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The closed-string 3-loop amplitude and S-duality

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Abstract: The low energy limit of the closed-string 3-loop amplitude (including its overall coefficient) is computed for four external massless states using the pure spinor formalism and the result is compared with a prediction of Green and Vanhove based on $\mathrm{SL}(2,\mathbb{Z})$ duality. Agreement is found provided the three-loop amplitude prescription includes a symmetry factor 1/3. We argue for its inclusion in order to compensate a $\mathbb{Z}_3$ symmetry present in genus-three Riemann surfaces.

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1 Introduction

Up to this day superstring amplitudes in ten-dimensional Minkowski space have never been computed at genus higher than two. In this paper the low energy limit of the genus three amplitude for four massless states in closed superstring theory is computed (including its overall coefficient) using the pure spinor formalism [1, 2].

After the relatively straightforward pure spinor derivation of the two-loop amplitude[1] in [4, 5], the natural question was how well the formalism would behave at higher genus. It is well-known by now that, in order to compute general amplitudes at genus higher than two the original BRST-invariant regulator of Berkovits [2] needs to be replaced by a more complicated scheme proposed by Berkovits and Nekrasov in [6]. Nevertheless, for four massless states at genus three one can still use the original regulator for the terms

[1]For the RNS derivation see the earlier works of D'Hoker and Phong, e.g. [3].

considered in this paper since they are F-terms and these were shown in [7] to be unaffected by the divergences which require the new regulator.

In addition to the regulator, there is one more point to consider though. As recently emphasized by Witten [8], to compute multiloop scattering amplitudes it is not sufficient to represent the external states by BRST-invariant vertex operators of definite conformal weight. The unintegrated vertex may have at most a simple pole singularity with the b-ghost whereas the integrated vertex must have no singularities at all. Unfortunately this is not the case for the massless pure spinor vertex operators of [1, 2] and one would probably need to use the vertices constructed in [9]. These vertices depend on the non-minimal variables and therefore require the concomitant use of the Berkovits-Nekrasov regulator.

Luckily, we will show that the low energy limit (of order D^6R^4) of the genus-three amplitude is not affected by these considerations because only the zero modes of the b-ghost enter in the derivation. Any subtlety is deferred to terms of order D^8R^4 and higher.

With that in mind, one can proceed with the three-loop computation using the formalism as described in [2]. And ever since the normalizations for the pure spinor measures were determined in [10] and systematically used in [11], keeping track of the overall normalization does not pose additional difficulties. In doing so, the precise normalization of the amplitude at order D^6R^4 is obtained and we can compare it with a prediction for this interaction made in 2005 by Green and Vanhove based on S-duality arguments [12]. We find that the results agree if the prescription for the three-loop amplitude includes an extra factor 1/3 and we argue that there is a $\mathbb{Z}_3$ symmetry of genus-three surfaces which should explain it.

2 Definitions and conventions

The non-minimal pure spinor formalism action for the left-moving sector reads [2]

$$S = \frac{1}{2\pi\alpha'}\int_{\Sigma_g} d^2z\,\left(\partial x^m\overline{\partial}x_m + \alpha' p_\alpha\overline{\partial}\theta^\alpha - \alpha' w_\alpha\overline{\partial}\lambda^\alpha - \alpha'\overline{w}^\alpha\overline{\partial}\overline{\lambda}_\alpha + \alpha' s^\alpha\overline{\partial}r_\alpha\right), \tag{2.1}$$

where λ^α and $\overline{\lambda}_\beta$ are bosonic pure spinors and r_α is a constrained fermionic variable,

$$(\lambda\gamma^m\lambda) = 0, \quad (\overline{\lambda}\gamma^m\overline{\lambda}) = 0, \quad (\overline{\lambda}\gamma^m r) = 0. \tag{2.2}$$

The fields in (2.1) have the following space-time dimensions [10]

$$[\alpha'] = 2,\ [x^m] = 1, \quad [\theta^\alpha, \lambda^\alpha, \overline{w}^\alpha, s^\alpha] = 1/2, \quad [p_\alpha, w_\alpha, \overline{\lambda}_\alpha, r_\alpha] = -1/2. \tag{2.3}$$

The genus-g OPEs for the matter variables following from (2.1) are [13]

$$x^m(z,\overline{z})\,x_n(w,\overline{w}) \sim \delta^m_n G(z,w), \qquad p_\alpha(z)\,\theta^\beta(w) \sim \delta^\beta_\alpha\eta(z,w), \tag{2.4}$$

where the Green's function $G(z,w)$ is written in terms of the prime form $E(z,w)$ and the global holomorphic 1-forms $w_I(z)$ as [14]

$$G(z_i,z_j) = -\frac{\alpha'}{2}\ln\left|E(z_i,z_j)\right|^2 + \alpha'\pi\left(\mathrm{Im}\int_{z_i}^{z_j} w_I\right)(\mathrm{Im}\,\Omega)^{-1}_{IJ}\left(\mathrm{Im}\int_{z_i}^{z_j} w_J\right), \tag{2.5}$$

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and satisfies $\frac{2}{\alpha'}\partial_{z_i}\overline{\partial}_{z_j}G(z_i,z_j) = 2\pi\delta^{(2)}(z_i-z_j) - \pi\Omega(z_i,z_j)$, where

$$\Omega(z_i,z_j) \equiv \sum_{I,J=1}^{3} w_I(z_i)(\mathrm{Im}\,\Omega)^{-1}_{IJ}\overline{w}_J(\overline{z}_j)\,, \tag{2.6}$$

and Ω_{IJ} is the period matrix which will be defined below. Furthermore,

$$\eta(z_i,z_j) = \eta_{ij} \equiv -\frac{2}{\alpha'}\frac{\partial}{\partial z_i}G(z_i,z_j). \tag{2.7}$$

The Green-Schwarz constraint $d_\alpha(z)$ and the supersymmetric momentum $\Pi^m(z)$ are

$$d_\alpha = p_\alpha - \frac{1}{\alpha'}(\gamma^m\theta)_\alpha\partial x_m - \frac{1}{4\alpha'}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta), \qquad \Pi^m = \partial x^m + \frac{1}{2}(\theta\gamma^m\partial\theta) \tag{2.8}$$

and satisfy the following OPEs [15]

$$\begin{aligned} d_\alpha(z)d_\beta(w) &\sim -\frac{2}{\alpha'}\frac{\gamma^m_{\alpha\beta}\Pi_m}{z-w}, & d_\alpha(z)f(x(w),\theta(w)) &\sim \frac{D_\alpha f}{z-w}, \\ d_\alpha(z)\Pi^m(w) &\sim \frac{\gamma^m_{\alpha\beta}\partial\theta^\beta}{z-w}, & \Pi^m(z)f(x(w),\theta(w)) &\sim -\frac{\alpha'}{2}\frac{k^m f}{z-w} \end{aligned} \tag{2.9}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha k_m$ is the supersymmetric derivative and $f(x,\theta)$ represents a generic superfield. The b-ghost is given by [2] (see also [16, 17])

$$\begin{aligned} b = {} & s^\alpha\partial\overline{\lambda}_\alpha + \frac{1}{4(\lambda\overline{\lambda})}\left[2\Pi^m(\overline{\lambda}\gamma_m d) - N_{mn}(\overline{\lambda}\gamma^{mn}\partial\theta) - J_\lambda(\overline{\lambda}\partial\theta) - (\overline{\lambda}\partial^2\theta)\right] \\ & + \frac{(\overline{\lambda}\gamma^{mnp}r)}{192(\lambda\overline{\lambda})^2}\left[\frac{\alpha'}{2}(d\gamma_{mnp}d) + 24N_{mn}\Pi_p\right] - \frac{\alpha'}{2}\frac{(r\gamma_{mnp}r)}{16(\lambda\overline{\lambda})^3}\left[(\overline{\lambda}\gamma^m d)N^{np} - \frac{(\overline{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{8(\lambda\overline{\lambda})}\right], \end{aligned} \tag{2.10}$$

and satisfies $\{Q,b(z)\} = T(z)$ where the BRST charge Q and the energy-momentum tensor $T(z)$ are

$$Q = \oint(\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha), \qquad T(z) = -\frac{1}{\alpha'}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + w_\alpha\partial\lambda^\alpha + \overline{w}^\alpha\partial\overline{\lambda}_\alpha - s^\alpha\partial r_\alpha.$$

From (2.3) it follows that $[Q] = [b] = [T] = 0$.

The massless vertex operators are given by $V(z,\overline{z}) = \kappa V(\theta)\otimes\tilde{V}(\overline{\theta})\,e^{ik\cdot x}$ and $\mathrm{U}(z,\overline{z}) = \kappa\mathrm{U}(\theta)\otimes\tilde{U}(\overline{\theta})\,e^{ik\cdot x}$, where

$$V(z) = \lambda^\alpha A_\alpha, \qquad \mathrm{U}(z) = \partial\theta^\alpha A_\alpha + A_m\Pi^m + \frac{\alpha'}{2}d_\alpha W^\alpha + \frac{\alpha'}{4}N_{mn}\mathcal{F}^{mn} \tag{2.11}$$

and $A_\alpha, A^m, W^\alpha, \mathcal{F}^{mn}$ are the $\mathcal{N}=1$ super-Yang-Mills superfields in ten dimensions satisfying [18]

$$\begin{aligned} D_\alpha A_\beta + D_\beta A_\alpha &= \gamma^m_{\alpha\beta}A_m, & D_\alpha A_m &= (\gamma_m W)_\alpha + k_m A_\alpha \\ D_\alpha\mathcal{F}_{mn} &= 2k_{[m}(\gamma_{n]}W)_\alpha, & D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathcal{F}_{mn}. \end{aligned} \tag{2.12}$$

The space-time dimensions of the superfields and the vertex operators are

$$[A_\alpha] = 1/2, \quad [A_m] = 0, \quad [W^\alpha] = -1/2, \quad [\mathcal{F}_{mn}] = -1, \qquad [V(z)] = [\mathrm{U}(z)] = 1. \tag{2.13}$$

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2.1 Integration on pure spinor space

The zero-mode measures for the non-minimal pure spinor variables in a genus-g surface have space-time dimension zero and are given by [2, 10, 11]

$$\begin{aligned}
&[d\lambda]\, T_{\alpha_1\ldots\alpha_5} = c_\lambda\, \epsilon_{\alpha_1\ldots\alpha_{16}} d\lambda^{\alpha_6}\ldots d\lambda^{\alpha_{16}} && [dw] = c_w\, T_{\alpha_1\ldots\alpha_5}\epsilon^{\alpha_1\ldots\alpha_{16}} dw_{\alpha_6}\ldots dw_{\alpha_{16}}\\
&[d\overline{\lambda}]\, \overline{T}^{\alpha_1\ldots\alpha_5} = c_{\overline{\lambda}}\, \epsilon^{\alpha_1\ldots\alpha_{16}} d\overline{\lambda}_{\alpha_6}\ldots d\overline{\lambda}_{\alpha_{16}} && [d\overline{w}]\, T_{\alpha_1\ldots\alpha_5} = c_{\overline{w}}\, \epsilon_{\alpha_1\ldots\alpha_{16}} d\overline{w}^{\alpha_6}\ldots d\overline{w}^{\alpha_{16}}\\
&[dr] = c_r\, \overline{T}^{\alpha_1\ldots\alpha_5}\epsilon_{\alpha_1\ldots\alpha_{16}}\partial_r^{\alpha_6}\ldots\partial_r^{\alpha_{16}} && [ds^I] = c_s\, T_{\alpha_1\ldots\alpha_5}\epsilon^{\alpha_1\ldots\alpha_{16}}\partial^{s^I}_{\alpha_6}\ldots\partial^{s^I}_{\alpha_{16}}\\
&[d\theta] = c_\theta\, d^{16}\theta && [dd^I] = c_d\, d^{16} d^I .
\end{aligned} \tag{2.14}$$

The normalizations are

$$\begin{aligned}
c_\lambda &= \left(\frac{\alpha'}{2}\right)^{-2}\frac{1}{11!}\left(\frac{A_g}{4\pi^2}\right)^{11/2} & c_w &= \left(\frac{\alpha'}{2}\right)^{2}\frac{(2\pi)^{-11}}{11!\,5!}\, Z_g^{-11/g}\\
c_{\overline{\lambda}} &= \left(\frac{\alpha'}{2}\right)^{2}\frac{2^6}{11!}\left(\frac{A_g}{4\pi^2}\right)^{11/2} & c_{\overline{w}} &= \left(\frac{\alpha'}{2}\right)^{-2}\frac{(\lambda\overline{\lambda})^3}{11!\,(2\pi)^{11}} Z_g^{-11/g}\\
c_r &= \left(\frac{\alpha'}{2}\right)^{-2}\frac{R}{11!5!}\left(\frac{2\pi}{A_g}\right)^{11/2} & c_s &= \left(\frac{\alpha'}{2}\right)^{2}\frac{(2\pi)^{11/2}R^{-1}}{2^6 11!\,5!\,(\lambda\overline{\lambda})^3} Z_g^{11/g}\\
c_\theta &= \left(\frac{\alpha'}{2}\right)^{4}\left(\frac{2\pi}{A_g}\right)^{16/2} & c_d &= \left(\frac{\alpha'}{2}\right)^{-4}(2\pi)^{16/2}\, Z_g^{16/g} .
\end{aligned} \tag{2.15}$$

where $A_g = \int d^2z\sqrt{g}$ is the area of the genus-g Riemann surface and

$$Z_g = \frac{1}{\sqrt{\det(2\,\mathrm{Im}\,\Omega)}}, \qquad g \geq 1. \tag{2.16}$$

The tensors $T_{\alpha_1\ldots\alpha_5}$ and $\overline{T}^{\alpha_1\ldots\alpha_5}$ in (2.14),

$$\begin{aligned}
T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} &= (\lambda\gamma^m)_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^p)_{\alpha_3}(\gamma_{mnp})_{\alpha_4\alpha_5}\\
\overline{T}^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} &= (\overline{\lambda}\gamma^m)^{\alpha_1}(\overline{\lambda}\gamma^n)^{\alpha_2}(\overline{\lambda}\gamma^p)^{\alpha_3}(\gamma_{mnp})^{\alpha_4\alpha_5}
\end{aligned} \tag{2.17}$$

are totally antisymmetric due to the pure spinor constraint (2.2) and satisfy $T\cdot\overline{T} = 5!\,2^6(\lambda\overline{\lambda})^3$. As explained in [11], setting $R^2 = \frac{\sqrt{2}}{2^{16}\pi}$ fixes the normalization of pure spinor tree-level amplitudes to be same as in the RNS computations of [19].

