Twisted supersymmetries in lattice $\mathcal{N} = 4$ super Yang-Mills theory

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ABSTRACT: Recently it has been shown how a topologically twisted version of $\mathcal{N} = 4$ super Yang-Mills may be discretized in such a way as to preserve one scalar supersymmetry at nonzero lattice spacing. The remaining fifteen supersymmetries are broken by terms of $\mathcal{O}(a)$ where a is the lattice spacing. One would like to know whether these remaining supersymmetries are regained in the continuum limit $a \to 0$ and, if not, how much tuning of the couplings in the lattice action is required. In this paper we derive the form of these additional twisted supersymmetries by combining a set of discrete R-symmetries of the continuum theory with the action of the scalar supersymmetry. We then argue that restoration of rotational symmetry in the continuum limit of the lattice theory likely implies restoration of R-symmetry and hence should lead to an automatic enhancement to the full $\mathcal{N} = 4$ supersymmetry without further fine-tuning.

Contents

1 Introduction

In the recent past there has been a lot of progress in the area of lattice supersymmetry — see [1–4] for a set of reviews and references therein. The key to these new approaches lies in a reformulation of the continuum theory in terms of *twisted* variables. The process of twisting[1] produces a nilpotent scalar supersymmetry $\mathcal{Q}$ and an action that can be written, quite generally, as a sum of $\mathcal{Q}$-exact and $\mathcal{Q}$-closed pieces. It is this structure that allows one to write down a $\mathcal{Q}$-invariant lattice theory [7–12]. Only theories with sufficiently large numbers of supersymmetries can be twisted or orbifolded in this manner and thus implemented on a lattice. Among them, we are particularly interested in studying the well known theory, four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) and this is the focus of the current paper.[2] It is has been the subject of a few numerical studies [14–17], with more in progress.

It should be noted that other complementary Monte Carlo approaches have been proposed for studying $\mathcal{N} = 4$ SYM in the planar large N limit see [18–24]. The lattice approach

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[1]The twisting technique leads to lattice theories which are essentially equivalent to those derived by orbifolding [5, 6].

[2]Another approach to $\mathcal{N} = 4$ SYM using overlap fermions can be found in [13]

we discuss here, based on exact lattice supersymmetry, works for any gauge group, but retains only one supercharge. These other approaches work only in the large-N limit, but retain larger numbers of exact supercharges. The results from the two approaches eventually should be compared in the region where they should coincide, namely at large N and close to the continuum limit.

In addition, lattice formulations of two-dimensional SYM theories with four and eight supercharges which preserve nilpotent supercharges are presented in [25, 26]. Although these formulations are based on a twisted supersymmetry different from the one described in this paper, evidence has been presented showing that the continuum limit of this formulation for the case of two-dimensional SYM with four supercharges [27, 28] yields the same physics [29].

The lattice action of $\mathcal{N} = 4$ SYM retains invariance under only the single scalar supercharge $\mathcal{Q}$. The remaining fifteen supercharges $\mathcal{Q}_a, \mathcal{Q}_{ab}, a, b = 1, \ldots, 5$, are not exact symmetries of the lattice action but are broken by terms of order the lattice spacing. Thus one must potentially tune a series of lattice couplings (counterterms) to achieve full supersymmetry in the continuum limit. The supersymmetric Ward-Takahashi (WT) identities corresponding to $\mathcal{Q}_a, \mathcal{Q}_{ab}$ would seem to be very important in this regard as they would provide insight into the degree of breaking and would serve as indicators that the full SUSY has been restored, during the process of fine-tuning. However, even in the continuum theory there are issues with these WT identities. There is no fully off-shell formulation of $\mathcal{N} = 4$ SYM, so when one goes through the normal process of deriving the WT identities, an essential ingredient is missing: the action is not invariant under the global SUSY transformation, unless the equations of motion are used. However, in the quantum theory (and in particular under the path integral) the classical equations of motion are not satisfied. This casts a pall over the whole procedure, and actually raises questions about whether or not $\langle 0|\partial_\mu S_A^\mu(x)\mathcal{O}(0)|0\rangle$ really vanishes for $x \neq 0$ and $A = a$ or ab, since these symmetries are not realized off-shell.[3] Another issue, which arises in the lattice theory, is that because the supercurrents are not conserved at finite lattice spacing (due to discretization errors that break these 15 supersymmetries), the bare supercurrent (i.e., the one simply transcribed from the continuum) will mix with other operators under renormalization. This has been extensively discussed in the context of $\mathcal{N} = 1$ SYM with Wilson fermions [30] and in another formulation of $\mathcal{N} = 4$ SYM [13]. Thus to use the SUSY WT identities in order to fine-tune the action, one must simultaneously determine these mixing coefficients. In the theory that we are discussing here, there are many such operators and mixing coefficients, rendering this a daunting task.

To our great relief, we have discovered a way to circumvent these difficulties! We have found an intimate relationship between the restoration of the $\mathrm{SU}(4)_R$ global R-symmetry of the continuum theory and the supersymmetric continuum limit of the lattice theory. In short, the exact $\mathcal{Q}$ SUSY, when combined with a restoration of the R-symmetry (or in fact, even a discrete subgroup of that symmetry), is sufficient to guarantee that $\mathcal{Q}_a, \mathcal{Q}_{ab}$ are restored and the long distance effective action is $\mathcal{N} = 4$ supersymmetric. So, rather

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[3]Here, S_A^μ is the supercurrent and $\mathcal{O}$ is any operator.

than measuring the SUSY WT identities, one can simply measure the $SU(4)_R$ WT identies. (See ref. [27] where the authors have constructed and numerically measured WT identities related to the supercurrent conservation in the two-dimensional four supercharge SYM.) Moreover, because of the twisting process, which combines this internal global symmetry with the spacetime symmetry group of Euclidean SO(4) rotations, we ultimately find the remarkable result that the recovery of Lorentz invariance in the continuum limit likely implies the full $\mathcal{N} = 4$ supersymmetry without any additional fine-tuning. However, it is an open question as to whether or not the SO(4) rotational symmetry requires fine-tuning of the lattice action in the present case. We show what condition must hold on coefficients in the long distance effective theory for this to be the case. Even if fine-tuning of these coefficients is necessary, at least it is relatively easy in a lattice simulation to test for SO(4) invariance at long distance.

The outline of the paper is as follows. We begin with a discussion of the continuum twisted $\mathcal{N} = 4$ SYM theory and its supersymmetries in section 2. The action of the theory and the scalar supersymmetry transformations of the fields are written down. We then derive the form of the fifteen additional twisted supersymmetries by combining a set of discrete R-symmetries of the continuum theory with the action of the scalar supersymmetry. In section 3 we discuss how invariance of the action under these discrete R-symmetries is already sufficient to guarantee invariance under the remaining fifteen twisted supersymmetries. In section 4 we briefly detail the difficulty with the Ward-Takahashi identities for the supercurrent in the twisted theory, due to the fact that there is no fully off-shell formulation.[4] We then discuss the lattice discretization of the theory in section 5. We discuss the fate of the discrete R-symmetries on the lattice in section 6 and show that while they are not invariances of the lattice theory they may be restored *automatically* in an appropriate rotationally symmetric continuum limit. Finally, we provide a summary of our main results in section 7.

2   The continuum $\mathcal{N} = 4$ theory and its supersymmetries

The four-dimensional Euclidean $\mathcal{N} = 4$ Yang-Mills theory in the continuum is obtained by dimensional reduction of ten-dimensional $\mathcal{N} = 1$ SYM. The ten-dimensional theory has a rotation group $SO(10)_{\text{Lorentz}}$ and in the four-dimensional theory it becomes the product of the four-dimensional rotation group and the R-symmetry group: $SO(4)_{\text{Lorentz}} \times SO(6)_{\text{R}}$. The four-dimensional theory can be rewritten in a useful way by transforming to a new set of twisted fields as shown originally by Marcus [31].[5] The action of the four-dimensional theory, after the twist, can be written in a compact form that can be seen as the dimensional reduction of a five-dimensional theory (see [6] for details of the construction):

$$S = \int \text{Tr} \left(-\overline{\mathcal{F}}_{mn}\mathcal{F}_{mn} + \frac{1}{2}[\overline{\mathcal{D}}_m, \mathcal{D}_m]^2 - \chi_{mn}\mathcal{D}_{[m}\psi_{n]} - \eta\overline{\mathcal{D}}_m\psi_m \right.$$
$$\left. -\frac{1}{4}\epsilon_{mnrpq}\chi_{pq}\overline{\mathcal{D}}_r\chi_{mn}\right), \tag{2.1}$$

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[4]There is a similar problem in the untwisted theory, again due to the lack of an off-shell formulation.
[5]This twist of $\mathcal{N} = 4$ SYM also plays a role in the geometric-Langlands program [32].

where the Roman indices run from $1, \cdots, 5$. Here and elsewhere below there is an implicit d^4x integration measure. The bosonic degrees of freedom of the theory, the four-dimensional gauge field and six scalars, are encoded in the five complexified gauge fields $\mathcal{A}_m$, with $\mathcal{A}_m = A_m + iB_m$. This encoding of fields leads to complexified covariant derivatives and field strengths in the theory. We have, for a generic field X

$$\mathcal{D}_m X \equiv \partial_m X + [\mathcal{A}_m, X], \quad \overline{\mathcal{D}}_m X \equiv \partial_m X + [\overline{\mathcal{A}}_m, X], \tag{2.2}$$

and the complexified field strengths

$$\mathcal{F}_{mn} \equiv [\mathcal{D}_m, \mathcal{D}_n] = \partial_m \mathcal{A}_n - \partial_n \mathcal{A}_m + [\mathcal{A}_m, \mathcal{A}_n], \tag{2.3}$$

$$\overline{\mathcal{F}}_{mn} \equiv [\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n] = \partial_m \overline{\mathcal{A}}_n - \partial_n \overline{\mathcal{A}}_m + [\overline{\mathcal{A}}_m, \overline{\mathcal{A}}_n]. \tag{2.4}$$

Of course, by definition $\partial_5 = 0$ since the theory has been reduced to four dimensions. The fermions of the original theory become p-forms after twisting. The 16 fermionic degrees of freedom are now packaged in the set $(\eta/2, \psi_m, \chi_{mn})$ of p-form fields.

