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Quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_s^5)$

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Abstract: We present the results of the first complete analytic calculation of the quark mass and field anomalous dimensions to $\mathcal{O}(\alpha_s^5)$ in QCD.

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Contents

1 Introduction

The quark masses depend on a renormalization scale. The dependence is usually referred to as "running" and is governed by the quark mass anomalous dimension, γ_m, defined as:

$$\mu^2 \frac{d}{d\mu^2} m|_{g^0,m^0} = m\gamma_m(a_s) \equiv -m \sum_{i\geq 0} \gamma_i \, a_s^{i+1}, \tag{1.1}$$

where $a_s = \alpha_s/\pi = g^2/(4\pi^2)$, g is the renormalized strong coupling constant and μ is the normalization scale in the customarily used $\overline{\text{MS}}$ renormalization scheme. Up to and including four loop level the anomalous dimension is known since long [1–5]. In this paper we will describe the results of calculation of γ_m and a related quantity — the quark field anomalous dimension — in the five-loop order.

The evaluation of the quark mass anomalous dimension with five-loop accuracy has important implications. The Higgs boson decay rate into charm and bottom quarks is proportional to the square of the respective quark mass at the scale of m_H and the uncertainty from the presently unknown 5-loop terms in the running of the quark mass is of order 10^{-3}. This is comparable to the precision advocated for experiments e.g. at TLEP [6]. Similarly, the issue of Yukawa unification is affected by precise predictions for the anomalous quark mass dimension.

The paper is organized as follows. The next section deals with the overall set-up of the calculations. Then we present our results (section 3), and a brief discussion (section 4) as well as a couple of selected applications (section 5). Our short conclusions are given in section 6.

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2 Technical preliminaries

To calculate γ_m one needs to find the so-called quark mass renormalization constant, Z_m, which is defined as the ratio of the bare and renormalized quark masses, viz.

$$Z_m = \frac{m^0}{m} = 1 + \sum_{i,j}^{0<j\leq i} (Z_m)_{ij} \frac{a_s^i}{\epsilon^j}. \tag{2.1}$$

Within the $\overline{\text{MS}}$ scheme [7, 8] the coefficients $(Z_m)_{ij}$ are just numbers [9]; $\epsilon \equiv 2 - D/2$ and D stands for the space-time dimension. Combining eqs. (1.1), (2.1) and using the RG-invariance of m^0, one arrives at the following formula for γ_m:

$$\gamma_m = \sum_{i\geq 0} (Z_m)_{i1}\, i\, a_s^i. \tag{2.2}$$

To find Z_m one should compute the vector and scalar parts of the quark self-energy $\Sigma_V(p^2)$ and $\Sigma_S(p^2)$. In our convention, the bare quark propagator is proportional to $\left[\not p\left(1+\Sigma_V^0(p^2)\right) - m_q^0\left(1-\Sigma_S^0(p^2)\right)\right]^{-1}$. Requiring the finiteness of the renormalized quark propagator and keeping only massless and terms linear in m_q, one arrives at the following recursive equations to find Z_m

$$Z_m Z_2 = 1 + K_\epsilon\left\{Z_m Z_2 \Sigma_S^0(p^2)\right\}, \quad Z_2 = 1 - K_\epsilon\left\{Z_2 \Sigma_V^0(p^2)\right\}, \tag{2.3}$$

where $K_\epsilon\{f(\epsilon)\}$ stands for the singular part of the Laurent expansion of $f(\epsilon)$ in ϵ near $\epsilon = 0$ and Z_2 is the quark wave function renormalization constant. Eqs. (2.3) express Z_m through massless propagator-type (that is dependent on one external momentum only) Feynman integrals (FI), denoted as *p-integrals* below.

Eqs. (2.3) require the calculation of a large number[1] of the *five*-loop p-integrals to find Z_m and Z_2 to $\mathcal{O}(\alpha_s^5)$.

At present there exists no direct way to analytically evaluate five-loop p-integrals. However, according to (2.1) for a given five-loop p-integral we need to know only its *pole* part in ϵ in the limit of $\epsilon \to 0$. A proper use of this fact can significantly simplify our task. The corresponding method — so-called Infrared Rearrangement (IRR)—first suggested in [11] and elaborated further in [12–14] allows to effectively decrease number of loops to be computed by one.[2] In its initial version IRR was not really universal; it was not applicable in some (though rather rare) cases of complicated FI's. The problem was solved by elaborating a special technique of subtraction of IR divergences — the R^*-operation [15, 16]. This technique succeeds in expressing the UV counterterm of every L-loop Feynman integral in terms of divergent and finite parts of some (L-1)-loop massless propagators.

