the divergence of this current and get[13]

$$\int_H <-l, J>$$
$$= \int_H \left(\frac{1}{2}R^{(1,1)}g_{tt} - \frac{1}{2}R_{cd}^{(1)}h^{cd}g_{tt} + \frac{1}{2}R_{cd}^{(0)}h^{ce}g_{ef}h^{fd}g_{tt} - R_{tt}^{(1,1)} + \frac{1}{2}R^{(1)}h_{tt} - \frac{1}{2}R_{cd}^{(0)}h^{cd}h_{tt}\right.$$
$$\left.+ 8\pi G T_{tt}^{F(2)} - \frac{1}{2}G_{cd}q^{cd}g_{tt} - q_t^b G_{bt} + \frac{1}{2}q_b^b G_{tt} - \Lambda q_{tt}\right)$$
$$= \int_H (-R_{ll}^{(1,1)} + 8\pi G T_{ll}^{F(2)})$$
$$= \int_{\mathrm{bdry}} <n, J>$$
$$= \int_{\mathrm{bdry}} \left(\frac{1}{2}R^{(1,1)}g_{nt} - \frac{1}{2}R_{cd}^{(1)}h^{cd}g_{nt} + \frac{1}{2}R_{cd}^{(0)}h^{ce}g_{ef}h^{fd}g_{nt} - R_{nt}^{(1,1)} + \frac{1}{2}R^{(1)}h_t^r - \frac{1}{2}R_{cd}^{(0)}h^{cd}h_t^r\right.$$
$$\left.+ 8\pi G T_{nt}^{F(2)} - \frac{1}{2}G_{cd}q^{cd}g_{nt} - q^{rb}G_{bt} + \frac{1}{2}q_b^b G_{nt} - \Lambda q_t^r\right)$$
$$= \int_{\mathrm{bdry}} (-R_{nt}^{(1,1)} + 8\pi G T_{nt}^{F(2)}). \tag{3.40}$$

Here the conditions[14] $h_{t\mu} \,\hat{=}\, q_{t\mu} \,\hat{=}\, 0$, $n_a \propto dr$ and $g^{rt} = h^{r\mu} = q^{r\mu} = 0$ are used. It is clear that the gravitational part in the right hand side is exactly the same as in the vacuum case. Now we consider the Maxwell part. The Maxwell part of the flux is the second order energy-momentum tensor component $T_{nt}^{F(2)}$. Given $F^{(0)} \propto dt \wedge dr$ and that g_{ab} is static, it is easy to show that the zeroth and first orders of the energy-momentum tensor component T_{nt} vanish, so we have up to second order

$$\int_{\mathrm{bdry}} T_{nt}^{F(2)} = \int_{\mathrm{bdry}} T_{nt} = \int_{\mathrm{bdry}} j^i E_i.$$

---

[13]Here $-l$ is the outer normal to the horizon [48], which coincides with $-\partial_t$ on the horizon.

[14]We use " $\hat{=}$ " to denote an equality that holds on the horizon and " $\tilde{=}$ " to denote an equality that holds on the boundary.

By the standard technique of the Raychaudhuri equation [48], the left hand side of (3.40) can be written as $T_H \delta S$, so we finally obtain the general relation

$$\delta S = \int_{\text{bdry}} \Sigma \hat{\epsilon}_{[d]}.$$

4 Entropy production in holographic superconductors/superfluids

The bulk theory of the original (and simplest) holographic superconductor/superfluid model [54] is a charged scalar field Φ minimally coupled to the Maxwell field A_μ:

$$I = \int \left[-\frac{1}{2} \nabla_\mu A_\nu F^{\mu\nu} - (\nabla_\nu - iA_\nu)\Phi(\nabla^\nu + iA^\nu)\Phi^* - m^2|\Phi|^2 \right] \sqrt{-g} d^{d+1}x, \qquad (4.1)$$

in the fixed Schwarzschild-AdS black brane background

$$ds_{d+1}^2 = \frac{dr^2}{f(r)} - f(r)dt^2 + r^2 d\boldsymbol{x}^2, \qquad f(r) = \frac{r^2}{\ell^2} - \frac{2M}{r^{d-2}}, \qquad (4.2)$$

i.e. without backreaction. The case with backreaction [55] or even more complicated models can also be considered, with the assumption that the equilibrium configuration is always asymptotic AdS.

4.1 Universal form of the holographic entropy production

The entropy production rate (3.2) for various transport processes can be covariantly written as

$$\Sigma(x) = -\frac{1}{T} \sum_A \pi_A \mathcal{L}_\xi \bar{\phi}^A, \qquad (4.3)$$

where capital Latin letters are used to index components of all fields, and

$$\pi_A(x) = \frac{1}{\sqrt{-g}} \frac{\delta I_{\text{bulk}}[\bar{\phi}]}{\delta \bar{\phi}^A(x)}$$

is the canonical conjugate momentum of $\bar{\phi}^A$, or from the field theory point of view the expectation value of the operator dual to ϕ^A. Or in other words, the energy dissipation rate is of the covariant form

$$E_{\text{diss}}(x) = -\sum_A \pi_A \mathcal{L}_\xi \bar{\phi}^A. \qquad (4.4)$$

Note that for the Maxwell field A_μ (or rather its boundary components $\bar{A}_a$), one has

$$j^a \mathcal{L}_\xi \bar{A}_a = j^a(\xi^b D_b \bar{A}_a + \bar{A}_b D_a \xi^b) = j^a \xi^b \bar{F}_{ba} + D_a(j^a \xi^b \bar{A}_b) \qquad (4.5)$$

if the conservation $D_a j^a = 0$ of the current j^a holds, which differs from the genuine Joule heat $j^i E_i$ by a divergence term and thus gives the correct total energy dissipation (or entropy production) upon integration on the whole boundary. However, if there are charged

fields in the system, then generically $D_a j^a \neq 0$, as we see explicitly later. For the moment, let us disregard this complexity as well as the divergence term in (4.5).