One of the most important aspects of the twisting process is the exposure of a nilpotent scalar supercharge $\mathcal{Q}$ that would make the theory compatible with lattice discretization. The remaining supersymmetric charges also transform as integer spin representations of the twisted rotation group. They are packaged as the p-forms $(\mathcal{Q}, \mathcal{Q}_m, \mathcal{Q}_{mn})$.

The nilpotent scalar supersymmetry $\mathcal{Q}$ acts on the twisted fields in a simple manner

$$\mathcal{Q}\mathcal{A}_m = \psi_m, \tag{2.5}$$

$$\mathcal{Q}\psi_m = 0, \tag{2.6}$$

$$\mathcal{Q}\overline{\mathcal{A}}_m = 0, \tag{2.7}$$

$$\mathcal{Q}\chi_{mn} = -[\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n], \tag{2.8}$$

$$\mathcal{Q}\eta = d, \tag{2.9}$$

$$\mathcal{Q}d = 0, \tag{2.10}$$

where d is an auxiliary field introduced for the off-shell completion of the action.[6] It obeys the equation of motion

$$d = \sum_m [\overline{\mathcal{D}}_m, \mathcal{D}_m]. \tag{2.11}$$

The off-shell form of the action can be written as the sum of a $\mathcal{Q}$-exact piece and a $\mathcal{Q}$-closed piece. It is

$$S = \int \mathrm{Tr}\, \mathcal{Q}\Lambda + S_{\mathcal{Q}-\mathrm{closed}}, \tag{2.12}$$

where

$$\Lambda = \chi_{mn}[\mathcal{D}_m, \mathcal{D}_n] + \eta[\overline{\mathcal{D}}_m, \mathcal{D}_m] - \frac{1}{2}\eta d, \tag{2.13}$$

and

$$S_{\mathcal{Q}-\mathrm{closed}} = -\frac{1}{4} \int \mathrm{Tr}\, \epsilon_{mnrpq} \chi_{pq} \overline{\mathcal{D}}_r \chi_{mn}. \tag{2.14}$$

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[6]It should be emphasized that the action is only off-shell with respect to $\mathcal{Q}$, whereas the other 15 supersymmetries require the equations of motion to yield invariance.

We recover the on-shell form of the action given in eq. (2.1) after integrating out the auxiliary field d. The off-shell action is invariant under the $\mathcal{Q}$ supersymmetry

$$\mathcal{Q}S = 0, \tag{2.15}$$

as the $\mathcal{Q}$-exact term in the off-shell action vanishes due to the nilpotent property of $\mathcal{Q}$ and the $\mathcal{Q}$-closed term is zero due to Bianchi identity for the complexified covariant derivatives.

The twisted action eq. (2.1) possesses global discrete symmetries that are subgroups of the R-symmetry group $\mathrm{SO}(6)_\mathrm{R}$ of the theory. We can make use of these discrete symmetries and the $\mathcal{Q}$ supercharge transformations of the fields eq. (2.5)–(2.10) to write down the field transformations associated with the $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries.

2.1 Action of the R-symmetries on the twisted fields

We will show that the twisted supersymmetries can be recovered from a knowledge of the scalar supersymmetry $\mathcal{Q}$ and a discrete subgroup of the R-symmetries. To this end let us first show how to find these discrete R-symmetries. Consider first the (kinetic part of) twisted continuum fermion action written in matrix form

$$S_F = \int \mathrm{Tr}\ \left(\overline{\Psi}\gamma_m\partial_m\Psi\right). \tag{2.16}$$

Here, Ψ is the 4×4 matrix that results from twisting the SO(4) rotational symmetry with a SO(4) subgroup of the SO(6) R-symmetry. Consider the following transformation on this matrix fermion

$$\Psi \to \Psi\Gamma^A, \tag{2.17}$$

where Γ^A with $A = 1, \cdots, 15$, belong to the set $\{i\gamma_m, \frac{1}{2}[\gamma_m, \gamma_n]\}$ of 4×4 matrices that form a basis for the generators of the $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$ R-symmetry. Furthermore, since the fermions are Majorana $\overline{\Psi} = C\Psi^T C^{-1}$ one can easily show that $\overline{\Psi} \to -\Gamma^A\overline{\Psi}$ and hence that the fermion action is invariant under the transformation. Expanding the matrix Ψ on the same basis (plus the unit matrix) yields the usual twisted fields $(\eta/2, \psi_m, \chi_{mn})$. It should be clear that these twisted fields are then permuted in a well defined manner after right multiplication by one of the Γ^A. If one subsequently demands that the Yukawa interactions are also invariant one finds that the complexified covariant derivatives must also transform according to the rule:

$$\mathrm{Im}\ \mathcal{D}_m \to \frac{1}{4}\mathrm{Tr}\left(-\Gamma^A\gamma_m\Gamma^A\right)\mathrm{Im}\ \mathcal{D}_m, \tag{2.18}$$

where $\mathrm{Im}\ \mathcal{D}_m \equiv (\mathcal{D}_m - \overline{\mathcal{D}}_m)/2i$. (See also [33] where the authors have shown that a solution of the above consistency condition exactly describes which charge can be defined on a lattice site.) In the next sections we will write out these discrete transformations explicitly and verify the invariance of the continuum action.

2.2 $\mathcal{Q}_a$ supersymmetries

Consider the following set of transformations of the fields and covariant derivatives corresponding to the discrete R-transformation associated with $i\gamma_a$ (the index a is considered

fixed in all that follows)

$$\eta \to 2\psi_a, \tag{2.19}$$

$$\psi_a \to \frac{1}{2}\eta, \tag{2.20}$$

$$\psi_b \to -\chi_{ab}, \tag{2.21}$$

$$\chi_{ab} \to -\psi_b, \tag{2.22}$$

$$\chi_{bc} \to \frac{1}{2}\epsilon_{bcagh}\chi_{gh}, \tag{2.23}$$

and

$$\mathcal{D}_a \to \mathcal{D}_a, \ \ \overline{\mathcal{D}}_a \to \overline{\mathcal{D}}_a, \tag{2.24}$$

$$\mathcal{D}_b \to \overline{\mathcal{D}}_b, \ \ \overline{\mathcal{D}}_b \to \mathcal{D}_b, \ \ b \neq a. \tag{2.25}$$

We denote this set of discrete transformations as R_a transformations. Let us check whether the action is invariant under R_a transformations. We have the bosonic action

$$S_B = \int \mathrm{Tr} \ -\sum_{m,n} \overline{\mathcal{F}}_{mn}\mathcal{F}_{mn} + \frac{1}{2}\Big(\sum_m [\overline{\mathcal{D}}_m, \mathcal{D}_m]\Big)^2. \tag{2.26}$$

Rewriting this to expose the index a,

$$S_B = \int \mathrm{Tr} \ -2\sum_{n\neq a}[\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_n][\mathcal{D}_a, \mathcal{D}_n] - \sum_{m\neq a, n\neq a}[\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n][\mathcal{D}_m, \mathcal{D}_n]$$
$$+\frac{1}{2}\Big([\overline{\mathcal{D}}_a, \mathcal{D}_a] + \sum_{m\neq a}[\overline{\mathcal{D}}_m, \mathcal{D}_m]\Big)^2. \tag{2.27}$$

Applying the above field transformations the bosonic action becomes

$$S'_B = \int \mathrm{Tr} \ -2\sum_{n\neq a}[\overline{\mathcal{D}}_a, \mathcal{D}_n][\mathcal{D}_a, \overline{\mathcal{D}}_n] - \sum_{\{m,n\}\neq a}[\mathcal{D}_m, \mathcal{D}_n][\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n]$$
$$+\frac{1}{2}\Big([\overline{\mathcal{D}}_a, \mathcal{D}_a] - \sum_{m\neq a}[\overline{\mathcal{D}}_m, \mathcal{D}_m]\Big)^2$$
$$= \int \mathrm{Tr} \ -2\sum_{n\neq a}[\overline{\mathcal{D}}_a, \mathcal{D}_n][\mathcal{D}_a, \overline{\mathcal{D}}_n] - \sum_{\{m,n\}\neq a}[\mathcal{D}_m, \mathcal{D}_n][\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n] + \frac{1}{2}[\overline{\mathcal{D}}_a, \mathcal{D}_a]^2$$
$$+\frac{1}{2}\sum_{m\neq a}[\overline{\mathcal{D}}_m, \mathcal{D}_m]^2 - \sum_{m\neq a}[\overline{\mathcal{D}}_a, \mathcal{D}_a][\overline{\mathcal{D}}_m, \mathcal{D}_m]. \tag{2.28}$$

Thus the change in the bosonic action under this discrete transformation is

$$S'_B - S_B = \int \sum_{n\neq a} \mathrm{Tr} \ \Big(-2[\overline{\mathcal{D}}_a, \mathcal{D}_n][\mathcal{D}_a, \overline{\mathcal{D}}_n] + 2[\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_n][\mathcal{D}_a, \mathcal{D}_n]$$
$$-2[\overline{\mathcal{D}}_a, \mathcal{D}_a][\overline{\mathcal{D}}_n, \mathcal{D}_n]\Big). \tag{2.29}$$

Making use of the following relation among complexified covariant derivatives under the trace

$$\mathrm{Tr}\left([\overline{\mathcal{D}}_a, \mathcal{D}_n][\mathcal{D}_a, \overline{\mathcal{D}}_n]\right) = \mathrm{Tr}\left([\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_n][\mathcal{D}_a, \mathcal{D}_n]\right) - \mathrm{Tr}\left([\overline{\mathcal{D}}_a, \mathcal{D}_a][\overline{\mathcal{D}}_n, \mathcal{D}_n]\right), \qquad (2.30)$$

with $n \neq a$, it is straightforward to show that $S_B' - S_B = 0$, i.e., the bosonic action is indeed invariant under this discrete transformation.

