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Duality in two-dimensional (2,2) supersymmetric non-Abelian gauge theories

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Abstract: We study the low energy behaviour of $\mathcal{N} = (2, 2)$ supersymmetric gauge theories in $1 + 1$ dimensions, with orthogonal and symplectic gauge groups and matters in the fundamental representation. We observe supersymmetry breaking in super-Yang-Mills theory and in theories with small numbers of flavors. For larger numbers of flavors, we discover duality between regular theories with different gauge groups and matter contents, where regularity refers to absence of quantum Coulomb branch. The result is applied to study families of superconformal field theories that can be used for superstring compactifications, with corners corresponding to three-dimensional Calabi-Yau manifolds. This work is motivated by recent development in mathematics concerning equivalences of derived categories.

Keywords: Supersymmetry and Duality, Supersymmetric gauge theory, Duality in Gauge Field Theories, Superstring Vacua

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Contents

1 Introduction

What is the space of all (2, 2) superconformal field theories in $1+1$ dimensions with $c = 9$ and integral R-charges? It is an interesting question in its own right and is also believed to be important. Related questions are: what is the space of all three-dimensional Calabi-Yau manifolds? and What is the space of all string compactifications to $3 + 1$ dimensions?

Linear sigma models provide a useful tool to find (2, 2) superconformal field theories, analyze them, and study their quantum moduli spaces [1]. Investigation so far had been centered on models with Abelian gauge groups. The low energy theory in a typical "phase" has simple interpretation as an orbifold of non-linear sigma model with a superpotential. The geometric phase is related to toric geometry, where well developed techniques are available. Models with non-Abelian gauge groups, on the other hand, had not yet been studied much. One reason is that such models typically have phases where a simple gauge group is unbroken, but the nature of the low energy behaviour of such theories had not been well understood. In [2], David Tong and the author attempted to understand the nature of a class of theories with special unitary gauge groups, and the result is applied to linear sigma models. In the present work, we study more about non-Abelian gauge interaction, with emphasis on orthogonal and symplectic gauge groups, and apply our findings to linear sigma models relevant for string compactifications. We hope that this work will eventually lead us to expand our perspective concerning the above questions.

One problem in theories with simple gauge groups is that there is often a non-compact Coulomb branch which makes it impossible to have a sensible conformal field theory with discrete spectrum in the infra-red limit. Coulomb branch is interesting in its own right, especially in relation to six dimensional theories associated with Neveu-Schwarz fivebranes. However, its presence is problematic in order to obtain theories relevant for string compactifications. This motivates us to look at *regular* theories, where Coulomb branch is lifted by quantum correction.

Let us describe the main results of this paper.

We first describe the result for the $\mathrm{O}(k)$ or $\mathrm{SO}(k)$ gauge theory with N massless fields in the vector representation, $x_1, \ldots, x_N$, with vanishing superpotential ($k = 1, 2, 3, \ldots$ and $N = 0, 1, 2, 3, \ldots,$). For $k \geq 3$, since these groups have $\mathbf{Z}_2$ fundamental group, we need to specify the mod 2 theta angle [3]. Also, an $\mathrm{O}(k)$ theory can be regarded as a $\mathbf{Z}_2$ orbifold of an $\mathrm{SO}(k)$ theory, and there are two possibilities additionally, denoted by $\mathrm{O}_+(k)$ and $\mathrm{O}_-(k)$. The theory with $k \geq 2$ is regular when $N - k$ is odd and the mod 2 theta angle is turned off, or when $N - k$ is even and the mod 2 theta angle is turned on. For $N \leq k-2$, whether regular or not, the theory has no normalizable supersymmetric ground state. That is, the supersymmetry is spontaneously broken. The rest applies only to regular theories. For $N = k-1$, the $\mathrm{SO}(k)$ and $\mathrm{O}_-(k)$ theories flow in the infra-red limit to the free theory of the scalar products, $(x_i x_j) = \sum_{a=1}^k x_i^a x_j^a$, the "mesons". The $\mathrm{O}_+(k)$ theory flows to two copies of such free theory. For $N \geq k$, we propose that there is a duality:

$$\begin{aligned} \mathrm{O}_+(k) &\longleftrightarrow \mathrm{SO}(N-k+1) \\ \mathrm{SO}(k) &\longleftrightarrow \mathrm{O}_+(N-k+1) \\ \mathrm{O}_-(k) &\longleftrightarrow \mathrm{O}_-(N-k+1). \end{aligned} \tag{1.1}$$

The theory with gauge group on the left hand side flows to the same infra-red fixed point as the theory with gauge group on the right hand side with N vectors $\widetilde{x}^1, \ldots, \widetilde{x}^N$ and $\frac{N(N+1)}{2}$ singlets $s_{ij} = s_{ji}$ with the superpotential

$$W = \sum_{i,j=1}^N s_{ij} (\widetilde{x}^i \widetilde{x}^j). \tag{1.2}$$

The mesons in the original theory correspond to the singlets in the dual, $(x_i x_j) = s_{ij}$. The symmetry $\mathbf{Z}_2 = \mathrm{O}(k)/\mathrm{SO}(k)$ in the $\mathrm{SO}(k)$ theory corresponds to the quantum $\mathbf{Z}_2$ symmetry of the dual $\mathrm{O}_+(N-k+1)$ theory (regarded as a $\mathbf{Z}_2$ orbifold). In particular, the "baryons" $[x_{i_1} \cdots x_{i_k}] = \det(x_{i_b}^a)$ in the $\mathrm{SO}(k)$ theory correspond to twist operators in the dual $\mathrm{O}(N-k+1)$ theory. There are similar correspondences between $\mathbf{Z}_2$ symmetries in the other dual pairs. The duality is tested against non-trivial checks, including 't Hooft anomaly matching, flow by complex mass deformation, vacuum counting with twisted mass deformation, comparison of (c,c) and (a,c) chiral rings.

We next describe the results for the $\mathrm{USp}(k)$ gauge theory with N massless fundamentals, $x_1, \ldots, x_N$, with vanishing superpotential ($k = 2, 4, 6, \ldots$ and $N = 0, 1, 2, 3, \ldots$). It is regular if and only if N is odd. For $N \leq k$, there is no normalizable supersymmetric ground state in both regular (N odd) and irregular (N even) theories. For $N = k+1$, the low energy theory is the free conformal field theory of the mesons, $[x_i x_j] = \sum_{a,b=1}^k x_i^a J_{ab} x_j^b$, where J_{ab} is the symplectic structure defining the gauge group. For higher odd $N \geq k+3$, there is a duality:

$$\mathrm{USp}(k) \quad \longleftrightarrow \quad \mathrm{USp}(N-k-1). \tag{1.3}$$

That is, the theory is dual to the $\mathrm{USp}(N-k-1)$ gauge theory with N fundamentals $\widetilde{x}^1, \ldots, \widetilde{x}^N$ and $\frac{N(N-1)}{2}$ singlets $a_{ij} = -a_{ji}$ with the superpotential

$$W = \sum_{i,j=1}^N a_{ij} [\widetilde{x}^i \widetilde{x}^j]. \tag{1.4}$$

The mesons in the original theory correspond to the singlets in the dual, $[x_i x_j] = a_{ij}$. Again, the duality is tested against non-trivial checks.

For completeness, we record the results obtained in [2] for the $\mathrm{SU}(k)$ gauge theory with N massless fundamentals, $x_1, \ldots, x_N$, with vanishing superpotential ($k = 2, 3, 4, \ldots$

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and $N = 0, 1, 2, \ldots$). The theory is regular when there is no k distinct N-th roots of unity that sum to zero. For $N \leq k$, there is no normalizable supersymmetric ground state. For $N = k+1$, the low energy theory is the free conformal field theory of the baryons $[x_{i_1} \cdots x_{i_k}]$. For $N \geq k+2$, there is a duality between regular theories:

$$\mathrm{SU}(k) \quad \longleftrightarrow \quad \mathrm{SU}(N-k). \tag{1.5}$$

That is, the theory is dual to the $\mathrm{SU}(N-k)$ gauge theory with N fundamentals $\widetilde{x}^1, \ldots, \widetilde{x}^N$ and vanishing superpotential. $[x_{i_1} \cdots x_{i_k}] = \epsilon_{i_1 \cdots i_k j_1 \cdots j_{N-k}} [\widetilde{x}^{j_1} \cdots \widetilde{x}^{j_{N-k}}]$ is the relation of the variables. This duality itself was not explicitly stated in [2] but can be proven rather trivially based on the relation $G(k, N) \cong G(N-k, N)$ of Grassmannians, just as in the series of dualities found in [2]. In a way, this is the most fundamental case from which the members in the series follows by addition of superpotential. It would be interesting to study $\mathrm{SU}(k)$ gauge theories with N fundamentals and M anti-fundamentals. We postpone the discussion of such theories for future works.

This pattern looks strikingly similar to the results found in $\mathcal{N} = 1$ gauge theories in $3+1$ dimensions [4–6] and [7, 8]; supersymmetry breaking for low (but non-zero) flavors, quantum deformed moduli space for a "critical" flavor, and duality for supercritical flavors:

$$\begin{aligned} \mathrm{SU}(k) &\longleftrightarrow \mathrm{SU}(N-k) \\ \mathrm{SO}(k) &\longleftrightarrow \mathrm{SO}(N-k+4) \\ \mathrm{USp}(k) &\longleftrightarrow \mathrm{USp}(N-k-4). \end{aligned} \tag{1.6}$$

For special unitary groups, the number of fundamentals must be equal to the number of antifundamentals to avoid the gauge anomaly. For symplectic groups, N must be even to avoid the global anomaly. There is also a similar pattern in $\mathcal{N} = 2$ gauge theories in $2+1$ dimensions [9–14]. In particular, for supercritical flavors, there is a duality between the following gauge groups:

$$\begin{aligned} \mathrm{U}(k) &\longleftrightarrow \mathrm{U}(N-k) \\ \mathrm{O}(k) &\longleftrightarrow \mathrm{O}(N-k+2) \\ \mathrm{USp}(k) &\longleftrightarrow \mathrm{USp}(N-k-2). \end{aligned} \tag{1.7}$$

There are also versions of duality between theories with Chern-Simons terms. The shift in rank for the orthogonal and symplectic groups decreases as $\pm 4, \pm 2, \pm 1$ as the dimension is reduced from four to two. This suggests an interpretation of our duality in terms of brane construction, as the charge of orientifold planes decreases by a factor of 2 for each reduction of dimension by 1.

The present work is motivated by recent development in mathematics concerning equivalences of derived categories (see for example [15] for an introduction). In fact, the direct motivation came from the paper by Hosono and Takagi [16] in which the authors suggested an equivalence of the derived categories of two distinct Calabi-Yau manifolds, X and Y, of dimension three. X is the intersection of five symmetric quadrics in $\mathbf{CP}^4 \times \mathbf{CP}^4$ divided by the exchange involution, while Y is a double cover over the degeneration locus of the

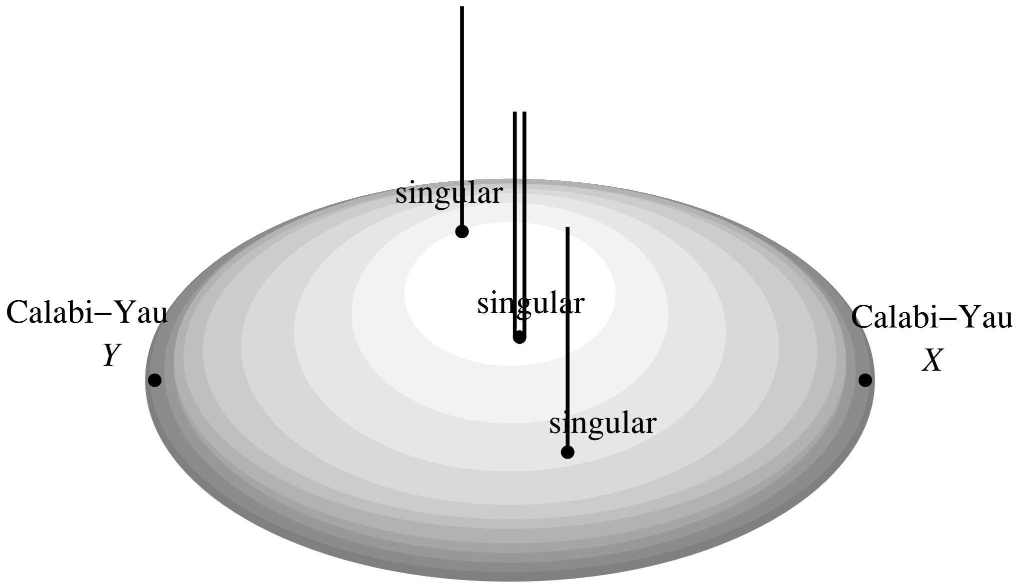

Figure 1. The quantum Kähler moduli space for Hosono-Takagi example.

associated quadratic form on $\mathbf{CP}^4$ (a determinantal quintic in $\mathbf{CP}^4$) which is ramified along a curve of higher degeneration. This suggests existence of a quantum Kähler moduli space of $(2, 2)$ superconformal field theories, which has a corner corresponding to X and another corner corresponding to Y. X naturally leads us to consider a linear sigma model with a non-Abelian gauge group

$$\frac{\mathrm{U}(1) \times \mathrm{O}(2)}{\{(\pm 1, \pm \mathbf{1}_2)\}}. \tag{1.8}$$

In the opposite phase, an O(2) subgroup of the gauge group is entirely unbroken, and we are forced to understand the low energy behaviour of such a quantum gauge theory. We managed to understand it to some extent, and the result tells us that we indeed have a ramified double cover over the determinantal quintic, but does not tell us how to construct it globally. This again forced us to have a better understanding and led us to discover the non-Abelian duality. In the dual model with gauge group $(\mathrm{U}(1) \times \mathrm{SO}(4))/\{(\pm 1, \pm \mathbf{1}_4)\}$, the global ramified double cover Y emerges naturally and completely classically. The dual pair of linear sigma models play complementary rôles at both of the two corners corresponding to X and Y. If the gauge symmetry is unbroken and a non-trivial quantum analysis is necessary in one theory, the gauge symmetry is completely Higgsed and purely classical analysis suffices in the dual theory.

Figure 1 shows the structure of the quantum Kähler moduli space obtained by the dual pair of linear sigma models. There are three singular points at which Coulomb branch emerges. One point is special in that there are *two copies* of a one-dimensional Coulomb branch. Comparing with the result of [16], this seems to be related to having *two* BPS particles that become massless at this point.

The example of Hosono-Takagi can be regarded as a sister of the example of Rødland [17] which was the motivation of the work [2]. In that work, we studied a linear sigma model with gauge group U(2), that is,[1]

$$\frac{\mathrm{U}(1)\times\mathrm{USp}(2)}{\{(\pm 1,\pm\mathbf{1}_2)\}}. \tag{1.9}$$

In this paper, we apply the duality also to this model and find a new picture. We will also apply the duality to linear sigma models relevant for intersection of quadrics, which was studied in mathematics [15, 21–23] as the basic example of non-trivial equivalence of derived categories. This exercise was also useful in refining our understanding of $\mathbf{Z}_2$ orbifolds and that resulted in shaping up the duality in the present form.

2 O(1) theories

In this section, we study several two-dimensional (2, 2) supersymmetric gauge theories with gauge group $\mathrm{O}(1)=\{1,-1\}$, i.e., orbifolds by the group $\mathbf{Z}_2=\mathbf{Z}/2\mathbf{Z}$.

We start with making general remarks concerning the definition of $\mathbf{Z}_2$ orbifolds in supersymmetric field theories.

2.1 General remarks on $\mathbb{Z}_2$ orbifolds

Let us consider a (2, 2) supersymmetric quantum field theory in $1+1$ dimensions with an involutive symmetry τ commuting with the supercharges. We would like to define an orbifold with respect to $\mathbf{Z}_2=\{1,\tau\}$.

For $g,h\in\{1,\tau\}$, we denote by ${}^{g}\underset{h}{\square}$ the g-twisted Witten index on the unprojected h-twisted Ramond-Ramond (RR) sector. It is equal to the partition function on a torus with h-twist in the "space" direction and g-twist in the "time" direction. As usual, it receives contribution only from supersymmetric ground states, and does not depend on the metric of the torus nor on the distinction between the space and time directions. In particular, we have

$$ {}^{\tau}\underset{\tau}{\square} \;=\; {}^{\tau\tau=1}\underset{\tau}{\square} \;=\; {}^{\tau}\underset{1}{\square}. \tag{2.1}$$

This provides constraints and relations concerning the action of τ and $(-1)^F$ on the twisted and untwisted sectors.

As an important example, let us consider a theory in which there is exactly a single supersymmetric ground state in each of the untwisted and the twisted sectors. The first equality (2.1) means that the twisted ground state must be τ invariant and hence survives the orbifold projection. The second equality means that, if τ acts as the sign ε_τ on the untwisted ground state, then the action of $(-1)^F$ on the twisted and untwisted sectors are related by

$$(-1)^F|_\tau=\varepsilon_\tau(-1)^F|_1. \tag{2.2}$$

The untwisted ground state survive the projection if $\varepsilon_\tau=1$ but is projected out if $\varepsilon_\tau=-1$. The sign ε_τ is a part of the data of the orbifold which has such a significant effect.

[1]It is interesting to see that the two simplest non-Abelian groups, USp(2) and O(2), appeared in this way. This reminds us of their rôle in discovering the M-theory lift of orientifold six planes [18–20].

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In general, $\mathbf{Z}_2$ orbifolds of theories with spinors always come in pairs — if we can define an orbifold by a symmetry τ then we can also consider the orbifold by $(-1)^{F_s}\tau$, where $(-1)^{F_s}$ is the operator that acts as the sign flip of all states in the untwisted RR sector. This was emphasized for superconformal field theories in [24, 25]. It follows from the operator product rule

$$\begin{aligned} \mathrm{NSNS}_g \times \mathrm{NSNS}_h &\longrightarrow \mathrm{NSNS}_{gh} \\ \mathrm{NSNS}_g \times \mathrm{RR}_h &\longrightarrow \mathrm{RR}_{gh} \\ \mathrm{RR}_g \times \mathrm{RR}_h &\longrightarrow \mathrm{NSNS}_{gh} \end{aligned} \tag{2.3}$$

that the switch from τ to $(-1)^{F_s}\tau$ does not change the orbifold projection in the untwisted NSNS sector but reverses the one in the twisted NSNS sector. For later use we record the effect:

$$(-1)^{F_s} = \begin{cases} 1 & \text{in untwisted NSNS and twisted RR} \\ -1 & \text{in twisted NSNS and untwisted RR.} \end{cases} \tag{2.4}$$

Dressing by $(-1)^{F_s}$ is different from the discrete torsion, which is absent for the orbifold group $\mathbf{Z}_2$ as $\mathrm{H}^2(\mathbf{Z}_2, \mathrm{U}(1))$ is trivial.

There is a "canonical" choice of orbifold in a class of theories. Let us consider the supersymmetric non-linear sigma model with a target Kähler manifold X. The spectral flow, or A-twist, provides a linear isomorphism between the space of RR ground states and the underlying space of the (a, c) ring, which is a deformation of the de Rham cohomology ring $\mathrm{H}_{\mathrm{dR}}(X, \mathbf{C})$. Suppose X has an involutive holomorphic isometry τ, with which we would like to define an orbifold. In the untwisted sector, there is a canonical choice of orbifold projection: the identity operator must be kept and hence, in the the untwisted (a, c) ring, those corresponding to the τ-invariant cohomology classes must remain. Now, the "canonical" choice would be the one that keeps the linear isomorphism between the untwisted RR ground states and the space of untwisted (a, c) ring elements. The non-canonical one would select the RR ground states corresponding to the anti-invariants in $\mathrm{H}_{\mathrm{dR}}(X, \mathbf{C})$. We shall sometimes denote the orbifold group by $\mathbf{Z}_2(-1)^{F_s}$ for the non-canonical choice. As an important example, let us consider the case where τ is the identity map of X. In this case, the unprojected twisted sector is isomorphic to the space of states of the original sigma model. For either choice of orbifold, the untwisted NSNS and twisted RR sector must survive the projection entirely. The untwisted RR as well as twisted NSNS sector survive entirely for the canonical choice, while they are all projected out for the non-canonical choice. Thus, the canonical orbifold $X/\mathbf{Z}_2$ is isomorphic to the sigma model whose target space is the disjoint union of two copies of X, while the non-canonical one $X/\mathbf{Z}_2(-1)^{F_s}$ is isomorphic to the original sigma model on X.

In the course of the paper, we shall introduce the notion of "canonical" or "standard" $\mathbf{Z}_2$ orbifold for other type of theories. This will also be extended to the definition of gauge theories with $\mathrm{O}(k)$ gauge group, which can be regarded as $\mathbf{Z}_2$ orbifolds of $\mathrm{SO}(k)$ gauge theories.

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2.2 Massive fields

As the first example, we study an orbifold of the theory of massive chiral multiplets with respect to the sign flip. Our main interest will be the spectrum of supersymmetric ground states.

One way to give a mass to a chiral multiplet $(x, \psi_\pm)$ is to introduce a superpotential

$$W = \frac{m}{2}x^2. \tag{2.5}$$

An alternative is *twisted mass* [26] which is given by the following procedure: gauge the phase rotation symmetry of x, give a value $-\widetilde{m}$ to the scalar component of the gauge multiplet, and then turn off the gauge interaction. A superpotential mass shall be called a *complex mass*. Note that a twisted mass is possible only when the phase rotation is a symmetry. In particular, we cannot give both complex and twisted masses at the same time, since the phase rotation is not a symmetry of (2.5). To be more explicit, the complex mass term reads

$$\mathcal{L}_m = -|m|^2|x|^2 - m\psi_+\psi_- - \overline{m}\overline{\psi}_-\overline{\psi}_+, \tag{2.6}$$

while the twisted mass term is

$$\mathcal{L}_{\widetilde{m}} = -|\widetilde{m}|^2|x|^2 - \widetilde{m}\psi_+\overline{\psi}_- - \overline{\widetilde{m}}\psi_-\overline{\psi}_+. \tag{2.7}$$

Let us consider a $\mathbf{Z}_2$ orbifold of the theory of $(x, \psi_\pm)$ with the usual kinetic term plus either of the two mass terms, by the sign flip symmetry,

$$\tau : (x, \psi_\pm) \to (-x, -\psi_\pm). \tag{2.8}$$

At first sight, the two orbifold theories, one with a complex mass and the other with a twisted mass, are isomorphic, as one can switch from one Lagrangian to the other by exchanging ψ_- and $\overline{\psi}_-$. However, we would like to define the orbifolds with respect to *a common τ action on the common space of states.* That is, we define them as two different mass deformations of a given orbifold of a massless chiral multiplet. Then, as we will see, the two are not isomorphic.

Each of the two theories have one untwisted and one twisted supersymmetric ground states before the orbifold projection. Let us compare the ground state wavefunctions in the two theories. There is literally no difference in the dependence on the bosonic field x and hence our focus will be the fermionic fields $\psi_\pm$. The fields, both bosons and fermions, are integer (*resp.* half-integer) moded in the untwisted (*resp.* twisted) RR sector. Let us first focus on the zero modes in the untwisted sector. The (fermionic part of) Hamiltonians of the two systems read

$$H_m = m\psi_{+0}\psi_{-0} + \overline{m}\overline{\psi}_{-0}\overline{\psi}_{+0}, \tag{2.9}$$

$$H_{\widetilde{m}} = \widetilde{m}\psi_{+0}\overline{\psi}_{-0} + \overline{\widetilde{m}}\psi_{-0}\overline{\psi}_{+0}. \tag{2.10}$$

The lowest energy states are respectively

$$|0\rangle_0 + \frac{\overline{m}}{|m|}\overline{\psi}_{+0}\overline{\psi}_{-0}|0\rangle_0, \tag{2.11}$$

$$\overline{\psi}_{+0}|0\rangle_0 + \frac{\widetilde{m}}{|\widetilde{m}|}\overline{\psi}_{-0}|0\rangle_0, \tag{2.12}$$

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where $|0\rangle_0$ is the state annihilated by ψ_{+0} and ψ_{-0}. The state $|0\rangle_{(0)}$ is transformed by τ to itself up to a sign, since the defining property is τ invariant. Therefore, τ transforms the two ground states, (2.11) and (2.12), to themselves but *with opposite signs*, as it flips the sign of $\overline{\psi}_{\pm 0}$. Non-zero modes are decoupled into infinite sectors labeled by the absolute value of the momentum. The lowest energy states in each sector are again different between the two theories but the $\mathbf{Z}_2$ orbifold action on them are the same. Therefore, the $\mathbf{Z}_2$ actions on the untwisted sector RR ground states are opposite between the two theories, while the actions on the twisted sector RR ground states are the same. The same computation can be used to study the $\mathbf{Z}_2$ action on NSNS sector states, where the fermions are half-integer (*resp.* integer) moded in the untwisted (*resp.* twisted) sector. The $\mathbf{Z}_2$ actions on the untwisted sector NSNS ground states are the same between the two theories, while the actions on the twisted sector NSNS ground states are the opposite.

As remarked in the previous subsection, the twisted sector RR ground state must survive the orbifold projection, in each of the two theories. On the other hand, whether the untwisted sector RR ground state survives or not is up to our choice. There are two possibilities (let $|\Omega\rangle_{\rm RR}$ *resp.* $|\widetilde{\Omega}\rangle_{\rm RR}$ be the untwisted ground state of the theory with a complex *resp.* twisted mass): $|\Omega\rangle_{\rm RR}$ is invariant and $|\widetilde{\Omega}\rangle_{\rm RR}$ is anti-invariant, or $|\Omega\rangle_{\rm RR}$ is anti-invariant and $|\widetilde{\Omega}\rangle_{\rm RR}$ is invariant. In the rest of the paper, we shall take the latter as our "standard" convention for the $\mathbf{Z}_2$ orbifold of a chiral multiplet by the sign flip. Namely: $|\Omega\rangle_{\rm RR}$ *is anti-invariant and is projected out, while* $|\widetilde{\Omega}\rangle_{\rm RR}$ *is invariant and survives the projection.* Note that $|\Omega\rangle_{\rm RR}$ and $|\widetilde{\Omega}\rangle_{\rm RR}$ have opposite statistics, as can be seen from (2.11) and (2.12). Let us assume that $|\Omega\rangle_{\rm RR}$ is fermionic, and hence $|\widetilde{\Omega}\rangle_{\rm RR}$ is bosonic. Then, it follows from the general constraint (2.2) that the twisted ground states in the two theories are both bosonic. Therefore, under this assignment, we have

$$\mathrm{Tr}(-1)^F = \begin{cases} 1 & \text{for complex mass} \\ 2 & \text{for twisted mass.} \end{cases} \tag{2.13}$$

What about the NSNS sector? As always, the NSNS ground state in the untwisted sector is invariant and survive the projection in each of the two theories. To determine the action in the twisted sector, we note that the theory with a twisted mass has one twisted and one untwisted supersymmetric ground states. This requires existence of a twist operator in the infra-red limit. On the other hand, we do not need such an operator for the theory with a complex mass. From this we conclude that the twisted NSNS ground state is anti-invariant and is projected out in the theory with complex mass while the one in the theory with twisted mass is invariant and survives the projection.

The definition and the result for theories with several massive multiplets is obtained simply by tensor product: let us consider the orbifold of N fields (i.e. N chiral multiplets) with complex masses and M fields with twisted masses, by the simultaneous sign flip of all the $N+M$ fields. Then it has one supersymmetric ground state from the twisted sector if N is odd, while it has two supersymmetric ground states, one twisted and one untwisted, if N is even. The states are all bosonic, and in particular,

$$\mathrm{Tr}(-1)^F = \begin{cases} 1 & \text{if } N \text{ is odd} \\ 2 & \text{if } N \text{ is even.} \end{cases} \tag{2.14}$$

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This result can be extended to the following periodicity phenomenon. We know that simply adding a massive field to a system does not change the infra-red behaviour. Does it hold also in orbifolds? Suppose we have a $\mathbf{Z}_2$ orbifold of a $2d$ $(2,2)$ supersymmetric quantum field theory $\mathcal{A}$ by its involutive symmetry $\tau_{\mathcal{A}}$. Let us add to $\mathcal{A}$ a single chiral multiplet x with a complex or twisted mass and mod out the combined system by $\tau = (\tau_{\mathcal{A}}, \tau_x)$ where τ_x is the sign flip symmetry considered above. At energies below the mass of x, relevant states in each sector are the tensor product of states of the $\mathcal{A}$ system with the ground state of the x system in that sector. By the transformation property of the ground states learned above, we find that the orbifold projection in this theory is the same as the one for $\mathcal{A}/\tau_{\mathcal{A}}$ if x has a twisted mass. On the other hand, if x has a complex mass, the projection is the same as $\mathcal{A}/\tau_{\mathcal{A}}$ in the untwisted NSNS and twisted RR sectors but is opposite to $\mathcal{A}/\tau_{\mathcal{A}}$ in the twisted NSNS and untwisted RR sectors. This lead us to claim that (at low energies) *the combined orbifold theory is equivalent to the original orbifold $\mathcal{A}/\tau_{\mathcal{A}}$ when x has a twisted mass, while it is equivalent to the other orbifold $\mathcal{A}/(-1)^{F_s}\tau_{\mathcal{A}}$ when x has a complex mass.* A similar conclusion holds if we combine $\mathcal{A}$ with N fields with complex mass and M fields with twisted mass: the combined orbifold system is equivalent at low energies to $\mathcal{A}/\tau_{\mathcal{A}}$ if N is even and to $\mathcal{A}/(-1)^{F_s}\tau_{\mathcal{A}}$ if N is odd.

Embedding into linear sigma models. We now show that the above "standard" choice of $\mathbf{Z}_2$ orbifold appears naturally as a part of linear sigma models. Let us consider a U(1) gauge theory consisting of a field p of charge -2 and fields $x_1, \ldots, x_N$ of charge 1. First let us set $W = 0$ and give no twisted mass. When the Fayet-Iliopoulos (FI) parameter r is negative, the D-term equation $-2|p|^2 + \sum_{i=1}^N |x_i|^2 = r$ requires p to have a non-zero value, breaking the U(1) gauge group to its $\mathbf{Z}_2$ subgroup. In the limit $r \to -\infty$, only x's remain as massless degrees of freedom and we obtain the free orbifold $\mathbf{C}^N/\mathbf{Z}_2$ by the simultaneous sign flip of $x_1, \ldots, x_N$. We study deformations of this linear sigma model that correspond to giving complex and twisted masses to x's in the orbifold.

Let us first consider turning on the superpotential

$$W = p(x_1^2 + \cdots + x_N^2). \tag{2.15}$$

The theory at $r \to -\infty$ is now the Landau-Ginzburg (LG) orbifold of the variables $x_1, \ldots, x_N$ with superpotential $W = x_1^2 + \cdots + x_N^2$ by the simultaneous sign flip. That is, a $\mathbf{Z}_2$ orbifold theory where all N variables have complex masses. The main question is: is it the "standard" one or the other one? To see this, let us analyze this linear sigma model in detail. In the regime $r \gg 0$, x's have non-zero values, breaking the gauge group completely, and the theory reduces to the non-linear sigma model on the quadric hypersurface $Q^{N-2} = \{x_1^2 + \cdots + x_N^2 = 0\}$ of $\mathbf{CP}^{N-1}$. We may also have Coulomb branch vacua. The effective twisted superpotential for the scalar component σ of the U(1) vector multiplet is

$$\widetilde{W}_{\text{eff}} = -(-2\sigma)(\log(-2\sigma) - 1) - N\sigma(\log\sigma - 1) - t\sigma, \tag{2.16}$$

for $t = r - i\theta$ where θ is the theta angle. The Coulomb branch vacua are found by solving $\partial_\sigma \widetilde{W}_{\text{eff}} \equiv 0 \pmod{2\pi i\mathbf{Z}}$, i.e.,

$$\sigma^{N-2} = 4\,\mathrm{e}^{-t}. \tag{2.17}$$

The detail of the theory depends on N. Let us begin with the case $N = 2$, where the axial U(1) R-symmetry is anomaly free and the FI parameter r does not run. We expect that the Witten index is constant if we move r from $r \ll 0$ to $r \gg 0$ as long as we avoid the point $\mathrm{e}^t = 4$ (which supports a non-compact Coulomb branch). At $r \gg 0$, we have the sigma model on the quadric $Q^0 = \{x_1^2 + x_2^2 = 0\}$, which is the set of two points. Its Witten index is of course 2. Hence our LG orbifold at $r \to -\infty$ should also have Witten index 2. Let us next discuss the case $N = 1$, in which the FI parameter runs from negative to positive under the renormalization group (toward longer distances). The theory describes a flow from the LG orbifold. The quadric $Q^{-1} = \{x_1^2 = 0\}$ is empty, but we have a single Coulomb branch vacuum at $\sigma = \mathrm{e}^t/4$. Thus the theory has a unique supersymmetric ground state. Finally, we discuss the case $N > 2$ where the FI parameter runs from positive to negative. The theory is a flow from the sigma model on Q^{N-2} to our LG orbifold or to one of the $N-2$ Coulomb branch vacua. Hodge number $h^{i,j}$ of the quadric Q^{N-2} is (see, for example [21])

$$N \text{ even: } \begin{cases} 1 & i = j \neq \frac{N-2}{2} \\ 2 & i = j = \frac{N-2}{2} \\ 0 & \text{otherwise,} \end{cases} \qquad N \text{ odd: } \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

In particular, the total number of supersymmetric ground states of the sigma model is N for even N and $N-1$ for odd N. Subtracting the number $N-2$ of the Coulomb branch vacua, we obtain 2 for even N and 1 for odd N. To summarize, for all N, the result of the linear sigma model agrees with the result, e.g. (2.14), for our $\mathbf{Z}_2$ orbifold. This means that the $\mathbf{Z}_2$ orbifold that appears at the $r \to -\infty$ limit of the linear sigma model is the "standard" one in our convention.