Using the above measures and the results of [10] one can show that the integration over an arbitrary number of pure spinors λ^α and $\overline{\lambda}_\beta$ is given by

$$\int [d\lambda][d\overline{\lambda}] e^{-(\lambda\overline{\lambda})}(\lambda\overline{\lambda})^m \lambda^{\alpha_1}\cdots\lambda^{\alpha_n}\overline{\lambda}_{\beta_1}\cdots\overline{\lambda}_{\beta_n} = \left(\frac{A_g}{2\pi}\right)^{11}\frac{12\,\Gamma(8+m+n)}{\Gamma(11)}\mathcal{T}^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}, \tag{2.18}$$

where $\mathcal{T}^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n}$ are the γ-matrix traceless tensors discussed in the appendix A and $\Gamma(x)$ is the gamma function. Using $\mathcal{T}^{\alpha_1\ldots\alpha_p}_{\alpha_1\ldots\alpha_p} = 1$ it follows that [11]

$$\int [d\lambda][d\overline{\lambda}](\lambda\overline{\lambda})^n e^{-(\lambda\overline{\lambda})} = \left(\frac{A_g}{2\pi}\right)^{11}\frac{\Gamma(8+n)}{7!\,60}\,. \tag{2.19}$$

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For an arbitrary superfield $M(\lambda,\overline{\lambda},\theta,r)$ we define

$$\langle M(\lambda,\overline{\lambda},\theta,r)\rangle_{(n,g)} \equiv \int [d\theta][dr][d\lambda][d\overline{\lambda}]\, \frac{\mathrm{e}^{-(\lambda\overline{\lambda})-(r\theta)}}{(\lambda\overline{\lambda})^{3-n}}\, M(\lambda,\overline{\lambda},\theta,r)\,, \tag{2.20}$$

which implies in particular that

$$\langle(\lambda^3\theta^5)\rangle_{(n,g)} = 2^7 R \left(\frac{2\pi}{A_g}\right)^{5/2}\left(\frac{\alpha'}{2}\right)^2 \frac{\Gamma(8+n)}{7!}\,\langle(\lambda^3\theta^5)\rangle\,, \tag{2.21}$$

where $(\lambda^3\theta^5)\equiv(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ and the pure spinor bracket $\langle\ldots\rangle$ in the right-hand side is normalized as $\langle(\lambda^3\theta^5)\rangle = 1$ [1]. The subscript g will be dropped whenever there is no chance for confusion.

From (2.9) and (2.8) follows that

$$\Pi^m(z_i)\overline{\Pi}^n(\overline{z}_j) \sim \frac{\alpha'}{2}\eta^{mn}\Big(2\pi\delta^{(2)}(z_i-z_j) - \pi\Omega(z_i,z_j)\Big)\,. \tag{2.22}$$

But using (2.22) directly leads to a mixing between left- and right-movers. Instead, one can keep the two sectors separate by expanding $\Pi^m(z) = \hat{\Pi}^m(z) + \sum_{I=1}^g \Pi^m_I w_I(z)$ and computing the holomorphic square with

$$\Pi^m_I\overline{\Pi}^n_J = -\frac{\alpha'}{2}\eta^{mn}\pi\,(\mathrm{Im}\,\Omega)^{-1}_{IJ}. \tag{2.23}$$

Using this prescription, contributions containing a single Π^m_I or $\overline{\Pi}^m_I$ vanish.

We use conventions where the (anti)symmetrization over n indices includes a factor of $1/n!$, the generalized Kronecker delta is $\delta^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} \equiv \delta^{[\alpha_1}_{\beta_1}\cdots\delta^{\alpha_n]}_{\beta_n}$ and satisfies $\delta^{\alpha_1\ldots\alpha_n}_{\alpha_1\ldots\alpha_n} = \binom{d}{n}$ where $d=10$ or $d=16$ for vector or spinor indices respectively. The integration over θ is given by $\int d^{16}\theta\,\theta^{\alpha_1}\cdots\theta^{\alpha_{16}} = \epsilon^{\alpha_1\ldots\alpha_{16}}$ and $\epsilon^{\alpha_1\ldots\alpha_{11}\sigma_1\ldots\sigma_5}\epsilon_{\alpha_1\ldots\alpha_{11}\beta_1\ldots\beta_5} = 11!5!\,\delta^{\sigma_1\ldots\sigma_5}_{\beta_1\ldots\beta_5}$,

The partition of 3-loop d_α zero-modes is denoted by $(N_1,N_2,N_3)_d$ and indicates that an expression contains N_I factors of d^I_α. Furthermore, we define

$$(\epsilon\cdot T\cdot d^I) \equiv \epsilon^{\alpha_1\ldots\alpha_{16}}T_{\alpha_1\ldots\alpha_5}d^I_{\alpha_6}\cdots d^I_{\alpha_{16}}\,, \quad (\overline{\lambda}rd^Id^J) \equiv (\overline{\lambda}\gamma^{mnp}r)(d^I\gamma_{mnp}d^J). \tag{2.24}$$

Two integrals frequently used in the next sections are summarized here,

$$\begin{aligned}\int [dd^I](\epsilon\cdot T\cdot d^I)\, d^I_{\alpha_1}d^I_{\alpha_2}d^I_{\alpha_3}d^I_{\alpha_4}d^I_{\alpha_5} &= 11!\,5!\,c_d\,T_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} \\ \int [dd^I](\epsilon\cdot T\cdot d^I)\, d^I_{\alpha_1}d^I_{\alpha_2}d^I_{\alpha_3}(d^I\gamma^{mnp}d^I) &= 11!\,5!\,96\,c_d\,(\lambda\gamma^{[m})_{\alpha_1}(\lambda\gamma^n)_{\alpha_2}(\lambda\gamma^{p]})_{\alpha_3}\,.\end{aligned} \tag{2.25}$$

2.2 Four-point SYM amplitude and kinematics

In [11, 19] the amplitudes in the Neveu-Schwarz (NS) sector were written using the kinematic factor K defined as

$$\begin{aligned}K \;=\; & F_1^{mn}F_2^{nm}F_3^{pq}F_4^{qp} + F_1^{mn}F_3^{nm}F_2^{pq}F_4^{qp} + F_1^{mn}F_4^{nm}F_2^{pq}F_3^{qp} \\ & -4\big(F_1^{mn}F_2^{np}F_3^{pq}F_4^{qm} + F_1^{mn}F_3^{np}F_2^{pq}F_4^{qm} + F_1^{mn}F_2^{np}F_4^{pq}F_3^{qm}\big)\end{aligned} \tag{2.26}$$

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where $F_{mn} = k_m e_n - k_n e_m$ is the field-strength. Since the amplitudes in the pure spinor formalism are manifestly supersymmetric one can rewrite K as follows

$$K = -2^3\, 2880\, A^{\rm YM}_{1234}\, s_{12} s_{23} \tag{2.27}$$

where $A^{\rm YM}_{1234}$ is the ten-dimensional SYM amplitude normalized as $A^{\rm YM}_{1234} = \langle V_1 E_{234} \rangle$ [20] and $s_{ij} = k^i \cdot k^j$ are the Mandelstam invariants. Furthermore $k^i \cdot k^i = 0$ is the massless condition and $k^1_m + \cdots + k^4_m = 0$ is the momentum conservation relation. In order to keep the momentum expansion formulæ of section 4 legible, we use the following definitions [21],

$$\sigma_2 = \left(\frac{\alpha'}{2}\right)^2 (s_{12}^2 + s_{13}^2 + s_{14}^2), \quad \sigma_3 = \left(\frac{\alpha'}{2}\right)^3 (s_{12}^3 + s_{13}^3 + s_{14}^3) \tag{2.28}$$

and note that $\sigma_3 = 3(\alpha'/2)^3 s_{12} s_{13} s_{14}$.

2.3 Riemann surfaces

A holomorphic field with conformal-weight one in a genus-g Riemann surface Σ_g can be expanded in a basis of holomorphic one-forms as $\phi(z) = \hat{\phi}(z) + \sum_{I=1}^g w_I(z)\phi^I$ and ϕ^I are the *zero modes* of $\phi(z)$. If $\{a_I, b_J\}$ are the generators of the $H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$ homology group, the holomorphic one-forms can be chosen such that for $I, J = 1, 2, \ldots, g$

$$\int_{a_I} w_J(z)\, dz = \delta_{IJ}, \quad \int_{b_I} w_J(z)\, dz = \Omega_{IJ}, \quad \int_{\Sigma_g} w_I\, \overline{w}_J\; d^2z = 2\ {\rm Im}\, \Omega_{IJ} \tag{2.29}$$

where Ω_{IJ} is the symmetric period matrix with $g(g+1)/2$ complex degrees of freedom [22] and $d^2z = idz \wedge d\overline{z} = 2\, d{\rm Re}(z) d{\rm Im}(z)$. For the three-loop amplitude we define

$$\begin{aligned} \Delta(z_i; z_j; z_k) &\equiv \epsilon^{IJK} w_I(z_i) w_J(z_j) w_K(z_k), \\ \Delta^m(z_i, z_j; z_k; z_l) &\equiv \epsilon^{IJK} (\Pi w)^m_I(z_i, z_j) w_J(z_k) w_K(z_l), \end{aligned} \tag{2.30}$$

where $(\Pi w)^m_I(z_i, z_j) \equiv \Pi^m_I w_I(z_i) w_I(z_j)$, (no sum in I). It follows that $\Delta^m(z_i, z_j; z_k; z_l)$ is symmetric in (ij) and antisymmetric in $[kl]$ and satisfies

$$\Pi^m_I w_I(z_i) \Delta(z_j; z_k; z_l) = \Delta^m(z_i, z_j; z_k; z_l) + \Delta^m(z_i, z_k; z_l; z_j) + \Delta^m(z_i, z_l; z_j; z_k)\,. \tag{2.31}$$

Furthermore, the period matrix extends a lattice called the Jacobian variety [14, 22], $J = \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g)$, which is invariant under the modular group ${\rm Sp}(2g, \mathbb{Z})$. And finally, we define

$$\int_{\Sigma_4} \equiv \int \prod_{i=1}^4 d^2 z_i\,. \tag{2.32}$$

2.3.1 Riemann surfaces of genus $g = 1, 2, 3$

It is well known that any Riemann surface of genus $g = 1, 2$ can be written as a hyperelliptic curve, i.e they admit a 2-1 map $f : \Sigma_g \to \mathbb{C}P^1$ to the Riemann sphere [22]. For instance, for genus $g = 1$ one has

$$y^2 = \prod_{i=1}^4 (z - \lambda_i), \qquad g = 1 \tag{2.33}$$

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and for genus $g = 2$ the curve is

$$y^2 = \prod_{i=1}^{6}(z - \lambda_i), \qquad g = 2, \tag{2.34}$$

where the $\{\lambda_i \in \mathbb{C}\}$ are the branch points. Since $PGL(2, \mathbb{C})$ is the automorphism group of $\mathbb{C}P^1$ three of the $\{\lambda_i\}$ branch points in (2.33) and (2.34) can be fixed. Setting $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda_3 = \infty$ leads to

$$\begin{aligned} y^2 &= z(z-1)(z-a), && g = 1, \\ y^2 &= z(z-1)(z-a_1)(z-a_2)(z-a_3), && g = 2, \end{aligned} \tag{2.35}$$

where the free complex parameters $\{a\}$ and $\{a_1, a_2, a_3\}$ parameterise the $g = 1$ and $g = 2$ moduli space, respectively. The two equations in (2.35) are clearly invariant under the $\mathbb{Z}_2$ symmetry $y \to -y$, implying that the one- [23, 24] and two-loop [11, 25] amplitude prescriptions must be multiplied by a symmetry factor 1/2.

