Exact results in $\mathcal{N} = 2$ gauge theories

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ABSTRACT: We derive exact formulae for the partition function and the expectation values of Wilson/'t Hooft loops, thus directly checking their S-duality transformations. We focus on a special class of $\mathcal{N} = 2$ gauge theories on S^4 with fundamental matter. In particular we show that, for a specific choice of the masses, the matrix model integral defining the gauge theory partition function localizes around a finite set of critical points where it can be explicitly evaluated and written in terms of generalized hypergeometric functions. From the AGT perspective the gauge theory partition function, evaluated with this choice of masses, is viewed as a four point correlator involving the insertion of a degenerated field. The well known simplicity of the degenerated correlator reflects the fact that for these choices of masses only a very restrictive type of instanton configurations contributes to the gauge theory partition function.

Contents

1 Introduction

The study of Wilson loops in supersymmetric gauge theories has received a lot of attention in recent years. According to holography, the expectation value of a supersymmetric Wilson loop in $\mathcal{N} = 4$ gauge theory at strong coupling is computed by the area of the minimal surface on AdS swept by an open string ending on the Wilson loop itself [1]. This result is confirmed by a perturbative computation in $\mathcal{N} = 4$ [2]. In fact the contribution of

each relevant Feynman integral is shown to be one and the multiplicities of the Feynman diagrams are evaluated by a gaussian matrix model. For circular Wilson loops the matrix model integral can be computed and it gives a Bessel function reproducing the right strong coupling asymptotics predicted by gravity. This formula was rigorously proven in [3] using localization techniques. The results were also extended to $\mathcal{N} = 2$, where the measure of the matrix model integral is completed by the one-loop, $Z_{\text{one-loop}}$, and instanton contributions, Z_{inst}, to the partition function of the gauge theory. There are two complications in extending the exact $\mathcal{N} = 4$ result to $\mathcal{N} = 2$. First, Z_{inst} can be computed only order by order in $q = e^{2\pi i\tau}$. Second, for a general form of the perturbative and non perturbative contributions the matrix model integral is not amenable to an analytic treatment. In this paper we show that for a special (but not very restrictive) choice of the gauge theory parameters both difficulties can be overcome and exact formulae can be derived.

We focus on $\mathcal{N} = 2$ gauge theories with gauge group U(N) and matter in the fundamental and anti-fundamental representations. The use of localization requires that the gauge theory be placed on a non-trivial ϵ-background lifting its Lorentz symmetries. From a physical point of view the introduction of the ϵ background can be viewed as a gravitational Ω-background [4] or as the result of type IIB RR fluxes [5–7]. We will consider both the cases of gauge theories on $\mathbb{R}^4$ and S^4. In the former case one should also specify the expectation values $a \in \text{SU}(N)$ of the scalar field at infinity. The partition function, $Z(a)$, depends then on the masses m, vevs a, ϵ_ℓ deformations and the gauge coupling q and it can be computed order by order in q via localization techniques [8–10]. In the limit where both ϵ_ℓ are small, it becomes $Z(a) \approx e^{-\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_{\text{SW}}}$, where $\mathcal{F}_{\text{SW}}$ is the Seiberg-Witten prepotential determining the two-derivative effective action of the gauge theory. On the other hand, in the limit when ϵ_1 is small but $\epsilon_2 = \epsilon$ is finite, the ϵ-deformed dynamics can be shown to be in correspondence with that of certain quantum integrable systems with ϵ playing the role of the Planck constant [11–13]. Both limits can be treated via saddle point techniques leading to a Seiberg-Witten curve (or its ϵ-deformed version) [14–16].[1]

In this paper we consider the case where both ϵ_1 and ϵ_2 are finite. Drawing inspiration from previous works in CFT's, we show that the masses can be chosen in such a way that only a very peculiar class of gauge instanton configurations can contribute to Z and that the full instanton sum can be explicitly evaluated (see [19, 20] for previous works in this direction). In the AGT dual description of the theory [21], where the gauge theory instanton partition function is described by a four-point conformal block of the Toda field theory, this specific choice of mass corresponds to the insertion of a degenerated field. The conformal block is then determined by a differential equation that, in the simplest case, can be solved in terms of generalized hypergeometric functions [22–24].

In the case of a gauge theory on S^4 the partition function is given by the integral $\int da |Z(a)|^2$ with Z and $\bar{Z}$ the contributions of the instantons and anti-instantons located at the north and south poles of the two charts in S^4 [3, 25]. The partition function is computed for $\epsilon_1 = \epsilon_2 = \frac{1}{r}$, with r the radius of the sphere. We will show that by restricting the overall sum of the masses in the fundamental representation, the integral over a localizes

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[1]The generalisation of this analysis to ADE quiver gauge theories can be found in [17, 18].

around some critical points where the full instanton partition function collapses to a very simple form. Exact formulae for the instanton partition function and for the expectation value of the circular Wilson/'t Hooft loops will then be derived. The results provide a direct check of S duality. In particular we will explicitly check that Wilson and 't Hooft loops are exchanged under S duality and that the gauge theory partition function is S-duality invariant.

We present also a qualitative study of the matrix model integral in the case where some of the masses become large but keeping finite the overall sum. In this limit the integral can be computed with a saddle point approximation giving the main qualitative features of the localization for critical masses. The analysis follows closely [26–29] and shares with these studies the leading behavior of the Wilson loops in the limit of a large number of colours.

This is the plan of the paper: In section 2 we derive the partition function of $\mathcal{N} = 2$ SYM on $\mathbb{C}^2$ with fundamental matter for some critical choices of the masses. In section 3 we treat the case of the gauge theory on S^4 and derive exact formulae for the partition function and the expectation value of the Wilson and 't Hooft loops. In section 4 we describe the AGT dual of the gauge theory. The technical material needed to follow our computations is confined in the appendices. In particular, appendix A collects the definitions and properties of the special functions used in the main text, appendix B the details of the four point function in Liouville theory associated to the SU(2) case, and appendix C is a detailed derivation of the main instanton partition function building blocks for the gauge theory.

2   The gauge theory on $\mathbb{C}^2$

We consider a four-dimensional U(N) gauge theory with $2N$ hypermultiplets, one half of these transforming in the fundamental representation and the other half in the anti-fundamental representation.
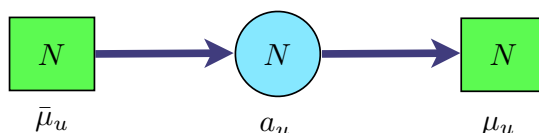
The partition function of the gauge theory is given by a product of the classical, one loop and instanton contributions

$$Z = Z_{\text{class}} \, Z_{\text{one-loop}} \, Z_{\text{inst}} \tag{2.1}$$

Denoting by $q = e^{2\pi i \tau}$ the gauge coupling and by $a = \{a_u\}$ the vevs, the classical contribution reads

$$Z_{\text{class}} = q^{-\frac{a \cdot a}{2\epsilon_1 \epsilon_2}} \tag{2.2}$$

The instanton partition function follows from the localization formula [8–10]

$$Z_{\text{inst}} = \sum_Y q^{|Y|} \prod_{u,v=1}^{N} \frac{Z_{\emptyset,Y_v}(\bar{\mu}_u - a_v) \, Z_{Y_u,\emptyset}(a_u - \mu_v)}{Z_{Y_u,Y_v}(a_u - a_v)} \tag{2.3}$$

with $a_u, \mu_u, \bar{\mu}_u$ parametrizing the vev's and masses and the sum running over the array $Y = \{Y_u\}$ of N Young tableaux specifying the positions of instantons around a_u. We denoted by $|Y|$ the instanton number given by the total number of boxes in Y. Finally Z_{Y_u,Y_v} is given by [9, 10] (see appendix C for details)[2]

$$
Z_{Y_u,Y_v}(x) = \prod_{(i,j)\in Y_u} (x + \epsilon_1(i - k_{vj}) - \epsilon_2 (j - 1 - \tilde{k}_{ui}))
$$
$$
\times \prod_{(i,j)\in Y_v} (x - \epsilon_1(i - 1 - k_{uj}) + \epsilon_2 (j - \tilde{k}_{vi})) \tag{2.4}
$$

with (i,j) running over rows and columns respectively of either the Y_u or Y_v tableaux. Here $\{k_{uj}\}$ and $\{\tilde{k}_{ui}\}$ are infinite and weakly decreasing sequences of positive integers giving the length of the rows and columns respectively of the tableau Y_u.[3] It is convenient to rewrite formula (2.4) in the infinite product form [30]

$$
Z_{Y_u,Y_v}(x) = \prod_{i,j=1}^{\infty} {}' \frac{x + \epsilon_1(i - k_{vj}) - \epsilon_2 (j - 1 - \tilde{k}_{ui})}{x + \epsilon_1 i - \epsilon_2 (j - 1)} \tag{2.5}
$$

where

$$
\prod_{i,j=1}^{\infty} {}' \equiv \lim_{L\to\infty} (-L\epsilon_1)^{|Y_v|} (-L\epsilon_2)^{|Y_u|} \prod_{i,j=1}^{L} \tag{2.6}
$$

One can check that the contributions coming from the numerator and denominator in (2.5) cancel against each other except for a finite number of terms in the numerator, which reproduce (2.4), and in the denominator, which cancel the prefactors in (2.6).[4] Moreover, the prefactors in (2.6) cancel in (2.3) between numerator and denominator and therefore the prime in the infinite product can be omitted.

Interestingly the denominator in (2.5) does not depend on the shape of the Young tableaux and therefore this infinite product can be factored out. The one-loop contribution is defined in such a way to cancel this Y-independent term. Up to a-constant terms one can write

$$
Z_{\text{one-loop}} = \prod_{i,j=1}^{\infty} \frac{\prod_{u,v=1}^{N}(\bar{\mu}_u - a_v + \epsilon_1 i - \epsilon_2 (j - 1))(a_u - \mu_v + \epsilon_1 i - \epsilon_2 (j - 1))}{\prod_{u<v}^{N}(a_{uv} + \epsilon_1 i - \epsilon_2 (j - 1))(-a_{uv} + \epsilon_1 i - \epsilon_2 (j - 1))}
$$
$$
= \frac{\prod_{u,v} \Gamma_2(\bar{\mu}_u - a_v + \epsilon)\Gamma_2(a_u - \mu_v + \epsilon)}{\prod_{u<v} \Gamma_2(a_{uv})\Gamma_2(a_{uv} + \epsilon)} \tag{2.7}
$$

where in the second line we used (A.4) and (A.9) to rewrite the infinite product in terms of the Barnes double Gamma function $\Gamma_2(x|\epsilon_1, \epsilon_2)$ (see appendix A for definitions and corresponding properties). Here and below $\epsilon = \epsilon_1 + \epsilon_2$.

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[2]We notice the reflection symmetry $Z_{Y_u Y_v}(-\epsilon - x) = Z_{Y_v Y_u}(x)$ that implies that Z_{inst} is symmetric under the exchange $\bar{\mu}_u \leftrightarrow \mu_u - \epsilon$.