Next, let us consider another deformation. Instead of turning on the superpotential, we give twisted masses with twisted masses $0, \widetilde{m}_1, \ldots, \widetilde{m}_N$ to $p, x_1, \ldots, x_N$. In the limit $r \to -\infty$, the theory reduces to the $\mathbf{Z}_2$ orbifold where all N fields have twisted masses. In the $r \gg 0$ regime, the classical vacuum equations read

$$\begin{aligned} -2|p|^2 + \sum_{i=1}^{N} |x_i|^2 &= r, \\ \sigma p = (\sigma - \widetilde{m}_1)x_1 &= \cdots = (\sigma - \widetilde{m}_N)x_N = 0, \\ \overline{\sigma} p = (\overline{\sigma} - \overline{\widetilde{m}}_1)x_1 &= \cdots = (\overline{\sigma} - \overline{\widetilde{m}}_N)x_N = 0. \end{aligned} \tag{2.19}$$

If the twisted masses are distinct, there are N solutions at $\sigma = \widetilde{m}_1, \ldots, \widetilde{m}_N$, each of which breaks the gauge symmetry completely. The vacuum equation on the Coulomb branch reads $\partial_\sigma \widetilde{W}_{\text{eff}} = 2\log(-2\sigma) - \sum_{i=1}^{N} \log(\sigma - \widetilde{m}_i) - t \equiv 0$, or

$$4\sigma^2 = \mathrm{e}^t(\sigma - \widetilde{m}_1) \cdots (\sigma - \widetilde{m}_N). \tag{2.20}$$

Let us first analyze the system with $N = 1$, which describes a flow from our $\mathbf{Z}_2$ orbifold. The equation (2.20) has two solutions, both of which go indeed to $\sigma = 0$ in the ultra-violet limit $t \to -\infty$. In the infra-red limit $t \to +\infty$, one solution goes to $\sigma = \widetilde{m}$ and the other solution goes away to infinity as $\sigma \sim \mathrm{e}^t/4$. The former supports the vacuum at $r \gg 0$

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corresponding to the single solution to (2.19), while the latter is a Coulomb branch vacuum. The theory indeed has two bosonic ground states. Next, let us consider the case $N = 2$ where r does not run. In the $r \gg 0$ regime, we observed two classical vacua solving (2.19). Let us check if that is everything. The equation (2.20) has two solutions (except at $\mathrm{e}^t = 4$); at $r \gg 0$ the two solutions are at $\sigma \sim \widetilde{m}_1$ and $\widetilde{m}_2$ and indeed correspond to the two classical vacua, and at $r \to -\infty$ the two solutions both go to $\sigma = 0$ which is the right value for our $\mathbf{Z}_2$ orbifold theory. Therefore, we did not miss anything and can conclude that the Witten index is 2 at any value of t (except $\mathrm{e}^t = 4$) and in particular at $t \to -\infty$. For $N > 2$, the N solutions to (2.20) are at $\sigma \sim \widetilde{m}_i$ in the ultra-violet limit $t \to +\infty$ and indeed correspond to the N classical vacua solving (2.19). In the infra-red limit, two of them go to $\sigma \to 0$ while other $(N-2)$ go away to infinity. This again confirms that our $\mathbf{Z}_2$ orbifold has two bosonic supersymmetric ground states. To summarize, for all N, the result of the linear sigma model agrees with the result for our $\mathbf{Z}_2$ orbifold. This ought to be the case as we have already confirmed that the $\mathbf{Z}_2$ orbifold that appears at $r \to -\infty$ limit of the linear sigma model is the "standard" one in our convention.

2.3 Corank 1 degeneration — branched double cover of $\mathbb{C}$ or its orbifolds

Let us next study the LG orbifold of $(N+1)$ variables, $x_1, \ldots, x_N$ and z, with the superpotential

$$W = zx_1^2 + x_2^2 + \cdots + x_N^2, \tag{2.21}$$

modulo the $\mathbf{Z}_2$ generated by

$$\tau : (z, x_1, \ldots, x_N) \longmapsto (z, -x_1, \ldots, -x_N). \tag{2.22}$$

By the periodicity mentioned earlier, the low energy behaviour of the theory depends only on N mod 2. The superpotential is quadratic in x_i's with coefficients depending on z. At $z \neq 0$ it is non-degenerate, i.e. the Hessian matrix is of maximal rank, but as z approaches 0 the rank goes down by 1. In the region where $|z|$ is large enough, the fields x_i are massive and can be integrated out first. This system of x's, as we have learned, has a single (N odd) or two (N even) massive vacua. Thus, in the region of z away from the origin $z = 0$, we have the theory of the variable z without potential if N is odd. If N is even, we have two copies of the free theory of z, that is, the sigma model whose target space is a double cover of the z-plane. Of course this argument does not tell anything about the behaviour near $z = 0$, which will be the main point of the discussion.

Even N. Let us first study the N even case. We take $N = 2$ for simplicity. We employ a certain deformation of the linear sigma model introduced in the previous subsection: the U(1) gauge theory with four fields of the following charges

$$\begin{array}{cccc} p & x_1 & x_2 & z \\ -2 & 1 & 1 & 0 \end{array} \tag{2.23}$$

with the superpotential

$$W = p(zx_1^2 + x_2^2). \tag{2.24}$$

The vector and axial U(1) R-symmetry exists and the FI parameter r does not run in this theory. In the limit $r \to -\infty$, we recover the $\mathbf{Z}_2$ orbifold under discussion. In the positive r regime, we have a sigma model whose target space is the hypersurface

$$zx_1^2 + x_2^2 = 0 \tag{2.25}$$

in $\mathbf{CP}^1 \times \mathbf{C} = \{([x_1 : x_2], z)\}$. The equation (2.25) and the D-term constraint $-2|p|^2 + |x|^2 = r > 0$ requires that x_1 must be non-zero, and we may use an inhomogeneous coordinate $\widetilde{z} = ix_2/x_1$. The equation (2.25) then reads

$$z = \widetilde{z}^2. \tag{2.26}$$

This means that $\widetilde{z}$ provides a global coordinate of the hypersurface. In particular the hypersurface is the complex plane $\mathbf{C}$ as a complex manifold. The metric is smooth everywhere and has an asymptotic form $\mathrm{d}s^2 \propto |\widetilde{z}\mathrm{d}\widetilde{z}|^2$ as $|\widetilde{z}| \to \infty$, which shows that the Euler density integral is -1, i.e., the curvature is mostly negative. We expect that the metric flattens under the renormalization group, and the theory flows to the free conformal field theory of the variable $\widetilde{z}$. This holds for any large positive values of r and hence for all values of r by the absence of singularity except at a point in the FI-theta parameter space. Therefore we conclude that *the LG orbifold for even N flows in the infra-red limit to the free conformal field theory of a single complex variable $\widetilde{z}$ that is related to z via (2.26).* Note that the $\widetilde{z}$-plane is indeed a double cover of the z-plane in the region away from $z = 0$. It is a branched double cover with the branch point $z = 0$. That there is a branched double cover was also argued in [28] using Berry's phase.

As in any other $\mathbf{Z}_2$ orbifold, our LG orbifold has the quantum $\mathbf{Z}_2$ symmetry. If we take the orbifold by this symmetry, we must get back the LG model before the orbifold (see, for example [27]). In that theory, the (x_1, x_2) system for a given non-zero z has a unique zero energy ground state with a mass gap. Thus, we expect a *single* cover of the z-plane at least away from $z = 0$. This is achieved only when the quantum $\mathbf{Z}_2$ acts on $\widetilde{z}$ as

$$\widetilde{z} \longmapsto -\widetilde{z}. \tag{2.27}$$

In particular, *the variable $\widetilde{z}$ is a twist field of the orbifold theory.*

Unfolding the $\mathbf{Z}_2$. At this occasion, let us discuss more about the LG model before the orbifold. In the absence of orbifolding, the models for all N are equivalent at low energies. Thus, we may assume $N = 1$ where we write $x = x_1$. From what we have just seen, we can say that it is dual to the orbifold of the free theory of $\widetilde{z}$ by (2.27). However, as always, we need to specify the orbifold action in the untwisted RR sector. We claim that it is the non-standard one: *the LG model of variables x and z with superpotential $W = zx^2$ flows in the infra-red limit to the free orbifold conformal field theory $\mathbf{C}/\mathbf{Z}_2(-1)^{F_s}$.* Or equivalently, *it is dual to the $\mathbf{Z}_2$ orbifold of the theory of two variables, $\widetilde{z}$ without mass and $\widetilde{y}$ with a complex mass, by the simultaneous sign flip $(\widetilde{z}, \widetilde{y}) \mapsto (-\widetilde{z}, -\widetilde{y})$.*

This can be derived as follows. We have seen that the LG orbifold $(W = zx_1^2 + x_2^2)/\mathbf{Z}_2$ is dual to the free theory of $\mathbf{C} = \{\widetilde{z}\}$ with the relation $z = \widetilde{z}^2$. Let us add one variable

ζ and perturb the system by the superpotential $\Delta W = \zeta z = \zeta \widetilde{z}^2$. This changes the free theory of $\widetilde{z}$ to the LG model with superpotential $W = \zeta\widetilde{z}^2$. In the LG orbifold side, we have the superpotential $W = zx_1^2 + x_2^2 + z\zeta$. If we integrate out z, we obtain the constraint $\zeta = -x_1^2$ and we are left with the orbifold theory of x_1 and x_2 with superpotential $W = x_2^2$. With the notation change $(\widetilde{z},\zeta) \to (x,z)$ and $(x_1,x_2) \to (i\widetilde{z},\widetilde{y})$, this is the claimed duality.

Let us do some consistency checks. First, let us give a complex mass to $\widetilde{z}$. This changes the dual theory to the LG orbifold $(W = \widetilde{z}^2 + \widetilde{y}^2)/\mathbf{Z}_2$ which has *two* supersymmetric ground states. In the LG side, this corresponds, under $z = \widetilde{z}^2$, to deforming the superpotential to $W = zx^2 + z$. We find two critical points, $(x,z) = (0,i)$ and $(0,-i)$, which means that there are *two* supersymmetric ground states, agreeing with the dual result. Next, let us give a twisted mass to $\widetilde{z}$. The dual orbifold theory, which has one field with a twisted mass and another with a complex mass, has *one* supersymmetric ground state. In the LG side, we give twisted masses associated with the symmetry where z has charge 2 and x has charge -1. The scalar potential is

$$U = |2zx|^2 + \left|x^2\right|^2 + |-\widetilde{m}x|^2 + |2\widetilde{m}z|^2\,. \tag{2.28}$$

It has a classical vacuum at the origin $(x,z) = (0,0)$. Let us see what happens when we turn off the superpotential $W = zx^2$, i.e., turn off the first two terms. The potential still has just one classical vacuum at the origin — the vacuum at the origin before turning off W stays there, and no other vacuum comes in from infinity. Thus, we expect that the number of ground states does not change if we set $W = 0$. We know that the $W = 0$ theory has a unique RR ground state and hence we expect that the number of supersymmetric ground states is *one* in the theory with $W = zx^2$ as well. This is confirmed by an exact analysis in appendix A. We again find that the result matches with the one in the dual. If we had chosen the dual to be the one without the massive field $\widetilde{y}$, we would have faced a problem: the number of ground states would be *one* (*resp.* *two*) if we give a complex (*resp.* twisted) mass to $\widetilde{z}$, which does not match with the LG result.

Odd N. Let us next discuss the N odd case. We take $N = 3$ and employ a chain of duality and standard relations as follows. We denote by $\mathcal{A}_N$ the system of the variables $(z, x_1, \ldots, x_N)$ with the superpotential (2.21) equipped with the symmetry (2.22), by $\mathcal{A}_N/\tau$ its orbifold equipped with the quantum symmetry $\widehat{\tau}$, by $\mathcal{B}$ the system of one massless variable $\widetilde{z}$ and one variable $\widetilde{y}$ with a complex mass equipped with the symmetry $\tau : (\widetilde{z},\widetilde{y}) \to (-\widetilde{z},-\widetilde{y})$, and by $\mathcal{C}$ the system of one massless variable $\widetilde{z}$ equipped with the symmetry $\tau : \widetilde{z} \to -\widetilde{z}$. We denote by $\mathcal{H}^{\rm inv}_{{\rm NSNS}_g}(\mathcal{A})$ *resp.* $\mathcal{H}^{\rm anti}_{{\rm NSNS}_g}(\mathcal{A})$ the space of invariants *resp.* anti-invariants in the g-twisted NSNS sector of a system $\mathcal{A}$, and similarly for the RR sectors. Then, we have the following equalities,

$$\begin{aligned}
\mathcal{H}^{\rm inv}_{{\rm NSNS}1}(\mathcal{A}_3) &= \mathcal{H}^{\rm inv}_{{\rm NSNS}1}(\mathcal{A}_2) = \mathcal{H}^{\rm inv}_{{\rm NSNS}1}(\mathcal{A}_2/\tau) = \mathcal{H}^{\rm inv}_{{\rm NSNS}1}(\mathcal{B}) = \mathcal{H}^{\rm inv}_{{\rm NSNS}1}(\mathcal{C})\\
\mathcal{H}^{\rm inv}_{{\rm NSNS}\tau}(\mathcal{A}_3) &= \mathcal{H}^{\rm anti}_{{\rm NSNS}\tau}(\mathcal{A}_2) = \mathcal{H}^{\rm anti}_{{\rm NSNS}\widehat{\tau}}(\mathcal{A}_2/\tau) = \mathcal{H}^{\rm anti}_{{\rm NSNS}\tau}(\mathcal{B}) = \mathcal{H}^{\rm inv}_{{\rm NSNS}\tau}(\mathcal{C})\\
\mathcal{H}^{\rm inv}_{{\rm RR}1}(\mathcal{A}_3) &= \mathcal{H}^{\rm anti}_{{\rm RR}1}(\mathcal{A}_2) = \mathcal{H}^{\rm inv}_{{\rm RR}\widehat{\tau}}(\mathcal{A}_2/\tau) = \mathcal{H}^{\rm inv}_{{\rm RR}\tau}(\mathcal{B}) = \mathcal{H}^{\rm inv}_{{\rm RR}\tau}(\mathcal{C})\\
\mathcal{H}^{\rm inv}_{{\rm RR}\tau}(\mathcal{A}_3) &= \mathcal{H}^{\rm inv}_{{\rm RR}\tau}(\mathcal{A}_2) = \mathcal{H}^{\rm anti}_{{\rm RR}1}(\mathcal{A}_2/\tau) = \mathcal{H}^{\rm anti}_{{\rm RR}1}(\mathcal{B}) = \mathcal{H}^{\rm inv}_{{\rm RR}1}(\mathcal{C})
\end{aligned} \tag{2.29}$$

The first equality comes from the relation $(\mathcal{A}_3,\tau)\cong(\mathcal{A}_2,(-1)^{F_s}\tau)$. The second equality is the standard relation between an orbifold $(\mathcal{A},\tau)$ and its quantum symmetry orbifold $(\mathcal{A}/\tau,\widehat{\tau})$ (see, e.g. [27]). The third equality follows from the duality found above, $(\mathcal{A}_2/\tau,\widehat{\tau})\cong(\mathcal{B},\tau)$. The fourth equality comes from the relation $(\mathcal{B},\tau)\cong(\mathcal{C},(-1)^{F_s}\tau)$. The conclusion is that *our LG orbifold with $N=3$ (and hence for any odd $N\geq 1$) flows in the infra-red limit to the free orbifold conformal field theory* $\mathbf{C}/\mathbf{Z}_2$ (the standard one). It matches with the expectation that the theory is a free theory of $z=\widetilde{z}^2$ in the region away from $z=0$. Note that the twisted and untwisted sectors are exchanged in the RR sector. This means that the quantum symmetry of the LG orbifold corresponds in the dual orbifold $\mathbf{C}/\mathbf{Z}_2$ to the quantum symmetry combined with the symmetry $(-1)^{\mathbf{F}}$ which is defined by

$$(-1)^{\mathbf{F}}=\begin{cases}1 & \text{in the NSNS sector}\\ -1 & \text{in the RR sector.}\end{cases}\tag{2.30}$$

2.4 Corank 2 degeneration — conifold with $r=0$ and $\theta=\pi$

As the final example in this section, we consider the $\mathbf{Z}_2$ LG orbifold

$$W=ax^2+2cxy+by^2\tag{2.31}$$
$$(x,y,a,b,c)\longmapsto(-x,-y,a,b,c).\tag{2.32}$$

As long as (a,b,c) is away from the degeneration locus

$$ab=c^2,\tag{2.33}$$

the fields (x,y) are massive an can be integrated out: the result of section 2.2 tells us that the sector of (x,y) mod $\mathbf{Z}_2$ has two massive vacua. That is, we have a double cover of the open subset $ab\neq c^2$ of the (a,b,c)-space. Near the degeneration locus (2.33) but away from the origin $(a,b,c)=(0,0,0)$, we may find a coordinate change, $(x,y)\to(x',y')$, so that the superpotential is expressed as

$$W=(c^2-ab)x'^2+y'^2.\tag{2.34}$$

The result of section 2.3 then tells us that the double cover is branched at the locus (2.33), i.e., of the form

$$c^2-ab=d^2.\tag{2.35}$$

The main question is the behaviour of the theory near the origin. Note that the equation (2.35) is the one for the conifold. Thus we expect that the theory is related in some way to that of the conifold. This is also what is observed in [28]. We would now like to know the precise relation to the conformal field theory associated with resolution or deformation of the conifold.

We consider a U(1) gauge theory with six fields of the following charges

$$\begin{array}{cccccc} p & x & y & a & b & c\\ -2 & 1 & 1 & 0 & 0 & 0\end{array}\tag{2.36}$$

with the superpotential

$$W = p(ax^2 + 2cxy + by^2). \tag{2.37}$$

In the $r \to -\infty$ limit, we recover the LG orbifold under question. The theory is singular at the value of $t = r - i\theta$ where there is a non-compact Coulomb branch. The latter exists when $t_{\rm eff} = \partial_\sigma \widetilde{W}_{\rm eff}$ vanishes modulo $2\pi i\mathbf{Z}$, where $\widetilde{W}_{\rm eff}(\sigma) = -(-2\sigma)(\log(-2\sigma)-1) - 2\sigma(\log\sigma - 1) - t\sigma$. The singular point is therefore

$$\mathrm{e}^t = 4. \tag{2.38}$$

In the $r \gg 0$ regime, we have a sigma model whose target space is the hypersurface

$$ax^2 + 2cxy + by^2 = 0 \tag{2.39}$$

in $\mathbf{CP}^1 \times \mathbf{C}^3$ where x, y are the homogeneous coordinates of the first factor $\mathbf{CP}^1$ and (a, b, c) are the coordinates of the second factor $\mathbf{C}^3$. This is indeed a resolved conifold: there are two solutions for (x, y) if (a, b, c) is away from (2.33) and one solution if it is at (2.33) except at the origin $(a, b, c) = (0, 0, 0)$ where arbitrary (x, y) solves the equation. That is, the entire $\mathbf{CP}^1$ sits on the hypersurface at the origin of $\mathbf{C}^3$. More explicitly, if we set

$$d = \begin{cases} ax/y + c & y \neq 0 \\ -by/x - c & x \neq 0, \end{cases} \tag{2.40}$$

then, a, b, c, d satisfy the conifold equation (2.35). Thus, we conclude that the LG orbifold (2.31)–(2.32) belongs to a one parameter family of theories that also includes a large volume limit of the resolved conifold.

At this point, we recall that there is another one parameter family that includes a large volume limit of the resolved conifold — in fact *two* large volume limits. It is obtained from the following U(1) gauge theory with vanishing superpotential with the following matter content:

$$\begin{array}{cccc} u_1 & u_2 & v_1 & v_2 \\ 1 & 1 & -1 & -1 \end{array} \tag{2.41}$$

This one parameter family has one singular point

$$\mathrm{e}^t = 1. \tag{2.42}$$

$r \gg 0$ and $r \ll 0$ are the two large volume regimes. If we set

$$a = u_1 v_1, \quad b = u_2 v_2, \quad c = \frac{u_1 v_2 + u_2 v_1}{2}, \quad d = \frac{u_1 v_2 - u_2 v_1}{2} \tag{2.43}$$

then, a, b, c, d obey the relation (2.35).

Now let us ask whether the LG orbifold (2.31)–(2.32) belongs to the second family. We propose that *it is the theory at* $\mathrm{e}^t = -1$. We give two evidences for this proposal. One is existence of a discrete symmetry. What is special about the theory at $\mathrm{e}^t = -1$ is that it has an extra $\mathbf{Z}_2$ symmetry. Let us consider the transformation

$$(u_1, u_2, v_1, v_2, V) \longmapsto (v_1, v_2, u_1, u_2, -V), \tag{2.44}$$

where V is the vector superfield for the U(1) gauge symmetry. This reverses the FI-theta parameter, $r \to -r$, $\theta \to -\theta$, and hence is a symmetry of the theory only at $(r, \theta) = (0, 0)$ and $(0, \pi)$. But $(r, \theta) = (0, 0)$ is the singular point, see (2.42). Thus, the theory at $(r, \theta) = (0, \pi)$ (i.e. $\mathrm{e}^t = -1$) is the only regular theory that possesses the $\mathbf{Z}_2$ symmetry. Note that it acts on the U(1) invariants (2.43) as

$$a \to a, \quad b \to b, \quad c \to c, \quad d \to -d. \tag{2.45}$$

We propose to identify this $\mathbf{Z}_2$ symmetry with the quantum $\mathbf{Z}_2$ symmetry of the LG orbifold (2.31)–(2.32). Indeed, as discussed in section 2.3, the quantum symmetry acts as the exchange of the two sheets over the (a, b, c) space away from the degeneration locus (2.33). That is, it must exchange the two solutions for d of the equation (2.35), which is nothing but the transformation (2.45).

To provide another evidence, let us discuss the relation between the two families. We write t_1 (*resp.* t_2) for the FI-theta parameter of the first (*resp.* second) family. Recall that the singular points are at $\mathrm{e}^{t_1} = 4$ in the first family and $\mathrm{e}^{t_2} = 1$ at the second family. We first find the relation between t_1 and t_2 by *assuming* the proposal. We expect that the relation is generically one to two, where one value of t_1 corresponds to two values of t_2 related by $t_2 \to -t_2$. This requires the relation of the form $\mathrm{e}^{t_1} = f(\mathrm{e}^{t_2} + \mathrm{e}^{-t_2})$ for some rational function $f(x)$ of degree 1. The relation between the singular points, large volume limits, and the $\mathbf{Z}_2$ symmetric points requires the function $f(x)$ to satisfy $f(2) = 4$, $f(\infty) = \infty$ and $f(-2) = 0$ respectively. This fixes the relation as

$$\mathrm{e}^{t_1} = \mathrm{e}^{t_2} + \mathrm{e}^{-t_2} + 2. \tag{2.46}$$

The main point of the second evidence is that this relation can be supported by the Picard-Fuchs equation for the central charges of B-type D-branes. Recall that the central charge is expressed as the period integral of some differential form for the corresponding A-branes in the mirror system, and satisfies Picard-Fuchs differential equation [29, 30]. The mirror for the second family is known and the equation reads as

$$\frac{\mathrm{d}^2}{\mathrm{d}t_2^2}\Pi = 0. \tag{2.47}$$

The mirror for the first family is not known but the dualization of the charged sector as in [30] leads to the following equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t_1^2}\Pi = \mathrm{e}^{-t_1}\left(2\frac{\mathrm{d}}{\mathrm{d}t_1} - 1\right)2\frac{\mathrm{d}}{\mathrm{d}t_1}\Pi. \tag{2.48}$$

The two equations (2.47) and (2.48) are equivalent provided that t_1 and t_2 are related by (2.46). This is a strong support for the relation (2.46), and in particular the proposed identification of the LG orbifold (2.31)–(2.32) as the theory of the second family at $\mathrm{e}^{t_2} = -1$.

3 O(2) theories

The group O(2) is isomorphic to the semi-direct product

$$\mathrm{O}(2) \cong \mathrm{SO}(2) \rtimes \mathbf{Z}_2, \tag{3.1}$$

where $\mathbf{Z}_2$ is generated by the reflection τ with respect to the first axis,

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2}$$

which acts on $\mathrm{SO}(2) \cong \mathrm{U}(1)$ by the group inversion. By this, an O(2) gauge theory can be regarded as a $\mathbf{Z}_2$ orbifold of a U(1) gauge theory. For example, if the O(2) theory consists of N fields in the fundamental representation (i.e. the doublet, **2**), $x_i = (x_i^1, x_i^2)^T$ ($i = 1, \ldots, N$), then, $u_i = x_i^1 + ix_i^2$ and $v_i = x_i^1 - ix_i^2$ have U(1) charges $+1$ and -1 respectively, and the generator τ of the orbifold group acts on the fields as

$$\tau : (u_1, v_1, \ldots, u_N, v_N, V) \longmapsto (v_1, u_1, \ldots, v_N, u_N, -V). \tag{3.3}$$

To be precise, the U(1) gauge theory is specified only when its FI-theta parameter $t = r - i\theta$ is specified. But (3.3) reverses its sign, $t \to -t$. Therefore, it is a symmetry only at $t = 0$ or $t = \pi i$ (mod $2\pi i\mathbf{Z}$). But one of them is a singular point in the one parameter family of U(1) theories because of the emergence of a non-compact Coulomb branch. By looking at the effective twisted superpotential, we find that the singular point is $t = \pi i N$ (mod $2\pi i\mathbf{Z}$). We decide not to consider such a theory with a non-compact flat direction. Thus, we take the other value

$$t = \pi i(N+1) \qquad \text{mod } 2\pi i\mathbf{Z}. \tag{3.4}$$

This applies whether or not there is a superpotential for the matter fields. When N is odd, we can take $t = 0$ as the tree level FI-theta parameter. When, N is even, we should take $t = \pi i$ (mod $2\pi i\mathbf{Z}$). Alternatively, in the latter case, we may take $t = 0$ but introduce one additional doublet x_{N+1} with a superpotential $W = \frac{m}{2}(x_{N+1}x_{N+1})$, as a regulator to prevent the singularity due to Coulomb branch. This works no matter how the mass m is large.

As always, we need to and will specify the $\mathbf{Z}_2$ orbifold action on the space of states. As an important point, the $\mathbf{Z}_2$ action depends on the choice of "regularizations" for the even N case — (i) setting $t = \pi i$ without introducing x_{N+1} or (ii) setting $t = 0$ while introducing x_{N+1}. Switching from one to the other has an effect of dressing the generator by $(-1)^{F_s}$ as we have learned.

3.1 Yang-Mills theory and QCD with complex mass — supersymmetry breaking

Let us first study the theory without a matter field, i.e., the O(2) "Yang-Mills" theory. According to our definition, it is the orbifold of the Maxwell theory with $r = 0$ and $\theta = \pi$ by the $\mathbf{Z}_2$ that flips the sign of the vector multiplet fields $V = (v_\mu, \sigma, \lambda)$. (Definition using

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a regulator field will be included in the massive QCD below.) We formulate the theory on the circle of length L. Let us first consider the untwisted RR sector, where we impose the periodic boundary condition on all fields. The energy spectrum from the U(1) gauge field v_μ is

$$E_n = \frac{e^2 L}{2}\left(-\frac{1}{2}+n\right)^2, \qquad n \in \mathbf{Z}, \tag{3.5}$$

where e is the gauge coupling. This is interpreted as the energy from the electric field $e^2\left(-\frac{1}{2}+n\right)$, where $-e^2/2$ is the background value associated with $\theta = \pi$ [31] and $e^2 n$ is from the conjugate momentum for the Wilson line. The remaining degrees of freedom, σ and λ, has the usual free massless Lagrangian. Thus the states of the Maxwell theory is decomposed into sectors labeled by $n \in \mathbf{Z}$. The orbifold generator τ must reverse the electric field, $e^2\left(-\frac{1}{2}+n\right) \to -e^2\left(-\frac{1}{2}+n\right)$. Thus, it acts on the sectors as

$$\tau : (n, \sigma, \lambda) \longmapsto (-n+1, -\sigma, -\lambda). \tag{3.6}$$

Note that the sectors are permuted and none of them is invariant. In particular, there is no subtlety concerning the definition of the $\mathbf{Z}_2$ orbifold action. The orbifold theory may be identified as the sum of sectors with the label n running only over

$$n = 0, 1, 2, 3, \ldots. \tag{3.7}$$

All the states have strictly positive energies. Let us next consider the twisted RR sector, where we impose the anti-periodic boundary condition for all fields. Gauss law constraint requires that the electric field is constant. Together with anti-periodicity, this means that the electric field is constantly vanishing. However, this is in conflict with the definition of $\theta = \pi$ as providing the background electric field $e^2/2$ (mod $e^2\mathbf{Z}$). That is, there is no twisted sector for this choice of the theta angle. To summarize, the theory has no zero energy state. In particular the supersymmetry is spontaneously broken.

Let us next consider "QCD" with quarks with complex masses, i.e., the theory with N fundamental matter fields, $x_1, \ldots, x_N$, with a non-degenerate quadratic superpotential, say,

$$W = (x_1 x_1) + \cdots + (x_N x_N). \tag{3.8}$$

Since all fields are massive, they can be integrated out and we are left with the O(2) "Yang-Mills" theory. Note that the theta angle for the U(1) has changed from the ultra-violet value (3.4) to the infra-red value $\theta_{\text{eff}} = \pi$. The treatment of the untwisted RR sector is as in the Yang-Mills theory. In particular, there is no zero energy state there. Unlike in the pure Yang-Mills, the theory does have a twisted RR sector. Gauss law constraint now takes the form

$$\partial_1\left(\frac{1}{e^2}F_{01}\right) = j^0 \tag{3.9}$$

where j^0 is the charge density. We may have a configuration as depicted in figure 2 that is compatible with Gauss law constraint, the anti-periodic boundary condition, and $\theta_{\text{eff}} = \pi$. All of such configurations have strictly positive energies. Hence, the supersymmetry is spontaneously broken.

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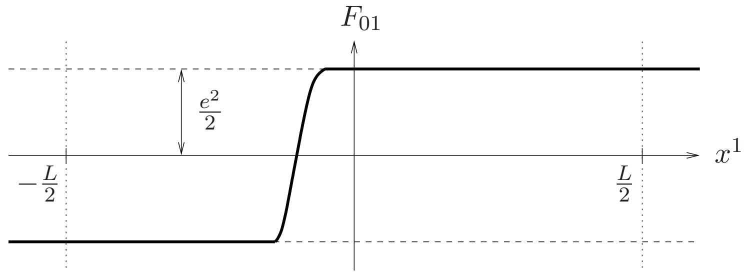

Figure 2. A consistent profile of the electric field in the twisted sector.

3.2 Twisted mass — defining the $\mathbb{Z}_2$ orbifold

Let us next discuss the O(2) gauge theory with N doublets $x_1, \dots, x_N$ now with twisted masses $\widetilde{m}_1, \dots, \widetilde{m}_N$. Our focus will be the spectrum of supersymmetric ground states. Through the course of the analysis, we specify the precise definition of the $\mathbf{Z}_2$ orbifold (for the massless theory as well).

The classical vacuum equation reads as follows:

$$\begin{aligned} \sum_{i=1}^{N} |u_i|^2 &= \sum_{i=1}^{N} |v_i|^2, \\ (\sigma - \widetilde{m}_i)u_i &= (-\sigma - \widetilde{m}_i)v_i = 0, \quad \forall i, \\ (\overline{\sigma} - \overline{\widetilde{m}}_i)u_i &= (-\overline{\sigma} - \overline{\widetilde{m}}_i)v_i = 0, \quad \forall i. \end{aligned} \tag{3.10}$$

We choose $\widetilde{m}_i$ to be generic. In particular, we assume $\widetilde{m}_i + \widetilde{m}_j \neq 0$ for all i and j. Then, there is no value of σ at which both u and v can be non-zero, and hence $u = v = 0$ is enforced by the first equation. In particular, the doublets are all massive at every value of σ. Hence we can integrate them out and study the effective theory for the vector multiplet. The effective twisted superpotential for σ is

$$\begin{aligned} \widetilde{W}_{\text{eff}} = &-\sum_{i=1}^{N}(\sigma - \widetilde{m}_i)(\log(\sigma - \widetilde{m}_i) - 1) - \sum_{i=1}^{N}(-\sigma - \widetilde{m}_i)(\log(-\sigma - \widetilde{m}_i) - 1) \\ &+\pi i(N+1)\sigma, \end{aligned} \tag{3.11}$$

and the vacuum equation reads

$$\prod_{i=1}^{N}(\sigma - \widetilde{m}_i) = (-1)^{N+1}\prod_{i=1}^{N}(-\sigma - \widetilde{m}_i). \tag{3.12}$$

The equation is of order N and is symmetric under the $\mathbf{Z}_2$ orbifold action

$$\sigma \longmapsto -\sigma. \tag{3.13}$$

If $\widetilde{m}_i$ are generic and in particular $\widetilde{m}_i + \widetilde{m}_j \neq 0$, the solutions are distinct and are away from the forbidden region $\sigma = \pm\widetilde{m}_i$ where (3.11) cannot be trusted.

