

## Calabi-Yau threefolds with large $h^{2,1}$

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**ABSTRACT:** We carry out a systematic analysis of Calabi-Yau threefolds that are elliptically fibered with section (“EFS”) and have a large Hodge number  $h^{2,1}$ . EFS Calabi-Yau threefolds live in a single connected space, with regions of moduli space associated with different topologies connected through transitions that can be understood in terms of singular Weierstrass models. We determine the complete set of such threefolds that have  $h^{2,1} \geq 350$  by tuning coefficients in Weierstrass models over Hirzebruch surfaces. The resulting set of Hodge numbers includes those of all known Calabi-Yau threefolds with  $h^{2,1} \geq 350$ , as well as three apparently new Calabi-Yau threefolds. We speculate that there are no other Calabi-Yau threefolds (elliptically fibered or not) with Hodge numbers that exceed this bound. We summarize the theoretical and practical obstacles to a complete enumeration of all possible EFS Calabi-Yau threefolds and fourfolds, including those with small Hodge numbers, using this approach.

**KEYWORDS:** F-Theory, Differential and Algebraic Geometry, Superstring Vacua

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## 1 Introduction

Since the early days of string theory, the geometry of Calabi-Yau (CY) threefolds has played an important role in understanding compactifications of the theory that give rise to four-dimensional effective physics [1–4]. Over the last three decades, progress from both mathematical and physical directions has led to the construction of a wide range of specific Calabi-Yau threefold geometries, and some general results on the structure of these manifolds. (See for example [5–8].) Many basic questions regarding this class of geometries remain unanswered, however, such as whether the number of distinct topological types of CY threefolds is finite.

The class of CY threefolds that admit an elliptic ( $T^2$  with complex structure) fibration with at least one section forms a subset of the full set of Calabi-Yau manifolds that is of interest both for mathematical and physical reasons. Mathematically, the existence of an elliptic fibration adds structure that simplifies the analysis and classification of possible CY geometries. It has been proven by Gross [9] that there are a finite number of distinct topological types (up to birational equivalence) of elliptically fibered Calabi-Yau threefolds. For an elliptically fibered Calabi-Yau threefold, the existence of a global section makes possible an explicit presentation as a Weierstrass model [10]

$$y^2 = x^3 + fx + g, \tag{1.1}$$

where  $f, g$  are functions (really, sections of line bundles) on the base  $B_2$  of the elliptic fibration  $\pi : X_3 \rightarrow B_2$ ,  $\pi^{-1}(p) \cong \mathbb{E} \cong T^2$ . Elliptically-fibered CY threefolds with section (henceforth “EFS CY3s”) have a role in physics as compactification spaces for F-theory [11, 12] that give rise to six-dimensional theories of supergravity. F-theory can be thought of as a nonperturbative description of type IIB string theory where the axiodilaton field  $\chi + ie^{-\phi}$  varies over a compact space (a complex surface  $B_2$  for supersymmetric 6D theories) and parameterizes the elliptic curve over this base. One particularly nice feature of the set of elliptically fibered Calabi-Yau threefolds is that their moduli spaces are all connected through singular Weierstrass models. On the physics side this unifies all the corresponding F-theory vacuum solutions of 6D supergravity into a single theory; tensionless string transitions [13, 14] connect the different branches of the theory. In addition to the special features that make them easier to analyze mathematically and connect them to the physics of F-theory, elliptically fibered threefolds may comprise a large fraction of the set of *all* Calabi-Yau threefolds, particularly those with large Hodge numbers. This paper contributes to a growing body of circumstantial evidence for this conclusion, which is discussed further in section 4.3.2.

The close connection between the physics of 6D supergravity theories and the geometry of EFS Calabi-Yau threefolds leads to a physically motivated approach to the classification of these geometries. From the work of Grassi [15] and the mathematical minimal model program for classifying surfaces [16, 17], it is known that all complex surfaces  $B_2$  that support elliptically fibered Calabi-Yau threefolds are in the set consisting of  $\mathbb{P}^2$ , the Enriques surface, the Hirzebruch surfaces  $\mathbb{F}_m$  for  $0 \leq m \leq 12$ , and blow-ups of the  $\mathbb{F}_m$  at one or more points. As argued in [18], the finiteness of the set of EFS CY3s can then be understood in

a constructive context from the finite number of topologically distinct tunings (strata) of the class of Weierstrass models over the minimal bases  $\mathbb{P}^2$  and  $\mathbb{F}_m$  (the Enriques surface is not as interesting since, up to torsion, the canonical class  $K$  vanishes, so the Weierstrass model is essentially trivial). From this point of view, in principle all EFS CY threefolds can be constructed by starting with the base surfaces  $\mathbb{P}^2$  and  $\mathbb{F}_m$ , and tuning the Weierstrass models over these bases in all possible ways consistent with the existence of a Calabi-Yau elliptic fibration. The set of such possible tunings can be described conveniently in terms of the spectra (gauge group and matter content) of the corresponding 6D supergravity theories. A complete classification of the types of intersection structures (corresponding to “non-Higgsable clusters” in the physical picture) that can appear in the base  $B_2$  was given in [19]. This was used in [20] to explicitly construct all toric bases  $B_2$  that support EFS CY3s, and in [21] to construct a broader class of bases admitting a single  $\mathbb{C}^*$  action. The generic elliptic fibrations over these toric and “semi-toric” bases were shown [22] to include the EFS CY3 with the largest possible value of the Hodge number  $h^{2,1}$  ( $= 491$ ), and to describe in outline the “shield-shaped” boundary on the set of known Hodge numbers found experimentally by Kreuzer and Skarke [23].

In this paper we pursue this line of inquiry further by explicitly constructing all EFS CY3s with large  $h^{2,1}$  through the tuning of Weierstrass models on  $\mathbb{F}_m$  for  $m \geq 7$ . We systematically construct all CY3s with  $h^{2,1} \geq 350$ , and compare with known data for threefolds with large  $h^{2,1}$ . The Hodge number pairs for EFS Calabi-Yau threefolds that have  $h^{2,1} \geq 350$  are plotted in figure 4; the detailed structure and construction of these threefolds is explained in the bulk of the paper. The arbitrary bound of 350 is chosen so that the number of possible threefolds is both limited enough to be manageable in a case-by-case analysis, and rich enough to illustrate the range of principles involved. A similar analysis can be used to systematically construct all elliptically fibered Calabi-Yau threefolds with section at increasingly small values of  $h^{2,1}$ . While this procedure becomes computationally intensive at lower values of  $h^{2,1}$ , and there are a number of practical and theoretical issues that must be resolved before a complete classification is possible, this program could in principle be pursued to enumerate *all* EFS CY3s.

The structure of this paper is as follows: in section 2 we review the basic structure of EFS CY3s and describe some aspects of the Weierstrass tunings needed to construct explicit CY3s. In section 3 we give a complete classification of all EFS CY3s that have  $h^{2,1} \geq 350$ . We conclude in section 4 with a description of the technical obstructions to classifying all EFS Calabi-Yau threefolds, and give a summary of results and discussion of further directions.

## 2 Classification of CY threefolds that are elliptically fibered with section

### 2.1 Elliptic fibrations and F-theory

As summarized in section 1, the Weierstrass form of an elliptic fibration  $y^2 = x^3 + fxz^4 + gz^6$  describes the total space of a Calabi-Yau threefold  $X$  in terms of information on the base surface (complex twofold)  $B_2$  by determining the complex structure of the elliptic fiber

(torus)  $\mathbb{E} \cong T^2$  over each point in the base in terms of a (complex) curve in  $\mathbb{P}^{2,3,1}$ . More precisely,  $f$  and  $g$ , as well as the discriminant  $\Delta = 4f^3 + 27g^2$ , are sections of line bundles

$$f \in \mathcal{O}(-4K) \quad g \in \mathcal{O}(-6K) \quad \Delta \in \mathcal{O}(-12K),$$

where  $K$  is the canonical class of the base  $B_2$ . Note that throughout this paper we will be somewhat informal about the distinction between divisors  $D$  in  $B_2$  and the associated homology classes  $[D]$  in  $H_2(B_2, \mathbb{Z})$ .

The singularity structure of  $X$  as an elliptic fibration over  $B_2$  is encoded in the vanishing loci of  $f, g$ , and  $\Delta$ . The close correspondence between the geometry of an elliptic fibration and the corresponding physical F-theory model illuminates both the mathematical and physical properties of these constructions. (Pedagogical introductions to F-theory compactifications can be found in [24–26].) The codimension one loci where  $\Delta$  vanishes to higher degree lead to singularities in the total space of the threefold that must be resolved to form a smooth Calabi-Yau total space. These singularities correspond physically to 7-branes in the F-theory picture, and the degrees of vanishing of  $f, g, \Delta$  (along with monodromy information in some cases) encode geometric structure that corresponds to the Lie algebra of the nonabelian gauge group  $G$  of the 6D theory according to the Kodaira-Tate classification of singularities summarized in table 1 [14, 27–31]. The codimension two vanishing loci of  $\Delta$  encode further singularities associated with hypermultiplet matter in the 6D theory transforming under some combination of irreps of the gauge group  $G$ . While the correspondence between matter and codimension two singularities is understood in the simplest and most generic cases, there is not yet a complete dictionary of this correspondence for general matter representations and arbitrary codimension two singularities. A further discussion of exotic matter representations and associated singularities appears in section 4.1.3.

The generic Weierstrass model over a given base  $B_2$  may have singularities in the elliptic fibration that are forced by the structure of irreducible effective divisors (curves) of negative self-intersection in  $B_2$  over which  $f, g$ , and  $\Delta$  must vanish. The possible configurations of curves that give rise to mandatory singularities — corresponding to “non-Higgsable clusters” of gauge groups and possible matter in the 6D F-theory picture — were classified in [19] and are depicted in figure 1, with the minimal gauge group and matter content for each cluster listed in table 2. By further tuning the coefficients in the Weierstrass model, higher degree singularities can be produced on  $f, g$ , and  $\Delta$ , corresponding to enhanced gauge groups in the 6D supergravity theory. In this way, a variety of topologically distinct Calabi-Yau threefolds can be constructed by tuning the Weierstrass model over a given base  $B_2$ .

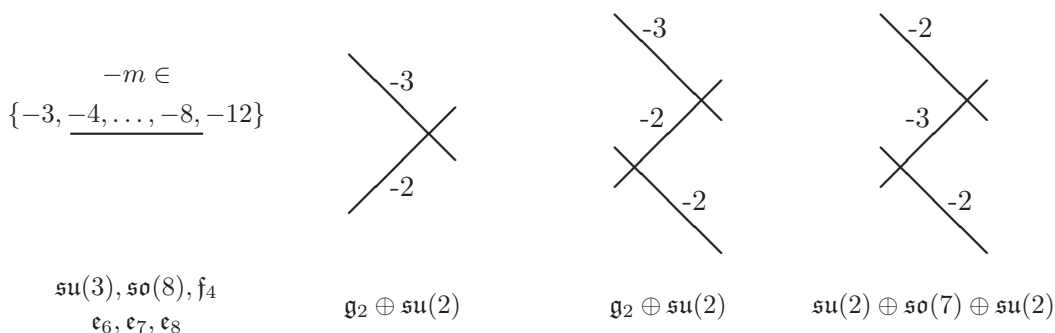
In general, a smooth Calabi-Yau threefold can be constructed by resolving all singularities in the total space of the elliptic fibration. If the vanishing of  $f, g, \Delta$  reaches degrees (4, 6, 12) over a divisor then there is no smooth resolution of the singular elliptic fibration as a Calabi-Yau threefold. If the vanishing of  $f, g, \Delta$  reaches degrees (4, 6, 12) on a codimension two locus in the base (a point in the surface  $B_2$ ), then the point must be blown up to form a new base  $B'_2$  over which there is a smooth CY3 after resolution of singularities (unless the blow-up leads to additional (4, 6, 12) vanishing on a divisor or point in the new base). Thus, to describe the possible EFS CY3s over a given base  $B_2$ , we need only

Type	ord ( $f$ )	ord ( $g$ )	ord ( $\Delta$ )	singularity	nonabelian symmetry algebra
$I_0$	$\geq 0$	$\geq 0$	0	none	none
$I_n$	0	0	$n \geq 2$	$A_{n-1}$	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor n/2 \rfloor)$
$II$	$\geq 1$	1	2	none	none
$III$	1	$\geq 2$	3	$A_1$	$\mathfrak{su}(2)$
$IV$	$\geq 2$	2	4	$A_2$	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$
$I_0^*$	$\geq 2$	$\geq 3$	6	$D_4$	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$
$I_n^*$	2	3	$n \geq 7$	$D_{n-2}$	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$
$IV^*$	$\geq 3$	4	8	$\mathfrak{e}_6$	$\mathfrak{e}_6$ or $\mathfrak{f}_4$
$III^*$	3	$\geq 5$	9	$\mathfrak{e}_7$	$\mathfrak{e}_7$
$II^*$	$\geq 4$	5	10	$\mathfrak{e}_8$	$\mathfrak{e}_8$
non-min	$\geq 4$	$\geq 6$	$\geq 12$	does not occur in F-theory	

**Table 1.** Table of codimension one singularity types for elliptic fibrations and associated non-abelian symmetry algebras. In cases where the algebra is not determined uniquely by the degrees of vanishing of  $f, g$ , the precise gauge algebra is fixed by monodromy conditions that can be identified from the form of the Weierstrass model.

Cluster	gauge algebra	$r$	$V$	$H_{\text{charged}}$
(-12)	$\mathfrak{e}_8$	8	248	0
(-8)	$\mathfrak{e}_7$	7	133	0
(-7)	$\mathfrak{e}_7$	7	133	28
(-6)	$\mathfrak{e}_6$	6	78	0
(-5)	$\mathfrak{f}_4$	4	52	0
(-4)	$\mathfrak{so}(8)$	4	28	0
(-3, -2, -2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	3	17	8
(-3, -2)	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	3	17	8
(-3)	$\mathfrak{su}(3)$	2	8	0
(-2, -3, -2)	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	3	23	16
(-2, -2, ..., -2)	no gauge group	0	0	0

**Table 2.** List of “non-Higgsable clusters” of irreducible effective divisors with self-intersection  $-2$  or below, and corresponding contributions to the gauge algebra and matter content of the 6D theory associated with F-theory compactifications on a generic elliptic fibration (with section) over a base containing each cluster. The quantities  $r$  and  $V$  denote the rank and dimension of the nonabelian gauge algebra, and  $H_{\text{charged}}$  denotes the number of charged hypermultiplet matter fields associated with intersections between the curves supporting the gauge group factors.



**Figure 1.** Clusters of intersecting curves that must carry a nonabelian gauge group factor. For each cluster the corresponding gauge algebra is noted and the gauge algebra and number of charged matter hypermultiplet are listed in table 2.

consider tunings where the degrees of vanishing of  $f, g, \Delta$  do not reach  $(4, 6, 12)$  at any point in the base.

For any given EFS CY threefold  $X$  with a Weierstrass description over a given base  $B_2$ , the Hodge numbers of  $X$  can be read off from the form of the singularities. A succinct description of the Hodge numbers of  $X$  can be given using the geometry-F-theory correspondence [12, 14, 22]

$$h^{1,1}(X) = r + T + 2 \tag{2.1}$$

$$h^{2,1}(X) = H_{\text{neutral}} - 1 = 272 + V - 29T - H_{\text{charged}}. \tag{2.2}$$

Here,  $T = h^{1,1}(B_2) - 1$  is the number of tensor multiplets in the 6D theory;  $r$  is the rank of the 6D gauge group and  $V$  is the number of vector multiplets in the 6D theory, while  $H_{\text{neutral}}$  and  $H_{\text{charged}}$  refer to the number of 6D matter hypermultiplets that are neutral/charged with respect to the gauge group  $G$ . The relation (2.1) is essentially the Shioda-Tate-Wazir formula [32]. The equality (2.2) follows from the gravitational anomaly cancellation condition in 6D supergravity,  $H - V = 273 - 29T$ , which corresponds to a topological relation on the Calabi-Yau side that has been verified for most matter representations with known nongeometric counterparts [31, 33]. The nonabelian part of the gauge group  $G$  can be read off from the Kodaira types of the singularities in the elliptic fibration according to table 1 (up to a discrete part that does not affect the Hodge numbers and that we do not compute in detail here.). The contribution to the rank  $r$  and the numbers of vector multiplets  $V$  and charged hypermultiplets  $H_{\text{neutral}}$  for the gauge fields and matter associated with non-Higgsable clusters (NHCs) are listed in table 2.

