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# Twisted $\mathcal{N} = 4$ super Yang-Mills theory in $\Omega$-background

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Abstract: We study the twisted $\mathcal{N} = 4$ super Yang-Mills theories in the $\Omega$-background with the constant R-symmetry Wilson line gauge field. Based on the classification of topological twists of $\mathcal{N} = 4$ supersymmetry (the half, the Vafa-Witten and the Marcus twists), we construct the deformed off-shell supersymmetry associated with the scalar supercharges for these twists. We find that the $\Omega$-deformed action is written in the exact form with respect to the scalar supercharges as in the undeformed case.

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## Contents



## 1 Introduction

The Ω-background deformation [1] of supersymmetric gauge theories provides a useful regularization procedure to study their non-perturbative effects [2–4]. The Ω-background is the curved geometry which admits the action of the U(1) vector fields. Since this background violates the translational symmetry, supersymmetry is explicitly broken. One can however introduce the constant R-symmetry Wilson line gauge field to retain a part of supersymmetry.

The scalar supercharge, which is obtained by the topological twist of supersymmetry [5], is a particularly important ingredient in the deformed theory because it is used to perform the path-integral exactly via the localization formula. For the Ω-deformed $\mathcal{N} = 2$ supersymmetric gauge theories [2, 3], the supercharge is nilpotent up to the gauge transformation and the $\mathrm{U}(1)^2$ rotation (see also [6] for their explicit off-shell transformations). The Ω-deformed $\mathcal{N} = 2$ super Yang-Mills theory is also realized by the dimensional reduction of the $\mathcal{N} = 1$ super Yang-Mills theory in the six-dimensional background [3].

It is interesting to explore the ten-dimensional Ω-background and their dimensional reduction to lower dimensions for studying the various Ω-deformed theories in a systematic way. In the previous paper [7], we studied the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory in the ten-dimensional Ω-background [8] (see also [9–12] for different generalization). Starting from the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory in the general curved background with torsion, we considered its dimensional reduction to four dimensions. We examined the parallel spinor conditions and the torsion constraints for the existence of the spinor associated with the scalar supercharges in the four-dimensional theory. The constant $\mathrm{SU}(4)_I$ R-symmetry Wilson line gauge field, which is necessary to preserve supersymmetry,



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is identified with the contorsion. We solved the parallel spinor conditions and the torsion constraints for the Ω-backgrounds. We obtained the on-shell deformed scalar supersymmetries associated with the three different topological twists, the half, the Vafa-Witten and the Marcus twists [13–15], which were classified by Yamron [13]. These twists correspond to the theories with the single scalar supercharge, the two charges with the same chirality, the two charges with opposite chirality, respectively.

In this paper we will study further the Ω-deformed $\mathcal{N} = 4$ super Yang-Mills theory. We are particularly interested in the off-shell supersymmetries, which are deformed non-trivially in the Ω-background. We will introduce the auxiliary fields to construct the scalar supersymmetry acting on them. It will be shown that the deformed scalar supercharges form the algebra, where they are nilpotent and their anti-commutator vanishes up to the gauge transformation and the Lorentz rotation associated with the U(1) vector fields.

Based on the construction of the off-shell supersymmetry algebra, we will study the cohomological properties of the Ω-deformed action. We will show that the deformed action is written in the exact form with respect to the deformed scalar supercharges. Our results show that the twisted $\mathcal{N} = 4$ super Yang-Mills theories are also well-defined in the Ω-backgrounds.

The organization of this paper is as follows. In section 2, we review the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory in the Ω-background. We will introduce three types of the topological twists and the on-shell scalar supersymmetries. In section 3, we will introduce the auxiliary fields to the theory and study the deformed supersymmetry transformations off-shell. We show that the action is written in the exact form with respect to the scalar supercharges. Section 4 is devoted for conclusion and discussions. In appendix we summarize the Dirac matrices in four and six dimensions.