When N is even, there are $\frac{N}{2}$ pairs of non-zero solutions. Since these solutions break the $\mathbf{Z}_2$ orbifold symmetry (3.13), there no need to consider the twisted sector, nor enters the subtlety of defining the $\mathbf{Z}_2$ orbifold. Thus, quite simply, there are $\frac{N}{2}$ supersymmetric ground states.

When N is odd, there are $\frac{N-1}{2}$ pairs of non-zero solutions, and one solution at $\sigma = 0$ at which the orbifold group is unbroken. There are $\frac{N-1}{2}$ supersymmetric ground states from the $\mathbf{Z}_2$ breaking solutions. The main issue is the spectrum at the $\mathbf{Z}_2$ symmetric solution $\sigma = 0$. The effective superpotential near that solution is

$$\widetilde{W}_{\text{eff}} = \left(\frac{1}{\widetilde{m}_1} + \cdots + \frac{1}{\widetilde{m}_N}\right)\sigma^2 + \cdots \tag{3.14}$$

where the ellipses stand for a constant and higher order terms. Even though we are considering the U(1) gauge multiplet, we can effectively treat the system as the $\mathbf{Z}_2$ orbifold of just a single twisted chiral multiplet of this superpotential. Before the orbifold projection there are two supersymmetric ground states, one twisted and one untwisted. As always, the one in the twisted sector is invariant and survives the orbifold projection. On the other hand, we must make a choice concerning the projection in the untwisted sector. We shall call the orbifold "standard" if the untwisted RR ground state is invariant and survives and "non-standard" if it is anti-invariant and is projected out. We shall denote the corresponding gauge group $\mathrm{O}_+(2)$ for the standard orbifold and $\mathrm{O}_-(2)$ for the non-standard one. Under this definition, the total number of ground states is

$$\begin{cases} \dfrac{N-1}{2} + 2 = \dfrac{N+3}{2} & \text{in } \mathrm{O}_+(2) \text{ theory} \\ \dfrac{N-1}{2} + 1 = \dfrac{N+1}{2} & \text{in } \mathrm{O}_-(2) \text{ theory.} \end{cases} \tag{3.15}$$

This definition of orbifolds for odd N can be extended, by continuity, to the theories where the twisted masses are turned off and then the superpotential is turned on. In particular, we have a definition for the theory with even N plus an additional doublet with a complex mass (the "regulator").

Embedding into linear sigma models. This definition can be compared with a geometrical setting where the notion of canonical or non-canonical orbifolds already exists (see section 2.1). This is done via a linear sigma model. Let us consider for odd N the theory with gauge group $(\mathrm{U}(1) \times \mathrm{O}(2))/\{(\pm 1, \pm \mathbf{1}_2)\}$ consisting of a field p in the representation $(-2, \mathbf{1})$ with zero twisted mass and fields $x_1, \ldots, x_N$ in the representation $(1, \mathbf{2})$ with twisted masses $\widetilde{m}_1, \ldots, \widetilde{m}_N$. In the regime where the FI parameter $r_{\mathrm{U}(1)}$ for the U(1) factor is negative, the field p must have a non-zero value and breaks the U(1) to $\{\pm 1\}$, that is, breaks the gauge group to simply O(2). In the limit $r_{\mathrm{U}(1)} \to -\infty$, the theory reduces to the O(2) gauge theory we are discussing. We would like to study this linear sigma model and in particular look what we have at the other regime $r_{\mathrm{U}(1)} \gg 0$.

We can write the gauge group as $(\mathrm{U}(1) \times \mathrm{U}(1))/\{(\pm 1, \pm 1)\} \rtimes \mathbf{Z}_2$. The FI-theta parameter of the first U(1) is unconstrained but the one for the second U(1) must be zero since we have an odd number of doublets. We shall reparametrize the continuous part of the group as

$$\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{\{(\pm 1,\pm 1)\}} \cong \mathrm{U}(1)_1\times\mathrm{U}(1)_2. \qquad (3.16)$$
$$[(g,h)] \longmapsto (gh, gh^{-1})$$

The FI-theta parameters must be equal between $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ and are denoted by $t = r - i\theta$; r is a free parameter of the theory when $N = 1$ while it runs from positive to negative when $N \geq 2$. The matter fields are p of charge $(-1,-1)$, u_i's of charge $(1,0)$ and v_i's of charge $(0,1)$ with respect to $\mathrm{U}(1)_1\times\mathrm{U}(1)_2$. The symmetry τ acts as the exchange of $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ as well as u's and v's.

At $r \gg 0$, the classical vacuum equations for the scalar fields are

$$\begin{aligned} -|p|^2+|u|^2 &= -|p|^2+|v|^2 = r, \\ (\sigma_1-\widetilde{m}_i)u_i &= (\sigma_2-\widetilde{m}_i)v_i = 0 \ \ \forall i, \quad (\sigma_1+\sigma_2)p = 0, \qquad (3.17)\\ (\overline{\sigma}_1-\overline{\widetilde{m}}_i)u_i &= (\overline{\sigma}_2-\overline{\widetilde{m}}_i)v_i = 0 \ \ \forall i, \quad (\overline{\sigma}_1+\overline{\sigma}_2)p = 0, \qquad (3.18)\end{aligned}$$

where σ_1 and σ_2 are the scalar components of the vector multiplets for $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$. If $\widetilde{m}_i$ are distinct, there are N^2 solutions: $(\sigma_1,\sigma_2) = (\widetilde{m}_{i_1},\widetilde{m}_{i_2})$ with $|u_{i_1}|^2 = |v_{i_2}|^2 = r$ and all other u's and v's and p are zero. Note that the $\mathbf{Z}_2$ orbifold group is broken at the N^2-N solution with $i_1 \neq i_2$ while unbroken at the N solutions with $i_1 = i_2$. Each pair of broken solutions yields one supersymmetric ground state while each unbroken solution yields two vacua (one twisted and one untwisted) for the canonical orbifold and one twisted vacuum for the non-canonical one, in the sense of section 2.1. Thus, the total number of vacua is

$$\begin{cases} \dfrac{N^2-N}{2} + 2N & \text{for the canonical orbifold,} \\ \dfrac{N^2-N}{2} + N & \text{for the non-canonical orbifold.} \end{cases} \qquad (3.19)$$

If $\widetilde{m}_i$ are not distinct, say all equal (write it $\widetilde{m}$), then, there is a continuum of solutions at $\sigma_1 = \sigma_2 = \widetilde{m}$ with $|u|^2 = |v|^2 = r$ and $p = 0$. The solution space is $\mathbf{CP}^{N-1}\times\mathbf{CP}^{N-1}$ on which the $\mathbf{Z}_2$ orbifold group acts as the exchange of the two $\mathbf{CP}^{N-1}$ factors. The supersymmetric ground states in the untwisted sector are in one to one correspondence with invariant *resp.* anti-invariant cohomology classes of $\mathbf{CP}^{N-1}\times\mathbf{CP}^{N-1}$ for the canonical *resp.* non-canonical orbifold. If H_1 and H_2 denote the hyperplane classes of the first and the second $\mathbf{CP}^{N-1}$ factor, invariant *resp.* anti-invariant cohomology classes are of the form $H_1^iH_2^j + H_1^jH_2^i$ *resp.* $H_1^iH_2^j - H_1^jH_2^i$, with $0 \leq i,j \leq N-1$. An elementary count finds that the total number of such classes is $\frac{N^2+N}{2}$ *resp.* $\frac{N^2-N}{2}$. On the other hand, the twisted sector ground states are in one to one correspondence with the cohomology classes of the diagonal $\mathbf{CP}^{N-1}$ (there are N of them), for both orbifolds. Thus, the total number of ground states is

$$\begin{cases} \dfrac{N^2+N}{2} + N & \text{for the canonical orbifold,} \\ \dfrac{N^2-N}{2} + N & \text{for the non-canonical orbifold.} \end{cases} \qquad (3.20)$$

The result of course matches with (3.19), including the separation into twisted and untwisted sectors.

In order to compare this result with the O(2) gauge theory at $r \to -\infty$, let us look into the Coulomb branch vacua. The effective twisted superpotential is $\widetilde{W}_{\rm eff} = -(-\sigma_1 - \sigma_2)(\log(-\sigma_1 - \sigma_2) - 1) - \sum_{a,i}(\sigma_a - \widetilde{m}_i)(\log(\sigma_a - \widetilde{m}_i) - 1) - t(\sigma_1 + \sigma_2)$, and the extremum equation reads,

$$\prod_{i=1}^{N}(\sigma_1 - \widetilde{m}_i) = \prod_{i=1}^{N}(\sigma_2 - \widetilde{m}_i) = -\,\mathrm{e}^{-t}(\sigma_1 + \sigma_2). \tag{3.21}$$

The are N^2 solutions — N of them have $\sigma_1 = \sigma_2$ ($\mathbf{Z}_2$ preserving) and $N^2 - N$ of them have $\sigma_1 \neq \sigma_2$ ($\mathbf{Z}_2$ breaking). In the limit $r \to +\infty$, the solutions behave as $\sigma_1 \to \widetilde{m}_{i_1}$ and $\sigma_2 \to \widetilde{m}_{i_2}$ for some i_1 and i_2. Thus, they all correspond to the classical vacua at $r \gg 0$ studied above. At $r \to -\infty$, some of the solutions have $\sigma_1 + \sigma_2 \to 0$ and correspond to vacua of the O(2) gauge theory under consideration, while the others have divergent values of $\sigma_1 + \sigma_2$ and have nothing to do with the O(2) gauge theory. Among the $\mathbf{Z}_2$ preserving solutions, one of them goes to $(0, 0)$ as $\sigma_1 = \sigma_2 \sim -\frac{1}{2}\,\mathrm{e}^t \prod_{i=1}^N(-\widetilde{m}_i)$, while the other $N - 1$ diverge as $\sigma_a^{N-1} \sim -2\,\mathrm{e}^{-t}$. Among the $\mathbf{Z}_2$ breaking solutions, $(N-1)^2$ of them are divergent as $\sigma_2 \sim \omega\sigma_1$ ($\omega^N = 1, \omega \neq 1$) and $\sigma_1^{N-1} \sim -\,\mathrm{e}^{-t}(1+\omega)$, while the rest ($N - 1$ solutions) are finite and hence has $\sigma_1 + \sigma_2 \to 0$ in the limit $r \to -\infty$. Among the ground states in (3.19) or (3.20), those from the solutions with $\sigma_1 + \sigma_2 \to 0$ in the limit $r \to -\infty$ are

$$\begin{cases} \dfrac{N-1}{2} + 2 & \text{for the canonical orbifold,} \\ \dfrac{N-1}{2} + 1 & \text{for the non-canonical orbifold.} \end{cases} \tag{3.22}$$

This is in perfect agreement with (3.15), provided that the "standard" $\mathrm{O}_+(2)$ theory corresponds to the canonical orbifold in the geometric setting at $r \gg 0$ and "non-standard" $\mathrm{O}_-(2)$ theory corresponds to the non-canonical orbifold.

3.3 Massless QCD — a dual description

Single flavor. Next, let us consider the massless "QCD" with one flavor, i.e., the theory with a single fundamental matter field $x = (x^1, x^2)^T$ with no superpotential nor twisted mass. It is a $\mathbf{Z}_2$ orbifold of the U(1) gauge theory with $r = \theta = 0$ consisting of fields u, v of charges 1, -1. We have a single generator of the O(2) invariants

$$a = (xx) := (x^1)^2 + (x^2)^2 = uv, \tag{3.23}$$

which is also the generator of the U(1) invariants.

Let us first study the U(1) theory before the $\mathbf{Z}_2$ orbifold. It has no Coulomb branch since the effective potential at large $|\sigma|$ is $e^2\pi^2/2$ by the one loop theta angle. Let us study the Higgs branch. The D-term equation reads

$$|u|^2 = |v|^2. \tag{3.24}$$

We find a one-dimensional Higgs branch, parametrized by the invariant $a = uv$, whose metric is classically of the form $\mathrm{d}s^2 = |\mathrm{d}a|^2/|a|$. It is the cone $\mathbf{C}/\mathbf{Z}_2$ and has a conical

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singularity at the origin $a = 0$. The singularity appears because the metric is obtained by integrating out the gauge field which is classically massless at $a = 0$. However, the Coulomb branch is lifted by the non-zero value of the effective theta angle and the gauge field may not be massless at $a = 0$. We propose from this that the conical singularity is smeared in the quantum theory and that the theory flows in the infra-red limit to the free conformal field theory of the single complex variable a. That is, the sigma model whose target space is the complex a-plane $\mathbf{C}$. Relevant mathematical fact is that the ring of $\mathbf{C}^\times$-invariant polynomials of u and v is generated by the invariant $a = uv$, which obeys no relation,

$$\mathbf{C}[u,v]^{\mathbf{C}^\times} = \mathbf{C}[a]. \tag{3.25}$$

The situation is quite similar to the case of SU(k) QCD with $(k + 1)$ flavors [2], where it was argued that it is the free theory of baryon variables. Relevant mathematical fact there was again that the the ring of SL($k,\mathbf{C}$)-invariant polynomials of $(k + 1)$ fundamentals is the polynomial ring (with no relation) of the baryonic variables [32].

Let us come back to the O(2) theory, which is obtained by taking a $\mathbf{Z}_2$ orbifold of the U(1) theory. In the effective description obtained above, the symmetry τ acts trivially on the variable a. Thus, the orbifold is either the sigma model on the disjoint union of two copies of $\mathbf{C} = \{a\}$ or $\mathbf{C} = \{a\}$ itself (see section 2.1). We claim that our standard choice yields the former and the non-standard one the latter,

$$\begin{aligned} &\mathrm{O}_+(2) \textit{ with one massless doublet} \longrightarrow \mathbf{C} \sqcup \mathbf{C} \\ &\mathrm{O}_-(2) \textit{ with one massless doublet} \longrightarrow \quad \mathbf{C}. \end{aligned} \tag{3.26}$$

This can be shown by the embedding into the linear sigma model discussed in section 3.2: if we set $\widetilde{m}_1 = 0$ in the $N = 1$ model, then the $r \gg 0$ theory is the $\mathbf{Z}_2$ orbifold of the sigma model whose target space is the total space of the line bundle $\mathcal{O}(-1,-1)$ over "$\mathbf{CP}^0 \times \mathbf{CP}^0$" (i.e. a complex line over a point), where the orbifold acts trivially. The $\mathrm{O}_+(2)$ theory corresponds, as we have learned, to the canonical orbifold and hence to two copies of the line $\mathbf{C}$ while the $\mathrm{O}_-(2)$ corresponds to the non-canonical one and to a single copy of $\mathbf{C}$.

Two flavors. Let us next consider the massless "QCD" with two flavors, i.e., the theory with two fundamentals, $x = (x^1, x^2)^T$ and $y = (y^1, y^2)^T$, having no superpotential nor twisted mass. It is defined as a $\mathbf{Z}_2$ orbifold of the U(1) gauge theory with $r = 0$ and $\theta = \pi$ consisting of fields u_1, u_2, v_1, v_2 of charge $1, 1, -1, -1$. The theory before the orbifold is something which we have already encountered in the present paper — it is the theory which is identified with the $\mathbf{Z}_2$ LG orbifold (2.31)–(2.32) discussed in section 2.4. Furthermore, the $\mathbf{Z}_2$ orbifold symmmetry is identified as the quantum symmetry of the $\mathbf{Z}_2$ LG orbifold. To be precise, there are two distinct orbifolds, corresponding to $\mathrm{O}_+(2)$ or $\mathrm{O}_-(2)$, and only one of them has that property. Without identifying which is the one, let us proceed for now assuming that the $\mathbf{Z}_2$ orbifold symmetry does correspond to the quantum $\mathbf{Z}_2$ of the LG orbifold (2.31)–(2.32). Then, the orbifold theory is the LG model before the orbifold i.e., (changing the notation to avoid possible confusion) the LG model of five variables

$\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}$ with the superpotential

$$W = \widetilde{a}\widetilde{x}^2 + 2\widetilde{c}\widetilde{x}\widetilde{y} + \widetilde{b}\widetilde{y}^2. \tag{3.27}$$

The relation to the O(2) invariants (xx), (xy), (yy) are

$$\widetilde{a} = u_1 v_1 = (xx), \qquad \widetilde{b} = u_2 v_2 = (yy), \qquad \widetilde{c} = \frac{u_1 v_2 + u_2 v_1}{2} = (xy). \tag{3.28}$$

The quantum $\mathbf{Z}_2$ symmetry (for the orbifold by the $\mathbf{Z}_2 = \mathrm{O}(2)/\mathrm{SO}(2)$) acts on the dual variables as

$$(\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}) \longmapsto (-\widetilde{x}, -\widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}). \tag{3.29}$$

This is because the quantum symmetry of the quantum symmetry orbifold is the original orbifold symmetry, which was (2.32) in the previous notation. In particular, the variables $\widetilde{x}$ and $\widetilde{y}$ are twist fields with respect to the $\mathbf{Z}_2 = \mathrm{O}(2)/\mathrm{SO}(2)$.

To find which of $\mathrm{O}_+(2)$ or $\mathrm{O}_-(2)$ we are discussing, let us perturb the system by giving a mass m to one of the two fundamentals, say y, by the tree level superpotential $W = m(yy)$. In the dual theory, this corresponds to deforming the superpotential to

$$W = \widetilde{a}\widetilde{x}^2 + 2\widetilde{c}\widetilde{x}\widetilde{y} + \widetilde{b}\widetilde{y}^2 + m\widetilde{b}. \tag{3.30}$$

If we integrate out $\widetilde{b}$, then we obtain the constraint $\widetilde{y}^2 + m = 0$, which has two solutions $\widetilde{y} = \pm i\sqrt{m}$. For each of them, plugging the value back to (3.30), we may integrate out $\widetilde{c}$ yielding the constraint $\widetilde{x} = 0$, which leaves us with the free theory of the single variable $\widetilde{a}$ that corresponds to the invariant (xx). That is, after the mass perturbation, we obtain the sigma model whose target space is two copies of $\mathbf{C}$. In view of (3.26), we see that this result is consistent if our theory was the $\mathrm{O}_+(2)$ theory, i.e., the $\mathrm{O}_+(2)$ theory with two massless doublets, x and y, and one regulator doublet x_3 with a complex mass. Indeed, if that is the case, after the mass perturbation, we have one massless doublet x and two doublets y and x_3 with complex masses. Addition of two fields with complex masses have no effect whatsoever, both in the theta angle and in the $\mathbf{Z}_2$ orbifold action. Thus, we are left with the $\mathrm{O}_+(2)$ theory with a single massless doublet x, whose low energy theory is indeed two copies of the free theory of the singlet (xx).

To summarize, we found that *the $\mathrm{O}_+(2)$ theory with two massless doublets x and y is dual to the Landau-Ginzburg model of five variables $\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}$ with the superpotential (3.27). $\widetilde{x}$ and $\widetilde{y}$ are twist fields with respect to the $\mathbf{Z}_2 = \mathrm{O}_+(2)/\mathrm{SO}(2)$ and the other variables are the gauge invariant composites, $\widetilde{a} = (xx)$, $\widetilde{b} = (yy)$ and $\widetilde{c} = (xy)$.*

Using the chain of duality and standard relations as in (2.29), we find the dual of the other theory as well: *the $\mathrm{O}_-(2)$ theory with two massless doublets is dual to the $\mathbf{Z}_2(-1)^{F_s}$ orbifold of the Landau-Ginzburg model of five variables $\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}$ with the superpotential (3.27), where the orbifold generator is $(\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}) \mapsto (-\widetilde{x}, -\widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c})$ combined with $(-1)^{F_s}$.*

Let us draw some conclusions from these duality relations.

3.4 Corank 1 degeneration — two or one massive vacua

We consider the theory with N doublets $x_1, \ldots, x_N$ and a singlet z with superpotential

$$W = z(x_1x_1) + \cdots + (x_Nx_N). \tag{3.31}$$

By the definition of the $O_\pm(2)$ theory, the low energy behaviour does not depend on N as long as $N \geq 1$: if N is odd, the theory is defined as it is. If N is even, it is defined as the $O_\pm(2)$ theory with one additional massive field (a regulator). Changing N by one either does not change anything (i.e. the regulator is reinterpreted as a physical massive field, or vice versa) or add/subtract two doublets with complex masses, which does not change the low energy behaviour.

Let us take $N = 1$ for simplicity. If z is fixed at a value away from zero, the superpotential for x_1 is regular and we have supersymmetry breaking. The question is what happens in a neighborhood of $z = 0$ and when the fluctuation of z is taken into account. To see this, we use the dual description. For the theory with gauge group $O_+(2)$ (*resp.* $O_-(2)$), the dual is two copies (*resp.* one copy) of the LG model of two variables, z and a, with the superpotential

$$W = za. \tag{3.32}$$

This superpotential has a unique critical point $z = a = 0$. Hence the theory has two (*resp.* one) massive supersymmetric ground states, with the expectation values

$$\langle z \rangle = 0, \quad \langle (x_1x_1) \rangle = 0. \tag{3.33}$$

One may also consider the theory with $N = 2$ and apply the duality obtained above. The $O_+(2)$ (*resp.* $O_-(2)$) theory is dual to the LG model (*resp.* $\mathbf{Z}_2(-1)^{F_s}$ LG orbifold) of six variables, $\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}$ and z, with the superpotential

$$W = \widetilde{a}\widetilde{x}^2 + 2\widetilde{c}\widetilde{x}\widetilde{y} + \widetilde{b}\widetilde{y}^2 + z\widetilde{a} + \widetilde{b}. \tag{3.34}$$

There are two (*resp.* one) critical points, $\widetilde{x} = \widetilde{a} = \widetilde{b} = \widetilde{c} = z = 0$ and $\widetilde{y} = \pm i$. That is, there are two (*resp.* one) massive supersymmetric ground states, with the expectation values $\langle z \rangle = 0$ and $\langle (x_1x_1) \rangle = \langle (x_2x_2) \rangle = \langle (x_1x_2) \rangle = 0$.

3.5 Corank 2 degeneration — ramified double cover of $\mathbb{C}^2$ or its orbifolds

Let us now consider the theory of two doublets, x and y, and three singlets, a, b and c, which are coupled via the superpotential

$$W = a(xx) + 2c(xy) + b(yy) \tag{3.35}$$

At values of (a, b, c) away from the degeneration locus

$$ab = c^2, \tag{3.36}$$

the superpotential gives masses to both x and y. As we learned in section 3.1, there is no zero energy state in such a theory. Therefore, the low energy theory will concentrate near the degeneration locus (3.36). Near that locus but away from the origin, $(a, b, c) = (0, 0, 0)$, we may change the variables to make the superpotential into the form

$$W = (c^2 - ab)(x'x') + (y'y'). \tag{3.37}$$

The result of section 3.4 then tells us that, for the gauge group $\mathrm{O}_+(2)$ (*resp.* $\mathrm{O}_-(2)$) we have two (*resp.* one) zero energy states along the locus (3.36), as far as (a, b, c) is away from the origin. Thus, we expect to have some kind of double (*resp.* single) cover over the degeneration locus (3.36). We would like to find what really is the low energy theory.

Let us apply the dual description for the $\mathrm{O}_+(2)$ theory. It is simply the LG model of five plus three variables, $\widetilde{x}, \widetilde{y}, \widetilde{a}, \widetilde{b}, \widetilde{c}$ and a, b, c, with the superpotential

$$W = \widetilde{a}\widetilde{x}^2 + 2\widetilde{c}\widetilde{x}\widetilde{y} + \widetilde{b}\widetilde{y}^2 + a\widetilde{a} + 2c\widetilde{c} + b\widetilde{b}. \tag{3.38}$$

Integrating out a, b, c, we obtain the constraints $\widetilde{a} = \widetilde{b} = \widetilde{c} = 0$ and we are left with the theory of $\widetilde{x}$ and $\widetilde{y}$ only, with vanishing superpotential. Thus, we obtain the conformal field theory of just two variables $\widetilde{x}$ and $\widetilde{y}$, with no constraint and no superpotential. Extremizing W with respect to $\widetilde{a}, \widetilde{b}, \widetilde{c}$ finds the relations

$$a = -\widetilde{x}^2, \qquad b = -\widetilde{y}^2, \qquad c = -\widetilde{x}\widetilde{y}. \tag{3.39}$$

Such an (a, b, c) indeed satisfies the equation (3.36). Conversely, for each non-zero (a, b, c) obeying (3.36), the equation (3.39) has two solutions for $(\widetilde{x}, \widetilde{y})$, related by the sign flip $(\widetilde{x}, \widetilde{y}) \to (-\widetilde{x}, -\widetilde{y})$. That is, the $(\widetilde{x}, \widetilde{y})$ space is a double cover of the degeneration locus $ab = c^2$, as expected. Recall that $ab = c^2$ is the equation defining the A_1 surface singularity, which is known to be realized by $\mathbf{C}^2/\mathbf{Z}_2$, and the relations (3.39) exhibit that realization,

$$\mathbf{C}^2 \longrightarrow \mathbf{C}^2/\mathbf{Z}_2. \tag{3.40}$$

Recall that this $\mathbf{Z}_2$ symmetry is the quantum symmetry of the orbifold by the $\mathbf{Z}_2 = \mathrm{O}_+(2)/\mathrm{SO}(2)$ (and hence $\widetilde{x}$ and $\widetilde{y}$ are twist fields). See (3.29). Note also that $\widetilde{a} = \widetilde{b} = \widetilde{c} = 0$ means

$$(xx) = (yy) = (xy) = 0. \tag{3.41}$$

We conclude that *the* $\mathrm{O}_+(2)$ *theory flows in the infra-red limit to to the free conformal field theory of two twist variables,* $\widetilde{x}$ *and* $\widetilde{y}$*, i.e., the sigma model with the target space* $\mathbf{C}^2$*. The singlets* a*,* b *and* c *are related to* $\widetilde{x}$ *and* $\widetilde{y}$ *by (3.39). The* $\mathrm{O}(2)$ *invariants,* (xx)*,* (yy) *and* (xy)*, vanish in the infra-red fixed point theory.*

We may unfold the $\mathbf{Z}_2 = \mathrm{O}_+(2)/\mathrm{SO}(2)$ by orbifolding the associated quantum symmetry. Also, we may use the chain of duality and standard relations as in (2.29). These lead us to conclude that *the* $\mathrm{SO}(2)$ *(*resp. $\mathrm{O}_-(2)$*) theory flows in the infra-red limit to the free orbifold conformal field theory* $\mathbf{C}^2/\mathbf{Z}_2$ *(*resp. $\mathbf{C}^2/\mathbf{Z}_2(-1)^{F_s}$*).* Here, "the $\mathrm{SO}(2)$ theory" of course stands for the $\mathrm{U}(1)$ theory with $r = 0$ and $\theta = \pi$.

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4 Orthogonal groups

In this section, we study low energy behaviour of theories with the orthogonal gauge group, $\mathrm{O}(k)$ or $\mathrm{SO}(k)$, with N chiral multiplets in the fundamental representation, i.e., the vector representation $\mathbf{k}$. We denote the chiral matter fields as $x_1, \ldots, x_N$ where each x_i is a column vector of length k, $x_i = (x_i^a)_{a=1,\ldots,k}$. Our main focus will be the theory with vanishing superpotential for $x_1, \ldots, x_N$. The group $\mathrm{O}(k)$ is the semi-direct product $\mathrm{SO}(k) \rtimes \mathbf{Z}_2$ for even k, as in $\mathrm{O}(2)$ discussed in the previous section, and the direct product $\mathrm{SO}(k) \times \mathbf{Z}_2$ for odd k, where $\{\mathbf{1}_k\} \times \mathbf{Z}_2$ corresponds to the subgroup generated by the central element $-\mathbf{1}_k$. In either case, the $\mathrm{O}(k)$ gauge theory can be treated as a $\mathbf{Z}_2$ orbifold of the $\mathrm{SO}(k)$ gauge theory. As always, there are two versions of the orbifold, related by $(-1)^{F_s}$. As the final important point, the groups $\mathrm{O}(k)$ and $\mathrm{SO}(k)$ have a non-trivial fundamental group

$$\pi_1(\mathrm{O}(k)) = \pi_1(\mathrm{SO}(k)) = \mathbf{Z}_2 \qquad \text{for } k \geq 3. \tag{4.1}$$

This means that there is a mod 2 theta angle: on a closed two-dimensional manifold, there are two topological types of principal G bundles for $G = \mathrm{O}(k)$ or $\mathrm{SO}(k)$, the trivial and the non-trivial. And the mod 2 theta angle assigns a phase (-1) to the path-integral weight for the non-trivial G bundle.

4.1 The space of classical vacua

Let us first describe the space of classical vacua — the space of scalar fields that annihilate the classical potential. We denote the scalar component of the vector multiplet by σ. It is a $k \times k$ antisymmetric complex matrix. We write x for the $k \times N$ matrix (x_i^a). The vacuum equation reads

$$\begin{aligned} [\sigma, \sigma^\dagger] &= 0, \\ x x^\dagger &= (x x^\dagger)^T, \\ \sigma x &= \sigma^\dagger x = 0. \end{aligned} \tag{4.2}$$

The first equation means that σ must lie in the complexification of the Lie algebra of a maximal torus. That is, up to gauge transformations, it is of the form

$$i \begin{pmatrix} & -\sigma_1 & & & \\ \sigma_1 & & & & \\ & & \ddots & & \\ & & & & -\sigma_\ell \\ & & & \sigma_\ell & \end{pmatrix} \quad \textit{resp.} \quad i \left(\begin{array}{ccccc|c} & -\sigma_1 & & & & \\ \sigma_1 & & & & & \\ & & \ddots & & & \\ & & & & -\sigma_\ell & \\ & & & \sigma_\ell & & \\ \hline & & & & & 0 \end{array}\right), \tag{4.3}$$

for $k = 2\ell$ *resp.* $k = 2\ell + 1$. The second equation means that $x x^\dagger$ is a real symmetric matrix, and hence it can be diagonalized using the gauge symmetry. With an appropriate

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U(N) flavor rotation, we can write the solution as

$$x = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_N \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & a_k \end{pmatrix} \tag{4.4}$$

depending on $N \leq k$ or $N \geq k$. The final equation requires that if the number of non-zero σ_a's is s, then the number of non-zero a_i's is at most $k-2s$. Let C_s (*resp.* H_r) be the set of gauge equivalence classes of solutions for σ of rank $2s$ or less (*resp.* x of rank r or less). It has complex dimension s (*resp.* $Nr - \frac{r(r-1)}{2}$). The space of classical vacua is

$$\mathcal{M} = \bigcup_{s=s_{\min}}^{\ell} \Big(C_s \times H_{k-2s}\Big), \tag{4.5}$$

where $s_{\min} = 0$ if $N \geq k-1$ and $s_{\min} = \left[\frac{k-N}{2}\right]$ if $N \leq k-2$. When $N \geq k-1$, there is a Higgs branch $C_0 \times H_k$ in which x is generically non-zero and breaks the gauge group completely (or to a $\mathbf{Z}_2$ subgroup for the O(k) theory with $N = k-1$). When k is even, there is a Coulomb branch $C_\ell \times H_0$ in which σ generically has the full rank k. Other components are the mixed Coulomb-Higgs branches where both x and σ are generically non-zero.

4.2 Regularity

Classically, the gauge theory reduces at low energies to the non-linear sigma model whose target space is the classical vacuum moduli space (4.5). This space is singular and non-compact, and hence we do not know if we have a sensible theory in the infra-red limit, or at least the fixed point theory must be described by something very much different from the sigma model for (4.5) [33, 34]. Here we would like to discuss the possibility that quantum corrections lift all non-compact flat directions in σ, i.e., Coulomb and all possible mixed branches. We shall refer to such a theory as a *regular* theory. Regularity is judged by the effective twisted superpotential $\widetilde{W}_{\rm eff}$ for σ_a's. By non-renormalization theorem, this is not affected even if the superpotential for the matter multiplets is introduced. In particular, if the theory is regular, by introducing a superpotential that lifts the Higgs branch, we can obtain a (2,2) superconformal field theory with discrete spectrum at the infra-red fixed point. We would like to compute $\widetilde{W}_{\rm eff}$ and find a criterion for regularity.