In our case $L = 5$ and, using IRR, one arrives at around 10^5 four-loop p-integrals. These can, subsequently, be reduced to 28 four-loop master-integrals, which are known analytically, including their finite parts, from [17, 18] as well as numerically from [19].

[1] We have used QGRAF [10] to produce around 10^5 FI's contributing to the quark self-energy at $\mathcal{O}(\alpha_s^5)$.

[2] With the price that resulting one-loop-less p-integrals should be evaluated up to and *including* their constant part in the small ϵ-expansion.

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We need, thus, to compute around 10^5 p-integrals. Their singular parts, in turn, can be algebraically reduced to only 28 master 4-loop p-integrals. The reduction is based on evaluating sufficiently many terms of the $1/D$ expansion [20] of the corresponding coefficient functions [21].

All our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of eight-cores Xeon computers using parallel MPI-based [22] as well as thread-based [23] versions of FORM [24].

3 Results

Our result for the anomalous dimension

$$\gamma_m = -\sum_{i=0}^{\infty} (\gamma_m)_i \, a_s^{i+1}$$

reads:

$$(\gamma_m)_0 = 1, \qquad (\gamma_m)_1 = \frac{1}{16}\left\{ \frac{202}{3} + n_f\left[-\frac{20}{9}\right]\right\}, \tag{3.1}$$

$$(\gamma_m)_2 = \frac{1}{64}\left\{ 1249 + n_f\left[-\frac{2216}{27} - \frac{160}{3}\zeta_3\right] + n_f^2\left[-\frac{140}{81}\right]\right\}, \tag{3.2}$$

$$\begin{aligned}(\gamma_m)_3 = \frac{1}{256}\bigg\{ &\frac{4603055}{162} + \frac{135680}{27}\zeta_3 - 8800\,\zeta_5 \\ &+ n_f\left[-\frac{91723}{27} - \frac{34192}{9}\zeta_3 + 880\,\zeta_4 + \frac{18400}{9}\zeta_5\right] \\ &+ n_f^2\left[\frac{5242}{243} + \frac{800}{9}\zeta_3 - \frac{160}{3}\zeta_4\right] + n_f^3\left[-\frac{332}{243} + \frac{64}{27}\zeta_3\right]\bigg\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}(\gamma_m)_4 = \frac{1}{4^5}\bigg\{ &\frac{99512327}{162} + \frac{46402466}{243}\zeta_3 + 96800\,\zeta_3^2 - \frac{698126}{9}\zeta_4 \\ &- \frac{231757160}{243}\zeta_5 + 242000\,\zeta_6 + 412720\,\zeta_7 \\ &+ n_f\bigg[-\frac{150736283}{1458} - \frac{12538016}{81}\zeta_3 - \frac{75680}{9}\zeta_3^2 + \frac{2038742}{27}\zeta_4 \\ &\quad + \frac{49876180}{243}\zeta_5 - \frac{638000}{9}\zeta_6 - \frac{1820000}{27}\zeta_7\bigg] \\ &+ n_f^2\left[\frac{1320742}{729} + \frac{2010824}{243}\zeta_3 + \frac{46400}{27}\zeta_3^2 - \frac{166300}{27}\zeta_4 - \frac{264040}{81}\zeta_5 + \frac{92000}{27}\zeta_6\right] \\ &+ n_f^3\left[\frac{91865}{1458} + \frac{12848}{81}\zeta_3 + \frac{448}{9}\zeta_4 - \frac{5120}{27}\zeta_5\right] + n_f^4\left[-\frac{260}{243} - \frac{320}{243}\zeta_3 + \frac{64}{27}\zeta_4\right]\bigg\}.\end{aligned} \tag{3.4}$$

Here ζ is the Riemann zeta-function ($\zeta_3 = 1.202056903\ldots$, $\zeta_4 = \pi^4/90$, $\zeta_5 = 1.036927755\ldots$, $\zeta_6 = 1.017343062\ldots$ and $\zeta_7 = 1.008349277\ldots$). Note that in four-loop order we exactly[3] reproduce well-known results obtained in [4, 5]. The n_f^3 and n_f^4

[3] This agreement can be also considered as an important check of all our setup which is completely different from the ones utilized at the four-loop calculations.

terms in (3.4) are in full agreement with the results derived previously on the basis of the $1/n_f$ method in [25–27].