[3]If i (j) is greater than the number of rows (columns) in Y_u, the value of k_{uj} ($\tilde{k}_{ui}$) is zero.

[4]For example for $Y_u = \square$, $Y_v = \emptyset$ one finds $\lim_{L\to\infty} (-L\epsilon_2) \left(\frac{x+\epsilon}{x+\epsilon-L\epsilon_2}\right) = x + \epsilon$ in agreement with (2.4).

The prepotential of the gauge theory in the ϵ-background is defined as [8]

$$\mathcal{F} = -\epsilon_1 \epsilon_2 \log Z \qquad (2.8)$$

We notice that the $Z_{\text{one-loop}}$ given by (2.7) has zeros in the moduli space of masses of the gauge theory. These zeros are located at the points where a flavor brane comes close to a gauge brane (for small ϵ_i's) and indicate that a fundamental matter is getting "massless".[5] In the following we will study the gauge theory in the nearby of these critical points.

2.1 Critical choices of masses

The contribution of the fundamental and anti-fundamental matter to the instanton partition function can be written according to (2.4) as

$$Z_{\emptyset,Y_v}(\bar{\mu}_u - a_v) = \prod_{(i,j)\in Y_v} (\bar{\mu}_u - a_v - \epsilon_1(i-1) - \epsilon_2(j-1))$$
$$Z_{Y_u,\emptyset}(a_u - \mu_v) = \prod_{(i,j)\in Y_u} (a_u - \mu_v + \epsilon_1 i + \epsilon_2 j) \qquad (2.9)$$

We notice that some eigenvalues in (2.9) become zero for the critical choice of masses

$$\mu_u = a_u + p_u \epsilon_1 + q_u \epsilon_2 \qquad p_u, q_u \in \mathbb{Z}_{\geq 1} \qquad (2.10)$$

or

$$\bar{\mu}_u = a_u + (p_u - 1)\epsilon_1 + (q_u - 1)\epsilon_2 \qquad p_u, q_u \in \mathbb{Z}_{\geq 1} \qquad (2.11)$$

with (p_u, q_u) some integers. Indeed for both choices, (2.9) vanishes if the tableaux Y_u contains the box $(i,j) = (p_u, q_u)$. In particular for $(p_u, q_u) = (1,1)$ no instanton contributions are allowed, for $(p_u, q_u) = (1,n)$ only those tableaux, Y_u, with less than n rows contribute and so on.

We will focus on the two simplest non trivial choices

$$\mathbf{I} \quad \mu_u = \mu_u^{\mathbf{I}} = a_u + \epsilon$$

$$\mathbf{II} \quad \mu_u = \mu_u^{\mathbf{II}} = a_u + \epsilon + \epsilon_2 \, \delta_{u,1} \qquad (2.12)$$

with $\bar{\mu}_u$ arbitrary. In the case $\mathbf{I}$ one finds that all non-trivial Young tableaux give a vanishing contribution leading to

$$Z_{\text{inst}}^{\mathbf{I}} = 1 \qquad (2.13)$$

On the other hand, for the choice $\mathbf{II}$ a non-trivial contribution arises only from an array of tableaux with a single non empty tableau Y_1 made of a unique row and $Y_{u\neq 1} = \emptyset$. For

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[5]In presence of an ϵ-background and vevs, the parameters $\mu, \bar{\mu}$ are related but not directly identified with the masses.

this simple configuration the instanton partition function can be explicitly evaluated and resummed. Indeed, denoting by k the length of Y_1, one finds

$$Z^{\mathbf{II}}_{\text{inst}} = \sum_{k=0}^{\infty} q^k \prod_{i=1}^{k} \frac{\prod_{u=1}^{N}(\bar{\mu}_u - a_1 - \epsilon_1(i-1))(a_1 - \mu_u^{\mathbf{II}} + \epsilon_1 i + \epsilon_2)}{(\epsilon_1(i-k) + \epsilon_2)(i\epsilon_1)\prod_{u=2}^{N}(a_{1u} + \epsilon_1 i + \epsilon_2)(a_{u1} - \epsilon_1(i-1))}$$

$$= \sum_{k=0}^{\infty} q^k \frac{\prod_{v=1}^{N}[A_v]_k}{k!\prod_{v=2}^{N}[B_v]_k} = {}_N F_{N-1}\binom{\mathbf{A}}{\mathbf{B}}|q) \tag{2.14}$$

with $[x]_k = \prod_{p=0}^{k-1}(x+p)$, ${}_N F_{N-1}\binom{\mathbf{A}}{\mathbf{B}}|q)$ the generalized hypergeometric function[6] and

$$\mathbf{A}_v = \frac{a_1 - \bar{\mu}_v}{\epsilon_1} = \frac{\mu_1 - \bar{\mu}_v - 2\epsilon_2}{\epsilon_1} - 1 \qquad\qquad v = 1, \dots N$$

$$\mathbf{B}_v = \frac{a_1 - a_v + \epsilon_2}{\epsilon_1} + 1 = \frac{\mu_1 - \mu_v}{\epsilon_1} + 1 \qquad\qquad v = 2, \dots N \tag{2.15}$$

Similar formulae can be found by replacing $\delta_{u,1}$ by $\delta_{u,j}$ in (2.12), i.e.

$$\mathbf{II} \qquad \mu_u^{(j)} = a_u + \epsilon + \epsilon_2\,\delta_{u,j} \quad \Rightarrow \quad Z^{\mathbf{II}}_{\text{inst,j}} = {}_N F_{N-1}\left(\begin{matrix}\mathbf{A}^{(j)}\\\mathbf{B}^{(j)}\end{matrix}|q\right) \tag{2.16}$$

with

$$A_k^{(j)} = 1 - B_j + A_k, \qquad\qquad B_k^{(j)} = \begin{cases} 1 - B_j + B_k & k \neq j \\ 2 - B_k & k = j \end{cases} \tag{2.17}$$

and $B_1 = 1$. Formulae (2.14) and (2.16) provide us with the simplest examples of $\mathcal{N} = 2$ gauge theories on $\mathbb{C}^2$ where non-trivial multi-instanton corrections can be computed in an analytic form. In section 3.4 we will see how this simplification can be understood from the point of view of the AGT dual Toda CFT where the instanton partition function is associated to correlators involving the insertion of degenerated fields. For example, the critical choice $(p_1, q_1) = (1,1)$, $(p_2, q_2) = (p, q)$ for the SU(2) gauge theory can be associated to the insertion of the so called $\phi_{(p,q)}$-degenerated field in Liouville theory.

3 The gauge theory on S^4

In this section we consider the gauge theory on S^4. The partition function in this case is obtained by squaring the Ω-deformed gauge theory partition function $Z(a)$ on $\mathbb{C}^2$ and integrating it over the vevs $a \in$ SU(N) of the scalar fields at infinity [3]. For a round sphere of radius r one takes $\epsilon_1 = \epsilon_2 = \frac{1}{r}$, while arbitrary ϵ_i represent the gauge theory on an ellipsoid [21].

In this section we derive exact formulae for the partition function of $\mathcal{N} = 2$ gauge theories on ellipsoids with critical masses. We will also compute the expectation value of circular supersymmetric Wilson/'t Hooft loops relying on the localization formulae [3, 25, 31].

3.1 The partition function

The partition function on the sphere S^4 (or ellipsoid in the case $\epsilon_1 \neq \epsilon_2$) is given by the integral [3]

$$Z_{S^4} = Z_{\text{flavor}} \int_\gamma da \, |Z_{\text{class}} Z_{\text{one-loop}} Z_{\text{inst}}|^2 \tag{3.1}$$

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[6]For the U(1) case the generalized hypergeometric solution reduces to $(1-q)^{-\mathbf{A_1}}$.
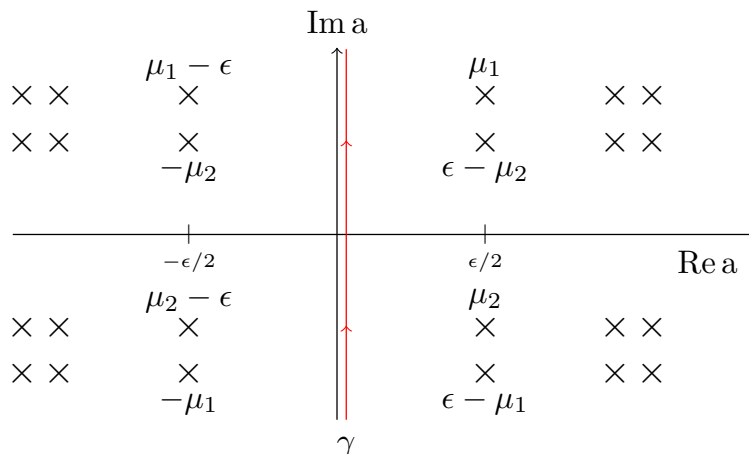
Figure 1. μ-dependent poles of $Z_{\text{pert}}(a)$ for the case of gauge group SU(2).

with $da = \prod_{u=1}^{N-1} da_u$, $a_N = -\sum_{u=1}^{N-1} a_u$. The integral runs over the lines $a_u \in i\mathbb{R}$. We take (consistently with the $\bar\mu \leftrightarrow \mu - \epsilon$ symmetry)

$$\bar\mu_u \in -\frac{\epsilon}{2} + i\mathbb{R} \qquad , \qquad \mu_u \in \frac{\epsilon}{2} + i\mathbb{R} \tag{3.2}$$

Notice that these conventions for the domain of $\bar\mu$'s are different from that in [21]. This leads to a different look for $Z_{\text{one}-\text{loop}}$ but identical results for the truly physical quantity $|Z_{\text{one}-\text{loop}}|^2$.

The term Z_{flavor} is a normalization factor associated to the $\text{SU}(N)^2 \times \text{U}(1)^2$ flavor symmetry

$$Z_{\text{flavor}} = \Upsilon(\kappa_0)\Upsilon(\kappa_1) \prod_{u<v} \Upsilon(\bar\mu_{uv})\Upsilon(-\mu_{uv}) \tag{3.3}$$

with

$$\kappa_0 = \sum_{u=1}^{N}(a_u - \bar\mu_u) \qquad \kappa_1 = \sum_{u=1}^{N}(a_u - \mu_u + \epsilon) \tag{3.4}$$

The function $\Upsilon(x)$ is defined in appendix A. Including or not this factor may be matter of taste since Z_{flavor} is independent of the $\text{SU}(N)$ gauge variables and it is therefore irrelevant to the dynamics of the non-abelian gauge theory. Still, the inclusion of this term, as we will see, guarantees the analyticity of the partition function Z_{S^4} over the $\mu, \bar\mu$ plane. In addition, the gauge theory partition function defined in this way precisely matches, as we will see, the AGT dual 4-point correlator in A_{N-1} Toda field theory up to $\mu, \bar\mu$-independent constants.