Let us first consider the Coulomb branch $C_\ell \times H_0$ for the k even case ($k = 2\ell$). If σ_a's are chosen so that $|\sigma_a|$ and $|\sigma_a \pm \sigma_b|$ for $a \neq b$ are all non-zero, we have either massive multiplets or the massless vector multiplets for the maximal torus U(1)$^\ell$. To obtain the effective theory for the latter, we integrate out the massive modes, consisting of the chiral multiplets x_i and the "off-diagonal" vector multiplets ${V^c}_d$. Let us first consider the vector multiplets. The contribution to $\widetilde{W}_{\rm eff}$ can be found [35] by looking at the mass terms for the gaugino,

$$-\,\mathrm{tr}\left(\overline{\lambda}_-[\sigma, \lambda_+] + \overline{\lambda}_+[\sigma^\dagger, \lambda_-]\right). \tag{4.6}$$

For $a \neq b$, those which are charged under $\mathrm{U}(1)_a \times \mathrm{U}(1)_b$ are in the 2×2 block, ${V^c}_d$ for $c = 2a-1, 2a$ and $d = 2b-1, 2b$, and have masses $\pm\sigma_a \pm \sigma_b$ (all the four possible sign combinations). The contribution to $\widetilde{W}_{\rm eff}$ of these four multiplets is $\pi i(\sigma_a - \sigma_b) + \pi i(\sigma_a + \sigma_b)$ which vanishes modulo $2\pi i$ times σ_a's. Computation of the contribution from the massive chiral multiplets is standard, $-\sum_{i,a} \sigma_a(\log\sigma_a - 1) - \sum_{i,a}(-\sigma_a)(\log(-\sigma_a) - 1)$, which is $\pi i N \sum_a \sigma_a$, again modulo $2\pi i$ times σ_a's. The total is

$$\widetilde{W}_{\rm eff} = \pi i N \sum_{a=1}^{\ell} \sigma_a. \tag{4.7}$$

Let us next consider the k odd case ($k = 2\ell + 1$) and look at the mixed branch $C_\ell \times H_1$. In this case, computation depends on the location of x in H_1. If it is zero, then, the entire gauge group, $\mathrm{O}(k)$ or $\mathrm{SO}(k)$, is unbroken and we can do the computation as usual. If it is non-zero, the gauge group is broken to its proper subgroup, $\mathrm{O}(k-1)$ or $\mathrm{SO}(k-1)$, and we need to take into account the Higgs effect. Let us first consider the former, expanding x around $x = 0$. The last components $x_1^k, \ldots, x_N^k$ are massless and we leave them in the effective theory. Integration over massive modes can be done in the same way as above, except that now, for odd k, we have the right-most off-diagonal components, ${V^c}_k$ for $c = 1, \ldots, k-1$. For $c = 2a-1, 2a$, they are charged only under $\mathrm{U}(1)_a$ and have masses $\pm\sigma_a$. They yield the non-trivial contribution $\pi i \sigma_a$ to $\widetilde{W}_{\rm eff}$. Contribution from other massive modes are the same, and the total is

$$\widetilde{W}_{\rm eff} = \pi i (N+1) \sum_{a=1}^{\ell} \sigma_a. \tag{4.8}$$

Let us next perform computation around a Higgsed point, say,

$$x_1 = \cdots = x_{N-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad x_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}. \tag{4.9}$$

In this case, we must treat the super-Higgs multiplet, consisting of the right-most off-diagonal vector multiplets ${V^a}_k$ and the complexified gauge orbit directions x_N^a, as one block. This block gives no contribution to $\widetilde{W}_{\rm eff}$ — the one from vectors and the one from chirals cancel against each other. The rest is as in $\mathrm{O}(k-1)$ or $\mathrm{SO}(k-1)$ theory with $N-1$ fundamentals. Note that $k-1$ is even and the above result (4.7) can be used. The result is $\widetilde{W}_{\rm eff} = \pi i(N-1) \sum_a \sigma_a$, which is equal to (4.8) modulo $2\pi i$ times σ_a's. Actually, we did not have to do the two computations in view of the decoupling between the chiral and twisted chiral multiplets — the result for $\widetilde{W}_{\rm eff}$ should not depend on where in H_1 you do the computation. Nevertheless, the fact that we indeed obtained the same result is gratifying.

Computation in various mixed branches should be obvious by now, thanks to the exercise given above that involves the super-Higgs multuplet. We obtain (4.7) or (4.8) depending on whether k is even or odd, where the sum over a must be reduced appropriately (for example, the sum is over $a \in \{1, \ldots, s\}$ on $C_s \times H_{k-2s}$).

The result is that we have an effective theta angle

$$\theta_{\text{eff}} = \begin{cases} \pi N & k \text{ even}, \\ \pi(N+1) & k \text{ odd}. \end{cases} \tag{4.10}$$

for each U(1) factor on the classical Coulomb or mixed branch. If it is zero modulo 2π, the energy density is zero and we do have non-compact Coulomb and mixed branches. If it is non-zero modulo 2π, then the energy density is $e^2_{\text{eff}}(\sigma)/8$ times the number of unbroken U(1) factors, where $e_{\text{eff}}(\sigma)$ is the effective gauge coupling at the given value of σ. The latter approaches the classical value e as $|\sigma_a|$ and $|\sigma_a \pm \sigma_b|$ for $a \neq b$ are all much larger compared to e. Thus, the non-compact Coulomb and mixed branches are all lifted in this case. To summarize, *the theory is regular if and only if $N - k$ is odd.*

Mod 2 theta angle. In the analysis so far, we have implicitly assumed that the mod 2 theta angle is set equal to zero. Let us now see its effect. First, the universal cover of the group SO(k) is Spin(k) which is realized as the subset of the Clifford algebra C($\mathbf{R}^k$),[2] generated multiplicatively by elements of the form $\exp\left(\sum_{a<b} t_{ab} e_a e_b\right)$. The conjugation action of Spin(k) on $\mathbf{R}^k \subset \mathrm{C}(\mathbf{R}^k)$ induces an isomorphism $\mathrm{SO}(k) \cong \mathrm{Spin}(k)/\mathbf{Z}_2$ where $\mathbf{Z}_2$ is the subgroup consisting of $\pm 1 \in \mathrm{C}(\mathbf{R}^k)$. That is why SO($k$) has the fundamental group $\mathbf{Z}_2$. An example of non-trivial loop in SO(k) is

$$t \in \mathbf{R}/2\pi\mathbf{Z} \;\longmapsto\; g_t = \begin{pmatrix} \cos(t) & -\sin(t) & \\ \sin(t) & \cos(t) & \\ & & \mathbf{1}_{k-2} \end{pmatrix} \in \mathrm{SO}(k). \tag{4.11}$$

Indeed it lifts to a path $\widetilde{g}_t = \exp\left(\frac{t}{2} e_1 e_2\right)$ from $\widetilde{g}_0 = 1$ to $\widetilde{g}_{2\pi} = -1$ in Spin(k). A topologically non-trivial SO(k) bundle over a closed surface Σ is the one having the transition function g_t along a circle in Σ parametrized by $t \in \mathbf{R}/2\pi\mathbf{Z}$ that separates Σ into two connected components. The mod 2 theta angle assigns the phase (-1) to such a principal SO(k) bundle. By this exercise, we see that it yields the theta angle

$$\theta = \pi, \tag{4.12}$$

for the subgroup $\mathrm{U}(1) \cong \mathrm{SO}(2) \subset \mathrm{SO}(k)$ of 2-dimensional rotations for each orthogonal decomposition $\mathbf{R}^k \cong \mathbf{R}^2 \oplus \mathbf{R}^{k-2}$. In particular, it yields a contribution π to the theta angle for each U(1) factor on the classical Coulomb or mixed branch. We can now state the complete criterion:

When $N - k$ is odd (resp. *even*)*, the theory is regular if and only if the tree level mod 2 theta angle is turned off* (resp. *turned on*)*.*

Thus, the theory with $N - k$ even, which is not by itself regular, can be made regular by turning on the mod 2 theta angle. Alternatively, we may consider adding a fundamental chiral multiplet with a complex mass m, as a regulator. In fact, in the limit $|m| \to \infty$ the effective action for the vector multiplet obtained by integrating it out is nothing but the

[2] It is generated by 1 and the basis $e_1, \ldots, e_k$ of $\mathbf{R}^k$ which obey the relation $e_a e_b + e_b e_a = -2\delta_{a,b}$. Note that $\exp(t e_a e_b) = \cos(t) + e_a e_b \sin(t)$ for $a < b$. In particular, $\exp(\pi e_a e_b) = -1$ and $\exp(2\pi e_a e_b) = 1$.

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mod 2 theta angle. This can be seen by noting that it yields the theta angle $\theta = \pi$ for each SO(2) subgroup of 2-dimensional rotation. When we consider the theory with gauge group O(k), the regulator field also has the effect of inverting the definition of the $\mathbf{Z}_2$ orbifold, i.e., dressing the generator by $(-1)^{F_s}$. The situation is exactly the same as what we have seen in the O(2) gauge theory.

In what follows, unless otherwise stated, we shall always assume that the theory is regular. When $N-k$ is even, either the mod 2 theta angle is turned on or a regulator field is introduced.

4.3 Twisted masses

Before discussing the theory with massless fundamentals $x_1, \ldots, x_N$, let us study the theory in which they have twisted masses $\widetilde{m}_1, \ldots, \widetilde{m}_N$. Along the way, we introduce a notation that distinguishes the two orbifold projections for the case where the gauge group is O(k). We assume that the masses are generic and in particular satisfy

$$\widetilde{m}_i + \widetilde{m}_j \neq 0 \quad \forall (i,j). \tag{4.13}$$

Then, the Higgs branch is lifted, and the theory is well behaved in any direction. Our focus is the spectrum of supersymmetric ground states of this regularized system.

We integrate out the fundamentals as they are massive in any field configuration. We also stay in the generic locus on the Coulomb branch and integrate out the massive off-diagonal components of the vector multiplet. The resulting effective twisted superpotential is

$$\begin{aligned}\widetilde{W}_{\text{eff}} = &-\sum_{i,a}(\sigma_a - \widetilde{m}_i)(\log(\sigma_a - \widetilde{m}_i) - 1) - \sum_{i,a}(-\sigma_a - \widetilde{m}_i)(\log(-\sigma_a - \widetilde{m}_i) - 1) \\ &+\pi i k \sum_a \sigma_a + \pi i (N-k+1) \sum_a \sigma_a .\end{aligned} \tag{4.14}$$

The first line is from the chiral multiplets, while the term $\pi i k \sum_a \sigma_a$ on the second line is from the massive vectors — as we have seen in the previous section, we have a non-zero contribution $\pi i \sum_a \sigma_a$ if and only if k is odd. The last term $\pi i(N-k+1)\sum_a \sigma_a$, which is non-zero when $N-k$ is even, is from either the mod 2 theta angle or the regulator field. The vacuum equation reads

$$\prod_{i=1}^{N}(\sigma - \widetilde{m}_i) = (-1)^{N+1}\prod_{i=1}^{N}(-\sigma - \widetilde{m}_i), \tag{4.15}$$

for $\sigma = \sigma_1, \ldots, \sigma_\ell$. The solutions are identified under the action of the Weyl group: permutations of σ_a's as well as the sign flips

$$\sigma_a \longmapsto \epsilon_a \sigma_a, \qquad \begin{cases} \epsilon_1 \cdots \epsilon_\ell = 1 & \text{SO}(k),\ k \text{ even}, \\ \text{no condition} & \text{otherwise.} \end{cases} \tag{4.16}$$

Note that the reflections with $\epsilon_1 \cdots \epsilon_\ell = -1$ is allowed for O(k), k even as well — they are from the disconnected component of O(k) and represent the $\mathbf{Z}_2$ orbifold generator. We require the solutions to obey

$$\begin{aligned} &\sigma_a \neq \pm\widetilde{m}_i, \\ &\sigma_a \neq \pm\sigma_b \quad a \neq b, \\ &\sigma_a \neq 0 \qquad \forall a \quad \text{if } k \text{ is odd.} \end{aligned} \tag{4.17}$$

In the forbidden region, there are massless degrees of freedom other than the $\mathrm{U}(1)^\ell$ vector multiplets, and the effective twisted superpotential (4.14) can not be trusted. We do not have to worry about the first condition, $\sigma_a \neq \pm\widetilde{m}_i$, as $\sigma = \pm\widetilde{m}_i$ are not among the roots of (4.15) thanks to the condition (4.13). We simply ignore solutions violating the other conditions. Namely, we assume that there is no supersymmetric ground state supported in the forbidden region. This point was examined in [2] in a specific class of models and consistent picture has emerged.

Let us first study the case where k is even ($k = 2\ell$) and N is even. The equation (4.15) have $\frac{N}{2}$ pairs of non-zero roots. Solutions for $\sigma_1, \ldots, \sigma_\ell$ obeying the condition (4.17) exists if and only if $\frac{N}{2} \geq \ell$. For $\mathrm{O}(k)$ gauge group, the number of inequivalent solutions is $\binom{\frac{N}{2}}{\ell}$. For $\mathrm{SO}(k)$ gauge group, the number is twice as much, $2\binom{\frac{N}{2}}{\ell}$, because of the constraint $\epsilon_1 \cdots \epsilon_\ell = 1$ on the Weyl group elements. As the Weyl group is completely broken at each of these solutions, these are the number of supersymmetric ground states. For the same reason, for the $\mathrm{O}(k)$ theory, the result does not depend on the choice of the orbifold.

When k is even ($k = 2\ell$) and N is odd, the equation (4.15) has $\frac{N-1}{2}$ pairs of non-zero roots and one root at $\sigma = 0$. Solutions for σ_a's obeying (4.17) exists if and only if $\frac{N-1}{2} + 1 \geq \ell$. The count from solutions for which σ_a's are all non-zero is as in the case above: $\binom{\frac{N-1}{2}}{\ell}$ for $\mathrm{O}(k)$ and $2\binom{\frac{N-1}{2}}{\ell}$ for $\mathrm{SO}(k)$. Let us consider solutions where one σ_a vanish, say, $\sigma_1 = 0$ (there are $\binom{\frac{N-1}{2}}{\ell-1}$ inequivalent solutions of this type). For $\mathrm{SO}(k)$ gauge group, the Weyl group is completely broken and we obtain $\binom{\frac{N-1}{2}}{\ell-1}$ as the number of ground states. For $\mathrm{O}(k)$ gauge group, exactly one Weyl group element is unbroken. It is the one that acts as the sign flip of σ_1 only, and is represented by

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}. \tag{4.18}$$

The spectrum of supersymmetric ground states from this sector is sensitive to the definition of the orbifold. As in the $\mathrm{O}(2)$ theory discussed in section 3.2, we denote the gauge group by $\mathrm{O}_+(k)$ if we receive two supersymmetric ground states (one twisted and one untwisted), and by $\mathrm{O}_-(k)$ if we receive one twisted supersymmetric ground state. The total number of states of this type is $2\binom{\frac{N-1}{2}}{\ell-1}$ for $\mathrm{O}_+(k)$ theory and $\binom{\frac{N-1}{2}}{\ell-1}$ for $\mathrm{O}_-(k)$ theory. To summarize, the number of ground states for even k and odd N case is $\binom{\frac{N-1}{2}}{\ell} + 2\binom{\frac{N-1}{2}}{\ell-1}$ for $\mathrm{O}_+(k)$, $\binom{\frac{N-1}{2}}{\ell} + \binom{\frac{N-1}{2}}{\ell-1} = \binom{\frac{N+1}{2}}{\ell}$ for $\mathrm{O}_-(k)$, and $2\binom{\frac{N-1}{2}}{\ell} + \binom{\frac{N-1}{2}}{\ell-1}$ for $\mathrm{SO}(k)$.

Let us next study the case where k is odd ($k = 2\ell+1$). Note that there is one component of each vector x_i that is neutral with respect to the maximal torus. If we choose the torus as in (4.3), then it is the last (k-th) component. Thus, the variables $x_1^k, \ldots, x_N^k$ are decoupled from the rest of the degrees of freedom on the generic locus of the Coulomb branch. Note that $\mathrm{O}(k)$ and $\mathrm{SO}(k)$ differ in the presence/absence of the group element

$$\left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \hline & & & -1 \end{array}\right), \tag{4.19}$$

that flips the sign of these components. For $\mathrm{SO}(k)$, the last component system is simply the model of N variables with twisted mass (possibly with one additional regulator field). This sector hence provides a unique supersymmetric ground state. For $\mathrm{O}(k)$, the last component system is the $\mathbf{Z}_2$ orbifold thereof. The spectrum in this sector again is sensitive to the choice of orbifold. We denote the gauge group by $\mathrm{O}_+(k)$ if the number of supersymmetric ground states is two for even N and one for odd N, and by $\mathrm{O}_-(k)$ if opposite, i.e., one for even N and two for odd N. We now turn to the sector of the first $k-1 = 2\ell$ components.

We first consider the case where N is even. The equation (4.15) has $\frac{N}{2}$ pairs of non-zero roots. Solutions obeying (4.17) exist if and only if $\frac{N}{2} \geq \ell$, and the number of inequivalent ones is $\binom{\frac{N}{2}}{\ell}$. This is for both $\mathrm{O}(k)$ and $\mathrm{SO}(k)$ since they share the same Weyl group when k is odd. Since the Weyl group is completely broken at each of them, this is the number of vacuum states from this sector. Combining with the last component sector, the total number of supersymmetric ground states is $2\binom{\frac{N}{2}}{\ell}$ for $\mathrm{O}_+(k)$, $\binom{\frac{N}{2}}{\ell}$ for $\mathrm{O}_-(k)$, and $\binom{\frac{N}{2}}{\ell}$ for $\mathrm{SO}(k)$.

Next, we consider the case where N is odd. The equation (4.15) has $\frac{N-1}{2}$ pairs of non-zero roots and one root at $\sigma = 0$. According to (4.17) we need to avoid the one at $\sigma = 0$ when k is odd. The solutions exist if and only if $\frac{N-1}{2} \geq \ell$, and the number of inequivalent ones is $\binom{\frac{N-1}{2}}{\ell}$ for both $\mathrm{O}(k)$ and $\mathrm{SO}(k)$. Since the Weyl group is completely broken at each of them, this is the number of vacuum states from this sector. Combining with the last component sector, the total number of supersymmetric ground states is $\binom{\frac{N-1}{2}}{\ell}$ for $\mathrm{O}_+(k)$, $2\binom{\frac{N-1}{2}}{\ell}$ for $\mathrm{O}_-(k)$, and $\binom{\frac{N-1}{2}}{\ell}$ for $\mathrm{SO}(k)$.

The definition of orbifold for $\mathrm{O}_\pm(k)$ can be extended by continuity to the theories where the twisted masses are turned off and then, possibly, the superpotential is turned on. For k even, this is defined originally for odd N and then is extended to the even N case via the regulator field. For k odd, this is already defined for both even and odd N, and we would like to check whether the two are continuously connected. We can focus on the last component sector in which the choice of orbifold is relevant. Let us start from the even N case, with no mod 2 theta angle nor the regulator field. The number of ground states is 2 for $\mathrm{O}_+(k)$, and this is the number in the "standard" $\mathbf{Z}_2$ orbifold in the sense of section 2.2. We then turn off the twisted mass for, say, x_N, and give it a complex mass. Then we have $N_{\text{eff}} = N - 1$ (odd) fundamentals with a regular field x_N. According to section 2.2, the number of supersymmetric ground states is 1, which is indeed the number we are assigning for $\mathrm{O}_+(k)$ with odd N_{eff}. This shows the continuity of our definition.

To summarize, for $N \leq k-2$ there is no supersymmetric ground state. For $N \geq k-1$, the number of supersymmetric ground states is given by:

group	k	N	number	group	k	N	number
$\mathrm{O}_+(k)$	even	even	$\binom{\frac{N}{2}}{\frac{k}{2}}$	$\mathrm{O}_+(k)$	odd	even	$2\binom{\frac{N}{2}}{\frac{k-1}{2}}$
$\mathrm{O}_-(k)$	even	even	$\binom{\frac{N}{2}}{\frac{k}{2}}$	$\mathrm{O}_-(k)$	odd	even	$\binom{\frac{N}{2}}{\frac{k-1}{2}}$
$\mathrm{SO}(k)$	even	even	$2\binom{\frac{N}{2}}{\frac{k}{2}}$	$\mathrm{SO}(k)$	odd	even	$\binom{\frac{N}{2}}{\frac{k-1}{2}}$
$\mathrm{O}_+(k)$	even	odd	$\binom{\frac{N-1}{2}}{\frac{k}{2}} + 2\binom{\frac{N-1}{2}}{\frac{k}{2}-1}$	$\mathrm{O}_+(k)$	odd	odd	$\binom{\frac{N-1}{2}}{\frac{k-1}{2}}$
$\mathrm{O}_-(k)$	even	odd	$\binom{\frac{N+1}{2}}{\frac{k}{2}}$	$\mathrm{O}_-(k)$	odd	odd	$2\binom{\frac{N-1}{2}}{\frac{k-1}{2}}$
$\mathrm{SO}(k)$	even	odd	$2\binom{\frac{N-1}{2}}{\frac{k}{2}} + \binom{\frac{N-1}{2}}{\frac{k}{2}-1}$	$\mathrm{SO}(k)$	odd	odd	$\binom{\frac{N-1}{2}}{\frac{k-1}{2}}$

(4.20)

4.4 $N \leq k-2$: supersymmetry breaking

Let us consider the (regular) theory with massless fundamentals where the number N is in the range $1 \leq N \leq k-2$. The observed fact that there is no supersymmetric ground state when the twisted masses are turn on implies that there is no normalizable supersymmetric ground state in the massless theory either. This is because [2], if there were a normalizable zero energy state in the massless theory, that would stay in the spectrum even if the masses are turned on, since the masses would only make better the behaviour of states at infinity in the field space.

In fact, there is no normalizable supersymmetric ground state also in irregular theory with $1 \leq N \leq k-2$ as well as in the pure Yang-Mills theory (regular or not) for $k \geq 3$.

To see that, let us continue from the previous subsection and take the limit where some of the twisted masses are sent to infinity. If an odd number of $\widetilde{m}_i$'s are sent to infinity, the behaviour of the superpotential (4.14) at large values of σ_a's is changed and a regular theory becomes an irregular theory. If an even number of $\widetilde{m}_i$'s are sent to infinity, the behaviour does not change and a regular theory becomes another regular theory. Note that a pure-Yang-Mills theory is obtained by sending all twisted masses to infinity — the regular one if N is even and the irregular one if N is odd. Let us look closely into the equation (4.15). If one twisted mass, say $\widetilde{m}_N$, is sent to infinity, then one pair of non-zero roots go away to infinity. To see that we rewrite the equation as

$$\left(-1 + \frac{\sigma}{\widetilde{m}_N}\right) \prod_{i=1}^{N-1} (\sigma - \widetilde{m}_i) = \left(-1 - \frac{\sigma}{\widetilde{m}_N}\right) \prod_{i=1}^{N-1} (\sigma + \widetilde{m}_i). \tag{4.21}$$

We see that the equation has a limit as $\widetilde{m}_N \to \infty$. It is an equation of order $(N-2)$. Since the order has decreased by 2, two roots must have gone away to infinity. If two twisted masses, say $\widetilde{m}_{N-1}$ and $\widetilde{m}_N$, are sent to infinity, the same argument shows that still one pair of non-zero roots go away to infinity.

This and the analysis of the previous subsection lead us to conclude that the irregular theory in the range $1 \leq N \leq k-2$ has no supersymmetric ground state, when generic twisted masses are turned on and hence also when they are turned off. We also find that the pure Yang-Mills theory with $k \geq 3$, whether regular or not, has no supersymmetric ground state.

4.5 $N = k-1$: free conformal field theory

Let us now consider the theory with $N = k-1$ massless fundamentals. In the regular theory, the Coulomb and mixed branches are lifted and we are left with the Higgs branch $H_k = H_{k-1}$ only. As a complex manifold, the Higgs branch is isomorphic to the affine space $\mathbf{C}^{\frac{(k-1)k}{2}}$ since the chiral ring of gauge invariants is isomorphic to the polynomial ring of the $\frac{k(k-1)}{2}$ scalar products $(x_i x_j)$ (the "mesons") with no relations, see [32]

$$\mathbf{C}[x_1, \ldots, x_{k-1}]^{\mathrm{SO}(k,\mathbf{C})} = \mathbf{C}\Big[\,(x_i x_j)\,\Big|\, 1 \leq i \leq j \leq k-1\Big]. \tag{4.22}$$

The classical metric is singular at the roots of Coulomb and mixed branches where parts of the gauge symmetry is unbroken. However, the singularity is expected to be smeared as these branches are lifted by quantum corrections. We claim that the theory flows in the infra-red limit to the free theory of the mesons. This is just as in the U(1) theory discussed in section 3.3 and in the SU(k) theory with $N = k+1$ massless fundamentals discussed in [2]. To be precise, this is for the gauge group SO(k). In the O(k) case, we must take the orbifold with respect to the $\mathbf{Z}_2$ symmetry that acts trivially on the mesons. This will make either two copies or one copy of the Higgs branch. We claim that the former is the case for the $\mathrm{O}_+(k)$ theory and the latter is the case for the $\mathrm{O}_-(k)$ theory. We do not provide a proof of this here, but consistency will be seen in what follows.

To summarize, we claim that *the* $\mathrm{O}_+(k)$ resp. $\mathrm{O}_-(k)$ resp. SO(k) *gauge theory with* $N = k-1$ *massless fundamentals flows in the infra-red limit to two copies* resp. *one copy* resp. *one copy of the free conformal field theory of the* $\frac{k(k-1)}{2}$ *mesonic variables.*

4.6 $N \geq k$: duality

Finally, let us consider the theory with $N \geq k$ massless fundamentals. We claim that there is a duality, where the correspondence of the gauge groups is

$$\begin{aligned} \mathrm{O}_+(k) &\longleftrightarrow \mathrm{SO}(N-k+1) \\ \mathrm{SO}(k) &\longleftrightarrow \mathrm{O}_+(N-k+1) \\ \mathrm{O}_-(k) &\longleftrightarrow \mathrm{O}_-(N-k+1). \end{aligned} \tag{4.23}$$

The theory with the gauge group on the left hand side and N *massless fundamentals* $x_1, \ldots, x_N$ *flows in the infra-red limit to the same fixed point as the theory with the* *gauge group on the right hand side and N fundamentals, $\widetilde{x}^1, \ldots, \widetilde{x}^N$, plus $\frac{N(N+1)}{2}$ singlets, $s_{ij} = s_{ji}$ $(1 \leq i, j \leq N)$, having the superpotential*

$$W = \sum_{i,j=1}^{N} s_{ij}(\widetilde{x}^i \widetilde{x}^j). \tag{4.24}$$

The mesons in the original theory correspond to the singlets in the dual,

$$(x_i x_j) = s_{ij}. \tag{4.25}$$

The baryons $[x_{i_1} \cdots x_{i_k}]$ in the original $\mathrm{SO}(k)$ theory correspond to twist operators in the dual $\mathrm{O}_+(N-k+1)$ theory regarded as a $\mathbf{Z}_2$ orbifold. More fundamentally, the order 2 symmetry $\mathrm{O}(k)/\mathrm{SO}(k)$ of the original theory corresponds to the quantum symmetry of the dual. Likewise for the baryons $[\widetilde{x}^{i_1} \cdots \widetilde{x}^{i_{N-k+1}}]$ and the order 2 symmetry of the dual $\mathrm{SO}(N-k+1)$ theory. The quantum symmetry of the $\mathrm{O}_-(k)$ theory corresponds to the quantum symmetry combined with $(-1)^{\mathbf{F}}$ in the dual $\mathrm{O}_-(N-k+1)$ theory, and vice versa.

The claimed relation is indeed a *duality*. I.e., the dual of the dual is the original. Let us start from $\mathrm{O}_+(k)$ with N massless fundamentals. The dual has gauge group $\mathrm{SO}(N-k+1)$ and its dual has $\mathrm{O}_+(N-(N-k+1)+1) = \mathrm{O}_+(k)$. The latter has N fundamentals, $\widetilde{\widetilde{x}}_i$, and $2\frac{N(N+1)}{2}$ singlets, $\widetilde{s}^{ij}$ and s_{ij}, having the superpotential

$$W = \sum \widetilde{s}^{ij}(\widetilde{\widetilde{x}}_i \widetilde{\widetilde{x}}_j) + \sum s_{ij}\widetilde{s}^{ij}. \tag{4.26}$$

The second term comes from the first dual superpotential and the relation $(\widetilde{x}^i \widetilde{x}^j) = \widetilde{s}^{ij}$. If $\widetilde{s}^{ij}$ is integrated out, we obtain the constraints $s_{ij} = -(\widetilde{\widetilde{x}}_i \widetilde{\widetilde{x}}_j)$. The resulting theory is simply the $\mathrm{O}_+(k)$ gauge theory with N fundamentals $\widetilde{\widetilde{x}}_i$ and no superpotential, which is indeed the theory we started with. Note that the constraints on s_{ij} and the meson/singlet correspondence implies the relation $(x_i x_j) = -(\widetilde{\widetilde{x}}_i \widetilde{\widetilde{x}}_j)$. The minus sign is typical for duality. The case where we start from $\mathrm{SO}(k)$ or $\mathrm{O}_-(k)$ is the same. Another way to see the duality nature is to couple the original system to singlets $\widetilde{s}^{ij}$ via the superpotential $W = \sum \widetilde{s}^{ij}(x_i x_j)$. This corresponds in the dual theory to the superpotential $W = \sum s_{ij}(\widetilde{x}^i \widetilde{x}^j) + \sum \widetilde{s}^{ij} s_{ij}$. Integrating out $\widetilde{s}^{ij}$ eliminates s_{ij} and we have the theory of the fundamentals $\widetilde{x}^i$ only. Note that we have the relation $\widetilde{s}^{ij} = -(\widetilde{x}^i \widetilde{x}^j)$, which is the singlet/meson correspondence (4.25) again up to a sign.

In what follows, we shall provide several evidences of the claimed duality.

Special cases. Special cases of the duality for small values of k and N had already been encountered and established in the earlier sections:

$\mathrm{SO}(1)/\mathrm{O}_\pm(1)$, $N=1$	$\mathrm{SO}(2)$, $N=2$	$\mathrm{O}_\pm(2)$, $N=2$	$\mathrm{SO}(1)/\mathrm{O}_\pm(1)$, $N=2$
2.3	2.4	3.3	3.5

In fact, the author used these cases as hints to find the general duality. The top row shows the left hand sides of the duality, and the number below shows the section in which the duality appeared. We define $O_+(1)$ *resp.* $O_-(1)$ theories as the standard *resp.* non-standard $\mathbf{Z}_2$ orbifold when $N-1$ is odd, and the definition is continued to the $N-1$ even case by introducing a "regulator" field. In other words, by periodicity, $O_+(1) = \mathbf{Z}_2$ *resp.* $\mathbf{Z}_2(-1)^{F_s}$ and $O_-(1) = \mathbf{Z}_2(-1)^{F_s}$ *resp.* $\mathbf{Z}_2$ when N is even *resp.* odd.

From O_+/SO duality to O_- duality. The duality between $O_-(k)$ and $O_-(N-k+1)$ can be derived using the chain of standard relations and the duality between $O_+(k)$ and $SO(N-k+1)$, as in (2.29). To see this, replace $(\mathcal{A}_3, \tau)$ in (2.29) by the $SO(k)$ gauge theory with N fundamentals equipped with the $\mathbf{Z}_2$ symmetry to define the $O_-(k)$ theory. Then, $(\mathcal{A}_2, \tau)$ should be replaced by the same theory equipped with the $\mathbf{Z}_2$ symmetry to define the $O_+(k)$ theory. $(\mathcal{A}_2/\tau, \widehat{\tau})$ is now the $O_+(k)$ theory with N fundamentals equipped with the quantum $\mathbf{Z}_2$ symmetry. According to the O_+/SO duality, this is equivalent to the dual $SO(N-k+1)$ theory equipped with the symmetry $\tau \in O(N-k+1)/SO(N-k+1)$, which replaces $(\mathcal{B}, \tau)$. Finally, the same theory equipped with $\tau(-1)^{F_s}$ replaces $(\mathcal{C}, \tau)$. The end result is the dual $O_-(N-k+1)$ theory, with the exchange of the twisted and untwisted sectors in the RR sector. The exchange shows that the quantum symmetry $\widehat{\tau}$ in the $O_-(k)$ theory corresponds to the symmetry $(-1)^{\mathbf{F}}\widehat{\tau}$ in the $O_-(N-k+1)$ theory.

Central charge. The dual pair of theories have the same symmetry other than the $\mathbf{Z}_2$ symmetry which was already mentioned: the $U(N)$ or $SU(N) \times U(1)_B$ flavor symmetry and the vector and axial $U(1)$ R-symmetries. Charge assignment compatible with the duality statement is

	$SU(N)$	$U(1)_B$	$U(1)_V$	$U(1)_A$
x	$\mathbf{N}$	1	0	0
$\widetilde{x}$	$\overline{\mathbf{N}}$	-1	1	0
s	$\mathbf{S}$	2	0	0

(4.27)

The R-charges could be modified by the $U(1)_B$ charge. The above choice is the unique one that assigns vanishing R-charges to x, and only with this choice, the two $U(1)$ R's can become parts of the $(2,2)$ superconformal symmetry in the infra-red fixed point of the original theory. The latter follows from the following argument [33]: for large values of x where the semi-classical sigma model analysis is valid, the R-currents can be chiral only if x does not rotate under the R-symmetries.

Assuming that these R-symmetries indeed become the parts of the superconformal symmetry, one can compute the central charge of the fixed point [2, 36–38]: each Dirac fermion with vector R-charge q and axial R-charge ∓ 1 contributes by $-q$ to the normalized central charge $\widehat{c} = c/3$. Recall that the fermionic component of a chiral multiplet of vector R-charge Q has $q = Q - 1$, and that the gaugino normally has $q = 1$. The central charge of the infra-red fixed point of the original theory is

$$\widehat{c} = kN - \frac{k(k-1)}{2}. \tag{4.28}$$

Note that this is the dimension of the Higgs branch H_k. This is consistent with the fact that the sigma model on a Kähler manifold of dimension d is classically a conformal field theory with central charge $\widehat{c} = d$. One the other hand, the central charge of the dual theory reads

$$\widehat{c}_{\rm dual} = \frac{N(N+1)}{2} - \frac{(N-k+1)(N-k)}{2}. \tag{4.29}$$

The two, (4.28) and (4.29), indeed agree. This comparison can be regarded as the 't Hooft anomaly matching for $\mathrm{U}(1)_R^2$, which is the only non-trivial one.