For the holographic superconductor/superfluid model (4.1), the entropy production corresponding to the Maxwell field A_μ is just included in (4.3) as discussed in the previous section, but there is also entropy production corresponding to the scalar field Φ. The latter is not a transport process in the usual sense, and so is not a familiar entropy production process. However, we argue that the entropy production corresponding to Φ is still given by the term with ϕ^A taken to be Φ in (4.3), i.e. the formula (4.3) gives the total entropy production rate if $\bar{\phi}^A$ runs over all components of all dynamic fields in the model.[15] In fact, from the thermodynamic point of view on the boundary side, $\Pi \mathcal{L}_\xi \bar{\Phi}$ (with Π the canonical conjugate momentum of $\bar{\Phi}$) does be the rate of work density done on the system. When some work is done, there is always the same amount of energy transformed from one form to another. In general physical systems, the energy is not necessarily transformed to heat. But in our case (in the dual boundary system), the complete dissipation of energy is eventually inevitable, since the system tends to settle down and then there is no macroscopic physical degree of freedom to "contain" the energy. On the other hand, from the bulk point of view, there is the same amount of energy flux going through the boundary into the bulk and eventually being absorbed by the black brane, as will be clear in the following discussion. In order to relate the entropy production (4.3) on the boundary to the entropy increase of the bulk black brane, one expects that (4.4) is the flux of some conserved current. Actually, one may recognize (4.4) as the flux of the Noether current corresponding to the Killing vector field ξ, i.e. the energy flux. Instead of writing down the Noether current corresponding to the Killing vector field, however, here we would like to give a general argument to relate $\pi_A \mathcal{L}_\xi \bar{\phi}^A$ to the flux of the current $T_\nu^\mu \xi^\nu$ used in the previous section, using only the diffeomorphism invariance for a general vector field ξ. Note that we will suppress the field index A hereafter.

Under the diffeomorphism induced by ξ, the invariance of the action (4.1) means

$$
\begin{aligned}
\mathcal{L}_\xi I &= \int_{\text{bulk}} \left(\frac{\delta I}{\delta \phi} \mathcal{L}_\xi \phi + \frac{\delta I}{\delta g_{\mu\nu}} \mathcal{L}_\xi g_{\mu\nu} \right) + \int_{\text{bdry}} (\pi \mathcal{L}_\xi \bar{\phi} - n_\mu \xi^\mu L) \\
&\triangleq \int_{\text{bulk}} \frac{1}{2} T^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu} + \int_{\text{bdry}} (\pi \mathcal{L}_\xi \bar{\phi} - n_\mu \xi^\mu L) \\
&= \int_{\text{bulk}} T^{\mu\nu} \nabla_\mu \xi_\nu + \int_{\text{bdry}} (\pi \mathcal{L}_\xi \bar{\phi} - n_\mu \xi^\mu L) \\
&= \int_{\text{bulk}} \nabla_\mu (T^{\mu\nu} \xi_\nu) + \int_{\text{bdry}} (\pi \mathcal{L}_\xi \bar{\phi} - n_\mu \xi^\mu L) \\
&= \int_{\text{bdry}} (n_\mu T^{\mu\nu} \xi_\nu + \pi \mathcal{L}_\xi \bar{\phi} - n_\mu \xi^\mu L) = 0,
\end{aligned}
\tag{4.6}
$$

where L is the Lagrangian of (4.1), $\triangleq$ stands for equating by the equation of motion $\frac{\delta I}{\delta \phi} = 0$, and the covariant conservation $\nabla_\mu T^{\mu\nu} = 0$ is used by virtue of the diffeomorphism

---

[15]We would like to conjecture that the formula (4.3) gives the total entropy production rate (up to divergence terms) generally, not only for the model considered here.

invariance without considering the boundary.[16] Note that we always require ξ to be tangent to the boundary, so $n_\mu \xi^\mu = 0$. Taking ξ as a local test function on the boundary, one obtains

$$n_\mu T^{\mu b} \xi_b + \pi \mathcal{L}_\xi \bar\phi = 0, \tag{4.7}$$

i.e. the local flux of the current $T^\mu_\nu \xi^\nu$ across the boundary is equal to the energy dissipation rate (4.4). Extending the above argument, we are allowed to investigate more general cases of holographic entropy production and other problems in a systematic way [56].

Then we turn to a more precise version of (4.5) in general cases, using the gauge invariance

$$\begin{aligned}
\mathcal{L}_\Lambda I &= \int_{\text{bulk}} \left(\frac{\delta I}{\delta \Phi} \mathcal{L}_\Lambda \Phi + \frac{\delta I}{\delta \Phi^*} \mathcal{L}_\Lambda \Phi^* + \frac{\delta I}{\delta A_\mu} \mathcal{L}_\Lambda A_\mu \right) + \int_{\text{bdry}} (\Pi \mathcal{L}_\Lambda \bar\Phi + \Pi^* \mathcal{L}_\Lambda \bar\Phi^* + j^a \mathcal{L}_\Lambda \bar A_a) \\
&\triangleq \int_{\text{bdry}} (i\Pi\Lambda\bar\Phi - i\Pi^*\Lambda\bar\Phi^* + j^a D_a \Lambda) = 0
\end{aligned}$$

of the action (4.1). Here Π and Π^* are the canonical conjugate momenta of Φ and Φ^*, respectively. Taking Λ as a local test function on the boundary, one obtains

$$D_a j^a = -i\Pi\bar\Phi + i\Pi^*\bar\Phi^*$$

locally, which means that j^a is conserved with either Dirichlet or Neumann boundary condition for the charged fields Φ and Φ^*. In the usual holographic superconductor/superfluid applications, the Dirichlet boundary condition is taken for the charged fields, so (4.5) holds in this case. Otherwise, (4.5) becomes

$$\begin{aligned}
j^a \mathcal{L}_\xi \bar A_a &= j^a \xi^b \bar F_{ba} + D_a(j^a \xi^b \bar A_b) - \xi^b \bar A_b D_a j^a \\
&= j^a \xi^b \bar F_{ba} + D_a(j^a \xi^b \bar A_b) + \xi^b \bar A_b (i\Pi\bar\Phi - i\Pi^*\bar\Phi^*). \tag{4.8}
\end{aligned}$$