Let us check the invariance of the fermionic action. We have

$$S_F = \int \mathrm{Tr} \; -\eta \sum_m \overline{\mathcal{D}}_m \psi_m - \sum_{m,n} \chi_{mn} \mathcal{D}_{[m} \psi_{n]} - \frac{1}{4} \sum_{m,n,c,d,e} \epsilon_{mncde} \chi_{de} \overline{\mathcal{D}}_c \chi_{mn}. \quad (2.31)$$

Exposing the index a,

$$\begin{aligned}
S_F = \int \mathrm{Tr} \Big(&- \eta \overline{\mathcal{D}}_a \psi_a - \sum_{n \neq a} \eta \overline{\mathcal{D}}_n \psi_n - 2 \sum_{n \neq a} \chi_{an} \mathcal{D}_a \psi_n + 2 \sum_{n \neq a} \chi_{an} \mathcal{D}_n \psi_a \\
&- \sum_{\{m,n\} \neq a} \chi_{mn} \mathcal{D}_m \psi_n + \sum_{\{m,n\} \neq a} \chi_{mn} \mathcal{D}_n \psi_m \\
&- \sum_{\{n,c,d,e\} \neq a} \epsilon_{ancde} \chi_{de} \overline{\mathcal{D}}_c \chi_{an} - \frac{1}{4} \sum_{\{m,n,d,e\} \neq a} \epsilon_{mnade} \chi_{de} \overline{\mathcal{D}}_a \chi_{mn} \Big). \qquad (2.32)
\end{aligned}$$

Under the R_a transformations it becomes

$$\begin{aligned}
S_F' = \int \mathrm{Tr} \Big(&- 2 \psi_a \overline{\mathcal{D}}_a \frac{1}{2} \eta + \sum_{n \neq a} 2 \psi_a \mathcal{D}_n \chi_{an} - 2 \sum_{n \neq a} \psi_n \mathcal{D}_a \chi_{an} - 2 \sum_{n \neq a} \psi_n \overline{\mathcal{D}}_n \frac{1}{2} \eta \\
&+ \sum_{\{m,n\} \neq a} \frac{1}{2} \epsilon_{mnagh} \chi_{gh} \overline{\mathcal{D}}_m \chi_{an} - \sum_{\{m,n\} \neq a} \frac{1}{2} \epsilon_{mnagh} \chi_{gh} \overline{\mathcal{D}}_n \chi_{am} \\
&+ \epsilon_{ancde} \frac{1}{2} \epsilon_{deapq} \chi_{pq} \mathcal{D}_c \psi_n - \frac{1}{4} \epsilon_{mnade} \frac{1}{2} \epsilon_{deapq} \chi_{pq} \overline{\mathcal{D}}_a \frac{1}{2} \epsilon_{mnars} \chi_{rs} \Big) \\
= \int \mathrm{Tr} \Big(&- \eta \overline{\mathcal{D}}_a \psi_a + \sum_{n \neq a} 2 \chi_{an} \mathcal{D}_n \psi_a - \sum_{n \neq a} 2 \chi_{an} \mathcal{D}_a \psi_n - \sum_{n \neq a} \eta \overline{\mathcal{D}}_n \psi_n \\
&+ \sum_{\{m,n\} \neq a} \epsilon_{mnagh} \chi_{gh} \overline{\mathcal{D}}_m \chi_{an} - \sum_{\{n,c\} \neq a} \chi_{nc} \mathcal{D}_n \psi_c + \sum_{\{n,c\} \neq a} \chi_{nc} \mathcal{D}_c \psi_n \\
&- \frac{1}{4} \epsilon_{pqars} \chi_{pq} \overline{\mathcal{D}}_a \chi_{rs} \Big) \\
= S_F. & \qquad (2.33)
\end{aligned}$$

Thus the fermionic action is also invariant under the R_a transformations.

Combining this discrete R-symmetry with the original $\mathcal{Q}$ supersymmetry we can now write down the one-form supersymmetries associated with the twisted supercharge $\mathcal{Q}_a$. After making the replacements eq. (2.19)–(2.25) in eq. (2.5)–(2.10) we have the supersym-

metry transformations associated with the supercharge $\mathcal{Q}_a$:

$$\mathcal{Q}_a \mathcal{A}_b = \frac{1}{2}\delta_{ab}\eta, \tag{2.34}$$

$$\mathcal{Q}_a \overline{\mathcal{A}}_b = -\chi_{ab}, \tag{2.35}$$

$$\mathcal{Q}_a \psi_b = \frac{1}{2}\delta_{ab}d_a + (1-\delta_{ab})[\overline{\mathcal{D}}_a, \mathcal{D}_b], \tag{2.36}$$

$$\mathcal{Q}_a \chi_{bc} = -\frac{1}{2}\epsilon_{abcgh}[\mathcal{D}_g, \mathcal{D}_h], \tag{2.37}$$

$$\mathcal{Q}_a \eta = 0, \tag{2.38}$$

$$\mathcal{Q}_a d_a = 0. \tag{2.39}$$

Here, $d_a = [\overline{\mathcal{D}}_a, \mathcal{D}_a] - \sum_{m \neq a}[\overline{\mathcal{D}}_m, \mathcal{D}_m]$, and the equation $\mathcal{Q}_a d_a = 0$ requires the equations of motion, since this is an on-shell supersymmetry.

From the above supersymmetry transformations we see that the $\mathcal{Q}_a$, like the scalar supercharge $\mathcal{Q}$, is also nilpotent: $\mathcal{Q}_a^2 = 0$. A straightforward, if tedious, calculation allows one to verify that the action is $\mathcal{Q}_a$ invariant on-shell (i.e., using the equations of motion), $\mathcal{Q}_a S = 0$.

2.3 $\mathcal{Q}_{ab}$ supersymmetries

We now consider the discrete subgroup of the R-symmetries associated with $\Gamma = \frac{1}{2}[\gamma_a, \gamma_b]$ corresponding to transformations of the twisted fields and covariant derivatives parametrized by the fixed indices a and b.

$$\eta \to 2\chi_{ab}, \tag{2.40}$$

$$\psi_a \to \psi_b, \tag{2.41}$$

$$\psi_b \to -\psi_a, \tag{2.42}$$

$$\psi_c \to \frac{1}{2}\epsilon_{cabgh}\chi_{gh}, \tag{2.43}$$

$$\chi_{ab} \to -\frac{1}{2}\eta, \tag{2.44}$$

$$\chi_{ac} \to \chi_{bc}, \tag{2.45}$$

$$\chi_{bc} \to -\chi_{ac}, \tag{2.46}$$

$$\chi_{gh} \to -\epsilon_{ghabc}\psi_c, \tag{2.47}$$

and

$$\mathcal{D}_{a,b} \to \overline{\mathcal{D}}_{a,b}, \quad \overline{\mathcal{D}}_{a,b} \to \mathcal{D}_{a,b}, \tag{2.48}$$

$$\mathcal{D}_c \to \mathcal{D}_c, \quad \overline{\mathcal{D}}_c \to \overline{\mathcal{D}}_c, \quad c \neq a, b. \tag{2.49}$$

We denote this set of discrete R-symmetries as R_{ab} transformations. Let us check if R_{ab} transformations are indeed a symmetry of the action. First consider the bosonic action

$$S_B = \int \text{Tr} \, -\sum_{m,n}\overline{\mathcal{F}}_{mn}\mathcal{F}_{mn} + \frac{1}{2}\Big(\sum_m [\overline{\mathcal{D}}_m, \mathcal{D}_m]\Big)^2. \tag{2.50}$$

Splitting this into pieces to expose the terms with indices a and b

$$S_B = \int \text{Tr} \left(-2[\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_b][\mathcal{D}_a, \mathcal{D}_b] - 2\sum_{c\neq a,b} [\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_c][\mathcal{D}_a, \mathcal{D}_c] - 2\sum_{c\neq a,b} [\overline{\mathcal{D}}_b, \overline{\mathcal{D}}_c][\mathcal{D}_b, \mathcal{D}_c] \right.$$
$$\left. -\sum_{\{c,d\}\neq a,b} [\overline{\mathcal{D}}_c, \overline{\mathcal{D}}_d][\mathcal{D}_c, \mathcal{D}_d] + \frac{1}{2}\Big([\overline{\mathcal{D}}_a, \mathcal{D}_a] + [\overline{\mathcal{D}}_b, \mathcal{D}_b] + \sum_{c\neq a,b} [\overline{\mathcal{D}}_c, \mathcal{D}_c]\Big)^2 \right). \quad (2.51)$$

Applying the field transformations

$$S_B'' = \int \text{Tr} \left(-2[\mathcal{D}_a, \mathcal{D}_b][\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_b] - 2\sum_{c\neq a,b} [\mathcal{D}_a, \overline{\mathcal{D}}_c][\overline{\mathcal{D}}_a, \mathcal{D}_c] - 2\sum_{c\neq a,b} [\mathcal{D}_b, \overline{\mathcal{D}}_c][\overline{\mathcal{D}}_b, \mathcal{D}_c] \right.$$
$$\left. -\sum_{\{c,d\}\neq a,b} [\overline{\mathcal{D}}_c, \overline{\mathcal{D}}_d][\mathcal{D}_c, \mathcal{D}_d] + \frac{1}{2}\Big(-[\overline{\mathcal{D}}_a, \mathcal{D}_a] - [\overline{\mathcal{D}}_b, \mathcal{D}_b] + \sum_{c\neq a,b} [\overline{\mathcal{D}}_c, \mathcal{D}_c]\Big)^2 \right). \quad (2.52)$$

Thus the change in the bosonic action under R_{ab} transformations is

$$S_B'' - S_B = \int \text{Tr} \left(-2\sum_{c\neq a,b} [\mathcal{D}_a, \overline{\mathcal{D}}_c][\overline{\mathcal{D}}_a, \mathcal{D}_c] - 2\sum_{c\neq a,b} [\mathcal{D}_b, \overline{\mathcal{D}}_c][\overline{\mathcal{D}}_b, \mathcal{D}_c] \right.$$
$$+2\sum_{c\neq a,b} [\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_c][\mathcal{D}_a, \mathcal{D}_c] + 2\sum_{c\neq a,b} [\overline{\mathcal{D}}_b, \overline{\mathcal{D}}_c][\mathcal{D}_b, \mathcal{D}_c]$$
$$\left. -2\sum_{c\neq a,b} [\overline{\mathcal{D}}_a, \mathcal{D}_a][\overline{\mathcal{D}}_c, \mathcal{D}_c] - 2\sum_{c\neq a,b} [\overline{\mathcal{D}}_b, \mathcal{D}_b][\overline{\mathcal{D}}_c, \mathcal{D}_c] \right). \quad (2.53)$$

Once again the relation

$$\text{Tr}\left([\overline{\mathcal{D}}_a, \mathcal{D}_b][\mathcal{D}_a, \overline{\mathcal{D}}_b]\right) = \text{Tr}\left([\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_b][\mathcal{D}_a, \mathcal{D}_b]\right) - \text{Tr}\left([\overline{\mathcal{D}}_a, \mathcal{D}_a][\overline{\mathcal{D}}_b, \mathcal{D}_b]\right), \quad (2.54)$$

allows us to show that in fact $S_B'' - S_B = 0$ and thus the bosonic action is invariant under this discrete field transformation.