Together with the classical and one loop contributions (2.2) and (2.7) one finds

$$Z_{\text{pert}}(a) = Z_{\text{flavor}}|Z_{\text{class}}Z_{\text{one}-\text{loop}}|^2$$
$$= |q|^{-\frac{a \cdot a}{\epsilon_1 \epsilon_2}} \frac{\Upsilon(\kappa_0)\Upsilon(\kappa_1) \prod_{u<v} \Upsilon(\bar\mu_{uv})\Upsilon(a_{uv})\Upsilon(a_{vu})\Upsilon(\mu_{vu})}{\prod_{u,v=1}^{N} \Upsilon(a_v - \bar\mu_u)\Upsilon(\mu_v - a_u)} \tag{3.5}$$

We notice that the integrand in (3.1) has an infinite number of poles in the a-plane. In figure 1, for the purpose of exemplification, we show the poles coming from the zeros

of $\Upsilon(\mu_v - a_u)$ in the case of gauge group SU(2). A similar sequence of poles comes from the zeros of $\Upsilon(a_v - \bar{\mu}_u)$. The integration in (3.1) is along the path, given by the imaginary axis, which is marked in red in figure 1. We notice that for $\mu, \bar{\mu}$ in the range (3.2) no poles fall along the contour γ and moreover the number of poles on the two sides of γ coincides.

The partition function Z_{S^4}, viewed as a function of the masses, can be extended analytically to the whole complex plane $\mu_u, \bar{\mu}_u \in \mathbb{C}$. In doing this the contour γ in (3.1) should be properly deformed in such a way that the poles of the integrand do not cross the integration path γ. Equivalently one can keep the contour always along the imaginary axis adding to the result of the integral the sum of the residues over the poles of the integrand crossing γ. This procedure is well known from the CFT side [22] (see also [25]).

In the following, we focus on special choices of the mass parameters of the gauge theory for which the integral over a in Z_{S^4} can be analytically computed. We consider two choices, that we refer as cases **I** and **II**. They are closely related to the cases **I** and **II** of the gauge theory on $\mathbb{C}^2$ considered in the last section. Indeed, as we will see, for the two choices the integral along γ localizes around critical values of a_u satisfying the relations (2.12) for which the full instanton sums Z_{inst} have been evaluated.

3.1.1 Critical masses: case I

We consider first the co-dimension one slice of the moduli space defined by masses μ_u satisfying the relation

$$\kappa_1 = \sum_{u=1}^{N} (\epsilon - \mu_u) = 0 \tag{3.6}$$

For this choice, $Z_{\text{one-loop}} \sim \Upsilon(\kappa_1) = 0$ all the way along the a-plane except at the points where the denominator of $Z_{\text{one-loop}}$ also vanishes. The integral is then given by the sum of the residues of those poles of $Z_{\text{one-loop}}$ crossing γ once we move from the region of definition (3.2). For the case **I** the relevant poles and the contour are displayed in figure 2 for the gauge group SU(2). This figure can be obtained from figure 1 by applying a rigid shift to μ_1 and μ_2 such that $\mu_1 + \mu_2 = 2\epsilon$. Under this shift the rows containing $\mu_{1,2}$ will be shifted towards the right while those containing $-\mu_{1,2}$ will be shifted towards the left. In the process the poles denoted by a bullet, $\bullet$, in the figure cross the imaginary axis, Im a, and contribute to the residues. On the other hand, the integrand in (3.1) contains the terms $\Upsilon^{-1}(\mu_2 - a_v)$ associated to the poles labelled $\epsilon - \mu_2$ in the figure and denoted by $\circ$, which come close to γ from the other side and cancel the zero coming from $\Upsilon(\kappa_1)$, leading to a finite result. A similar analysis can be performed for the SU(N) case. For simplicity, we will take $\mu_u, \bar{\mu}_u$ from now on given by real numbers (with small imaginary parts) and therefore all the poles in the figures are located near the real line.

Summarizing, for the gauge group SU(N) the partition function gets contributions only from the residue at the poles

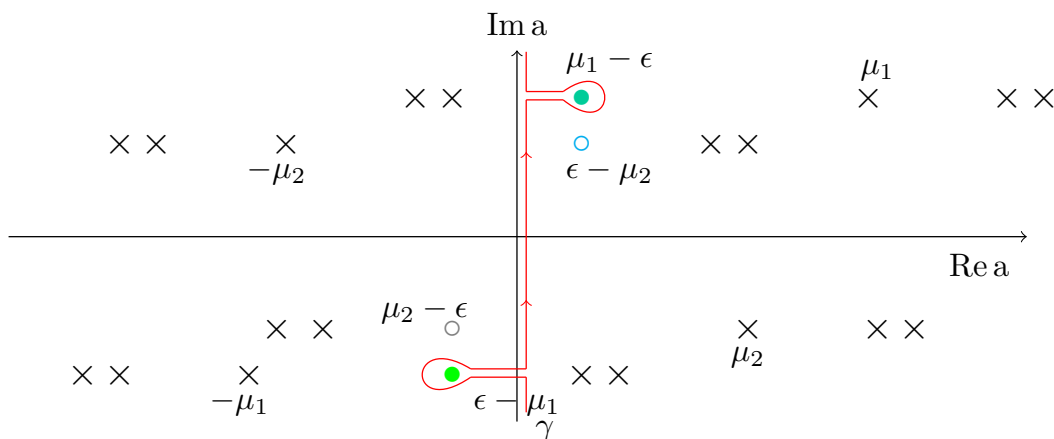$$a_u = a_u^{\text{crit}} = \mu_u - \epsilon \tag{3.7}$$

Figure 2. To obey to the condition (3.6) the poles of figure 1 must be shifted and the integration contour γ needs to be deformed. In this figure we depict the situation for the gauge group SU(2).

up to a permutation of the μ_u's. The residue is given by

$$
Z_{S^4}^{\mathbf{I}} = N\, N!\, \mathrm{Res}_{a=a^{\mathrm{crit}}} \left(Z_{\mathrm{pert}}(a)\, |Z_{\mathrm{inst}}(a)|^2 \right)
$$

$$
= |q|^{-\frac{(\mu-\epsilon)^2}{\epsilon_1 \epsilon_2}}\, \frac{N\, N!\, \Upsilon(\kappa_0)}{\Upsilon'(0)^{N-1}}\, \frac{\prod_{u<v} \Upsilon(\bar{\mu}_{uv})\Upsilon(\mu_{vu})}{\prod_{u,v} \Upsilon(\mu_v - \bar{\mu}_u - \epsilon)} \tag{3.8}
$$

where we have used $Z_{\mathrm{inst}}(a_{\mathrm{crit}}^{\mathbf{I}}) = 1$ according to (2.13). The $N!$ comes from the sum over the permutations of the μ_u's, while the extra N counts their combinations in the N-tuples which define each pole. Formula (3.8) is obvious from a CFT perspective since the critical choice **I** of masses corresponds to the insertion of an identity operator leading to a three (rather than four) point function that is clearly independent from the coordinates. On the other hand, the result is highly non-trivial from the gauge theory point of view, since it provides an exact formula for the gauge theory partition function on that slice of the moduli space defined by (3.6).

3.1.2 Critical masses: case II

The next simplest (but non-trivial) case corresponds to the choice

$$
\kappa_1 = \sum_{u=1}^{N} (\epsilon - \mu_u) = -\epsilon_2 \tag{3.9}
$$

Once again, $Z_{\mathrm{one-loop}} \sim \Upsilon(\kappa_1) = 0$ almost everywhere except at the poles in (3.5). The situation is very similar to the previous case. The relevant poles can now be written as

$$
a_u = a_{u,j}^{\mathrm{crit}} = \mu_u - \epsilon - \delta_{uj}\, \epsilon_2 \tag{3.10}
$$

up to permutations of the μ_u's. The partition function becomes

$$
Z_{S^4}^{\mathbf{II}} = \sum_{j=1}^{N} c_j |Z_j(q)|^2 \tag{3.11}
$$

with

$$Z_j = Z_{\text{inst}}(a_{\text{crit}}^{(j)}) = {}_N F_{N-1}\left(\begin{matrix} \mathbf{A}^{(j)} \\ \mathbf{B}^{(j)} \end{matrix}\Big| q\right) \tag{3.12}$$

written in terms of the $\mathbf{A}^{(j)}, \mathbf{B}^{(j)}$ defined in (2.17) and (2.15), and

$$c_j = N! \operatorname{Res}_{a=a_{\text{crit}}^{(j)}} Z_{\text{flavor}}(a)|Z_{\text{class}} Z_{\text{one}-\text{loop}}(a)|^2 \tag{3.13}$$

$$= N! \, |q|^{-\frac{a_{\text{crit}}^{(j)} \cdot a_{\text{crit}}^{(j)}}{\epsilon_1 \epsilon_2}} \frac{\Upsilon(\kappa_0) \prod_{u<v} \Upsilon(\bar\mu_{uv}) \Upsilon(\mu_{vu})}{\Upsilon'(0)^{N-1} \prod_{u,v} \Upsilon(\mu_v - \bar\mu_u - \epsilon - \epsilon_2 \delta_{vj})} \prod_{u \neq j} \frac{\Upsilon(\mu_{uj} - \kappa_1)}{\Upsilon(\mu_{uj})}$$

Alternatively one can write

$$Z_{S^4}^{\mathbf{II}} = c_1 \sum_{j=1}^{N} r_j |Z_j(q)|^2 \tag{3.14}$$

with

$$r_j = \frac{c_j}{c_1} = |q^{1-B_j}|^2 \prod_{u=1}^{N} \frac{\gamma(B_u)\,\gamma(B_j - B_u)}{\gamma(A_u)\,\gamma(B_j - A_u)} \tag{3.15}$$

and the $\gamma(x)$ function defined in (A.14). Written in this form, one can check that the Z_{S^4} defined by (3.14) with r_j given by (3.15) is single valued on the whole q-complex plane. This fact is obvious around the point $q = 0$ given that Z_j picks up a phase when going around the origin. The single valuedness around $q = \infty$ follows by sending $q \to q^{-1}$ and using the trigonometric relation (A.20) and the form of r_j to show that Z_{S^4} is again given by a sum of squares around $q = \infty$. This provides a highly non-trivial consistency check of the results for Z_{S^4}. In the section 3.4 we will present a further test of this result by matching the gauge theory partition function with the AGT dual correlator in the A_{N-1}Toda Field Theory.

3.2 Wilson loops

The expectation value of a supersymmetric circular Wilson loop (at the equator in the first plane) is given by the localization formula [3, 25]

$$\langle W \rangle = \left\langle e^{i \int_C A_M dx^M} \right\rangle = \frac{1}{Z_{S^4}} \int_\gamma da \, Z_{\text{pert}}(a)|Z_{\text{inst}}(a)|^2 \text{Tr}_{\mathbf{R}} \, e^{\frac{2\pi i a}{\epsilon_1}} \tag{3.16}$$

Plugging (3.8) and (3.14) into (3.16) one finds for the two choices of critical masses and for $\mathbf{R}$ being the fundamental representation

$$\langle W \rangle^{\mathbf{I}} = \sum_{u=1}^{N} e^{2\pi i \frac{(\mu_u - \epsilon)}{\epsilon_1}}$$

$$\langle W \rangle^{\mathbf{II}} = \frac{\sum_{j,u=1}^{N} e^{2\pi i \frac{(\mu_u - \epsilon - \epsilon_2 \delta_{uj})}{\epsilon_1}} r_j |Z_j|^2}{\sum_{j=1}^{N} r_j \, |Z_j|^2} \tag{3.17}$$

Formulae (3.17) are exact and show that $\ln\langle W \rangle$ grows linearly with the perimeter $2\pi/\epsilon_1$ of the Wilson loop. Interestingly on a round sphere both results reduce to the classical contribution $\langle W \rangle = \sum_{u=1}^{N} e^{2\pi i \mu_u}$.