Although not all Riemann surfaces of genus three are hyperelliptic, this surface is still a special case. Every genus-three Riemann surface can be embedded in $\mathbb{C}P^2$ as a quartic curve, i.e.

$$G_{IJKL} Z^I Z^J Z^K Z^L = 0, \tag{2.36}$$

where $Z^I = (x, y, z)$ are the homogeneous coordinates of $\mathbb{C}P^2$ [22]. This means a Riemann surface of $g = 3$ is a global holomorphic section on the $\mathcal{O}(4)$ line bundle over $\mathbb{C}P^2$. The number of the global holomorphic sections on $\mathcal{O}(4)$ is given by the dimension of the $H^0(\mathbb{C}P^2, \mathcal{O}(4))$ Čech cohomology group, which is

$$h^0(\mathbb{C}P^2, \mathcal{O}(4)) = \binom{6}{2} = 15. \tag{2.37}$$

So, the number of free parameters in (2.36) is $15 - 1 = 14$, where the number 1 accounts for the fact that (2.36) is invariant under the scale symmetry $Z^I \to tZ^I$, $t \in \mathbb{C}^*$. Since the automorphism group of $\mathbb{C}P^2$ is $PGL(3, \mathbb{C})$, which has dimension $\dim(PGL(3, \mathbb{C})) = 8$, we can fix 8 of the 14 free parameters. Therefore every Riemann surface of genus $g = 3$ depends on 6 parameters and its algebraic curve (2.36) can be written, without loss of generality, as

$$xy^3 + a_1 zy^3 + a_2 x^4 + a_3 x^3 z + a_4 x^2 z^2 + a_5 x z^3 + a_6 z^4 = 0, \tag{2.38}$$

where $\{a_i\}$ parameterise the moduli space. This curve is invariant under the $\mathbb{Z}_3$ symmetry $y \to e^{2\pi i n/3} y$, with $n = 0, 1, 2$, which implies that the three-loop scattering amplitude prescription must be multiplied by the 1/3 factor.

2.3.2 Moduli space

The moduli space $\mathcal{M}_g$ is defined as the space of inequivalent complex structures on the Riemann surface Σ_g. It is well known that its complex dimension is $\dim_{\mathbb{C}}(\mathcal{M}_g) = 3g - 3$, for $g > 1$. We denote the complex coordinates on this space by τ_i for $i = 1, \ldots, 3g - 3$.

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For genus two and three the dimension of the moduli space is the same as the dimension of the period matrices, i.e., $3g-3 = g(g+1)/2$ for $g = 2, 3$. So there is a one-to-one map between inequivalent complex structures and inequivalent period matrices. This means that for genus $g = 2, 3$ the scattering amplitude can be written in terms of the period matrix instead of the moduli coordinates and Beltrami differentials. This rewriting can be achieved using the identities [14]

$$\int d^2z\, w_I(z)\, w_J(z)\mu_i(z) = \frac{\delta\Omega_{IJ}}{\delta\tau_i}, \quad \int \prod_{j=1}^{6} d^2\tau_j \left|\epsilon_{i_1\ldots i_6}\frac{\delta\Omega_{11}}{\delta\tau_{i_1}}\ldots\frac{\delta\Omega_{33}}{\delta\tau_{i_6}}\right|^2 = \int \prod_{I\leq J}^{3} d^2\Omega_{IJ}, \tag{2.39}$$

where the Beltrami differential is given by $\mu^{\bar{z}}_{iz} = \partial_z v_i^{\bar{z}}$ and $v_i^{\bar{z}}(z,\bar{z})$ is a small complex structure deformation.

However the factor $\prod_{I\leq J}^{3} d^2\Omega_{IJ}$ is not invariant under the modular transformation $\mathrm{Sp}(6,\mathbb{Z})$. In general, the $\mathrm{Sp}(2g,\mathbb{Z})$-invariant measure for the genus-g moduli space is

$$d\mu_g \equiv \frac{d^2\Omega_{IJ}}{(\det \mathrm{Im}\,\Omega_{IJ})^{g+1}}, \tag{2.40}$$

and this is precisely the measure that will be obtained from first principles in the next section for genus $g = 3$.

The corresponding volume of the inequivalent period matrix space is given in [26]

$$\mathrm{Vol}_g \equiv \int_{\mathcal{M}_g} d\mu_g = 2^{g^2+1}(2\pi)^{g(g+1)/2}\prod_{k=1}^{g}\frac{(k-1)!}{(2k)!}|B_{2k}|\,, \tag{2.41}$$

where B_{2k} are the Bernoulli numbers and the extra $2^{g(g+1)/2}$ factor in (2.41) compared to the original formula in [26] is due to a different convention for $d^2\Omega_{IJ}$ (see e.g. [19]). In particular,

$$\mathrm{Vol}_1 = \frac{2\pi}{3}, \quad \mathrm{Vol}_2 = \frac{4\pi^3}{3^3\,5}, \quad \mathrm{Vol}_3 = \frac{2^6\pi^6}{3^6\,5^2\,7}\,. \tag{2.42}$$

2.4 The amplitude prescription

The prescription to compute the multiloop n-point closed-string amplitude was given in [2] and it becomes

$$\mathcal{A}_3 = \frac{1}{3}\kappa^4 e^{4\lambda}\int_{\mathcal{M}_3}\prod_{j=1}^{6} d^2\tau_j \int_{\Sigma_4}\left|\langle \mathcal{N}(b,\mu_j)\, U^1(z_1)\ldots U^4(z_4)\rangle\right|^2\,, \tag{2.43}$$

for three loops and four points. $\mathcal{M}_3$ is the fundamental domain of the genus-three Riemann surface and the symmetry factor $1/3$ has been argued for in the previous section. The b-ghost insertion is

$$(b,\mu_j) = \frac{1}{2\pi}\int d^2y_j\, b_{zz}\mu^z_{j\,\bar{z}}, \quad j = 1,\ldots,6. \tag{2.44}$$

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After the non-zero modes are integrated out using their OPEs, the pure spinor bracket $\langle\ldots\rangle$ denotes the integration over the zero-modes[2]

$$\langle\ldots\rangle = \int [d\theta][dr][d\lambda][d\overline{\lambda}] \prod_{I=1}^{3} [dd^I][ds^I][d\overline{w}^I][dw^I] \tag{2.45}$$

and $\mathcal{N}$ is the BRST regulator discussed in [2] which can be written as

$$\mathcal{N} = \sum_{I=1}^{3} e^{-(\lambda\overline{\lambda})-(w^I\overline{w}^I)-(r\theta)+(s^I d^I)}. \tag{2.46}$$

After the integration over $[dd^I][ds^I][dw^I][d\overline{w}^I]$ is performed, the remaining variables $\lambda^\alpha, \overline{\lambda}_\beta, \theta^\delta$ and r_α have conformal weight zero and therefore are the same ones which need to be integrated in the prescription of the tree-level amplitudes. Using the Theorem 1 from appendix A all correlators at this stage of the computation reduce to pure spinor superspace expressions [27] whose component expansions can be straightforwardly computed [28, 29]. In particular, the last correlator to evaluate is a combination of the zero mode integration of tree-level pure spinor variables (2.21) and x^m,

$$|\langle(\lambda^3\theta^5)\rangle_{(n,g)}|^2 \Big\langle \prod_{j=1}^{4} e^{ik^j\cdot x^j} \Big\rangle = (2\pi)^{10}\delta^{(10)}(k)\frac{\sqrt{2}}{2^7\pi^6}\left(\frac{\alpha'}{2}\right)^{-1}\left(\frac{\Gamma(8+n)}{7!}\right)^2 |\langle(\lambda^3\theta^5)\rangle|^2\,\mathcal{I}(s_{ij}), \tag{2.47}$$

where $\delta^{(10)}(k) \equiv \delta^{(10)}(\sum_i k_i^m)$ and $\mathcal{I}(s_{ij})$ is the Koba-Nielsen factor

$$\mathcal{I}(s_{ij}) = \exp\Big(-\sum_{i<j} s_{ij}G(z_i,z_j)\Big). \tag{2.48}$$

Given the above conventions, the space-time dimension of the closed-string n-point amplitude is independent of the genus; $[\mathcal{A}_g] = n(2+[\kappa])$. One can show that unitarity [30] requires $\kappa^2 e^{-2\lambda} = \left(\frac{\alpha'}{2}\right)^{-2}\sqrt{2}2^8\pi^7$, so $[\kappa] = -2$ and the amplitudes are dimensionless.

3 The closed-string 3-loop amplitude

At genus three there are $(16,16,16)_d$ zero-modes of d_α and $(11,11,11)_s$ zero-modes of s^α. The factor $e^{(d^I s^I)}$ in the regulator (2.46) is the only source of s^α zero-modes so the integration over $[ds^I]$ brings down $(11,11,11)_d$ zero-modes,

$$\int \prod_{I=1}^{3}[ds^I]\, e^{-(d^I s^I)} = \left(\frac{\alpha'}{2}\right)^6 \frac{(2\pi)^{33/2} Z_3^{11}}{R^3 2^{18}(11!\,5!)^3(\lambda\overline{\lambda})^9} \prod_{I=1}^{3}(\epsilon\cdot T\cdot d^I)\,. \tag{3.1}$$

The remaining $(5,5,5)_d$ must come from the b-ghosts and the external vertices. Since the number of d_α zero-modes from the external vertices and from each b-ghost can be at most

[2]The definition of the pure spinor bracket here should not be confused with the standard zero mode integration $\langle(\lambda^3\theta^5)\rangle = 1$ of [1]. Since the context makes the distinction clear, we chose not to distinguish the notation.

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four and two respectively, there are only two possibilities for the b-ghosts: they provide 11 or 12 d_α zero modes. Note that these possibilities lead to integrations over pure spinor variables which can be regularized using the original procedure of Berkovits [2].

In the following we decompose the amplitude (2.43) according to the two different b-ghost sectors as $\mathcal{A}_3 = A_{11} + A_{12}$ and evaluate each sector in turn.

3.1 12 d_α zero-modes from the b-ghosts

In this sector there is no chance for OPE singularities between the b-ghosts and the external vertices and therefore (b, μ) is still a well-defined measure [8]. To see this note that if six b-ghosts provide twelve d_α zero modes, each one of them must pick the term $(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)/(192(\lambda\overline{\lambda})^2)$ in (2.10). The zero-mode part of each $(d\gamma_{mnp}d)(y)$ factor is

$$(d^1\gamma_{mnp}d^1)w_1(y)w_1(y) + 2(d^1\gamma_{mnp}d^2)w_1(y)w_2(y) + 2(d^1\gamma_{mnp}d^3)w_1(y)w_3(y)$$
$$+ (d^2\gamma_{mnp}d^2)w_2(y)w_2(y) + 2(d^2\gamma_{mnp}d^3)w_2(y)w_3(y) + (d^3\gamma_{mnp}d^3)w_3(y)w_3(y),$$

and a short computation using (2.39) gives,

$$\int \prod_{j=1}^{6} d^2\tau_j \Big|(b,\mu_j)\Big|^2 = c_{b_1}^2 \int d^2\Omega_{IJ} \left|\frac{B_{(4,4,4)}}{(\lambda\overline{\lambda})^{12}}\right|^2, \tag{3.2}$$

where $c_{b_1} = (\frac{\alpha'}{2})^6 \frac{2^3}{(2^7\,3\pi)^6}$ and

$$B_{(4,4,4)} \equiv (\overline{\lambda}rd^1d^1)(\overline{\lambda}rd^1d^2)(\overline{\lambda}rd^1d^3)(\overline{\lambda}rd^2d^2)(\overline{\lambda}rd^2d^3)(\overline{\lambda}rd^3d^3). \tag{3.3}$$

Note that $B_{(4,4,4)}$ is totally symmetric in the zero-mode labels (123). Since w^I and $\overline{w}^I$ appear only in the regulator $\mathcal{N}$ their integration is straightforward

$$\int \prod_{I=1}^{3} [dw^I][d\overline{w}^I]\, \mathrm{e}^{-(w^I\overline{w}^I)} = \frac{(\lambda\overline{\lambda})^9}{(2\pi)^{33}} Z_3^{-22}\,. \tag{3.4}$$

Defining (if $B_{(p,q,r)}$ does not contain an index m one omits it altogether on both sides)

$$D^m_{(p+11,q+11,r+11)} \equiv \int \prod_{I=1}^{3} [dd^I](\epsilon\cdot T\cdot d^I) B^m_{(p,q,r)} \tag{3.5}$$

and gathering the above results,

$$A_{12} = \frac{\sqrt{2}\,2^{-101}\kappa^4 e^{4\lambda}}{\pi^{42}\,3^{13}(11!\,5!)^6}\left(\frac{\alpha'}{2}\right)^{24} \int \frac{d^2\Omega_{IJ}}{Z_3^{22}} \int_{\Sigma_4} \Big|\big\langle D_{(15,15,15)}U^1U^2U^3U^4\big\rangle_{(-9)}\Big|^2 \big\langle \prod_{j=1}^{4} \mathrm{e}^{ik^j\cdot x^j}\big\rangle\,. \tag{3.6}$$

The only non-vanishing contribution to the integral in (3.6) contains three d_α from the external vertices,

$$U_1U_2U_3U_4 = \left(\frac{\alpha'}{2}\right)^4 \big[(dW_{12})(dW_3)(dW_4)\,\eta_{12} + (dW_{13})(dW_2)(dW_4)\,\eta_{13}$$
$$+(dW_{14})(dW_2)(dW_3)\,\eta_{14} + (dW_{23})(dW_1)(dW_4)\,\eta_{23} \tag{3.7}$$