Turning to the fermionic part of the action we have

$$S_F = \int \text{Tr} \left(-\eta \sum_m \overline{\mathcal{D}}_m \psi_m - \sum_{m,n} \chi_{mn} \mathcal{D}_{[m} \psi_{n]} - \frac{1}{4} \sum_{m,n,c,d,e} \epsilon_{mncde} \chi_{de} \overline{\mathcal{D}}_c \chi_{mn} \right). \quad (2.55)$$

Splitting this into pieces to expose the terms with indices a and b

$$S_F = \int \text{Tr} \left(-\eta \overline{\mathcal{D}}_a \psi_a - \eta \overline{\mathcal{D}}_b \psi_b - \sum_{m\neq a,b} \eta \overline{\mathcal{D}}_m \psi_m - 2\chi_{ab} \mathcal{D}_a \psi_b + 2\chi_{ab} \mathcal{D}_b \psi_a \right.$$
$$-2\sum_{n\neq a,b} \chi_{an} \mathcal{D}_a \psi_n + 2\sum_{n\neq a,b} \chi_{an} \mathcal{D}_n \psi_a - 2\sum_{n\neq a,b} \chi_{bn} \mathcal{D}_b \psi_n + 2\sum_{n\neq a,b} \chi_{bn} \mathcal{D}_n \psi_b$$
$$-\sum_{\{m,n\}\neq a,b} \chi_{mn} \mathcal{D}_m \psi_n + \sum_{\{m,n\}\neq a,b} \chi_{mn} \mathcal{D}_n \psi_m - \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$$
$$\left. -2\epsilon_{ancbe} \chi_{an} \overline{\mathcal{D}}_c \chi_{be} - \epsilon_{deabn} \chi_{de} \overline{\mathcal{D}}_a \chi_{bn} - \epsilon_{deban} \chi_{de} \overline{\mathcal{D}}_b \chi_{an} \right). \quad (2.56)$$

Applying the field transformations given in eq. (2.40)–(2.49) the fermionic action becomes

$$
\begin{aligned}
S_F'' = \int \mathrm{Tr} \Big(& -2\chi_{ab}\mathcal{D}_a\psi_b + 2\chi_{ab}\mathcal{D}_b\psi_a - \chi_{ab}\overline{\mathcal{D}}_m\epsilon_{mabgh}\chi_{gh} - \eta\overline{\mathcal{D}}_a\psi_a - \eta\overline{\mathcal{D}}_b\psi_b \\
& -\chi_{bn}\overline{\mathcal{D}}_a\epsilon_{nabgh}\chi_{gh} + 2\chi_{bn}\mathcal{D}_n\psi_b + \chi_{an}\overline{\mathcal{D}}_b\epsilon_{nabgh}\chi_{gh} + 2\chi_{an}\mathcal{D}_n\psi_a \\
& +\frac{1}{2}\epsilon_{mnabc}\psi_c\mathcal{D}_m\epsilon_{nabrs}\chi_{rs} - \frac{1}{2}\epsilon_{mnabh}\psi_h\mathcal{D}_n\epsilon_{mabuv}\chi_{uv} - \frac{1}{2}\epsilon_{abcde}\eta\overline{\mathcal{D}}_c\epsilon_{deabh}\psi_h \\
& +2\epsilon_{ancbe}\chi_{bn}\overline{\mathcal{D}}_c\chi_{ae} - \epsilon_{deabn}\epsilon_{deabc}\psi_c\mathcal{D}_a\chi_{an} + \epsilon_{deban}\epsilon_{deabp}\psi_p\mathcal{D}_b\chi_{bn} \Big).
\end{aligned}
\tag{2.57}
$$

Simplifying and rearranging the terms

$$
\begin{aligned}
S_F'' = \int \mathrm{Tr} \Big(& -\eta\overline{\mathcal{D}}_a\psi_a - \eta\overline{\mathcal{D}}_b\psi_b - 2\eta\overline{\mathcal{D}}_m\psi_m - 2\chi_{ab}\mathcal{D}_a\psi_b + 2\chi_{ab}\mathcal{D}_b\psi_a \\
& -2\chi_{an}\mathcal{D}_a\psi_n + 2\chi_{an}\mathcal{D}_n\psi_a - 2\chi_{bn}\mathcal{D}_b\psi_n + 2\chi_{bn}\mathcal{D}_n\psi_b \\
& -\chi_{mn}\mathcal{D}_m\psi_n + \chi_{mn}\mathcal{D}_n\psi_m - \epsilon_{abcde}\chi_{ab}\overline{\mathcal{D}}_c\chi_{de} \\
& -2\epsilon_{ancbe}\chi_{an}\overline{\mathcal{D}}_c\chi_{be} - \epsilon_{deabn}\chi_{de}\overline{\mathcal{D}}_a\chi_{bn} - \epsilon_{deban}\chi_{de}\overline{\mathcal{D}}_b\chi_{an} \Big) \\
= \; & S_F.
\end{aligned}
\tag{2.58}
$$

Thus we have recovered the original fermionic part of the action.

After making the replacements eq. (2.40)–(2.49) in the $\mathcal{Q}$ supersymmetries for the fields we have the supersymmetry transformations associated with the supercharge $\mathcal{Q}_{ab}$:

$$
\mathcal{Q}_{ab}\mathcal{A}_c = \frac{1}{2}\sum_{g,h}\epsilon_{abcgh}\chi_{gh},
\tag{2.59}
$$

$$
\mathcal{Q}_{ab}\overline{\mathcal{A}}_c = \delta_{ac}\psi_b - \delta_{bc}\psi_a,
\tag{2.60}
$$

$$
\mathcal{Q}_{ab}\psi_c = \sum_{g,h}\epsilon_{abcgh}[\overline{\mathcal{D}}_g, \overline{\mathcal{D}}_h],
\tag{2.61}
$$

$$
\mathcal{Q}_{ab}\chi_{cd} = \frac{1}{2}\delta_{ac}\delta_{bd}d_{ab} - \delta_{bc}[\mathcal{D}_a, \overline{\mathcal{D}}_d] + \delta_{ac}[\mathcal{D}_b, \overline{\mathcal{D}}_d],
\tag{2.62}
$$

$$
\mathcal{Q}_{ab}\eta = 2[\mathcal{D}_a, \mathcal{D}_b],
\tag{2.63}
$$

$$
\mathcal{Q}_{ab}d_{ab} = 0.
\tag{2.64}
$$

Here, $d_{ab} = -[\overline{\mathcal{D}}_a, \mathcal{D}_a] - [\overline{\mathcal{D}}_b, \mathcal{D}_b] + \sum_{m\neq a,b}[\overline{\mathcal{D}}_m, \mathcal{D}_m]$, and the equation $\mathcal{Q}_{ab}d_{ab} = 0$ requires the equations of motion, since this is an on-shell supersymmetry.

From the above supersymmetry transformations we see that the supercharge $\mathcal{Q}_{ab}$ is also nilpotent: $\mathcal{Q}_{ab}^2 = 0$ and the action invariant under $\mathcal{Q}_{ab}$ on-shell.

2.4 Combined expressions for supersymmetry transformations

The supersymmetry transformations can be combined to express them in a compact form once we introduce the (Grassmann odd) supersymmetry parameters $(\delta\kappa_0, \delta\kappa_a, \delta\kappa_{ab})$ associated with the supercharges $(\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab})$. Thus the supersymmetry transformations of

the fields are

$$\delta_\kappa \mathcal{A}_m = \delta\kappa_0 \psi_m + \frac{1}{2}\delta\kappa_b \delta_{bm}\eta + \frac{1}{2}\delta\kappa_{ab}\epsilon_{abmgh}\chi_{gh}, \tag{2.65}$$

$$\delta_\kappa \overline{\mathcal{A}}_m = -\delta\kappa_b \chi_{bm} + \delta\kappa_{ab}(\delta_{am}\psi_b - \delta_{bm}\psi_a), \tag{2.66}$$

$$\delta_\kappa \psi_m = \delta\kappa_b\Big(\frac{1}{2}\delta_{bm}d_b + (1-\delta_{bm})[\overline{\mathcal{D}}_b, \mathcal{D}_m]\Big) + \delta\kappa_{ab}\epsilon_{abmgh}[\overline{\mathcal{D}}_g, \overline{\mathcal{D}}_h], \tag{2.67}$$

$$\delta_\kappa \chi_{mn} = -\delta\kappa_0[\overline{\mathcal{D}}_m, \overline{\mathcal{D}}_n] - \frac{1}{2}\delta\kappa_b \epsilon_{bmngh}[\mathcal{D}_g, \mathcal{D}_h]$$
$$+\delta\kappa_{ab}\Big(\frac{1}{2}\delta_{am}\delta_{bn}d_{ab} - \delta_{bm}[\mathcal{D}_a, \overline{\mathcal{D}}_n] + \delta_{am}[\mathcal{D}_b, \overline{\mathcal{D}}_n]\Big), \tag{2.68}$$

$$\delta_\kappa \eta = \delta\kappa_0 d + 2\delta\kappa_{ab}[\mathcal{D}_a, \mathcal{D}_b], \tag{2.69}$$

$$\delta_\kappa d = \delta_\kappa d_a = \delta_\kappa d_{ab} = 0 \tag{2.70}$$

where the operator δ_κ is

$$\delta_\kappa = \delta\kappa_0 \mathcal{Q} + \delta\kappa_a \mathcal{Q}_a + \delta\kappa_{ab}\mathcal{Q}_{ab}. \tag{2.71}$$

The above set of transformations are clearly a symmetry of the action if the parameters $(\delta\kappa_0, \delta\kappa_a, \delta\kappa_{ab})$ are strictly singlets under the gauge group.