3.3 't Hooft loops

The results in the last sections are exact and therefore can be used to study the strong coupling behavior of the partition function and Wilson loops in $\mathcal{N} = 2$ gauge theories with critical masses. Indeed, using standard transformation properties of the generalized hypergeometric functions the results can be rewritten as an expansion around $q = 1$ rather than $q = 0$. Here we illustrate this analysis for the SU(2) case.

Let us consider first the partition function. Using (A.21) and (A.22) (and dropping the subscript 2 for B) one can write

$$
Z_1 = \frac{\pi \Gamma(B)}{\sin(\pi(B - A_1 - A_2))} \left(\frac{\tilde{Z}_1}{\Gamma(B - A_1)\Gamma(B - A_2)\Gamma(A_1 + A_2 - B + 1)} \right.
$$
$$
\left. - \frac{(1 - q)^{B - A_1 - A_2} \tilde{Z}_2}{\Gamma(A_1)\Gamma(A_2)\Gamma(B - A_1 - A_2 + 1)} \right)
$$
$$
Z_2 = \frac{\pi \Gamma(2 - B)\, q^{B-1}}{\sin(\pi(B - A_1 - A_2))} \left(\frac{\tilde{Z}_1}{\Gamma(1 - A_1)\Gamma(1 - A_2)\Gamma(A_1 + A_2 - B + 1)} \right. \tag{3.18}
$$
$$
\left. - \frac{(1 - q)^{B - A_1 - A_2} \tilde{Z}_2}{\Gamma(1 + A_1 - B)\Gamma(1 + A_2 - B)\Gamma(B - A_1 - A_2 + 1)} \right)
$$

with

$$
Z_1 = {}_2F_1 \left({A_1, A_2 \atop B} | q \right) \qquad\qquad Z_2 = {}_2F_1 \left({1 - B + A_1, 1 - B + A_2 \atop 2 - B} | q \right)
$$
$$
\tilde{Z}_1 = {}_2F_1 \left({A_1, A_2 \atop A_1 + A_2 - B + 1} | 1 - q \right) \qquad \tilde{Z}_2 = {}_2F_1 \left({B - A_1, B - A_2 \atop 1 + B - A_1 - A_2} | 1 - q \right) \tag{3.19}
$$

We notice that the quantities with the tilde are obtained from those without via the replacements[7]

$$
q \to 1 - q \qquad B \to A_1 + A_2 - B + 1 \tag{3.20}
$$

Plugging (3.18) into (3.14) one finds

$$
Z_{S^4} = c_1 |Z_1|^2 + c_2 |Z_2|^2 = \tilde{c}_1 |\tilde{Z}_1|^2 + \tilde{c}_2 |\tilde{Z}_2|^2 \tag{3.21}
$$

with

$$
\tilde{c}_1 = \frac{\gamma(B)\gamma(1 - B + A_1)}{\gamma(B - A_2)\gamma(1 - B + A_1 + A_2)}\, c_1 \quad \tilde{c}_2 = \frac{|1 - q|^{2(B - A_1 - A_2)}\, \gamma(B)\gamma(A_1 + A_2 - B)}{\gamma(A_1)\gamma(A_2)\gamma(B - A_2)\gamma(1 - B + A_2)}\, c_1 \tag{3.22}
$$

The equivalent descriptions in the left and right hand sides of (3.21) provide the expansions of the gauge theory partition function in the weak ($q \approx 0$) and strong ($q \approx 1$) coupling regimes. The natural variables in the two regimes are related by the map (3.20).

Finally, we consider the expectation value of a circular Wilson loop in the fundamental representation (spin $\frac{1}{2}$) of SU(2). Following [3, 25] the evaluation of the circular Wilson

---

[7]The replacement $q \to 1 - q$ corresponds to send $\{\infty, 1, q, 0\}$ into $\{\infty, 0, q, 1\}$, while $B \to A_1 + A_2 - B + 1$ amounts to exchange $\alpha_2 \leftrightarrow \alpha_4$ on the CFT side, according to the notation in B.3.

loop boils down to insert the operator $2\cos\left(\frac{2\pi a}{\epsilon_1}\right)$ in the gauge theory partition function. After localization around $a = a_{\mathrm{crit}}^{(j)} = \{\frac{1}{2}(B-1)\epsilon_1 \mp \frac{1}{2}\epsilon_2\}$ this results into

$$\langle W \rangle = \frac{1}{Z_{S^4}}\left[-2\,c_1\cos\left(\pi B - \frac{\pi\epsilon_2}{\epsilon_1}\right)|Z_1|^2 - 2\,c_2\cos\left(\pi B + \frac{\pi\epsilon_2}{\epsilon_1}\right)|Z_2|^2\right] \qquad (3.23)$$

Using (3.18) to rewrite (3.23) in terms of the $\tilde{Z}_i$ one finds for the expectation value of the 't Hooft loop

$$\langle H_{1/2}\rangle = \frac{c_{ij}\tilde{Z}_i\overline{\tilde{Z}_j}}{Z_{S^4}} \qquad (3.24)$$

with

$$c_{11} = \frac{2\tilde{c}_1\left(\cos\left(\pi B + \frac{\pi\epsilon_2}{\epsilon_1}\right)\sin\pi A_1\sin\pi A_2 - \cos\left(\pi B - \frac{\pi\epsilon_2}{\epsilon_1}\right)\sin\pi(B-A_1)\sin\pi(B-A_2)\right)}{\sin\pi B\sin\pi(B-A_1-A_2)}$$

$$c_{22} = \frac{2\tilde{c}_2\left(\cos\left(\pi B - \frac{\pi\epsilon_2}{\epsilon_1}\right)\sin\pi A_1\sin\pi A_2 - \cos\left(\pi B + \frac{\pi\epsilon_2}{\epsilon_1}\right)\sin\pi(B-A_1)\sin\pi(B-A_2)\right)}{\sin\pi B\sin\pi(B-A_1-A_2)}$$

$$c_{21} = \frac{(1-q)^{B-A_1-A_2}\,2\pi^3\,\tilde{c}_1\left(\cos\left(\pi B - \frac{\pi\epsilon_2}{\epsilon_1}\right) - \cos\left(\pi B + \frac{\pi\epsilon_2}{\epsilon_1}\right)\right)}{\sin\pi B\sin^2\pi(B-A_1-A_2)\,\Gamma(B-A_1-A_2)\Gamma(1+B-A_1-A_2)\prod_{i=1}^2\Gamma(A_i)\Gamma(1-B+A_i)}$$

$$c_{12} = \frac{(1-q)^{A_1+A_2-B}\,2\pi^3\,\tilde{c}_2\left(\cos\left(\pi B - \frac{\pi\epsilon_2}{\epsilon_1}\right) - \cos\left(\pi B + \frac{\pi\epsilon_2}{\epsilon_1}\right)\right)}{\sin\pi B\sin^2\pi(B-A_1-A_2)\,\Gamma(A_1+A_2-B)\Gamma(1-B+A_1+A_2)\prod_{i=1}^2\Gamma(1-A_i)\Gamma(B-A_i)}$$

$$(3.25)$$

The quantities c_{ij} are related to the $H_\pm$, H_0 obtained in [25] from a CFT analysis (see [31] for direct computations of 't Hooft loops in the gauge theory side). In appendix B.4 we provide a detailed comparison of the results. The matching provides a strong consistency test of our results here and it is an explicit check of S duality among Wilson and 't Hooft loops.

3.3.1 A saddle point analysis

The qualitative behavior of the results we found for the expectation value of the Wilson loop in $\mathcal{N} = 2$ gauge theories with critical masses can be confirmed by considering the limit in which some of the masses are large keeping the overall sum small. This limit can be treated with a simple saddle point analysis along the lines of the large N analysis in [26–29]. As we will see, this rather crude approximation already captures the localization of the a-integral and the qualitative behaviour of our result for gauge theories with critical masses. For simplicity, we focus on the SU(2) case. We write

$$Z_{\mathrm{pert}}(a) = e^{-S_{\mathrm{eff}}} \qquad (3.26)$$

with

$$S_{\mathrm{eff}} \approx -\ln\Upsilon(2a)\Upsilon(-2a) + \sum_{u=1}^2\ln\Upsilon(a-\bar{\mu}_u)\Upsilon(\mu_u-a)\Upsilon(-a-\bar{\mu}_u)\Upsilon(\mu_u+a) + \frac{a^2}{\epsilon_1\epsilon_2}\ln|q| \quad (3.27)$$
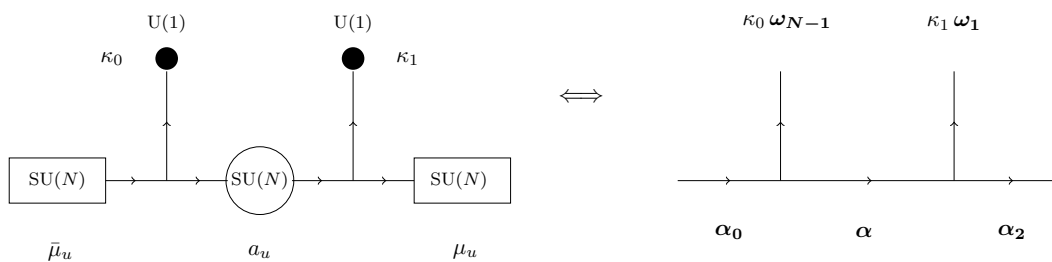
Figure 3. On the left the quiver diagram for the conformal $SU(N)$ gauge theory: in evidence the $SU(N)^2 \times U(1)^2$ flavor factors. On the right the diagram of the conformal block for the dual Toda field theory.

For $\epsilon_1 = \epsilon_2 = 1$, and $\mu_1 = -\mu_2 = \mu$ with $\mu \gg \bar{\mu}_u$, the expansion (A.18) leads to

$$S_{\text{eff}} = -2\, a^2 \ln(2a)^2 + (\mu - a)^2 \ln(\mu - a)^2 + (\mu + a)^2 \ln(\mu + a)^2 + \ldots \quad (3.28)$$

The leading contribution to the integral of $|Z_{\text{pert}} Z_{\text{inst}}|^2$ comes from the a_{crit} extremizing S_{eff}, i.e. from the solutions of $S'_{\text{eff}}(a_{\text{crit}}) = 0$. One finds $a_{\text{crit}} = \pm \mu$. Plugging into (3.16) one finds for the expectation value of a Wilson loop in the fundamental representation

$$\langle W \rangle \approx 2 \cos 2\pi \mu \quad (3.29)$$

in agreement with (3.17) for a round sphere $\epsilon_1 = \epsilon_2 = 1$.