It is easy to see that the last term in the above equation can be combined with $\Pi \mathcal{L}_\xi \bar\Phi + \Pi^* \mathcal{L}_\xi \bar\Phi^*$ to form a gauge invariant extension

$$\Pi(\mathcal{L}_\xi - i\bar A_\xi)\bar\Phi + \Pi^*(\mathcal{L}_\xi + i\bar A_\xi)\bar\Phi^*$$

of the latter, where $\bar A_\xi := \xi^b \bar A_b$. The appearance of this gauge invariant combination is expected as the scalar field contribution in the energy dissipation rate (4.4), since physically the total energy dissipation should be gauge invariant, while the Maxwell field contribution $j^a \xi^b \bar F_{ba}$ in (4.8) is already gauge invariant. The gauge invariance of (4.4) is also confirmed from (4.7) (up to divergence terms), where the energy-momentum tensor $T^{\mu\nu}$ is gauge invariant. It is easy to see, however, that $\bar A_\xi = 0$ is a rather convenient gauge choice, which means that one can simply identify the local flux $n_\mu T^{F\mu b} \xi_b$ of the Maxwell field and that of the scalar field with $-j^a \mathcal{L}_\xi \bar A_a$ and $-(\Pi \mathcal{L}_\xi \bar\Phi + \Pi^* \mathcal{L}_\xi \bar\Phi^*)$, respectively.

---

[16]There are important subtleties for this general argument in the backreacted case, which can be remedied in a full framework of perturbative effective action [56].

4.2 The second order conserved current

In the so-called broken phase, there is a non-vanishing profile of the scalar field Φ in the (equilibrium) holographic superconductor/superfluid configurations. In this case, the perturbations of the scalar, electromagnetic and gravitational fields are coupled to one another, in contrast to the unbroken phase where the perturbation of the scalar field is decoupled. However, even in the broken phase a second order conserved current can also be used to prove the entropy production formula. In section 3.3, for the case with cross-transportation, we have already constructed a general conserved current as

$$J_a = -G_{at}^{(0,2)}[g, q] + q_a^b G_{bt} + \frac{1}{2} G_{bc} q^{cb} (\partial_t)_a - \frac{1}{2} q_b^b G_{at}, \tag{4.9}$$

based on the Bianchi identity. Consider the following perturbation of the Einstein-Maxwell-Scalar system:

$$\begin{aligned}
\tilde{g}_{ab} &= g_{ab} + h_{ab} + q_{ab} + \cdots, \\
F_{ab} &= F_{ab}^{(0)} + F_{ab}^{(1)} + F_{ab}^{(2)} + \cdots, \\
\Phi &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots.
\end{aligned} \tag{4.10}$$

The second order perturbation of the Einstein equation is

$$G^{(0,2)} + \Lambda q_{ab} = 8\pi G (T_{ab}^G + T_{ab}^{F(2)} + T_{ab}^{\Phi(2)}), \tag{4.11}$$

where T_{ab}^G, $T_{ab}^{F(2)}$ have been given in section III and $T_{ab}^{\Phi(2)}$ denotes the second order perturbation of the energy-momentum tensor of Φ. We also consider the Stokes theorem for the current J_a. On the horizon, with the help of gauge, it is

$$\int_H <-l, J> = \int_H \left[-\frac{1}{8\pi G} R_{ll}^{(1,1)} + T_{ll}^{F(2)} + T_{ll}^{\Phi(2)} \right]. \tag{4.12}$$

Using the second order Raychaudhuri equation, it can be shown (see appendix B) that this integral equals $T_H \delta S$. On the boundary, the flux of this current is

$$\int_{\text{bdry}} <n, J> = \int_{\text{bdry}} \left[\frac{1}{8\pi G} R_{tn}^{(1,1)} - T_{tn}^{F(2)} - T_{tn}^{\Phi(2)} \right]. \tag{4.13}$$

The integral of the Ricci part and the Maxwell part have been considered separately in the previous section, so here we only focus on the scalar field part. The energy-momentum tensor of Φ up to second order is

$$\begin{aligned}
T_{ab}^\Phi &= (\partial_a + iA_a)\Phi(\partial_b - iA_b)\Phi^* + \text{c.c.} - g_{ab}(\partial_c + iA_c)\Phi(\partial^c - iA^c)\Phi^* \\
&= [\partial_a + i(A_a^{(0)} + A_a^{(1)} + A_a^{(2)} + \cdots)](\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots) \\
&\quad \times [\partial_b - i(A_b^{(0)} + A_b^{(1)} + A_b^{(2)} + \cdots)](\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots)^* + \text{c.c.} \\
&\quad - \tilde{g}_{ab}[\partial_c + i(A_c^{(0)} + A_c^{(1)} + A_c^{(2)} + \cdots)](\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots) \\
&\quad \times [\partial^c - i(A^{(0)c} + A^{(1)c} + A^{(2)c} + \cdots)](\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots)^*. \tag{4.14}
\end{aligned}$$

On the boundary, the t-n component of T^Φ is

$$
\begin{aligned}
T^\Phi(\partial_t, n) \\
= [\partial_t + i(A_t^{(0)} + A_t^{(1)} + A_t^{(2)} + \cdots)](\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots) \\
\times [\partial_n - i(A_n^{(0)} + A_n^{(1)} + A_n^{(2)} + \cdots)](\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots)^* + \text{c.c.} \quad (4.15)
\end{aligned}
$$

Using the gauge choice $A_r = 0$ and $A_t \stackrel{\cdot}{=} 0$ (as mentioned at the end of the previous subsection), the above equation becomes

$$
\begin{aligned}
T^\Phi(\partial_t, n) \\
\stackrel{\cdot}{=} \partial_t(\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots)\partial_n(\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots)^* + \text{c.c.} \\
\stackrel{\cdot}{=} - (\Pi \mathcal{L}_\xi \bar{\Phi} + \Pi^* \mathcal{L}_\xi \bar{\Phi}^*).
\end{aligned}
$$