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$$+(dW_{24})(dW_1)(dW_3)\,\eta_{24}+(dW_{34})(dW_1)(dW_2)\,\eta_{34}\big]$$
$$+\left(\frac{\alpha'}{2}\right)^3\sum_{I=1}^{3}\Pi_I^m w_I(z_1)A_m^1(dW^2)(dW^3)(dW^4)+(1\leftrightarrow 2,3,4)\,,$$

where W_{ij} denotes the BRST block [31],

$$W_{ij}^\alpha=\frac{1}{4}(\gamma_{pq}W_j)^\alpha\mathcal{F}_i^{pq}+(k_j\cdot A_i)W_j^\alpha-(i\leftrightarrow j)\,. \tag{3.8}$$

Since now only the zero modes contribute, each $d_\alpha(z_i)$ becomes $d_\alpha^I w_I(z_i)$. Note that $D_{(15,15,15)}d_\alpha^I d_\beta^J d_\gamma^K w_I(z_i)w_J(z_j)w_K(z_k)=D_{(15,15,15)}d_\alpha^1 d_\beta^2 d_\gamma^3\Delta(z_i;z_j;z_k)$ because the only non-vanishing contribution has $(1,1,1)_d$ zero-modes and $\prod_{I=1}^3[dd^I](\epsilon\cdot T\cdot d^I)$ is totally antisymmetric in the zero-mode labels [123]. Thus,

$$\begin{aligned}D_{(15,15,15)}(dW_{ij})(dW_k)(dW_l)&=(11!\,5!)^3 96^3\,c_d^3\,T_{ij,k,l}(\lambda,\overline{\lambda},r)\Delta(z_j;z_k;z_l)\\ D_{(15,15,15)}(dW_i)(dW_j)(dW_k)A_l^m&=(11!\,5!)^3 96^3\,c_d^3\,L_{ijkl}^m(\lambda,\overline{\lambda},r)\Delta(z_j;z_k;z_l)\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}T_{ij,k,l}(\lambda,\overline{\lambda},r)&=(\overline{\lambda}\gamma^{abc}r)(\overline{\lambda}\gamma^{def}r)(\overline{\lambda}\gamma^{ghi}r)(\overline{\lambda}\gamma^{mnp}r)(\overline{\lambda}\gamma^{qrs}r)(\overline{\lambda}\gamma^{tuv}r)\\&\quad\times(\lambda\gamma^{adefm}\lambda)(\lambda\gamma^{bghit}\lambda)(\lambda\gamma^{uqrsn}\lambda)(\lambda\gamma^c W_{ij})(\lambda\gamma^p W_k)(\lambda\gamma^v W_l)\,,\\ L_{ijkl}^m(\lambda,\overline{\lambda},r)&=(\overline{\lambda}\gamma^{abc}r)(\overline{\lambda}\gamma^{def}r)(\overline{\lambda}\gamma^{ghi}r)(\overline{\lambda}\gamma^{mnp}r)(\overline{\lambda}\gamma^{qrs}r)(\overline{\lambda}\gamma^{tuv}r)\\&\quad\times(\lambda\gamma^{adefm}\lambda)(\lambda\gamma^{bghit}\lambda)(\lambda\gamma^{uqrsn}\lambda)(\lambda\gamma^c W_i)(\lambda\gamma^p W_j)(\lambda\gamma^v W_k)A_l^m.\end{aligned} \tag{3.10}$$

As shown in the appendix A, it is always possible to rewrite the $\overline{\lambda}^n\lambda^{n+3}$ dependence in (3.10) as $(\lambda\overline{\lambda})^n\lambda^3$ when performing the zero mode integrals and therefore we write $T_{ij,k,l}(\lambda,\overline{\lambda},r)=(\lambda\overline{\lambda})^6 T_{ij,k,l}(\lambda,r)$ and drop the (λ,r) arguments from now on. Note that $T_{ij,k,l}$ is antisymmetric in $[ij]$ and $[kl]$ and L_{ijkl}^m is antisymmetric in $[ijk]$.

Plugging the above results in (3.6) and using the identity (2.31) together with the definitions $M_{ij,k,l}=s_{ij}^{-1}T_{ij,k,l}$, $X_{ij}=(\alpha'/2)s_{ij}\eta_{ij}$ yields

$$A_{12}=\tfrac{\sqrt{2}\pi^6}{2^{23}3^7}\kappa^4 e^{4\lambda}\left(\tfrac{\alpha'}{2}\right)^6\int\tfrac{d^2\Omega_{IJ}}{Z_3^{-10}}\int_{\Sigma_4}\langle|\mathcal{K}_{12}|^2\rangle_{(-3)}\big\langle\textstyle\prod_{j=1}^4 \mathrm{e}^{ik^j\cdot x^j}\big\rangle \tag{3.11}$$

where

$$\begin{aligned}\mathcal{K}_{12}&=M_{12,3,4}\Delta(z_2;z_3;z_4)\,X_{12}+M_{13,2,4}\Delta(z_3;z_2;z_4)\,X_{13}+M_{14,2,3}\Delta(z_4;z_2;z_3)\,X_{14}\\&\quad+M_{23,1,4}\Delta(z_3;z_1;z_4)\,X_{23}+M_{24,1,3}\Delta(z_4;z_1;z_3)\,X_{24}+M_{34,1,2}\Delta(z_4;z_1;z_2)\,X_{34}\\&\quad+\Delta^m(z_1,z_2;z_3;z_4)\big[L_{1342}^m+L_{2341}^m\big]+\Delta^m(z_1,z_3;z_2;z_4)\big[L_{1243}^m+L_{3241}^m\big]\\&\quad+\Delta^m(z_1,z_4;z_2;z_3)\big[L_{1234}^m+L_{4231}^m\big]+\Delta^m(z_2,z_3;z_1;z_4)\big[L_{2143}^m+L_{3142}^m\big]\\&\quad+\Delta^m(z_2,z_4;z_1;z_3)\big[L_{2134}^m+L_{4132}^m\big]+\Delta^m(z_3,z_4;z_1;z_2)\big[L_{3124}^m+L_{4123}^m\big].\end{aligned} \tag{3.12}$$

3.2 11 d_α zero-modes from the b-ghosts

By not using the Siegel-gauge vertex operators of [9] one could in principle face problems with the consistency condition for (b,μ) discussed in [8]. However, in the low energy limit discussed here, this potential complication can be ignored since the only contribution comes from terms in which the b ghost does not have singular OPEs.[3] To see this note that one

[3]We are grateful to Nathan Berkovits for discussions and for his comments on the draft at this point.

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possibility to obtain 11 d_α zero modes out of six b-ghosts is given by

$$b^6 \longrightarrow \frac{(\lambda\overline{\lambda})^{-12}}{(192)^6}\left(\frac{\alpha'}{2}\right)^6\left[(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)\right]^6, \tag{3.13}$$

where one of the $d_\alpha(y_i)$ is the non-zero-mode part $\hat{d}_\alpha(y_i)$ and contracts through the OPE (2.9) with an external vertex (y_i denotes the position of the corresponding b-ghost). However this term is of order D^8R^4 and there are no inverse powers of Mandelstam invariants coming from the integration over the vertex positions since there are no simple pole singularities among them [32], as they must contribute the four remaining d_α zero modes. The claim that (3.13) leads to terms of order D^8R^4 is easy to verify. The external vertices contribute W^4 superfields, the OPE between $\hat{d}_\alpha(y)$ and one superfield W^β gives $D_\alpha W^\beta$ and each r_α from (3.13) counts as a covariant derivative D_α because of the factor $e^{-(r\theta)}$ in the regulator $\mathcal{N}$. This gives kinematic terms proportional to $\langle D_\alpha^7 W^4\rangle = \langle k^3W^3\mathcal{F}\rangle = k^4F_{mn}^4$ whose holomorphic square is D^8R^4. The other possibilities of b-ghost singularities are similarly analyzed. Therefore the terms which might be affected by the issues pointed out in [8] do not affect the leading order terms D^6R^4 and will not be considered in the following.

When the b-ghosts have no singularities with the vertices and contribute 11 zero-modes of d_α, one possibility is

$$b^6 \longrightarrow \frac{(\lambda\overline{\lambda})^{-13}}{16(192)^5}\left(\frac{\alpha'}{2}\right)^6(r\gamma_{qrs}r)(\overline{\lambda}\gamma^q d)N^{rs}\left[(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)\right]^5, \tag{3.14}$$

but it vanishes upon integration over $[dw]$ because $\int[dw][d\overline{w}]w_\alpha e^{-(w^I\overline{w}_I)} = 0$. The other possibility is

$$b^6 \longrightarrow \frac{(\lambda\overline{\lambda})^{-11}}{2(192)^5}\left(\frac{\alpha'}{2}\right)^5(\overline{\lambda}\gamma_q d)\Pi^q\left[(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)\right]^5 \tag{3.15}$$

where the $\Pi^m(y)$ field is proportional to its zero modes $\Pi^I_m w_I(y)$. In this case the integration over the positions of the b-ghosts can be carried out,

$$\int\prod_{j=1}^6 d^2\tau_j\Big|(b,\mu_j)\Big|^2 = c_{b_2}^2\int d^2\Omega_{IJ}\left|\frac{1}{(\lambda\overline{\lambda})^{11}}\left(\Pi^1_m B^m_{(3,4,4)} + \Pi^2_m B^m_{(4,3,4)} + \Pi^3_m B^m_{(4,4,3)}\right)\right|^2 \tag{3.16}$$

where $c_{b_2} = \frac{2^2(\alpha'/2)^5}{4\pi\,(2^7\,3\pi)^5} = 48(\alpha'/2)^{-1}\,c_{b_1}$ and

$$\begin{aligned} B^m_{(3,4,4)} = {} & +2(\overline{\lambda}\gamma^m d^1)(\overline{\lambda}rd^1d^2)(\overline{\lambda}rd^1d^3)(\overline{\lambda}rd^2d^2)(\overline{\lambda}rd^2d^3)(\overline{\lambda}rd^3d^3) \\ & -(\overline{\lambda}\gamma^m d^2)(\overline{\lambda}rd^1d^1)(\overline{\lambda}rd^1d^3)(\overline{\lambda}rd^2d^2)(\overline{\lambda}rd^2d^3)(\overline{\lambda}rd^3d^3) \\ & +(\overline{\lambda}\gamma^m d^3)(\overline{\lambda}rd^1d^1)(\overline{\lambda}rd^1d^2)(\overline{\lambda}rd^2d^2)(\overline{\lambda}rd^2d^3)(\overline{\lambda}rd^3d^3)\,, \end{aligned} \tag{3.17}$$

while $B_{(4,3,4)}$ and $B_{(4,4,3)}$ are obtained from $B_{(3,4,4)}$ by swapping $d^1_\alpha \leftrightarrow d^2_\alpha$ and $d^1_\alpha \leftrightarrow d^3_\alpha$, respectively. Furthermore, note that $B_{(3,4,4)}$ is symmetric under $d^2_\alpha \leftrightarrow d^3_\alpha$.

Taking into account that $c_{b_2} = 48(\alpha'/2)^{-1}\,c_{b_1}$ and using the definition (3.5) leads to the following expression for A_{11},

$$\begin{aligned} A_{11} = {} & \frac{\sqrt{2}\,2^{-93}\kappa^4 e^{4\lambda}}{\pi^{42}\,3^{11}(11!\,5!)^6}\left(\frac{\alpha'}{2}\right)^{22}\int\frac{d^2\Omega_{IJ}}{Z_3^{22}}\int_{\Sigma_4}\langle\prod_{j=1}^4 e^{ik^j\cdot x^j}\rangle \\ & \times\left|\left\langle(\Pi^1_m D^m_{(14,15,15)} + \Pi^2_m D^m_{(15,14,15)} + \Pi^3_m D^m_{(15,15,14)})U^1U^2U^3U^4\right\rangle_{(-8)}\right|^2. \end{aligned} \tag{3.18}$$