## 2 Ω-deformed $\mathcal{N} = 4$ super Yang-Mills theory and on-shell supersymmetries

In this section, we review the Ω-deformation of $\mathcal{N} = 4$ super Yang-Mills theory. This theory is obtained by the dimensional reduction of the $\mathcal{N} = 1$ super Yang-Mills theory in the ten-dimensional spacetime with the metric [8]

$$ds_{10}^2 = \left(dx^{a+4}\right)^2 + \left(dx^\mu + \Omega_a^\mu dx^{a+4}\right)^2 . \tag{2.1}$$

Here $x^\mu, x^{a+4}$ $(\mu = 1, \cdots, 4,\ a = 1, \cdots, 6)$ are the spacetime coordinates. $\Omega^\mu{}_a = \Omega^{\mu\nu}{}_a x_\nu$ are the $\mathrm{U}(1)^6$ vector fields acting on $x^\mu$. The constant matrices $\Omega_{\mu\nu a}$ are anti-symmetric and satisfy the commutation relations:

$$\Omega_\mu{}^\rho{}_a \Omega_{\rho\nu b} - \Omega_\mu{}^\rho{}_b \Omega_{\rho\nu a} = 0 . \tag{2.2}$$

In the following we use the representation of the matrices $\Omega_{\mu\nu a}$, which are parametrized as

$$\Omega_{\mu\nu a} = \begin{pmatrix} 0 & \epsilon_a^1 & 0 & 0 \\ -\epsilon_a^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_a^2 \\ 0 & 0 & \epsilon_a^2 & 0 \end{pmatrix} , \tag{2.3}$$

where $\epsilon_a^1$, $\epsilon_a^2$ are real constants.



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We now compactify the $x^5, \cdots, x^{10}$ directions on the six-torus $\mathbf{T}^6$ and perform the dimensional reduction to four dimensions. The ten-dimensional Lorentz group SO(10) becomes $\mathrm{SO}(4) \times \mathrm{SO}(6)_I$ where SO(4) is the four-dimensional Lorentz group and $\mathrm{SO}(6)_I \simeq \mathrm{SU}(4)_I$ is the R-symmetry group. We further introduce the constant $\mathrm{SU}(4)_I$ R-symmetry Wilson line gauge field $(\mathcal{A}_a)^A{}_B$. Here the index $a$ labels the vector representation of $\mathrm{SO}(6)_I$ while $A$ labels the (anti)fundamental representation of $\mathrm{SU}(4)_I$. Then the dimensionally reduced action is given by [8]

$$
\begin{aligned}
S = \frac{1}{\kappa g^2} \int d^4x \ \mathrm{Tr} \Bigg[ & \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \Lambda^A \sigma^\mu D_\mu \bar{\Lambda}_A + \frac{1}{2} \left( D_\mu \varphi_a - F_{\mu\nu} \Omega_a^\nu \right)^2 \\
& - \frac{1}{2} \left( \Sigma_a \right)^{AB} \bar{\Lambda}_A \left[ \varphi_a, \bar{\Lambda}_B \right] - \frac{1}{2} \left( \bar{\Sigma}_a \right)_{AB} \Lambda^A \left[ \varphi_a, \Lambda^B \right] \\
& - \frac{1}{4} \Bigg( \left[ \varphi_a, \varphi_b \right] + i \Omega_a^\mu D_\mu \varphi_b - i \Omega_b^\mu D_\mu \varphi_a - i F_{\mu\nu} \Omega_a^\mu \Omega_b^\nu \\
& \qquad - \frac{1}{2} \left( \left( \Sigma_b \bar{\Sigma}_c \right)^A{}_B \varphi_c \left( \mathcal{A}_a \right)^B{}_A - \left( \Sigma_a \bar{\Sigma}_c \right)^A{}_B \varphi_c \left( \mathcal{A}_b \right)^B{}_A \right) \Bigg)^2 \\
& - \frac{i}{2} \Omega_a^\mu \left( \left( \Sigma_a \right)^{AB} \bar{\Lambda}_A D_\mu \bar{\Lambda}_B + \left( \bar{\Sigma}_a \right)_{AB} \Lambda^A D_\mu \Lambda^B \right) \\
& + \frac{i}{4} \Omega_{\mu\nu a} \left( \left( \Sigma_a \right)^{AB} \bar{\Lambda}_A \bar{\sigma}^{\mu\nu} \bar{\Lambda}_B + \left( \bar{\Sigma}_a \right)_{AB} \Lambda^A \sigma^{\mu\nu} \Lambda^B \right) \\
& + \frac{1}{2} \left( \Sigma_a \right)^{AB} \bar{\Lambda}_A \bar{\Lambda}_D \left( \mathcal{A}_a \right)^D{}_B - \frac{1}{2} \left( \bar{\Sigma}_a \right)_{AB} \Lambda^A \left( \mathcal{A}_a \right)^B{}_D \Lambda^D \Bigg] .
\end{aligned}
\tag{2.4}
$$