In principle,  $G$  can also have abelian  $(U(1))$  factors, corresponding to additional sections of the elliptic fibration, which contribute to  $h^{1,1}(X)$  through  $r$ . Mathematically, these sections are associated with a higher rank Mordell-Weil group of the fibration. There can also be torsion in the Mordell-Weil group [34], which corresponds to the discrete part of the gauge group in the gravity theory, but does not contribute to the Hodge numbers of the elliptic fibration. Because abelian factors arise from global, rather than local, aspects of the total space of the elliptic fibration, it is difficult to systematically describe  $U(1)$  factors

in the gauge group. Though there has been substantial progress on this problem in recent years, as we show in section 3.1, abelian  $U(1)$  factors in the gauge group cannot arise for EFS CY3s with  $h^{2,1} \geq 350$ , so we do not need to consider them in this paper, and  $r$  and  $V$  in (2.1), (2.2) can be read off directly from the codimension one singularities in the elliptic fibration. The representations and multiplicity of charged matter needed to compute  $H_{\text{charged}}$  in (2.2) can also be read off directly from the form of the local singularities in the absence of abelian gauge group factors. While the types of singularities associated with completely general matter representations have not yet been classified, the codimension two singularities that arise in EFS CY3s of large  $h^{2,1}$  belong to the simple categories of well-understood matter representations and associated singularities.

We now describe some of the details of the steps needed to systematically classify EFS Calabi-Yau threefolds starting at large  $h^{2,1}$ .

## 2.2 Systematic classification of EFS Calabi-Yau threefolds

A complete classification and enumeration of Calabi-Yau threefolds that are elliptically fibered with section can in principle be carried out in three steps:

1. Classify and enumerate all bases  $B_2$  that support a smooth elliptically fibered Calabi-Yau threefold with section.
2. Classify and enumerate all codimension one gauge groups that can be “tuned” over a given base, giving enhanced gauge groups in the 6D theory.
3. Given the gauge group structure, classify and enumerate the set of compatible matter representations — in some cases this may involve further tuning of codimension two singularities.

In the remainder of this section we describe some general aspects of the procedures involved in these steps 1–3 for the construction of EFS CY3s with large  $h^{2,1}$ . Some of the technical limitations to carrying out these three steps for all EFS Calabi-Yau threefolds are discussed in section 4.1.

A key principle that enables efficient classification of the threefolds of interest through the structure of their singularities is the decomposition of an effective divisor (curve)  $D$  in  $B_2$  into a *base locus* of irreducible effective curves  $C_i$  of negative self-intersection, and a residual part  $X$ , which satisfies  $X \cdot C \geq 0$  for all effective curves  $C$ . Treated over the rational numbers  $\mathbb{Q}$ , this gives the *Zariski decomposition* [35]

$$D = \sum_i \gamma_i C_i + X, \quad \gamma_i \in \mathbb{Q}. \tag{2.3}$$

This decomposition determines the minimal degree of vanishing of a section of a line bundle over curves  $C_i$  in the base. For example, on  $\mathbb{F}_{12}$  we have an irreducible effective divisor  $S$  with  $S \cdot S = -12$ ,  $-K \cdot S = -10$ . Thus,  $-K$  has a Zariski decomposition  $-K = (5/6)C + X$ . It follows that  $-4K = (10/3)C + X$ ,  $-6K = 10C + X$ . Since  $f, g$  are sections of  $\mathcal{O}(-4K), \mathcal{O}(-6K)$  respectively,  $f$  must vanish to degree 4 ( $= \lceil 10/3 \rceil$ ) on  $S$ , and  $g$  must



vanish to degree 5 on  $C$ , implying that there is an  $\epsilon_8$  type singularity associated with the generic elliptic fibration over  $\mathbb{F}_{12}$ . This matches the well-known fact that the gauge group of the generic F-theory model on  $\mathbb{F}_{12}$  is  $E_8$  [12]. This general principle was used in the classification of all non-Higgsable clusters in [19], and will be used as a basic tool throughout this paper. Note that the Zariski decomposition (2.3) determines the *minimal* degree of singularity of  $f, g$  over a given curve, but the actual degree of vanishing can be made higher for specific models by tuning the coefficients in the Weierstrass representation.

### 2.3 Bases $B_2$ for EFS Calabi-Yau threefolds with large $h^{2,1}$

The bases  $B_2$  that can support an elliptically fibered Calabi-Yau threefold are complex surfaces, which can be characterized by the structure of effective divisors (complex curves) on the surface. Divisors on  $B_2$  are formal integral linear combinations of algebraic curves, which map to homology classes in  $H_2(B_2, \mathbb{Z})$ . The effective divisors are those where the expansion in terms of algebraic curves has nonnegative coefficients; the effective divisors generate a cone (the Mori cone, dual to the Kähler cone on cohomology classes) in  $H_2(B_2, \mathbb{Z})$ .

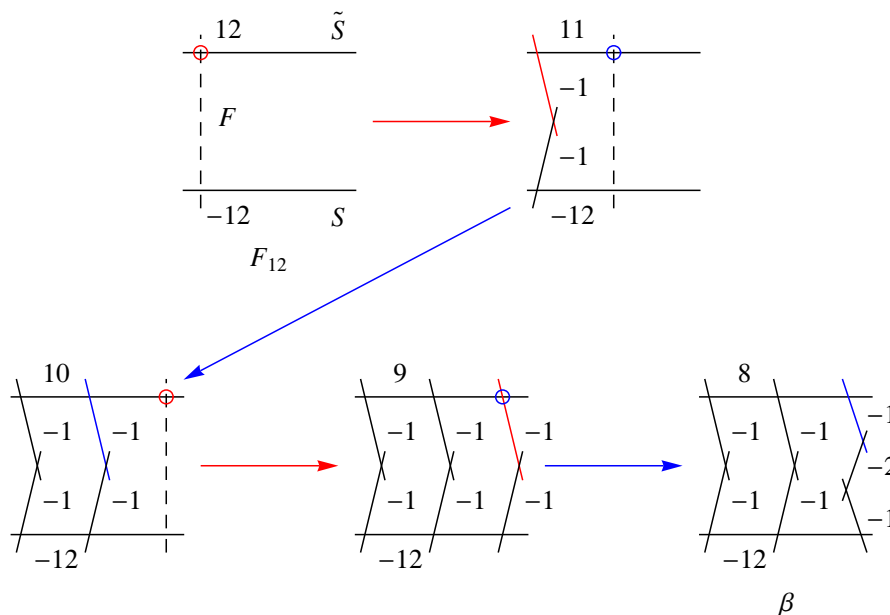
As summarized in section 1, the minimal model program for classification of complex surfaces and the results of Grassi show that the only bases  $B_2$  that can support an elliptically fibered Calabi-Yau threefold are  $\mathbb{P}^2, \mathbb{F}_m (0 \leq m \leq 12)$ , the Enriques surface, and blow-ups of these spaces. The values of  $h^{2,1}$  for the generic elliptic fibration over each of these surfaces can be read off from the intersection structure of each base using table 2 and equations (2.1) and (2.2). The intersection structure of divisors on the bases  $\mathbb{F}_m$  is quite simple.  $\mathbb{F}_m$  is a  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$ , with  $h^{1,1}(\mathbb{F}_m) = 2$ , so  $T = 1$ . The cone of effective divisor classes on each of these surfaces is generated by  $S, F$ , where  $S$  is a section of the  $\mathbb{P}^1$  bundle with  $S \cdot S = -m$ , and  $F$  is a fiber with  $F \cdot F = 0, F \cdot S = 1$ .<sup>1</sup>

The  $-12$  curve on  $\mathbb{F}_{12}$  carries an  $E_8$  gauge group, so the generic elliptic fibration over this base has  $r = 8, V = 248$  and  $h^{1,1} = 11, h^{2,1} = 491$ . Similarly, for  $\mathbb{F}_8$  and  $\mathbb{F}_7$  we have  $h^{1,1} = 10, h^{2,1} = 376$ , and for  $\mathbb{F}_6, h^{1,1} = 11, h^{2,1} = 321$ , with decreasing values of  $h^{2,1}$  for  $\mathbb{F}_m, m < 6$  (see [20] for a complete list). Since tuning Weierstrass coefficients to increase the size of the gauge group or blow up points in the base entails a reduction in  $h^{2,1}$ , to construct all EFS CY3s with  $h^{2,1} \geq 350$ , we need only consider the minimal bases  $\mathbb{F}_{12}, \mathbb{F}_8$ , and  $\mathbb{F}_7$ . Note that, as discussed, for example, in [19],  $\mathbb{F}_m$  for  $m = 9, 10, 11$  contain points on the  $-m$  curve where  $f, g$  must vanish to degrees 4, 6, which must be blown up leading to a new base of the form of  $\mathbb{F}_{12}$  or a blow-up thereof, so the Hirzebruch surfaces  $\mathbb{F}_9, \mathbb{F}_{10}, \mathbb{F}_{11}$  are not good bases for an EFS CY3.

The irreducible effective divisors on  $\mathbb{F}_m$  are those of the form  $D = aS + bF, b \geq ma$ , since if  $b < ma$ , then  $D \cdot S < 0$  and  $D$  contains  $S$  as a component (and is therefore reducible). Blowing up a base  $B_2 = \mathbb{F}_m$  at a point  $p$  produces a new  $-1$  curve, the *exceptional divisor*  $E$  of the blow-up. Each curve  $C$  in  $B_2$  that passes once smoothly through  $p$  gives a *proper transform*  $C' \sim C - E$ , with  $E \cdot C' = 1$ . Since  $\mathbb{F}_m$  is a  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$ , each  $p \in \mathbb{F}_m$  lies on some fiber in the class of  $F$ .

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<sup>1</sup>Really,  $[F]$  is a class in  $H_2$ , and the fibers are a continuous family of divisors in this class that foliate the total space; as mentioned earlier, we will generally go back and forth freely between divisors and their associated classes.



**Figure 2.** A general F-theory base  $B_2$  is formed by a sequence of blow-ups on a Hirzebruch surface  $\mathbb{F}_m$ . In this example, three generic points are blown up sequentially on  $\mathbb{F}_{12}$ , and a fourth blow-up point is chosen to be on the exceptional divisor from the third blow-up. These points are all blown up on fibers in such a way that a global  $\mathbb{C}^*$  structure is preserved. The final base  $\beta$  enters the discussion in the text in several places.

We can describe a sequence of blow-ups on  $\mathbb{F}_m$  by tracking the cone of effective divisor classes after each blow-up. The result of a single blow-up at a generic point on  $\mathbb{F}_m$  gives a new base  $B'_2$ , with an exceptional divisor  $E$  having  $E \cdot E = -1$  extending the cone of effective divisors in a new direction. If we denote the specific fiber of  $\mathbb{F}_m$  containing  $p$  as  $F_1$ , then  $F'_1 \sim F_1 - E$  is also in the new cone of effective divisors, with  $F'_1 \cdot E = 1$ . There is also an effective divisor in the class  $\tilde{S} = S + mF$  (with  $\tilde{S} \cdot \tilde{S} = +m$ ) that passes through the generic point  $p$ , and this gives a new curve  $\tilde{S}'$  in  $B'_2$  with  $\tilde{S}' \cdot \tilde{S}' = m - 1$ . In this way, we can sequentially blow up points on  $\mathbb{F}_m$  to achieve any allowable base  $B_2$  for an EFS Calabi-Yau threefold. An example of a sequence of bases formed from four consecutive blow-ups of  $\mathbb{F}_{12}$  is shown in figure 2.

A point in the base must be blown up whenever there is a  $(4, 6)$  vanishing of  $f, g$  at that point. In general, such a singularity can be arranged at a point in the base by tuning 29 parameters in the Weierstrass model [36]. This matches with the gravitational anomaly cancellation condition  $H - V = 273 - 29T$  (see (2.9)), since a single new tensor field arises when the point in base is blown up. From (2.1) and (2.2) we thus see that, generically, blowing up a point will cause a change in the Hodge numbers of a base by

$$\Delta h^{1,1} = +1, \quad \Delta h^{2,1} = -29. \tag{2.4}$$

As an example, the final base  $\beta$  depicted in figure 2 is associated with four blow-ups of  $\mathbb{F}_{12}$ , and thus has Hodge numbers  $h^{1,1} = 11 + 4 = 15, h^{2,1} = 491 - 4 \times 29 = 375$ . In some situations, when there is a gauge group involved along divisors containing the blow-up

point, there is also a change in  $V$  that modifies the number of moduli removed by the blow-up, and correspondingly affects the Hodge numbers of the new Calabi-Yau threefold.

In general, the combinatorial structure of the cone of effective divisors on  $B_2$  can become quite complicated. A simple subclass of the set of bases that are formed when multiple points on  $\mathbb{F}_m$  are blown up consists of those bases where the points blown up lie on  $\phi$  distinct fibers, and those blown up on each fiber are at the intersection of irreducible effective divisors of negative self intersection lying within that fiber or intersections between such divisors and the sections  $S, \tilde{S}$  of the original  $\mathbb{F}_m$ . In this case, a global  $\mathbb{C}^*$ -structure is preserved on the base  $B_2$ ; bases of this type were classified in [21]. When  $\phi \leq 2$ , so that all points blown up lie on two or fewer fibers, the base is toric; the set of toric bases was classified in [20]. In cases where the number of fibers blown up satisfies  $\phi \leq m$ , the initial point  $p_i$  blown up on each fiber can be a generic point and a representative of the class  $\tilde{S}$  can be found that passes through all these points, so that the base has a global  $\mathbb{C}^*$  structure. Almost all the bases we consider in this paper will have this structure, and can be represented as  $\mathbb{C}^*$ -bases with  $\phi$  nontrivial fibers. We will discuss particular situations where we need to go beyond this framework as they arise.

For the toric and  $\mathbb{C}^*$ -bases, an explicit representation of the monomials in the Weierstrass model can easily be given, as described in [20, 21]. This representation is useful for explicit calculations, as discussed further below in section 2.6.

One issue that must be addressed in enumerating distinct bases for EFS CY3s is the role of  $-2$  curves in the base. In general, isolated  $-2$  curves, or connected clusters of  $-2$  curves that do not carry a gauge group, are realized at specific points in the moduli space of fibrations over bases without those  $-2$  curves. For example, blowing up  $\mathbb{F}_{12}$  at two distinct generic points  $p_1, p_2$  gives rise to two nontrivial fibers, each containing two connected curves of self-intersection  $(-1, -1)$  (like the left two fibers in the base  $\beta$  from figure 2). In the limit where  $p_2$  approaches  $p_1$ , this becomes two blow-ups on a single fiber, containing three connected curves of self-intersection  $(-1, -2, -1)$  (e.g., the right-most fiber in figure 2). This can be seen in the toric and  $\mathbb{C}^*$  cases directly through the enumeration of monomials, as discussed in [20, 21]; the  $-2$  curves in clusters not associated with Kodaira singularities giving nonabelian gauge groups correspond to extra elements of  $h^{2,1}$  not visible in the explicit monomial count, and the corresponding Calabi-Yau is most effectively described by the more generic base where the blow-up points are kept distinct. On the other hand, when a  $-2$  curve supports a nontrivial gauge group either due to an NHC or a tuning, this curve is “held in place” by the singularity structure, which would not be possible in the given form without the  $-2$  curve. Thus, when enumerating all distinct possible EFS CY3s, we should only include  $-2$  curves in bases where  $(f, g, \Delta)$  have nonzero vanishing degrees over these curves.<sup>2</sup>

By following these principles, we can systematically enumerate the bases associated with EFS CY3s with large  $h^{2,1}$ . In almost all cases, the bases have a  $\mathbb{C}^*$  structure and can be described as  $\mathbb{F}_{12}$  blown up at a sequence of points along one or more fibers. The precise sequences of possible blow-ups are detailed in section 3.

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<sup>2</sup>Note that there is one additional subtlety, which arises when a configuration of  $-2$  curves describes a degenerate elliptic fiber [21], but this situation does not arise for any bases considered in this paper.