The dual theory in some detail. Let us study the low energy behaviour of the dual theory. We first look at the classical flat directions. The D-term equations are as in (4.2), and we also have the F-term equations from the superpotential (4.24),

$$(\widetilde{x}^i \widetilde{x}^j) = 0 \quad \forall (i,j), \tag{4.30}$$

$$s_{ij}\widetilde{x}^j_{\tilde{a}} = 0 \quad \forall (i,\tilde{a}). \tag{4.31}$$

The first equations and the D-term equations require $\widetilde{x}^1 = \cdots = \widetilde{x}^N = 0$, which makes the second equation vacuous. Thus, the space of classical vacua is just the space S_N parametrized by the singlets s_{ij}. It is the space of all symmetric $N \times N$ matrices and has dimension $\frac{N(N+1)}{2}$.

The gauge group is entirely unbroken everywhere in the flat directions, and therefore, quantum effects of gauge interactions should be taken into account. Let us first consider the case where the original gauge group is $\mathrm{O}_+(k)$ and the dual gauge group is $\mathrm{SO}(\widetilde{k})$, with $\widetilde{k} = N - k + 1$. As we have done many times in sections 2 and 3, we work in the Born-Oppenheimer approximation, treating the singlets s_{ij} as slow variables and $\mathrm{SO}(\widetilde{k})$ gauge fields and the fundamentals $\widetilde{x}^i$ as fast variables. From this view point, we may regard $s = (s_{ij})$ as a mass matrix for $\widetilde{x}^i$'s, and its corank (N minus its rank) is the effective number $N_{\rm eff}$ of massless fundamentals. We have learned that, if $N_{\rm eff} \leq \widetilde{k} - 2$, the supersymmetry is spontaneously broken, i.e., there is no zero energy state. Thus, unless the supersymmetry is entirely broken, the low energy dynamics will concentrate on the locus of s_{ij}'s where the matrix s has corank $\widetilde{k} - 1$ or higher. That is, rank $N - (\widetilde{k} - 1) = k$ or lower,

$$S_{N,\leq k} = \left\{ s \in S_N \,\middle|\, \mathrm{rank}(s) \leq k \right\}. \tag{4.32}$$

Let us look at the behaviour of the theory near such a locus. For concreteness, let us look at the region of S_N where the last $k \times k$ block of (s_{ij}) has rank k. We separate $\widetilde{x}^i$'s into two groups: the first $N - k$ of them, $\widetilde{x}^\alpha$ for $\alpha = 1, \ldots, N-k$, and the last k of them, $\widetilde{x}^\mu$ for $\mu = N-k+1, \ldots, N$. Fields from the latter group are massive and can be integrated out. This leaves us with the superpotential

$$W = \sum_{\alpha,\beta=1}^{N-k} \left(s_{\alpha\beta} - \sum_{\mu,\nu=N-k+1}^{N} s_{\alpha\mu} s^{\mu\nu} s_{\nu\beta} \right) (\widetilde{x}^\alpha \widetilde{x}^\beta) \tag{4.33}$$

In the above, $(s^{\mu\nu})$ is the inverse of the last $k \times k$ block $(s_{\mu\nu})$ of (s_{ij}). At this point, we again use what we have learned: the $\mathrm{SO}(\widetilde{k})$ theory with $N_{\rm eff} = N - k = \widetilde{k} - 1$ massless fundamentals is the free theory of the mesons at low energies. Then, the composites $(\widetilde{x}^\alpha \widetilde{x}^\beta)$

in (4.33) should be regarded as elementary fields and can be integrated out. This yields the constraints $s_{\alpha\beta} = \sum s_{\alpha\mu} s^{\mu\nu} s_{\nu\beta}$, which means that s is of rank k since

$$A = BC^{-1}B^T \implies \left(\begin{array}{c|c} A & B \\ B^T & C \end{array}\right) = \left(\begin{array}{c|c} \mathbf{1}_{N-k} & BC^{-1} \\ \hline & \mathbf{1}_k \end{array}\right) \left(\begin{array}{c|c} \mathbf{0}_{N-k} & \\ \hline & C \end{array}\right) \left(\begin{array}{c|c} \mathbf{1}_{N-k} & \\ \hline C^{-1}B^T & \mathbf{1}_k \end{array}\right). \tag{4.34}$$

Therefore, the low energy theory is the sigma model whose target space is the submanifold $S_{N,\leq k}$, in the region of the field space where the rank of s is at least k. The space $S_{N,\leq k}$ has codimension $\frac{(N-k)(N-k+1)}{2}$ in S_N,[3] and that explains the central charge (4.29). It can be regarded as the same space as the the Higgs branch $H_k = H_{\mathrm{O}(k),N}$ of the original theory, in the sense that both spaces are parametrized by $N \times N$ symmetric matrices of rank k or less: $(x_i x_j)$ for $H_{\mathrm{O}(k),N}$ and s_{ij} for $S_{N,\leq k}$, which indeed correspond to each other under the duality (4.25).

The analysis for the case where the original gauge group is $\mathrm{O}_-(k)$ and the dual group is $\mathrm{O}_-(\widetilde{k})$ proceeds in the same way. We find that the singlet s_{ij} is constrained to be in the subspace $S_{N,\leq k}$ and the dual theory reduces to the sigma model on $S_{N,\leq k}$ in the open domain of rank exactly k. Of course, this dual pair is different from the one above. In the original side, they differ in the orbifold projections in the twisted NSNS and untwisted RR sectors. In the dual side, the non-standard orbifold should be at work in the $\mathrm{O}_-(\widetilde{k})$ theory.

Finally, let us study the case where the original gauge group is $\mathrm{SO}(k)$ and the dual group is $\mathrm{O}_+(\widetilde{k})$. The analysis of the dual theory proceeds in the same way until the point where we use the low energy description of the theory with $N_{\text{eff}} = \widetilde{k} - 1$. In the present case, where the gauge group is $\mathrm{O}_+(\widetilde{k})$, there are two copies of the free theory of invariants $(\widetilde{x}^\alpha \widetilde{x}^\beta)$. Therefore, we have a *double cover* of $S_{N,\leq k}$ at least over the open subset consisting of matrices s of maximal rank k. The two sheets are exchanged under the $\mathbf{Z}_2$ quantum symmetry of the orbifold. Let us compare it with the Higgs branch of the original theory, $H_k = H_{\mathrm{SO}(k),N}$. Since the Higgs branch for the $\mathrm{O}(k)$ theory is obtained by a $\mathbf{Z}_2$ quotient of the one for the $\mathrm{SO}(k)$ theory, $H_{\mathrm{SO}(k),N}$ is indeed a double cover of $H_{\mathrm{O}(k),N}$. The baryons $[x_{i_1} \cdots x_{i_k}]$ are the ones that distinguish the two sheets above $H_{\mathrm{O}(k),N}$, and they are indeed claimed to be twist fields in the dual theory.

Flow by complex mass. Let us consider the $\mathrm{O}_+(k)$ *resp.* $\mathrm{SO}(k)$ *resp.* $\mathrm{O}_-(k)$ gauge theory with N fuandamental matter fields with a superpotential mass term for one of them, say the last one, $W = (x_N x_N)$. This introduces a term s_{NN} in the dual superpotential,

$$W = \sum_{i,j=1}^{N} s_{ij} \widetilde{x}^i \widetilde{x}^j + s_{NN}. \tag{4.35}$$

[3]The subspace of S_N consisting of matrices of corank i or higher has codimension $\frac{i(i+1)}{2}$: to choose such a matrix, we first choose a subspace of codimension i and then choose a symmetric bilinear form in that subspace. The first choice involves $i(N-i)$ parameters, as it corresponds to choosing a point of the Grassmannian $G(N-i, N)$, and the second choice involves $\frac{(N-i)(N-i+1)}{2}$ parameters. Therefore the codimension is $\frac{N(N+1)}{2} - \{i(N-i) + \frac{(N-i)(N-i+1)}{2}\} = \frac{i(i+1)}{2}$.

If we integrate out s_{NN}, we obtain the constraint $(\widetilde{x}^N)^2 + 1 = 0$ which can be solved by

$$\widetilde{x}^N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm i \end{pmatrix}. \tag{4.36}$$

It breaks the dual gauge group to the subgroup $SO(\widetilde{k}-1)$ *resp.* $O_+(\widetilde{k}-1)$ *resp.* $O_-(\widetilde{k}-1)$. Note that the solution is unique except in the case where $\widetilde{k} = 1$ and the dual gauge group is $SO(1) = \{1\}$, i.e., $N = k$ and the original gauge group is $O_+(k)$, in which the two solutions $+i$ and $-i$ are inequivalent. Plugging (4.36) to the superpotential we have terms of the form $\pm 2i s_{j'N} \widetilde{x}^{j'}_{\widetilde{k}}$ for $j' = 1, \ldots, N-1$. Integrating out $s_{j'N}$, we obtain the constraint $\widetilde{x}^{j'}_{\widetilde{k}} = 0$. Thus, we are left with the $SO(\widetilde{k}-1)$ *resp.* $O_+(\widetilde{k}-1)$ *resp.* $O_-(\widetilde{k}-1)$ gauge theory with $N-1$ fundamentals $\widetilde{x}'^1, \ldots, \widetilde{x}'^{N-1}$ and $\frac{(N-1)N}{2}$ singlets $s_{i'j'}$, having the remaining superpotential. This is indeed the dual of the $O_+(k)$ *resp.* $SO(k)$ *resp.* $O_-(k)$ theory with $N-1$ massless fuandamentals.

For the case where the starting point is $N = k$, the dual gauge group $SO(1)$ or $O_\pm(1)$ is trivial or completely broken by (4.36) and the "fundamentals" $\widetilde{x}^{j'}$'s are completely gone. If the original gauge group is $SO(k)$ or $O_-(k)$ (the dual group $O_+(1)$ or $O_-(1)$), we have the free theory of only the singlets $s_{i'j'}$, which correspond to the mezons $(x_{i'}x_{j'})$ in the original. If the original gauge group is $O_+(k)$ (the dual group $SO(1)$), then, since the two solutions (4.36), i.e., $\widetilde{x}^N = \pm i$, are inequivalent, we have two copies of the free theory of the singlets. To summarize, the duality reproduces the effective theory for the $N = k-1$ theory obtained in section 4.5.

Vacuum counting with twisted mass. As another test of duality, let us compare the number of supersymmetric ground states, or more precisely the Witten index, of the dual pair perturbed by twisted masses. The counting for the original theory, where $x_1, \ldots, x_N$ are given twisted masses $\widetilde{m}_1, \ldots, \widetilde{m}_N$, has been done in section 4.3 under the genericity assumption including (4.13). As this is associated with the $U(1)^N \subset U(N)$ global symmetry, this corresponds in the dual side to giving twisted mass $\widetilde{m}_i + \widetilde{m}_j$ to s_{ij} and $-\widetilde{m}_i$ to $\widetilde{x}^i$. Note that the masses for s_{ij} are all non-zero under (4.13). We discussed in section 2.3 the vacuum counting problem in such a system. As argued there, the spectrum of supersymmetric ground states, or at least the Witten index, is expected not to change if we turn off the superpotential, since no vacuum runs off to nor come in from infinity. (This was confirmed in a simple example by an exact analysis in appendix A.) Then, since the singlet sector provides "one" as the number of ground states, the total number is the same as the theory of $\widetilde{x}$'s only. For this the result of section 4.3 is applicable, though of course for the dual group. Thus, the comparison is a straightforward task — just stare the table (4.20). A complete match!

4.7 Chiral rings

We discuss the chiral rings, both (c, c) and (a, c) rings, of the models we are studying. In some cases, the duality can be used to determine them. In some other cases, we can determine the rings in both sides of the dual pair and the result can be used to test the duality.

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The (c,c) ring. The classical (c,c) ring is the ring of gauge invariant polynomials of the chiral multiplet fields. For the O(k) theory (in the untwisted sector) and for the SO(k) theory, it is respectively [32]

$$\mathbf{C}[x_1,\ldots,x_N]^{\mathrm{O}(k,\mathbf{C})} = \mathbf{C}\big[\,(x_ix_j)\,\big]/J_1, \tag{4.37}$$

and

$$\mathbf{C}[x_1,\ldots,x_N]^{\mathrm{SO}(k,\mathbf{C})} = \mathbf{C}\big[\,(x_ix_j),\,[x_{i_1}\cdots x_{i_k}]\,\big]/(J_1,J_2,J_3), \tag{4.38}$$

where J_1, J_2, J_3 denote relations of the form

$$J_1: \qquad \det\begin{pmatrix}(x_{i_0}x_{j_0}) & \cdots & (x_{i_0}x_{j_k})\\ \vdots & & \vdots\\ (x_{i_k}x_{j_0}) & \cdots & (x_{i_k}x_{j_k})\end{pmatrix} = 0, \tag{4.39}$$

$$J_2: \qquad [x_{i_1}\cdots x_{i_k}][x_{j_1}\cdots x_{j_k}] = \det\begin{pmatrix}(x_{i_1}x_{j_1}) & \cdots & (x_{i_1}x_{j_k})\\ \vdots & & \vdots\\ (x_{i_k}x_{j_1}) & \cdots & (x_{i_k}x_{j_k})\end{pmatrix}, \tag{4.40}$$

$$J_3: \qquad \sum_{p=0}^{k}(-1)^p[x_{i_0}\cdots\widehat{x_{i_p}}\cdots x_{i_k}](x_{i_p}x_j) = 0. \tag{4.41}$$

These relations must be satisfied in the semiclassical regime where the gauge group is completely broken, and hence must be the exact chiral ring relations, as a potential parameter of correction does not exist. In the dual side, the corresponding relations are not visible in the classical theory and appear only in the infra-red limit of the quantum theory. Indeed the relations J_1 are consistent with the constraint $\mathrm{rank}(s)\leq k$ obtained in the paragraph including (4.32)–(4.34). In the case of O(k) gauge group, we also have (c,c) ring elements from the twisted sector. For the O$_+(k)$ theory, the twist fields correspond to $\mathrm{O}(\widetilde{k})/\mathrm{SO}(\widetilde{k})$ anti-invariants in the dual theory and are generated by the baryons $[\widetilde{x}^{i_1}\cdots\widetilde{x}^{i_{\widetilde{k}}}]$. We can find the relations involving these fields using the above relations J_2 as well as the classical constraint from the F-term equations (4.30)–(4.31). These lead to the following conclusions. We write $s_{ij}=(x_ix_j)$, $b_{i_1\cdots i_k}=[x_{i_1}\cdots x_{i_k}]$, $\widetilde{b}^{i_1\cdots i_{\widetilde{k}}}=[\widetilde{x}^{i_1}\cdots\widetilde{x}^{i_{\widetilde{k}}}]$ whenever appropriate.

The (c,c) ring of the O$_+(k)$ theory is the polynomial ring of s_{ij} and $\widetilde{b}^{i_1\cdots i_{\widetilde{k}}}$ modulo the relations

$$\det\begin{pmatrix}s_{i_0j_0} & \cdots & s_{i_0j_k}\\ \vdots & & \vdots\\ s_{i_kj_0} & \cdots & s_{i_kj_k}\end{pmatrix} = 0, \qquad \widetilde{b}^{i_1\cdots i_{\widetilde{k}}}\,\widetilde{b}^{j_1\cdots j_{\widetilde{k}}} = 0, \qquad \sum_{j_1=1}^{N} s_{ij_1}\,\widetilde{b}^{j_1j_2\cdots j_{\widetilde{k}}} = 0. \tag{4.42}$$

The (c,c) ring of the SO(k) theory is the polynomial ring of s_{ij} and $b_{i_1\cdots i_k}$ modulo the relations

$$\det\begin{pmatrix}s_{i_0j_0} & \cdots & s_{i_0j_k}\\ \vdots & & \vdots\\ s_{i_kj_0} & \cdots & s_{i_kj_k}\end{pmatrix} = 0, \qquad b_{i_1\cdots i_{\widetilde{k}}}\,b_{j_1\cdots j_{\widetilde{k}}} = \det\begin{pmatrix}s_{i_1j_1} & \cdots & s_{i_1j_k}\\ \vdots & & \vdots\\ s_{i_kj_1} & \cdots & s_{i_kj_k}\end{pmatrix},$$

$$\sum_{p=0}^{k}(-1)^p\,s_{ij_p}\,b_{j_0\cdots\widehat{j_p}\cdots j_k} = 0. \tag{4.43}$$

The (c,c) ring of the untwisted elements of the $\mathrm{O}_-(k)$ theory is the polynomial ring of s_{ij} modulo the relations

$$\det\begin{pmatrix} s_{i_0j_0} & \cdots & s_{i_0j_k} \\ \vdots & & \vdots \\ s_{i_kj_0} & \cdots & s_{i_kj_k} \end{pmatrix} = 0. \tag{4.44}$$

The (a,c) ring. Classically, the (a,c) ring is the ring of gauge invariant polynomials of the scalar components σ of the gauge multiplets, which is isomorphic to the ring of Weyl invariant polynomials of the components $\sigma_1,\ldots,\sigma_\ell$ for the maximal torus. Examples of elements are

$$c_{2i} = \mathrm{tr}(\sigma^{2i}) = \sum_{a=1}^{\ell}\sigma_a^{2i}, \qquad p_\ell = \mathrm{Pf}(\sigma) = \sigma_1\cdots\sigma_\ell \ \ (\text{for } \mathrm{SO}(2\ell)). \tag{4.45}$$

As the generators, we can take $c_2,\ldots,c_{2\ell}$ for $\mathrm{SO}(2\ell+1)$, $\mathrm{O}(2\ell+1)$ and $\mathrm{O}(2\ell)$, and $c_2,\ldots,c_{2\ell-2},p_\ell$ for $\mathrm{SO}(2\ell)$. There are no relations among them and thus the ring is the polynomial ring of these generating variables. The underlying vector space is of course infinite dimensional.

The story is different in the quantum theory. Coulomb branch is lifted by quantum corrections if the theory is regular. This implies that the underlying vector space of the (a,c) ring of the infra-red conformal field theory is finite dimensional. Thus, we expect to have quantum relations among the generators, $c_2,\ldots,c_{2\ell}$, or $c_2,\ldots,c_{2\ell-2},p_\ell$. In addition, we also have (a,c) ring elements from the twisted sector in the theories with gauge group $\mathrm{O}(k)$ or $\mathrm{O}(\widetilde{k})$. In the $\mathrm{SO}(k)$ and $\mathrm{O}_+(k)$ theories, where we have the spectral flow between (a,c) ring elements and RR ground states, the dimension is expected to be equal to the number (4.20) of supersymmetric ground states in the model perturbed by twisted masses.

The key to find the quantum relations is the relations (4.15) for the mass deformed system. In the massless limit, $\widetilde{m}_i \to 0$, they become

$$(\sigma_a)^N = 0 \qquad a = 1,\ldots,\ell. \tag{4.46}$$

From these we would like to extract relations among the gauge invariants. Analogous problem has been discussed in [39] (see also [30, 35]). We apply the "change of variables method" from that reference to the case when N is odd, which proceeds as follows. The relations (4.46) are the Jacobi relations of the function $\frac{1}{N+1}(\sigma_1^{N+1}+\cdots+\sigma_\ell^{N+1})$, which is invariant under the $\mathrm{SO}(k)$ and $\mathrm{O}(k)$ Weyl group when N is odd. We express this function in terms of the generators of the $\mathrm{SO}(k)$ Weyl invariants, and call it W;

$$\frac{1}{N+1}(\sigma_1^{N+1}+\cdots+\sigma_\ell^{N+1}) = \begin{cases} W(c_2,\ldots,c_{2\ell-2},p) & k \text{ even}, \\ W(c_2,\ldots,c_{2\ell}) & k \text{ odd}, \end{cases} \tag{4.47}$$

For the $\mathrm{SO}(k)$ theory, the (a,c) ring is identified with the Jacobi ring of W, that is, the (c,c) ring of the Landau-Ginzburg model with the superpotential W. For the $\mathrm{O}_\pm(k)$ theory, the (a,c) ring is identified with the (c,c) ring of the Landau-Ginzburg *orbifold* with the superpotential W with respect to $\mathrm{O}_\pm(k)/\mathrm{SO}(k) \cong \mathbf{Z}_2$. We are not claiming that the conformal field theory is dual, or mirror to be precise, to the Landau-Ginzburg model/orbifold. We are simply identifying the (a,c) ring of our theory with the (c,c) ring of the LG. The two theories cannot be mirror to each other as the (c,c) ring of our theory is infinite dimensional while the (a,c) ring of the LG is finite dimensional.

This applies both to the original $\mathrm{O}(k)$ or $\mathrm{SO}(k)$ theory and to the dual $\mathrm{O}(\widetilde{k})$ or $\mathrm{SO}(\widetilde{k})$ theory. Let us compute the ring in some examples, and check against the duality.

$\mathbf{SO(2)}, \boldsymbol{N}=\mathbf{5}$ versus $\mathbf{O_+(4)}, \boldsymbol{N}=\mathbf{5}$. The (a,c) ring of the SO(2) theory is the Jacobi ring of $W=\frac{1}{6}p_1^6$, i.e. the polynomial ring of p_1 modulo the relation

$$p_1^5=0. \tag{4.48}$$

Under the $\mathbf{Z}_2=\mathrm{O}(2)/\mathrm{SO}(2)$, three elements, $1,p^2,p^4$, are even and two elements, p,p^3, are odd. The (a,c) ring of the $\mathrm{O}_+(4)$ theory is the (c,c) ring of a LG orbifold with the superpotential $W=\frac{1}{6}c_2^3-\frac{1}{2}p_2^2c_2$ with respect to $\mathbf{Z}_2:(c_2,p_2)\to(c_2,-p_2)$. The spectra of (c,c) elements and RR ground states for the two cases are as follows (We follow the notation of [25]. In particular, $K_\tau\in\mathbf{Z}/2\mathbf{Z}$ is the parameter that distinguishes two possible orbifold projections): for $K_\tau=1$, the states surviving the orbifold projection are

$$(c,c):\quad\begin{cases}|0\rangle^1_{(c,c)},\ c_2|0\rangle^1_{(c,c)},\ c_2^2|0\rangle^1_{(c,c)} & \text{from untwisted}\\ |0\rangle^\tau_{(c,c)},\ c_2|0\rangle^\tau_{(c,c)} & \text{from twisted}\end{cases} \tag{4.49}$$

$$\mathrm{RR}:\quad\begin{cases}|0\rangle^1_{\mathrm{R}},\ c_2|0\rangle^1_{\mathrm{R}},\ c_2^2|0\rangle^1_{\mathrm{R}} & \text{from untwisted}\\ |0\rangle^\tau_{\mathrm{R}},\ c_2|0\rangle^\tau_{\mathrm{R}} & \text{from twisted.}\end{cases} \tag{4.50}$$

For $K_\tau=0$, the surviving states are

$$(c,c):\quad\begin{cases}|0\rangle^1_{(c,c)},\ c_2|0\rangle^1_{(c,c)},\ c_2^2|0\rangle^1_{(c,c)} & \text{from untwisted}\\ \qquad\text{none} & \text{from twisted}\end{cases} \tag{4.51}$$

$$\mathrm{RR}:\quad\begin{cases}p_2|0\rangle^1_{\mathrm{R}} & \text{from untwisted}\\ |0\rangle^\tau_{\mathrm{R}},\ c_2|0\rangle^\tau_{\mathrm{R}} & \text{from twisted.}\end{cases} \tag{4.52}$$

We see that we need to take $K_\tau=1$ for the $\mathrm{O}_+(4)$ theory, in order to have 3 untwisted and 2 twisted (c,c) ring elements, as expected from the vacuum counting in section 4.3. (Then $K_\tau=0$ should correspond to the $\mathrm{O}_-(4)$ theory. Note that the number of RR ground states also matches with (4.20) also for $\mathrm{O}_-(4)$.) The ring relation is standard for the untwisted sector elements. Relations involving twist operators can be found using the recent result by Krawitz [40].[4] Let 1_τ be an element corresponding to the state $|0\rangle^\tau_{(c,c)}$. The ring relation is then

$$1_\tau\cdot 1_\tau=-c_2,\quad c_2^2\cdot 1_\tau=0, \tag{4.53}$$

in addition to $c_2^3=0$ that comes from the Jacobi relations $c_2^2=p_2^2$, $p_2c_2=0$. The rings for the dual pair are indeed isomorphic under the correspondence

$$1,p_1,p_1^2,p_1^3,p_1^4\longleftrightarrow 1,1_\tau,-c_2,-c_2\cdot 1_\tau,c_2^2. \tag{4.54}$$

[4]See Definition 13, eq. (2.7). We thank Tyler Jarvis for instruction on the ring structure and also pointing out that arXiv:0906.0796 has an error in Definition 2, eq. (10).

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We can also perform the test for the other versions of dual pair: $O_+(2)$ versus $SO(4)$ as well as $O_-(2)$ versus $O_-(4)$. The ring of $O_+(2)$ is again found from [40]. It is generated by 1_τ and p_1^2 which obey the relations $p_1^2 \cdot 1_\tau = 0$ and $1_\tau \cdot 1_\tau = p_1^4$. The ring of $SO(4)$ is the Jocobi ring of $W = c_2^3/6 - p_2^2 c_2/2$, i.e., the polynomial ring of c_2 and p_2 modulo the relation $c_2^2 = p_2^2, p_2 c_2 = 0$. The two rings are isomorphic under

$$1, p_1^2, 1_\tau, p_1^4 \longleftrightarrow 1, c_2, p_2, c_2^2. \tag{4.55}$$

The ring of $O_-(2)$ is the ring of even polynomials of p_1 modulo $p_1^5 = 0$ while the ring of $O_-(4)$ is the ring of polynomials of c_2 with $c_2^3 = 0$. They are obviously isomorphic.

$SO(3), N = 7$ versus $O_+(5), N = 7$. The (a, c) ring of the $SO(3)$ theory is the Jacobi ring of $W = \frac{1}{8}c_2^4$, that is, the polynomial ring of c_2 modulo the relation $c_2^3 = 0$. The (a, c) ring of the $O_+(5)$ theory is the (c, c) ring of the orbifold of the LG model with $W = \frac{1}{8}c_2^2 c_4 + \frac{1}{16}c_4^2 - \frac{1}{16}c_2^4$ by the $\mathbf{Z}_2$ that acts trivially on the variables. In order to be consistent with (4.20), the orbifold must be the one which is isomorphic to the model without the orbifold. The ring is therefore the polynomial ring of c_2 and c_4 modulo the relation $c_2^2 + c_4 = c_2^3 - c_2 c_4 = 0$, that is, the polynomial ring of c_2 modulo the relation $c_2^3 = 0$. The two rings are indeed isomorphic to each other. Nothing changes for the dual pair $O_+(3)$ versus $SO(5)$. For the pair $O_-(3)$ versus $O_-(5)$, the ring is doubled on both sides — for each untwisted element, there is a copy in the twisted sector, and the ring relation is the obvious one. The two rings are isomorphic to each other.

In this paper, we do not try to give full rational for the above procedure to determine the (a, c) ring for odd N case, nor even to propose the ring for even N case. Also, we do not attempt to prove the isomorphism for general dual pair. It is possible that the would be proven isomorphism is related to the level-rank duality for the fusion ring of the Wess-Zumino-Witten models or for the chiral ring of Kazama-Suzuki models. We postpone the full account on these for future work.

5 Symplectic groups

In this section, we study low energy behaviour of theories with the symplectic gauge group $USp(k)$ with N chiral multiplets, $x_1, \ldots, x_N$ in the fundamental representation $\mathbf{k}$. Here k is an even integer, $k = 2\ell$, for $\ell = 1, 2, 3, \ldots$. We recall that the group $USp(k)$ is the group of $k \times k$ unitary matrix that preserves the symplectic structure defined by the matrix

$$J_k = \left(\begin{array}{c|c} & -\mathbf{1}_\ell \\ \hline \mathbf{1}_\ell & \end{array} \right). \tag{5.1}$$

That is, $USp(k)$ consists of $k \times k$ matrix g satisfying $g^\dagger g = 1_k$ and $g^T J_k g = J_k$. It is simply connected and hence any principal $USp(k)$ bundle on a closed surface is topologically trivial. In particular, there is no room for theta angle.

Note that $USp(k)$ and $SU(k)$ coincide at $k = 2$. Some of the results below for the $k = 2$ case had been obtained in [2] as results for $SU(2)$ gauge theories.

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5.1 The space of classical vacua

Let us first describe the space of classical vacua. We denote the scalar component of the vector multiplet by σ. It is a $k \times k$ matrix such that $J_k \sigma$ is symmetric. We write x for the $k \times N$ matrix (x_i^a), and denote by $x_\uparrow$ and $x_\downarrow$ its upper and lower $\ell \times N$ submatrices. The vacuum equation reads

$$\begin{aligned} [\sigma, \sigma^\dagger] &= 0, \\ x_\uparrow x_\uparrow^\dagger &= (x_\downarrow x_\downarrow^\dagger)^T, \quad x_\uparrow x_\downarrow^\dagger = -(x_\uparrow x_\downarrow^\dagger)^T, \\ \sigma x &= \sigma^\dagger x = 0. \end{aligned} \tag{5.2}$$

The first equation means that, up to gauge transformations, σ is of the form

$$\left(\begin{array}{c|c} \begin{matrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\ell \end{matrix} & \\ \hline & \begin{matrix} -\sigma_1 & & \\ & \ddots & \\ & & -\sigma_\ell \end{matrix} \end{array}\right) \tag{5.3}$$

Let us introduce $\ell \times N$ quaternion matrix $\mathbf{x} = x_\uparrow + j x_\downarrow$. The combination $\mathbf{x}\mathbf{x}^\dagger$ is a self-adjoint quaternion matrix and can be diagonalized using a $\mathrm{U}(\mathbf{H}^k) \cong \mathrm{USp}(k)$ conjugation. By the equations on the second line, this means that (after the gauge rotation) $x_\uparrow x_\uparrow^\dagger = (x_\downarrow x_\downarrow^\dagger)^T$ is a real diagonal matrix and $x_\uparrow x_\downarrow^\dagger$ vanishes. This implies that, with an appropriate $\mathrm{U}(N)$ flavor rotation, the solution can be made into the form

$$x = \left(\begin{array}{c|c|c} \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_m \end{matrix} & & \\ \hline & \begin{matrix} a_1 & & \\ & \ddots & \\ & & a_m \end{matrix} & \end{array}\right), \tag{5.4}$$

where $r = 2m$ can range over even numbers from 0 to $\min\{k, N\}$ (*resp.* $\min\{k, N-1\}$) for even (*resp.* odd) N. The final equation requires that if the number of non-zero σ_a's is s, then the number of non-zero a_i's is at most $\ell - s$. Let C_s (*resp.* H_r) be the set of gauge equivalence class of solutions for σ of rank $2s$ or less (*resp.* x of rank r or less). It has complex dimension s (*resp.* $Nr - \frac{r(r+1)}{2}$). The space of classical vacua is

$$\mathcal{M} = \bigcup_{s=s_{\min}}^{\ell} \Big(C_s \times H_{k-2s} \Big), \tag{5.5}$$

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where $s_{\min} = 0$ if $N \geq k$ and $s_{\min} = [\frac{k-N+1}{2}]$ if $N \leq k-1$. When $N \geq k$ there is a Higgs branch $C_0 \times H_k$. There is always a Coulomb branch $C_\ell \times H_0$. Other components are the mixed Coulomb-Higgs branches.

5.2 Regularity

We are interested in regular theory where all the Coulomb and mixed branches are lifted by quantum corrections. The computation of the effective twisted superpotential is very simple compared to the orthogonal groups. Let us consider the classical Coulomb branch, $C_\ell \times H_0$. The massive vector mutiplets give no contribution to the twisted superpotential — there are four multiplets that are charged under $\mathrm{U}(1)_a \times \mathrm{U}(1)_b$ and yields $\pi i(\sigma_a+\sigma_b)+\pi i(\sigma_a-\sigma_b) \equiv 0$. The massive chiral multiplets give the usual contribution, and the result is

$$\widetilde{W}_{\text{eff}} = -\sum_{i,a} \sigma_a(\log \sigma_a - 1) - \sum_{i,a} (-\sigma_a)(\log(-\sigma_a) - 1) = \pi i N \sum_{a=1}^{\ell} \sigma_a. \tag{5.6}$$

Computation on the mixed branches gives the same result except that the sum is over $a = 1, \ldots, s$ for $C_s \times H_{k-2s}$. The conclusion is that *the theory is regular if and only if N is odd.*

The theory with even N is not regular. Unlike in the orthogonal groups, the symplectic group is simply connected and does not allow any theta angle. Also, the trick using complex mass does not work here — the gauge invariants $[x_i x_j]$ are antisymmetric in $i \leftrightarrow j$ and any mass reduces the degrees of freedom by *even* number.