Recall that "$\stackrel{\cdot}{=}$" means equating on the boundary, as defined in footnote 14. Since the background bulk space-time is stationary, we have $\partial_t \Phi^{(0)} = 0$. Then we find

$$
\begin{aligned}
T^\Phi(\partial_t, n) \\
\stackrel{\cdot}{=} \partial_t(\Phi^{(1)} + \Phi^{(2)} + \cdots)\partial_n(\Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots)^* + \text{c.c.} \\
\stackrel{\cdot}{=} (\partial_t \Phi^{(1)} \partial_n \Phi^{*(0)} + \partial_t \Phi^{(1)} \partial_n \Phi^{*(1)} + \partial_t \Phi^{(2)} \partial_n \Phi^{*(0)} + \cdots) + \text{c.c.} \\
\stackrel{\cdot}{=} [\partial_t \Phi^{(1)} \partial_n \Phi^{*(1)} + \partial_t(\Phi^{(1)} \partial_n \Phi^{*(0)} + \Phi^{(2)} \partial_n \Phi^{*(0)}) + \cdots] + \text{c.c.} \quad (4.16)
\end{aligned}
$$

So we know up to second order,

$$
\int_{\text{bdry}} T^\Phi(\partial_t, n) = \int_{\text{bdry}} T^{\Phi(2)}(\partial_t, n). \quad (4.17)
$$

Combining the above result with previous results, we know that the entropy production formula also holds for the case of holographic superconductors/superfluids.

4.3 From finite cutoff to the conformal boundary

In standard AdS/CFT, as well as in AdS/CMT or AdS/QCD, the dual field theory is defined at the conformal boundary of the asymptotic AdS space-time. In order to clarify the entropy production in these holographic systems, we need to extend the above discussion to the conformal boundary. Naively, the boundary in our discussion can be put at any place, so we may just take its position r_c tending to the conformal infinity, where the entropy production of the conformal field theory should be viewed as the limit of that of the finite-cutoff theory. However, the so-called holographic renormalization procedure (see, e.g. [33]) is generically required for this limit to obtain finite physical quantities in the dual conformal field theory. This procedure is complicated in general, but it turns out that the total entropy production of the dual boundary theory is not affected by this procedure, as will be explained below.

In fact, in the holographic renormalization, one first introduces a cutoff scale ϵ (with $\epsilon \to 0$ the conformal boundary) for the radial coordinate $z := \frac{1}{r}$, separates the divergent terms of physical quantities (usually the on-shell action) when $\epsilon \to 0$, and then adds a

counter-term I_{CT}, which is purely composed of fields within the cutoff surface $z = \epsilon$, to the action I to render it finite when on-shell. The freedom to add additional finite counter-terms leads to different renormalization schemes. With our notations, the new action

$$\tilde{I} = I + I_{\text{CT}}[\bar{g}_{ab}, \bar{\phi}],$$

so (with $n_\mu \xi^\mu = 0$)

$$\begin{aligned}
\mathcal{L}_\xi \tilde{I} &\triangleq \int_{\text{bdry}} \left(n_\mu T^{\mu b} \xi_b + \pi \mathcal{L}_\xi \bar{\phi} + \frac{\delta I_{\text{CT}}}{\delta \bar{\phi}} \mathcal{L}_\xi \bar{\phi} + \frac{\delta I_{\text{CT}}}{\delta \bar{g}_{ab}} \mathcal{L}_\xi \bar{g}_{ab} \right) \\
&= \int_{\text{bdry}} \left(n_\mu T^{\mu b} \xi_b + \pi \mathcal{L}_\xi \bar{\phi} + \frac{\delta I_{\text{CT}}}{\delta \bar{\phi}} \mathcal{L}_\xi \bar{\phi} + 2 \frac{\delta I_{\text{CT}}}{\delta \bar{g}_{ab}} D_a \xi_b \right) \\
&= \int_{\text{bdry}} \left(n_\mu T^{\mu b} \xi_b + \tilde{\pi} \mathcal{L}_\xi \bar{\phi} + D_a \frac{2 \delta I_{\text{CT}}}{\delta \bar{g}_{ab}} \xi_b \right) = 0,
\end{aligned}$$

where $\tilde{\pi} = \pi + \frac{\delta I_{\text{CT}}}{\delta \bar{\phi}}$ is the renormalized conjugate momentum (or more familiarly the expectation value $\langle O_\phi \rangle_{\text{CFT}}$ of the operator dual to ϕ), and taking ξ as a local test function on the boundary allows us to do the integration by parts freely. Recall here that $\triangleq$ stands for equating by the equation of motion, which has been used in (4.6). Thus we obtain

$$n_\mu T^{\mu b} \xi_b + D_a \frac{2 \delta I_{\text{CT}}}{\delta \bar{g}_{ab}} \xi_b + \tilde{\pi} \mathcal{L}_\xi \bar{\phi} = 0.$$

Recalling that ξ should be eventually taken to be the time-like Killing vector field on the boundary,[17] we know

$$n_\mu T^{\mu b} \xi_b + D_a \left(\frac{2 \delta I_{\text{CT}}}{\delta \bar{g}_{ab}} \xi_b \right) + \tilde{\pi} \mathcal{L}_\xi \bar{\phi} = 0,$$

i.e. the local flux of the current $T^\mu_\nu \xi^\nu$ across the boundary is different from the renormalized energy dissipation rate $-\tilde{\pi} \mathcal{L}_\xi \bar{\phi}$ only by a divergence term on the cutoff surface, which gives the same total entropy production upon integration over the whole boundary.