## 2.4 Constraints on codimension one singularities and associated gauge groups

In this and the following sections, we describe in more detail how codimension one and two singularities in the elliptic fibration of the Calabi-Yau threefold  $X$  over a given base  $B_2$  can be understood and classified. In this analysis we use the physical language of F-theory; though in principle the arguments here could be understood purely mathematically without reference to gauge groups or matter, the physical F-theory picture is extremely helpful in clarifying the geometric structures involved.

As we have described already, the NHCs of intersecting irreducible effective divisors of negative self-intersection tabulated in table 2 give rise to nonabelian gauge groups and, in some cases, charged matter over any base  $B_2$  that contains these clusters. These physical features of the EFS CY3s encode the topological structure of  $X$  through equations (2.1) and (2.2). Additional and/or enhanced gauge groups and matter can also be realized, giving rise to a range of different EFS CY3s over a given base  $B_2$ , by tuning the parameters in the Weierstrass model (1.1). Over simple bases like  $\mathbb{P}^2$ , the range of possible tunings is enormous, giving rise to many thousands of topologically distinct CY3s elliptically fibered over the fixed base [37, 38]. For the CY3s with large  $h^{2,1}$  that we consider here, however, the range of possible tunings over the relevant bases  $B_2$  is quite small.

Some general constraints on when codimension one singularities can be tuned beyond the minimal values required on a given base follow from the Zariski decompositions of  $-4K$  and  $-6K$ . These constraints provide strong bounds on the set of possible gauge groups that can be tuned over any given  $B_2$ . These constraints, which we analyze in general terms in this section, do not, however, guarantee the existence of a given tuned model with specific gauge groups. To confirm that a Weierstrass model can be realized, a more detailed analysis is needed, as discussed in the subsequent sections.

Consider a rational curve<sup>3</sup>  $C$  of self-intersection  $C \cdot C = -k$ . From  $(K + C) \cdot C = 2g - 2 = -2$ , we have  $K \cdot C = k - 2$ . Consider a divisor  $D = -nK$  that contains as irreducible components a set of curves  $B_i$  with multiplicities  $b_i$  that each intersect  $C$  simply at a single point:  $B_i \cdot C = 1$ . Then we have

$$D = cC + \sum_i b_i B_i + X, \quad \text{with } X \cdot C \geq 0, \quad (2.5)$$

where  $D \cdot C = -nK \cdot C = -n(k - 2)$ , so a section of (the line bundle associated with)  $D$  must vanish at least  $c$  times on  $C$ , where

$$X \cdot C = \left( D - cC - \sum_i b_i B_i \right) \cdot C = -n(k - 2) + kc - \sum_i b_i \geq 0. \quad (2.6)$$

$$\Rightarrow c \geq \frac{1}{k} \left( \sum_i b_i + n(k - 2) \right). \quad (2.7)$$

This result has a number of specific consequences for where codimension one singularities can be tuned on bases with a given configuration of non-Higgsable clusters from table 2,

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<sup>3</sup>A rational curve is a complex curve of genus 0; it is shown in [19] that an effective divisor in the base of an elliptically fibered Calabi threefold cannot be a higher genus curve of negative self-intersection without forcing a (4, 6) vanishing of  $f, g$ .

which are connected in any given base by a network of  $-1$  curves. We give some specific examples:

**No  $(f, g)$  tuning can give a nonabelian gauge algebra on  $-1$  curves connected to any singular cluster other than a single  $-3$  curve.** Consider, for example, a  $-4$  curve  $C$  that intersects a  $-1$  curve  $B$ . The minimal  $(f, g)$  tuning on  $B$  needed to get a nontrivial Kodaira singularity (*i.e.*, one which gives rise to a nonabelian gauge algebra) is  $(1, 2)$ . Applying (2.7) with  $k = 4$  for  $n = 4, b_1 = 1$  gives  $c_{(4)} \geq 9/4$ , and for  $n = 6, b_1 = 2$  gives  $c_{(6)} \geq 13/4$ , so tuning a  $(1, 2)$  vanishing on a  $-1$  curve  $B$  that intersects a  $-4$  curve  $C$  forces a  $(3, 4)$  vanishing on  $C$ , which means that  $f, g$  vanish to degrees  $(4, 6)$  at the point  $B \cdot C$ , which cannot happen on a good base  $B_2$  for an EFS CY3. A similar argument shows that  $(f, g)$  cannot be tuned to vanish to degrees  $(1, 2)$  on a  $-1$  curve that intersects any of the other NHCs that carry a nontrivial gauge group other than one or two isolated  $-3$  curves. A  $-1$  curve that intersects a  $-3$  curve can carry an  $(f, g)$  vanishing of  $(1, 2)$ , while the  $-3$  curve carries a  $(2, 3)$  vanishing. Note that a  $-1$  curve  $C$  intersecting three  $-3$  curves each with vanishing  $(2, 3)$  would have by (2.7)  $c_{(4)} \geq 2, c_{(6)} \geq 3$ , so  $C \cdot B_i$  would correspond to points of  $(4, 6)$  vanishing.

**Vanishing of  $f, g, \Delta$  on a  $-2$  curve.** From (2.7), the degrees of vanishing of  $f, g$ , or  $\Delta$  on any  $-2$  curve  $C$  must be

$$c \geq \sum_i \frac{b_i}{2} \tag{2.8}$$

where  $b_i$  are the degrees of vanishing of  $f, g$ , or  $\Delta$  on curves  $B_i$  that intersect  $C$ . We refer to this rule for degrees of vanishing on  $-2$  curves as the “averaging rule” in later arguments, where it will be of use in the analysis of toric and  $\mathbb{C}^*$  bases, for which each divisor in a fiber intersects precisely two neighboring divisors.

As an example, in the non-Higgsable  $(-3, -2)$  cluster, the degrees of vanishing of  $(f, g)$  on the  $-3$  and  $-2$  curves are, respectively,  $(2, 3)$ , and  $(1, 2)$ , which satisfy the above inequality (e.g., for  $g, b_i = 3, c_{(6)} = 2 \geq 3/2$ ).

These rules, and applications of (2.7) in a variety of other cases, strongly constrain the places where extra codimension one singularities can be tuned over EFS CY3s with large  $h^{2,1}$ . In general, a tuning is only possible when sections of  $\mathcal{O}(-4K), \mathcal{O}(-6K)$  can be found in the form  $D = \sum_i c_i C_i + X$  with no divisor or points where  $(f, g)$  vanish to degrees  $(4, 6)$ .

## 2.5 Anomalies and matter content

Another set of geometric constraints are encoded in the detailed anomaly cancellation equations of 6D supergravity theories. For an F-theory compactification on a base  $B_2$  with canonical class  $K$  and nonabelian gauge group factors  $G_i$  associated with codimension one singularities on divisors  $S_i$ , the anomaly cancellation conditions are [18, 39–42]

$$H - V = 273 - 29T \tag{2.9}$$

$$0 = B_{\text{adj}}^i - \sum_{\mathbf{R}} x_{\mathbf{R}}^i B_{\mathbf{R}}^i \tag{2.10}$$

$$K \cdot K = 9 - T \tag{2.11}$$

Group	Rep	$A_{\mathbf{R}}$	$B_{\mathbf{R}}$	$C_{\mathbf{R}}$
SU(2)	<b>2</b>	1	—	$\frac{1}{2}$
	<b>3</b>	4	—	8
SU(3)	<b>3</b>	1	—	$\frac{1}{2}$
	<b>8</b>	6	—	9
$G_2$	<b>7</b>	1	—	$\frac{1}{4}$
	<b>14</b>	4	—	$\frac{5}{2}$

**Table 3.** Group theory coefficients  $A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}$  for fundamental and adjoint matter representations of gauge groups relevant for the analysis of this paper. Note that the gauge groups SU(2), SU(3),  $G_2$  have no fourth order Casimir so there are no coefficients  $B_{\mathbf{R}}$ .

$$-K \cdot S_i = \frac{1}{6} \lambda_i \left( \sum_{\mathbf{R}} x_{\mathbf{R}}^i A_{\mathbf{R}}^i - A_{\text{adj}}^i \right) \tag{2.12}$$

$$S_i \cdot S_i = \frac{1}{3} \lambda_i^2 \left( \sum_{\mathbf{R}} x_{\mathbf{R}}^i C_{\mathbf{R}}^i - C_{\text{adj}}^i \right) \tag{2.13}$$

$$S_i \cdot S_j = \lambda_i \lambda_j \sum_{\mathbf{R}\mathbf{S}} x_{\mathbf{R}\mathbf{S}}^{ij} A_{\mathbf{R}}^i A_{\mathbf{S}}^j \tag{2.14}$$

where  $A_{\mathbf{R}}, B_{\mathbf{R}}, C_{\mathbf{R}}$  are group theory coefficients defined through

$$\text{tr}_{\mathbf{R}} F^2 = A_{\mathbf{R}} \text{tr} F^2 \tag{2.15}$$

$$\text{tr}_{\mathbf{R}} F^4 = B_{\mathbf{R}} \text{tr} F^4 + C_{\mathbf{R}} (\text{tr} F^2)^2, \tag{2.16}$$

$\lambda_i$  are numerical constants associated with the different types of gauge group factors ( $\lambda = 1$  for SU( $N$ ), 2 for SO( $N$ ) and  $G_2$ ), and where  $x_{\mathbf{R}}^i$  and  $x_{\mathbf{R}\mathbf{S}}^{ij}$  denote the number of matter fields that transform in each irreducible representation  $\mathbf{R}$  of the gauge group factor  $G_i$  and  $(\mathbf{R}, \mathbf{S})$  of  $G_i \otimes G_j$  respectively. (The unadorned “tr” above denotes a trace in the fundamental representation.) Note that for groups such as SU(2) and SU(3), which lack a fourth order invariant,  $B_{\mathbf{R}} = 0$  and there is no condition (2.10). The group theory coefficients for the representations relevant for this paper are compiled for convenience in table 3

The 6D anomaly cancellation conditions provide additional constraints on the set of possible structures for EFS Calabi-Yau threefolds. For any set of possible gauge groups satisfying the Zariski conditions described in the previous section, the anomaly cancellation conditions can be used to further check the consistency of the model and to compute the possible matter spectra, giving  $H_{\text{charged}}$ , which can then be used in (2.2) to compute  $h^{2,1}$ . For example, consider tuning a gauge group SU(2) on a curve  $C$  of genus  $g$  and self-intersection  $-n$ . Assuming only fundamental (**2**) and adjoint (**3**) matter, the spectrum of fields charged under this gauge group is uniquely determined by the anomaly cancellation conditions

$$K \cdot C = 2g + n - 2 = \frac{1}{6} (A_{\mathbf{3}}(1 - x_{\mathbf{3}}) - A_{\mathbf{2}}x_{\mathbf{2}}) = 2/3 - x_{\mathbf{2}}/6 - 2x_{\mathbf{3}}/3 \tag{2.17}$$

	matter	$\Delta h^{1,1}$	$\Delta h^{2,1}$
$\mathfrak{su}(2)$	$(16 - 6n) \times \mathbf{2}$	+1	$-29 + 12n$
$\mathfrak{su}(3)$	$(18 - 6n) \times \mathbf{3}$	+2	$-46 + 18n$
$\mathfrak{g}_2$	$(10 - 3n) \times \mathbf{7}$	+2	$-56 + 21n$

**Table 4.** Table of matter content and Hodge number shifts for tuned gauge algebra summands on a  $-n$  curve  $C$ . Shifts are computed assuming the curve carries no original gauge group; for  $n \geq 3$  the contribution from the associated non-Higgsable cluster must be subtracted. These shifts also do not include any necessary modifications for bifundamental matter, which must be taken into account when  $C$  intersects other curves carrying a gauge group.

$$C \cdot C = -n = \frac{1}{3}(C_3(x_3 - 1) + C_2x_2) = 8(x_3 - 1)/3 + x_2/6, \quad (2.18)$$

to be  $x_3 = g, x_2 = 16 - 6n - 16g$ . For a rational curve  $C$  with  $g = 0$ , there are simply  $16 - 6n$  fields in the fundamental ( $\mathbf{2}$ ) representation. This matches with the expectation that when  $-n \leq -3$  there is a larger gauge group and an  $SU(2)$  is impossible. For higher genus curves  $g$  the number of fields in the adjoint is generically  $g$  with no higher-dimensional matter representations. For specially tuned models, higher matter representations are possible, but for  $\mathfrak{su}(2)$  all representations other than  $\mathbf{2}$  contribute to the genus [37, 43]. Gauge groups on higher genus curves and associated exotic matter representations of this type do not appear in the models considered here at large  $h^{2,1}$ , and are discussed further in section 4.1.3.

From the gauge group and matter content associated with a given tuned Weierstrass model, the Hodge numbers can be computed from (2.1), (2.2). Continuing with the preceding example, tuning an  $SU(2)$  gauge group on a divisor of self-intersection  $-n$  that does not intersect any other curves carrying gauge groups leads to a change in Hodge numbers of

$$\Delta h^{1,1} = \Delta r = +1, \quad (2.19)$$

$$\Delta h^{2,1} = \Delta V - \Delta H_{\text{charged}} = +3 - 2(16 - 6n) = -29 + 12n. \quad (2.20)$$

It is straightforward to compute the contribution to the Hodge numbers from tuning any of the other gauge groups associated with a Kodaira singularity type on a rational curve of given self-intersection. Table 4 tabulates these values for the gauge group factors that are relevant for this paper.

Finally, the anomaly cancellation condition (2.14) indicates that when two curves  $C, D$  intersect and both carry gauge groups, a certain part of the matter is charged under both gauge group factors. This bi-charged matter is a subset of the total charged matter content in each case, and must be taken into account when computing the Hodge numbers of a threefold with this structure in the base. For example, two  $SU(2)$  factors tuned on two intersecting  $-2$  curves each have, from table 4, 4 fundamental matter fields. From (2.14), there is one bifundamental matter field transforming in the  $\mathbf{2} \times \mathbf{2}$  representation. This field, which contains 4 complex scalars, is counted in the matter charged under each of the  $SU(2)$ . Thus, while the change in  $h^{2,1}$  from tuning each of these  $SU(2)$  factors individually is  $\Delta h^{2,1} = -5$ , the net change from tuning both of these factors is  $-6$ ; i.e., the second  $SU(2)$  requires tuning only a single additional Weierstrass modulus.

## 2.6 Weierstrass models

While the Zariski decomposition of  $f, g, \Delta$ , and the anomaly cancellation conditions described in the last two sections place strong constraints on the set of possible gauge groups and matter fields that can be tuned in a Weierstrass model over any given base  $B_2$ , these constraints are necessary but not sufficient for the existence of a consistent geometry. To prove that a given Calabi-Yau geometry exists, it is helpful to consider an explicit construction of the Weierstrass model. This can be done in a straightforward way for toric bases using the explicit realization of the monomials in the Weierstrass model as elements of the lattice  $N^*$  dual to the lattice  $N$  in which the toric fan is described. This approach generalizes in a simple way to bases that admit only a single  $\mathbb{C}^*$  action. The details of this analysis are worked out in detail in [20, 21]. It is also possible to describe Weierstrass models explicitly for bases that are not toric or  $\mathbb{C}^*$ , though there is at present no general method for doing this and the analysis must be done on a case-by-case basis. Explicit construction of the monomials in a given Weierstrass model plays two important roles in analyzing the Calabi-Yau threefolds we consider in this paper. First, by imposing the desired vanishing conditions for  $f, g, \Delta$  on all curves carrying gauge groups, we can check the explicit Weierstrass model to confirm that no additional vanishing conditions are forced on any curves or points that would produce additional gauge groups or force a blow-up or invalidate the model due to (4, 6) points or curves. Second, we can perform an explicit check on the value of  $h^{2,1}$  computed using the last term in (2.2) by relating the number of free degrees of freedom in the Weierstrass model to the number of neutral scalar fields. This analysis can, among other things, reveal the presence of additional  $U(1)$  gauge group factors that contribute to  $V$  and  $r$ . In [21], for example, it was found using this type of analysis that a small subset of the possible  $\mathbb{C}^*$ -bases for EFS Calabi-Yau threefolds give rise to generic nonzero Mordell-Weil rank.