3 R-symmetries and fine-tuning of the action

In ref. [34] it has been shown that the only relevant counterterms that can modify the action through radiative corrections correspond to renormalizations of marginal operators already present in the bare action. Thus the most general form for the renormalized action is

$$S = \int \mathrm{Tr}\, \mathcal{Q}\Big(\alpha_1 \chi_{mn}\mathcal{F}_{mn} + \alpha_2 \eta[\overline{\mathcal{D}}_m, \mathcal{D}_m] - \frac{1}{2}\alpha_3 \eta d\Big) - \frac{1}{4}\alpha_4 \epsilon_{mncde}\chi_{de}\overline{\mathcal{D}}_c\chi_{mn}, \tag{3.1}$$

where α_i with $i = 1, \cdots, 4$ are dimensionless numbers taking values $(1, 1, 1, 1)$ in the classical theory. Indeed one would think that at most three dimensionless ratios of these couplings might need to be tuned in the quantum theory. In this section we show that the discrete R-symmetries provide a powerful constraint on the relations between α_i's since the operators appear through radiative corrections to the action must respect these additional symmetries.

After integrating out the auxiliary field d, we have the form of the action with general coefficients

$$S = \int \mathrm{Tr}\, \Big(-\alpha_1\overline{\mathcal{F}}_{mn}\mathcal{F}_{mn} + \frac{1}{2}\Big(\frac{\alpha_2^2}{\alpha_3}\Big)[\overline{\mathcal{D}}_m, \mathcal{D}_m]^2 - \alpha_1\chi_{mn}\mathcal{D}_{[m}\psi_{n]} - \alpha_2\eta\overline{\mathcal{D}}_m\psi_m$$
$$-\frac{1}{4}\alpha_4\epsilon_{mnrpq}\chi_{pq}\overline{\mathcal{D}}_r\chi_{mn}\Big). \tag{3.2}$$

We have shown that the action is invariant under the R_a transformations. The invariance under any one of the R_a transformations in the bosonic sector yields the following constraint

$$\alpha_1 = \frac{\alpha_2^2}{\alpha_3}. \tag{3.3}$$

In the fermionic sector invariance under any one of the R_a transformation gives another constraint

$$\alpha_1 = \alpha_2 = \alpha_4. \tag{3.4}$$

Combining these two constraints we find

$$\alpha_i = \alpha, \ i = 1, \cdots 4, \tag{3.5}$$

with α an arbitrary parameter, which can be absorbed by redefining the overall gauge coupling of the theory.

A similar set of relations among α_i and the same conclusion as in eq. (3.5) can be obtained from examining the invariance of the action under any one of the R_{ab} interchange symmetries.

Thus we can write the quantum action of the theory as

$$S = \int \text{Tr} \ \alpha \Big(- \overline{\mathcal{F}}_{mn} \mathcal{F}_{mn} + \frac{1}{2} [\overline{\mathcal{D}}_m, \mathcal{D}_m]^2 - \chi_{mn} \mathcal{D}_{[m} \psi_{n]} - \eta \overline{\mathcal{D}}_m \psi_m$$
$$- \frac{1}{4} \epsilon_{mnrpq} \chi_{pq} \overline{\mathcal{D}}_r \chi_{mn} \Big). \tag{3.6}$$

These results imply that invariance under the single scalar supercharge $\mathcal{Q}$ together with invariance under any one of the 15 R_a or R_{ab} symmetries implies that the theory is invariant under *all* additional twisted supersymmetries. We will use this fact to argue later that the supersymmetries of the lattice theory will be enhanced at any point in the parameter space where the R-symmetries of the theory are restored.

It is interesting to note that if SO(4) rotational symmetry (the Euclidean version of Lorentz invariance) is restored in the continuum limit, then the supersymmetry must be enhanced to at least $\mathcal{N} = 1$. In that case it is quite possible that a discrete R-symmetry will be present since $\mathcal{N} = 1$ theories often have a $\text{U}(1)_R$ R-symmetry — and even if it is anomalous a discrete subgroup survives. We would then expect to have one or more of the 15 R_a or R_{ab} symmetries in effect. In that case the full $\mathcal{N} = 4$ supersymmetry emerges because of the above arguments.

4 Ward-Takahashi identities

In this section we illustrate the difficulty that occurs when we attempt to derive the WT identities involving the supercurrent and any local (or non-local) operator by making a change of variables in the functional integral which defines the expectation value of the operator. We will find that the fact that we do not have a fully off-shell formulation presents an obstruction.

Let us consider a composite operator $O(y)$ constructed out of the component fields Φ of the theory. The expectation value of this operator is given by

$$\langle O(y) \rangle = \frac{1}{\mathcal{Z}} \int [d\Phi] e^{-S[\Phi]} O(y), \tag{4.1}$$

where $\mathcal{Z}$ denotes the partition function and $[d\Phi]$ the measure of the functional integral. Now consider some infinitesimal transformation of the fields defined by

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \delta_\kappa \Phi(x), \qquad (4.2)$$

where $\delta_\kappa \Phi(x) = \kappa \Delta \Phi(x)$, with κ a Grassmann parameter and Δ the deformation on the field.

The functional integral itself is independent of any relabeling of integration variables. This gives,

$$
\begin{aligned}
0 &= \langle O'(y) \rangle - \langle O(y) \rangle \\
&= \delta_\kappa \langle O(y) \rangle \\
&= \frac{1}{\mathcal{Z}} \int [d\Phi] \Big(e^{-S[\Phi]} (-\delta_\kappa S[\Phi]) O(y) + e^{-S[\Phi]} \delta_\kappa O(y) \Big) \\
&= -\langle (\delta_\kappa S) O(y) \rangle + \langle \delta_\kappa O(y) \rangle.
\end{aligned}
\qquad (4.3)
$$

That is,

$$\langle (\delta_\kappa S) O(y) \rangle = \langle \delta_\kappa O(y) \rangle. \qquad (4.4)$$

For the symmetry transformation $\delta_\kappa = \kappa_A \mathcal{Q}_A$, with the index A labeling the 16 supercharges, it takes the form

$$\langle (\mathcal{Q}_A S) O(y) \rangle = \langle \mathcal{Q}_A O(y) \rangle. \qquad (4.5)$$

So far, the identity presents no problem. However, in the continuum one would like to argue that $\mathcal{Q}_A S = 0$, and so one obtains the "global" WT identity, $\langle \mathcal{Q}_A O(y) \rangle = 0$. However, $\mathcal{Q}_A S = 0$ requires the use of the equations of motion. These are not valid for the fields under the path integral, since we are not just summing over classical paths. So we have no right to set $\mathcal{Q}_A S = 0$, and so $\langle \mathcal{Q}_A O(y) \rangle = 0$ does not follow. An exception is the scalar supercharge $\mathcal{Q}$: we do have an off-shell formulation, involving d, for which $\mathcal{Q}S = 0$ without using the equations of motion. Thus we find that we can prove the identity $\langle \mathcal{Q}O(y) \rangle = 0$.

Another comment is in order here. While it is true that in the quantum theory we cannot use the classical equations of motion, we can use their quantum analogue, the Schwinger-Dyson equations. Then the equations of motion are satisfied up to contact terms. However $\mathcal{Q}_A S$ involves an integral over all spacetime, and so this will always include the point y, where the operator $O(y)$ sits. Thus we will pick up a nonzero contribution from the contact term when we evaluate $\langle (\mathcal{Q}_A S) O(y) \rangle$. It can be shown that this contact term is precisely the right-hand side, $\langle \mathcal{Q}_A O(y) \rangle$. Thus we end up with a trivial identity that contains no new information.

For the purpose of finding the critical values of the parameters for which fine-tuning is successful, and the SUSY long distance theory emerges, it is more useful to study the "local" WT identities — i.e., the ones that involve the supercurrent. The analogy that one should make is in finding the critical bare mass (equivalently, the critical κ) when using Wilson fermions in lattice QCD. There one studies the axial current WT identities, such as $\langle 0 | \partial_\mu J_{5\mu}^a(x) P^b(0) | 0 \rangle$, where P^b is the pseudoscalar density, and demands that they

vanish. To obtain these sorts of identities, we generalize the transformation of the fields given above to be a spacetime dependent transformation:

$$\Phi(x) \to \Phi'(x) = \Phi(x) + \delta_{\kappa(x)}\Phi(x), \tag{4.6}$$

where $\delta_{\kappa(x)}\Phi(x) = \kappa(x)\Delta\Phi(x)$, with $\kappa(x)$ a spacetime dependent, arbitrary Grassman function. In our twisted theory we have

$$\delta_{\kappa(x)} = \kappa_A(x)\mathcal{Q}_A, \tag{4.7}$$

with A denoting the sixteen possibilities corresponding to $\mathcal{Q}, \mathcal{Q}_a, \mathcal{Q}_{ab}$.