5.3 Twisted masses

Let us give twisted masses $\widetilde{m}_1, \ldots, \widetilde{m}_N$ to the chiral matter fields $x_1, \ldots, x_N$, and study the spectrum of supersymmetric ground states. We assume genericity of $\widetilde{m}_i$'s including $\widetilde{m}_i + \widetilde{m}_j \neq 0$ so that the Higgs branch is lifted everywhere on the σ-space. The effective twisted superpotential is

$$\widetilde{W}_{\text{eff}} = -\sum_{i,a} (\sigma_a - \widetilde{m}_i)(\log(\sigma_a - \widetilde{m}_i) - 1) - \sum_{i,a} (-\sigma_a - \widetilde{m}_i)(\log(-\sigma_a - \widetilde{m}_i) - 1). \tag{5.7}$$

and the vacuum equation for $\sigma = \sigma_1, \ldots, \sigma_\ell$ reads

$$\prod_{i=1}^{N} (\sigma - \widetilde{m}_i) = (-1)^{N+1} \prod_{i=1}^{N} (-\sigma - \widetilde{m}_i). \tag{5.8}$$

The solutions are identified under the action of the Weyl group: permutations and independent sign flips of σ_a's. We require the solutions to obey

$$\begin{aligned} \sigma_a &\neq \pm \widetilde{m}_i, \\ \sigma_a &\neq \sigma_b \quad a \neq b, \\ \sigma_a + \sigma_b &\neq 0 \quad \forall (a,b). \end{aligned} \tag{5.9}$$

The same remark made for the $\mathrm{O}(k)$ or $\mathrm{SO}(k)$ gauge theory after (4.17) applies here without modification.

The equation (5.8) has a single root at $\sigma = 0$ and $\frac{N-1}{2}$ pairs of non-zero roots. There are solutions for σ_a's obeying (5.9) if and only if $\frac{N-1}{2} \geq \ell$, i.e., $N \geq k+1$. The number of inequivalent solutions is

$$\binom{\frac{N-1}{2}}{\frac{k}{2}}. \tag{5.10}$$

The Weyl group is completely broken at each solution, and hence this is the number of supersymmetric ground states.

5.4 $N \leq k$: supersymmetry breaking

By the result of the previous subsection and applying the same argument as in the orthogonal gauge groups, we conclude the following: there is no normalizable supersymmetric ground state in pure USp(k) Yang-Mills theory (an irregular theory) as well as in the theory with $N \leq k$ massless fundamentals, for both N odd (regular theory) and N even (irregular theory).

5.5 $N = k+1$: free conformal field theory

Let us now consider the theory with $N = k+1$ massless fundamentals. In the regular theory, the Coulomb and mixed branches are lifted and we are left with the Higgs branch H_k only. As a complex manifold, the Higgs branch is isomorphic to the affine space $\mathbf{C}^{\frac{(k+1)k}{2}}$ since the chiral ring of gauge invariants is isomorphic to the polynomial ring of the $\binom{k+1}{2}$ symplectic products $[x_i x_j]$ (the "mesons") with no relations,

$$\mathbf{C}[x_1, \ldots, x_{k+1}]^{\mathrm{Sp}(k,\mathbf{C})} = \mathbf{C}\Big[\, [x_i x_j] \,\Big|\, 1 \leq i < j \leq k+1 \Big] \tag{5.11}$$

The metric is singular at the roots of Coulomb and mixed branches, but the singularity is expected to be smeared as these branches are lifted. We claim that *the* USp(k) *gauge theory with $N = k+1$ massless fundamentals flows in the infra-red limit to the free conformal field theory of the $\frac{(k+1)k}{2}$ mesonic variables.*

5.6 $N \geq k+3$: duality

Finally, let us consider the theory with an odd $N \geq k+3$ massless fundamentals. We claim that there is a duality:

The USp(k) *gauge theory with N fundamentals $x_1, \ldots, x_N$ flows in the infra-red limit to the same fixed point as the* USp($N-k-1$) *gauge theory with N fundamentals, $\widetilde{x}^1, \ldots, \widetilde{x}^N$, and $\frac{N(N-1)}{2}$ singlets, $a_{ij} = -a_{ji}$ ($1 \leq i, j \leq N$), having the superpotential*

$$W = \sum_{i,j=1}^{N} a_{ij} [\widetilde{x}^i \widetilde{x}^j]. \tag{5.12}$$

The mesons in the original theory correspond to the singlets in the dual,

$$[x_i x_j] = a_{ij}. \tag{5.13}$$

It is a duality — the dual of the dual is the same as the original. We omit the detail here, except showing the equality, $N - (N-k-1) - 1 = k$.

The central charge. The two theories has the same symmetry: the $\mathrm{SU}(N) \times \mathrm{U}(1)_B$ flavor symmetry and the vector and axial U(1) R-symmetries:

	$\mathrm{SU}(N)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_A$
x	$\mathbf{N}$	1	0	0
$\widetilde{x}$	$\overline{\mathbf{N}}$	-1	1	0
s	$\mathbf{A}$	2	0	0

(5.14)

We have assigned vanishing R-charges to x, so that the two U(1) R's can become parts of the $(2, 2)$ superconformal symmetry in the infra-red fixed point of the original theory. Assuming that they indeed do correspond to the parts of the superconformal symmetry, let us compute the central charge of the original theory. The one for the original theory is

$$\widehat{c} = kN - \frac{k(k+1)}{2}, \tag{5.15}$$

which is also the dimension of the Higgs branch H_k. The one for the dual theory is

$$\widehat{c}_{\mathrm{dual}} = \frac{N(N-1)}{2} - \frac{(N-k-1)(N-k)}{2}. \tag{5.16}$$

The two, (5.15) and (5.16), indeed agree.

The dual theory in some detail. Let us study the low energy behaviour of the dual theory with gauge group $\mathrm{USp}(\widetilde{k})$, with $\widetilde{k} = N - k - 1$. The D-term equations are like (5.2) and the F-term equations from (5.12) are

$$[\widetilde{x}^i \widetilde{x}^j] = 0 \quad \forall (i, j), \tag{5.17}$$

$$a_{ij} \widetilde{x}^j_{\tilde{a}} = 0 \quad \forall (i, \tilde{a}). \tag{5.18}$$

They force $\widetilde{x}^1 = \cdots = \widetilde{x}^N = 0$ but no condition on the singlets a_{ij}. The space of classical vacua is the space $A_N = \{(a_{ij})\} \cong \mathbf{C}^{\frac{N(N-1)}{2}}$ of $N \times N$ antisymmetric matrices. The gauge group is unbroken everywhere, and quantum effects of gauge interactions must be taken into account. When we view $a = (a_{ij})$ as the mass matrix for $\widetilde{x}^i$ and study the gauge sector first, the nature of the low energy theory depends very much on the corank of a, as it is equal to the effective number N_{eff} of massless fundamentals. If $N_{\mathrm{eff}} \leq \widetilde{k} - 1$, there is no zero energy state. Thus, the low energy dynamics will concentrate on the locus where the matrix a has corank $\widetilde{k} + 1$ or higher, i.e., rank $N - (\widetilde{k} + 1) = k$ or lower,

$$A_{N,\leq k} = \left\{ \, a \in A_N \,\middle|\, \mathrm{rank}(a) \leq k \, \right\}. \tag{5.19}$$

Let us look at the behaviour of the theory near such a locus. For concreteness, let us look at the region of A_N where the last $k \times k$ block of (a_{ij}) has rank k. We separate $\widetilde{x}^i$'s into two groups: the first $N - k$ and the last k of them, $\widetilde{x}^\alpha$ and $\widetilde{x}^\mu$. Integrating out the massive fields from the latter, we obtain the superpotential

$$W = \sum_{\alpha,\beta=1}^{N-k} \left(a_{\alpha\beta} - \sum_{\mu,\nu=N-k+1}^{N} a_{\alpha\mu} a^{\mu\nu} a_{\nu\beta} \right) [\widetilde{x}^\alpha \widetilde{x}^\beta], \tag{5.20}$$

where $(a^{\mu\nu})$ is the inverse of the last $k \times k$ block $(a_{\mu\nu})$ of (a_{ij}). The $\mathrm{USp}(\widetilde{k})$ theory with $N_{\text{eff}} = N-k = \widetilde{k}+1$ massless fundamentals is the free theory of the mesons at low energies. Then, the composites $[\widetilde{x}^\alpha \widetilde{x}^\beta]$ in (5.20) can be integrate them out as elementary fields and we obtain the constraints $a_{\alpha\beta} = \sum a_{\alpha\mu} a^{\mu\nu} a_{\nu\beta}$. This means that a is of rank k since

$$A = -BC^{-1}B^T \Longrightarrow \left(\begin{array}{c|c} A & B \\ \hline -B^T & C \end{array}\right) = \left(\begin{array}{c|c} \mathbf{1}_{N-k} & BC^{-1} \\ \hline & \mathbf{1}_k \end{array}\right) \left(\begin{array}{c|c} \mathbf{0}_{N-k} & \\ \hline & C \end{array}\right) \left(\begin{array}{c|c} \mathbf{1}_{N-k} & \\ \hline -C^{-1}B^T & \mathbf{1}_k \end{array}\right). \tag{5.21}$$

Therefore, the low energy theory is the sigma model whose target space is the submanifold $A_{N,\leq k}$, in the region of the field space where the rank of a is at least k. The space $A_{N,\leq k}$ has codimension $\frac{(N-k)(N-k-1)}{2}$ in A_N,[5] which explains the central charge (5.16). It can be regarded as the same space as the Higgs branch $H_k = H_{\mathrm{USp}(k),N}$ of the original theory, in the sense that both spaces are parametrized by $N \times N$ antisymmetric matrices of rank k or less: $[x_i x_j]$ for $H_{\mathrm{USp}(k),N}$ and a_{ij} for $A_{N,\leq k}$, which indeed correspond to each other under the duality (5.13).

Flow by complex mass. Let us consider the theory with a superpotential mass term for two of the N fundamentals, say the last two, $W = [x_{N-1} x_N]$. This introduces a term $a_{(N-1)\,N}$ in the dual superpotential,

$$W = \sum_{i,j=1}^{N} a_{ij} [\widetilde{x}^i \widetilde{x}^j] + a_{(N-1)\,N}. \tag{5.22}$$

If we integrate out $a_{(N-1)\,N}$, we obtain the constraint $[\widetilde{x}^{N-1} \widetilde{x}^N] + 1 = 0$, which can be solved by

$$\widetilde{x}^{N-1} = \left(\begin{array}{c} \mathbf{e}_\ell \\ \hline \mathbf{0}_\ell \end{array}\right), \quad \widetilde{x}^N = \left(\begin{array}{c} \mathbf{0}_\ell \\ \hline \mathbf{e}_\ell \end{array}\right) \quad \text{where} \quad \mathbf{e}_\ell = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{5.23}$$

They break the dual gauge group to the subgroup $\mathrm{USp}(N-k-3)$. Plugging them back to the superpotential, we have terms of the form $a_{j'N} \widetilde{x}^{j'}_\ell - a_{j'(N-1)} \widetilde{x}^{j'}_{2\ell}$ for $j' = 1, \ldots, N-2$. Integrating out $a_{j'(N-1)}$ and $a_{j'N}$, we obtain the constraint $\widetilde{x}^{j'}_a = 0$ for $j' = 1, \ldots, N-2$ and $a = \ell, 2\ell$. Thus, we are left with the $\mathrm{USp}(N-k-3)$ gauge theory with $N-2$ fundamentals $\widetilde{x}'^1, \ldots, \widetilde{x}'^{N-2}$ and $\frac{(N-2)(N-3)}{2}$ singlets $a_{i'j'}$, having the remaining superpotential. This is indeed the dual of the $\mathrm{USp}(k)$ theory with $N-2$ massless fundamentals.

If the starting point was $N = k+3$, the dual gauge group $\mathrm{USp}(2)$ is completely broken and the fundamentals are all gone. What remains is the free theory of the singlets $a_{i'j'}$ for $i, j = 1, \ldots, k+1$, which correspond to the mesons $[x_i x_j]$. The duality indeed reproduces the effective theory for the $N = k+1$ theory obtained in section 5.5.

[5]The subspace of A_N consisting of matrices of an odd corank i or higher has codimension $\frac{i(i-1)}{2}$: to choose such a matrix, we first choose a subspace of codimension i and then choose an antisymmetric bilinear form in that subspace. The first choice involves $i(N-i)$ parameters, as it corresponds to choosing a point of the Grassmannian $G(N-i, N)$, and the second choice involves $\frac{(N-i)(N-i-1)}{2}$ parameters. Therefore the codimension is $\frac{N(N-1)}{2} - \{i(N-i) + \frac{(N-i)(N-i-1)}{2}\} = \frac{i(i-1)}{2}$.

Vacuum counting with twisted mass. Let us compare the Witten index of the dual pair perturbed by twisted masses. The counting for the original theory, where $x_1, \ldots, x_N$ are given twisted masses $\widetilde{m}_1, \ldots, \widetilde{m}_N$, has been done in section 5.3 under a certain genericity assumption. This corresponds, in the dual side, to non-zero and generic twisted masses $\widetilde{m}_i + \widetilde{m}_j$ for a_{ij} and $-\widetilde{m}_i$ for $\widetilde{x}^i$. We expect that the Witten index does not change as we turn off the superpotential, since no vacuum runs off to nor come in from infinity, Then, it is the same as the theory of $\widetilde{x}$'s only. For this the result of section 5.3 is applicable, though of course for the dual group. Thus, we only have to compare (5.10) for USp(k) and for USp($N-k-1$). They indeed agree with each other.

5.7 Chiral rings

The (c,c) ring. The classical (c,c) ring is the ring of gauge invariant polynomials of the chiral multiplet fields, which is known to be [32]

$$\mathbf{C}[x_1, \ldots, x_N]^{\mathrm{Sp}(k,\mathbf{C})} = \mathbf{C}\Big[\,[x_i x_j]\,\Big]/(J_0, J_1, \ldots, J_\ell) \tag{5.24}$$

where the relations are

$$\begin{aligned}
J_0 :&\quad \sum_{\sigma\in\mathfrak{S}_{k+1}} (-1)^{l(\sigma)}[x_{j_0}x_{i_{\sigma(0)}}][x_{i_{\sigma(1)}}x_{i_{\sigma(2)}}]\cdots[x_{i_{\sigma(k-1)}}x_{i_{\sigma(k)}}] = 0,\\
J_1 :&\quad \sum_{\sigma\in\mathfrak{S}_{k+1}} (-1)^{l(\sigma)}[x_{j_0}x_{i_{\sigma(0)}}][x_{j_1}x_{i_{\sigma(1)}}][x_{j_2}x_{i_{\sigma(2)}}][x_{i_{\sigma(3)}}x_{i_{\sigma(4)}}]\cdots[x_{i_{\sigma(k-1)}}x_{i_{\sigma(k)}}] = 0,\\
&\quad\cdots\\
J_\ell :&\quad \sum_{\sigma\mathfrak{S}_{k+1}} (-1)^{l(\sigma)}[x_{j_0}x_{i_{\sigma(0)}}][x_{j_1}x_{i_{\sigma(1)}}]\cdots[x_{j_k}x_{i_{\sigma(k)}}] = 0.
\end{aligned} \tag{5.25}$$

As these must be satisfied in the semiclassical regime and as there is no parameter for corrections, this must be the (c,c) ring of the theory. In the dual side, the corresponding relations for a_{ij} appear only in the infra-red limit. These are consistent with the constraint $\mathrm{rank}(a) \leq k$ obtained in the paragraph including (5.19)–(5.21).

The (a,c) ring. The (a,c) ring of the classical theory is the ring of gauge invariant polynomials of σ, or equivalently, the ring of Weyl invariant polynomials of $\sigma_1, \ldots, \sigma_\ell$. It is the polynomial ring of the invariants $c_2, \ldots, c_{2\ell}$, where

$$c_{2i} = \mathrm{tr}(\sigma^{2i}) = \sum_{a=1}^{\ell} \sigma_a^{2i}. \tag{5.26}$$

The underlying vector space is of course infinite dimensional. In the quantum theory, since the Coulomb branch is lifted if it is regular (i.e. for odd N), we expect to have relations among $c_2, \ldots, c_{2\ell}$, so that the underlying vector space has a finite dimension which is equal to the number (5.10). The relations are found via the $\widetilde{m}_i \to 0$ limit of (5.8),

$$(\sigma_a)^N = 0 \qquad a = 1, \ldots, \ell, \tag{5.27}$$

which are the Jacobi relations of the function $\frac{1}{N+1}(\sigma_1^{N+1}+\cdots+\sigma_\ell^{N+1})$. We express this function, which is invariant under the Weyl group when N is odd, in terms of $c_2,\ldots,c_{2\ell}$ and call it W. Then, the (a,c) ring is the Jacobi ring of $W(c_2,\ldots,c_{2\ell})$.

Notice that it is isomorphic to the ring for the $\mathrm{SO}(2\ell+1)$ or $\mathrm{O}_+(2\ell+1)$ theory with the same N. Of course this does not mean that the theory is equivalent to the SO or O_+ theories. They are different in many other ways, such as the central charge and the (c,c) ring.

This holds for both the original $\mathrm{USp}(k)$ theory and for the dual $\mathrm{USp}(\widetilde{k})$ theory. It is straightforward to check that the rings for the dual pair are isomorphic to each other, at least for low values of (k,N). We do not try to give a proof for general (k,N) in this paper. (In fact, a proof in this case is equivalent to a proof in the SO(odd) or O_+(odd) theories with odd N, by the isomorphism mentioned above.) It would be interesting to see if there is a relation to level-rank duality in Wess-Zumino-Witten fusion rings or in the chiral rings in Kazama-Suzuki models.

6 Motivation, test, and application

6.1 A linear sigma model including $\mathrm{O}_+(2)$

The present work started as an attempt to understand, from the quantum field theory point of view, the relation discussed in [16] between two different Calabi-Yau manifolds. One of the two naturally leads us to consider the following linear sigma model.

It is the theory with the gauge group $G=(\mathrm{U}(1)\times\mathrm{O}(2))/\{(\pm1,\pm\mathbf{1}_2)\}$, with the matter fields

$$\underbrace{p^1\ \ p^2\ \ p^3\ \ p^4\ \ p^5}_{(-2,\mathbf{1})}\ \underbrace{x_1\ \ x_2\ \ x_3\ \ x_4\ \ x_5}_{(1,\mathbf{2})} \tag{6.1}$$

and the superpotential

$$W=\sum_{i,j,k=1}^{5}S_k^{ij}p^k(x_ix_j). \tag{6.2}$$

$S_k^{ij}=S_k^{ji}$ are complex numbers which are generic in the sense specified soon. The theta parameter for the $\mathrm{SO}(2)\subset\mathrm{O}(2)$ is turned off for the theory to be regular, as we have 5 (odd) doublets. We often use the parametrization introduced in section 3.2:

$$G=\frac{\mathrm{U}(1)\times\mathrm{U}(1)}{\{(\pm1,\pm1)\}}\rtimes\mathbf{Z}_2\cong(\mathrm{U}(1)_1\times\mathrm{U}(1)_2)\rtimes\mathbf{Z}_2 \tag{6.3}$$
$$([(g,h)],*)\longmapsto(gh,gh^{-1},*).$$

The FI-theta parameters for $\mathrm{U}(1)_1$ are equal to those of $\mathrm{U}(1)_2$ and are denoted by (r,θ), or $t=r-i\theta$. The matter fields are p^k's of charge $(-1,-1)$, u_i's of charge $(1,0)$ and v_i's of charge $(0,1)$ with respect to $\mathrm{U}(1)_1\times\mathrm{U}(1)_2$, and the superpotential (6.2) reads as

$$W=\sum_{i,j,k=1}^{5}S_k^{ij}p^k(u_iv_j+v_iu_j). \tag{6.4}$$

The generator τ of $\mathbf{Z}_2$ exchanges $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$ as well as u_i and v_i. The D-term equations read

$$-|p|^2 + |u|^2 = -|p|^2 + |v|^2 = r, \tag{6.5}$$

and the F-term equations are

$$\sum_{ij} S_k^{ij} u_i v_j = 0, \qquad k = 1, \dots, 5, \tag{6.6}$$

$$\sum_{j,k} S_k^{ij} p^k u_j = \sum_{j,k} S_k^{ij} p^k v_j = 0, \quad i = 1, \dots, 5. \tag{6.7}$$

The low energy theory at $r \gg 0$. Let us analyze the low energy theory at $r \gg 0$. The D-term equations require $u \neq 0$ and $v \neq 0$, thus $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ is completely broken. The space of (u, v) can be identified as $\mathbf{CP}^4 \times \mathbf{CP}^4$ on which τ acts by the exchange of the two $\mathbf{CP}^4$ factors. Let $\widetilde{X}_S$ be the subspace of $\mathbf{CP}^4 \times \mathbf{CP}^4$ consisting of (u, v) satisfying the first set of F-term equations, (6.6). We assume that $\widetilde{X}_S$ is a smooth submanifold of $\mathbf{CP}^4 \times \mathbf{CP}^4$ of codimension 5. Namely, we require that the differential of the five equations has the maximal rank,

(C): *If (u, v) represents a point of $\widetilde{X}_S$, then the 5×10 matrix (Su, Sv) is of rank 5.*

Here, Su is the square matrix whose $(k, i)^{\text{th}}$ entry is $\sum_j S_k^{ij} u_j$. This condition also forbids the exchange τ to have a fixed point: a fixed point would be represented by (u, u) where $u \neq 0$ satisfies $\sum_{i,j} S_k^{ij} u_i u_j = 0$, but the matrix (Su, Su) has rank 4 or less, as Su has u in the kernel, contradicting (C). In particular, the gauge group is completely broken. Again by the condition (C), the second set of F-term equations (6.7) requires that all p^k's vanish. We conclude that the vacuum manifold is the free quotient $X_S = \widetilde{X}_S / \mathbf{Z}_2$, which may also be written simply as

$$X_S = \left\{ x \in (\mathbf{C}^2)^{\oplus 5} \;\middle|\; x \neq 0, \;\; \sum_{i,j} S_k^{ij} (x_i x_j) = 0 \;\; \forall k \right\} \Big/ (\mathbf{C}^\times \times \mathrm{O}(2, \mathbf{C})) / \mathbf{Z}_2. \tag{6.8}$$

$\widetilde{X}_S$ and X_S are three dimensional Calabi-Yau manifolds, with $h^{1,1}(\widetilde{X}_S) = 2$, $h^{2,1}(\widetilde{X}_S) = 52$ and $h^{1,1}(X_S) = 1$, $h^{2,1}(X_S) = 26$ [16]. The modes transverse to $\widetilde{X}_S$ are all massive and hence the low energy theory is the non-linear sigma model whose target space is X_S.

As always, we need to make a choice of the $\mathbf{Z}_2$ orbifold, which is a part of the definition of the $\mathrm{O}(2)$ gauge theory. We fix this by requiring the sigma model on X_S to be the standard one, where RR ground states are in one to one correspondence with the cohomology classes of X_S, i.e., $\mathbf{Z}_2$ invariant (rather than anti-invariant) cohomology classes of $\widetilde{X}_S$, with Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 26 & & 26 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{6.9}$$

For this we need to choose the $\mathbf{Z}_2$ orbifold to be the standard one. This is achieved if we choose the $\mathrm{O}(2)$ factor of the gauge group to be the $\mathrm{O}_+(2)$.

Singularity. Let us find the location of the singular points. We denote by σ_1 and σ_2 the fieldstrength for the groups $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$. They are exchanged by the symmetry τ,

$$\sigma_1 \stackrel{\tau}{\longleftrightarrow} \sigma_2. \tag{6.10}$$

The effective twisted superpotential is $\widetilde{W}_{\text{eff}} = 5(\sigma_1+\sigma_2)(\log(-\sigma_1-\sigma_2)-1) - 5\sigma_1(\log\sigma_1 - 1) - 5\sigma_2(\log\sigma_2 - 1) - t(\sigma_1+\sigma_2)$. The theory is singular if there is a non-compact Coulomb branch determined by the equations $\partial_{\sigma_1}\widetilde{W}_{\text{eff}} = \partial_{\sigma_2}\widetilde{W}_{\text{eff}} = 0 \bmod 2\pi i\mathbf{Z}$, i.e.,

$$\mathrm{e}^t = \frac{(-\sigma_1-\sigma_2)^5}{\sigma_1^5} = \frac{(-\sigma_1-\sigma_2)^5}{\sigma_2^5}. \tag{6.11}$$

There are three inequivalent solutions, $(\sigma_1,\sigma_2) \propto (1,1)$, $(1, \mathrm{e}^{\frac{2\pi i}{5}})$, $(1, \mathrm{e}^{\frac{4\pi i}{5}})$, for the values of t given respectively by

$$\mathrm{e}^t \;=\; -2^5, \;\; -(1+\mathrm{e}^{\frac{2\pi i}{5}})^5, \;\; -(1+\mathrm{e}^{\frac{4\pi i}{5}})^5. \tag{6.12}$$

Note that the first point is special in that the symmetry (6.10) is unbroken. There we must take into account the $\mathbf{Z}_2$ orbifold that acts trivially. Since we choose the gauge group to be $\mathrm{O}_+(2)$, the $\mathbf{Z}_2$ orbifold yields *two copies* of the Coulomb branch at $\mathrm{e}^t = -2^5$. At each of the symmetry breaking points, there is just one copy of Coulomb branch.

Location of singular points agree with the result of [16] (under the relation $x = -\mathrm{e}^{-t}$ to the parameter x of [16]). In that work, the monodromies of the Picard-Fuchs system are also computed — at the three points they are conjugate to

$$\begin{pmatrix} 1 & & & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{6.13}$$

The difference is clear. A spacetime interpretation would be that there are *two* massless hypermultiplets of charge 1 at the first point, while there is just one massless hypermultiplet of charge 1 at each of the latter two. This observation tempts us to make a

Conjecture: *Suppose a linear sigma model gives rise to a family of 4d $\mathcal{N}=2$ compactifications of Type II superstrings. If a disjoint union of n copies of Coulomb branch is supported at a locus of the moduli space, then there are n massless hypermultiplets of the same charge along the locus.*

The low energy theory at $r \ll 0$. Let us now study the low energy theory at $r \ll 0$. This time the D-term equations require $p \neq 0$. We then find that the equations $|u|^2 = |v|^2$ and $p\cdot(Su, Sv) = 0$ force $u = v = 0$ under the condition (C). The space of classical vacua is therefore the space of p obeying $|p|^2 = |r|$ modulo phase rotations, that is, a $\mathbf{CP}^4$. However, the low energy theory is not just the sigma model on this vacuum manifold, since a non-trivial subgroup of the gauge group is unbroken: non-zero values of p break the $\mathrm{U}(1)$ factor to $\{\pm 1\}$ and hence the unbroken gauge group is $(\{\pm 1\}\times \mathrm{O}_+(2))/\{(\pm 1, \pm \mathbf{1}_2)\} \cong \mathrm{O}_+(2)$. We may consider the theory as the $O_+(2)$ gauge theory with five doublets, $x_1, \ldots, x_5$, fibred over the $\mathbf{CP}^4$, with the superpotential (6.2), which may be rewritten as

$$W = \sum_{i,j=1}^{5} S^{ij}(p)(x_i x_j), \tag{6.14}$$

where $S^{ij}(p) = \sum_{k=1}^{5} S_k^{ij} p^k$. The low energy behaviour of the $O_+(2)$ gauge theory for a given value of p depends very much on the rank of the mass matrix $S(p) = (S^{ij}(p))$. It follows from the condition (C) that $S(p)$ has at least rank 3 if $p \neq 0$. That is, a possible rank of $S(p)$ is 3, 4 or 5. Let us put

$$Y_S = \left\{ [p] \in \mathbf{CP}^4 \,\middle|\, \mathrm{rank} S(p) \leq 4 \right\}, \tag{6.15}$$

$$C_S = \left\{ [p] \in \mathbf{CP}^4 \,\middle|\, \mathrm{rank} S(p) = 3 \right\}. \tag{6.16}$$

Y_S is a hypersurface of $\mathbf{CP}^4$ given by the equation $\det S(p) = 0$. C_S is of codimension 3 in $\mathbf{CP}^4$ and hence is a curve. Y_S has A_1 singularity along C_S (see section 3.5). The mass matrix $S(p)$ has corank 1 along $Y_S \setminus C_S$ and corank 2 along C_S. In this situation, we may apply the analysis of section 3.4 and 3.5 concerning the $O_+(2)$ gauge theories with doublets having superpotential with corank 1 and 2 degenerations. The result there implies that, at least locally, the low energy theory is the sigma model whose target space is a double cover of Y_S ramified along C_S — the cover is of the type $\mathbf{C}^2 \to \mathbf{C}^2/\mathbf{Z}_2$ as in (3.40), in the direction of Y_S transverse to C_S. The question is whether such a ramified double cover of Y_S exists globally. This is in fact proven to be the case in [16]. Thus, we can say that the low energy theory is indeed the sigma model on that double cover.

However, it is quite unsatisfactory in that the local understanding cannot tell anything about the existence of the global cover, let alone the construction of such a cover. It would have been disastrous if there were more than one covers or if there were none (though neither is the case fortunately). This was the actual motivation to look for the dual description of the O(2) theory with more than two massless flavors, which resulted in the discovery of the non-Abelian duality.

6.2 The dual linear sigma model including SO(4)

As described in section 3.3, the $O_+(2)$ gauge theory with five doublets $x_1, \ldots, x_5$ is dual to the SO(4) gauge theory with five quartets $\widetilde{x}^1, \ldots, \widetilde{x}^5$ and fifteen singlets s_{ij} with the superpotential $W = \sum_{i,j} s_{ij}(\widetilde{x}^i \widetilde{x}^j)$. The singlets are related to the gauge invariants by $s_{ij} = (x_i x_j)$. This duality can be incorporated into the full linear sigma model and leads us to consider the following theory.

It is the $(\mathrm{U}(1) \times \mathrm{SO}(4))/\{(\pm 1, \pm \mathbf{1}_4)\}$ gauge theory with the following field content

$$\underbrace{p^1 \;\; p^2 \;\; p^3 \;\; p^4 \;\; p^5}_{(-2,\mathbf{1})} \; \underbrace{\widetilde{x}^1 \;\; \widetilde{x}^2 \;\; \widetilde{x}^3 \;\; \widetilde{x}^4 \;\; \widetilde{x}^5}_{(-1,\mathbf{4})} \; \underbrace{(s_{ij})_{1 \leq i \leq j \leq 5}}_{(2,\mathbf{1})} \tag{6.17}$$

and the superpotential

$$W = \sum_{i,j} s_{ij}(\widetilde{x}^i \widetilde{x}^j) + \sum_{i,j,k} S_k^{ij} p^k s_{ij}. \tag{6.18}$$

The sum of U(1) charges vanish, $-2 \times 5 - 1 \times 20 + 2 \times 15 = 0$. Hence, the axial U(1) R-symmetry is not anomalous and the FI-theta parameters $(\widetilde{r}, \widetilde{\theta})$ is a free parameter of the theory. As we will show later, we can normalize the latter so that the D-term equation for the U(1) subgroup reads as

$$-2|p|^2 - |\widetilde{x}|^2 + 2|s|^2 = 2\widetilde{r}. \tag{6.19}$$

Let us consider the theory at $\widetilde{r} \ll 0$. The D-term and F-term equations require $p \neq 0$, which breaks the gauge group to the subgroup $(\{\pm 1\} \times \mathrm{SO}(4))/\{(\pm 1, \pm \mathbf{1}_4)\} \cong \mathrm{SO}(4)$. Thus, we have an SO(4) gauge theory fibred over $\mathbf{CP}^4 = \{p\}$. Since this fibration of SO(4) theories is dual to the fibration of $\mathrm{O}_+(2)$ theories discussed in the previous subsection, we conclude that this linear sigma model at $\widetilde{r} \ll 0$ is dual to the original one (6.1)–(6.2) at $r \ll 0$. Comparing (6.19) and (6.5), we expect that $\widetilde{r}$ agrees with r in the limit $r \to -\infty$. The precise relation will be determined momentarily.

The low energy theory at $\widetilde{r} \ll 0$. Let us further study the low energy behaviour of this theory at $\widetilde{r} \ll 0$. Integrating out the fields s_{ij} we obtain the constraints

$$(\widetilde{x}^i \widetilde{x}^j) + S^{ij}(p) = 0 \quad \forall (i,j). \tag{6.20}$$

Since $S(p)$ has rank at least three for $p \neq 0$ by the condition (C), we find that the 4×5 matrix $(\widetilde{x}^i_a)$ has rank at least 3 if it obeys the constraints (6.20). This completely breaks the residual SO(4) gauge group. We find that the manifold of classical vacua is the free quotient

$$\widetilde{Y}_S = \left\{ (p, \widetilde{x}) \in \mathbf{C}^{\oplus 5} \oplus (\mathbf{C}^4)^{\oplus 5} \,\middle|\, p \neq 0,\ (6.20) \right\} \Big/ \frac{\mathbf{C}^\times \times \mathrm{SO}(4, \mathbf{C})}{\{(\pm 1, \pm \mathbf{1}_4)\}}. \tag{6.21}$$

If it is a smooth manifold, we identify the low energy theory as the non-linear sigma model with this target space. Since our theory is dual to (6.1)–(6.2), this must be the double cover of Y_S which we were longing for! This is indeed the case, as we show now.