We summarize here the relationship between  $h^{2,1}$  and the number of Weierstrass monomials  $W$  for a generic elliptic fibration over a  $\mathbb{C}^*$  base:

$$h^{2,1}(X) = H_{\text{neutral}} - 1 = W - w_{\text{aut}} + N - 4 + N_{-2} - G_1, \quad (2.21)$$

where  $w_{\text{aut}} = 1 + \max(0, 1 + n_0, 1 + n_\infty)$  is the number of automorphism symmetries, with  $n_0, n_\infty$  the self-intersections of the divisors coming from  $S, \tilde{S}$ ,  $N$  is the number of fibers containing blow-ups,  $N_{-2}$  is the number of  $-2$  curves that can be removed by moving to a generic point in the moduli space of the associated threefold, and  $G_1$  is the number of  $-2$  curve combinations that represent a degenerate elliptic fiber. The relation (2.21) is a slight refinement of the relation determined in [21] to include tuned Weierstrass models; in particular, when considering tuned (non-generic) elliptic fibrations over a given base the set of  $-2$  curves contributing to  $N_{-2}$  does not include certain  $-2$  curves where  $\Delta$  vanishes to some order, even if this  $-2$  curve is not in a non-Higgsable cluster supporting a nonabelian gauge group. We encounter an example of this in the following section.

### 2.6.1 Weierstrass models: some subtleties

As mentioned earlier, and also discussed in [20, 21], curves of self-intersection  $-2$  must be treated carefully when analyzing the Weierstrass monomials and corresponding Hodge



numbers.  $-2$  curves that do not carry vanishing degrees of  $f, g, \Delta$  in most circumstances are associated with special codimension one loci in Calabi-Yau moduli space, and indicate additional elements of  $h^{2,1}$  that are not visible in the Weierstrass monomials for the model with the  $-2$  curve. To consistently distinguish different topological types of Calabi-Yau threefolds, we should generally only consider the most generic bases in each moduli space component, which have no  $-2$  curves on which  $f, g, \Delta$  do not vanish to some degree. For example, the Weierstrass model describing the base  $\beta$  appearing in figure 2 has one fewer parameter than expected for the given Calabi-Yau threefold, corresponding to a contribution of  $N_{-2} = 1$  in (2.21). The generic base for this threefold is given by blowing up four completely generic points in  $\mathbb{F}_{12}$ , which gives four distinct  $(-1, -1)$  fibers; the base  $\beta$  that contains a  $-2$  curve in one fiber arises at limit points of the moduli space where one of the blow-up points lies on the exceptional divisor produced by one of the other blow-ups. The generic elliptic fibration over  $\beta$  thus lives on the same moduli space as the generic elliptic fibration over the base with four  $(-1, -1)$  fibers. If, on the other hand, we tune an  $SU(2)$  factor on the top  $-1$  curve of the  $(-1, -2, -1)$  fiber, then the  $-2$  curve acquires a degree of vanishing of  $\Delta$  of at least 1, and it is fixed in place by the structure of the singularity. This  $SU(2)$  factor cannot be tuned in the bulk of the moduli space of the generic four-times blown up  $\mathbb{F}_{12}$ . In this situation,  $N_{-2} = 0$ , and it can be checked that the  $\mathbb{C}^*$  Weierstrass model contains the correct number of monomials.<sup>4</sup>

Another subtlety that must be taken into account when computing the number of free parameters for a Weierstrass model with given codimension one singularity types is the appearance of each of the gauge group factors  $SU(2), SU(3)$  in two distinct ways in the Kodaira classification. In a generic situation, in the absence of other gauge groups, an  $SU(2)$  or  $SU(3)$  gauge group tuned by a Kodaira type III or IV singularity, as listed in table 1 is simply a special case of a type  $I_2$  or  $I_3$  singularity, and the complete set of degrees of freedom needed to compute  $h^{2,1}$  should be computed by imposing only the latter conditions. In other cases, however, such as in the context of non-Higgsable clusters, the type III or IV singularity type may be forced by the structure of other gauge groups or divisors. In this case the specified gauge group structure may not be possible with an  $I_n$  singularity type, in which case there are no monomials associated with such additional freedom.

Finally, for those gauge algebra types that depend not only on the degrees of vanishing of  $f, g, \Delta$ , but also on monodromy, the correct counting of degrees of freedom in the Weierstrass model depends on the monodromy conditions. The monodromy conditions for each of the gauge group choices in type  $IV, I_0^*$ , and  $IV^*$  Kodaira singlets are described in [29, 30], and can easily be characterized in terms of the structure of monomials in the Weierstrass model [44].

For all the models considered here, we have carried out an explicit construction of the Weierstrass monomials, and confirmed that the appropriate geometric structure exists and that the number of monomials properly matches the value of  $h^{2,1}$ , when the proper shifts according to  $-2$  curves and automorphisms as described in (2.21) are taken into account.

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<sup>4</sup>The fact that additional structure can appear associated with  $-2$  curves also arises in a related context in 4D heterotic theories based on elliptically fibered Calabi-Yau threefolds over bases containing these curves [44].

For all the models considered here, the blow-ups on the different fibers are independent, since the gauge groups on  $S, \tilde{S}$  do not change. This means that the monomial analysis can be performed in a local chart around each fiber independently, without loss of generality.

### 2.6.2 Constraints on Weierstrass models: an example

As an example of the utility of the explicit Weierstrass monomial construction, we consider a simple example of a situation in which the Zariski and anomaly analyses suggest that a tuning may be possible, but it is ruled out by explicit consideration of the Weierstrass model.

Consider again the base  $B_2 = \beta$  depicted in figure 2. We can ask if an  $SU(2)$  can be tuned through an  $I_2(0, 0, 2)$  singularity on the top  $-1$  curve  $C$  of one of the  $(-1, -1)$  fibers. (In fact, this analysis is equivalent for any such fiber on  $\mathbb{F}_{12}$ , since as discussed above the analysis is essentially local on each fiber in this situation where there is no change in the degree of vanishing of  $f, g, \Delta$  on  $S, \tilde{S}$ .) The  $SU(2)$  that we might tune in this fashion does not violate any conditions visible from the Zariski analysis, since we can take  $\Delta = 2C + 10S + X$ , and still satisfy  $X \cdot D = 0$  where  $D$  is the lower  $-1$  curve connecting  $C$  and  $S$ . (Note, however, that we cannot have a type *III* or *IV*  $SU(2)$  on  $C$ , since this would force a vanishing of  $\Delta$  on  $D$ .) Tuning an  $SU(2)$  on  $C$  also does not present any problems involving anomalies, since we have sufficient hypermultiplets to have an  $SU(2)$  with the requisite 10 fundamental matter fields. This configuration is, however, ruled out by an explicit Weierstrass analysis. In the toric language [20, 45], we can take  $\mathbb{F}_{12}$  to have a toric fan given by vectors  $v_i \in N = \mathbb{Z}^2$ :  $v_1 = (0, 1), v_2 = (1, 0), v_3 = (0, -1), v_4 = (-1, -12)$ . The allowed Weierstrass monomials for the generic elliptic fibration over  $\mathbb{F}_{12}$  are then  $u \in N^*, \langle u, v_i \rangle \geq -n$  with  $n = 4, 6$  for  $f, g$  respectively. Taking a local coordinate system where  $z = 0$  on the fiber  $F$  associated with  $v_2$ , and  $w = 0$  on  $\tilde{S}$ , the allowed monomials in  $f = f_{k,m} z^k w^m, g = g_{k,m} z^k w^m$  are those with  $k, m \geq 0, 12(m - n) + (k - n) \leq n$ ; these degrees of freedom are depicted in figure 3. The only monomial that keeps  $S$  from having a  $(4, 6)$  singularity is the  $w^7$  term in  $g$ , so the coefficient  $g_{0,7}$  cannot vanish without breaking the Calabi-Yau structure. Blowing up the point of intersection between  $F$  and  $\tilde{S}$  adds the vector  $v_5 = (1, 1)$  to the toric fan, so we must remove the monomials  $u$  with  $\langle u, v_5 \rangle < -n$  from  $f, g$ ; in the chosen coordinates, this amounts to removing all monomials such that  $m + k < n$ , as depicted by the red diagonal line in the figure. With a change of coordinates  $z = \zeta x, w = x, f = \hat{f} x^4, g = \hat{g} x^6$ , we have a local expansion around  $E, F' = F - E$  with coordinate  $x = 0$  on  $E$ . We can then expand

$$\hat{f}(\zeta, x) = \hat{f}_0(\zeta) + \hat{f}_1(\zeta)x + \dots \quad (2.22)$$

$$= (\hat{f}_{0,0} + \hat{f}_{1,0}\zeta + \dots \hat{f}_{4,0}\zeta^4) + (\hat{f}_{1,1}\zeta + \hat{f}_{1,1}\zeta^2 + \dots \hat{f}_{5,1}\zeta^5)x + \dots \quad (2.23)$$

$$\hat{g}(\zeta, x) = \hat{g}_0(\zeta) + \hat{g}_1(\zeta)x + \dots \quad (2.24)$$

$$= (\hat{g}_{0,0} + \hat{g}_{1,0}\zeta + \dots \hat{g}_{6,0}\zeta^6) + (\hat{g}_{0,1} + \hat{g}_{1,1}\zeta + \dots \hat{g}_{7,1}\zeta^7)x + \dots \quad (2.25)$$

The condition that  $\Delta$  vanish at order  $x^0$  requires that  $4\hat{f}_0^3 + 27\hat{g}_0^2 = 0$ , which we can satisfy by setting  $\hat{f}_0(\zeta) = -3\alpha^2, \hat{g}_0(\zeta) = 2\alpha^3$  for some quadratic function  $\alpha(\zeta)$ . The condition that

$\Delta$  vanish at order  $x$  then requires that

$$2\hat{f}_0^2\hat{f}_1 + 9\hat{g}_0\hat{g}_1 = 0. \tag{2.26}$$

This condition cannot, however, be satisfied when  $\alpha \neq 0$ , without setting  $\hat{g}_{0,1} = 0$ , since  $\hat{f}_1$  contains no term of order  $\zeta^0$ . But  $\hat{g}_{0,1} = g_{0,7} = 0$  forces  $g$  to vanish to degree 6 on  $S$  so there would be a degree (4, 6) singularity on  $S$ , which is incompatible with the Calabi-Yau structure. Thus, we cannot tune an  $I_2$   $SU(2)$  singularity on  $C$ . Note that while the coordinates  $\zeta, x$  make this computation particularly transparent, the same result can be derived directly in the  $z, w$  coordinates. In particular, this means that an  $SU(2)$  cannot be tuned on the curve in question even if further points on the base are blown up.

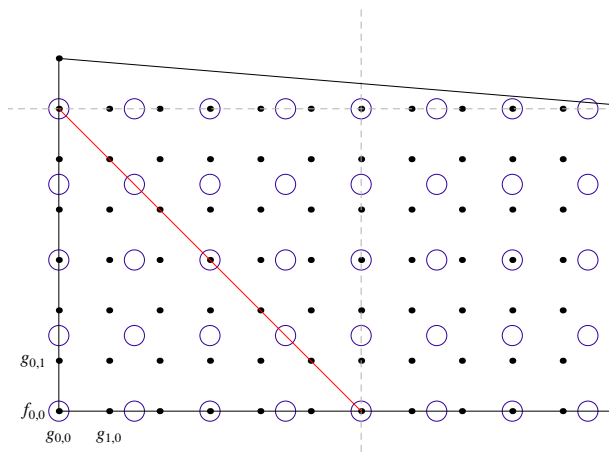
Note also that while this analysis rules out an  $SU(2)$  on the  $-1$  curve  $C$  in question, it is still possible to tune  $\Delta$  to vanish to second order on this curve. If  $\hat{f}_0 = \hat{g}_0 = 0$ , then (2.26) is automatically satisfied. This allows for the possibility of a (1, 1, 2) vanishing of  $(f, g, \Delta)$  on  $C$ . Indeed, such a vanishing — which does not lead to any gauge group — arises in some configurations for EFS CY threefolds, as we see below.

This kind of analysis can be used to check explicitly whether a Weierstrass model exists for any given combination of gauge group tunings that satisfy the Zariski and anomaly cancellation conditions. This is straightforward for the gauge groups that are imposed by particular orders of vanishing of  $f, g$ , since this corresponds simply to setting the coefficients of certain monomials in these functions to vanish. The analysis is more subtle, however, for type  $I_n$  and  $I_n^*$  singularities, such as the  $I_2$  example considered here, where vanishing on  $\Delta$  requires more complicated polynomial conditions on the coefficients. For large  $n$ , the algebra involved in explicitly imposing an  $I_n$  singularity can be quite involved. This is not an issue for any of the threefolds considered in this paper, but presents a technical obstacle to a systematic analysis for general  $h^{2,1}$ . We return to this issue in section 4.1.2.

Finally, note that the fact that an  $SU(2)$  cannot be tuned on the top  $-1$  curve of a  $(-1, -1)$  fiber matches with the example described in section 2.6.1, where an  $SU(2)$  tuned on the top curve of a  $(-1, -2, -1)$  fiber fixes the middle  $(-2)$  curve in place. The lower  $-1$  curve cannot be moved to a different location on the  $-12$  curve  $S$ , which would remove the  $-2$  curve, since this would leave behind precisely the configuration we have just ruled out. This confirms that this  $-2$  curve does not represent a missing modulus and does not contribute to  $N_{-2}$  in (2.21), even though it does not itself support a gauge group.

### 3 Systematic construction of EFS CY threefolds with large $h^{2,1}$

We now systematically describe how all Calabi-Yau threefolds that are elliptically fibered with section (EFS) and have  $h^{2,1} \geq 350$  are constructed by tuning gauge groups on  $\mathbb{F}_{12}, \mathbb{F}_8, \mathbb{F}_7$ , and blow-ups thereof. We begin with the Hirzebruch surfaces and consider all possible tunings that would give a threefold with  $h^{2,1} \geq 350$ . For those tunings that are possible by the Zariski and anomaly cancellation conditions we check the Weierstrass models explicitly using the toric monomial method. For each set of valid Hodge numbers we compare with the Kreuzer-Skarke database [23] of Hodge numbers for Calabi-Yau threefolds realized as hypersurfaces in toric varieties using the Batyrev construction [46]. The



**Figure 3.** Monomials in the generic Weierstrass model over  $\mathbb{F}_{12}$  are of the form  $f_{k,m}z^k w^m, g_{k,m}z^k w^m$ , and can be associated with points depicted above in the lattice  $N^*$  dual to the lattice  $N$  carrying the rays in the toric fan for  $\mathbb{F}_{12}$ . Circles denote monomials in  $f$ , and dots denote monomials in  $g$ . Blowing up a generic point in  $\mathbb{F}_{12}$  can be described in a local coordinate system by setting all monomials below the red line to vanish. As described in the text, an  $SU(2)$  gauge group cannot be tuned on the exceptional divisor from the blow-up without forcing the monomial coefficient  $g_{0,7}$  to vanish, which makes it impossible to form a Calabi-Yau due to a  $(4, 6)$  vanishing on the divisor  $S$ .

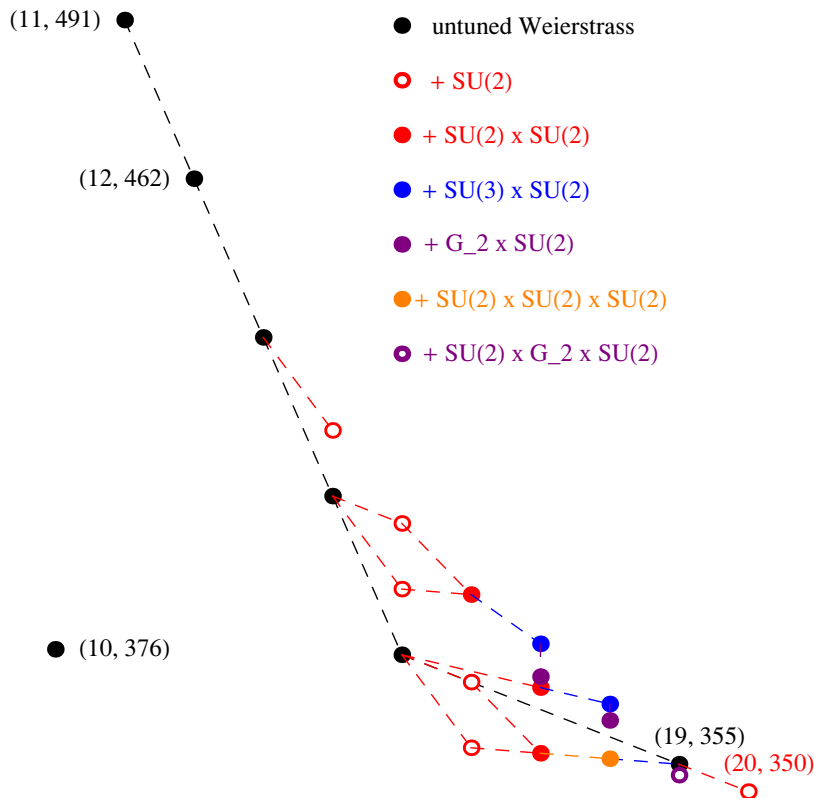
final results of our analysis are compiled in figure 4, and the full set of constructions is listed in table 5.