3.4 The AGT dual

In this section we describe the AGT dual of the gauge theory partition function with critical masses. According to the AGT dictionary [25, 32], the partition function of an $SU(N)$ gauge theory with N fundamental and N anti-fundamental hypermultiplets is mapped to a four point function on the sphere in A_{N-1} Toda field theory with two $U(1)$ and two $SU(N)$ punctures, see figure 3. The choice of critical masses corresponds to the insertion of a degenerated field at one of the $U(1)$ punctures. The corresponding correlator was computed in [23, 24]. Here we collect some background material on the Toda Field theory and the results for the relevant correlator showing the matching with the gauge theory answer.

3.4.1 Toda field theory

Let us briefly recall few facts concerning the Toda field theory. The action is given by

$$\mathcal{A} = \int d^2 x \left(\frac{1}{8\pi} (\partial \varphi)^2 + m \sum_{i=1}^{N-1} e^{b(e_i, \varphi)} \right) \quad (3.30)$$

where φ is a N dimensional vector equipped with the usual Euclidean scalar product, whose components sum to zero and e_i are the simple roots of the algebra A_{N-1}. Explicitly

$$(e_i)_k = \delta_{i,k} - \delta_{i+1,k} \quad (3.31)$$

with $k = 1, 2, \ldots N$. The fundamental weights ω_i, by definition, constitute the dual basis: $(e_i, \omega_j) = \delta_{i,j}$. Explicitly

$$(\omega_i)_k = \begin{cases} 1 - \dfrac{i}{N}, & k \le i \\ -\dfrac{i}{N}, & k > i \end{cases} \tag{3.32}$$

We also need the set of weights of the fundamental representation

$$h_k = \omega_1 - e_1 - \cdots - e_{k-1} \tag{3.33}$$

for $k > 1$ and $h_1 = \omega_1$. In particular, for an arbitrary vector a one finds

$$(h_k, a) = a_k - \frac{1}{N} \sum_{k=1}^{N} a_k \qquad (h_k, h_l) = \delta_{k,l} - \frac{1}{N} \tag{3.34}$$

Let us define $\mathbf{Q} = Q\rho$, where $Q = b + 1/b$ and

$$\rho = \frac{1}{2} \sum_{k=1}^{N} (N + 1 - 2k) h_k \tag{3.35}$$

is the Weyl vector (half sum of all positive roots). The central charge of the Virasoro algebra is

$$c = N - 1 + 12(\mathbf{Q}, \mathbf{Q}) = (N - 1)(1 + N(N + 1)Q^2) \tag{3.36}$$

The primary fields, $V_{\boldsymbol{\alpha}} = \exp(\boldsymbol{\alpha}, \phi)$, in the Toda theory have conformal dimension

$$\Delta_{\boldsymbol{\alpha}} = \frac{(\boldsymbol{\alpha}, 2\mathbf{Q} - \boldsymbol{\alpha})}{2} = \frac{(\mathbf{Q} - P, \mathbf{Q} + P)}{2}$$

with P the momentum defined by

$$P = \boldsymbol{\alpha} - \mathbf{Q} \tag{3.37}$$

Conventionally the states are parameterized in terms of P and the fields via $\boldsymbol{\alpha}$. It is important that this parametrization is modulo Weyl reflections, which simply permute the components of the momenta. We also notice that $\boldsymbol{\alpha}$ can always be chosen such that $\sum_u \alpha_u = 0$ and then

$$(\boldsymbol{\alpha} - \mathbf{Q}, h_u) = (\boldsymbol{\alpha} - \mathbf{Q})_u \tag{3.38}$$

The W-Ward identities (there are holomorphic currents of spin s, $W^{(s)}$, with $s = 3, \ldots, N$) show that a two point function $\langle V_1(z_1) V_2(z_2) \rangle$ is nonzero only if $P_1 = -P_2$. An ingoing state should then be accompanied by the reflection $P \to -P$ or equivalently $\boldsymbol{\alpha} \to 2\mathbf{Q} - \boldsymbol{\alpha}$.

3.4.2 The four point conformal block

The instanton partition function of the gauge theory on $\mathbb{C}^2$ can be related (see B.3 for details) to the chiral four point function conformal block[8]

$$\langle V_{2\mathbf{Q}-\boldsymbol{\alpha}_0}(\infty)V_{\kappa_0\omega_{N-1}}(1)V_{\kappa_1\omega_1}(z)V_{\boldsymbol{\alpha}_2}(0)\rangle = z^{(h_1,\boldsymbol{\alpha}_2)b}(1-z)^{\frac{\kappa_0}{N}b}G(z) \tag{3.39}$$

with κ_1 given by

$$\kappa_1 = (1-p)\frac{1}{b} + (1-q)b \tag{3.40}$$

for some $p, q \in \mathbb{Z}_{\geq 0}$. These operators are associated to degenerated fields in the Toda field theory. A correlator involving the insertion of a degenerated field satisfies a differential equation of order pq. In particular the case $p = q = 1$ corresponds to the identity operator and the associated conformal block is trivial

$$\mathbf{I} \quad \kappa_1 = 0 \qquad G(z) = 1 \tag{3.41}$$

The first interesting case occurs at $(p, q) = (1, 2)$ associated to the so called ϕ_{12} field. The function $G(z)$ in this case satisfies the Pochhammer differential equation (see [23, 24])

$$\left[z\prod_{i=1}^{N}\left(z\frac{\partial}{\partial z} + A_i\right) - z\frac{\partial}{\partial z}\prod_{i=2}^{N}\left(z\frac{\partial}{\partial z} + B_i - 1\right)\right]G(z) = 0 \tag{3.42}$$

with

$$A_j = \frac{\kappa_0}{N}b + \frac{(1-N)}{N}b^2 + (h_1, \boldsymbol{\alpha_2} - \mathbf{Q})b - (h_j, \boldsymbol{\alpha_0} - \mathbf{Q})b \qquad j = 1, \ldots N$$

$$B_j = 1 + (h_1 - h_j, \boldsymbol{\alpha_2} - \mathbf{Q})b \qquad\qquad\qquad j = 2, \ldots N \tag{3.43}$$

The N independent solutions can be written as

$$\mathbf{II} \quad \kappa_1 = -b: \qquad G_j(z) = z^{1-B_j}\,_N F_{N-1}\left(\begin{matrix}\mathbf{A}^{(j)}\\\mathbf{B}^{(j)}\end{matrix}\Big| z\right) \tag{3.44}$$

with $j = 1, \ldots, N$, $B_1 = 1$ and $\mathbf{A}^{(j)}, \mathbf{B}^{(j)}$ given by (2.17).

3.4.3 The four point correlator

The full correlator can be written as

$$\mathcal{G}(z, \bar{z}) = \sum_{j=1}^{N} C_j \left| z^{(h_1,\boldsymbol{\alpha}_2)b}(1-z)^{\frac{\kappa_0}{N}b}G_j(z)\right|^2 \tag{3.45}$$

with C_j determined by the requirement that $\mathcal{G}(z, \bar{z})$ is single valued over the z-plane. Alternatively the coefficients C_j can be built out of the three point vertices

$$C_j = \frac{1}{(2\pi i)^{N-1}}\oint_{\boldsymbol{\alpha}_{\mathbf{crit}}^{(j)}} d^{N-1}\alpha\, C(2\mathbf{Q} - \boldsymbol{\alpha_0}, \boldsymbol{\alpha}, \kappa_0 w_{N-1})\, C(2\mathbf{Q} - \boldsymbol{\alpha}, \boldsymbol{\alpha_2}, \kappa_1 w_1) \tag{3.46}$$

---

[8]We notice that $V_{\kappa_1\omega_1} \sim V_{(N\epsilon-\kappa_1)\omega_{N-1}}$. Indeed the momenta $\kappa_1\omega_1 - \mathbf{Q}$ and $(N\epsilon - \kappa_1)\omega_{N-1} - \mathbf{Q}$ are related by a cyclic permutation of components.

with[9]

$$C(2\mathbf{Q} - \boldsymbol{\alpha}, \boldsymbol{\beta}, \kappa w_{N-1}) = \frac{\Upsilon(b)^{N-1}\Upsilon(\kappa)\prod_{u<v}\Upsilon((\boldsymbol{\alpha} - \mathbf{Q}, h_u - h_v))\Upsilon((\mathbf{Q} - \boldsymbol{\beta}, h_u - h_v))}{\prod_{u,v}\Upsilon(\frac{\kappa}{N} + (\mathbf{Q} - \boldsymbol{\alpha}, h_u) + (\boldsymbol{\beta} - \mathbf{Q}, h_v))} \tag{3.47}$$

and the same formula for the three point function involving the insertion of $V_{\kappa w_1}$ with $h_u \to -h_u$, $h_v \to -h_v$ in the denominator of (3.47). The contour integral picks up the residue at the pole

$$\boldsymbol{\alpha}_{\mathbf{crit}}^{(j)} = \boldsymbol{\alpha_2} + \kappa_1 h_j \tag{3.48}$$

with $\kappa_1 = -b$. One finds

$$\begin{aligned}
C_j &= \operatorname{Res}_{\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\mathbf{crit}}^{(j)}} C(2\mathbf{Q} - \boldsymbol{\alpha_0}, \boldsymbol{\alpha}, \kappa_0 w_{N-1}) C(2\mathbf{Q} - \boldsymbol{\alpha}, \boldsymbol{\alpha_2}, \kappa_1 w_1) \tag{3.49} \\
&= \frac{\Upsilon(b)^{2N-2}}{\Upsilon'(0)^{N-1}} \frac{\Upsilon(\kappa_0)\prod_{u<v}\Upsilon((\boldsymbol{\alpha_0} - \mathbf{Q})_{uv})\Upsilon((\mathbf{Q} - \boldsymbol{\alpha_2})_{uv})\prod_{u\neq v}\Upsilon((\boldsymbol{\alpha_2} - \mathbf{Q})_{uv} + \kappa_1\delta_{uj} - \kappa_1\delta_{vj})}{\prod_{u,v}\Upsilon(\frac{(\kappa_0 - \kappa_1)}{N} + (\mathbf{Q} - \boldsymbol{\alpha_0}, h_u) + (\boldsymbol{\alpha_2} - \mathbf{Q}, h_v) + \kappa_1\delta_{jv})\prod_{u\neq v}\Upsilon((\boldsymbol{\alpha_2} - \mathbf{Q})_{uv} + \kappa_1\delta_{uj})} \\
&= \frac{\Upsilon(b)^{2N-2}}{\Upsilon'(0)^{N-1}} \frac{\Upsilon(\kappa_0)\prod_{u<v}\Upsilon((\boldsymbol{\alpha_0} - \mathbf{Q})_{uv})\Upsilon((\mathbf{Q} - \boldsymbol{\alpha_2})_{uv})}{\prod_{u,v}\Upsilon(\frac{(\kappa_0 - \kappa_1)}{N} + (\mathbf{Q} - \boldsymbol{\alpha_0}, h_u) + (\boldsymbol{\alpha_2} - \mathbf{Q}, h_v) + \kappa_1\delta_{jv})}\prod_{u\neq j}\frac{\Upsilon((\boldsymbol{\alpha_2} - \mathbf{Q})_{uj} - \kappa_1)}{\Upsilon((\boldsymbol{\alpha_2} - \mathbf{Q})_{uj})}
\end{aligned}$$

Using the dictionary below (3.50) one can see that the result (3.45) with C_j given by (3.49) perfectly matches the gauge theory partition function up to an irrelevant constant. In particular, the ratios $R_j = \frac{C_j}{C_1}$ are given again by (3.15) with A_j, B_j now given by (3.43): $R_j = r_j|q^{B_j-1}|^2$. This ensures the single valuedness of $G(z, \bar{z})$ and provides us with a highly non-trivial consistence check of the results for C_j.