Here $A_\mu$ $(\mu = 1, 2, 3, 4)$ is a gauge field, $\Lambda^A_\alpha$, $\bar{\Lambda}_{\dot{\alpha} A}$ $(\alpha, \dot{\alpha} = 1, 2, A = 1, 2, 3, 4)$ are Weyl fermions and $\varphi_a$ $(a = 1, \cdots, 6)$ are real scalar fields. The fields are in the adjoint representation of a gauge group. The constant $\kappa$ denotes the normalization of the Lie algebra of the gauge group and $g$ is the gauge coupling constant. The indices $\alpha$, $\dot{\alpha}$ represent left and right spinors of the Lorentz group $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. These indices are raised and lowered by the anti-symmetric symbols $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ with $\epsilon^{12} = -\epsilon_{12} = 1$. The conventions of four- and six-dimensional Dirac matrices $\sigma^\mu$, $\bar{\sigma}^\mu$, $(\Sigma_a)^{AB}$, $(\bar{\Sigma}_a)_{AB}$ are given in appendix. The gauge covariant derivative is defined by $D_\mu * = \partial_\mu * + i[A_\mu, *]$. The field strength of the gauge field is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$.

The action (2.4) is also obtained by replacing the following terms in the undeformed action as

$$
\begin{aligned}
\left[ \varphi_a, \varphi_b \right] & \longrightarrow \left[ \varphi_a, \varphi_b \right] + i \Omega_a^\mu D_\mu \varphi_b - i \Omega_b^\mu D_\mu \varphi_a - i F_{\mu\nu} \Omega_a^\mu \Omega_b^\nu \\
& \qquad - \frac{1}{2} \left( \left( \Sigma_b \bar{\Sigma}_c \right)^A{}_B \varphi_c \left( \mathcal{A}_a \right)^B{}_A - \left( \Sigma_a \bar{\Sigma}_c \right)^A{}_B \varphi_c \left( \mathcal{A}_b \right)^B{}_A \right) , \\
D_\mu \varphi_a & \longrightarrow D_\mu \varphi_a - F_{\mu\nu} \Omega_a^\nu , \\
\left[ \varphi_a, \Lambda^A \right] & \longrightarrow \left[ \varphi_a, \Lambda^A \right] + i \Omega_a^\mu D_\mu \Lambda^A - \frac{i}{2} \Omega_{\mu\nu a} \sigma^{\mu\nu} \Lambda^A + \left( \mathcal{A}_a \right)^A{}_B \Lambda^B , \\
\left[ \varphi_a, \bar{\Lambda}_A \right] & \longrightarrow \left[ \varphi_a, \bar{\Lambda}_A \right] + i \Omega_a^\mu D_\mu \bar{\Lambda}_A - \frac{i}{2} \Omega_{\mu\nu a} \bar{\sigma}^{\mu\nu} \bar{\Lambda}_A - \bar{\Lambda}_B \left( \mathcal{A}_a \right)^B{}_A .
\end{aligned}
\tag{2.5}
$$