If we make use of the equations of motion, the variation of the action can be brought into the form

$$\delta_\kappa S = \int d^4x \ \partial_m \kappa_A(x) \mathcal{S}_A^m(x), \tag{4.8}$$

where the index m is summed over and $\mathcal{S}_A^m(x)$ is the supercurrent (Noether current) associated with $\mathcal{N} = 4$ SUSY. We know that this is true because if the κ_A are constant, the action is invariant with the equations of motion. Note that the supercurrent is decomposed into scalar, one-form and two-form parts

$$\mathcal{S}_A^m = (\mathcal{S}_0^m, \mathcal{S}_a^m, \mathcal{S}_{ab}^m). \tag{4.9}$$

We write the WT identity relations given in eq. (4.4) as

$$\langle (\delta_\kappa S)O(y) \rangle = \langle \delta_\kappa O(y) \rangle. \tag{4.10}$$

Plugging in the variation of the action,

$$-\left\langle \int d^4z \kappa_A(z)\Big(\partial_m \mathcal{S}_A^m(z)\Big)O(y) \right\rangle = \left\langle \kappa_A(y)\mathcal{Q}_A O(y)\right\rangle,$$

$$-\frac{\delta}{\delta\kappa_A(x)}\left\langle \int d^4z \kappa_A(z)\Big(\partial_m \mathcal{S}_A^m(z)\Big)O(y) \right\rangle = \frac{\delta}{\delta\kappa_A(x)}\left\langle \kappa_A(y)\mathcal{Q}_A O(y)\right\rangle,$$

$$-\left\langle \int d^4z \delta^{(4)}(x-z)\partial_m \mathcal{S}_A^m(z)O(y) \right\rangle = \left\langle \delta^{(4)}(x-y)\mathcal{Q}_A O(y)\right\rangle. \tag{4.11}$$

Thus we would have the supersymmetric WT identities

$$\langle \partial_m \mathcal{S}_A^m(x)O(y) \rangle = -\delta^{(4)}(x-y)\langle \mathcal{Q}_A O(y) \rangle. \tag{4.12}$$

Setting $y = 0$ they take the form

$$\partial_m \langle \mathcal{S}_A^m(x)O(0) \rangle = -\delta^{(4)}(x)\langle \mathcal{Q}_A O(0) \rangle. \tag{4.13}$$

However, note that we made use of the equations of motion in deriving (4.8). This was necessary in order to eliminate terms which do not involve $\partial_m \kappa_A$, but rather just κ_A without derivatives, since the symmetry is only realized on-shell. Inside the path integral in which we derived the WT identity, the fields do not generically lie on classical paths and so this step was not valid. Thus the proof has a fatal flaw and it is not at all clear that the WT identity will be satisfied. However, as above, $\mathcal{Q}$ is realized off-shell, so its corresponding identity will in fact be satisfied. But this is of no use to us in fine-tuning because $\mathcal{Q}$ is exact on the lattice anyhow.

5 Twisted supersymmetries on the lattice

5.1 The lattice action

The twisted continuum theory can be discretized in a straightforward manner. The complex continuum gauge fields $\mathcal{A}_a$, $a = 1, \cdots, 5$, are mapped to complexified Wilson gauge links $\mathcal{U}_a$ living on links $(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a)$ of a four-dimensional abstract hypercubic lattice, with $\mathbf{n}$ denoting the integer valued site on the lattice and $\widehat{\boldsymbol{\mu}}_a$ the basis vector along a direction. Since our theory has five complexified gauge links we need a four-dimensional lattice with five basis vectors. We can take all these basis vectors to be equivalent to ensure that the lattice theory enjoys the maximal symmetry. This requirement imposes a condition on the lattice theory that it must possess an S_5 point group symmetry. It is indeed consistent with the continuum symmetry of the theory as the S_5 point group symmetry is a discrete subgroup of the SO(4) twisted rotational symmetry. A lattice arrangement in which all the five basis vectors are treated equally is called the A_4^* lattice.

The five basis vectors of the A_4^* lattice are

$$
\begin{aligned}
\widehat{\boldsymbol{e}}_1 &= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right), \\
\widehat{\boldsymbol{e}}_2 &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right), \\
\widehat{\boldsymbol{e}}_3 &= \left(0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right), \\
\widehat{\boldsymbol{e}}_4 &= \left(0, 0, -\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{20}} \right), \\
\widehat{\boldsymbol{e}}_5 &= \left(0, 0, 0, -\frac{4}{\sqrt{20}} \right),
\end{aligned}
\tag{5.1}
$$

and they are aligned such that they connect the center of a 4-simplex to its five corners. It should be noted that these basis vectors are related to the weights of the defining representation of SU(5). The point group symmetry S_5 of the lattice action is the Weyl subgroup of SU(5).

We begin with placing the fields of the lattice theory on the links of an abstract hypercubic lattice with integer site $\mathbf{n}$ and basis vectors $\widehat{\boldsymbol{\mu}}_a$:

$$
\begin{aligned}
\widehat{\boldsymbol{\mu}}_1 &= (1, 0, 0, 0), \\
\widehat{\boldsymbol{\mu}}_2 &= (0, 1, 0, 0), \\
\widehat{\boldsymbol{\mu}}_3 &= (0, 0, 1, 0), \\
\widehat{\boldsymbol{\mu}}_4 &= (0, 0, 0, 1), \\
\widehat{\boldsymbol{\mu}}_5 &= (-1, -1, -1, -1),
\end{aligned}
\tag{5.2}
$$

obeying the constraint $\sum_m \widehat{\boldsymbol{\mu}}_m = 0$. These abstract vectors do not correspond to positions in spacetime of sites and links of the A_4^* lattice. They can be related to the physical location in spacetime using the A_4^* basis vectors $\widehat{\boldsymbol{e}}_a$. We relate the integer site $\mathbf{n}$ to the spacetime

location $\mathbf{R}$ in the following way:

$$\mathbf{R} = a \sum_{\nu=1}^{4} (\mu_\nu \cdot \mathbf{n}) \widehat{\boldsymbol{e}}_\nu = a \sum_{\nu=1}^{4} n_\nu \widehat{\boldsymbol{e}}_\nu, \tag{5.3}$$

where a is the lattice spacing. We can also show that a small lattice displacement of the form $d\mathbf{n} = \widehat{\boldsymbol{\mu}}_m$ on the abstract lattice corresponds to a spacetime translation by $(a\widehat{\boldsymbol{e}}_m)$ on the A_4^* lattice:

$$d\mathbf{R} = a \sum_{\nu=1}^{4} (\mu_\nu \cdot d\mathbf{n}) \widehat{\boldsymbol{e}}_\nu = a \sum_{\nu=1}^{4} (\widehat{\boldsymbol{\mu}}_\nu \cdot \widehat{\boldsymbol{\mu}}_m) \widehat{\boldsymbol{e}}_\nu = a\widehat{\boldsymbol{e}}_m. \tag{5.4}$$

We place the twisted fields on the links of the hypercubic lattice the following way. The field $\psi_m(\mathbf{n})$ is the superpartner of $\mathcal{U}_m(\mathbf{n})$ and thus it must also reside on the same link. The field $\overline{\mathcal{U}}_m(\mathbf{n})$ resides on the oppositely oriented link from $\mathbf{n} \to \mathbf{n} - \widehat{\boldsymbol{\mu}}_m$. The ten fermions $\chi_{mn}(\mathbf{n})$ are then chosen to reside on new fermionic links $\mathbf{n} + \widehat{\boldsymbol{\mu}}_m + \widehat{\boldsymbol{\mu}}_n \to \mathbf{n}$ while the singlet fermionic field $\eta(\mathbf{n})$ is assigned to the degenerate link consisting of a single site $\mathbf{n}$. The nilpotent scalar supersymmetry $\mathcal{Q}$ acts on the lattice fields in the following way

$$\mathcal{Q}\mathcal{U}_m(\mathbf{n}) = \psi_m(\mathbf{n}) \tag{5.5}$$

$$\mathcal{Q}\psi_m(\mathbf{n}) = 0 \tag{5.6}$$

$$\mathcal{Q}\overline{\mathcal{U}}_m(\mathbf{n}) = 0 \tag{5.7}$$

$$\mathcal{Q}\chi_{mn}(\mathbf{n}) = -\left(\mathcal{D}_m^{(+)}\mathcal{U}_n(\mathbf{n})\right)^\dagger = -\mathcal{F}_{mn}^\dagger(\mathbf{n}) \tag{5.8}$$

$$\mathcal{Q}\eta(\mathbf{n}) = d(\mathbf{n}) \tag{5.9}$$

$$\mathcal{Q}d(\mathbf{n}) = 0 \tag{5.10}$$

We see that the $\mathcal{Q}$ supersymmetry transforms a bosonic field of one type to a fermionic field of the same type at the same place on the lattice. The complexified field strength takes the following form on the abstract lattice

$$\mathcal{F}_{mn}(\mathbf{n}) = \mathcal{D}_m^{(+)}\mathcal{U}_n(\mathbf{n}) = \mathcal{U}_m(\mathbf{n})\mathcal{U}_n(\mathbf{n} + \widehat{\boldsymbol{\mu}}_m) - \mathcal{U}_n(\mathbf{n})\mathcal{U}_m(\mathbf{n} + \widehat{\boldsymbol{\mu}}_n). \tag{5.11}$$

In the naive continuum limit it reduces to the continuum (complex) field strength and is automatically antisymmetric in the indices.