3.4.4 The Toda/gauge theory dictionary

CFT and gauge theory variables are related via the dictionary

$$(P_m)_u = (\boldsymbol{\alpha_m} - \mathbf{Q})_u = a_{m,u} - \frac{1}{N}\sum_{u=1}^{N} a_{m,u} \qquad a_{m,u} = (a_{0,u}, a_{1,u}, a_{2,u}) = (\bar{\mu}_u, a_u, \mu_u)$$

$$\kappa_0 = \sum_u (a_u - \bar{\mu}_u) \qquad \kappa_1 = \sum_u (a_u - \mu_u + \epsilon) \qquad G_j(q) = q^{\frac{\mu_j - \mu_1}{\epsilon_1}} Z_j(q) \tag{3.50}$$

with $\epsilon_1 = \frac{1}{b}$, $\epsilon_2 = b$. Plugging (3.50) into (3.44) one finds perfect agreement between the Toda conformal blocks $G_j(q)$ and the instanton partition functions Z_j for the critical choices of masses **II** (2.16). On the other hand the residues C_j in the Toda field theory reproduce the one-loop contributions in the gauge theory on S^4. Case **I** corresponds to the insertion of an identity operator and therefore the conformal block is trivial consistently with the fact that $Z_{\mathrm{inst}}^I = 1$ in the gauge theory side.

4 Summary of results

In this paper we have derived exact formulae for the partition function of a special class of $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^4$ and S^4 with fundamental and anti-fundamental matter. On

---

[9]For simplicity we work in units where $\pi\mu\gamma(b^2)b^{2-2b^2} = 1$.

$\mathbb{R}^4$ we have showed that choosing the difference between the masses and the vev's to be a combination of the ϵ_ℓ with integer coefficients, only a very restrictive type of instanton configurations (Young tableaux shape) can contribute to the gauge theory partition function. The critical choices of masses are in correspondence with the choices of degenerated field insertions in the AGT dual Liouville or Toda field Theory. The simplest non-trivial cases correspond to either no instantons, or instantons associated to Young tableaux made of a single row. In the latter case the full instanton partition function can be evaluated and written in terms of generalized hypergeometric functions.

On S^4, a critical mass corresponds to discrete choices of the overall mass associated to the center U(1) of the U(N) flavor group acting on the fundamental matter. For these choices the integral $\int da |Z(a)|^2$ localizes around a finite set of critical points where it can be explicitly evaluated as a sum of residues of the integrand. The gauge theory computation mimics that of a very well known CFT correlator in Toda field theory involving the insertion of a degenerated field. Exact formulae for the expectation values of circular supersymmetric Wilson and 't Hooft loops in this class of gauge theories are also derived.

Acknowledgments

A The Barnes double gamma function

The Barnes double gamma function can be defined by the integral[10]

$$\log \Gamma_2(x|\epsilon_1, \epsilon_2) = \frac{d}{ds} \left(\frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} \frac{t^s e^{-xt}}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})} \right) \Big|_{s=0} \tag{A.1}$$

Using the representation of the logarithm

$$\log \frac{x}{\Lambda} = -\frac{d}{ds} \left(\frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} t^s e^{-xt} \right) \Big|_{s=0} \tag{A.2}$$

one can think of $\Gamma_2(x)$ as a regularization of the infinite product (for $\epsilon_1, \epsilon_2 > 0$)

$$\Gamma_2(x) = \prod_{i,j=0}^\infty \left(\frac{\Lambda}{x + i\epsilon_1 + j\epsilon_2} \right) \tag{A.3}$$

---

[10]In the following and in the main text we used the shorthand notation $\Gamma_2(x)$ when it is not necessary to specify its dependence on the ϵ_i. The related function $\gamma_{\epsilon_1, \epsilon_2}(x) = \Gamma_2(x + \epsilon)$ is often used in the literature.

with poles in the non-negative integers (i, j). Using the same manipulations one finds (for $\epsilon_1 > 0$, $\epsilon_2 < 0$)

$$\prod_{i,j=1}^{\infty} \left(\frac{x + i\epsilon_1 - (j-1)\epsilon_2}{\Lambda} \right) = \Gamma_2(x + \epsilon) \tag{A.4}$$

The following identity holds

$$\Gamma_2(x + \epsilon_1)\,\Gamma_2(x + \epsilon_2) = x\,\Gamma_2(x)\,\Gamma_2(x + \epsilon_1 + \epsilon_2) \tag{A.5}$$

For large $\operatorname{Re} x$ the expansion of (A.1) can be written as

$$\log \Gamma_2(x) = c_0\, x^2 \left(-\frac{1}{4} \log \left(\frac{x}{\Lambda} \right)^2 + \frac{3}{4} \right) + c_1\, x \left(\frac{1}{2} \log \left(\frac{x}{\Lambda} \right)^2 - 1 \right)$$
$$- \frac{c_2}{4} \log \left(\frac{x}{\Lambda} \right)^2 + \sum_{n=3}^{\infty} \frac{c_n x^{2-n}}{n(n-1)(n-2)} \tag{A.6}$$

with c_n defined by

$$\frac{1}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})} = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^{n-2} \tag{A.7}$$

i.e.

$$c_0 = \frac{1}{\epsilon_1 \epsilon_2} \qquad c_1 = \frac{\epsilon_1 + \epsilon_2}{2\epsilon_1 \epsilon_2} \qquad c_2 = \frac{\epsilon_1^2 + 3\epsilon_1 \epsilon_2 + \epsilon_2^2}{6\epsilon_1 \epsilon_2} \tag{A.8}$$

For $\operatorname{Re} x < 0$ we defined $\Gamma_2(x)$ via the (A.6), so we have the simple reflection property

$$\Gamma_2(-x|\epsilon_1, \epsilon_2) = \Gamma_2(x| -\epsilon_1, -\epsilon_2) = \Gamma_2(x + \epsilon|\epsilon_1, \epsilon_2) \tag{A.9}$$

where the last equality follows from (A.1).

The case $\epsilon_1 = \frac{1}{b}$, $\epsilon_2 = b$. We will mainly consider the case

$$\epsilon_1 = \frac{1}{b} \qquad \epsilon_2 = b \qquad Q = \epsilon = \epsilon_1 + \epsilon_2 = \frac{1}{b} + b \tag{A.10}$$

and define

$$\Upsilon(x) = \frac{1}{\Gamma_2(x|\frac{1}{b}, b)\Gamma_2(Q - x|\frac{1}{b}, b)} \tag{A.11}$$

The function $\Upsilon(x)$ is an entire function with zeros at

$$\Upsilon\left(-mb - \frac{n}{b} \right) = \Upsilon\left((m+1)b + \frac{n+1}{b} \right) = 0 \qquad \text{for} \qquad m, n \in \mathbb{Z}_{\geq 0} \tag{A.12}$$

which satisfies the properties

$$\Upsilon(x) = \Upsilon(Q - x) \qquad\qquad \Upsilon\left(\frac{Q}{2} \right) = 1$$

$$\Upsilon(x + b) = \gamma(b\,x)b^{1 - 2\,b\,x}\,\Upsilon(x) \qquad \Upsilon\left(x + \frac{1}{b} \right) = \gamma\left(\frac{x}{b} \right) b^{\frac{2x}{b} - 1}\,\Upsilon(x)$$

$$\tag{A.13}$$

with

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \tag{A.14}$$

The function $\Upsilon(x)$ admits the integral representation

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] \tag{A.15}$$

The case $b = 1$ is particularly simple. The double Gamma function reduces to the G-Barnes function $G(x)$

$$G(x) = \frac{1}{\Gamma_2(x|1,1)} \qquad \Upsilon_{b=1}(x) = G(x)G(2-x) \tag{A.16}$$

satisfying $G(x+1) = \Gamma(x)G(x)$, $G(1) = 1$. The G-function can be expanded around $x \approx 0$

$$\log G(x+1) = \frac{x}{2}(\log 2\pi - 1) - (1+\gamma)\frac{x^2}{2} + \sum_{n=3}^\infty (-)^{n-1}\zeta(n-1)\frac{x^n}{n} \tag{A.17}$$

or for large x

$$\log G(x+1) = x^2 \left(\frac{1}{4}\ln x^2 - \frac{3}{4} \right) + \frac{x}{2}\ln 2\pi - \frac{1}{24}\ln x^2 + \zeta'(-1) + \sum_{n=1}^\infty \frac{B_{2n+2}}{4n(n+1)x^{2n}}$$

leading to

$$\begin{aligned} \log \Upsilon(x+1|1,1) &= \log G(1+x) + \log G(1-x) \\ &= x^2 \left(\frac{1}{2}\ln x^2 - \frac{3}{2} \right) - \frac{1}{12}\ln x^2 + 2\zeta'(-1) + \sum_{n=1}^\infty \frac{B_{2n+2}}{2n(n+1)x^{2n}} \end{aligned} \tag{A.18}$$

for large x and

$$\log \Upsilon(x+1|1,1) = -\sum_{n=2}^\infty \zeta(2n-1)\frac{x^{2n}}{2n} - (\gamma+1)x^2 \tag{A.19}$$

for x small. In the previous formulae $\zeta(n)$ is the Riemann zeta function, the B_n are the Bernoulli numbers, while γ is the Euler-Mascheroni constant.