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Each external vertex U^i contribute through the term $(\alpha'/2)(dW^i)(z_i)$ and the integration over the d zero-modes can be carried out by using the following formulae,

$$\begin{aligned}
D^m_{(14,15,15)}(d^1W^1)(d^1W^2)(d^2W^3)(d^3W^4) &= +(11!\,5!)^3 96^2 c_d^3\, S^m_{1234}(\lambda,\overline{\lambda},r)\\
D^m_{(15,14,15)}(d^2W^1)(d^2W^2)(d^1W^3)(d^3W^4) &= -(11!\,5!)^3 96^2 c_d^3\, S^m_{1234}(\lambda,\overline{\lambda},r)\\
D^m_{(15,15,14)}(d^3W^1)(d^3W^2)(d^1W^3)(d^2W^4) &= +(11!\,5!)^3 96^2 c_d^3\, S^m_{1234}(\lambda,\overline{\lambda},r)
\end{aligned} \tag{3.19}$$

where

$$S^m_{1234}(\lambda,\overline{\lambda},r) = S^{(1)\,m}_{1234}(\lambda,\overline{\lambda},r) + S^{(2)\,m}_{1234}(\lambda,\overline{\lambda},r) - S^{(2)\,m}_{1243}(\lambda,\overline{\lambda},r) \tag{3.20}$$

and

$$\begin{aligned}
S^{(1)\,m}_{1234}(\lambda,\overline{\lambda},r) &= 2\,(\overline{\lambda}\gamma^m\gamma^{a_1}\lambda)(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\overline{\lambda}\gamma_{m_3n_3p_3}r)(\overline{\lambda}\gamma_{m_4n_4p_4}r)(\overline{\lambda}\gamma_{m_5n_5p_5}r)\\
&\quad\times(\lambda\gamma^{a_2m_1n_1p_1m_3}\lambda)(\lambda\gamma^{a_3m_2n_2p_2m_5}\lambda)(\lambda\gamma^{n_3m_4n_4p_4n_5}\lambda)\\
&\quad\times(W^1\gamma^{a_1a_2a_3}W^2)(\lambda\gamma^{p_3}W^3)(\lambda\gamma^{p_5}W^4)\\
S^{(2)\,m}_{1234}(\lambda,\overline{\lambda},r) &= 96\,(\overline{\lambda}\gamma^m\gamma^{m_3}\lambda)(\overline{\lambda}\gamma_{m_1n_1p_1}r)(\overline{\lambda}\gamma_{m_2n_2p_2}r)(\overline{\lambda}\gamma_{m_3n_3p_3}r)(\overline{\lambda}\gamma_{m_4n_4p_4}r)(\overline{\lambda}\gamma_{m_5n_5p_5}r)\\
&\quad\times(\lambda\gamma^{m_1m_2n_2p_2m_5}\lambda)(\lambda\gamma^{n_3m_4n_4p_4n_5}\lambda)\\
&\quad\times(\lambda\gamma^{n_1}W^1)(\lambda\gamma^{p_1}W^2)(\lambda\gamma^{p_3}W^3)(\lambda\gamma^{p_5}W^4)\,.
\end{aligned} \tag{3.21}$$

Note from the definition (3.19) that $S^m_{1234}(\lambda,\overline{\lambda},r)$ is symmetric in the particle labels (12) and antisymmetric in [34]. The explicit expression for $S^{(1)}_{1234}(\lambda,\overline{\lambda},r)$ is symmetric in (12) and antisymmetric in [34] whereas $S^{(2)}_{1234}(\lambda,\overline{\lambda},r)$ is symmetric in (12), so (3.20) indeed has the required symmetries. According to the procedure of appendix A we write $S^m_{1234}(\lambda,\overline{\lambda},r) = (\lambda\overline{\lambda})^6 S^m_{1234}(\lambda,r)$ and drop the arguments (λ,r) in the following.

After expanding $d_\alpha(z) = \hat{d}_\alpha(z) + d^I_\alpha w_I(z)$ and using (3.19) together with the symmetry properties of S^m_{1234} it is a matter of bookkeeping the permutations to arrive at

$$(\Pi^1_m D^m_{(14,15,15)} + \Pi^2_m D^m_{(15,14,15)} + \Pi^3_m D^m_{(15,15,14)})U^1U^2U^3U^4 = \left(\frac{\alpha'}{2}\right)^4 (11!5!)^3\, 96^2\, c_d^3\, (\lambda\overline{\lambda})^6 \mathcal{K}_{11} \tag{3.22}$$

where

$$\begin{aligned}
\mathcal{K}_{11} = &+S^m_{1234}\Delta^m(z_1,z_2;z_3;z_4) + S^m_{1324}\Delta^m(z_1,z_3;z_2;z_4) + S^m_{1423}\Delta^m(z_1,z_4;z_2;z_3)\\
&+S^m_{2314}\Delta^m(z_2,z_3;z_1;z_4)+S^m_{2413}\Delta^m(z_2,z_4;z_1;z_3)+S^m_{3412}\Delta^m(z_3,z_4;z_1;z_2)\,.
\end{aligned} \tag{3.23}$$

Finally,

$$A_{11} = \kappa^4 e^{4\lambda}\,\frac{\sqrt{2}\pi^6}{2^{25}3^7}\left(\frac{\alpha'}{2}\right)^6 \int \frac{d^2\Omega_{IJ}}{Z_3^{-10}} \int_{\Sigma_4} \langle |\mathcal{K}_{11}|^2\rangle_{(-2)} \langle \prod_{j=1}^4 e^{ik^j\cdot x^j}\rangle. \tag{3.24}$$

Therefore from (3.11) and (3.24) the three-loop amplitude $\mathcal{A}_3 = A_{12} + A_{11}$ becomes

$$\mathcal{A}_3 = \kappa^4 e^{4\lambda}\,\frac{\sqrt{2}\pi^6}{2^{23}3^7}\left(\frac{\alpha'}{2}\right)^6 \int_{\mathcal{M}_3} \frac{d^2\Omega_{IJ}}{(\det(2\,\mathrm{Im}\,\Omega))^5} \int_{\Sigma_4} \Big[\langle|\mathcal{F}|^2\rangle_{(-3)} + \langle|\mathcal{T}|^2\rangle_{(-3)}\Big] \langle \prod_{j=1}^4 e^{ik^j\cdot x^j}\rangle \tag{3.25}$$

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where (2.16) has been used,

$$\begin{aligned}\mathcal{F} =& + M_{12,3,4}\Delta(z_2;z_3;z_4)\,X_{12} + M_{13,2,4}\Delta(z_3;z_2;z_4)\,X_{13} + M_{14,2,3}\Delta(z_4;z_2;z_3)\,X_{14}\\ &+M_{23,1,4}\Delta(z_3;z_1;z_4)\,X_{23} + M_{24,1,3}\Delta(z_4;z_1;z_3)\,X_{24} + M_{34,1,2}\Delta(z_4;z_1;z_2)\,X_{34}\\ \mathcal{T} =& + T^m_{1234}\Delta^m(z_1,z_2;z_3;z_4) + T^m_{1324}\Delta^m(z_1,z_3;z_2;z_4) + T^m_{1423}\Delta^m(z_1,z_4;z_2;z_3)\\ &+ T^m_{2314}\Delta^m(z_2,z_3;z_1;z_4) + T^m_{2413}\Delta^m(z_2,z_4;z_1;z_3) + T^m_{3412}\Delta^m(z_3,z_4;z_1;z_2)\end{aligned} \tag{3.26}$$

and (the other T^m_{ijkl} follow from relabelling),

$$T^m_{1234} = L^m_{1342} + L^m_{2341} + \frac{5}{2}S^m_{1234}\,. \tag{3.27}$$

The factor 5 in (3.27) is due to $\langle\ldots\rangle_{(-2)} = 5\langle\ldots\rangle_{(-3)}$ and follows from (2.21). The factor 1/2 accounts for the different overall normalizations of (3.24) and (3.11).

After using (2.47) the three-loop amplitude (3.25) becomes

$$\mathcal{A}_3 = (2\pi)^{10}\delta^{(10)}(k)\frac{\kappa^4 e^{4\lambda}}{2^{31}3^9\,5^2\,7^2}\left(\frac{\alpha'}{2}\right)^5\int_{\mathcal{M}_3}\frac{d^2\Omega_{IJ}}{(\det(2\,\mathrm{Im}\,\Omega))^5}\int_{\Sigma_4}\Big[\langle|\mathcal{F}|^2\rangle + \langle|\mathcal{T}|^2\rangle\Big]\,\mathcal{I}(s_{ij})\,. \tag{3.28}$$

3.3 The low energy limit D^6R^4

Since the superfields in $\mathcal{F}$ and $\mathcal{T}$ have component expansions terms of order $k^4F^4_{mn}$ and $k^3F^4_{mn}$ one might naively expect that only $\langle|\mathcal{T}|^2\rangle$ in (3.25) contributes to the low energy limit of order D^6R^4. However the integration by parts identities of appendix B show that,

$$\begin{aligned}\int_{\Sigma_4}\langle|\mathcal{F}|^2\rangle\mathcal{I}(s_{ij}) &= -\pi\left(\frac{\alpha'}{2}\right)\langle\mathcal{K}\rangle\int_{\Sigma_4}\Omega_{12}\Delta(z_2;z_3;z_4)\overline{\Delta}(z_1;z_3;z_4) + \mathcal{O}(\alpha'^2)\\ &= -36\pi\left(\frac{\alpha'}{2}\right)\langle\mathcal{K}\rangle\det(2\,\mathrm{Im}\,\Omega) + \mathcal{O}(\alpha'^2)\end{aligned} \tag{3.29}$$

where

$$\mathcal{K} = \frac{|T_{23,1,4}|^2}{s_{23}} + \frac{|T_{24,1,3}|^2}{s_{24}} + \frac{|T_{34,1,2}|^2}{s_{34}} + \frac{|T_{12,3,4}|^2}{s_{12}} + \frac{|T_{13,2,4}|^2}{s_{13}} + \frac{|T_{14,2,3}|^2}{s_{14}} \tag{3.30}$$

is such that $\langle\mathcal{K}\rangle$ is also of order D^6R^4.

The low energy limit of $\int_{\Sigma_4}|\mathcal{T}|^2\,\mathcal{I}(s_{ij})$ can be obtained by dropping the Koba-Nielsen factor and carrying out the integrations using Riemann's relation (2.29) in the form,

$$\begin{aligned}\int_{\Sigma_4}\Delta^m(z_1,z_2;z_3;z_4)\overline{\Delta}^n(\overline{z}_1,\overline{z}_2;\overline{z}_3;\overline{z}_4) &= -12\pi\eta^{mn}\left(\frac{\alpha'}{2}\right)\det(2\,\mathrm{Im}\,\Omega),\\ \int_{\Sigma_4}\Delta^m(z_1,z_2;z_3;z_4)\overline{\Delta}^n(\overline{z}_1,\overline{z}_3;\overline{z}_2;\overline{z}_4) &= +12\pi\eta^{mn}\left(\frac{\alpha'}{2}\right)\det(2\,\mathrm{Im}\,\Omega),\\ \int_{\Sigma_4}\Delta^m(z_1,z_2;z_3;z_4)\overline{\Delta}^n(\overline{z}_3,\overline{z}_4;\overline{z}_1;\overline{z}_2) &= 0\,.\end{aligned} \tag{3.31}$$

Since every integral in $\int_{\Sigma_4}|\mathcal{T}|^2$ can be obtained from relabellings of (3.31) using the symmetries of $\Delta^m(z_1,z_2;z_3;z_4)$, straightforward algebra gives

$$\int_{\Sigma_4}|\mathcal{T}|^2 = -36\pi\left(\frac{\alpha'}{2}\right)\det(2\,\mathrm{Im}\,\Omega)\,\mathcal{L}\cdot\tilde{\mathcal{L}}\,, \tag{3.32}$$

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where

$$\mathcal{L}\cdot\tilde{\mathcal{L}} = \frac{1}{3}\sum_{i,j=1}^{6} M_{ij} L_i\cdot\tilde{L}_j \tag{3.33}$$

and M_{ij} is the symmetric 6×6 matrix

$$M_{ij} = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -1 & -1 & 1 \end{pmatrix},$$

and we defined the set $L_i^m = \{T_{1234}^m, T_{1324}^m, T_{1423}^m, T_{2314}^m, T_{2413}^m, T_{3412}^m\}$.

Therefore the low energy limit of the closed-string three-loop amplitude is given by

$$\begin{aligned}\mathcal{A}_3 &= -(2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^6 \left\langle \mathcal{K}+\mathcal{L}\cdot\tilde{\mathcal{L}}\right\rangle \kappa^4 e^{4\lambda}\,\frac{\pi}{2^{29}3^7\,5^2\,7^2}\int_{\mathcal{M}_3}\frac{d^2\Omega_{IJ}}{(\det(2\,\mathrm{Im}\,\Omega))^4} \\ &= -(2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^6 \left\langle \mathcal{K}+\mathcal{L}\cdot\tilde{\mathcal{L}}\right\rangle \kappa^4 e^{4\lambda}\,\frac{\pi\zeta_6}{2^{35}\,3^{10}\,5^3\,7^2}\end{aligned} \tag{3.34}$$

where the integral is $2^{-12}\,\mathrm{Vol}_3$ and we used $\zeta_6 = \pi^6/945$. A long calculation gives [28]

$$\left\langle \mathcal{K}+\mathcal{L}\cdot\tilde{\mathcal{L}}\right\rangle = -2^{35}\,3^7\,5^3\,7^2\,(s_{12}^3+s_{13}^3+s_{14}^3)\,K\overline{K} \tag{3.35}$$

irrespective of whether it is for type IIA or IIB, confirming the theorem of [7].

Therefore

$$\mathcal{A}_3 = (2\pi)^{10}\delta^{(10)}(k)\,\kappa^4 e^{4\lambda}\,\frac{\pi\,\zeta_6}{3^3}\left(\frac{\alpha'}{2}\right)^6 (s_{12}^3+s_{13}^3+s_{14}^3)\,K\overline{K} \tag{3.36}$$

is low energy limit of the type IIA and IIB three-loop amplitude.