In the $\epsilon \to 0$ limit, another subtlety is that $\bar{\phi}$ (as well as the the induced boundary metric $\bar{g}_{ab}$ itself) is generically vanishing or divergent. In order to obtain well-defined field quantities on the conformal boundary, one should do a field redefinition, which can be viewed as a (constant) Weyl transformation on the boundary. Concretely, for the holographic superconductor/superfluid model discussed here, the relevant field redefinition is

$$g_{\mu\nu} = z^{-2} \hat{g}_{\mu\nu}, \tag{4.18}$$
$$\Phi = z^{d-\Delta} \hat{\Phi} \tag{4.19}$$

with $\Delta = (d + \sqrt{d^2 + 4m^2 \ell^2})/2$. Replacing the fields in the previous discussion with the above redefined fields, the dual boundary theory now has well-defined field quantities and the entropy production is of the same form (with the renormalized conjugate momentum $\tilde{\pi}$ undergoing a corresponding Weyl transformation, including the possible conformal

---

[17]Note that if the bulk space-time is asymptotic anti-de Sitter, there is always a boundary surface tending to the conformal infinity that becomes the flat Minkowski space-time in the limit.

anomaly), while the flux of the current $T_\nu^\mu \xi^\nu$ has nothing to do with the redefinition. Note that the flux of this conserved current across the horizon leads to the increase of the horizon area, as before. The entropy production formula

$$\Sigma = -\frac{1}{T} \sum_A \tilde{\pi}_A \mathcal{L}_\xi \bar{\phi}^A \tag{4.20}$$

is thus justified, whose integral over the boundary space-time coincides with the entropy increase of the bulk black hole. Remarkably, the conjugate momentum $\tilde{\pi}$ (the expectation value $\langle O_\phi \rangle_{\text{CFT}}$) is scheme-dependent, so is the entropy production rate (4.20), but the total entropy production is scheme-independent.

5 Conclusion and discussion

We have shown, based on a general holographic principle, the validity of the general bulk/boundary correspondence at least at the level of thermodynamics and hydrodynamics, where in particular, a perfect matching between the bulk gravity and boundary system is exactly derived for near-equilibrium entropy production on both sides by resorting to the conserved current. Compared to the standard AdS/CFT, the bulk/boundary correspondence discussed here is more general in the following sense. First, the bulk space-time is not required to be asymptotically AdS but can also be asymptotically flat or dS. Second, the boundary is not required to be located at the conformal infinity (or the asymptotic region). When pushing the cutoff surface to the conformal infinity of the asymptotic AdS space-time, we have shown that the near-equilibrium entropy production in the simplest holographic superconductor/superfluid model can be clearly understood. Furthermore, we also believe that our strategy together with our statements can apply to more general space-time with other asymptotic behaviors such as Lifshitz or Schrödinger, which has received much attention in AdS/CMT.

Our boundary system, by construction, is not necessarily conformal, so the entropy can also be produced by the bulk viscosity on the boundary [57], which has been included in our discussion. It should be noted, however, that the validity of the holographic interpretation of the entropy production without considering bulk viscosity does not rely on the isotropy of the background space. Or in other words, the case with bulk viscosity requires one more constraint on the background (equilibrium) configuration than the case without bulk viscosity. This interesting phenomenon should be investigated further.

An important open problem is the possible holographic interpretation of entropy production in the far-from-equilibrium case. As briefly mentioned in the Introduction, in this case there are both conceptual and technical difficulties for a holographic picture. For the conceptual side, there is no well-established holographic principle or dictionary in the far-from-equilibrium case. A typical example is the debate on whether the entropy from the bulk side corresponds to the apparent horizon or the event horizon [58, 59]. For the technical side, the bulk space-time dual to a far-from-equilibrium boundary system is fully dynamic, which is difficult to explore analytically. However, some interesting analytic works have been done along a similar direction [63, 64]. We hope to come back to this problem later.

Acknowledgments

A Black-hole thermodynamics with the topological charge

In this appendix, we will consider the black-hole thermodynamics with the topological charge ε in the general Lovelock-Maxwell theory, following the formulation proposed in ref. [60]. In this formulation, we assume a standard form

$$ds_{d+1}^2 = \frac{dr^2}{f(r)} - f(r)dt^2 + r^2 d\Omega_{d-1}^{(k)2}$$

of the metric and focus on an "equal-potential" surface $f = $ const, and reinterpret the Lovelock equations of motion as a generalized first law, which gives the traditional black-hole first law in the case $f = 0$ (the horizon).

As described in ref. [60], we can read from the generalized first law the ADM mass

$$M = \frac{d-1}{16\pi}\Omega_{d-1}\left(r^d \sum_j \tilde{\alpha}_j \left(\frac{k-f}{r^2}\right)^j + \frac{Q^2}{r^{d-2}}\right) \tag{A.1}$$

and generalized Wald entropy

$$S = \frac{d-1}{4}\Omega_{d-1}r^{d-1}\sum_j \frac{\tilde{\alpha}_j j}{d-2j+1}\left(\frac{k-f}{r^2}\right)^{j-1}$$

of the maximally symmetric charged black hole, where $\tilde{\alpha}_j$ is proportional to the j-th Lovelock coupling constant. Viewing M as a function of (f, Q, k) or (r, Q, k), differentiation of (A.1) gives

$$dM = -\frac{f'}{4\pi}\frac{\partial M}{\partial f}4\pi dr + \frac{\partial M}{\partial Q}dQ + \frac{d-1}{16\pi}\Omega_{d-1}r^{d-2}\sum_j \tilde{\alpha}_j j\left(\frac{k-f}{r^2}\right)^{j-1}dk. \tag{A.2}$$

However, we need the differentiation of M as a function of (S, Q, k) to obtain the generalized first law with the topological charge, so we should consider the substitution

$$(r, k) \rightarrow (S, k)$$

of variables. In fact, we have[18]

$$\begin{pmatrix} \frac{\partial r}{\partial S} & \frac{\partial k}{\partial S} \\ \frac{\partial r}{\partial k} & \frac{\partial k}{\partial k} \end{pmatrix} = \begin{pmatrix} \frac{\partial S}{\partial r} & \frac{\partial k}{\partial r} \\ \frac{\partial S}{\partial k} & \frac{\partial k}{\partial k} \end{pmatrix}^{-1} = \begin{pmatrix} -4\pi\frac{\partial M}{\partial f} & 0 \\ \frac{\partial S}{\partial k} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -(4\pi\frac{\partial M}{\partial f})^{-1} & 0 \\ (4\pi\frac{\partial M}{\partial f})^{-1}\frac{\partial S}{\partial k} & 1 \end{pmatrix},$$

which leads to

$$dr = \frac{\partial r}{\partial S}dS + \frac{\partial r}{\partial k}dk = -\left(4\pi\frac{\partial M}{\partial f}\right)^{-1}dS + \left(4\pi\frac{\partial M}{\partial f}\right)^{-1}\frac{\partial S}{\partial k}dk.$$