For completeness we present below the result for the quark field anomalous dimension $\gamma_2 = -\sum_{i=0}^{\infty} (\gamma_2)_i \, a_s^{i+1}$:

$$
\begin{aligned}
(\gamma_2)_4 = \frac{1}{4^5}\Bigg\{ & \frac{2798900231}{7776} + \frac{17969627}{864}\zeta_3 + \frac{13214911}{648}\,\zeta_3^2 + \frac{16730765}{864}\zeta_4 - \frac{832567417}{3888}\zeta_5 \\
& + \frac{40109575}{1296}\,\zeta_6 + \frac{124597529}{1728}\,\zeta_7 \\
& + n_f \Bigg[-\frac{861347053}{11664} - \frac{274621439}{11664}\zeta_3 + \frac{1960337}{972}\,\zeta_3^2 + \frac{465395}{1296}\zeta_4 \\
& + \frac{22169149}{5832}\zeta_5 + \frac{1278475}{1944}\,\zeta_6 + \frac{3443909}{216}\,\zeta_7 \Bigg] \\
& + n_f^2 \Bigg[\frac{37300355}{11664} + \frac{1349831}{486}\zeta_3 - \frac{128}{9}\,\zeta_3^2 - \frac{27415}{54}\zeta_4 - \frac{12079}{27}\zeta_5 - \frac{800}{9}\,\zeta_6 - \frac{1323}{2}\,\zeta_7 \Bigg] \\
& + n_f^3 \Bigg[-\frac{114049}{8748} - \frac{1396}{81}\zeta_3 + \frac{208}{9}\zeta_4 \Bigg] + n_f^4 \Bigg[\frac{332}{729} - \frac{64}{81}\zeta_3 \Bigg] \Bigg\}. \qquad (3.5)
\end{aligned}
$$

The above result is presented for the Feynman gauge; the coefficients $(\gamma_2)_i$ with $i \leq 3$ can be found in [28] (for the case of a general covariant gauge and SU(N) gauge group).

4 Discussion

In numerical form γ_m reads

$$
\begin{aligned}
\gamma_m = \; & -a_s - a_s^2\,(4.20833 - 0.138889 n_f) \\
& - a_s^3\left(19.5156 - 2.28412 n_f - 0.0270062 n_f^2\right) \\
& - a_s^4\left(98.9434 - 19.1075 n_f + 0.276163 n_f^2 + 0.00579322 n_f^3\right) \\
& - a_s^5\left(559.7069 - 143.6864\, n_f + 7.4824\, n_f^2 + 0.1083\, n_f^3 - 0.000085359\, n_f^4\right) \qquad (4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_m \underset{n_f=3}{=\!=} \; & -a_s - 3.79167\, a_s^2 - 12.4202\, a_s^3 - 44.2629\, a_s^4 - 198.907\, a_s^5, \\
\gamma_m \underset{n_f=4}{=\!=} \; & -a_s - 3.65278\, a_s^2 - 9.94704\, a_s^3 - 27.3029\, a_s^4 - 111.59\, a_s^5, \\
\gamma_m \underset{n_f=5}{=\!=} \; & -a_s - 3.51389\, a_s^2 - 7.41986\, a_s^3 - 11.0343\, a_s^4 - 41.8205\, a_s^5, \\
\gamma_m \underset{n_f=6}{=\!=} \; & -a_s - 3.37500\, a_s^2 - 4.83867\, a_s^3 + 4.50817\, a_s^4 + 9.76016\, a_s^5. \qquad (4.2)
\end{aligned}
$$

Note that significant cancellations between n_f^0 and n_f^1 terms for the values of n_f around 3 or so persist also at five-loop order. As a result we observe a moderate growth of the series in a_s appearing in the quark mass anomalous dimension at various values of active quark flavours (recall that even for scales as small as 2 GeV $a_s \equiv \frac{\alpha_s}{\pi} \approx 0.1$).

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n_f	3	4	5	6
$(\gamma_m)_4^{\rm exact}$	198.899	111.579	41.807	-9.777
$(\gamma_m)_4^{\rm APAP}$ [29]	162.0	67.1	-13.7	-80.0
$(\gamma_m)_4^{\rm APAP}$ [30]	163.0	75.2	12.6	12.2
$(\gamma_m)_4^{\rm APAP}$ [31]	164.0	71.6	-4.8	-64.6

Table 1. The exact results for $(\gamma_m)_4$ together with the predictions made with the help of the original APAP method and its two somewhat modified versions.