4 Perturbative calculations versus S-duality predictions

We first review the one- and two-loop comparisons between S-duality predictions and perturbative amplitude calculations of [19, 21] using our conventions. After that we extend their analysis to include the three-loop result (3.36). We will find that the amplitude we computed in (3.36) agrees with the prediction of Green and Vanhove [12].

4.1 One- and two-loops

The closed-string massless four-point amplitudes at genus 0, 1 and 2 computed in [11] (including their overall coefficients) are given by (see also [19]),

$$\mathcal{A}_0 = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{-2\lambda}\,\frac{\sqrt{2}}{2^{16}\pi^5}\,\mathcal{B}_0(s_{ij}) \tag{4.1}$$

$$\mathcal{A}_1 = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4\,\frac{1}{2^{14}\pi^2}\int_{\mathcal{M}_1}\frac{d^2\tau}{\tau_2^2}\mathcal{B}_1(s_{ij}|\tau)$$
$$\mathcal{A}_2 = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{2\lambda}\,\frac{\sqrt{2}}{2^{15}}\int_{\mathcal{M}_2}\frac{d^2\Omega_{IJ}}{(\det(\mathrm{Im}\,\Omega))^3}\mathcal{B}_2(s_{ij}|\Omega)$$

where[4] [19, 21],

$$\begin{aligned}\mathcal{B}_0(s_{ij}) &= \frac{\Gamma(-\alpha' s_{12}/2)\Gamma(-\alpha' s_{13}/2)\Gamma(-\alpha' s_{14}/2)}{\Gamma(1+\alpha' s_{12}/2)\Gamma(1+\alpha' s_{13}/2)\Gamma(1+\alpha' s_{14}/2)} \\ &= \frac{3}{\sigma_3} + 2\zeta_3 + \zeta_5\sigma_2 + \frac{2}{3}\zeta_3^2\sigma_3 + \cdots \\ \mathcal{B}_1(s_{ij}|\tau) &= \int\prod_{i=2}^{4}\frac{d^2z_i}{\tau_2}\,\mathcal{I}(s_{ij}) = 2^3\Big(1 + \frac{\zeta_3}{3}\sigma_3 + \cdots \\ \mathcal{B}_2(s_{ij}|\Omega) &= \int_{\Sigma^4}\frac{|\mathcal{Y}|^2}{(\det(\mathrm{Im}\,\Omega))^2}\,\mathcal{I}(s_{ij}) = 2^7\sigma_2 + \cdots\end{aligned} \tag{4.2}$$

Plugging in the volume of the moduli spaces (2.42) one obtains the following low energy expansions,

$$\mathcal{A}_0 = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{-2\lambda}\,\frac{\sqrt{2}}{2^{16}\pi^5}\left[\frac{3}{\sigma_3} + 2\zeta_3 + \zeta_5\sigma_2 + \frac{2}{3}\zeta_3^2\sigma_3 + \cdots\right. \tag{4.3}$$
$$\mathcal{A}_1 = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4\,\frac{1}{2^{10}\,3\pi}\left[1 + \frac{\zeta_3}{3}\sigma_3 + \cdots\right. \tag{4.4}$$
$$\mathcal{A}_2 = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{2\lambda}\,\frac{\sqrt{2}\pi^3}{2^6\,3^3\,5}\left[\sigma_2 + \cdots\right. \tag{4.5}$$

The SL$(2,\mathbb{Z})$-duality predictions for the perturbative effective action are [12, 35–37]

$$S^{\alpha'^3} = C_1\int d^{10}x\sqrt{-g}\,\mathcal{R}^4\left(2\zeta_3 e^{-2\phi} + \frac{2\pi^2}{3}\right), \tag{4.6}$$
$$S^{\alpha'^5} = C_2\int d^{10}x\sqrt{-g}\,D^4\mathcal{R}^4\left(2\zeta_5 e^{-2\phi} + \frac{8}{3}\zeta_4 e^{2\phi}\right), \tag{4.7}$$
$$S^{\alpha'^6} = C_3\int d^{10}x\sqrt{-g}\,D^6\mathcal{R}^4\left(4\zeta_3^2 e^{-2\phi} + 8\zeta_2\zeta_3 + \frac{48}{5}\zeta_2^2 e^{2\phi} + \frac{8}{9}\zeta_6 e^{4\phi}\right). \tag{4.8}$$

where the precise definitions of R^4, $D^4\mathcal{R}^4$ and $D^6\mathcal{R}^4$ and the constants $C_{\{1,2,3\}}$ will not be needed in the following discussion since only the ratios of the interactions at different loop orders will be important.

Matching the ratios of the α'^3 interactions at one-loop and tree-level leads to a relation between e^ϕ and e^λ,

$$\frac{\sqrt{2}2^4\pi^4 e^{2\lambda}}{3\zeta_3} = \frac{e^{2\phi}\pi^2}{3\zeta_3} \quad\rightarrow\quad e^{2\phi} = e^{2\lambda}\sqrt{2}2^4\pi^2, \tag{4.9}$$

[4] $\mathcal{B}_0(s_{ij})$ here is $\frac{1}{2\pi}C(s,t,u)$ from [11]. Furthermore, see [33] for a recent attempt to evaluate non-leading terms at two-loops and [34] for an elegant way to rewrite the tree-level expansion.

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where the left-hand side follows from the amplitudes while the right-hand side from the effective action (4.6). Now one can compare the S-duality predictions for the amplitudes at order α'^5 and α'^6 (denoted with a Latin capital $A^{\alpha'^n}$) and the perturbative results.

For the α'^5 interaction, the ratio between the two-loop and tree-level interactions in the effective action (4.7) is $\frac{4\zeta_4}{3\zeta_5}e^{4\phi}$ and leads to the prediction

$$A_2^{\alpha'^5} = \mathcal{A}_0^{\alpha'^5}\frac{2^{11}\pi^4\zeta_4}{3\zeta_5}e^{4\lambda} = (2\pi)^{10}\delta^{(10)}(k)\,\kappa^4 e^{2\lambda}\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\frac{\sqrt{2}\zeta_4}{2^5\,3\,\pi}\,\sigma_2\,, \tag{4.10}$$

which agrees with the two-loop perturbative calculation (4.5) (recall that $\zeta_4 = \pi^4/90$).

For the α'^6 interaction, the ratio between the one-loop and tree-level terms following from the effective action (4.8) is $2\zeta_2/\zeta_3 e^{2\phi} = \zeta_2/\zeta_3\sqrt{2}2^5\pi^2 e^{2\lambda}$ and implies

$$A_1^{\alpha'^6} = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4\,\frac{\zeta_2\zeta_3}{2^9 3\pi^3}\,\sigma_3\,, \tag{4.11}$$

which agrees with the one-loop perturbative calculation (4.4), in accord with the analysis of [21].

4.2 Three-loops

Similarly, the ratio between the three-loop and tree-level terms of (4.8)

$$\frac{2\zeta_6}{9\zeta_3^2}\,e^{6\phi} = \frac{\sqrt{2}2^{14}\pi^6\zeta_6}{9\zeta_3^2}\,e^{6\lambda} = \frac{\mathcal{A}_3^{\alpha'^6}}{\mathcal{A}_0^{\alpha'^6}}$$

predicts the following three-loop amplitude

$$A_3^{\alpha'^6} = (2\pi)^{10}\delta^{(10)}(k)\left(\frac{\alpha'}{2}\right)^3 K\overline{K}\,\kappa^4 e^{4\lambda}\,\frac{\pi\zeta_6}{3^3}\,\sigma_3\,, \tag{4.12}$$

in complete agreement with the first principles perturbative calculation (3.36).

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A A general formula for integration of pure spinors

Component expansions of a general ghost-number-three pure spinor superspace expression of the form $\lambda^{\alpha_1}\dots\lambda^{\alpha_{n+3}}\overline{\lambda}_{\beta_1}\dots\overline{\lambda}_{\beta_n}f^{\beta_1\dots\beta_n}_{\alpha_1\dots\alpha_{n+3}}(\theta,r)$ can be computed most conveniently by first rewriting it in the form $(\lambda\overline{\lambda})^n\lambda^{\alpha_1}\dots\lambda^{\alpha_3}f_{\alpha_1\dots\alpha_3}(\theta,r)$. Doing that allows the straightforward application of the formula (2.21) and the identities listed in the appendix of [38]. The case $n=1$ was discussed in [39], now the solution for general n will be presented.

Let $\mathcal{T}^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_n}$ denote a SO(10)-invariant tensor which is symmetric and γ-traceless in both sets of indices. When n is even there is $(n/2+1)$-dimensional basis [40],

$$\begin{aligned}\mathcal{T}^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_n} &= \sum_{k=0}^{n/2} c_k^{(n)} T_k^{(n)}\,, \\ T_k^{(n)} &= \delta^{(\alpha_1}_{(\beta_1}\cdots\delta^{\alpha_{n-2k}}_{\beta_{n-2k}}(\gamma\cdot\gamma)^{\alpha_{n-2k+1}\alpha_{n-2k+2}}_{\beta_{n-2k+1}\beta_{n-2k+2}}\cdots(\gamma\cdot\gamma)^{\alpha_{n-1}\alpha_n)}_{\beta_{n-1}\beta_n)}\,,\end{aligned} \tag{A.1}$$

and $(\gamma\cdot\gamma)^{\alpha_1\alpha_2}_{\beta_1\beta_2}\equiv\gamma^{\alpha_1\alpha_2}_m\gamma^m_{\beta_1\beta_2}$. Imposing the γ-traceless condition leads to a recurrence relation for the coefficients $c_k^{(n)}$ [41] and the normalization condition $T^{\alpha_1\dots\alpha_n}_{\alpha_1\dots\alpha_n}=1$ relates the coefficient $c_0^{(n)}$ with the dimension of the pure spinor representation $N_n\equiv\dim([0000n])$,

$$c_{k+1}^{(n)} = -\frac{(n-2k)(n-2k-1)}{8(k+1)(n-k+2)}c_k^{(n)}\,,\quad c_0^{(n)}=1/N_n\,, \tag{A.2}$$

where[5]

$$\begin{aligned}N_n &= \frac{1}{302400}(n+7)(n+6)(n+5)^2(n+4)^2(n+3)^2(n+2)(n+1) \\ &= 16, 126, 672, 2772, 9504, 28314\dots\end{aligned} \tag{A.3}$$

When $m=n-1$ is odd, the tensor $\mathcal{T}^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_m}$ can be obtained from (A.1) by contracting a pair of indices; $\mathcal{T}^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_m}=\mathcal{T}^{\alpha_1\dots\alpha_m\alpha_{m+1}}_{\beta_1\dots\beta_m\alpha_{m+1}}$.

The explicit expressions for the first few tensors read as follows,

$$\begin{aligned}\mathcal{T}^{\alpha_1}_{\beta_1} &= \frac{1}{16}\delta^{\alpha_1}_{\beta_1}\,,\\ \mathcal{T}^{\alpha_1\alpha_2}_{\beta_1\beta_2} &= \frac{1}{126}\left[\delta^{(\alpha_1}_{\beta_1}\delta^{\alpha_2)}_{\beta_2}-\frac{1}{16}(\gamma\cdot\gamma)^{\alpha_1\alpha_2}_{\beta_1\beta_2}\right],\\ \mathcal{T}^{\alpha_1\dots\alpha_3}_{\beta_1\dots\beta_3} &= \frac{1}{672}\left[\delta^{(\alpha_1}_{\beta_1}\cdots\delta^{\alpha_3)}_{\beta_3}-\frac{3}{20}\delta^{(\alpha_1}_{(\beta_1}(\gamma\cdot\gamma)^{\alpha_2\alpha_3)}_{\beta_2\beta_3)}\right],\\ \mathcal{T}^{\alpha_1\dots\alpha_4}_{\beta_1\dots\beta_4} &= \frac{1}{2772}\left[\delta^{(\alpha_1}_{\beta_1}\cdots\delta^{\alpha_4)}_{\beta_4}-\frac{1}{4}\delta^{(\alpha_1}_{(\beta_1}\delta^{\alpha_2}_{\beta_2}(\gamma\cdot\gamma)^{\alpha_3\alpha_4)}_{\beta_3\beta_4)}+\frac{1}{160}(\gamma\cdot\gamma)^{(\alpha_1\alpha_2}_{(\beta_1\beta_2}(\gamma\cdot\gamma)^{\alpha_3\alpha_4)}_{\beta_3\beta_4)}\right].\end{aligned} \tag{A.4}$$

Using the integration formula of [10, 11] and the above γ-traceless tensors it follows that[6]

$$\int[d\lambda][d\overline{\lambda}]e^{-(\lambda\overline{\lambda})}(\lambda\overline{\lambda})^m\lambda^{\alpha_1}\cdots\lambda^{\alpha_n}\overline{\lambda}_{\beta_1}\cdots\overline{\lambda}_{\beta_n} = \left(\frac{A_g}{2\pi}\right)^{11}\frac{12\,\Gamma(8+m+n)}{\Gamma(11)}\mathcal{T}^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_n}. \tag{A.5}$$

[5] N_n can be obtained from $(1+t)(1+4t+t^2)(1-t)^{-11}=1+\sum_{n\geq1}N_nt^n$ [42, 43].

[6] Note that all numbers in (A.5) have a geometrical meaning. The number 8 is the ghost anomaly (the first Chern class of the projective pure spinor space), 11 is the complex dimension of the pure spinor space and 12 is the degree of the projective pure spinor space [10, 42].