Substitution of the above equation into (A.2) gives

$$\begin{aligned} dM &= \frac{f'}{4\pi}dS + \varphi dQ + \left[\frac{d-1}{16\pi}\Omega_{d-1}r^{d-2}\sum_j \tilde{\alpha}_j j\left(\frac{k-f}{r^2}\right)^{j-1} + \left(4\pi\frac{\partial M}{\partial f}\right)^{-1}\frac{\partial S}{\partial k}\left(-f'\frac{\partial M}{\partial f}\right)\right]dk \\ &= \frac{f'}{4\pi}dS + \varphi dQ + \Omega_{d-1}\left[\frac{d-1}{16\pi}r^{d-2}\sum_j \tilde{\alpha}_j j\left(\frac{k-f}{r^2}\right)^{j-1} - \frac{f'}{4\pi}\frac{d-1}{4}r^{d-3}\sum_j\frac{\tilde{\alpha}_j j(j-1)}{d-2j+1}\left(\frac{k-f}{r^2}\right)^{j-2}\right]dk \\ &= \frac{f'}{4\pi}dS + \varphi dQ + \frac{\Omega_{d-1}k^{\frac{3-d}{2}}}{8\pi}\left[r^{d-2}\sum_j\tilde{\alpha}_j j\left(\frac{k-f}{r^2}\right)^{j-1} - f'r^{d-3}\sum_j\frac{\tilde{\alpha}_j j(j-1)}{d-2j+1}\left(\frac{k-f}{r^2}\right)^{j-2}\right]dk^{\frac{d-1}{2}} \\ &= T_{UV}dS + \varphi dQ + \frac{\varepsilon^{\frac{3-d}{d-1}}S'}{2\pi(d-1)}d\varepsilon \end{aligned}$$

with $T_{UV} = \frac{f'}{4\pi}$ the so-called Unruh-Verlinde temperature,[19] $\varphi = \frac{\partial M}{\partial Q}$ the electric potential and $\varepsilon = k^{\frac{d-1}{2}}$ the topological charge. In comparison to (2.23), the chemical potential μ in (2.10) is just the difference of φ between the horizon and the holographic screen (up to a redshift factor). Thus it is natural to conjecture that the conjugate quantity ς [of the form (2.21) in the Einstein-Maxwell theory] is just the difference of

$$\varpi = \frac{\varepsilon^{\frac{3-d}{d-1}}S'}{2\pi(d-1)}$$

between the horizon and the holographic screen (up to a redshift factor), which is indeed the case in the Einstein-Maxwell theory. But it is still unclear whether ϖ can be viewed as some kind of potential.

B The increase of horizon area from the Raychaudhuri equation

In this appendix, we will show that the entropy increase of the bulk black hole is equal to the total entropy production on the boundary system, i.e.

$$T_H\frac{\delta A_H}{4G} = \int_{\text{bdry}}\Sigma, \tag{B.1}$$

where Σ is defined in eq. (3.5). In section III, we have shown that the flux of the conserved current $-\frac{1}{8\pi G}G_{\mu t}^{(1,1)}$ on the holographic screen is just the entropy production of the

---

[18]Here we need the relation $\frac{\partial S}{\partial r} = -4\pi\frac{\partial M}{\partial f}$ discovered in ref. [60].

[19]This temperature tends to the Hawking temperature of the black hole when the holographic screen approaches the horizon.

boundary system. In this section, we will show that the total flux of the same current on the horizon is equal to $T_H \frac{\delta A_H}{4G}$.

From eq. (3.7), the integral on the horizon is

$$\int_H T^\mu_t \lambda_\mu \tilde{\epsilon}_{[d]} = -\frac{1}{8\pi G} \int_H R^{(1,1)}_{ll} \tilde{\epsilon}_{[d]}. \tag{B.2}$$

The concrete form of $R^{(1,1)}_{ll}$ is

$$
\begin{aligned}
R^{(1,1)}_{ll} \hat{=} &-\partial_u \left[\frac{1}{2} h^{\rho\sigma}(h_{\rho\sigma,u} + h_{u\sigma,\rho} - h_{u\rho,\sigma})\right] + \partial_u \left[\frac{1}{2} h^{u\sigma}(h_{\alpha\sigma,u} + h_{u\sigma,\alpha} - h_{u\alpha,\sigma})\right] \\
&+\partial_i \left[\frac{1}{2} h^{i\sigma}(h_{u\sigma,u} + h_{u\sigma,u} - h_{uu,\sigma})\right] + \partial_r \left[\frac{1}{2} h^{r\sigma}(h_{u\sigma,u} + h_{u\sigma,u} - h_{uu,\sigma})\right] \\
&-\frac{1}{2} g^{\rho\sigma}(h_{u\sigma,\eta} + h_{\eta\sigma,u} - h_{u\eta,\sigma})\frac{1}{2} g^{\eta\lambda}(h_{\rho\lambda,u} + h_{u\lambda,\rho} - h_{u\rho,\lambda}) \\
&-\frac{1}{2} g^{\rho\sigma}(h_{u\sigma,\eta} + h_{\eta\sigma,u} - h_{u\eta,\sigma})\frac{1}{2} h^{\eta\lambda}(g_{\rho\lambda,u} + g_{u\lambda,\rho} - g_{u\rho,\lambda}) \\
&-\frac{1}{2} h^{\rho\sigma}(g_{u\sigma,\eta} + g_{\eta\sigma,u} - g_{u\eta,\sigma})\frac{1}{2} g^{\eta\lambda}(h_{\rho\lambda,u} + h_{u\lambda,\rho} - h_{u\rho,\lambda}) \\
&-\frac{1}{2} h^{\rho\sigma}(g_{u\sigma,\eta} + g_{\eta\sigma,u} - g_{u\eta,\sigma})\frac{1}{2} h^{\eta\lambda}(g_{\rho\lambda,u} + g_{u\lambda,\rho} - g_{u\rho,\lambda}) \\
&-\frac{1}{2} h^{\rho\sigma}(h_{u\sigma,\eta} + h_{\eta\sigma,u} - h_{u\eta,\sigma})\frac{1}{2} g^{\eta\lambda}(g_{\rho\lambda,u} + g_{u\lambda,\rho} - g_{u\rho,\lambda}) \\
&-\frac{1}{2} g^{\rho\sigma}(g_{u\sigma,\eta} + g_{\eta\sigma,u} - g_{u\eta,\sigma})\frac{1}{2} h^{\eta\lambda}(h_{\rho\lambda,u} + h_{u\lambda,\rho} - h_{u\rho,\lambda}) \\
&+\frac{1}{2} g^{\rho\sigma}(h_{\rho\sigma,\eta} + h_{\eta\sigma,\rho} - h_{\eta\rho,\sigma})\frac{1}{2} g^{\eta\lambda}(h_{u\lambda,u} + h_{u\lambda,u} - h_{uu,\lambda}) \\
&+\frac{1}{2} g^{\rho\sigma}(h_{\rho\sigma,\eta} + h_{\eta\sigma,\rho} - h_{\eta\rho,\sigma})\frac{1}{2} h^{\eta\lambda}(g_{u\lambda,u} + g_{u\lambda,u} - g_{uu,\lambda}) \\
&+\frac{1}{2} h^{\rho\sigma}(g_{\rho\sigma,\eta} + g_{\eta\sigma,\rho} - g_{\eta\rho,\sigma})\frac{1}{2} g^{\eta\lambda}(h_{u\lambda,u} + h_{u\lambda,u} - h_{uu,\lambda}) \\
&+\frac{1}{2} h^{\rho\sigma}(g_{\rho\sigma,\eta} + g_{\eta\sigma,\rho} - g_{\eta\rho,\sigma})\frac{1}{2} h^{\eta\lambda}(g_{u\lambda,u} + g_{u\lambda,u} - g_{uu,\lambda}) \\
&+\frac{1}{2} h^{\rho\sigma}(h_{\rho\sigma,\eta} + h_{\eta\sigma,\rho} - h_{\eta\rho,\sigma})\frac{1}{2} g^{\eta\lambda}(g_{u\lambda,u} + g_{u\lambda,u} - g_{uu,\lambda}) \\
&+\frac{1}{2} g^{\rho\sigma}(g_{\rho\sigma,\eta} + g_{\eta\sigma,\rho} - g_{\eta\rho,\sigma})\frac{1}{2} h^{\eta\lambda}(h_{u\lambda,u} + h_{u\lambda,u} - h_{uu,\lambda}). \tag{B.3}
\end{aligned}
$$