### 3.1 Tuning models over $\mathbb{F}_{12}$

To systematically construct all Calabi-Yau threefolds that are elliptically fibered with section, beginning with the largest value of  $h^{2,1}$  and preceding downward, we begin with the generic elliptic fibration over  $\mathbb{F}_{12}$ . As described above and in [22], this Calabi-Yau threefold has Hodge numbers  $(h^{1,1}, h^{2,1}) = (11, 491)$ , and has the largest value of  $h^{2,1}$  possible for any EFS CY threefold.

There are few ways available to tune an enhanced gauge group over the base  $B_2 = \mathbb{F}_{12}$ . The gauge algebra on the curve  $S$  with  $S \cdot S = -12$  is  $\mathfrak{e}_8$  and cannot be enhanced. Tuning a gauge algebra on any fiber  $F$  would increase the degree of vanishing at the point  $S \cdot F$  beyond  $(4, 5, 10)$ , which is not allowed since such a point lies on  $S$  and cannot be blown up to give a valid base. The only option for tuning is on the curve  $\tilde{S} = S + 12F$ , which has self-intersection  $+12$  (or on curves with a multiple of this divisor class, which would have self-intersection  $\geq 48$ ). Tuning an  $\mathfrak{su}(2)$  factor on the curve  $\tilde{S}$  gives 88 fundamental matter fields, from table 4, so the Hodge numbers are  $(12, 318)$ . A threefold with these Hodge numbers is in the Kreuzer-Skarke database, but has  $h^{2,1} < 350$ , so we do not concern ourselves further with it here. Tuning any larger gauge group factor reduces  $h^{2,1}$  still further; for example, tuning an  $\mathfrak{su}(3)$  gives Hodge numbers  $(13, 229)$ .

This example illustrates the basic paradigm: on curves of higher self-intersection, there are fewer restrictions on the possible tunings, but more charged matter is required to fulfill



**Figure 4.** In this paper we explicitly construct all elliptically fibered Calabi-Yau threefolds with section having  $h^{2,1} \geq 350$ . The Hodge numbers of these threefolds are shown here, with the detailed construction explained in the bulk of the text. Black points represent generic elliptic fibrations over different bases  $B_2$ , and colored points represent tuned Weierstrass models over these bases with enhanced gauge groups. The three purple data points appear to be new Calabi-Yau manifolds not found in the Kreuzer-Skarke database (see section 4.3.3). All elliptically fibered Calabi-Yau threefolds with section are connected by geometric transitions associated with tuning Weierstrass moduli over a particular base (“Higgsing/unHiggsing”) and/or blowing up and down points in the base (corresponding to tensionless string transitions in the physical F-theory context). Note that the point  $(10, 376)$ , corresponding to generic elliptic fibrations over  $\mathbb{F}_7, \mathbb{F}_8$ , is connected to the other threefolds shown through a sequence of blow-up and blow-down transitions on the base that pass through the set of threefolds with smaller Hodge numbers  $h^{2,1} < 350$ . Note also that there are two distinct constructions that give the Hodge numbers  $(19, 355)$ ; in addition to an untuned Weierstrass model with generic gauge group  $G_2 \times \text{SU}(2)$  there is a tuning of the generic  $(15, 375)$  Weierstrass model with a gauge group  $\text{SU}(2) \times \text{SU}(3) \times \text{SU}(2)$ .

anomaly cancelation conditions. As a rule of thumb, it is often easy to increase  $h^{1,1}$  via tuning so long as one is willing to accept a large decrease in  $h^{2,1}$ .

There is one other possibility that should be discussed here, and that is the possibility of tuning an abelian gauge group factor. As shown in [47], any  $U(1)$  factor can be seen as arising from a Higgsed  $SU(2)$  gauge group factor (which may be a subgroup of a larger nonabelian group), under which some matter transforms in the adjoint representation. The  $U(1)$  factor is associated with the divisor class  $C$  in the base that supports the  $SU(2)$  gauge group after unHiggsing; to have an adjoint, irreducible curves in this divisor class must have nonzero genus. In the case of  $B_2 = \mathbb{F}_{12}$ , the divisor class  $C$  cannot intersect  $S$  without producing a  $(4, 6)$  singularity, so it must be a multiple  $C = n\tilde{S}$  of the curve of self-intersection  $+12$  in  $B_2$ . For  $n = 2$ , the curve  $2\tilde{S}$  has genus  $g = 11$ , and the resulting  $SU(2)$  model would have 11 adjoint matter fields and 128 fundamental matter fields. Although this model should exist, it has a substantially reduced number of Weierstrass moduli corresponding to uncharged matter fields, even after breaking of the  $SU(2)$  by a single adjoint. Similarly, a discrete abelian group would involve further breaking of the  $U(1)$  that would maintain a relatively small value of  $h^{2,1}$ . Thus, while in principle it may be possible to tune an abelian factor, for this base and the others considered here the resulting Calabi-Yau threefold has relatively small  $h^{2,1}$ , and we do not need to consider abelian factors in constructing threefolds with  $h^{2,1} \geq 350$ . We discuss abelian factors further in section 4.1.4.

### 3.2 Tuning models over $\mathbb{F}_8$ and $\mathbb{F}_7$

The generic elliptically fibered Calabi-Yau threefolds over the Hirzebruch bases  $\mathbb{F}_7$  and  $\mathbb{F}_8$  have Hodge numbers  $(10, 376)$ . The discussion of tuning over these bases is precisely analogous to the preceding discussion for the base  $\mathbb{F}_{12}$ , and there are no tuned models over these bases with  $h^{2,1} \geq 350$ . Since  $376 - 29 < 350$ , there are also no threefolds formed over bases that are blow-ups of  $\mathbb{F}_7$  or  $\mathbb{F}_8$  that have  $h^{2,1} \geq 350$ . The threefolds with Hodge numbers  $(10, 376)$  over these bases are, however, continuously connected to the threefolds over  $\mathbb{F}_{12}$  and blow-ups thereof; for example, blowing up  $\mathbb{F}_8$  at four generic points on the curve  $S$  of self-intersection  $-8$  gives a base that is equivalent to the one reached by blowing up  $\mathbb{F}_{12}$  at four generic points. It is not immediately clear whether the threefolds formed from generic elliptic fibrations over  $\mathbb{F}_7$  and  $\mathbb{F}_8$  are equivalent. We discuss this issue further in section 4.3.4.

### 3.3 Decomposition into fibers

To find further EFS CY threefolds with large  $h^{2,1}$  we must blow up one or more points in the base  $B = \mathbb{F}_{12}$  to get further bases over which a variety of Weierstrass models can be tuned. We can blow up any point on  $\mathbb{F}_{12}$  that does not lie on the curve  $S$  of self-intersection  $-12$ . Any such point lies on a fiber  $F$  that intersects  $S$  and  $\tilde{S}$  each at one point. After blowing up one point we can blow up another point on the same fiber or on another fiber. Until the number of blow-ups is large ( $> 12$ ), blow-ups on distinct fibers do not interact, so that we may analyze the sequence of blow-ups possible along one given fiber, and then we can combine such sequences to construct threefolds involving the blow-ups of multiple fibers. Along any given fiber, as long as each blow-up occurs at an intersection of curves

of negative self-intersection or at the point of intersection of the fiber with  $\tilde{S}$ , we can use toric methods for describing the monomials, as in section 2.6. After a sufficient number of blow-ups, it is also possible to construct fibers that do not fit into the toric framework, though we need to consider only one example of this in the analysis for threefolds with  $h^{2,1} \geq 350$ . For fibers that simply consist of a linear sequence of mutually intersecting curves, such as those in figure 2, for convenience we label the curves  $C_1, C_2, \dots$ , with  $C_1$  the curve that intersects the  $-12$  curve  $C$  (so we always have  $C_1 \cdot C_1 = -1$ ).

### 3.4 $\mathbb{F}_{12}$ blown up at one point ( $\mathbb{F}_{12}^{[1]}$ )

We now consider the sequence of fiber geometries that can arise when we blow up consecutive points in  $\mathbb{F}_{12}$  that lie in a single fiber. Blowing up a generic point on  $\mathbb{F}_{12}$  gives a toric base with a single nontrivial fiber  $(-1, -1)$  containing curves  $C_2, C_1$ , as in the first step in figure 2. As discussed in section 2.3, blowing up a point when no gauge groups are involved leads to a shift in Hodge numbers of  $+1, -29$ . The generic elliptic fibration over the base  $\mathbb{F}_{12}$  with a single blow-up, which we denote  $\mathbb{F}_{12}^{[1]}$ , thus has Hodge numbers  $(12, 462)$ .

For the base  $\mathbb{F}_{12}^{[1]}$ , as for  $\mathbb{F}_{12}$ , there is no place that we can tune a gauge group other than the  $+11$  curve; as described in section 2.6, tuning an  $\mathfrak{su}(2)$  factor on either  $-1$  curve raises the degree of vanishing of  $f, g$  on  $S$  to  $(4, 6)$ , and is not possible. Any other tuning on the  $-1$  curves increases the degree of vanishing still further and is not allowed. The model with an  $\mathfrak{su}(2)$  on the  $+11$  curve is just the blow-up of the case with Hodge numbers  $(12, 318)$  and has Hodge numbers  $(13, 301)$  (note that the number of fundamental matter fields is reduced by 6 compared to the  $+12$  curve in  $\mathbb{F}_m$ ).

### 3.5 Threefolds over the base $\mathbb{F}_{12}^{[2]}$

Now consider blowing up a second point on  $\mathbb{F}_{12}$  by blowing up a point on  $\mathbb{F}_{12}^{[1]}$ . If the second point is a generic point that does not lie on the first blown-up fiber, we can take it to be on a separate fiber. The shift in Hodge numbers just adds between the two fibers and is then  $2 \times (+1, -29)$ , giving an EFS threefold with Hodge numbers  $(13, 433)$ .

Now, consider which points in the  $(-1, -1)$  fiber can be blown up and give a consistent model. We cannot blow up a point in  $C_1$  (the  $-1$  curve intersecting the  $-12$  curve), since then it would become a  $-2$  intersecting a  $-12$ , which is not allowed by the intersection rules of [19]. A representative  $\tilde{S}'$  of the (non-rigid)  $+11$  class passes through each point on  $C_2$  (this is one of the degrees of freedom in  $w_{\text{aut}}$  in (2.21)), so without loss of generality we can blow up any point in  $C_2$ , and we get a fiber of the form  $(-1, -2, -1)$ , which now connects a  $+10$  curve  $\tilde{S}'$  to a  $-12$  curve. In the absence of tuning, the corresponding Calabi-Yau threefold simply lies in a codimension one locus in the moduli space of complex structures of the threefold with Hodge numbers  $(13, 433)$  having two  $(-1, -1)$  fibers. Now, however, we consider what can be tuned on the  $(-1, -2, -1)$  fiber. For the same reason, described in the example in section 2.6, that we could not tune any gauge group on the upper  $-1$  of a  $(-1, -1)$  fiber, we cannot tune a gauge group on the  $-2$  curve ( $C_2$ ). Thus, the only curve on which we can tune any gauge group is the top  $-1$  curve  $C_3$ . It is easy to check that we can tune an  $\mathfrak{su}(2)$  on this top curve, either by tuning a type  $I_2$  singularity or the more specialized type  $III$ . This does not violate the Zariski or anomaly conditions, and explicit examination

of the Weierstrass model shows that this configuration is allowed. From table 4, we see that this tuning shifts the Hodge numbers by  $(+1, -17)$ , giving a Calabi-Yau with Hodge numbers  $(14, 416)$ . No other Calabi-Yau with  $h^{2,1} \geq 350$  can be formed by tuning a gauge group over  $\mathbb{F}_{12}^{[2]}$ . Some checking is needed, however, to confirm that no other gauge group can be tuned on  $C_3$ . From the analysis of section 2.6, the only allowed degrees of vanishing on  $C_2$  are  $(0, 0, 1)$  or  $(1, 1, 2)$ , so by the averaging rule the only way in which the degrees of vanishing on  $C_3$  could give any larger gauge algebra than  $\mathfrak{su}(2)$  is for a type *IV*  $(2, 2, 4)$  singularity carrying an  $\mathfrak{su}(3)$  gauge algebra. Expanding  $g = g_0(w) + g_1(w)\zeta + \dots$  in powers of a coordinate  $\zeta$  that vanishes on  $C_3$  (with  $w = 0$  on  $\tilde{S}$ ), the condition for an  $\mathfrak{su}(3)$  gauge group at a type *IV* singularity is that  $g_2$  be a perfect square. The highest power of  $w$  appearing in  $g_2$ , however, is  $w^7$ , corresponding to the single monomial of degree 5 in  $g$  over  $S$ . If  $g_2$  is a perfect square then this coefficient would have to vanish, giving a  $(4, 6)$  vanishing on  $S$ . Thus, there is only one possible tuning of  $\mathbb{F}_{12}^{[2]}$ , with a single  $\mathfrak{su}(2)$  on  $C_3$ .

### 3.6 Threefolds over the base $\mathbb{F}_{12}^{[3]}$

Now we consider blowing up a third point on  $\mathbb{F}_{12}$ . Unless all three points are on the same fiber, we simply have a combination of the previously considered configurations. On the twice blown up fiber  $(-1, -2, -1)$ , we cannot blow up on  $C_1$  or  $C_2$ , or we would have a cluster that is not allowed in such close proximity to the  $-12$  through the rules of [19]. So we can only blow up on the initial  $-1$  curve  $C_3$ . As above, a representative of the  $+10$  curve on  $\mathbb{F}_{12}^{[2]}$  passes through each point on  $C_3$ , so a blow-up at any such point gives the base  $\mathbb{F}_{12}^{[3]}$  with fiber  $(-1, -2, -2, -1)$ . This is on the same moduli space as the Calabi-Yau with Hodge numbers  $(14, 404)$  having three  $(-1, -1)$  fibers. We can, however, tune various gauge groups on  $\mathbb{F}_{12}^{[3]}$  that fix the  $-2$  structure in place. From the analysis of previous cases we know that we cannot tune a gauge group on  $C_1$  or  $C_2$ , and the only possible gauge algebra on  $C_3$  is  $\mathfrak{su}(2)$ . (Note that the argument from the previous section constraining the gauge group on  $C_3$  remains valid even when additional points are blown up). By the averaging rule, the largest possible vanishing orders of  $f, g, \Delta$  that are possible on  $C_4$  are  $(3, 3, 6)$ . A systematic analysis shows that we can tune the following gauge algebra combinations on the initial  $(-1, -2)$  curves  $C_4$  and  $C_3$ :

$$\cdot \oplus \mathfrak{su}(2) \rightarrow (h^{1,1}, h^{2,1}) = (15, 399) \tag{3.1}$$

$$\mathfrak{su}(2) \oplus \cdot \rightarrow (h^{1,1}, h^{2,1}) = (15, 387) \tag{3.2}$$

$$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \rightarrow (h^{1,1}, h^{2,1}) = (16, 386) \tag{3.3}$$

$$\mathfrak{su}(3) \oplus \mathfrak{su}(2) \rightarrow (h^{1,1}, h^{2,1}) = (17, 377) \tag{3.4}$$

$$\mathfrak{g}_2 \oplus \mathfrak{su}(2) \rightarrow (h^{1,1}, h^{2,1}) = (17, 371) \tag{3.5}$$

Note that in the last three cases, there is bifundamental matter. For example, in the case (3.3) the shift in  $h^{2,1}$  corresponds to the net change in  $V - H_{\text{ch}}$ . From table 4 we would expect  $-5 - 17 = -22$ , but there is a bifundamental  $\mathbf{2} \times \mathbf{2}$  from the intersection between the  $-1$  and  $-2$  curves so that 4 of the matter hypermultiplets have been counted twice, and the actual change to  $h^{2,1}$  is  $404 - 18 = 386$ .