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To see this it is enough to check that the right-hand side of (A.5) has the same symmetries of the left-hand side and it is correctly normalized.

Let us define the tensor $\overline{T}^{\alpha\beta\gamma;\sigma_1\ldots\sigma_5}$ by [44]

$$\overline{\lambda}_\alpha\overline{\lambda}_\beta\overline{\lambda}_\gamma\overline{T}^{\alpha\beta\gamma;\sigma_1\ldots\sigma_5} = \overline{T}^{\sigma_1\ldots\sigma_5}, \tag{A.6}$$

where $\overline{T}^{\sigma_1\ldots\sigma_5}$ is given in (2.17). Since one can take $\overline{T}^{\alpha\beta\gamma;\sigma_1\ldots\sigma_5}$ to be γ-traceless in the $(\alpha\beta\gamma)$ indices it follows from (A.4) that

$$\mathcal{T}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\overline{T}^{\beta_1\beta_2\beta_3;\sigma_1\ldots\sigma_5} = \frac{1}{672}\overline{T}^{\alpha_1\alpha_2\alpha_3;\sigma_1\ldots\sigma_5}. \tag{A.7}$$

Theorem 1. *Let $f(\lambda^{n+3},\overline{\lambda}^n,\theta)$ be a general superfield with ghost-number +3, then*

$$\big\langle f(\lambda^{n+3},\overline{\lambda}^n,\theta)\big\rangle_{(m,g)} = \big\langle(\lambda\overline{\lambda})^n\hat{f}(\lambda^3,\theta^5)\big\rangle_{(m,g)} \tag{A.8}$$

where

$$\hat{f}(\lambda^3,\theta^5) = 672\,\lambda^{\beta_1}\lambda^{\beta_2}\lambda^{\beta_3}\,\mathcal{T}^{\sigma_1\ldots\sigma_{n+3}}_{\beta_1\ldots\beta_{n+3}}\,f^{\beta_4\ldots\beta_{n+3}}_{\sigma_1\ldots\sigma_{n+3};\delta_1\ldots\delta_5}\theta^{\delta_1}\ldots\theta^{\delta_5}.$$

Proof. Integrating the right-hand side of (A.8) over $[dr]$ and $[d\theta]$ using the measures of (2.14) and the definition (2.20) yields

$$\begin{aligned}\text{r.h.s.} = 11!\,5!\,c_r c_\theta\int[d\lambda][d\overline{\lambda}]e^{-(\lambda\overline{\lambda})}\,672(\lambda\overline{\lambda})^{n+m-3}\lambda^{\beta_1}\lambda^{\beta_2}\lambda^{\beta_3}\,\overline{\lambda}_{\gamma_1}\overline{\lambda}_{\gamma_2}\overline{\lambda}_{\gamma_3}\\ \times\,\overline{T}^{\gamma_1\gamma_2\gamma_3;\delta_1\ldots\delta_5}\mathcal{T}^{\sigma_1\ldots\sigma_{n+3}}_{\beta_1\ldots\beta_{n+3}}\,f^{\beta_4\ldots\beta_{n+3}}_{\sigma_1\ldots\sigma_{n+3};\delta_1\ldots\delta_5}.\end{aligned}$$

Given that the $\mathcal{T}$ tensors are normalized such that $\mathcal{T}^{\alpha_1\ldots\alpha_p}_{\beta_1\ldots\beta_p} = 1$ the integration over the pure spinors λ and $\overline{\lambda}$ using (A.5) leads to

$$\begin{aligned}\text{r.h.s.} &= 11!\,5!\,c_r c_\theta\left(\frac{A_g}{2\pi}\right)^{11}\frac{\Gamma(8+m+n)}{302400}\,672\mathcal{T}^{\beta_1\beta_2\beta_3}_{\gamma_1\gamma_2\gamma_3}\overline{T}^{\gamma_1\gamma_2\gamma_3;\delta_1\ldots\delta_5}\mathcal{T}^{\sigma_1\ldots\sigma_{n+3}}_{\beta_1\ldots\beta_{n+3}}\,f^{\beta_4\ldots\beta_{n+3}}_{\sigma_1\ldots\sigma_{n+3};\delta_1\ldots\delta_5}\\ &= 11!\,5!\,c_r c_\theta\left(\frac{A_g}{2\pi}\right)^{11}\frac{\Gamma(8+m+n)}{302400}\,\overline{T}^{\beta_1\beta_2\beta_3;\delta_1\ldots\delta_5}\mathcal{T}^{\sigma_1\ldots\sigma_{n+3}}_{\beta_1\ldots\beta_{n+3}}\,f^{\beta_4\ldots\beta_{n+3}}_{\sigma_1\ldots\sigma_{n+3};\delta_1\ldots\delta_5}\end{aligned} \tag{A.9}$$

where (A.7) has been used in the second line. However it is easy to show that the evaluation of the left-hand side of (A.8) is equal to (A.9), finishing the proof. □

For completeness, note that $\overline{T}^{\alpha\beta\gamma;\sigma_1\ldots\sigma_5}$ defined in (A.6) is proportional to the pure spinor correlator $\langle\lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\sigma_1}\ldots\theta^{\sigma_5}\rangle_{(n,g)}$. Indeed, a short computation shows that

$$\langle\lambda^\alpha\lambda^\beta\lambda^\gamma\theta^{\sigma_1}\ldots\theta^{\sigma_5}\rangle_{(n,g)} = \left(\frac{\alpha'}{2}\right)^2\left(\frac{2\pi}{A_g}\right)^{5/2}\frac{\Gamma(8+n)}{302400}\frac{R}{672}\,\overline{T}^{\alpha\beta\gamma;\sigma_1\ldots\sigma_5}. \tag{A.10}$$

As a consistency check, multiplying both sides by $\gamma^m_{\alpha\delta_1}\gamma^n_{\beta\delta_2}\gamma^p_{\gamma\delta_3}(\gamma_{mnp})_{\delta_4\delta_5}$ recovers (2.21),

$$\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle_{(n,g)} = \left(\frac{\alpha'}{2}\right)^2\left(\frac{2\pi}{A_g}\right)^{5/2}2^7R\,\frac{\Gamma(8+n)}{7!}$$

where we used that $\overline{T}^{\alpha\beta\gamma;\delta_1\ldots\delta_5}\gamma^m_{\alpha\delta_1}\gamma^n_{\beta\delta_2}\gamma^p_{\gamma\delta_3}(\gamma_{mnp})_{\delta_4\delta_5} = 5160960$ [45].

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As an example, the function $f(\lambda^4,\overline{\lambda},\theta)\equiv\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\lambda^{\alpha_4}\overline{\lambda}_{\beta_1}f^{\beta_1}_{\alpha_1\alpha_2\alpha_3\alpha_4}(\theta)$ can be easily rewritten according to the Theorem 1 by using

$$\begin{aligned}672\,\lambda^{\sigma_1}\lambda^{\sigma_2}\lambda^{\sigma_3}\mathcal{T}^{\alpha_1\alpha_2\alpha_3\alpha_4}_{\sigma_1\sigma_2\sigma_3\beta_1} &= \frac{2}{33}\Big[\big\{\delta^{\alpha_1}_{\beta_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\lambda^{\alpha_4}+(\alpha_1\leftrightarrow\alpha_2,\alpha_3,\alpha_4)\big\} \\ &\quad -\frac{1}{12}(\lambda\gamma^m)_{\beta_1}\big\{\gamma_m^{\alpha_1\alpha_2}\lambda^{\alpha_3}\lambda^{\alpha_4}+\gamma_m^{\alpha_1\alpha_3}\lambda^{\alpha_2}\lambda^{\alpha_4}+\gamma_m^{\alpha_1\alpha_4}\lambda^{\alpha_2}\lambda^{\alpha_3} \\ &\quad +\gamma_m^{\alpha_2\alpha_3}\lambda^{\alpha_1}\lambda^{\alpha_4}+\gamma_m^{\alpha_2\alpha_4}\lambda^{\alpha_1}\lambda^{\alpha_3}+\gamma_m^{\alpha_3\alpha_4}\lambda^{\alpha_1}\lambda^{\alpha_2}\big\}\Big]\,.\end{aligned} \tag{A.11}$$

A.1 Factoring $(\lambda\overline{\lambda})^6$ from $L^m_{1234}(\lambda,\overline{\lambda},r)$ and $T_{12,3,4}(\lambda,\overline{\lambda},r)$

Because of the constraint (2.2) the definition (3.10) can be written as

$$\begin{aligned}L^x_{1234}(\lambda,\overline{\lambda},r) &= (\overline{\lambda}\gamma^a\gamma^b\gamma^c r)(\overline{\lambda}\gamma^{def}r)(\lambda\gamma^{adefm}\lambda)(\overline{\lambda}\gamma^n\gamma^m\gamma^p r)(\overline{\lambda}\gamma^{qrs}r)(\lambda\gamma^{nqrsu}\lambda) \\ &\quad\times(\overline{\lambda}\gamma^t\gamma^u\gamma^v r)(\overline{\lambda}\gamma^{ghi}r)(\lambda\gamma^{tghib}\lambda)\big[(\lambda\gamma^cW_1)(\lambda\gamma^pW_2)(\lambda\gamma^vW_3)A^x_4\big]\end{aligned} \tag{A.12}$$

Applying the identity $(\overline{\lambda}\gamma_{mnp}r)(\lambda\gamma^{amnpb}\lambda)=48(\lambda\overline{\lambda})(\lambda\gamma^a\gamma^b r)-48(\lambda\gamma^a\gamma^b\overline{\lambda})(\lambda r)$ and using the pure spinor constraint gives

$$\begin{aligned}L^x_{1234}(\lambda,\overline{\lambda},r) &= -48^3(\lambda\overline{\lambda})^3(\lambda\gamma^d\gamma^a r)(\lambda\gamma^g\gamma^e r)(\lambda\gamma^b\gamma^i r) \\ &\quad\times(\overline{\lambda}\gamma^a\gamma^c\gamma^b r)(\lambda\gamma^cW_1)(\overline{\lambda}\gamma^e\gamma^f\gamma^d r)(\lambda\gamma^fW_2)(\overline{\lambda}\gamma^i\gamma^h\gamma^g r)(\lambda\gamma^hW_3)A^x_4\end{aligned} \tag{A.13}$$

where we also renamed indices. Using

$$(\overline{\lambda}\gamma^m\gamma^p\gamma^n r)(\lambda\gamma^pW^2)=-(\lambda\gamma^p\gamma^n r)(\overline{\lambda}\gamma^m\gamma^pW^2)-(\lambda\gamma^p\gamma^m\overline{\lambda})(r\gamma^n\gamma^pW^2), \tag{A.14}$$

in the last three factors and doing straightforward algebra yields,

$$\begin{aligned}L^x_{1234}(\lambda,\overline{\lambda},r) &= 48^3\,8(\lambda\overline{\lambda})^5Q\Big[\\ &\quad+(\lambda\gamma^a\gamma^d r)(\lambda\gamma^c\gamma^e r)(\lambda\gamma^b\gamma^f r)(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(\overline{\lambda}\gamma^{cf}W^3) \\ &\quad+(\lambda\gamma^a\gamma^d r)(\lambda\gamma^c\gamma^e r)(\lambda\gamma^b\gamma^f r)(r\gamma^{ab}W^1)(\overline{\lambda}\gamma^{de}W^2)(r\gamma^{cf}W^3) \\ &\quad+(\lambda\gamma^a\gamma^d r)(\lambda\gamma^c\gamma^e r)(\lambda\gamma^b\gamma^f r)(\overline{\lambda}\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3) \\ &\quad-(\lambda\gamma^c\gamma^e r)(\lambda\gamma^b\gamma^f r)(\lambda\gamma^a\gamma^d\overline{\lambda})(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3) \\ &\quad+(\lambda\gamma^c\gamma^e r)(\lambda\gamma^a\gamma^d r)(\lambda\gamma^b\gamma^f\overline{\lambda})(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3) \\ &\quad+(\lambda\gamma^a\gamma^d r)(\lambda\gamma^b\gamma^f r)(\lambda\gamma^c\gamma^e\overline{\lambda})(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3)\Big] \\ &\quad+48^3\,8(\lambda\overline{\lambda})^6(\lambda\gamma^a\gamma^d r)(\lambda\gamma^c\gamma^e r)(\lambda\gamma^b\gamma^f r)(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3)\end{aligned} \tag{A.15}$$

It is easy to check that (A.15) is totally antisymmetric in [123] as required. The terms proportional to $(\lambda\overline{\lambda})^5\,Q$ will be rewritten using (A.11) and we identified $(\lambda r)=Q$ because of the factor $e^{-(r\theta)}$ in $\mathcal{N}$. Despite the explicit appearance of the BRST charge in some terms, they are not BRST-trivial because of the remaining factor $\overline{\lambda}$. However, since $Q^2=0$ and the difference between $(\lambda\gamma^a\gamma^d r)$ and $(\lambda\gamma^{ad}r)$ is proportional to Q, one can replace all factors of $(\lambda\gamma^a\gamma^d r)$ by $(\lambda\gamma^{ad}r)$. Doing this replacement is also allowed in the last term because there are no factors of $\overline{\lambda}_\alpha$, so the BRST charge vanishes in the cohomology.