Using the gauge in section III, $h^{r\mu} = 0$, it is easy to see $h_{u\mu} \hat{=} 0$. Now we need to calculate all terms in above equation. Recall that "$\hat{=}$" means equating on the horizon, as defined in footnote 14.

Obviously, the first and second terms are zero because we can do time integral to make them to be boundary term. The boundary term vanish because of the zero initial data and the Price law [61, 62].

The third term can be expressed as

$$\partial_i \left[\frac{1}{2} h^{i\sigma}(h_{u\sigma,u} + h_{u\sigma,u} - h_{uu,\sigma})\right]$$

$$\hat{=} \partial_i \left[\frac{1}{2} h^{ir}(-h_{uu,r}) \right]$$

$$\hat{=} 0. \tag{B.4}$$

In the first step, we have used the fact $h_{u\mu} \hat{=} 0$. In the second step, we have used the gauge condition $h^{ir} = 0$.

The fourth term is zero because the gauge condition $h^{r\mu} = 0$.

Let us consider the non-derivative terms. The first non-derivative term in eq. (B.3) is

$$\frac{1}{2} g^{\rho\sigma}(h_{u\sigma,\eta} + h_{\eta\sigma,u} - h_{u\eta,\sigma}) \frac{1}{2} g^{\eta\lambda}(h_{\rho\lambda,u} + h_{u\lambda,\rho} - h_{u\rho,\lambda})$$

$$\hat{=} \frac{1}{2} g^{\rho\sigma} h_{u\sigma,\eta} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u} + \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u} - \frac{1}{2} g^{\rho\sigma} h_{u\eta,\sigma} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u}$$

$$+ \frac{1}{2} g^{\rho\sigma} h_{u\sigma,\eta} \frac{1}{2} g^{\eta\lambda} h_{u\lambda,\rho} + \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{u\lambda,\rho} - \frac{1}{2} g^{\rho\sigma} h_{u\eta,\sigma} \frac{1}{2} g^{\eta\lambda} h_{u\lambda,\rho}$$

$$- \frac{1}{2} g^{\rho\sigma} h_{u\sigma,\eta} \frac{1}{2} g^{\eta\lambda} h_{u\rho,\lambda} - \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{u\rho,\lambda} + \frac{1}{2} g^{\rho\sigma} h_{u\eta,\sigma} \frac{1}{2} g^{\eta\lambda} h_{u\rho,\lambda}$$

$$\hat{=} \frac{1}{2} g^{\rho\sigma} h_{u\sigma,r} \frac{1}{2} g^{r\lambda} h_{\rho\lambda,u} + \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u} - \frac{1}{2} g^{\rho r} h_{u\eta,r} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u}$$

$$+ \frac{1}{2} g^{\rho\sigma} h_{u\sigma,r} \frac{1}{2} g^{r\lambda} h_{u\lambda,\rho} + \frac{1}{2} g^{r\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{u\lambda,r} - \frac{1}{2} g^{r\sigma} h_{u\eta,\sigma} \frac{1}{2} g^{\eta\lambda} h_{u\lambda,r}$$

$$- \frac{1}{2} g^{\rho\sigma} h_{u\sigma,r} \frac{1}{2} g^{r\lambda} h_{u\rho,\lambda} - \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta r} h_{u\rho,r} + \frac{1}{2} g^{\rho\sigma} h_{u\eta,\sigma} \frac{1}{2} g^{\eta r} h_{u\rho,r}$$

$$\hat{=} \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u} + \frac{1}{2} g^{ru} h_{uu,r} \frac{1}{2} g^{ru} h_{uu,r} + \frac{1}{2} g^{ur} h_{uu,r} \frac{1}{2} g^{ur} h_{uu,r}$$