A.1 Hypergeometric identities

The large z expansion of the hypergeometric functions can be found from the Taylor expansion around $z = 0$ via the identity

$$_N F_{N-1}(^{\mathbf{A}}_{\mathbf{B}}|z) \prod_{i=1}^N \frac{\Gamma(A_i)}{\Gamma(B_i)} = \sum_{k=1}^N (-z)^{-A_k} \Gamma(A_k) \frac{\prod_{i\neq k}^N \Gamma(A_i - A_k)}{\prod_{i=2}^N \Gamma(B_i - A_k)} {}_N F_{N-1}\left(^{1+A_k-\hat{\mathbf{B}}}_{1+A_k-\hat{\mathbf{A}}^{(k)}} \Big| \frac{1}{z} \right) \tag{A.20}$$

with $\hat{\mathbf{A}}^{(k)} = \{A_{u \neq k}\}_{u=1,\ldots,N}$ and $\hat{\mathbf{B}} = \{1, B_2, \ldots, B_N\}$. A similar relation allows to write the expansion around $z = 1$. Explicitly, for $N = 2$ one finds

$$\frac{\sin(\pi(B - A_1 - A_2))}{\pi \Gamma(B)} \, {}_2F_1\left({}^{A_1,A_2}_{\quad B}\Big| z\right) \tag{A.21}$$

$$= \frac{1}{\Gamma(B - A_1)\Gamma(B - A_2)\Gamma(A_1 + A_2 - B + 1)} \, {}_2F_1\left({}^{A_1,A_2}_{A_1+A_2-B+1}\Big| 1 - z\right)$$

$$- \frac{(1 - z)^{B - A_1 - A_2}}{\Gamma(A_1)\Gamma(A_2)\Gamma(B - A_1 - A_2 + 1)} \, {}_2F_1\left({}^{B-A_1,B-A_2}_{B-A_1-A_2+1}\Big| 1 - z\right)$$

We will also use in the main text the Euler's transformation

$$ {}_2F_1\left({}^{A_1,A_2}_{\quad B}\Big| z\right) = (1 - z)^{B - A_1 - A_2} {}_2F_1\left({}^{B-A_1,B-A_2}_{\qquad B}\Big| z\right) \tag{A.22}$$

B The correlator in Liouville theory

In this appendix we derive the four point function in the dual 2d Liouville CFT involving the insertion of a null state. We consider a two dimension conformal field theory with central charge

$$c = 1 + 6Q^2 = 1 + 6\left(b + \frac{1}{b}\right)^2 \tag{B.1}$$

stress energy tensor T and chiral fields ϕ_h. The operator product expansions (OPE) read

$$T(w)\phi_h(z) \approx \frac{h\phi_h}{(w - z)^2} + \frac{\partial_z \phi_h(z)}{(w - z)} + \ldots$$

$$T(w)T(z) \approx \frac{c}{(z - w)^4} + \frac{2T(z)}{(w - z)^2} + \frac{\partial_z T(z)}{(w - z)} + \ldots \tag{B.2}$$

We define the Virasoro operators

$$L_n^z \mathcal{O}(z) = \oint_{\gamma_z} \frac{dw}{2\pi i} (w - z)^{n+1} T(w)\, \mathcal{O}(z) \tag{B.3}$$

with γ_z a contour around z. Virasoro generators satisfy the algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \tag{B.4}$$

It is convenient to parametrize the dimension of a primary field as

$$h(\alpha) = \alpha(Q - \alpha) \tag{B.5}$$

and denote the corresponding field as ϕ_α. It should be noted that this parametrization is not one to one, since the fields ϕ_α and $\phi_{Q-\alpha}$ are identified up to a numerical multiplier, called reflection amplitude.

A null state $\mathcal{O}_{\text{null}}$ is defined by the condition

$$L_n^z \mathcal{O}_{\text{null}}(z) = 0 \qquad n > 0 \tag{B.6}$$

The general null state is labeled by two integers and can be written as

$$\mathcal{O}_{mn} = \mathbf{L}_{mn}\phi_{mn} \tag{B.7}$$

with $\mathbf{L}_{mn}$ being some polynomial of the Virasoro generators at the total level $N = nm$ acting on a primary field ϕ_{mn} of conformal dimension

$$h_{mn} = \alpha_{mn}(Q - \alpha_{mn}) \tag{B.8}$$

with two equally admissible choices for the parameters α_{mn}

$$\alpha_{mn} \equiv \frac{1}{2b}(1 - m) + \frac{b}{2}(1 - n) \tag{B.9}$$

or

$$\alpha_{mn} \equiv \frac{1}{2b}(1 + m) + \frac{b}{2}(1 + n) \tag{B.10}$$

For instance

$$\mathcal{O}_{12} = \left(L_{-1}^2 + b^2 L_{-2}\right)\phi_{12} \qquad\qquad h_{12} = -\frac{1}{2} - \frac{3b^2}{4} \tag{B.11}$$

$$\mathcal{O}_{13} = \left(L_{-1}^3 + 4b^2 L_{-1}L_{-2} + (4b^4 - 2b^2)L_{-3}\right)\phi_{13} \qquad h_{13} = -1 - 2b^2$$

and so on.

B.1 The degenerated conformal blocks

We are interested in four point functions involving the insertion of a ϕ_{nm} primary state. We denote the four-point function by

$$F(z_i) = \langle\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle \tag{B.12}$$

Conformal invariance strongly restricts the form of F. Using the OPE (B.2) one expects

$$\langle T(w)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\phi_4(z_4)\rangle = \sum_{i=1}^{4}\left(\frac{h_i F}{(w - z_i)^2} + \frac{F_i}{(w - z_i)}\right) \tag{B.13}$$

with $F_i = \partial_{z_i}F$. On the other hand at large w, the correlator (B.13) is expected to fall as w^{-4} since the vacuum is annihilated by L_0, and L_{-1}, i.e. $\langle 0|T(w) = \sum_{n\in\mathbb{Z}}\langle 0|L_n w^{n-2} \sim w^{-4}$. Equating to zero the coefficients of w^{-n} with $n = 1, 2, 3$ one finds

$$\sum_{i=1}^{4} F_i = 0$$

$$\sum_{i=1}^{4}(z_i F_i + h_i F) = 0$$

$$\sum_{i=1}^{4}(z_i^2 F_i + 2z_i h_i F) = 0 \tag{B.14}$$

These equations imply in particular that F is a function only of the harmonic ratio $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Moreover equations (B.14) can be used to expressed F_1, F_2, F_4 in terms of $F(z)$ and $F'(z)$. In particular setting

$$z_1 = \infty \qquad z_2 = 1 \qquad z_3 = z \qquad z_4 = 0 \tag{B.15}$$

one finds

$$F_1 = 0 \qquad F_2 = -z\,F' + (2\,h_1 - \delta)F \qquad F_4 = (z-1)F' + (\delta - 2h_1)F \tag{B.16}$$

with $\delta = \sum_i h_i$. We are interested in correlators of the type

$$F(z) = \langle \phi_{\alpha_1}(\infty)\phi_{\alpha_2}(1)\phi_{\alpha_3}(z)\phi_{\alpha_4}(0) \rangle \tag{B.17}$$

with

$$\alpha_3 = -\frac{b}{2} \tag{B.18}$$

such that $\phi_{\alpha_3} \equiv \phi_{(12)}$ is a degenerated field at the second level.[11] Using the fact that the null state $\mathcal{O}_{12}$ is orthogonal to all the other states in the theory, one can write

$$0 = \langle \phi_{\alpha_1}(\infty)\phi_{\alpha_2}(1)\mathcal{O}_{12}(z)\phi_{\alpha_4}(0) \rangle = \mathbf{L}_{12}F = \left(L_{-1}^2 + b^2\,L_{-2} \right)F(z) \tag{B.19}$$

where

$$
\begin{aligned}
L_{-1}^n\,F &= \partial_z^n\,F \\
L_{-2}\,F &= \sum_{i \in \{1,2,4\}} \left[\frac{h_i F}{(z_3 - z_i)^2} + \frac{F_i}{(z_3 - z_i)} \right]
\end{aligned}
\tag{B.20}
$$

The forms of $L_{-1,-2}$ follow from (B.3) after using the OPE (B.2). Writing

$$F(z) = z^{\alpha_4 b}\,(1-z)^{\alpha_2 b}f(z) \tag{B.21}$$

the differential equation (B.19) reduces to the hypergeometric equation

$$z(1-z)f''(z) + (B - (A_1 + A_2 + 1)z)f'(z) - A_1 A_2\,f(z) = 0 \tag{B.22}$$

with

$$
\begin{aligned}
A_1 &= -\frac{b^2}{2} + (\alpha_2 + \alpha_4 - \alpha_1)b, \;\; A_2 = -1 - \frac{3b^2}{2} + (\alpha_1 + \alpha_2 + \alpha_4)b, \\
B &= 2\alpha_4 b - b^2
\end{aligned}
\tag{B.23}
$$

The two independent solutions of (B.19)[12]

$$F^{(+)}(z) = z^{\alpha_4 b}\,(1-z)^{\alpha_2 b}\,{}_2F_1\big({}^{\mathbf{A}}_{B}\big|z\big) \tag{B.24}$$

$$F^{(-)}(z) = z^{1+b^2-\alpha_4 b}\,(1-z)^{\alpha_2 b}\,{}_2F_1\left({}^{1-B+\mathbf{A}}_{2-B}\Big|z\right)$$

---

[11]We have chosen here to exploit the expression (B.9) for the parameter $\alpha_{1,2}$. As we will see later, it is this choice which corresponds to the case in which only tableaux with one row contribute to the gauge theory, as discussed in section 2.1.

[12]The two solutions $z^{-\alpha_4 b}\,(1-z)^{-\alpha_2 b}F^{(\pm)}(z)$ correspond to $G_j(z)$ in (3.44) with $j = 1, 2$.

correspond to the exchange in the *s*-channel of the field with dimension specified by the parameters

$$\alpha_\pm = \alpha_4 \mp \frac{b}{2} \tag{B.25}$$

(these are the only primary fields which result in the fusion of the fields $\phi_{1,2}(z)$ and $\phi_{\alpha_4}(0)$ at $z = 0$).

B.2 The physical correlator

The physical correlation function can be built out of the conformal blocks $F^{(\pm)}$ and the three point function $C(\alpha_1, \alpha_2, \alpha_3)$ by the so standard gluing algorithm. For the Liouville theory the three point function is given by

$$C(\alpha_1, \alpha_2, \alpha_3) = \left(\pi\mu\gamma(b^2)b^{2-2b^2} \right)^{(Q-\alpha_1-\alpha_2-\alpha_3)/b} \tag{B.26}$$

$$\times \frac{\Upsilon'(0)\Upsilon(2\alpha_1)\Upsilon(2\alpha_2)\Upsilon(2\alpha_3)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)\Upsilon(\alpha_1 + \alpha_2 - \alpha_3)\Upsilon(\alpha_1 - \alpha_2 + \alpha_3)\Upsilon(-\alpha_1 + \alpha_2 + \alpha_3)}$$

We will work in units where $\pi\mu\gamma(b^2)b^{2-2b^2} = 1$. The four point function involving a degenerated ϕ_{12}-field can then be written as

$$F(z, \bar{z}) = C_+ \left| F^{(+)}(z) \right|^2 + C_- \left| F^{(-)}(z) \right|^2 \tag{B.27}$$

with

$$C_\pm = \frac{1}{2\pi i} \lim_{\alpha_3 \to -\frac{b}{2}} \oint_{\gamma_\pm} d\alpha \, C(\alpha_1^*, \alpha_2, \alpha) C(\alpha^*, \alpha_3, \alpha_4) \tag{B.28}$$

and $\gamma_\pm$ being a contour around $\alpha_\pm = \alpha_4 \pm \alpha_3$. Explicitly

$$
\begin{aligned}
C_+ &= C(\alpha_1^*, \alpha_2, \alpha_+) \\
C_- &= \frac{C_+ \, \gamma(B)\gamma(B-1)}{\gamma(A_1) \, \gamma(A_2) \, \gamma(B - A_1)\gamma(B - A_2)}
\end{aligned} \tag{B.29}
$$

with A_i, B defined in (B.23). One can check that $F(z, \bar{z})$ defined by (B.27) is single valued over the whole complex plane.