Since $\widetilde{x}^i$ are in the quartet $\mathbf{4}$, the 5×5 matrix $(\widetilde{x}^i \widetilde{x}^j)$ has at most rank 4. Thus, (6.20) yields the constraint on p that $S(p)$ must have rank at most 4. I.e., we obtained the constraint that p must represent a point of Y_S, this time, by a completely classical argument. Therefore, $(p, \widetilde{x}) \mapsto p$ defines a map

$$f : \widetilde{Y}_S \longrightarrow Y_S. \tag{6.22}$$

Let us find the fibre of this map. A choice of $p \neq 0$ that represents a point of Y_S breaks the group of the quotient (6.21) to the subgroup $(\{\pm 1\} \times \mathrm{SO}(4, \mathbf{C}))/\mathbf{Z}_2 \cong \mathrm{SO}(4, \mathbf{C})$. The symmetric matrix $S(p)$ can be diagonalized using the $GL(5, \mathbf{C})$ coordinate change and we may assume $S(p) = -\mathrm{diag}(c_1, c_2, c_3, c_4, 0)$, where three of $c_1, \ldots, c_4$ must be non-zero. A solution to the equation (6.20) for such $S(p)$ can be made into the following form using the $\mathrm{SO}(4, \mathbf{C})$ symmetry,

$$(\widetilde{x}^1, \ldots, \widetilde{x}^5) = \begin{pmatrix} \pm\sqrt{c_1} & & & & \\ & \pm\sqrt{c_2} & & & \\ & & \pm\sqrt{c_3} & & \\ & & & \pm\sqrt{c_4} & 0 \end{pmatrix}. \tag{6.23}$$

Even number of sign flip of the non-zero entries is a part of the SO$(4,\mathbf{C})$ symmetry, while odd number of sign flip is not a part of it if $c_1,\ldots,c_4$ are all non-zero — $\det(\widetilde{x}^1\cdots\widetilde{x}^4)$ distinguishes the orbits. If one of $c_1,\ldots,c_4$ vanishes, any number of sign flip is a part of SO$(4,\mathbf{C})$. Therefore, the fibre $f^{-1}([p])$ consists of two points if rank$S(p)=4$ while it is a single point if rank$S(p)=3$ (i.e. if $[p]$ belongs to C_S). Let us look at the behaviour of f near the curve C_S. For example, take a point p_* where $S(p_*)=-\mathrm{diag}(0,0,1,1,1)$. If we assume that p_* is a smooth point of C_S, we may assume that we can find three coordinates (a,b,c) of $\mathbf{CP}^4=\{p\}$ transverse to C_S so that

$$S(p)=-\begin{pmatrix} a & c & & & \\ c & b & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}. \tag{6.24}$$

$[p]$ belongs to Y_S if and only if $ab=c^2$. The equation (6.20) is solved by

$$(\widetilde{x}^1,\ldots,\widetilde{x}^5)=\begin{pmatrix} \xi & \eta & & \\ & & 1 & \\ & & & 1 \\ & & & & 1 \end{pmatrix},\qquad \xi^2=a,\;\;\eta^2=b,\;\;\xi\eta=c. \tag{6.25}$$

This is indeed the expected local behaviour. Namely, we found that the forgetful map (6.22) is a double cover of Y_S ramified along the curve C_S.

We confirmed that the space $\widetilde{Y}_S$ in (6.21) is the double cover of Y_S which we were looking for. Note that we were able to obtain the target space $\widetilde{Y}_S$ by a completely classical analysis in this dual model, since the gauge group is completely Higgsed. This construction of $\widetilde{Y}_S$ is strikingly explicit compared to the one given in [16] based on the relative spectrum of a sheaf of algebra over Y_S.

Smoothness of $\widetilde{Y}_S$. Let us discuss the condition for smoothness of $\widetilde{Y}_S$. In view of the fact that it is a double cover of Y_S ramified over C_S, with the local structure as above, it is smooth if C_S is a smooth submanifold of $\mathbf{CP}^4$. The curve C_S is locally an intersection of three hypersurfaces. To see this, take a point $[p_*]\in C_S$ and choose a coordinate system such that $S(p_*)=-\mathrm{diag}(0,0,1,1,1)$. Then, C_S can be defined by $\Delta_{11}(p)=\Delta_{12}(p)=\Delta_{22}(p)=0$ in the region where the matrix $S(p)_{345}=(S^{ij}(p))_{3\le i,j\le 5}$ has rank 3. Here $\Delta_{ij}(p)$ is the determinant of the 4×4 obtained by deleting i-th raw and j-the column of $S(p)$. The curve C_S is smooth if and only if the 5×3 matrix of the differentials of these three equations has rank 3. The matrix is equal to $(S^{11}_k\,S^{12}_k\,S^{22}_k)$ times $\det S(p)_{345}\neq 0$. Therefore, the condition for the smoothness of the curve C_S is

(D): *If $S(p_*)$ is of the form* $\left(\begin{array}{c|c} \mathbf{0}_2 & \\ \hline & *_3 \end{array}\right)$, *the 5×3 matrix $(S^{11}_k\,S^{12}_k\,S^{22}_k)$ has rank 3.*

The condition (D) is also a necessary condition for the smoothness of $\widetilde{Y}_S$. To see this, let us recall that $\widetilde{Y}_S$ is defined by the free quotient (6.21). The number of variables is $5+20=25$

while the number of equations is 15 and the dimension of the group is $1+6=7$. And the difference $25-15-7=3$ matches the dimension of $\widetilde{Y}_S$. Thus, $\widetilde{Y}_S$ is smooth if and only if the 25×15 matrix of differentials of the equations has rank 15. At the point p_* in the discussion above, and for $\widetilde{x}$ given by (6.25) with $\xi=\eta=0$, the matrix takes the form

$$\begin{pmatrix} \mathbf{0}_{20\times 3} & *_{20\times 12} \\ S_k^{11}\, S_k^{12}\, S_k^{22} & *_{5\times 12} \end{pmatrix}. \tag{6.26}$$

It has rank 15 only if the 5×3 part $(S_k^{11}\, S_k^{12}\, S_k^{22})$ has rank 3. In summary, (D) is the condition for smoothness of $\widetilde{Y}_S$.

This condition follows from the condition (C) for smoothness of $\widetilde{X}_S$. To see this, suppose (D) fails. That is, there is some p_* such that $S(p_*)=-\mathrm{diag}(0,0,1,1,1)$ but $(S_k^{11}\, S_k^{12}\, S_k^{22})$ has rank 2 or less, i.e., there is some $(\alpha,\beta,\gamma)\neq(0,0,0)$ such that $\alpha S_k^{11}+\beta S_k^{22}+\gamma S_k^{12}=0$ for all k. We can find $(u_1,u_2)\neq(0,0)$ and $(v_1,v_2)\neq(0,0)$ such that $\alpha=u_1v_1$, $\beta=u_2v_2$ and $\gamma=u_1v_2+u_2v_1$. Then, $u=(u_1,u_2,0,0,0)$ and $v=(v_1,v_2,0,0,0)$ represent a point of $\widetilde{X}_S$. Note that (Su,Sv) is annihilated by $p_*\neq 0$, which means that (Su,Sv) has rank 4 or less. I.e., (C) fails.

The converse also holds if we assume that the τ action on $\widetilde{X}_S$ has no fixed point. Suppose that (C) fails under that assumption. Then, there are linearly independent two 5-vectors u and v satisfying $\sum_{i,j}S_k^{ij}u_iv_j=0$ for all k such that (Su,Sv) has rank 4 or less. That means that there is some p_* that annihilates this 5×10 matrix. With a choice of coordinates, we may assume $u=(1,0,0,0,0)$ and $v=(v_1,1,0,0,0)$. That p_* annihilates (Su,Sv) means that the matrix $S(p_*)$ is of the form in the set-up of (D). However, the equation $\sum_{i,j}S_k^{ij}u_iv_j=0$ reads $S_k^{11}v_1+S_k^{12}=0$, which means that $(S_k^{11}\, S_k^{12}\, S_k^{22})$ cannot have rank 3. Thus, (D) fails.

To summarize, "$\widetilde{X}_S$ is smooth" is equivalent to "$\widetilde{Y}_S$ is smooth and the τ action on $\widetilde{X}_S$ is fixed point free". $\widetilde{X}_S$, X_S and $\widetilde{Y}_S$ are all smooth under the condition (C).

The FI-theta parameters. Let us now carry out the promise concerning the FI-theta parameters $(\widetilde{r},\widetilde{\theta})$. The Lie algebra of the gauge group is the direct sum of $\mathfrak{u}(1)$ and $\mathfrak{so}(4)$, where $\alpha\in\mathfrak{u}(1)$, regarded as a real number, generates the one parameter subgroup $\{[(\,\mathrm{e}^{it\alpha},\mathbf{1}_4)]\}_{t\in\mathbf{R}}$. We denote by $F_{\mathfrak{u}(1)}$ the $\mathfrak{u}(1)$ component of the curvature. On a closed worldsheet Σ, the flux $\int_\Sigma F_{\mathfrak{u}(1)}$ obeys a certain quantization condition. For the usual U(1) gauge group the condition is that the flux can take all values of $2\pi\mathbf{Z}$. For the present gauge group, it can take all values of $\pi\mathbf{Z}$. For example, we may decompose Σ into two parts by a circle S^1 parametrized by $t\in\mathbf{R}/2\pi\mathbf{Z}$ and consider the principal bundle determined by the transition function along S^1, given by $[(\,\mathrm{e}^{\frac{it}{2}},h_t)]$, where h_t is some SO(4)-valued function such that $h_{t+2\pi}=-h_t$. Then, the flux for any connection of this bundle is $\pm\pi$ (the sign depends on the orientation of Σ versus that of S^1). Therefore, the theta term must be of the form $\int_\Sigma\frac{2\widetilde{\theta}}{2\pi}F_{\mathfrak{u}(1)}$, rather than the usual $\int_\Sigma\frac{\widetilde{\theta}}{2\pi}F_{\mathfrak{u}(1)}$, in order to have the periodicity $\widetilde{\theta}\equiv\widetilde{\theta}+2\pi$. The corresponding twisted superpotential is

$$\widetilde{W}_{tree}=-2\,\widetilde{t}\,\sigma_{\mathfrak{u}(1)}, \tag{6.27}$$

for $\widetilde{t}=\widetilde{r}-i\widetilde{\theta}$. The FI parameter $\widetilde{r}$ enters into the D-term equation indeed as (6.19).

Let us find the singular points in the parameter space, in order to find the precise relation between $(\widetilde{r}, \widetilde{\theta})$ and (r, θ). A maximal torus of the gauge group is $(\mathrm{U}(1) \times \mathrm{SO}(2) \times \mathrm{SO}(2))/\{(\pm 1, \pm \mathbf{1}_2, \pm \mathbf{1}_2)\}$ and we identify it as $\mathrm{U}(1)_0 \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$, where the element $[(z, h_1, h_2)]$ of the former group is identified with the element (g_0, g_1, g_2) of the latter group by

$$g_0 = z^2, \quad g_1 = zh_1, \quad g_2 = zh_2. \tag{6.28}$$

Here we abuse the notation: the rotation of $\mathbf{R}^2$ by an angle α is identified with the element $\mathrm{e}^{i\alpha} \in \mathrm{U}(1)$. The SO(4) Weyl group action $(h_1, h_2) \mapsto (h_2, h_1)$, (h_1^{-1}, h_2^{-1}) corresponds to

$$(g_0, g_1, g_2) \mapsto (g_0, g_2, g_1), \;\; (g_0, g_0 g_1^{-1}, g_0 g_2^{-1}). \tag{6.29}$$

By $z^2 = g_0$, the tree level twisted superpotential (6.27) is written as $\widetilde{W}_{tree} = -\widetilde{t}\sigma_0$. The fields have the following charges under $\mathrm{U}(1)_0 \times \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$:

$$\underbrace{p^1 \;\; p^2 \;\; p^3 \;\; p^4 \;\; p^5}_{(-1,0,0)} \; \underbrace{\widetilde{x}^1 \;\; \widetilde{x}^2 \;\; \widetilde{x}^3 \;\; \widetilde{x}^4 \;\; \widetilde{x}^5}_{\substack{(-1,1,0)\\(0,-1,0)\\(-1,0,1)\\(0,0,-1)}} \; \underbrace{(s_{ij})_{1 \le i \le j \le 5}}_{(1,0,0)} \tag{6.30}$$

Writing down the effective twisted superpotential and extremizing it, we obtain the equations determining the Coulomb branch,

$$\frac{(-\sigma_0)^5(-\sigma_0+\sigma_1)^5(-\sigma_0+\sigma_2)^5}{(\sigma_0)^{15}} = -\,\mathrm{e}^{\widetilde{t}}, \quad \frac{(-\sigma_1)^5}{(-\sigma_0+\sigma_1)^5} = \frac{(-\sigma_2)^5}{(-\sigma_0+\sigma_2)^5} = 1. \tag{6.31}$$

The sign in $-\,\mathrm{e}^{\widetilde{t}}$ comes from integrating out the off diagonal components of the vector multiplet, as in the $\pi i \sigma_a$ shift (4.8) in section 4.2. We need to avoid solutions such that $\sigma_1 = \sigma_2$ and $\sigma_0 = \sigma_1 + \sigma_2$ at which the unbroken subgroup is bigger than the maximal torus. By the second set of equations, we find $\sigma_a = \frac{\sigma_0}{1+\omega_a}$, with $\omega_a^5 = 1$, for $a = 1, 2$, where $\omega_1 = \omega_2^{\pm 1}$ needs to be avoided. The Weyl group action (6.29) is translated into $(\omega_1, \omega_2) \mapsto (\omega_2, \omega_1)$, $(\omega_1^{-1}, \omega_2^{-1})$. The are four inequivalent possibilities, $(\omega_1, \omega_2) = (\,\mathrm{e}^{\frac{2\pi i}{5}}, \mathrm{e}^{\frac{4\pi i}{5}})$, $(\,\mathrm{e}^{-\frac{2\pi i}{5}}, \mathrm{e}^{\frac{4\pi i}{5}})$, $(1, \mathrm{e}^{\frac{2\pi i}{5}})$, $(1, \mathrm{e}^{\frac{4\pi i}{5}})$. Correspondingly, $\widetilde{t}$ has values

$$\mathrm{e}^{\widetilde{t}} = -1, \quad -1, \quad -2^{-5}(1+\mathrm{e}^{\frac{4\pi i}{5}})^5, \quad -2^{-5}(1+\mathrm{e}^{\frac{2\pi i}{5}})^5. \tag{6.32}$$

The Weyl group is completely broken at each of the four Coulomb branches. Therefore, there are two copies of one-dimensional Coulomb branch at $\mathrm{e}^{\widetilde{t}} = -1$, while there is one at each of the other two values of $\mathrm{e}^{\widetilde{t}}$. Comparing it with (6.12), we find the relation between t and $\widetilde{t}$: it is $\mathrm{e}^{t} = 2^5\,\mathrm{e}^{\widetilde{t}}$, i.e.,

$$r = \widetilde{r} + 5\log 2, \quad \theta = \widetilde{\theta}. \tag{6.33}$$

We have reproduced, in a very different way, the observation that $\mathrm{e}^{t} = -2^5$ has a double Coulomb branch.

The low energy theory at $\widetilde{r} \gg 0$. Nothing stops us from studying the low energy behaviour of the model in the opposite regime $\widetilde{r} \gg 0$. The D-term equation (6.19) requires that $s = (s_{ij})$ is non-zero. This breaks the gauge group to SO(4). We have an SO(4) gauge theory fibred over $\mathbf{CP}^{14} = \{s\}$. The 5 quartets of this theory has mass matrix s_{ij} and the nature of the low energy theory depends on its rank. Analysis of such a system has been carried out in section 4.6, which can be applied here without modification. By the supersymmetry breaking for the SO(4) theory with $N_{\text{eff}} \leq 2$, we find that the low energy dynamics concentrates near the locus of $\mathbf{CP}^{14}$ where s_{ij} is of rank 2 or less. The low energy description of the theory with $N_{\text{eff}} = 3$ in terms of composite mesons tells us that, inside the open subset of $\mathbf{CP}^{14}$ where s has rank at least 2, the theory reduces to the sigma model whose target space is the locus of rank exactly 2. In the present case, we also have the F-term constraints

$$\sum_{i,j} S_k^{ij} s_{ij} = 0, \quad k = 1, \ldots, 5. \tag{6.34}$$

Thus, the low energy theory is a simple non-linear sigma model whose target space is

$$\left\{ [s] \in \mathbf{CP}^{14} \;\middle|\; \operatorname{rank} s = 2, \quad (6.34) \right\}. \tag{6.35}$$

To be precise, we may need to worry about the rank 1 locus. Here we simply assume genericity of S_k^{ij} so that there is no rank 1 solution to (6.34). This is equivalent to the assumption that no $u \neq 0$ solves $\sum_{i,j} S_k^{ij} u_i u_j = 0$, i.e., the τ action on $\widetilde{X}_S$ is free, which is guaranteed by the condition (C). The space (6.35) is isomorphic to $X_S = \widetilde{X}_S/\mathbf{Z}_2$ via the correspondence $s_{ij} \propto (x_i x_j)$ — compare (6.35) with (6.8). We have reproduced the low energy theory of the original linear sigma model at $r \gg 0$.

Summary. In the original $(\mathrm{U}(1) \times \mathrm{O}_+(2))/\mathbf{Z}_2$ theory, the $\mathrm{O}_+(2)$ gauge symmetry is completely Higgsed and the classical analysis suffices in the $r \gg 0$ phase while it is entirely unbroken and its strong quantum effect is essential in the $r \ll 0$ phase. In the dual $(\mathrm{U}(1) \times \mathrm{SO}(4))/\mathbf{Z}_2$ theory, the SO(4) gauge symmetry is unbroken in the $r \gg 0$ phase while it is completely Higgsed in the $r \ll 0$ phase. The exchange between Higgs/weak/classical and confinement/strong/quantum is a typical feature of duality.

6.3 SO(2) and $\mathrm{O}_-(2)$ versions

Purely from curiosity, we study the linear sigma model of section 6.1 in which the $\mathrm{O}_+(2)$ factor of the gauge group is replaced by SO(2) or $\mathrm{O}_-(2)$.

SO(2). Let us consider the model with the gauge group $(\mathrm{U}(1) \times \mathrm{SO}(2))/\{(\pm 1, \pm \mathbf{1}_2)\} \cong \mathrm{U}(1)_1 \times \mathrm{U}(1)_2$. The FI-theta parameters t_1 and t_2 of the two U(1) factors are independent in this model. The D-term equations read

$$-|p|^2 + |u|^2 = r_1, \quad -|p|^2 + |v|^2 = r_2, \tag{6.36}$$

and the F-term equations remain the same as (6.6)–(6.7). The model has three classical phases: (I) $r_1 > 0$, $r_2 > 0$, (II) $r_2 < 0$, $r_1 > r_2$, and (III) $r_1 < 0$, $r_1 < r_2$. The D-term equations require $u \neq 0$ and $v \neq 0$ in Phase I, $u \neq 0$ and $p \neq 0$ in Phase II, and $v \neq 0$ and

$p \neq 0$ in Phase III. The low energy theory in the respective phase is the sigma model with the target space

$$
\begin{aligned}
X_{\rm I} &= \left\{ (u,v) \in \mathbf{CP}^4 \times \mathbf{CP}^4 \,\middle|\, \sum_{i,j} S_k^{ij} u_i v_j = 0 \ \forall k \right\}, \\
X_{\rm II} &= \left\{ (u,p) \in \mathbf{CP}^4 \times \mathbf{CP}^4 \,\middle|\, \sum_{i,j} S_k^{ij} p^k u_i = 0 \ \forall j \right\}, \\
X_{\rm III} &= \left\{ (v,p) \in \mathbf{CP}^4 \times \mathbf{CP}^4 \,\middle|\, \sum_{i,j} S_k^{ij} p^k v_j = 0 \ \forall i \right\}.
\end{aligned}
\tag{6.37}
$$

Note that $X_{\rm I} = \widetilde{X}_S$ and $X_{\rm II} \cong X_{\rm III}$. Hodge diamond of RR ground states is

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
1 & & 52 & & 52 & & 1 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\tag{6.38}
$$

The quantum Kähler moduli space can be found by looking for Coulomb branch vacua. Writing down the effective twisted superpotential and extremizing it, we obtain the vacuum equations for σ_1 and σ_2, which read for $z := \sigma_2/\sigma_1$ as

$$
\mathrm{e}^{t_1} = -(1+z)^5, \qquad \mathrm{e}^{t_2} = -(1+z^{-1})^5. \tag{6.39}
$$

This provides a parametric representation of the singular locus. We indeed see the three phase boundaries: the I-II phase boundary corresponds to $z \to \infty$ where $\mathrm{e}^{t_1} \to \infty$ and $\mathrm{e}^{t_2} \to -1$, the I-III boundary corresponds to $z \to 0$ where $\mathrm{e}^{t_2} \to \infty$ and $\mathrm{e}^{t_1} \to -1$, and the II-III boundary corresponds to $z \to -1$ where $\mathrm{e}^{t_1} \to 0$, $\mathrm{e}^{t_2} \to 0$ and $\mathrm{e}^{t_1 - t_2} \to -1$.

Let us look at the locus

$$
r_1 = r_2 \ll 0, \quad \text{and} \quad \theta_1 = \theta_2. \tag{6.40}
$$

It is on the classical II-III phase boundary but avoids the quantum phase boundary which is the line on the opposite side $\theta_1 - \theta_2 = \pi$ (mod $2\pi\mathbf{Z}$). The theory is regular and we may apply our understanding of the low energy behaviour of the SO(2) gauge theory or its dual $\mathrm{O}_+(4)$ gauge theory. This tells us that the low energy theory is the orbifold conformal field theory

$$
\widetilde{Y}_S/\mathbf{Z}_2, \tag{6.41}
$$

where $\widetilde{Y}_S$ is defined by (6.21) and $\mathbf{Z}_2$ is the symmetry associated with $\mathrm{O}_+(4)/\mathrm{SO}(4)$. Note that the map $f : \widetilde{Y}_S \to Y_S$ in (6.22) is the mathematical quotient with respect to this $\mathbf{Z}_2$.

We would like to make two remarks concerning the dual model with gauge group $(\mathrm{U}(1) \times \mathrm{O}_+(4))/\{(\pm 1, \pm \mathbf{1}_4)\}$. The center of this group is U(1) and it may appear that the model has just one Kähler parameter. This corresponds to the $t_1 = t_2$ subspace of the original quantum Kähler moduli space. The missing exactly marginal (a,c) operator should come from the twisted sector with respect to the $\mathbf{Z}_2 \cong \mathrm{O}_+(4)/\mathrm{SO}(4)$. Indeed, this must be the twist operator denoted by 1_τ in the study of the (a,c) ring of the relevant $\mathrm{O}_+(4)$ theory in section 4.7. Another remark is about the local analysis of the dual in the $\widetilde{r} \gg 0$ phase. The difference from the SO(4) case occurs at the point where we use the low energy description of the theory with $N_{\text{eff}} = 3$ massless fundamentals: for SO(4) we had one copy of the free theory of composite mesons but for $\mathrm{O}_+(4)$ we have two copies. This gives us a double cover of (6.35), and that must be the double cover $\widetilde{X}_S$ of X_S.

$\mathrm{O}_-(2)$. Next, we consider the model with the gauge group $(\mathrm{U}(1)\times\mathrm{O}_-(2))/\{(\pm 1, \pm\mathbf{1}_2)\} \cong (\mathrm{U}(1)_1 \times \mathrm{U}(1)_2) \rtimes \mathbf{Z}_2(-1)^{F_s}$. This is a one parameter model, $t_1 = t_2 = t$, with the same D- and F-term equations as in section 6.1. The difference is that the orbifold group is the non-standard one. Accordingly the low energy theory is the non-standard orbifold: at $r \gg 0$ we have

$$\widetilde{X}_S/\mathbf{Z}_2(-1)^{F_s}, \tag{6.42}$$

and at $r \ll 0$ we have

$$\widetilde{Y}_S/\mathbf{Z}_2(-1)^{F_s} \tag{6.43}$$

Hodge diamond of RR ground states is

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 26 & & 26 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \tag{6.44}$$

The singular points are $\mathrm{e}^t = -2^5$, $-(1+\mathrm{e}^{\frac{2\pi i}{5}})^5$ and $-(1+\mathrm{e}^{\frac{4\pi i}{5}})^5$ as in (6.12) but this time, there is only *one* copy of Coulomb branch at the first point as well as in the other two.

The dual model with gauge group $(\mathrm{U}(1) \times \mathrm{O}_-(4))/\{(\pm 1, \pm\mathbf{1}_4)\}$ has no other exactly marginal operator than the one that generates $\delta\widetilde{t}$. The orbifold with respect to $\mathrm{O}_-(4)/\mathrm{SO}(4)$ is opposite to that of $\mathrm{O}_+(4)/\mathrm{SO}(4)$ in the twisted NSNS sector, and we have indeed seen in (4.51) that there is no twisted (a,c) ring element in the relevant $\mathrm{O}_-(4)$ theory. In the $\widetilde{r} \gg 0$ phase, we find that the target space is (6.35), i.e. X_S, rather than the double cover $\widetilde{X}_S$, since the $\mathrm{O}_-(4)$ theory with $N_{\text{eff}} = 3$ massless fundamentals flows to *one* copy of the free theory of composite mesons. It is not the standard sigma model as it is obtained after the non-standard orbifold operated locally.

6.4 Rødland's example — USp(2) versus USp(4)

The linear sigma model studied in [2] has gauge group $(\mathrm{U}(1)\times\mathrm{USp}(2))/\{(\pm, \pm\mathbf{1}_2)\} \cong \mathrm{U}(2)$ and the following matter fields, superpotential and twisted superpotential:

$$\underbrace{p^1 \ \cdots \ p^7}_{\det^{-1}} \ \underbrace{x_1 \ \cdots \ x_7}_{\mathbf{2}} \tag{6.45}$$

$$W = \sum_{i,j,k=1}^{7} A_k^{ij} p^k [x_i x_j], \tag{6.46}$$

$$\widetilde{W} = -t\,\mathrm{tr}_{\mathbf{2}}\sigma. \tag{6.47}$$

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$A_k^{ij} = -A_k^{ji}$ are complex numbers which are generic in a suitable sense. The low energy theory at $r \gg 0$ is the non-linear sigma model whose target space is the complete intersection of seven planes in the Grassmannian $G(2,7)$,

$$X_A = \left\{ [x] \in G(2,7) \;\middle|\; \sum_{i,j} A_k^{ij}[x_i x_j] = 0 \;\; \forall k \right\}. \tag{6.48}$$

The low energy theory at $r \ll 0$ is the non-linear sigma model whose target space is the Pfaffian Calabi-Yau manifold,

$$Y_A = \left\{ [p] \in \mathbf{CP}^6 \;\middle|\; \mathrm{rank} A(p) = 4 \right\}, \tag{6.49}$$

where $A^{ij}(p) = \sum_k A_k^{ij} p^k$. This was demonstrated in [2] by finding and employing the low energy description of USp(2) gauge theories with 1 or 3 massless fundamentals. Both X_A and Y_A have Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 50 & & 50 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{6.50}$$

but they are topologically and birationally inequivalent. Finally, the theory is singular at

$$\mathrm{e}^t \;=\; (1+\omega)^7,\;\; (1+\omega^2)^7,\;\; (1+\omega^3)^7, \tag{6.51}$$

where $\omega := \mathrm{e}^{\frac{2\pi i}{7}}$. There is a single one dimensional Coulomb branch at each point. Note that (6.51) differs from [2] by a sign. This is because [2] missed the possible effect of integrating out the off-diagonal components of the vector multiplet discussed in section 4.2. In the present case, the effect is the π shift of the theta angle of the central U(1), which contributes to the sign change.[6]

As in the model including O(2), we may employ the duality for the USp(2) part, and consider the dual linear sigma model. It has gauge group $(\mathrm{U}(1) \times \mathrm{USp}(4))/\{(\pm, \pm \mathbf{1}_4)\}$ and the following matter fields, superpotential and twisted superpotential:

$$\underbrace{p^1 \;\cdots\; p^7}_{(-2,\mathbf{1})} \; \underbrace{\widetilde{x}^1 \;\cdots\; \widetilde{x}^7}_{(-1,\mathbf{4})} \; \underbrace{(a_{ij})_{1\le i<j\le 7}}_{(2,\mathbf{1})} \tag{6.52}$$

$$W = \sum_{i,j=1}^{7} a_{ij}[\widetilde{x}^i \widetilde{x}^j] + \sum_{i,j,k=1}^{7} A_k^{ij} p^k a_{ij}, \tag{6.53}$$

$$\widetilde{W} = -2\,\widetilde{t}\,\sigma_{\mathrm{u}(1)}. \tag{6.54}$$

At $\widetilde{r} \ll 0$, the gauge group is completely Higgsed. We have F-term equations

$$[\widetilde{x}^i \widetilde{x}^j] + A^{ij}(p) = 0 \quad \forall (i,j). \tag{6.55}$$

[6] This effect was also missed in the first version of the present paper.

The low energy theory is the non-linear sigma model whose target space is the free quotient

$$\left\{ (p,\widetilde{x}) \in \mathbf{C}^{\oplus 7} \oplus (\mathbf{C}^4)^{\oplus 7} \;\middle|\; p \neq 0, \quad (6.55) \right\} \Big/ \frac{\mathbf{C}^\times \times \mathrm{Sp}(4,\mathbf{C})}{\{(\pm 1, \pm \mathbf{1}_4)\}}. \tag{6.56}$$

There is a map to Y_A given by $(p,\widetilde{x}) \mapsto p$. Indeed if $(p,\widetilde{x})$ solves (6.55), then $A(p)$ has rank 4. In fact it is an isomorphism. To see that, let us choose a point $[p] \in Y_A$ and use the $GL(7,\mathbf{C})$ coordinate change to make $A(p)$ into the form

$$\begin{pmatrix} J_2 & & \\ & J_2 & \\ & & \mathbf{0}_3 \end{pmatrix}. \tag{6.57}$$

A solution to (6.55) is given by

$$\widetilde{x} = (\mathbf{1}_4, \mathbf{0}_{4\times 3}), \tag{6.58}$$

and any solution is in its $\mathrm{Sp}(4,\mathbf{C})$ orbit. Thus, the fibre of the map at each point of Y_A consists of a unique point. We have seen that (6.56) is another representation of the Pfaffian variety Y_A. It is obtained in a completely classical manner in this dual model, while (6.49) is obtained by a very non-trivial analysis of the quantum theory in the original model.

At $\widetilde{r} \gg 0$, the D-term equation requires $a \neq 0$ and the gauge group is broken to USp(4). This leaves us with a USp(4) gauge theory fibred over $\mathbf{CP}^{20} = \{a\}$. The 7 quartets have mass matrix a_{ij} and the nature of the theory depends on its rank. Analysis of such a system has been carried out in section 5.6, which can be applied here without modification: by the supersymmetry breaking for the USp(4) theory with $N_{\text{eff}} \leq 3$ and by the low energy description of the theory with $N_{\text{eff}} = 5$ in terms of composite mesons, we find that the theory reduces at low energies to the sigma model on the locus of $\mathbf{CP}^{20}$ where a_{ij} has rank 2. (Note that a cannot have rank 0 as that would violate the D-term equation for the U(1).) We also have the F-term constraints,

$$\sum_{i,j} A_k^{ij} a_{ij} = 0, \qquad k = 1, \ldots, 7. \tag{6.59}$$

The low energy theory is a simple non-linear sigma model with the target space

$$\left\{ [a] \in \mathbf{CP}^{20} \;\middle|\; \mathrm{rank}\, a = 2, \quad (6.59) \right\}. \tag{6.60}$$

This is isomorphic to X_A under the correspondence $a_{ij} \propto [x_i x_j]$. We have reproduced the low energy theory of the original linear sigma model at $r \gg 0$.

Finally, let us identify the singular points and find the relation to the FI-theta parameters of the original model. Identification and parametrization of the maximal torus can be done in almost the same way as in the model including SO(4). One difference is that the Weyl group is slightly bigger, $(h_1,h_2) \mapsto (h_2,h_1)$, (h_1^{-1},h_2), (h_1,h_2^{-1}), and there are more forbidden loci: $\sigma_0 = 2\sigma_1$ and $\sigma_0 = 2\sigma_2$ in addition to $\sigma_1 = \sigma_2$ and $\sigma_0 = \sigma_1 + \sigma_2$. The equation determining the Coulomb branch is

$$\frac{(-\sigma_0)^7(-\sigma_0+\sigma_1)^7(-\sigma_0+\sigma_2)^7}{(\sigma_0)^{21}} = -\,\mathrm{e}^{\widetilde{t}}, \quad \frac{(-\sigma_1)^7}{(-\sigma_0+\sigma_1)^7} = \frac{(-\sigma_2)^7}{(-\sigma_0+\sigma_2)^7} = 1. \tag{6.61}$$

The sign in $-\mathrm{e}^{\widetilde{t}}$ comes from the effect of integrating out the off diagonal components of the vector multiplet. We find $\sigma_a = \frac{\sigma_0}{1+\omega_a}$, with $\omega_a^7 = 1$, for $a = 1, 2$, where we need to avoid $\omega_1 = 1$ and $\omega_2 = 1$ in addition to $\omega_1 = \omega_2^{\pm 1}$. The Weyl group action becomes $(\omega_1, \omega_2) \mapsto (\omega_2, \omega_1)$, $(\omega_1^{-1}, \omega_2)$, $(\omega_1, \omega_2^{-1})$. There are three inequivalent possibilities $(\omega_1, \omega_2) = (\omega, \omega^2)$, (ω, ω^3) and (ω^2, ω^3) (again, $\omega := \mathrm{e}^{\frac{2\pi i}{7}}$), for which $\mathrm{e}^{\widetilde{t}} = (1 + \omega^3)^7$, $(1 + \omega^2)^7$ and $(1 + \omega)^7$ respectively. The Weyl group is completely broken and there is a single Coulomb branch at each of these points. Comparing with (6.51), we may set

$$\widetilde{r} = r, \quad \widetilde{\theta} = \theta. \tag{6.62}$$

To summarize, we obtained completely consistent results from the dual pair of linear sigma models. The two play complementary rôles. If the gauge symmetry is unbroken and a non-trivial quantum analysis is needed in one theory, the gauge group is Higgsed and the result is obtained by purely classical analysis in the dual. This happens both at $r \gg 0$ and $r \ll 0$.