The link mappings and orientations of the lattice fields are conveniently summarized by giving their gauge transformation properties on the lattice

$$\eta(\mathbf{n}) \to G(\mathbf{n})\eta(\mathbf{n})G^\dagger(\mathbf{n}) \tag{5.12}$$

$$\psi_m(\mathbf{n}) \to G(\mathbf{n})\psi_m(\mathbf{n})G^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_m) \tag{5.13}$$

$$\chi_{mn}(\mathbf{n}) \to G(\mathbf{n} + \widehat{\boldsymbol{\mu}}_m + \widehat{\boldsymbol{\mu}}_n)\chi_{mn}(\mathbf{n})G^\dagger(\mathbf{n}) \tag{5.14}$$

$$\mathcal{U}_m(\mathbf{n}) \to G(\mathbf{n})\mathcal{U}_m(\mathbf{n})G^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_m) \tag{5.15}$$

$$\overline{\mathcal{U}}_m(\mathbf{n}) \to G(\mathbf{n} + \widehat{\boldsymbol{\mu}}_m)\overline{\mathcal{U}}_m(\mathbf{n})G^\dagger(\mathbf{n}) \tag{5.16}$$

where for instance $G \in \mathrm{U}(N)$. The action on the lattice takes the form

$$S = S_{\mathcal{Q}-\text{exact}} + S_{\mathcal{Q}-\text{closed}}, \tag{5.17}$$

where

$$S_{\mathcal{Q}-\text{exact}} = \sum_{\mathbf{n}} \text{Tr} \left(-\mathcal{F}_{mn}^{\dagger}(\mathbf{n})\mathcal{F}_{mn}(\mathbf{n}) + \frac{1}{2}\left(\overline{\mathcal{D}}_m^{(-)}\mathcal{U}_m(\mathbf{n})\right)^2 \right.$$
$$\left. - \chi_{mn}(\mathbf{n})\mathcal{D}_{[m}^{(+)}\psi_{n]}(\mathbf{n}) - \eta(\mathbf{n})\overline{\mathcal{D}}_m^{(-)}\psi_m(\mathbf{n}) \right), \qquad (5.18)$$

and

$$S_{\mathcal{Q}-\text{closed}} = \sum_{\mathbf{n},m,n,c,d,e} -\frac{1}{4}\text{Tr}\,\epsilon_{mncde}\chi_{de}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_m+\widehat{\boldsymbol{\mu}}_n+\widehat{\boldsymbol{\mu}}_c)\overline{\mathcal{D}}_c^{(-)}\chi_{mn}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_c). \qquad (5.19)$$

We see that the terms in the action form closed loops on the lattice ensuring gauge invariance.

The covariant forward and backward difference operators appearing in the lattice action are given by

$$\mathcal{D}_m^{(+)}f(\mathbf{n}) = \mathcal{U}_m(\mathbf{n})f(\mathbf{n}+\widehat{\boldsymbol{\mu}}_m) - f(\mathbf{n})\mathcal{U}_m(\mathbf{n}), \qquad (5.20)$$

$$\mathcal{D}_m^{(+)}f_n(\mathbf{n}) = \mathcal{U}_m(\mathbf{n})f_n(\mathbf{n}+\widehat{\boldsymbol{\mu}}_m) - f_n(\mathbf{n})\mathcal{U}_m(\mathbf{n}+\widehat{\boldsymbol{\mu}}_n), \qquad (5.21)$$

$$\overline{\mathcal{D}}_m^{(-)}f_m(\mathbf{n}) = f_m(\mathbf{n})\overline{\mathcal{U}}_m(\mathbf{n}) - \overline{\mathcal{U}}_m(\mathbf{n}-\widehat{\boldsymbol{\mu}}_m)f_m(\mathbf{n}-\widehat{\boldsymbol{\mu}}_m), \qquad (5.22)$$

$$\overline{\mathcal{D}}_c^{(-)}f_{mn}(\mathbf{n}) = f_{mn}(\mathbf{n})\overline{\mathcal{U}}_c(\mathbf{n}-\widehat{\boldsymbol{\mu}}_c) - \overline{\mathcal{U}}_c(\mathbf{n}+\widehat{\boldsymbol{\mu}}_m+\widehat{\boldsymbol{\mu}}_n-\widehat{\boldsymbol{\mu}}_c)f_{mn}(\mathbf{n}-\widehat{\boldsymbol{\mu}}_c). \qquad (5.23)$$

It can be shown that the $\mathcal{Q}$-closed term satisfies an exact Bianchi identity on the lattice just as for the continuum

$$\epsilon_{mncde}\overline{\mathcal{D}}_c^{(-)}\overline{\mathcal{F}}_{mn}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_c) = 0. \qquad (5.24)$$

5.2 $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supersymmetries on the lattice

If we try to implement eq. (2.65)–(2.70) on the lattice, according to the discretization prescription given above we encounter an immediate problem - the lattice fields connected by $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ are located on distinct links in the lattice and hence transform differently under gauge transformation. Thus these supersymmetries do not commute with gauge transformations. Indeed, the condition that the parameters $(\delta\kappa_0, \delta\kappa_a, \delta\kappa_{ab})$ should be singlets under gauge transformations singles out $\mathcal{Q}$ as the only supercharge that be exactly preserved on the lattice.

Below, we show the supersymmetry transformations associated with $\mathcal{Q}_a$ and $\mathcal{Q}_{ab}$ supercharges. To emphasize the differing link characters of the fields involved we have adopted the expanded notation where a lattice field f_a associated with link $\mathbf{n} \to \mathbf{n}+\widehat{\boldsymbol{\mu}}_a$ is denoted $f_a(\mathbf{n}, \mathbf{n}+\widehat{\boldsymbol{\mu}}_a)$

$$\mathcal{Q}_a\mathcal{U}_b(\mathbf{n}, \mathbf{n}+\widehat{\boldsymbol{\mu}}_b) = \frac{1}{2}\delta_{ab}\eta(\mathbf{n}, \mathbf{n}), \qquad (5.25)$$

$$\mathcal{Q}_a\overline{\mathcal{U}}_b(\mathbf{n}+\widehat{\boldsymbol{\mu}}_b, \mathbf{n}) = -\chi_{ab}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_a+\widehat{\boldsymbol{\mu}}_b, \mathbf{n}), \qquad (5.26)$$

$$\mathcal{Q}_a\psi_b(\mathbf{n}, \mathbf{n}+\widehat{\boldsymbol{\mu}}_b) = \frac{1}{2}\delta_{ab}d_a(\mathbf{n}, \mathbf{n}) + (1-\delta_{ab})(\overline{\mathcal{D}}_a^{(+)}\mathcal{U}_b)(\mathbf{n}, \mathbf{n}+\widehat{\boldsymbol{\mu}}_a+\widehat{\boldsymbol{\mu}}_b), \quad (5.27)$$

$$\mathcal{Q}_a\chi_{bc}(\mathbf{n}+\widehat{\boldsymbol{\mu}}_b+\widehat{\boldsymbol{\mu}}_c, \mathbf{n}) = -\frac{1}{2}\epsilon_{abcgh}(\mathcal{D}_g^{(+)}\mathcal{U}_h)(\mathbf{n}, \mathbf{n}+\widehat{\boldsymbol{\mu}}_g+\widehat{\boldsymbol{\mu}}_h), \qquad (5.28)$$

$$\mathcal{Q}_a\eta(\mathbf{n}, \mathbf{n}) = 0, \qquad (5.29)$$

$$\mathcal{Q}_a d_a(\mathbf{n}, \mathbf{n}) = 0. \qquad (5.30)$$

$$\mathcal{Q}_{ab}\mathcal{U}_c(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_c) = \frac{1}{2}\epsilon_{abcgh}\chi_{gh}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_g + \widehat{\boldsymbol{\mu}}_h, \mathbf{n}), \tag{5.31}$$

$$\mathcal{Q}_{ab}\overline{\mathcal{U}}_c(\mathbf{n} + \widehat{\boldsymbol{\mu}}_c, \mathbf{n}) = \delta_{ac}\psi_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b) - \delta_{bc}\psi_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a), \tag{5.32}$$

$$\mathcal{Q}_{ab}\psi_c(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_c) = \epsilon_{abcgh}(\overline{\mathcal{D}}_g^{(+)}\overline{\mathcal{U}}_h)(\mathbf{n} + \widehat{\boldsymbol{\mu}}_g + \widehat{\boldsymbol{\mu}}_h, \mathbf{n}), \tag{5.33}$$

$$\mathcal{Q}_{ab}\chi_{cd}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_c + \widehat{\boldsymbol{\mu}}_d) = \frac{1}{2}\delta_{ac}\delta_{bd}d_{ab}(\mathbf{n}, \mathbf{n}) - \delta_{bc}(\mathcal{D}_a^{(+)}\overline{\mathcal{U}}_d)(\mathbf{n}, \mathbf{n}) - \delta_{ac}\mathcal{D}_b^{(+)}\overline{\mathcal{U}}_d(\mathbf{n}, \mathbf{n}), \tag{5.34}$$

$$\mathcal{Q}_{ab}\eta(\mathbf{n}, \mathbf{n}) = 2(\mathcal{D}_a^{(+)}\mathcal{U}_b)(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b), \tag{5.35}$$

$$\mathcal{Q}_{ab}d_{ab}(\mathbf{n}, \mathbf{n}) = 0. \tag{5.36}$$

Of course the translation between continuum and lattice expressions is not unique; one can always change the arguments of the lattice fields by terms of order the lattice spacing and add terms which are subleading in the lattice spacing to the supersymmetry transformations. However, this cannot change the essential result that any supersymmetry transformation other than that associated with the scalar $\mathcal{Q}$ will necessarily relate lattice objects of differing link character and hence these supersymmetries will *not* commute with gauge transformations. This has an immediate consequence; the variation of any gauge invariant closed loop under either $\mathcal{Q}_a$ or $\mathcal{Q}_{ab}$ will lead to a gauge variant quantity whose expectation value will vanish by Elitzur's theorem [35]. Thus Ward identities constructed this way will be satisfied trivially because of gauge invariance. Indeed, the only way to construct non-vanishing supersymmetry variations is to start from a non-gauge invariant operator corresponding to an open loop which closes under supersymmetry variation. We have been able to construct only a extremely small number of such operators and they are all of short length. This hints perhaps that the supersymmetric Ward identities associated with the non-scalar twisted supersymmetries supply very few additional constraints on the structure of the quantum theory over and above gauge invariance.

6 Discrete R-symmetries and the lattice theory

In spite of the fact that the lattice action has the same form as the continuum theory, with a straightforward transcription of fields and covariant derivatives, we will show below that it does not enjoy the discrete R-symmetries R_a and R_{ab} that were enumerated above. Thus we cannot draw the same implication as in the continuum, that under renormalization (e.g., in the determination of the low energy effective action under Wilsonian renormalization group flow), the coefficients α_i that were discussed above will all be equal. However, if any of the 15 discrete R-symmetries emerge in the long distance theory, this equality is sufficient to yield the full $\mathcal{N} = 4$ supersymmetry at low energies. Thus apart from a renormalization of the overall coefficient, which is just the inverse gauge coupling squared, no fine-tuning of the lattice action would in this case be required.