All of the tunings (3.1)–(3.5) give consistent constructions of EFS Calabi-Yau threefolds. Note, however, that the threefold realized through (3.2) is not a generic threefold in the given branch of the moduli space. For this construction, the curve  $C_2$  is a  $-2$  curve without vanishing degree for  $\Delta$ . Thus, the threefold can be deformed by moving  $C_1$  to a different point on  $S$ . This gives a  $\mathbb{C}^*$  base with a single  $(-1, -1)$  fiber and a  $(-1, -2, -1)$  fiber with a single  $\mathfrak{su}(2)$  as can be tuned on  $\mathbb{F}_{12}^{[2]}$ . Checking the Hodge numbers, we see that indeed the resulting model is equivalent to the blow-up of the  $(14, 416)$  threefold at a generic point, so we do not list this construction separately in table 5.

The final case (3.5) is of particular interest, as it appears to give a Calabi-Yau threefold that did not arise in the complete classification by Kreuzer and Skarke of threefolds based on hypersurfaces in toric varieties. In this case there is a matter field charged under the  $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$  transforming in the  $(7, \frac{1}{2}\mathbf{2})$  (half-hypermultiplet in the fundamental of  $\mathfrak{su}(2)$ ), which raises  $h^{2,1}$  by 7:  $404 - 5 - 35 + 7 = 371$ . Given the apparent novelty of this construction, for this particular threefold we spell out some of the details of the Weierstrass monomial calculation that we have performed as a cross-check. After requiring that  $(f, g, \Delta)$  vanish to degree  $(2, 3, 6)$  on the  $(-1)$ -curve  $C_4$  and  $(2, 2, 4)$  on the adjacent  $(-2)$ -curve  $C_3$  ( $\Delta$  must vanish to degree 4 on  $C_3$  and to degree 2 on  $C_2$ , by the averaging rule), the number of Weierstrass monomials in  $f, g$  becomes

$$W_f = 125, \quad W_g = 260. \tag{3.6}$$

With  $w_{\text{aut}} = 1 + (9 + 1) = 11$ ,  $N_{-2} = G_1 = 0$ , we have then  $h^{2,1} = 125 + 260 - 11 - 3 = 371$ , in agreement with the expectation from anomaly cancellation. It is also straightforward to check that this set of Weierstrass monomials does not impose any unexpected  $(4, 6)$  vanishing on curves or points in the base which would invalidate the threefold construction. Because a  $(2, 3, 6)$  tuning is ambiguous, we consider the possible monodromies associated with the gauge group on  $C_4$ , which can be analyzed in terms of monomials in a local coordinate system. Expanding  $f = \sum_i \hat{f}_i \zeta^i$  and  $g = \sum_i \hat{g}_i \zeta^i$  in a coordinate  $\zeta$  that vanishes on  $C_4$ , the monodromy that determines the choice of gauge algebra  $\mathfrak{g}_2, \mathfrak{so}(7)$  or  $\mathfrak{so}(8)$  is determined by the form the polynomial containing the leading order terms in  $\zeta$  from the Weierstrass equation

$$x^3 + \hat{f}_2 x + \hat{g}_3, \tag{3.7}$$

where the coefficients  $\hat{f}_2$  and  $\hat{g}_3$  are functions on the  $-1$  curve  $C_4$  only of the usual coordinate  $w$ , which vanishes on  $\tilde{S}$ . The monodromy condition that selects the gauge group can be found from the factorization structure of (3.7),

$$\begin{aligned} x^3 + Ax + B \quad (\text{generic}) &\Rightarrow \mathfrak{g}_2 \\ (x - A)(x^2 + Ax + B) &\Rightarrow \mathfrak{so}(7) \end{aligned} \tag{3.8}$$

$$(x - A)(x - B)(x + (A + B)) \Rightarrow \mathfrak{so}(8). \tag{3.9}$$

From an analysis similar to that described in section section 2.6 (which can also be read off directly from figure 3, noting that the monomials  $\zeta^j w^k$  correspond to  $z^{j+3(n-k)} w^k$ , for  $n = 4, 6$  for  $f, g$  respectively), we find that  $\hat{f}_2, \hat{g}_3$  have the form

$$\hat{f}_2(w) = \hat{f}_{2,0} + \hat{f}_{2,1}w + \hat{f}_{2,2}w^2 + \hat{f}_{2,3}w^3 + \hat{f}_{2,4}w^4$$

$$\hat{g}_3(w) = \hat{g}_{3,0} + \hat{g}_{3,1}w + \hat{g}_{3,2}w^2 + \hat{g}_{3,3}w^3 + \hat{g}_{3,4}w^4 + \hat{g}_{3,5}w^5 + \hat{g}_{3,6}w^6 + \hat{g}_{3,7}w^7. \quad (3.10)$$

The  $w^7$  term in  $g_3$  corresponds to the monomial  $w^7$  with coefficient  $g_{0,7}$  in the original  $z, w$  coordinates, which as discussed above cannot be tuned to zero since this would force a  $(4, 6, 12)$  singularity on  $S$ . This implies, however, that (3.7) cannot have a nontrivial factorization. Any tuning of an  $\mathfrak{so}(7)$  gauge algebra, for example, must, upon expanding (3.8), yield  $\hat{f}_2 = B - A^2$ , which would imply that  $A$  must be no more than quadratic and  $B$  no more than quartic (a higher-order cancellation with  $A$  cubic and  $B$  sextic is not possible since this would lead to 9th order terms in  $g$ ). This means, however, that  $\hat{g}_3 = AB$  could be at most of order six; in other words, this factorization cannot be achieved without tuning the  $w^7$  term in  $\hat{g}_3$  to zero. A similar argument demonstrates that an  $\mathfrak{so}(8)$  cannot be tuned on  $C_4$ , but it is clear already that any tuning of  $\mathfrak{so}(8)$  involves at least the restrictions of  $\mathfrak{so}(7)$  on the monomials in question, hence the impossibility of  $\mathfrak{so}(7)$  implies the impossibility of  $\mathfrak{so}(8)$ . Thus, the presence of the  $w^7$  term in  $g$  guarantees that the monodromy associated with a Kodaira type  $I_0^*$  singularity over  $C_4$  must give an  $\mathfrak{g}_2$  gauge algebra, as in (3.5).

The upshot of this analysis is that the tuning (3.5) seems to give a threefold with Hodge numbers  $(17, 371)$  while no tuning beyond  $\mathfrak{g}_2$  is possible on  $C_4$ . Some possible subtleties in the interpretation of the  $(17, 371)$  threefold are discussed in section 4.3.3.

One other issue that should also be explained explicitly is the reason that it is not possible to tune an  $\mathfrak{su}(3)$  algebra on  $C_4$  without tuning a gauge group on  $C_3$ . It is straightforward to check using monomials that tuning  $g$  to vanish to order 2 on  $C_4$  forces  $g$  to also vanish to order 2 on  $C_3$ , so a type *IV*  $(2, 2, 4)$  vanishing on  $C_4$  forces a type *III*  $(1, 2, 3)$  vanishing at least on  $C_3$ , which must always be associated with a nonabelian gauge group. And tuning a  $(0, 0, 3)$  vanishing on  $C_4$  produces at least a  $(0, 0, 2)$  vanishing on  $C_3$  by the averaging rule, but by checking monomials we can verify that no vanishing is imposed on  $f$  or  $g$  on  $C_3$  so again tuning an  $\mathfrak{su}(3)$  on  $C_4$  necessarily imposes at least an  $\mathfrak{su}(2)$  on  $C_3$ .

This completes the classification of possible tuning structures that are possible on  $\mathbb{F}_{12}^{[3]}$ ; the resulting Calabi-Yau threefolds are tabulated in table 5.

### 3.7 Threefolds over the base $\mathbb{F}_{12}^{[4]}$

At the next stage, again we can only blow up on the first  $-1$  curve ( $C_4$ ) in the  $(-1, -2, -2, -1)$  fiber in  $\mathbb{F}_{12}^{[3]}$ , since for example a  $-12$  curve cannot be connected by a  $-1$  curve to a  $(-2, -3)$  cluster so we cannot blow up on the second  $(-2)$  curve  $C_3$ . Again, the Calabi-Yau threefold  $\mathbb{F}_{12}^{[4]}$  over the base with the resulting  $(-1, -2, -2, -2, -1)$  fiber is in the same moduli space as the one with four  $(-1, -1)$  fibers and has Hodge numbers  $(15, 375)$ . But there are an increasing number of possible gauge groups that can be tuned on the initial three curves  $C_5, C_4$ , and  $C_3$ .

All of the analysis performed for tunings on  $C_4-C_1$  in  $\mathbb{F}_{12}^{[3]}$  holds for tunings on these same curves in  $\mathbb{F}_{12}^{[4]}$ . Thus, each of the gauge groups tuned over  $\mathbb{F}_{12}^{[3]}$  can be tuned in a parallel fashion on  $\mathbb{F}_{12}^{[4]}$ . The only difference is that  $C_4$  is now a  $-2$  curve, so the gauge groups on that curve have reduced matter content and the change in Hodge numbers from the tuning decreases accordingly. For example, while tuning an  $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$  on  $C_4$  and

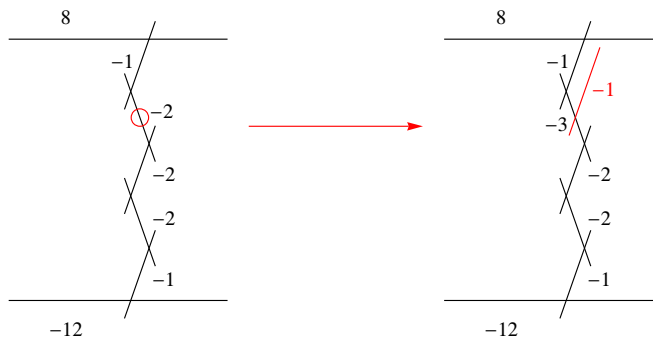
$C_3$  over  $\mathbb{F}_{12}^{[3]}$  shifts the Hodge numbers by  $(\Delta h^{1,1}, \Delta h^{2,1}) = (+2, -18)$  as discussed above, the shift for the same gauge group tuning on  $\mathbb{F}_{12}^{[4]}$  is  $(+2, -6)$  since there are 6 fewer matter fields in the fundamental  $\mathbf{2}$  representation of the  $\mathfrak{su}(2)$  over  $C_4$ . We can also confirm directly that none of the allowed tunings on  $C_4$ – $C_1$  impose any mandatory vanishing condition on  $C_5$ . Thus, the tunings (3.3)–(3.5) can all be done in a similar fashion, giving another set of threefolds tabulated in table 5, including another apparently new threefold not in the CY database at (18, 363). Note that the tuning (3.1) of a single  $\mathfrak{su}(2)$  on  $C_3$  in  $\mathbb{F}_{12}^{[4]}$  gives a threefold on the same moduli space as the blow-up at a generic point of this tuning on  $\mathbb{F}_{12}^{[3]}$ , with Hodge numbers (16, 370).

Finally, we can consider tuning a gauge group on  $C_5$  in combination with any other gauge groups on the other curves. As in the analysis in the previous section, if no gauge group is tuned on  $C_3$ , the threefold is non-generic since the curve  $C_1$  can be moved on  $S$ . By the averaging rule, tuning an  $\mathfrak{su}(2)$  on both  $C_3$  and  $C_5$  will also force an  $\mathfrak{su}(2)$  on  $C_4$ . An  $\mathfrak{su}(2)$  can be tuned on  $C_5$ , along with  $\mathfrak{su}(2)$  factors on  $C_4$ , and  $C_3$ , giving a threefold with Hodge numbers (18, 356). Enhancement of the  $\mathfrak{su}(2)$  on  $C_4$  to  $\mathfrak{su}(3)$  is then still possible, which yields a threefold with the Hodge numbers (19, 355). Note that these Hodge numbers are identical to those of a generic fibration over a five-times blown up  $\mathbb{F}_{12}$  (discussed below); this provides the first example of a situation where two apparently distinct constructions produce threefolds with identical Hodge numbers. The possible relationship between such models is discussed in section 4.3.4. Finally, the middle  $\mathfrak{su}(3)$  (on  $C_4$ ) can again be enhanced to  $\mathfrak{g}_2$ , yielding a model with Hodge numbers (19, 353), another new construction that does not appear in the Kreuzer-Skarke database. There are also a number of possible configurations where  $\mathfrak{su}(3)$  and larger gauge groups are tuned on  $C_5$ , but since  $C_5$  is a  $-1$  curve and gauge groups tuned on such divisors carry more matter, these all give threefolds with smaller Hodge numbers  $h^{2,1} < 350$ . One such tuning that is worth mentioning, however, is given by imposing the condition that  $\Delta$  vanish to degree 4 on  $C_5$ . This can be arranged, giving for example a model with gauge group  $\mathfrak{sp}(2) \oplus \mathfrak{su}(2)$  and Hodge numbers (20, 340), which arises in the Kreuzer-Skarke database. A more detailed exploration of these and other models with  $h^{2,1} < 350$  is left to further work. This completes the summary of threefolds based on tuning of  $\mathbb{F}_{12}^{[4]}$ .

### 3.8 Five blow-ups

At this stage the story becomes even more interesting. We can blow up the fiber  $(-1, -2, -2, -2, -1)$  again at an arbitrary point on  $C_5$  to get (A)  $\mathbb{F}_{12}^{[5]}$  with a resulting  $(-1, -2, -2, -2, -2, -1)$  fiber. We can also, however, blow up in two other ways. We can blow up the point of intersection between  $C_5$  and  $C_4$  giving a chain (B)  $(-2, -1, -3, -2, -2, -1)$ . Alternatively, we can blow up a generic point in the curve  $C_4$ , giving the fiber (C) shown in figure 5. In the latter case, the fiber and associated base no longer have a toric description. Let us consider these three cases in turn:

(A) This is the straightforward generalization of the previous examples,  $\mathbb{F}_{12}^{[5]}$  has Hodge numbers (16, 346), and is on the same moduli space as the base with five  $(-1, -1)$  fibers. There are a variety of tunings, which all have  $h^{2,1} < 350$  and which therefore for the present purposes we omit. It bears mentioning that tunings on the multiple  $-2$  curves



**Figure 5.** Blowing up on the top  $-2$  curve ( $C_4$ ) on a  $(-1, -2, -2, -2, -1)$  chain results in a divisor structure giving a non-toric base, with a  $(-3, -2, -2)$  non-Higgsable cluster (case (C) in the text). In the limit of moduli space where the intersection points of the two  $-1$  curves with the  $-3$  curve coincide, the fiber becomes  $(-2, -1, -3, -2, -2, -1)$  and the base becomes toric (case (B)).

in this base give a rich variety of possible threefolds, and it is at this point that larger algebras such as  $\mathfrak{f}_4$  and  $\mathfrak{e}_6$  can be tuned.

(B) In this case, as discussed in [20], the appearance of the non-Higgsable cluster  $(-3, -2)$  requires a non-Higgsable gauge algebra  $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ . The associated rank 3 gauge algebra with 17 vector multiplets and 8 charged matter multiplets raise the Hodge numbers of this base to  $(19, 355)$ . There are various tunings on  $C_6$ , but all go below  $h^{2,1} = 350$ , except for a single  $\mathfrak{su}(2)$  on  $C_6$  that gives a standard shift to a threefold with Hodge numbers  $(19, 355) + (1, -5) = (20, 350)$ . Note that without tuning, the initial  $-2$  curve in this fiber represents an extra Weierstrass modulus, so this is not a generic configuration, as discussed in the following case. Note also, however, that the analysis of section 2.4 shows that no gauge group can be tuned on the  $-1$  curve  $C_5$  since it is adjacent to a non-Higgsable cluster that is not a single  $-3$  curve.