6.5 Intersection of quadrics

Let $S_1(x), \ldots, S_M(x)$ be quadratic polynomials of N variables $x = (x_1, \ldots, x_N)$. We denote by Q_S the intersection of M quadrics in $\mathbf{CP}^{N-1}$

$$S_1(x) = \cdots = S_M(x) = 0. \tag{6.63}$$

We assume that Q_S is a smooth submanifold of dimension $N - 1 - M$.

A linear sigma model for Q_S is the U(1) gauge theory with fields $p^1, \ldots, p^M$ of charge -2 and fields $x_1, \ldots, x_N$ of charge 1, with the superpotential

$$W = p^1 S_1(x) + \cdots p^M S_M(x). \tag{6.64}$$

For large positive values of the FI parameter, $r \gg 0$, the D-term equation forces x to have non-zero values and the gauge group is completely broken. The theory reduces at low energies to the non-linear sigma model whose target space is Q_S. For $r \ll 0$, on the other hand, $p = (p^1, \ldots, p^M)$ must have non-zero values, and the U(1) gauge group is broken to the $\mathbf{Z}_2$ subgroup. The low energy theory is the so called hybrid model. It is a Landau-Ginzburg model on the $\mathbf{Z}_2$ orbibundle $\mathcal{O}(-\frac{1}{2})^{\oplus N}$ over $\mathbf{CP}^{M-1}$. The equation for Coulomb branch vacua is

$$(-2\sigma)^{2M}/\sigma^N = \mathrm{e}^t. \tag{6.65}$$

When $N = 2M$, where Q_S is a Calabi-Yau manifold, the axial U(1) R symmetry is anomaly free and t is a parameter of the theory. There is a singular point at $\mathrm{e}^t = 2^N$. When $N > 2M$, the theory is a flow from the sigma model on Q_S to the $r \ll 0$ hybrid model or one of the $(N - 2M)$ Coulomb branch vacua. When $N < 2M$, the theory is a flow from the $r \ll 0$ hybrid model to the sigma model on Q_S or one of the $(2M - N)$ Coulomb branch vacua.

Our main interest is the nature of the hybrid model at $r \ll 0$. (See [28] for an earlier study.) We may rewrite the superpotential (6.64) as

$$W = \sum_{i,j=1}^{N} S^{ij}(p) x_i x_j, \tag{6.66}$$

where $S^{ij}(p) = \sum_k S_k^{ij} p^k$ for $S_k(x) = \sum_{i,j} S_k^{ij} x_i x_j$. The model can be regarded as the $\mathbf{Z}_2$ Landau-Ginzburg orbifold of x with this quadratic superpotential fibred over $\mathbf{P} := \mathbf{CP}^{M-1}$. We denote by $\mathbf{P}_{(i)}$ the locus of $p \in \mathbf{P}$ where $S(p)$ has rank $N - i$. It has codimension $\frac{i(i+1)}{2}$. Over the generic locus $\mathbf{P}_{(0)}$, the fields x_i are all massive and can be integrated out. The $\mathbf{Z}_2$ orbifold is the standard one, so that the number of zero energy states in the x sector is two *resp.* one if N is even *resp.* odd (section 2.2). Thus, we have a double *resp.* single cover over $\mathbf{P}_{(0)}$. Near the first degeneration locus $\mathbf{P}_{(1)}$ (codimension 1), the result of section 2.3 can be applied: the double cover for the N even case is branched along $\mathbf{P}_{(1)}$, while the cover for the odd N case is of the form of the orbifold $\mathbf{C}/\mathbf{Z}_2$ in the transverse direction to $\mathbf{P}_{(1)}$. Near the second degeneration locus $\mathbf{P}_{(2)}$ (codimension 3), the result of section 2.4 and 3.3 can be applied: in the transverse direction with coordinate (a, b, c), the double cover for the N even case is the conifold $c^2 - ab = d^2$ with $\theta = \pi$ where the $\mathbf{Z}_2$ deck transformation is $d \to \pm d$, while the cover for the N odd case is the $\mathbf{Z}_2(-1)^{F_s}$ orbifold thereof. We would like to see how such local behaviour may be glued together and find a global picture. For this purpose we turn to the dual model.

A key to find the dual is to rewrite the gauge group as

$$\mathrm{U}(1) = \frac{\mathrm{U}(1) \times \mathrm{O}(1)}{\{(\pm 1, \pm 1)\}}, \tag{6.67}$$

and apply the duality to the O(1) sector that appears in the $r \ll 0$ hybrid model. Since we have the standard $\mathbf{Z}_2$ orbifold with N fields that transform by sign flip, the O(1) is $\mathrm{O}_+(1)$ when N is even while it is $\mathrm{O}_-(1)$ when N is odd (see *Special Cases* in section 3.3). For even *resp.* odd N, the dual model has gauge group

$$\frac{\mathrm{U}(1) \times \mathrm{SO}(N)}{\{(\pm 1, \pm \mathbf{1}_N)\}} \quad \textit{resp.} \quad \frac{\mathrm{U}(1) \times \mathrm{O}_-(N)}{\{(\pm 1, \pm \mathbf{1}_N)\}}, \tag{6.68}$$

the matter fields

$$\underbrace{p^1 \ \cdots \ p^M}_{(-2, \mathbf{1})} \ \underbrace{\widetilde{x}^1 \ \cdots \ \widetilde{x}^N}_{(-1, \mathbf{N})} \ \underbrace{(s_{ij})_{1 \le i \le j \le N}}_{(2, \mathbf{1})} \tag{6.69}$$

and the superpotential

$$W = \sum_{j,j=1}^{N} s_{ij}(\widetilde{x}^i \widetilde{x}^j) + \sum_{i,j=1}^{N} S^{ij}(p) s_{ij}. \tag{6.70}$$

The mod 2 theta angle for the SO(N) factor is turned on (as $N - k = 0$ (even)). We write $\widetilde{t}_{\mathrm{u}(1)} = \widetilde{r}_{\mathrm{u}(1)} - i\widetilde{\theta}_{\mathrm{u}(1)}$ for the FI-theta parameter for the U(1) factor.

Let us be more precise about the theta angle. There is no subtlety for odd N since the gauge group is simply isomorphic to $\mathrm{U}(1)\times\mathrm{SO}(N)$, as the element $(-1,-\mathbf{1}_N)$ identifies the two connected components. In particular, the theta parameter has the standard periodicity $\widetilde{\theta}_{\mathrm{u}(1)}\equiv\widetilde{\theta}_{\mathrm{u}(1)}+2\pi$. For even N, the fundamental group of the gauge group is isomorphic to $\mathbf{Z}\oplus\mathbf{Z}_2$ but not canonically so. For example, a loop associated to $(n,0)\in\mathbf{Z}\oplus\mathbf{Z}_2$ may be chosen as $t\in\mathbf{R}/2\pi\mathbf{Z}\mapsto g_t=[(\mathrm{e}^{\frac{int}{2}},h_t)]$ where h_t is represented by $\widetilde{h}_t=\exp\left(\frac{nt}{4}(e_1e_2+\cdots+e_{N-1}e_N)\right)])$ in $\mathrm{Spin}(N)$, but we could equally well choose the one where the sign of e_ae_{a+1} in the exponent is flipped. To be specific, let us define $\widetilde{\theta}_{\mathrm{u}(1)}$ so that the path-integral weight is given the phase $\mathrm{e}^{\frac{in}{2}\widetilde{\theta}_{\mathrm{u}(1)}}$ for the gauge bundle defined by this particular transition function g_t. It has the extended periodicity $\widetilde{\theta}_{\mathrm{u}(1)}\equiv\widetilde{\theta}_{\mathrm{u}(1)}+4\pi$. Recall that $2\widetilde{\theta}$ in (6.27) also has the extended periodicity for the same reason. However, unlike in that case, the theories with $\widetilde{\theta}_{\mathrm{u}(1)}$ and $\widetilde{\theta}_{\mathrm{u}(1)}+2\pi$ are equivalent — the symmetry $\tau\in\mathrm{O}(N)/\mathrm{SO}(N)$ makes the shift

$$\widetilde{\theta}_{\mathrm{u}(1)}\longrightarrow\widetilde{\theta}_{\mathrm{u}(1)}+2\pi. \tag{6.71}$$

To see this, note that conjugation by τ changes h_t by multiplication of a non-contractible loop in $\mathrm{SO}(N)$. For example, if τ is represented by $\mathrm{diag}(-1,1,\ldots,1)$, then $\tau\widetilde{h}_t\tau^{-1}=\exp\left(\frac{nt}{4}(-e_1e_2+\cdots+e_{N-1}e_N)\right)=\exp\left(-\frac{nt}{2}e_1e_2\right)\widetilde{h}_t$. Since we have a non-trivial mod 2 theta angle for $\mathrm{SO}(N)$, the path-integral weight changes by $(-1)^n$. This change is nothing but the shift (6.71). In the model of (6.27), the symmetry $\tau\in\mathrm{O}(4)/\mathrm{SO}(4)$ exists but does not shift $\widetilde{\theta}_{\mathrm{u}(1)}=2\widetilde{\theta}$ since the mod 2 theta angle for $\mathrm{SO}(4)$ is turned off. Note that the shift (6.71) is related to the ambiguity in the choice of isomorphism of the fundamental group to $\mathbf{Z}\oplus\mathbf{Z}_2$.

Let us now analyze the theory at $\widetilde{r}_{\mathrm{u}(1)}\ll 0$. The D- and F-term equations require p to have non-zero values, and the gauge group is broken to $\mathrm{SO}(N)$ (even N) or $\mathrm{O}_-(N)$ (odd N). Integrating out the fields s_{ij} we obtain the constraints

$$(\widetilde{x}^i\widetilde{x}^j)+S^{ij}(p)=0\quad\forall(i,j). \tag{6.72}$$

Suppose $S(p)$ has rank at least $N-1$ for all $p\neq 0$, i.e., $\mathbf{P}=\mathbf{P}_{(0)}\cup\mathbf{P}_{(1)}$, which would be the case if $\dim\mathbf{P}\leq 2$. Then, $\widetilde{x}$ has rank at least $N-1$ for every solution to (6.72). For even N the residual gauge group $\mathrm{SO}(N)$ is completely broken at any solution to (6.72). Therefore, the low energy theory is the sigma model whose target space is the free quotient

$$\widetilde{\mathbf{P}}_S=\left\{(p,\widetilde{x})\in\mathbf{C}^M\oplus(\mathbf{C}^N)^{\oplus N}\,\middle|\,p\neq 0,\ (6.72)\right\}\Big/\frac{\mathbf{C}^\times\times\mathrm{SO}(N,\mathbf{C})}{\{(\pm1,\pm\mathbf{1}_N)\}}. \tag{6.73}$$

The map $(p,\widetilde{x})\in\widetilde{\mathbf{P}}_S\mapsto p\in\mathbf{P}$ is a double cover that is branched along the degeneration locus $\mathbf{P}_{(1)}$. This can be seen as in the argument to show that (6.22) is a ramified double cover. Indeed, if $S(p)=-\mathrm{diag}(z,1,\ldots,1)$, the solution to (6.72) is given by $\widetilde{x}=\mathrm{diag}(\widetilde{z},1,\ldots,1)$ with $z=\widetilde{z}^2$. For non-zero z, the two solutions with opposite signs of $\widetilde{z}$ are distinct. $\widetilde{\mathbf{P}}_S$ provides an explicit global realization of the branched double cover that is expected in the local analysis of the original linear sigma model. For odd N, the residual gauge group

$\mathrm{O}_-(N)$ is completely broken at maximal rank solutions over $\mathbf{P}_{(0)}$ but a $\mathbf{Z}_2$ subgroup remains unbroken at corank 1 solutions over $\mathbf{P}_{(1)}$. The low energy theory is the sigma model on an orbifold

$$\mathbf{P}_S = \left\{ (p, \widetilde{x}) \in \mathbf{C}^M \oplus (\mathbf{C}^N)^{\oplus N} \,\middle|\, p \neq 0,\ (6.72) \right\} \Big/ \frac{\mathbf{C}^\times \times \mathrm{O}_-(N, \mathbf{C})}{\{(\pm 1, \pm \mathbf{1}_N)\}}. \tag{6.74}$$

For $S(p) = -\mathrm{diag}(z, 1, \ldots, 1)$ and $\widetilde{x} = \mathrm{diag}(\widetilde{z}, 1, \ldots, 1)$ with $z = \widetilde{z}^2$, the relevant unbroken gauge group is $\mathrm{O}_-(1)$ that acts on a single variable $\widetilde{z}$ as $\widetilde{z} \to -\widetilde{z}$. Therefore, the orbifold is the standard one $\mathbf{C}/\mathbf{Z}_2$ in the direction transverse to $\mathbf{P}_{(1)}$. Again, $\mathbf{P}_S$ provides an explicit global realization of the orbifold that is expected in the local analysis of the original linear sigma model. In a general case, especially when $\mathbf{P}$ has dimension three or higher, $\mathbf{P}_{(i)}$ with $i \geq 2$ are non-empty. At solutions to (6.72) over such higher degeneration locus, continuous subgroups of the gauge group remain unbroken. In particular, the quotients (6.73) and (6.74) are not smooth manifolds nor orbifolds, and we no longer have a sigma model description of the low energy theory. But we do see what we have locally in the direction transverse to $\mathbf{P}_{(i)}$: for even *resp.* odd N it is the $\mathrm{SO}(i)$ *resp.* $\mathrm{O}_-(i)$ gauge theory with i massless fundamentals, which flows to a superconformal field theory with central charge $\widehat{c} = \frac{i(i+1)}{2}$. The case $i = 2$ is indeed (orbifold of) the conifold at $\theta = \pi$. It would be interesting to understand the total system better, using the original and the dual models.

The model at $\widetilde{r}_{\mathfrak{u}(1)} \gg 0$ can be analyzed as follows. The D-term equation requires $s \neq 0$ and the gauge group is broken to $\mathrm{SO}(N)$ (even N) or $\mathrm{O}_-(N)$ (odd N). By the supersymmetry breaking for the theory with $N_{\text{eff}} \leq N - 2$ and the low energy description of the theory with $N_{\text{eff}} = N - 1$ in terms of the composite mesons, we find that the theory reduces to the sigma model on the locus of $s \in \mathbf{CP}^{\frac{N(N+1)}{2}}$ of corank $N - 1$, i.e., rank 1, so that one may write

$$s_{ij} = x_i x_j. \tag{6.75}$$

We also have the F-term constraints

$$\sum_{ij} S_k^{ij} s_{ij} = 0, \quad k = 1, \ldots, M. \tag{6.76}$$

Namely, we have the sigma model whose target space is the complete intersection of the quadrics (6.63).

Finally, let us analyze the Coulomb branch vacua.

Even N. We use the following parametrization of the maximal torus of the gauge group;

$$\begin{aligned} \frac{\mathrm{U}(1) \times \mathrm{SO}(2)_1 \times \cdots \times \mathrm{SO}(2)_{\frac{N}{2}}}{\{(\pm 1, \pm \mathbf{1}_2, \ldots, \pm \mathbf{1}_2)\}} &\cong \mathrm{U}(1)_0 \times \mathrm{U}(1)_1 \times \cdots \times \mathrm{U}(1)_{\frac{N}{2}} \\ (z, h_1, \ldots, h_{\frac{N}{2}}) &\longmapsto (z^2, zh_1, \ldots, zh_{\frac{N}{2}}). \end{aligned} \tag{6.77}$$

For the theta parameter $\widetilde{\theta}_{\mathfrak{u}(1)}$ (period 4π) as defined in the paragraph including (6.71), the tree level twisted superpotential on the Coulomb branch is

$$\widetilde{W}_{tree} = -\widetilde{t}_{\mathfrak{u}(1)} \sigma_{\mathfrak{u}(1)} + \pi i \sum_{a=1}^{\frac{N}{2}} (\sigma_{\mathfrak{so}(2)_a} - \sigma_{\mathfrak{u}(1)}) = -t_0 \sigma_0 + \pi i \sum_{a=1}^{\frac{N}{2}} \sigma_a. \tag{6.78}$$

Here we write $\sigma_0 = 2\sigma_{\mathfrak{u}(1)}$ and $\sigma_a = \sigma_{\mathfrak{u}(1)} + \sigma_{\mathfrak{so}(2)_a}$ following (6.77) and we put $t_0 := \frac{1}{2}\widetilde{t}_{\mathfrak{u}(1)} + \frac{N}{2}\pi i$ (period $2\pi i$). The πi terms in (6.78) come from the mod 2 theta angle for the SO(N) gauge group. The effective twisted superpotential is

$$\begin{aligned}\widetilde{W}_{\text{eff}} = &-\frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2}\pi i\sigma_0 - M(-\sigma_0)(\log(-\sigma_0)-1)\\ &-N\sum_{a=1}^{\frac{N}{2}}\Big\{(-\sigma_0+\sigma_a)(\log(-\sigma_0+\sigma_a)-1)+(-\sigma_a)(\log(-\sigma_a)-1)\Big\}\\ &-\frac{N(N+1)}{2}\sigma_0(\log\sigma_0-1) - t_0\sigma_0 + \pi i\sum_{a=1}^{\frac{N}{2}}\sigma_a, \end{aligned} \tag{6.79}$$

where the first term results from integrating out the off diagonal components of the vector multiplet. The vacuum equations read

$$\frac{(-\sigma_0)^M\prod_{a=1}^{\frac{N}{2}}(-\sigma_0+\sigma_a)^N}{\sigma_0^{\frac{N(N+1)}{2}}} = (-1)^{\frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2}}\,\mathrm{e}^{t_0}, \qquad \frac{\sigma_a^N}{(\sigma_0-\sigma_a)^N} = -1 \quad \forall a. \tag{6.80}$$

Writing $\sigma_a = \sigma_0\left(\frac{1}{2}+u_a\right)$ (i.e., $u_a := \sigma_{\mathfrak{so}(2)_a}/\sigma_0$), we see that each u_a must solve

$$\left(\frac{1}{2}+u\right)^N + \left(\frac{1}{2}-u\right)^N = 0 \tag{6.81}$$

and σ_0 is then determined by

$$\sigma_0^{M-\frac{N}{2}}(-1)^M\prod_{a=1}^{\frac{N}{2}}\left(\frac{1}{2}-u_a\right)^N = (-1)^{\frac{\frac{N}{2}\left(\frac{N}{2}-1\right)}{2}}\,\mathrm{e}^{t_0}. \tag{6.82}$$

The equation (6.81) has $\frac{N}{2}$ pairs of non-zero roots, and we must find solutions such that $u_a \neq \pm u_b$ ($a \neq b$) modulo the SO(N) Weyl group action — permutations and sign flips of u_a's preserving the product $u_1 \cdots u_{\frac{N}{2}}$. There are two inequivalent solutions. When $N \neq 2M$, the equation (6.82) for given u_a's has $\left|M-\frac{N}{2}\right|$ solutions for σ_0. Thus, there are total of $|2M-N|$ solutions, matching with the result in the original linear sigma model. When $N = 2M$, the equation (6.82) should be regarded as the one that determines the location of singular points in the parameter space. Corresponding to the two solutions for u_a's, we have *two* singular points, related by $\mathrm{e}^{t_0} \to -\mathrm{e}^{t_0}$, i.e., $t_0 \to t_0 + \pi i$. Thus our one parameter family of theories cannot be the same as the one parameter family from the original model which has one singular point ($\mathrm{e}^t = 2^N$). At this point, we recall that we have a symmetry $\tau \in \mathrm{O}(N)/\mathrm{SO}(N)$ which shifts $\widetilde{t}_{\mathfrak{u}(1)}$ by $2\pi i$ (6.71), that is, $t_0 \to t_0 + \pi i$. Thus, our dual family can be regarded as a double cover of the original family. A precise covering map is given by

$$\mathrm{e}^t = 2^N\prod_{a=1}^{\frac{N}{2}}\left(1+\mathrm{e}^{\frac{\pi i(2a-1)}{N}}\right)^{2N}\cdot\mathrm{e}^{2t_0}. \tag{6.83}$$

This also relates t and $\widetilde{t}_{\mathfrak{u}(1)}$ via $\mathrm{e}^{2t_0} = \mathrm{e}^{\widetilde{t}_{\mathfrak{u}(1)}}$.

Odd N. This case is more straightforward as the gauge group is simply isomorphic to $U(1) \times SO(N)$. We denote the scalar components of the vector multuplet by σ for the U(1) part and by $\sigma_1, \ldots, \sigma_{\frac{N-1}{2}}$ for the maximal torus of $SO(N)$. The tree level twisted superpotential is

$$\widetilde{W}_{tree} = -\tilde{t}_{\mathfrak{u}(1)}\sigma + \pi i(\sigma_1 + \cdots + \sigma_{\frac{N-1}{2}}). \tag{6.84}$$

Again, the πi terms come from the mod 2 theta angle of the $SO(N)$ gauge group. Computation of the effective twisted superpotential is straightforward and the vacuum equation reads

$$\frac{(-2\sigma)^{2M}(-\sigma)^N \prod_{a=1}^{\frac{N-1}{2}}(-\sigma+\sigma_a)^N(-\sigma-\sigma_a)^N}{(2\sigma)^{N(N+1)}} = e^{\tilde{t}_{\mathfrak{u}(1)}}, \qquad \frac{(-\sigma-\sigma_a)^N}{(-\sigma+\sigma_a)^N} = 1 \quad \forall a. \tag{6.85}$$

We see that each $z_a = \sigma_a/\sigma$ must solve the equation

$$(1+z)^N - (1-z)^N = 0, \tag{6.86}$$

and σ is then determined by

$$(-\sigma)^{2M-N}2^{2M-N(N+1)}\prod_{a=1}^{\frac{N-1}{2}}(1-z_a^2)^N = e^{\tilde{t}_{\mathfrak{u}(1)}}. \tag{6.87}$$

The equation (6.86) has one root at $z = 0$ and $\frac{N-1}{2}$ pairs of non-zero roots. We look for solutions such that $z_a \neq \pm z_b$ $(a \neq b)$ and $z_a \neq 0$ modulo the $SO(N)$ Weyl group action — permutations and independent sign flips of z_a's. There is a unique solution. Thus, we find $|2M - N|$ Coulomb branch vacua, matching with the result in the original linear sigma model.

6.6 Equivalences of D-brane categories

As mentioned earlier, the present work is motivated by recent development in mathematics concerning equivalences of derived categories of certain pairs of algebraic varieties. Such categories are realized as the categories of B-branes in the supersymmetric non-linear sigma models. If two varieties X and Y sits on a common quantum Kähler moduli space, it is expected from the general principle of $(2,2)$ supersymmetry that X and Y have equivalent derived categories. Our task was to promote the equivalences found in mathematics to statements in quantum field theories. Here we summarize the relevant equivalences, give some references, and make some comments.

The study of section 6.1 and 6.2 is directly related to the work [16] by Hosono and Takagi. The relevant equivalence is

$$D^b(X_S) \cong D^b(\widetilde{Y}_S). \tag{6.88}$$

The proof is being done by the authors of [16].

From the linear sigma model with gauge group $(\mathrm{U}(1) \times \mathrm{SO}(2))/\{(\pm 1, \pm \mathbf{1}_2)\}$ studied in section 6.3, we have equivalences

$$\begin{array}{ccccc} & & D^b(X_{\mathrm{II}}) & & \\ & {\scriptstyle\cong}\nearrow & & \searrow{\scriptstyle\cong} & \\ D^b(\widetilde{X}_S) & & & & D^b_{\mathbf{Z}_2}(\widetilde{Y}_S) \\ & {\scriptstyle\cong}\searrow & & \nearrow{\scriptstyle\cong} & \\ & & D^b(X_{\mathrm{III}}) & & \end{array} \tag{6.89}$$

The two arrows on the left (i.e. those not involving the orbifold category $D^b_{\mathbf{Z}_2}(\widetilde{Y}_S)$) as well as the unwritten vertical arrow in the middle are already mentioned in [16]. They can be promoted to the relations between $\mathcal{N} = 2$ theories with boundaries [41].

From the linear sigma model with gauge group $(\mathrm{U}(1) \times \mathrm{O}_-(2))/\{(\pm 1, \pm \mathbf{1}_2)\}$ studied also in section 6.3, we have

$$D^b_{\mathbf{Z}_2(-1)^{F_s}}(\widetilde{X}_S) \cong D^b_{\mathbf{Z}_2(-1)^{F_s}}(\widetilde{Y}_S). \tag{6.90}$$

We invented a notation $D^b_{\mathbf{Z}_2(-1)^{F_s}}(-)$ for the category of the non-standard $\mathbf{Z}_2$ orbifold, not knowing the well accepted notation in mathematics.

The equivalence relevant for section 6.4 is

$$D^b(X_A) \cong D^b(Y_A). \tag{6.91}$$

This was first pointed out by E. Witten as a consequence of the work [2]. Proofs are given by Borisov-Caldararu [42] and Kuznetsov [43].

The study of section 6.5 is related to the following equivalences, found by Bondal-Orlov [15, 22, 23]:

$$\begin{aligned} D^b(Q_S) &\cong \langle \mathcal{C}_S, \mathcal{O}_1, \ldots, \mathcal{O}_{N-2M} \rangle \quad \text{if } N \geq 2M, \\ \big\langle \mathcal{B}_{N-2M}, \ldots, \mathcal{B}_{-1}, D^b(Q_S) \big\rangle &\cong \mathcal{C}_S \quad \text{if } N \leq 2M, \end{aligned} \tag{6.92}$$

where $\mathcal{C}_S$ is a category that corresponds to the linear sigma model at $\sigma_{\mathrm{u}(1)} = 0$ for $r \ll 0$. For $M \leq 3$, $\mathcal{C}_S = D^b(\widetilde{\mathbf{P}}_S)$ for even N and $\mathcal{C}_S = D^b(\mathbf{P}_S)$ for odd N. $\mathcal{O}_i$ and $\mathcal{B}_{-j}$ are "exceptional objetcs" of the category on the other side of the equivalence. $\langle -, -, \ldots, - \rangle$ stands for "semi-orthogonal decomposition". In fact, this equivalence for odd N was the motivation to refine our understanding of $\mathbf{Z}_2$ orbifolds at an earlier stage, and the refinement resulted in finding the O_- duality (the earlier understanding gave us only the SO/O_+ duality). Linear sigma models relevant for the equivalence for the case $N = 2M$ were studied earlier in [28].

A point of view that seems to underlie all of these equivalences is *projective duality* and its categorical counterpart proposed by A. Kuznetsov, called Homological Projective Duality [44]. Indeed, comparison of the presentation (6.15) of Y_S and (6.35) of X_S suggests projective duality. Note that these two presentations have been found in two linear sigma models which are dual to each other. The same applies to (6.49) of Y_A and (6.60) of X_A. It would be interesting to see if there is a relation between the gauge theory duality and projective duality at a more fundamental level.

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A Supersymmetric quantum mechanics with both superpotential and twisted masses

We study the spectrum of supersymmetric ground states of the theory of two variables z and x having the superpotential

$$W = zx^2, \tag{A.1}$$

and twisted masses associated with the symmetry

$$x \longmapsto \lambda^{-1}x, \quad z \longmapsto \lambda^2 z. \tag{A.2}$$

That is, x and z have twisted masses $-\widetilde{m}$ and $2\widetilde{m}$ respectively.

For the purpose of finding the ground states, we may only consider the zero mode sector. The supercharges are given by

$$\begin{aligned}
\overline{Q}_+ &= \overline{\psi}^x_+\frac{\partial}{\partial\overline{x}} + \overline{\psi}^z_+\frac{\partial}{\partial\overline{z}} - \left(2zx\psi^x_- + x^2\psi^z_- - \widetilde{m}x\overline{\psi}^x_- + 2\widetilde{m}z\overline{\psi}^z_-\right) && \text{(A.3)}\\
\overline{Q}_- &= \overline{\psi}^x_-\frac{\partial}{\partial\overline{x}} + \overline{\psi}^z_-\frac{\partial}{\partial\overline{z}} + \left(2zx\psi^x_+ + x^2\psi^z_+ + \overline{\widetilde{m}}x\overline{\psi}^x_+ - 2\overline{\widetilde{m}}z\overline{\psi}^z_+\right) && \text{(A.4)}\\
Q_+ &= -\psi^x_+\frac{\partial}{\partial x} - \psi^z_+\frac{\partial}{\partial z} - \left(2\overline{z}\overline{x}\overline{\psi}^x_- + \overline{x}^2\overline{\psi}^z_- - \overline{\widetilde{m}}\overline{x}\psi^x_- + 2\overline{\widetilde{m}}\overline{z}\psi^z_-\right) && \text{(A.5)}\\
Q_- &= -\psi^x_-\frac{\partial}{\partial x} - \psi^z_-\frac{\partial}{\partial z} + \left(2\overline{z}\overline{x}\overline{\psi}^x_+ + \overline{x}^2\overline{\psi}^z_+ + \widetilde{m}\overline{x}\psi^x_+ - 2\widetilde{m}\overline{z}\psi^z_+\right). && \text{(A.6)}
\end{aligned}$$

Non-zero anticommutators are

$$\begin{aligned}
\{\overline{Q}_+, Q_+\} &= \{\overline{Q}_-, Q_-\} = H, && \text{(A.7)}\\
\{\overline{Q}_+, Q_-\} &= \widetilde{m}J, \quad \{\overline{Q}_-, Q_+\} = \overline{\widetilde{m}}J, && \text{(A.8)}
\end{aligned}$$

where H and J are the Hamiltonian and the generator of (A.2) respectively,

$$\begin{aligned}
H = &-\frac{\partial^2}{\partial x\partial\overline{x}} - \frac{\partial^2}{\partial z\partial\overline{z}} + |2zx|^2 + |x^2|^2 + |\widetilde{m}x|^2 + |2\widetilde{m}z|^2 && \text{(A.9)}\\
&+ \left[\left(2z\psi^x_+\psi^x_- + 2x\psi^x_+\psi^z_- + 2x\psi^z_+\psi^x_-\right) + h.c.\right] + \left[\left(-\widetilde{m}\psi^x_+\overline{\psi}^x_- + 2\widetilde{m}\psi^z_+\overline{\psi}^z_-\right) + h.c.\right],\\
J = &-\left(x\frac{\partial}{\partial x} - \overline{x}\frac{\partial}{\partial\overline{x}} - \overline{\psi}^x_+\psi^x_+ + \psi^x_-\overline{\psi}^x_-\right) + 2\left(z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial\overline{z}} - \overline{\psi}^z_+\psi^z_+ + \psi^z_-\overline{\psi}^z_-\right). && \text{(A.10)}
\end{aligned}$$

The space of states can be chosen to be the space

$$\mathcal{H} = \bigoplus_{p,q=0}^{2} \Omega^{0,p}(\mathbf{C}^2, \wedge^q T_{\mathbf{C}^2}) \tag{A.11}$$

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of differential forms on $\mathbf{C}^2 = \{(x, z)\}$ with values in polyvector fields. The fermions are represented on it as

$$\overline{\psi}^x_+ + \overline{\psi}^x_- = \mathrm{d}\overline{x}\wedge, \qquad \psi^x_+ + \psi^x_- = \imath\left(\frac{\partial}{\partial\overline{x}}\right),$$

$$\psi^x_+ - \psi^x_- = \imath\,(\mathrm{d}x)\,, \qquad \overline{\psi}^x_+ - \overline{\psi}^x_- = \frac{\partial}{\partial x}\wedge,$$

and similarly for $\psi^z_\pm$ and $\overline{\psi}^z_\pm$.

If we choose $\widetilde{m}$ to be pure imaginary, $\widetilde{m} + \overline{\widetilde{m}} = 0$, the operator $Q = \overline{Q}_+ + \overline{Q}_-$ obey the relation,

$$Q^2 = 0, \quad \{Q, Q^\dagger\} = 2H. \tag{A.12}$$

In particular, there is a one to one correspondence between supersymmetric ground states and Q-cohomology classes. Under the same condition, $\widetilde{m} + \overline{\widetilde{m}} = 0$, Q is represented by the operator $Q = \overline{\partial} + Q_{\rm hol}$ on the space $\mathcal{H}$ in (A.11), where

$$Q_{\rm hol} = \imath(\mathrm{d}W) - \widetilde{m}K\wedge, \tag{A.13}$$

with $dW = 2zx\mathrm{d}x + x^2\mathrm{d}z$ and $K = -x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z}$. Using the standard argument, one can show that Q-cohomology classes are in one to one correspondence with the $Q_{\rm hol}$-cohomology classes where $Q_{\rm hol}$ is regarded as the differential on the space of holomorphic polyvector fields,

$$\mathcal{H}_{\rm hol} = \bigoplus_{q=0}^{2} \Gamma_{\rm hol}(\mathbf{C}^2, \wedge^q T_{\mathbf{C}^2}). \tag{A.14}$$

Note that

$$Q_{\rm hol}(1) = -\widetilde{m}\left(-x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z}\right),$$

$$Q_{\rm hol}\left(\frac{\partial}{\partial x}\right) = 2zx + 2\widetilde{m}z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}, \qquad Q_{\rm hol}\left(\frac{\partial}{\partial z}\right) = x^2 + \widetilde{m}x\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z},$$

$$Q_{\rm hol}\left(\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}\right) = x\left(-x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z}\right).$$

It is easy to see that there is only one cohomology class, which is represented by

$$x + \widetilde{m}\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z}. \tag{A.15}$$

This proves that the system has a unique supersymmetric ground state.

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