For generic $\Omega_{\mu\nu a}$ and $(\mathcal{A}_a)^A{}_B$, supersymmetry of the theory is broken explicitly. However, a part of supersymmetry is recovered when $\Omega_{\mu\nu a}$ and $(\mathcal{A}_a)^A{}_B$ satisfy certain conditions. When the matrices $\Omega_{\mu\nu a}$ are (anti)self-dual and $(\mathcal{A}_a)^A{}_B = 0$, the theory has



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anti-chiral (chiral) half of the $\mathcal{N}=4$ supersymmetry [8]. When $\Omega_{\mu\nu a}$ are neither self-dual nor anti-self-dual, the supersymmetry condition can be studied by the parallel spinor conditions in ten-dimensional $\Omega$-background with the torsion, where the torsion is identified with the R-symmetry Wilson line gauge field. We can solve the parallel spinor conditions and the constraints for the torsion in the $\Omega$-background, which preserve gauge symmetry in four dimensions and supersymmetry associated with the topological twist. These conditions are satisfied for special $\Omega_{\mu\nu a}$ and $(\mathcal{A}_a)^A{}_B$. Then we obtain the topologically twisted supersymmetries deformed by $\Omega_{\mu\nu a}$ and $(\mathcal{A}_a)^A{}_B$ [7].

The topological twist is to pick an embedding of the Lorentz group $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ into the R-symmetry group $\mathrm{SU}(4)_I$ and to define a new Lorentz group. Let us take the $\mathrm{SU}(2)_{L'} \times \mathrm{SU}(2)_{R'}$ subgroup of $\mathrm{SU}(4)_I$ such that the index $A$ is decomposed into $A' = 1, 2$ and $\hat{A} = 3, 4$. Here $A'$ and $\hat{A}$ are indices for the two-dimensional representations of $\mathrm{SU}(2)_{R'}$ and $\mathrm{SU}(2)_{L'}$, respectively. The vector index $a$ is also decomposed into $a' = 1, 2$ which corresponds to the two $\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_{L'}$ singlets, and $\hat{a} = 3, \cdots, 6$ which labels the $\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_{L'}$ bifundamental representation.

Topological twists in $\mathcal{N}=4$ super Yang-Mills theory are classified into three types [13]. They are called the half twist, the Vafa-Witten twist [14] and the Marcus twist [15] (or the GL twist in [16]). In the following subsections, we summarize the three types of topological twists and the deformed supersymmetries.

### 2.1 The half twist

In the case of the half twist, we replace $\mathrm{SU}(2)_R$ by the diagonal subgroup of $\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_R$. The new Lorentz group becomes $\mathrm{SU}(2)_L \times [\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_R]_{\mathrm{diag}}$. Here the subscript "diag" stands for the diagonal subgroup which means that the spinor index $\dot{\alpha}$ and the R-symmetry index $A'$ are identified. The Weyl fermions $\Lambda_\alpha^{A'}$ and $\bar{\Lambda}_{\dot{\alpha}}^{A'}$ are rewritten as follows,

$$\bar{\Lambda}_{\dot{\alpha}}^{A'} = \frac{1}{2}\delta_{\dot{\alpha}}^{A'}\bar{\Lambda} + \frac{1}{2}\left(\bar{\sigma}^{\mu\nu}\right)^{A'}{}_{\dot{\alpha}}\bar{\Lambda}_{\mu\nu}, \qquad \Lambda_\alpha^{A'} = \frac{1}{2}\epsilon^{A'B'}\left(\sigma^\mu\right)_{\alpha B'}\Lambda_\mu\,. \tag{2.6}$$

We redefine the scalar field as

$$\varphi^{AB} = \frac{i}{\sqrt{2}}\left(\Sigma_a\right)^{AB}\varphi_a, \qquad \bar{\varphi}_{AB} = -\frac{i}{\sqrt{2}}\left(\bar{\Sigma}_a\right)_{AB}\varphi_a\,. \tag{2.7}$$