There is some reason to hope that the discrete R-symmetries do in fact emerge at low energies. The reason is that the twisting process has combined the SO(4) spacetime symmetry with an SO(4) subgroup of the R-symmetry group. If the SO(4) rotational symmetry of the continuum emerges at low energies, then we also expect to obtain the SO(4) subgroup of R-symmetry, since they are basically on the same footing as far as twisting is

concerned. In that case some of the discrete R-symmetries would also be emergent, which is sufficient to guarantee the equality of coefficients.

To see why the discrete R-symmetries are not invariances of the lattice theory, note that they relate site fields to link fields and vice versa. This breaks the gauge invariance of the lattice theory: closed loops will transform to expressions that are no longer closed loops. As an example consider the following choice for R_a:

$$\eta(\mathbf{n}, \mathbf{n}) \to 2\psi_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a), \tag{6.1}$$

$$\psi_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a) \to \frac{1}{2}\eta(\mathbf{n}, \mathbf{n}), \tag{6.2}$$

$$\psi_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b) \to -\chi_{ab}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b, \mathbf{n}), \tag{6.3}$$

$$\chi_{ab}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b, \mathbf{n}) \to -\psi_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b), \tag{6.4}$$

$$\chi_{bc}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_b + \widehat{\boldsymbol{\mu}}_c, \mathbf{n}) \to \frac{1}{2}\epsilon_{bcagh}\chi_{gh}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_g + \widehat{\boldsymbol{\mu}}_h, \mathbf{n}), \tag{6.5}$$

and

$$\mathcal{U}_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a) \to \mathcal{U}_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a), \quad \mathcal{U}_a^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a, \mathbf{n}) \to \mathcal{U}_a^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a, \mathbf{n}), \tag{6.6}$$

$$\mathcal{U}_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b) \to \mathcal{U}_b^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_b, \mathbf{n}), \quad \mathcal{U}_b^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_b, \mathbf{n}) \to \mathcal{U}_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b), \quad b \neq a. \tag{6.7}$$

Then one of the terms in our lattice action is

$$S_4 = -\alpha_2 \sum_{\mathbf{n}, m} \text{Tr}\, \eta(\mathbf{n}, \mathbf{n})\Big(\psi_m(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_m)\mathcal{U}_m^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_m, \mathbf{n})$$

$$-\mathcal{U}_m^\dagger(\mathbf{n}, \mathbf{n} - \widehat{\boldsymbol{\mu}}_m)\psi_m(\mathbf{n} - \widehat{\boldsymbol{\mu}}_m, \mathbf{n})\Big). \tag{6.8}$$

After exposing the index a

$$S_4 = -\alpha_2 \sum_{\mathbf{n}} \text{Tr}\, \eta(\mathbf{n}, \mathbf{n})\Big(\psi_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a)\mathcal{U}_a^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a, \mathbf{n})$$

$$-\mathcal{U}_a^\dagger(\mathbf{n}, \mathbf{n} - \widehat{\boldsymbol{\mu}}_a)\psi_a(\mathbf{n} - \widehat{\boldsymbol{\mu}}_a, \mathbf{n})\Big)$$

$$-\alpha_2 \sum_{\mathbf{n}, b} \text{Tr}\, \eta(\mathbf{n}, \mathbf{n})\Big(\psi_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b)\mathcal{U}_b^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_b, \mathbf{n})$$

$$-\mathcal{U}_b^\dagger(\mathbf{n}, \mathbf{n} - \widehat{\boldsymbol{\mu}}_b)\psi_b(\mathbf{n} - \widehat{\boldsymbol{\mu}}_b, \mathbf{n})\Big). \tag{6.9}$$

Applying the field interchanges

$$S_4 \to S_4' = -\alpha_2 \sum_{\mathbf{n}} \text{Tr}\, 2\psi_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a)\Big(\frac{1}{2}\eta(\mathbf{n}, \mathbf{n})\mathcal{U}_a^\dagger(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a, \mathbf{n})$$

$$-\mathcal{U}_a^\dagger(\mathbf{n}, \mathbf{n} - \widehat{\boldsymbol{\mu}}_a)\frac{1}{2}\eta(\mathbf{n} - \widehat{\boldsymbol{\mu}}_a, \mathbf{n} - \widehat{\boldsymbol{\mu}}_a)\Big)$$

$$-\alpha_2 \sum_{\mathbf{n}, b} \text{Tr}\, 2\psi_a(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_a)\Big(-\chi_{ab}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a + \widehat{\boldsymbol{\mu}}_b, \mathbf{n})\mathcal{U}_b(\mathbf{n}, \mathbf{n} + \widehat{\boldsymbol{\mu}}_b)$$

$$+\mathcal{U}_b(\mathbf{n} - \widehat{\boldsymbol{\mu}}_b, \mathbf{n})\chi_{ab}(\mathbf{n} + \widehat{\boldsymbol{\mu}}_a, \mathbf{n} - \widehat{\boldsymbol{\mu}}_b)\Big), \tag{6.10}$$

we see that none of the terms closed loops on the lattice, thus breaking gauge invariance. One can make alternate choices for R_a that will render one of the terms a closed loop, but then none of the other terms will be closed loops. By these sorts of arguments it is easy to convince oneself that no definition of R_a or R_{ab} will be symmetries of the lattice theory.

7 Conclusions

In this article we have derived the full set of supersymmetry transformations relevant to the Marcus twist of $\mathcal{N} = 4$ super Yang-Mills. The recent supersymmetric lattice construction of $\mathcal{N} = 4$ super Yang-Mills is based on a discretization of this twisted theory in which just the scalar supersymmetry is preserved in the lattice theory while the other fifteen supersymmetries are broken by lattice effects at a generic point in parameter space. We have shown how to derive the explicit form of these additional supersymmetries by combining the scalar supersymmetry with a series of discrete R-symmetries.

In a previous paper [34] we have given general arguments, based on exact lattice symmetries, that the restoration of these additional supersymmetries may potentially require the tuning of three independent marginal couplings in the classical lattice action. However, in this paper we find yet another remarkable property of the $\mathcal{Q}$-exact twisted formulation: if any one of the 15 discrete R-symmetries are emergent in the infrared theory, the full supersymmetry of the target $\mathcal{N} = 4$ theory is recovered *without any additional fine tuning*. Since Lorentz invariance implies at least $\mathcal{N} = 1$ supersymmetry, it is quite reasonable that a $U(1)_R$ symmetry or a discrete subgroup would emerge in the Lorentz invariant continuum limit. In that case we would expect one or more of the 15 discrete R-symmetries to be present. Furthermore, we have argued that if the SO(4) Euclidean (twisted) rotational invariance is emergent in the low energy theory, as is necessary to obtain a Lorentz invariant continuum limit, then the discrete R-symmetries will most likely be in force. This is because the twisting process treats the SO(4) spacetime symmetry and the SO(4) R-symmetry in an even-handed way, so we expect that restoration of one will imply restoration of the other. However, this is not a rigorous statement. However, if it is true, the crucial question of supersymmetry restoration is determined by whether or not Lorentz invariance is automatic in the continuum limit. Even if it is not true, checking for the restoration of the discrete R-symmetries is quite a bit easier than attempting to check SUSY WIs, because of the issue of operator mixing in the latter case.

Clearly the question of the restoration of the SO(4) spacetime symmetry has straightforward tests in term of measurements of correlation functions. The S_5 symmetry group of the A_4^* lattice goes a long way toward ensuring the desired limit. For instance, the bosonic term that appear in eq. (3.2) will give a Lorentz invariant continuum limit in that sector. Unfortunately, the fermionic terms that appear in eq. (3.2) require $\alpha_1 = \alpha_2 = \alpha_4$ in order for Lorentz invariance to emerge. Thus, a more detailed study of the quantum continuum limit in this theory needs to be performed both perturbatively and nonperturbatively, in order to understand the fate of Lorentz invariance and the discrete R-symmetries. The fermions that we are using are equivalent to staggered fermions, and so restoration of Lorentz invariance would proceed as it does in lattice QCD if we had the full hypercubic

symmetry group. However, S_5 is less restrictive, as is seen from the independent coefficients on the fermionic terms in eq. (3.2).

Thus we see that it is not really a matter of there being *new* operators that break the discrete R-symmetry or Lorentz symmetry, since the operators in eq. (3.2) are the same as in the bare action. As we have just stated, in order to have Lorentz symmetry, the coefficients of all the fermion terms have to be the same, $\alpha_1 = \alpha_2 = \alpha_4$ and to have in addition the discrete R-symmetries, one also requires α_3 to match these coefficients. We are not aware of any mechanism that would enforce equality of these coefficients under renormalization group flow, so generically we expect logarithmic violations of both symmetries. However, whether or not this generic situation (which is the worst case scenario) actually occurs can only be determined by perturbative and/or nonperturbative calculations that need to be performed. In the case of perturbative calculations we have already checked at one-loop in ref. [34] and found equality of the coefficients. To compute the same effects at two loops in lattice perturbation theory is notoriously difficult, because of the integrals which must be evaluated numerically. We think that it will be quite a bit easier to check the discrete R-symmetries by computing correlation functions with Monte Carlo simulations, which is what we are currently planning to do as the next step, based on the results of this paper.

Even if a positive answer to this question is obtained, which would indeed be remarkable, there would still be important items to be done. One thing is that irrelevant operators may break the supersymmetry by $\mathcal{O}(a)$ effects, where a is the lattice spacing. Thus it is still necessary to take the continuum limit. In a theory that is supposed to be conformal at all scales, taking the continuum limit is quite a bit different from in a QCD-like theory where one has asymptotic freedom and dimensional transmutation. That is, vanishing bare coupling does not correspond to the continuum limit in the lattice theory that we are studying. Rather, what one must do is study physics at scales x much larger than the lattice spacing, which amounts to long distance correlation functions. It still needs to be studied how large x must be relative to a in order to obtain the continuum physics. How would we know if we are obtaining the continuum physics, considering that the results are nonperturbative? It is here that it is crucial that we know that the restoration of SO(4) Lorentz invariance and $SU(4)_R$ symmetry, together with the exact $\mathcal{Q}$ supersymmetry, is sufficient. These are properties that can be checked through the measurement of correlation functions, which should be related to each other through the symmetries.

Acknowledgments

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