$$\hat{=} \frac{1}{2} g^{\rho\sigma} h_{\eta\sigma,u} \frac{1}{2} g^{\eta\lambda} h_{\rho\lambda,u} + \frac{1}{2} (h_{uu,r})^2$$

$$\hat{=} \frac{1}{2} g^{ji} h_{qi,u} \frac{1}{2} g^{qk} h_{jk,u} + \frac{1}{2} (h_{uu,r})^2$$

$$\hat{=} \frac{1}{d-1} (\theta^{(1)})^2 + (\sigma^{(1)})^2 + \frac{1}{2} (h_{uu,r})^2, \tag{B.5}$$

where Greek indices run from 0 to d and Latin indices run from 1 to $d-1$. In the second line, we have used $h_{u\mu} \hat{=} 0$. In the following three steps, we have used the gauge $h^{r\mu} = 0$, $h_{u\mu} \hat{=} 0$, $g_{ur} \hat{=} g^{ur} \hat{=} 1$, $g_{uu} \hat{=} g_{ui} \hat{=} g_{rr} \hat{=} g_{ri} \hat{=} 0$ and $g^{uu} \hat{=} g^{ui} \hat{=} g^{rr} \hat{=} g^{ri} \hat{=} 0$. In the last step, we have used the definition $\frac{1}{2} h_{ij,u} =: \frac{1}{d-1} \theta^{(1)} g_{ij} + \sigma_{ij}^{(1)}$.

Using similar analysis, one can get

$$R_{ll}^{(1,1)} \hat{=} -\frac{1}{d-1} (\theta^{(1)})^2 - (\sigma^{(1)})^2 + \frac{1}{2} (h_{uu,r})^2 - \theta^{(1)} h_{uu,r} - \partial_u \left[\frac{1}{4} h^{ij} h_{ij} g_{uu,r} \right]$$

$$- \partial_u \left[\frac{1}{2} h^{\rho\sigma} (h_{\rho\sigma,u} + h_{u\sigma,\rho} - h_{u\rho,\sigma}) \right]$$

$$+ \partial_u \left[\frac{1}{2} h^{u\sigma} (h_{\alpha\sigma,u} + h_{u\sigma,\alpha} - h_{u\alpha,\sigma}) \right]. \tag{B.6}$$

In order to consider the value of $R_{ll}^{(1,1)}$ on the horizon, we need the value of first order perturbation of horizon expansion $\theta^{(1)}$. To do this, we need to consider the perturbation of Raychaudhuri equation. The first order perturbation of this equation is

$$\dot{\theta}^{(1)} = \kappa^{(0)} \theta^{(1)} - \theta^{(0)} \theta^{(1)} - 2\sigma^{(0)} \cdot \sigma^{(1)} - R_{ll}^{(1,1)}$$

$$= \kappa^{(0)}\theta^{(1)}, \tag{B.7}$$

where we have used the fact $\theta^{(0)} \hat{=} \sigma^{(0)} \hat{=} 0$ and the linearized vacuum Einstein equation $R_{ll}^{(1,1)} = 0$. For the non-vacuum case, one needs to consider the concrete form of $R_{ll}^{(1,1)}$. For the Einstein-Maxwell-Scalar system, the linearized Einstein equation implies

$$R_{ll}^{(1,1)} = 8\pi G F_{li}^{(0)} F_l^{(1)i} + 8\pi G \nabla_l \Phi^{(0)} \nabla_l \Phi^{(1)}. \tag{B.8}$$

Because the back ground is a stationary black hole, one can show that $F_{li}^{(0)} \hat{=} \nabla_l \Phi^{(0)} \hat{=} 0$ based on the zeroth order Raychaudhuri equation. This means that $R_{ll}^{(1,1)} \hat{=} 0$ also holds for the Einstein-Maxwell-Scalar system, so as eq. (B.7). Eq. (B.7) is an ordinary differential equation on the horizon. With the zero initial data, one can get $\theta^{(1)} \hat{=} 0$, so eq. (B.6) becomes

$$\begin{aligned}
R_{ll}^{(1,1)} \hat{=} &-(\sigma^{(1)})^2 - \frac{1}{2}(h_{uu,r})^2 - \partial_u \left[\frac{1}{4}h^{ij}h_{ij}g_{uu,r}\right] \\
&-\partial_u \left[\frac{1}{2}h^{\rho\sigma}(h_{\rho\sigma,u} + h_{u\sigma,\rho} - h_{u\rho,\sigma})\right] \\
&+\partial_u \left[\frac{1}{2}h^{u\sigma}(h_{\alpha\sigma,u} + h_{u\sigma,\alpha} - h_{u\alpha,\sigma})\right].
\end{aligned} \tag{B.9}$$

Taking a suitable coordinate transformation, one can show that $(h_{uu,r})^2$ vanishes on the horizon. We have shown that the first order perturbation $\theta^{(1)}$ of the horizon expansion vanishes. This means that the variation of horizon area is contributed by the second order perturbation. The second order Raychaudhuri equation is

$$\begin{aligned}
\dot{\theta}^{(2)} - \kappa^{(0)}\theta^{(2)} &= -\sigma_{ij}^{(1)}\sigma^{(1)ij} - R_{ll}^{(2)} \\
&= -\sigma_{ij}^{(1)}\sigma^{(1)ij} - 8\pi G T_{ll}^{(2)},
\end{aligned} \tag{B.10}$$

where the second order Einstein equation is used. Based on Wald's standard technique [47, 48], we know that

$$\int_H \dot{\theta}^{(2)} - \kappa^{(0)}\theta^{(2)} = -\kappa^{(0)}\delta A_H = -\int_H [(\sigma^{(1)})^2 + 8\pi G T_{ll}^{(2)}]. \tag{B.11}$$

Combining the above equation with eq. (B.9), it is easy to see that up to second order perturbation

$$\begin{aligned}
T_H \delta S &= \frac{1}{8\pi G} \int_H [-R_{ll}^{(1,1)} + 8\pi G T_{ll}^{(2)}] \\
&= \frac{1}{8\pi G} \int_{\text{bdry}} [-R_{nt}^{(1,1)} + 8\pi G T_{nt}^{(2)}] \\
&= \int_{\text{bdry}} \Sigma.
\end{aligned} \tag{B.12}$$

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