(C) In this non-toric case we again have the same non-Higgsable cluster as in the previous case, and the same Hodge numbers  $(19, 355)$ . In this case there are also no tunings possible. This construction represents the generic class of threefolds of which the untuned model (B) above represents a codimension one limit. The final blow-up of a point in  $C_4$  in this non-toric construction can be taken to approach the point which was blown up to form  $C_5$  in  $\mathbb{F}_{12}^{[4]}$ , producing the  $-2$  curve found in (B).

### 3.9 More blow-ups

Further blow-ups raise the Hodge number  $h^{2,1}$  below 350. As the number of blow-ups increases, the number of fiber configurations also increases. We leave a systematic analysis of tuned models over further blown up bases for further work.

## 4 Conclusions

In this paper we have initiated a systematic analysis of the set of all elliptically fibered Calabi-Yau threefolds, starting with those having large Hodge number  $h^{2,1}$ . These Calabi-Yau threefolds fit together into a single connected space, with the continuous moduli spaces

$h^{11}$	$h^{21}$	Base	K-S #	$(\Delta h^{11}, \Delta h^{21})$	Fiber	$\mathcal{G}_{\text{extra}}$
11	491	$\mathbb{F}_{12}$	1	(0,0)	0	
12	462	$\mathbb{F}_{12}$	2	(1, -29)	11	
13	433	$\mathbb{F}_{12}$	4	$2 \times (1, -29)$	$2 \times (11)$	
14	416	$\mathbb{F}_{12}$	2	(3, -75)	<u>121</u>	$\mathfrak{su}(2)$
14	404	$\mathbb{F}_{12}$	6	$3 \times (1, -29)$	$3 \times (11)$	
15	399	$\mathbb{F}_{12}$	1	(4, -92)	<u>1221</u>	$\mathfrak{su}(2)$
15	387	$\mathbb{F}_{12}$	4	(1, -29) + (3, -75)	11 + <u>121</u>	$\mathfrak{su}(2)$
16	386	$\mathbb{F}_{12}$	1	(5, -105)	<u>1221</u>	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
17	377	$\mathbb{F}_{12}$	3	(6, -114)	<u>1221</u>	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$
10	376	$\mathbb{F}_8$	2	(0,0)	0	
		$\mathbb{F}_7$		(0,0)	0	
15	375	$\mathbb{F}_{12}$	9	$4 \times (1, -29)$	$4 \times (11)$	
17	371	$\mathbb{F}_{12}$	0	(6, -120)	<u>1221</u>	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$
16	370	$\mathbb{F}_{12}$	3	(1, -29) + (4, -92)	11 + <u>1221</u>	$\mathfrak{su}(2)$
17	369	$\mathbb{F}_{12}$	1	(6, -122)	<u>12221</u>	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
18	366	$\mathbb{F}_{12}$	2	(7, -125)	<u>12221</u>	$\mathfrak{su}(3) \oplus \mathfrak{su}(2)$
18	363	$\mathbb{F}_{12}$	0	(7, -128)	<u>12221</u>	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$
16	358	$\mathbb{F}_{12}$	7	$2 \times (1, -29) + (3, -75)$	$2 \times (11) + \underline{121}$	$\mathfrak{su}(2)$
17	357	$\mathbb{F}_{12}$	2	(1, -29) + (5, -105)	11 + <u>1221</u>	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
18	356	$\mathbb{F}_{12}$	1	(7, -135)	<u>12221</u>	$\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$
19	355	$\mathbb{F}_{12}$	3	(8, -136)	<u>213221</u>	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$
						(generically non-toric)
		$\mathbb{F}_{12}$	3	(8, -136)	<u>12221</u>	$\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2)$
19	353	$\mathbb{F}_{12}$	0	(8, -138)	<u>12221</u>	$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$
20	350	$\mathbb{F}_{12}$	1	(9, -141)	<u>213221</u>	$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$

**Table 5.** Table of all possible Calabi-Yau threefolds that are elliptically fibered with section and have  $h^{21} \geq 350$ . For each pair of Hodge numbers, the number of distinct constructions found by Kreuzer and Skarke giving these Hodge numbers is listed (0= new construction). The data for explicit construction through a tuned elliptic fibration over a blow-up of  $\mathbb{F}_{12}$  is given for each threefold. In each case, the fiber types and extra tuned gauge groups (beyond those forced from the structure of the original Hirzebruch base —  $\mathfrak{e}_8$  in all cases except the (10, 376) CY's) is indicated. Each fiber is given by a sequence of the (negative of the) self-intersection numbers of the curves in the fiber; underlined curves carry tuned gauge group factors, while overlined curves carry gauge group factors associated with non-Higgsable clusters.

associated with different topologies connected together through transitions between singular points in the different components of the moduli space. This structure is clearly and explicitly described in the framework of Weierstrass models. In principle, the approach taken here could be used to classify all EFS CY threefolds. There are, however, a number of practical and technical limitations to carrying out this analysis for the set of all threefolds with arbitrary Hodge numbers given the current state of knowledge. We describe these issues in section 4.1. A similar analysis could in principle be carried out for Calabi-Yau fourfolds, though in this context there are even larger unresolved mathematical questions, discussed in section 4.2. Some other comments on future directions are given in section 4.3.

#### 4.1 Classifying all EFS Calabi-Yau threefolds

In order to classify the complete set of EFS Calabi-Yau threefolds, some specific technical problems that begin to arise at smaller Hodge numbers need to be resolved. The primary outstanding issues seem to be the following 4 items:

**General bases.** A systematic means for explicitly enumerating the complete set of possible bases  $B_2$ , including bases that are neither toric nor “semi-toric” has not yet been developed.

**Tuning classical groups.** A general rule for determining when the gauge groups  $SU(N)$ ,  $SO(N)$ , and  $Sp(N)$  can be tuned on a given divisor is not known.

**Codimension two singularities.** A complete classification of codimension two singularities and associated matter representations has not yet been realized.

**Extra sections and abelian gauge group factors.** There is no general approach available yet for determining when an elliptic fibration of arbitrary Mordell-Weil rank can be tuned over a given base  $B_2$ .

We describe these issues in some further detail and summarize the current state of understanding for each issue in the remainder of this section. If all these issues can be resolved, it seems that the complete classification and enumeration of EFS CY threefolds may be a problem of tractable computational complexity, as discussed further in section 4.1.1.

##### 4.1.1 General bases

As discussed in section 2.3, the set of possible bases is constrained by the set of allowed non-Higgsable clusters of intersecting divisors with negative self-intersection [19], and a complete enumeration of all bases with toric and semi-toric ( $\mathbb{C}^*$ ) structure has been completed [20, 21]. In principle, there is no conceptual obstruction to explicitly enumerating the finite set of possible bases  $B_2$  that support an elliptically fibered CY threefold, but in practice this becomes rather difficult since the intersection structure can become rather complicated as more points are blown up. For bases with smaller values of  $h^{2,1}$  than those considered here, there are more ways in which points can be blown up without preserving a toric or  $\mathbb{C}^*$  structure. This leads to increasingly complicated branching structures in the set of intersecting divisors. It is a difficult combinatorial problem to track the new divisors

of negative self intersection that may appear as non-generic points are blown up in a base that has no  $\mathbb{C}^*$  symmetry. For example, new curves of negative self intersection may appear from curves of positive or vanishing self intersection that pass through multiple blown up points; in more extreme cases, a set of points may be blown up that lie on a highly singular codimension one curve, complicating the divisor intersection structure. A related issue is that the number of generators of the Mori cone of effective divisors can become large — indeed, for the del Pezzo surface  $dP_9$  formed by blowing up  $P^2$  at 9 generic points, the cone of effective divisors is generated by an infinite family of distinct  $-1$  curves.

While the combinatorics of this problem may seem forbidding, several pieces of evidence suggest that a complete enumeration may be a tractable problem. The analysis of  $\mathbb{C}^*$  bases in [21] shows that allowing certain kinds of branching and corresponding loops in the web of effective rigid divisors (associated with multiple fibers intersecting  $S, \tilde{S}$ ) does not dramatically increase the range of possible bases;<sup>5</sup> the full set of  $\mathbb{C}^*$  bases is several times larger than the number of toric bases ( $\sim 160,000$  vs.  $\sim 60,000$ ), but not exponentially larger. It also seems that as the complexity of  $\mathbb{C}^*$  bases increases, the range and complexity of non- $\mathbb{C}^*$  structures that can be added by further blow-ups decreases. Further work in this direction is in progress, but it seems likely that the total number of possible bases may not exceed the number already identified as toric or  $\mathbb{C}^*$ -bases by more than one or two orders of magnitude.

#### 4.1.2 Tuning $I_n$ and $I_n^*$ codimension one singularities

As described in section 2.4, section 2.5, and section 2.6, though the intersection structure of divisors, Zariski-type decomposition, and 6D anomalies can strongly restrict which gauge groups can be tuned over any given configuration of curves, these conditions are necessary but not sufficient, and to verify that a valid threefold with given structure exists a more direct method such as an explicit Weierstrass construction is needed. For gauge algebras such as  $\mathfrak{e}_7$  and  $\mathfrak{e}_8$  that are realized by tuning coefficients in  $f$  and  $g$  to get the desired Kodaira singularity types, it is fairly straightforward to confirm that Weierstrass models with the desired properties can be constructed. For algebras like  $\mathfrak{g}_2$ ,  $\mathfrak{e}_6$ , or  $\mathfrak{f}_4$  that involve monodromy but are still realized by tuning  $f, g$ , it is also possible to check the Weierstrass model directly by considering the set of allowed monomials in the specific model; examples of this were described in section 3. There are some types of gauge algebra, however, namely those realized by Kodaira type  $I_n$  and  $I_n^*$ , where the tuning required is on the discriminant  $\Delta$  and not directly on  $f, g$ . This leads to a more difficult algebra problem, since as  $n$  becomes larger, the set of required conditions become complicated polynomial conditions on the coefficients of  $f, g$ , rather than linear conditions as arise in all other cases.

An example of this kind of difficulty arises in considering the tuning of a Kodaira type  $I_n$  singularity giving an  $\mathfrak{su}(n)$  gauge algebra over a simple curve of degree one in the base  $B_2 = \mathbb{P}^2$ . In this case, anomaly cancellation conditions restrict the rank of the group so that  $n \leq 24$ . But explicit construction of the models for large  $n$  is algebraically somewhat complicated. In this case,  $f$  is a polynomial of degree 12 in local coordinates  $z, w$  in the base, and  $g$  is a polynomial of degree 18. If we consider a curve  $C$  defined by the locus

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<sup>5</sup>Some simple branching structures of this kind are also encountered in the classification of 6D superconformal field theories [48].

where  $z = 0$ , we can expand  $f, g, \Delta$  in the form e.g.  $f = f_{12}(w) + f_{11}(w)z + \dots + f_0z^{12}$ , where  $f_m$  is a polynomial of degree  $m$  in  $w$ . An explicit analysis of  $\mathfrak{su}(n)$  models in this context was carried out in [43], and Weierstrass models for these theories were found for  $n \leq 20, n = 22$ , and  $n = 24$ , but no models were identified for  $n = 21, 23$ . Similarly, in [49], Weierstrass models for elliptic fibrations over bases  $B_2 = \mathbb{F}_1, \mathbb{F}_2$  with Kodaira type  $I_n$  singularities over the curves  $S$  of self-intersection  $-1, -2$  in these bases were analyzed. Anomaly considerations suggest that in each case there are enough degrees of freedom to tune an  $I_{15}$  singularity, but solutions were only found algebraically up to  $n = 14$ .

In general, such algebraically complicated problems arise whenever one attempts to tune an  $I_n$  or  $I_n^*$  singularity. For a complete classification and enumeration of all elliptically fibered Calabi-Yau threefolds with section, either a direct method is needed for constructing a solution for the resulting set of polynomial equations on the coefficients of  $f, g$ , or some more general theorem is needed stating when this algebra problem has a solution. This problem is also in some cases apparently intertwined with the issue of determining the discrete part of the gauge group, associated with torsion in the Mordell-Weil group, as discussed in section 4.1.4.

### 4.1.3 Codimension two singularities

The possible singularity types at codimension two are not completely classified. In most simple cases, a local rank one enhancement of the gauge algebra gives matter that can be simply interpreted [29, 33, 50]. For example, at a point where an  $I_n$  singularity locus crosses a  $(0, 0, 1)$  component of the discriminant locus  $\Delta$  there is an enhancement to  $I_{n+1}$  corresponding to matter in the fundamental representation of the associated  $\mathfrak{su}(n)$ . In other cases, however, the singularities can be more complicated. Despite much recent progress in understanding codimension two singularities and associated matter content [31, 43, 51–57], there are still many aspects of codimension two singularities, even for Calabi-Yau threefolds, that are still not well understood or completely classified. In principle, however, there should be a systematic way of relating codimension two singularity types to representation theory in the same way that the Kodaira-Tate classification relates codimension one singularity types to Lie algebras.

One particular class of codimension two singularities that is not as yet systematically understood or classified are cases where the curve  $C$  that supports a Kodaira type singularity is itself singular. For simple singularity types, such as an intersection between two curves — which gives bifundamental matter — or a simple intersection of the curve with itself — which for  $\mathfrak{su}(n)$  gives an adjoint representation or a symmetric + antisymmetric representation — the connection between representation theory and geometric singularities is understood [42, 43]. For more exotic singularity types of  $C$ , however, there is as yet no full understanding. Analysis of anomalies in 6D theories [37] indicates that for any given representation  $\mathbf{R}$ , there is a corresponding singularity that contributes to the arithmetic genus of the curve  $C$  through

$$g_{\mathbf{R}} = \frac{1}{12}(2\lambda^2 C_{\mathbf{R}} + \lambda B_{\mathbf{R}} - \lambda A_{\mathbf{R}}), \tag{4.1}$$



where the anomaly coefficients  $A_R, B_R, C_R$  are defined through (2.16). For example, the **20** “box” representation of  $SU(4)$  should correspond to a singularity with arithmetic genus contribution 3 on the curve  $C$ ; while a potential realization of this representation through an embedding of an  $A_3$  singularity into a  $D_6$  singularity was suggested at the group theory level in [43], the explicit geometry of the associated singularity has not been worked out. Without a general theory for this kind of singularity structure, a complete classification of EFS CY threefolds will not be possible.

#### 4.1.4 Mordell-Weil group and abelian gauge factors

One of the trickiest issues that needs to be resolved for a complete classification of EFS Calabi-Yau threefolds to be possible is the problem of determining when additional non-trivial global sections of an elliptic fibration over a given base  $B_2$  can be constructed, and explicitly constructing them when possible. The construction of an explicit Weierstrass model depends on the existence of a single global section. Using the fiber-wise addition operation on elliptic curves (which corresponds to the usual addition law on  $T^2$ ), the set of global sections forms an abelian group known as the *Mordell-Weil* group. The Mordell-Weil group contains a free part  $\mathbb{Z}^k$  of rank  $k$ , and can also have discrete torsion associated with sections for which a finite multiple gives the identity (0 section). The rank of the Mordell-Weil group determines the number of abelian  $U(1)$  factors in the corresponding 6D gauge group [14]. In recent years there has been quite a bit of progress in understanding the role of the Mordell-Weil group and  $U(1)$  factors in F-theory constructions and corresponding supergravity theories [47, 58–74]. We review briefly here some of the parts of this story relevant for constructing EFS CY threefolds, and summarize some outstanding questions.

For a single  $U(1)$  factor (rank 1 Mordell-Weil group), a general form for the corresponding Weierstrass model was described by Morrison and Park in [62]. It was shown in [47] that a Weierstrass model with a single section of this type can always be tuned so that the global section, corresponding to a nontrivial four-cycle in the total space of the Calabi-Yau threefold, becomes “vertical” and is associated with a codimension one Kodaira type singularity giving a nonabelian gauge group factor in the 6D theory with matter in the adjoint representation. From the point of view of this paper, this means that any model with a rank one Mordell-Weil group can be constructed by first tuning an  $SU(2)$  or higher-rank non-abelian group on a curve of nonzero genus, and then Higgsing the group using the adjoint matter to give a residual  $U(1)$  gauge group factor. This should in principle make it possible to systematically construct all Calabi-Yau threefolds with rank one Mordell-Weil group.