Similar behavior shows up for γ_2:

$$\begin{aligned}\gamma_2 = \; & -0.33333 a_s - a_s^2\,(-1.9583 + 0.08333\, n_f) \\ & - a_s^3\,(-10.3370 + 1.0877\, n_f - 0.01157\, n_f^2) \\ & - a_s^4\,(-53.0220 + 10.1090\, n_f - 0.27703\, n_f^2 - 0.0023\, n_f^3) \\ & - a_s^4\,(-310.0700 + 76.3260\, n_f - 4.6339\, n_f^2 + 0.0085\, n_f^3 + 0.00048\, n_f^4)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\gamma_2 &\underset{n_f=3}{=\!=} -0.33333\, a_s - 1.7083\, a_s^2 - 7.1779\, a_s^3 - 25.2480\, a_s^4 - 122.5300\, a_s^5, \\ \gamma_2 &\underset{n_f=4}{=\!=} -0.33333\, a_s - 1.6250\, a_s^2 - 6.1712\, a_s^3 - 17.1610\, a_s^4 - 78.2430\, a_s^5, \\ \gamma_2 &\underset{n_f=5}{=\!=} -0.33333\, a_s - 1.5417\, a_s^2 - 5.1877\, a_s^3 - 9.6824\, a_s^4 - 42.9240\, a_s^5, \\ \gamma_2 &\underset{n_f=6}{=\!=} -0.33333\, a_s - 1.4583\, a_s^2 - 4.2274\, a_s^3 - 2.8251\, a_s^4 - 16.4710\, a_s^5.\end{aligned} \tag{4.4}$$

It is instructive to compare our numerical result for $(\gamma_m)_4$

$$(\gamma_m)_4 = 559.71 - 143.6\, n_f + 7.4824\, n_f^2 + 0.1083\, n_f^3 - 0.00008535\, n_f^4 \tag{4.5}$$

with a 15 years old prediction based on the "Asymptotic Páde Approximants" (APAP) method [29] (the n_f^4 term below was used as the input)

$$(\gamma_m)_4^{\rm APAP} = 530 - 143\, n_f + 6.67\, n_f^2 + 0.037\, n_f^3 - 0.00008535\, n_f^4 \;. \tag{4.6}$$

Unfortunately, this impressively good agreement does *not* survive for fixed values of n_f due to severe cancellations between different powers of n_f as one can see from the table 1.

The solution of eq. (1.1) reads:

$$\frac{m(\mu)}{m(\mu_0)} = \frac{c(a_s(\mu))}{c(a_s(\mu_0))}, \quad c(x) = \exp\left\{\int dx' \frac{\gamma_m(x')}{\beta(x')}\right\}, \tag{4.7}$$

$$\begin{aligned}c(x) = \; & (x)^{\bar\gamma_0}\left\{1 + d_1 x + (d_1^2/2 + d_2)\, x^2 + (d_1^3/6 + d_1 d_2 + d_3)\, x^3 \right. \\ & \left. + (d_1^4/24 + d_1^2 d_2/2 + d_2^2/2 + d_1 d_3 + d_4)\, x^4 + \mathcal{O}(x^5)\right\},\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
d_1 &= -\bar\beta_1\,\bar\gamma_0 + \bar\gamma_1, &(4.9)\\
d_2 &= \bar\beta_1^2\,\bar\gamma_0/2 - \bar\beta_2\,\bar\gamma_0/2 - \bar\beta_1\,\bar\gamma_1/2 + \bar\gamma_2/2, &(4.10)\\
d_3 &= -\bar\beta_1^3\,\bar\gamma_0/3 + 2\,\bar\beta_1\,\bar\beta_2\,\bar\gamma_0/3 - \bar\beta_3\,\bar\gamma_0/3 + \bar\beta_1^2\,\bar\gamma_1/3 - \bar\beta_2\,\bar\gamma_1/3 - \bar\beta_1\,\bar\gamma_2/3 + \bar\gamma_3/3, &(4.11)\\
d_4 &= \bar\beta_1^4\,\bar\gamma_0/4 - 3\,\bar\beta_1^2\,\bar\beta_2\,\bar\gamma_0/4 + \bar\beta_2^2\,\bar\gamma_0/4 + \bar\beta_1\,\bar\beta_3\,\bar\gamma_0/2 - \bar\beta_4\,\bar\gamma_0/4 - \bar\beta_1^3\,\bar\gamma_1/4\\
&\quad + \bar\beta_1\,\bar\beta_2\,\bar\gamma_1/2 - \bar\beta_3\,\bar\gamma_1/4 + \bar\beta_1^2\,\bar\gamma_2/4 - \bar\beta_2\,\bar\gamma_2/4 - \bar\beta_1\,\bar\gamma_3/4 + \bar\gamma_4/4. &(4.12)
\end{aligned}
$$