Then using the matrix representations of $(\Sigma_a)^{AB}$ and $(\bar{\Sigma}_a)_{AB}$, the scalar fields are decomposed as

$$\varphi^{AB} = \begin{pmatrix} \epsilon^{A'B'}\varphi & \varphi^{A'\hat{B}} \\ \varphi^{\hat{A}B'} & -\epsilon^{\hat{A}\hat{B}}\bar{\varphi} \end{pmatrix}, \qquad \bar{\varphi}_{AB} = \begin{pmatrix} \epsilon_{A'B'}\bar{\varphi} & \bar{\varphi}_{A'\hat{B}} \\ \bar{\varphi}_{\hat{A}B'} & -\epsilon_{\hat{A}\hat{B}}\varphi \end{pmatrix}, \tag{2.8}$$

where $\varphi$, $\bar{\varphi}$ are the $\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_{L'}$ singlets and $\varphi^{A'\hat{B}}$ belongs to the $\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_{L'}$ bifundamental representation.

The supercharges $\bar{Q}_{\dot{\alpha}}^{A'}$ and $Q_\alpha^{A'}$ are decomposed into the scalar $\bar{Q}$ , the tensor $\bar{Q}_{\mu\nu}$ and the vector $Q_\mu$. In [7] we examined the torsion and the parallel spinor conditions for



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preserving $\bar{Q}$. The solution to the conditions is given by

$$\Omega_{\mu\nu a'} = \begin{pmatrix} 0 & \epsilon^1_{a'} & 0 & 0 \\ -\epsilon^1_{a'} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon^2_{a'} \\ 0 & 0 & \epsilon^2_{a'} & 0 \end{pmatrix}, \qquad (\mathcal{A}_{a'})^A{}_B = \begin{pmatrix} \frac{1}{4}(\epsilon^1_{a'} + \epsilon^2_{a'})\tau^3 & 0 \\ 0 & m_{a'}\tau^3 \end{pmatrix},$$

$$\Omega_{\mu\nu\hat{a}} = (\mathcal{A}_{\hat{a}})^A{}_B = 0, \qquad \big(a' = 1, 2,\ \hat{a} = 3, 4, 5, 6\big), \tag{2.9}$$

where $m_{a'}$ are real parameters. The Wilson line gauge fields $(\mathcal{A})^{\hat{A}}{}_{\hat{B}}, (\bar{\mathcal{A}})^{\hat{A}}{}_{\hat{B}}$ are identified with the mass matrices $M^{\hat{A}}{}_{\hat{B}} = m\tau^3$, $\bar{M}^{\hat{A}}{}_{\hat{B}} = \bar{m}\tau^3$ of the adjoint hypermultiplet of the $\mathcal{N} = 2^*$ theory [4, 8]. Here we defined

$$\begin{aligned} \mathcal{A} &= \frac{1}{\sqrt{2}}(\mathcal{A}_1 - i\mathcal{A}_2), \qquad \bar{\mathcal{A}} = \frac{1}{\sqrt{2}}(\mathcal{A}_1 + i\mathcal{A}_2), \\ m &= \frac{1}{\sqrt{2}}(m_1 - im_2), \qquad \bar{m} = \frac{1}{\sqrt{2}}(m_1 + im_2). \end{aligned} \tag{2.10}$$