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Similarly, $(\lambda\gamma^a\gamma^d\overline{\lambda})$ can be substituted by $(\lambda\gamma^{ad}\overline{\lambda})$ since the difference is BRST-trivial due to the resulting factor of $Q(\lambda\overline{\lambda})^6$. Therefore (A.15) becomes

$$L^x_{1234}(\lambda,\overline{\lambda},r) = 48^3\,8\big(F_{123}+F_{312}+F_{231}-G_{123}-G_{312}-G_{231}+H_{123}\big)A^x_4 \tag{A.16}$$

where

$$F_{123} = (\lambda\overline{\lambda})^5 Q(\lambda\gamma^{ad}r)(\lambda\gamma^{ce}r)(\lambda\gamma^{bf}r)(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(\overline{\lambda}\gamma^{cf}W^3)\,, \tag{A.17}$$

$$G_{123} = (\lambda\overline{\lambda})^5 Q(\lambda\gamma^{ce}r)(\lambda\gamma^{bf}r)(\lambda\gamma^{ad}\overline{\lambda})(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3)\,, \tag{A.18}$$

$$H_{123} = (\lambda\overline{\lambda})^6(\lambda\gamma^{ad}r)(\lambda\gamma^{ce}r)(\lambda\gamma^{bf}r)(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3)\,. \tag{A.19}$$

It is not difficult to show that H_{123} is totally antisymmetric in [123] whereas F_{123} and G_{123} are antisymmetric in [12].

Let us rewrite the superfield F_{123} using the Theorem 1. Since the γ-matrix traceless tensors are normalized such that $\mathcal{T}^{\alpha_1\ldots\alpha_n}_{\alpha_1\ldots\alpha_n}=1$, the factor $(\lambda\overline{\lambda})^5$ is inert under the application of the theorem and one can use (A.11) directly. Furthermore, all terms which still contain an explicit BRST charge after using (A.11) will be BRST-trivial because of the factor $(\lambda\overline{\lambda})^6$. So in fact only four terms in (A.11) are non-vanishing when applied to F_{123}. After straightforward algebra and discarding BRST-exact terms,

$$\begin{aligned} F_{123} = \frac{2}{33}(\lambda\overline{\lambda})^6\Big[&-(\lambda\gamma^{ad}r)(\lambda\gamma^{ce}r)(\lambda\gamma^{bf}r)(r\gamma^{ab}W^1)(r\gamma^{de}W^2)(r\gamma^{cf}W^3)\\ &-\frac{1}{4}(\lambda\gamma^{de}r)(\lambda\gamma^{af}r)(\lambda\gamma^{bc}r)(r\gamma^{ab}W^3)(r\gamma^{cd}W^1)(r\gamma^{ef}W^2)\\ &+\frac{1}{4}(\lambda\gamma^{de}r)(\lambda\gamma^{af}r)(\lambda\gamma^{bc}r)(r\gamma^{ab}W^3)(r\gamma^{cd}W^2)(r\gamma^{ef}W^1)\Big] \end{aligned} \tag{A.20}$$

which implies that $F_{123}=-\frac{1}{11}H_{123}$ (and similarly $G_{123}=\frac{1}{11}H_{123}$). Plugging these results into (A.16) and taking into account the total antisymmetry of H_{ijk} one finally obtains $L_{123}=(48^3\,40/11)H_{123}$. Identical manipulations apply to $T_{12,3,4}$, so

$$\begin{aligned} L^m_{1234}(\lambda,\overline{\lambda},r) &= \frac{48^3\,40}{11}(\lambda\overline{\lambda})^6(\lambda\gamma^{af}r)(\lambda\gamma^{bc}r)(\lambda\gamma^{de}r)(r\gamma^{ab}W^1)(r\gamma^{cd}W^2)(r\gamma^{ef}W^3)A^m_4\,,\\ T_{12,3,4}(\lambda,\overline{\lambda},r) &= \frac{48^3\,40}{11}(\lambda\overline{\lambda})^6(\lambda\gamma^{af}r)(\lambda\gamma^{bc}r)(\lambda\gamma^{de}r)(r\gamma^{ab}W_{12})(r\gamma^{cd}W_3)(r\gamma^{ef}W_4)\,. \end{aligned} \tag{A.21}$$

Similar manipulations can be used in $S^m_{1234}(\lambda,\overline{\lambda},r)$ but for historic reasons we computed the five covariant derivatives before rewriting it with the factor $(\lambda\overline{\lambda})^6$. The resulting expression is not particularly illuminating and was therefore omitted.

B Integration by parts

Noting that one can replace $\Delta(z_j;z_k;z_l)X_{1j}$ by $\Delta(z_1;z_k;z_l)X_{1j}$ in (3.26) because $U_iU_j\sim(dW_{ij})(z_j)\eta_{ij}=(dW_{ij})(z_i)\eta_{ij}$ it is straightforward to show that the identities which eliminate X_{1i} and $\overline{X}_{1j}$ are given by

$$\begin{aligned} \Delta(1,3,4)\overline{\Delta}(2,3,4)X_{12}\overline{X}_{12} = {}&\Delta(1,3,4)\overline{\Delta}(1,3,4)(X_{23}+X_{24})(\overline{X}_{23}+\overline{X}_{24})\\ &+\Delta(1,3,4)\overline{\Delta}(2,3,4)s_{12}\tilde{\Omega}_{21}-\Delta(1,2,4)\overline{\Delta}(1,3,4)s_{23}\tilde{\Omega}_{32}\\ &+\Delta(1,2,3)\overline{\Delta}(1,3,4)s_{24}\tilde{\Omega}_{42}\,, \end{aligned}$$

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$$
\begin{aligned}
\Delta(1,3,4)\overline{\Delta}(1,2,4)X_{12}\overline{X}_{13} &= (X_{23}+X_{24})(-\overline{X}_{23}+\overline{X}_{34})\Delta(1,3,4)\overline{\Delta}(1,2,4)\\
&\quad +s_{23}\tilde{\Omega}_{23}\Delta(1,3,4)\overline{\Delta}(1,2,4)\,,\\
\Delta(1,3,4)\overline{\Delta}(1,2,4)X_{12}\overline{X}_{23} &= \Big[(X_{23}+X_{24})\overline{X}_{23}-s_{23}\tilde{\Omega}_{23}\Big]\,\Delta(1,3,4)\overline{\Delta}(1,2,4)\,,\\
\Delta(1,3,4)\overline{\Delta}(1,2,3)X_{12}\overline{X}_{34} &= (X_{23}+X_{24})\overline{X}_{34}\Delta(1,3,4)\overline{\Delta}(1,2,3),
\end{aligned}
\tag{B.1}
$$

where we used that

$$
\overline{\partial}_i X_{ji}=s_{ij}\tilde{\Omega}_{ji},\quad \partial_i\overline{X}_{ji}=s_{ij}\tilde{\Omega}_{ij},\quad \overline{\partial}_i X_{ij}=-s_{ij}\tilde{\Omega}_{ji},\quad \partial_i\overline{X}_{ij}=-s_{ij}\tilde{\Omega}_{ij}\,,
\tag{B.2}
$$

and defined $\tilde{\Omega}_{ij}=(\alpha'/2)\pi\Omega(z_i,z_j)$. All other identities needed to write $|\mathcal{F}|^2$ in a basis of integrals follow from the above by relabelling. Applying them together with

$$
\tilde{\Omega}_{ij}\Delta(j,k,l)\overline{\Delta}(i,k,l)=\tilde{\Omega}_{ji}\Delta(i,k,l)\overline{\Delta}(j,k,l)\,,
\tag{B.3}
$$

implies that $|\mathcal{F}|^2$ is equal to

$$
\begin{aligned}
&+|C_{32,1,4}|^2(X_{23}\overline{X}_{23}-s_{23}\tilde{\Omega}_{23})\Delta(1,3,4)\overline{\Delta}(1,2,4)\\
&-|C_{24,1,3}|^2(X_{24}\overline{X}_{24}-s_{24}\tilde{\Omega}_{24})\Delta(1,3,4)\overline{\Delta}(1,2,3)\\
&+|C_{34,1,2}|^2(X_{34}\overline{X}_{34}-s_{34}\tilde{\Omega}_{34})\Delta(1,2,4)\overline{\Delta}(1,2,3)\\
&+C_{32,1,4}\tilde{C}_{24,1,3}X_{23}\overline{X}_{24}\Delta(1,2,4)\overline{\Delta}(1,3,4)+C_{32,1,4}\tilde{C}_{34,1,2}X_{23}\overline{X}_{34}\Delta(1,3,4)\overline{\Delta}(1,2,4)\\
&+C_{24,1,3}\tilde{C}_{32,1,4}X_{24}\overline{X}_{23}\Delta(1,3,4)\overline{\Delta}(1,2,4)+C_{24,1,3}\tilde{C}_{34,1,2}X_{24}\overline{X}_{34}\Delta(1,3,4)\overline{\Delta}(1,2,3)\\
&+C_{34,1,2}\tilde{C}_{32,1,4}X_{34}\overline{X}_{23}\Delta(1,2,4)\overline{\Delta}(1,3,4)+C_{34,1,2}\tilde{C}_{24,1,3}X_{34}\overline{X}_{24}\Delta(1,2,4)\overline{\Delta}(1,3,4)\\
&+s_{12}\tilde{\Omega}_{12}|M_{12,3,4}|^2\Delta(2,3,4)\overline{\Delta}(1,3,4)-s_{13}\tilde{\Omega}_{13}|M_{13,2,4}|^2\Delta(2,3,4)\overline{\Delta}(1,2,4)\\
&+s_{14}\tilde{\Omega}_{14}|M_{14,2,3}|^2\Delta(2,3,4)\overline{\Delta}(1,2,3)+s_{23}\tilde{\Omega}_{23}|M_{23,1,4}|^2\Delta(1,3,4)\overline{\Delta}(1,2,4)\\
&-s_{24}\tilde{\Omega}_{24}|M_{24,1,3}|^2\Delta(1,3,4)\overline{\Delta}(1,2,3)+s_{34}\tilde{\Omega}_{34}|M_{34,1,2}|^2\Delta(1,2,4)\overline{\Delta}(1,2,3)
\end{aligned}
\tag{B.4}
$$

where we defined (the others follow from relabelling)

$$
C_{24,1,3}\equiv M_{24,1,3}+M_{14,2,3}+M_{12,3,4}.
\tag{B.5}
$$

The α'-expansion of the above integrals has not been derived but one can argue from the results of [46] that $\eta_{ij}\overline{\eta}_{ij}-\tilde{\Omega}_{ij}s_{ij}^{-1}$ and $\eta_{ij}\overline{\eta}_{ik}$ have no kinematic poles. Therefore the leading-order contribution from (B.4) is given by the $\tilde{\Omega}_{ij}$ terms and it follows from relabelling of integration variables that they are all equal to $\pm\int\tilde{\Omega}_{12}\Delta(2,3,4)\overline{\Delta}(1,3,4)$ (the sign is easy to obtain). Thus the low energy limit of $|\mathcal{F}|^2$ in (3.25) corresponds to[7]

$$
-\frac{\sqrt{2}\pi^7}{2^{23}3^7}\kappa^4 e^{4\lambda}\left(\frac{\alpha'}{2}\right)^7\int\frac{d^2\Omega_{IJ}}{Z_3^{-10}}\int\prod_{i=1}^4 d^2z_i\,\Omega_{12}\Delta(2,3,4)\overline{\Delta}(1,3,4)\langle\mathcal{K}\rangle_{(-3)}\langle\prod_{j=1}^4 e^{ik^j\cdot x^j}\rangle
\tag{B.6}
$$

where we used $\tilde{\Omega}_{12}=\pi(\alpha'/2)\Omega_{12}$ and defined,

$$
\mathcal{K}=\frac{|T_{23,1,4}|^2}{s_{23}}+\frac{|T_{24,1,3}|^2}{s_{24}}+\frac{|T_{34,1,2}|^2}{s_{34}}+\frac{|T_{12,3,4}|^2}{s_{12}}+\frac{|T_{13,2,4}|^2}{s_{13}}+\frac{|T_{14,2,3}|^2}{s_{14}}\,.
\tag{B.7}
$$

[7]The minus sign compensates the "convention" $ik^m\to k^m$ in the Koba-Nielsen factor.

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B.1 Open superstring

In the case of the open superstring it is not difficult to argue that the corresponding low energy limit is

$$\mathcal{K}^{(\text{open})} = \frac{T_{23,4,1} + T_{41,2,3}}{s_{23}} + \frac{T_{34,1,2} + T_{12,3,4}}{s_{34}}. \tag{B.8}$$

The component expansion of (B.8) provides a good consistency check for the methods of [28] since one recovers the α'^3 interaction of the open superstring tree-level amplitude [47, 48],

$$\frac{\langle T_{23,4,1} + T_{41,2,3}\rangle}{s_{23}} + \frac{\langle T_{34,1,2} + T_{12,3,4}\rangle}{s_{34}} = 1344 \cdot 40 \cdot 48^3 \cdot 2880\, A^{\text{YM}}_{1234} s_{12} s_{13} s_{23}. \tag{B.9}$$

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