Here $\bar\gamma_i = (\gamma_m)_i/\beta_0$, $\bar\beta_i = \beta_i/\beta_0$ and

$$
\beta(a_s) = -\sum_{i\geq 0} \beta_i\, a_s^{i+2} = -\beta_0 \left\{ \sum_{i\geq 0} \bar\beta_i\, a_s^{i+2} \right\}
$$

is the QCD β-function. Unfortunately, the coefficient d_4 in eq. (4.12) does depend on the yet unknown *five-loop* coefficient β_4 (up to four loops the β-function is known from [14, 32–39]).

Numerically, the c-function reads:

$$
c(x) \underset{n_f=3}{=\!=} x^{4/9}\, c_s(x),\;\; c(x) \underset{n_f=4}{=\!=} x^{12/25}\, c_c(x),\;\; c(x) \underset{n_f=5}{=\!=} x^{12/23}\, c_b(x),\;\; c(x) \underset{n_f=6}{=\!=} x^{4/7}\, c_t(x),
$$

with

$$
\begin{aligned}
c_s(x) &= 1 + 0.8950\,x + 1.3714\,x^2 + 1.9517\,x^3 + (15.6982 - 0.11111\,\bar\beta_4)\,x^4,\\
c_c(x) &= 1 + 1.0141\,x + 1.3892\,x^2 + 1.0905\,x^3 + (9.1104 - 0.12000\,\bar\beta_4)\,x^4,\\
c_b(x) &= 1 + 1.1755\,x + 1.5007\,x^2 + 0.17248\,x^3 + (2.69277 - 0.13046\,\bar\beta_4)\,x^4,\\
c_t(x) &= 1 + 1.3980\,x + 1.7935\,x^2 - 0.68343\,x^3 + (-3.5130 - 0.14286\,\bar\beta_4)\,x^4. &(4.13)
\end{aligned}
$$

5 Applications

5.1 RGI mass

Eq. (4.7) naturally leads to an important concept: the RGI mass

$$
m^{\mathrm{RGI}} \equiv m(\mu_0)/c(a_s(\mu_0)), \tag{5.1}
$$

which is often used in the context of lattice calculations. The mass is μ and *scheme* independent; in *any* (mass-independent) scheme

$$
\lim_{\mu\to\infty} a_s(\mu)^{-\bar\gamma_0}\; m(\mu) = m^{\mathrm{RGI}}.
$$

The function $c_s(x)$ is used, e.g, by the *ALPHA* lattice collaboration to find the $\overline{\mathrm{MS}}$ mass of the strange quark at a lower scale, say, $m_s(2\ \mathrm{GeV})$ from the m_s^{RGI} mass determined from lattice simulations (see, e.g. [40]). For example, setting $a_s(\mu = 2\,\mathrm{GeV}) = \frac{\alpha_s(\mu)}{\pi} = 0.1$, we arrive at ($h$ counts loops):

$$
\begin{aligned}
m_s(2\,\mathrm{GeV}) &= m_s^{\mathrm{RGI}}\,(a_s(2\,\mathrm{GeV}))^{\frac{4}{9}}\, \Big(1 + 0.0895\,h^2 + 0.0137\,h^3 + 0.00195\,h^4\\
&\qquad\qquad + (0.00157 - 0.000011\,\overline{\beta}_4)\,h^5\Big) &(5.2)
\end{aligned}
$$

In order to have an idea of effects due the five-loop term in (5.2) one should make a guess about $\bar{\beta}_4$. By inspecting lower orders in

$$\beta(n_f = 3) = -\left(\frac{4}{9}\right)\left(a_s + 1.777\, a_s^2 + 4.4711\, a_s^3 + 20.990\, a_s^4 + \bar{\beta}_4\, a_s^5\right)$$

one can assume a natural estimate of $\overline{\beta}_4$ as laying in the interval $50 - 100$. With this choice we conclude that the (apparent) convergence of the above series is quite good even at a rather small energy scale of 2 GeV.