For higher rank, the story becomes more complicated. Elliptic fibrations with Mordell-Weil groups of rank two and three can be realized by constructing threefolds where the fiber is realized in different ways from the Weierstrass form (1.1) [75, 76]. Explicit constructions of Weierstrass models for general classes of threefolds with rank two and three Mordell-Weil group were identified in [66–68] and [72] respectively, but there is no general construction for models with Mordell-Weil rank higher than three. CY threefolds with much larger Mordell-Weil rank have been constructed; it was shown in [21], in particular, that for certain  $\mathbb{C}^*$  bases there is an automatic (“non-Higgsable”) Mordell-Weil group of higher rank, with ranks up to  $k = 8$ . It must be possible to construct an elliptically fibered

Calabi-Yau threefold over the base  $\mathbb{P}^2$  with Mordell-Weil rank seven; this follows from the explicit construction in [43] of an  $SU(8)$  model with adjoint matter (with an  $I_8$  singularity on a cubic curve), which can be Higgsed to give  $U(1)^7$  (though the explicit Higgsed model has not been constructed). It is also possible that an  $SU(9)$  model with adjoint matter may exist on  $\mathbb{P}^2$ , which would give a Mordell-Weil rank of 8. It is not known whether all higher rank Mordell-Weil groups can be constructed by Higgsing higher rank nonabelian gauge groups; this would mean that the results of [47] or a single section could be generalized to an arbitrary number of sections, so that all global sections could simultaneously be tuned to vertical sections without changing  $h^{1,1}$ . If this were true, it would lead to a systematic approach to constructing all EFS CY threefolds with arbitrary Mordell-Weil rank, but more work is needed to understand this structure for higher rank models. It is also known that the Mordell-Weil rank cannot be arbitrarily high; for example, anomaly cancellation conditions in 6D impose the constraint that the rank satisfies  $k \leq 17$  when the base is  $\mathbb{P}^2$  [60], and this constraint can probably be strengthened considerably.

Beyond the rank of the Mordell-Weil group, which affects the Hodge numbers of the threefold formed by a particular Weierstrass model through (2.1), the torsion part of the Mordell-Weil group is also as yet incompletely understood. For a complete classification of EFS CY threefolds from Weierstrass models, a better understanding is needed of what kinds of torsion in the Mordell-Weil group are possible and how they can be tuned explicitly in Weierstrass models. In particular, while the Kodaira type dictates the Lie algebra of the corresponding 6D theory, the gauge group  $G$  may take the form  $\prod_i G_i/\Gamma$  where  $G_i$  are the associated simply connected groups and  $\Gamma$  is a discrete subgroup dictated by the torsion in the Mordell-Weil group. We have not studied this discrete structure in this work, but understanding it is necessary for a full classification of EFS CY threefolds. A systematic discussion of Mordell-Weil torsion is given in [34]. Some examples of Mordell-Weil groups with torsion are given, for example, in [47].

## 4.2 Classifying all EFS Calabi-Yau fourfolds

The methods of this paper can be used to analyze elliptically fibered Calabi-Yau manifolds of higher dimensionality, though there are more serious technical and conceptual obstacles to a complete classification of fourfolds or higher. Elliptically fibered Calabi-Yau fourfolds are of particular interest for F-theory compactifications to the physically relevant case of four space-time dimensions.

The classification of minimal bases  $B_2$  that support EFS Calabi-Yau threefolds, which formed the starting point of the analysis here of EFS CY3s with large  $h^{2,1}$ , depended upon the mathematical analysis of minimal surfaces and Grassi's result for minimal surfaces that support an elliptically fibered CY threefold. The analogous results for fourfolds are less well understood. In principle, the mathematics of Mori theory [77] can be used to determine the minimal set of threefold bases that support EFS Calabi-Yau fourfolds, but this story appears to be somewhat more complicated than the case of complex base surfaces. For fourfolds, the set of possible transitions associated with tuning Weierstrass models include blowing up curves as well as divisors, which further complicates the process of analyzing the set of bases, even given the set of minimal bases. Some basic aspects of

these transitions are explored in [75, 78, 79]. At least in the toric context, however, an analysis of CY fourfolds along the lines of this paper seems tractable. There has been some exploration of the space of Calabi-Yau fourfolds with a toric description [75, 80–83], and a complete enumeration of toric bases  $\mathcal{B}_3$  with a  $\mathbb{P}^1$  bundle structure that support elliptic fibrations for F-theory models with smooth heterotic duals was carried out in [44], along with a complete classification of non-toric threefold bases with this structure. A systematic analysis using methods analogous to [19, 20] of the space of all toric bases that support elliptically fibered CY fourfolds seems tractable, if computationally demanding. Note that since over many bases there are a vast number of different tunings, classifying the bases and associated generic Weierstrass models is a much more tractable problem than a complete classification of CY fourfold geometries.

### 4.3 Further directions

The analysis initiated in this paper can in principle be continued to substantially lower values of  $h^{2,1}$  before any of the issues described in section 4.1 become serious problems. Even outside the set of toric and  $\mathbb{C}^*$  bases, the number of ways that the Hirzebruch surfaces  $\mathbb{F}_m$  with large  $m$  can be blown up is fairly restricted. Algebraic problems with  $I_n$  and  $I_n^*$ , nontrivial Mordell-Weil groups, and exotic matter content are all issues that become relevant only at lower values of  $h^{2,1}$ . Further work in this direction is in progress, which may both reveal more about the structure of elliptically fibered Calabi-Yau threefolds and may also help provide specific situations in which the issues described in section 4.1 can be systematically addressed. There are a number of more general conceptual issues that can be addressed in the context of this program, which we discuss briefly here.

#### 4.3.1 Hodge number structures

The approach taken here, which in principle can systematically identify all elliptically fibered Calabi-Yau threefolds that admit a global section, is complementary to methods involving toric constructions that have been used in many earlier studies of the global space of CY threefolds. The systematic analysis by Kreuzer and Skarke [23] of CY threefolds that can be realized as hypersurfaces in toric varieties through the Batyrev construction [46] gives an enormous sample of Calabi-Yau threefolds whose Hodge numbers have clear structure and boundaries. The analysis of elliptically fibered threefolds through Weierstrass models groups the threefolds according to the base  $B_2$  of the elliptic fibration, and both simplifies the classification and enumeration of models and enables the systematic study of non-toric elliptically fibered CY threefolds. The fact, observed in [21, 22], that generic elliptic fibrations over both toric bases and a large class of non-toric bases span a similar range of Hodge numbers, with similar substructure and essentially the same boundary, suggests that these sets of threefolds are not just a small random subset, but may in some sense be a representative sample of all Calabi-Yau threefolds. In [84], Candelas, Constantin, and Skarke used the Batyrev construction and the method of “tops” [85] to analyze Calabi-Yau threefolds with an elliptic (K3) fibration structure and identified certain patterns in the set of associated Hodge numbers. Some of these patterns are clearly related to the transitions described through Weierstrass models as blow-ups and tuning of gauge groups.

For example, the characteristic shift by Hodge numbers of  $(+1, -29)$  is clearly seen from the Weierstrass based analysis as the set of blow-up transitions between distinct bases  $B_2$ . Similarly, shifts such as  $(+1, -17)$  can be seen as arising from transitions on the full threefold geometry associated with tuning an  $\mathfrak{su}(2)$  algebra on a  $-1$  curve, *etc.* In [84], another structure noted is the classification of fibrations into “ $E_8$ ,” “ $E_7$ ,” *etc.* types based on the way in which the elliptic fiber degenerates along the base. These correspond precisely in the Weierstrass/base picture to the families of threefolds that can be realized by blowing up points and tuning additional gauge groups over the bases  $\mathbb{F}_{12}, \mathbb{F}_8$ , *etc.*.

One structure that is manifest in the Hodge numbers of Calabi-Yau threefolds, however, which is not as transparent from the Weierstrass point of view is the mirror symmetry of the set of threefolds, which exchanges the Hodge numbers  $h^{1,1}$  and  $h^{2,1}$ . From the Batyrev point of view, mirror symmetry has a simple interpretation in terms of the dual polytope defining a toric variety used to construct a Calabi-Yau manifold. From the point of view of Weierstrass models of elliptic fibrations over fixed bases, however, it seems harder to understand, for example, how a blow-up transition with change in Hodge numbers  $(+1, -29)$  is related to a sequence of blow-ups that give a shift  $(+29, -1)$  and typically generate a full chain of divisors in the base associated with a gauge group factor  $E_8 \times F_4^2 \times (G_2 \times \text{SU}(2))^2$  [20, 22]. Understanding how these two different approaches of toric constructions based on reflexive polytopes and Weierstrass models on general bases can be brought together to improve our understanding of mirror symmetry and the structure of Hodge numbers for CY threefolds is an exciting direction for further work.

### 4.3.2 General Calabi-Yau’s with large Hodge numbers

The results presented here add to a growing body of evidence that the set of elliptically fibered CY threefolds with section may provide a useful guide in studying general Calabi-Yau threefolds, and may in fact dominate the set of possible Calabi-Yau threefolds. While there is as yet no clear argument that places any bound on the Hodge numbers of a general Calabi-Yau threefold, several pieces of empirical evidence seem to suggest that the CY threefolds with the largest Hodge numbers may in fact be those that are elliptically fibered. In this paper we have shown that all Hodge numbers for known Calabi-Yau manifolds that have  $h^{2,1} \geq 350$  are realized by elliptically fibered threefolds. It seems natural to speculate that the threefolds constructed here may constitute *all* Calabi-Yau threefolds (elliptically fibered or not) that lie above this bound. The results of [22] suggest that more generally, the outer boundary of the set of Hodge numbers for possible CY threefolds may be realized in a systematic way by elliptically fibered threefolds, and further empirical evidence from [84] also suggests that a large fraction of the models in the Kreuzer Skarke database with large Hodge numbers are elliptically fibered. Since the methods of this paper do not depend on toric geometry, it seems that this set of observations is not an artifact of the toric approach, but rather that those threefolds constructed using toric methods form a good sample, at least of those threefolds that admit elliptic fibrations. Other independent approaches to constructing Calabi-Yau manifolds have recently given further supporting evidence for the dominance of elliptically fibered manifolds in the set of Calabi-Yaus. In [86, 87], the complete set of Calabi-Yau fourfolds constructed as complete intersections in products of

projective spaces were constructed. From almost one million distinct constructions it was found that 99.95% admit at least one elliptic fibration; a similar analysis finds that 99.3% of threefolds that are complete intersections admit an elliptic fibration [88]. Taken together, these results suggest that it may be possible to prove that all Calabi-Yau threefolds have Hodge numbers that satisfy the inequality  $h^{1,1} + h^{2,1} \leq 491 + 11 = 502$ . Some initial exploration of one approach to finding such a bound from the point of view of the conformal field theory on the superstring world sheet has been undertaken in [89, 90].

### 4.3.3 New Calabi-Yau threefolds

In this paper we have identified three apparently new Calabi-Yau threefolds, with Hodge numbers  $(17, 371)$ ,  $(18, 363)$ , and  $(19, 353)$ . We have performed a number of checks to confirm that these models are consistent, which all work out, so naively these appear to represent a new set of Calabi-Yau threefolds. Continuing the analysis of this paper to lower Hodge numbers generates an increasingly large number of other new threefolds, particularly as the bases involved themselves become non-toric. There are several questions that might be studied related to the Hodge numbers found here that apparently describe new Calabi-Yau threefolds. One question is why these models do not appear in the Kreuzer-Skarke database. In particular, the new threefolds identified here are elliptic fibrations over toric bases, so we might expect in principle that they should appear in the Batyrev construction. One possible explanation may be that the structure of the tuned  $\mathfrak{g}_2$  gauge algebra that is common to all these constructions somehow takes the full space outside the context of hypersurfaces in toric varieties even though the base is still toric.

Another possible explanation for why they these new threefolds do not appear in the Kreuzer-Skarke database, however, may be related to the fact that they arise from tuning moduli in other Calabi-Yau threefolds that have the same value of  $h^{1,1}$  (the threefolds with Hodge numbers  $(17, 377)$ ,  $(18, 366)$ , and  $(19, 355)$  respectively), associated with the enhancement of  $\mathfrak{su}(3)$  to  $\mathfrak{g}_2$ . This means that the geometric transitions associated with these tunings are less dramatic than the other tunings and blow-ups since they do not actually change the dimension of  $H^{1,1}$ . One possible scenario is that these apparently new Calabi-Yau threefolds may actually represent special loci in the moduli spaces of the corresponding  $\mathfrak{su}(3)$  structure threefolds, and might not actually represent topologically distinct Calabi-Yau manifolds. This situation might be analogous to the tuning of moduli in a base to give a  $-2$  curve at a codimension one space in the moduli space, which changes the structure of the Mori cone but not the topology of the manifold. Further study of the detailed structure of these apparently new threefold constructions goes beyond the methods developed in this paper but should in principle be able to clarify this issue.

It would also be interesting to analyze the mirrors of these apparently new threefolds. A cursory check indicates that it is difficult to construct the threefolds with the mirror Hodge numbers where  $h^{1,1}$  and  $h^{2,1}$  are exchanged from tuned Weierstrass models as elliptic fibrations. This suggests either that the mirrors may not be elliptically fibered or that the second explanation given above is correct and that these are not actually topologically distinct Calabi-Yau threefolds. The former scenario would indicate that the dominance of elliptic fibrations may be asymmetric in the Hodge numbers. Further understanding of

these issues and construction of additional new Calabi-Yau threefolds using these methods are an interesting direction for further work.

#### 4.3.4 Uniqueness and equivalence of Calabi-Yau threefolds

Another difficult problem on which the methods of this paper may be able to shed some light is the question of when two Calabi-Yau threefolds, given by different data, are identical. In the Kreuzer-Skarke database there are many examples of Hodge numbers for which multiple toric constructions provide CY threefolds, as illustrated in table 5. A priori, it is difficult to tell when these threefolds represent the same complex manifold. Wall's theorem [91] states that when the Hodge numbers, triple intersection numbers, and first Pontryagin class of the threefolds are the same the spaces are the same as real manifolds, but even this does not guarantee that two manifolds live in the same complex structure moduli space. The problem of telling whether two sets of triple intersection numbers given in different bases are equivalent is also by itself a difficult computational problem. Thus, it is difficult to tell whether two Calabi-Yau manifolds given, for example, by the toric data in the Kreuzer-Skarke list, are identical.

The methods of this paper provide an approach that can resolve this kind of question in some cases. When the construction of an elliptically fibered Calabi-Yau threefold over a given base with specified Hodge numbers can be shown to be unique (up to moduli deformation) using the Weierstrass methods implemented here, this guarantees that any two CY threefolds that are both elliptically fibered and share these Hodge numbers must be identical as complex manifolds. In particular, with the exceptions of the Hodge number pairs (10, 376) and (19, 355), for all the Hodge numbers found in this paper with  $h^{2,1} > 350$ , there was a unique EFS CY threefold construction. It follows that any EFS CY threefolds with these Hodge numbers should be geometrically identical as Calabi-Yau manifolds. As an example, consider the elliptically-fibered Calabi-Yau threefold with Hodge numbers (12, 462). There are two distinct toric constructions of threefolds with these Hodge numbers in the Kreuzer-Skarke database. Both admit elliptically fibrations. As we have proved here in section 3.4, however, there is a unique construction of such a CY threefold, which is realized by considering the generic elliptic fibration over a base  $\mathbb{F}_{12}^{[1]}$  given by blowing up the Hirzebruch surface  $\mathbb{F}_{12}$  at any point not lying on the  $-12$  curve  $S$ . In principle, continuing this kind of argument to lower Hodge numbers might be able to significantly constrain the number of possible distinct Calabi-Yau threefolds that can be realized using known constructions. In principle this line of reasoning can also be applied at a more refined level by computing the triple intersection numbers for the manifolds in question. This approach may be able to distinguish some pairs of elliptic fibration constructions with identical Hodge numbers, such as the two constructions found here for threefolds with Hodge numbers (19, 353), or the generic elliptic fibrations over  $\mathbb{F}_7$  and  $\mathbb{F}_8$ , which both have Hodge numbers (10, 376). Of course, however, many CY threefolds are likely to admit multiple distinct elliptic fibrations (as found for fourfolds in [87]), so in some cases apparently distinct constructions of elliptic fibrations will still give equivalent Calabi-Yau threefolds. We leave further exploration of these interesting questions to future work.

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