B.3 The CFT/gauge theory dictionary

The gauge theory and CFT parameters are related by the dictionary (3.50). For $N = 2$ this reduces to

$$
\begin{array}{llll}
\epsilon_1 = \dfrac{1}{b} & \epsilon_2 = b & \epsilon = Q = \dfrac{1}{b} + b & c = 1 + 6Q^2 \\[2mm]
\alpha_1 = (P_0 + \mathbf{Q})_1 = \dfrac{\epsilon}{2} + \dfrac{1}{2}(\bar{\mu}_1 - \bar{\mu}_2) & \alpha_2 = \dfrac{\kappa_0}{2} = -\dfrac{1}{2}(\bar{\mu}_1 + \bar{\mu}_2) \\[2mm]
\alpha = (P_1 + \mathbf{Q})_1 = \dfrac{\epsilon}{2} + a & a = a_1 = -a_2 \\[2mm]
\alpha_4 = (P_2 + \mathbf{Q})_1 = \dfrac{\epsilon}{2} + \dfrac{1}{2}(\mu_1 - \mu_2) & \alpha_3 = \dfrac{\kappa_1}{2} = \epsilon - \dfrac{1}{2}(\mu_1 + \mu_2) & \text{(B.30)}
\end{array}
$$

Using this dictionary one can check that

$$Z_{\text{inst}} = {}_N F_{N-1}(\begin{smallmatrix}\mathbf{A}\\B\end{smallmatrix}|q) = (1-q)^{-\alpha_2 b} q^{-\alpha_4 b} F^{(+)}(q) \qquad (\text{B.31})$$

with the left hand side given by the gauge theory result (2.14) and the right hand side by the Liouville conformal block (B.24). The term $(1-q)^{-\alpha_2 b} = (1-q)^{2\alpha_2 \alpha_3}$ gives the contribution of the U(1) part while $q^{\alpha_4 b} = q^{\Delta_\alpha - \Delta_{\alpha_3} - \Delta_{\alpha_4}}$.

Analogously $Z_{\text{inst},j=2}$ in (2.16) is given by

$$Z_{\text{inst},j=2} = {}_N F_{N-1}(\begin{smallmatrix}\mathbf{A}\\B\end{smallmatrix}|q) = (1-q)^{-\alpha_2 b} q^{1-b^2+\alpha_4 b} F^{(-)}(q). \qquad (\text{B.32})$$

B.4 The 't Hooft loop coefficients

The 't Hooft loop coefficients $H_\pm$, H_0 are defined as [25][13]

$$H_0(a) = \frac{4 \cos\left(\pi b^2\right)\left[\cos\left(2\pi b\, P_4\right)\cos\left(2\pi b P_3\right) + \cos\left(2\pi b P_1\right)\cos\left(2\pi b\, P_2\right)\right]}{\cos\left(4\pi b\, P\right) - \cos\left(2\pi b^2\right)}$$

$$+ \frac{4\cos\left(2\pi b\, P\right)\left[\cos\left(2\pi b\, P_1\right)\cos\left(2\pi b\, P_3\right) + \cos\left(2\pi b\, P_4\right)\cos\left(2\pi b\, P_2\right)\right]}{\cos\left(4\pi b\, P\right) - \cos\left(2\pi b^2\right)}$$

$$H_\pm(a) = -\frac{2\pi^2\,\Gamma\left(1 + 2b^2 \pm 2b\, P\right)\Gamma\left(b^2 \pm 2b\, P\right)\Gamma\left(\pm 2b\, P\right)\Gamma\left(1 + b^2 \pm 2b\, P\right)}{\Gamma\left(-b^2\right)\Gamma\left(1+b^2\right)\sin\left(\pi b^2\right)\prod_{j,k=1}^{2}\Gamma\left(\frac{1}{2} + \frac{b^2}{2} \pm b\, P \pm (-1)^j\, b\, P_1 \pm (-1)^k\, b\, P_4\right)}$$

$$\times \frac{1}{\prod_{j,k=1}^{2}\Gamma\left(\frac{1}{2} + \frac{b^2}{2} \pm b\, P \pm (-1)^j b\, P_3 \pm (-1)^k\, b\, P_2\right)} \qquad (\text{B.33})$$

with

$$P_i = \alpha_i - \frac{Q}{2} \qquad\qquad P = a - \frac{Q}{2} \qquad (\text{B.34})$$

Specializing to $\alpha_3 = -\frac{b}{2}$ and using the dictionary (B.23) one can check the highly non-trivial relations

$$c_{11} = H_0\left(\alpha_2 - \frac{b}{2}\right)\tilde{c}_1 \qquad\qquad c_{12} = 2\pi H_-\left(\alpha_2 + \frac{b}{2}\right)\tilde{c}_2\,(1-q)^{A_1+A_2-B}$$

$$c_{22} = H_0\left(\alpha_2 + \frac{b}{2}\right)\tilde{c}_2 \qquad\qquad c_{21} = 2\pi H_+\left(\alpha_2 - \frac{b}{2}\right)\tilde{c}_1\,(1-q)^{B-A_1-A_2} \qquad (\text{B.35})$$

between the c_{ij} functions defined in (3.25) and the H-functions in (B.33).

C Instanton partition function

In this appendix we review the derivation of the formula (2.4) for the instanton partition function $Z_{Y_u Y_v}$ associated to the pair (Y_u, Y_v) of Young tableaux. We refer the reader to [9, 10, 33] for the original references and details.

In a D(-1)-D3 brane realization of the instanton moduli space, the tableaux Y_u and Y_v describe the distributions of D(-1)-instantons around the D3-branes at positions a_u

---

[13]After the exchange $\alpha_2 \leftrightarrow \alpha_4$.

and a_v respectively. The partition function (determinant) $Z_{Y_u Y_v}$ can be associated to the character (trace)

$$\mathbf{T}_{uv} = -V_u V_v^* (1 - T_1)(1 - T_2) + W_u V_v^* + V_u W_v^* T_1 T_2 \tag{C.1}$$

with

$$V_u = \sum_{(i,j) \in Y_u} T_{a_u} T_1^{i-1} T_2^{j-1} \qquad W_u = T_{a_u} = e^{i a_u} \qquad T_1 = e^{i \epsilon_1} \qquad T_2 = e^{i \epsilon_2} \tag{C.2}$$

The monomials in (C.1) coming from the $V_u V_v^*$ terms are associated to D(-1)-D(-1) strings, those proportional to $W_u V_v^*$ from D3-D(-1) strings etc. Powers of T_ℓ trace the $U(1)^2 \in SO(4)$ Lorentz charges of the instanton moduli. Negative contributions subtract the degrees of freedom associated to the ADHM constraints.

C.1 Evaluation of $\mathbf{T}_{uv}$

In the following we show that the character $\mathbf{T}_{uv}$ in (C.1) can be written as a sum of $|Y_u| + |Y_v|$ terms with no negative contributions. We write

$$V_u = \sum_{i=1}^{k_{u,1}} \sum_{j=1}^{\tilde{k}_{u,i}} T_{a_u} T_1^{i-1} T_2^{j-1} = \sum_{j=1}^{\tilde{k}_{u,1}} \sum_{i=1}^{k_{u,j}} T_{a_u} T_1^{i-1} T_2^{j-1} \tag{C.3}$$

with $(k_{u1}, k_{u2}, \ldots)$ and $(\tilde{k}_{u1}, \tilde{k}_{u2}, \ldots)$ denoting the length of the rows and columns of Y_u respectively. By definition we set $k_{uj} = 0$ for $j > \tilde{k}_{u1}$. One finds

$$V_u(1 - T_2) = \sum_{i=1}^{k_{u,1}} T_{a_u} T_1^{i-1} (1 - T_2^{\tilde{k}_{u,i}})$$

$$V_v^*(1 - T_1) = \sum_{j=1}^{\tilde{k}_{v,1}} T_{-a_v} T_2^{1-j} (T_1^{1-k_{v,j}} - T_1) \tag{C.4}$$

Plugging this into (C.1) one finds

$$\mathbf{T}_{uv} = -V_u V_v^* (1 - T_1)(1 - T_2) + W_u V_v^* + V_u W_v^* T_1 T_2 \tag{C.5}$$

$$= T_{a_{uv}} \left[\sum_{i=1}^{k_{u,1}} \sum_{j=1}^{\tilde{k}_{v,1}} (T_1^i - T_1^{i-k_{v,j}})(T_2^{1-j} - T_2^{1-j+\tilde{k}_{u,i}}) \right.$$

$$\left. + \sum_{i=1}^{k_{v,1}} \sum_{j=1}^{\tilde{k}_{v,i}} T_1^{1-i} T_2^{1-j} + \sum_{i=1}^{k_{u,1}} \sum_{j=1}^{\tilde{k}_{u,i}} T_1^i T_2^j \right]$$

We notice that the sum over j in the first term of (C.5) can be extended to infinity since $k_{v,j} = 0$ for $j > \tilde{k}_{v,1}$. Thinking of $\mathbf{T}_{uv}$ as a polynomial in T_2, it is easy to extract the part of $\mathbf{T}_{uv}$ with positive powers in T_2. Writing $\mathbf{T}_{uv} = \sum_{k \in \mathbb{Z}} c_k T_2^k$, we define $\mathbf{T}_{uv}^{>0} = \sum_{k>0} c_k T_2^k$ its positive part. One finds

$$\mathbf{T}_{uv}^{>0} = \sum_{i=1}^{k_{u,1}} \sum_{j=1}^{\tilde{k}_{u,i}} T_{a_{uv}} T_1^{i-k_{v,j}} T_2^{1-j+\tilde{k}_{u,i}} = \sum_{s_a \in Y_u} T_{a_{uv}} T_1^{-h_v(s)} T_2^{v_u(s)+1} \tag{C.6}$$

with

$$h_v(s) = k_{v,j} - i \qquad v_u(s) = \tilde{k}_{u,i} - j \tag{C.7}$$

On the other hand

$$\mathbf{T}_{uv} = T_1 T_2 \mathbf{T}_{vu}^* \tag{C.8}$$

as follows from the first line in (C.1). Combining (C.8) with (C.6) one finds

$$\mathbf{T}_{uv} = \mathbf{T}_{uv}^{>0} + T_1 T_2 \left(\mathbf{T}_{vu}^{>0}\right)^* \tag{C.9}$$

leading to

$$\mathbf{T}_{uv} = T_{a_{uv}} \left(\sum_{s \in Y_u} T_1^{-h_v(s)} T_2^{v_u(s)+1} + \sum_{s \in Y_v} T_1^{h_u(s)+1} T_2^{-v_v(s)} \right) \tag{C.10}$$

Finally, writing $\mathbf{T}_{uv} = \sum_i e^{i\lambda_i}$ and collecting the eigenvalues λ_i one finds for the determinant $Z_{Y_u Y_v} = \prod_i \lambda_i^{-1}$ the formula (2.4).

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