On the other hand, the authors of [30] estimate $\bar{\beta}_4$ in the $n_f = 3$ QCD as large as -850! With such a huge and negative value of $\bar{\beta}_4$ the five loop term in (5.2) would amount to 0.01092 and, thus, would significantly exceed the four-loop contribution (0.00195).

5.2 Higgs decay into quarks

The decay width of the Higgs boson into a pair of quarks can be written in the form

$$\Gamma(H \to \bar{f} f) = \frac{G_F\, M_H}{4\sqrt{2}\pi}\, m_f^2(\mu)\, R^S(s = M_H^2, \mu) \tag{5.3}$$

where μ is the normalization scale and R^S is the spectral density of the scalar correlator, known to α_s^4 from [41]

$$\begin{aligned} R^S(s = M_H^2, \mu = M_H) &= 1 + 5.667\, a_s + 29.147\, a_s^2 + 41.758\, a_s^3 - 825.7\, a_s^4 \\ &= 1 + 0.2041 + 0.0379 + 0.0020 - 0.00140 \end{aligned} \tag{5.4}$$

where we set $a_s = \alpha_s/\pi = 0.0360$ (for the Higgs mass value $M_H = 125$ GeV and $\alpha_s(M_Z) = 0.118$).

Expression (5.3) depends on two phenomenological parameters, namely, $\alpha_s(M_H)$ and the quark running mass m_q. In what follows we consider, for definiteness, the dominant decay mode $H \to \bar{b}b$. To avoid the appearance of large logarithms of the type $\ln \mu^2/M_H^2$ the parameter μ is customarily chosen to be around M_H. However, the starting value of m_b is usually determined at a much smaller scale (typically around 5-10 GeV [42]). The evolution of $m_b(\mu)$ from a lower scale to $\mu = M_h$ is described by a corresponding RG equation which is completely fixed by the quark mass anomalous dimension $\gamma(\alpha_s)$ and the QCD beta function $\beta(\alpha_s)$ (for QCD with $n_f = 5$). In order to match the $\mathcal{O}(\alpha_s^4)$ accuracy of (5.4) one should know *both* RG functions β and γ_m in the five-loop approximation. Let us proceed, assuming conservatively that $0 \leq \bar{\beta}_4^{n_f=5} \leq 200$.

The value of $m_b(\mu = M_H)$ is to be obtained with RG running from $m_b(\mu = 10\,\text{GeV})$ and, thus, depends on β and γ_m. Using the Mathematica package RunDec[4] [43] and eq. (4.13) we find for the shift from the five-loop term

$$\frac{\delta m_b^2(M_H)}{m_b^2(M_H)} = -1.3 \cdot 10^{-4}(\bar{\beta}_4 = 0)| - 4.3 \cdot 10^{-4}(\bar{\beta}_4 = 100)| - 7.3 \cdot 10^{-4}(\bar{\beta}_4 = 200)$$

[4] We have extended the package by including the five-loop effects to the running of α_s and quark masses.

If we set $\mu = M_H$, then the combined effect of $\mathcal{O}(\alpha_s^4)$ terms as coming from the five-loop running and four-loop contribution to R^S on

$$\Gamma(H \to \bar{b}b) = \frac{G_F \, M_H}{4\sqrt{2}\pi} \, m_f^2(M_H) \, R^S(s = M_H^2, M_H) \tag{5.5}$$

is around -2‰ (for $\bar{\beta}_4 = 100$). This should be contrasted to the parametric uncertainties coming from the input parameters $\alpha_s(M_Z) = 0.1185(6)$ [44] and $m_b(m_b) = 4.169(8)$ GeV [45] which correspond to ± 1‰ and ± 4‰ respectively.

We conclude, that the $\mathcal{O}(\alpha_s^4)$ terms in (5.4), (5.5)) are of no phenomenological relevancy at present. But, the situation could be different if the project of TLEP [6] is implemented. For instance, the uncertainty in $\alpha_s(M_Z)$ could be reduced to ±2‰ and Higgs boson branching ratios with precisions in the permille range are advertised.

6 Conclusions

We have analytically computed the anomalous dimensions of the quark mass γ_m and field γ_2 in the five loop approximation. The self-consistent description of the quark mass evolution at five loop requires the knowledge of the QCD β-function to the same number of loops. The corresponding, significantly more complicated calculation is under consideration.

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