We note that the matrices

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & \epsilon^1 & 0 & 0 \\ -\epsilon^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon^2 \\ 0 & 0 & \epsilon^2 & 0 \end{pmatrix}, \quad \bar{\Omega}_{\mu\nu} = \begin{pmatrix} 0 & \bar{\epsilon}^1 & 0 & 0 \\ -\bar{\epsilon}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\epsilon}^2 \\ 0 & 0 & \bar{\epsilon}^2 & 0 \end{pmatrix},$$

$$\epsilon^i = \frac{1}{\sqrt{2}}\left(\epsilon^i_1 - i\epsilon^i_2\right), \qquad \bar{\epsilon}^i = \frac{1}{\sqrt{2}}\left(\epsilon^i_1 + i\epsilon^i_2\right), \quad (i = 1, 2) \tag{2.11}$$

characterize the $\Omega$-background defined in six dimensions [1, 6]. The $\bar{Q}$-transformations are given by

$$\begin{aligned} \bar{Q}A_\mu &= \Lambda_\mu, \\ \bar{Q}\Lambda_\mu &= -2\sqrt{2}\,(D_\mu\varphi - F_{\mu\nu}\Omega^\nu), \\ \bar{Q}\varphi &= \Omega^\mu\Lambda_\mu, \\ \bar{Q}\bar{\varphi} &= -\sqrt{2}\bar{\Lambda} + \bar{\Omega}^\mu\Lambda_\mu, \\ \bar{Q}\bar{\Lambda} &= -2i\left([\varphi, \bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right), \\ \bar{Q}\bar{\Lambda}_{\mu\nu} &= -2F^-_{\mu\nu} - i\,(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\left[\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}\right], \\ \bar{Q}\varphi^{\dot{\alpha}\hat{A}} &= -\sqrt{2}\bar{\Lambda}^{\dot{\alpha}\hat{A}}, \\ \bar{Q}\bar{\Lambda}^{\dot{\alpha}\hat{A}} &= -2i\left(\left[\varphi, \varphi^{\dot{\alpha}\hat{A}}\right] + i\Omega^\mu D_\mu\varphi^{\dot{\alpha}\hat{A}} + M^{\hat{A}}{}_{\hat{B}}\varphi^{\dot{\alpha}\hat{B}}\right) - \Omega^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\varphi^{\dot{\beta}\hat{A}}, \\ \bar{Q}\Lambda^{\hat{A}}_\alpha &= \sqrt{2}\,(\sigma^\mu)_{\dot{\alpha}\alpha}\,D_\mu\varphi^{\dot{\alpha}\hat{A}}. \end{aligned} \tag{2.12}$$

Here $\Omega^\mu$ and $\bar{\Omega}^\mu$ are defined by

$$\Omega^\mu = \Omega^{\mu\nu}x_\nu, \quad \bar{\Omega}^\mu = \bar{\Omega}^{\mu\nu}x_\nu. \tag{2.13}$$

The superscript $\pm$ stands for the (anti)self-dual part of a tensor $X_{\mu\nu}$:

$$X^\pm_{\mu\nu} = \frac{1}{2}\left(X_{\mu\nu} \pm \tilde{X}_{\mu\nu}\right), \tag{2.14}$$



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where $\tilde{X}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}X^{\rho\sigma}$ is the dual of $X_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon_{1234} = 1$. The supercharge $\bar{Q}$ is nilpotent up to the gauge transformation, the Lorentz rotation associated with the $\mathrm{U}(1)\times\mathrm{U}(1)$ rotation generated by $\Omega_\mu$ and the $\mathrm{SU}(2)_{L'}$ rotation. In fact, the nilpotency of $\bar{Q}$ on the fermions $\bar{\Lambda}_{\mu\nu}$ and $\Lambda^{\hat{A}}_{\alpha}$ holds by imposing the equations of motion.

### 2.2 The Vafa-Witten twist

In the case of the Vafa-Witten twist, we replace $\mathrm{SU}(2)_R$ by the diagonal subgroup of $\mathrm{SU}(2)_{L'}\times\mathrm{SU}(2)_{R'}\times\mathrm{SU}(2)_R$. The new Lorentz group becomes $\mathrm{SU}(2)_L\times[\mathrm{SU}(2)_{L'}\times\mathrm{SU}(2)_{R'}\times\mathrm{SU}(2)_R]_{\mathrm{diag}}$. The spinor index $\dot{\alpha}$ is identified with the R-symmetry index $A'$ and $\dot{\alpha}$ is also identified with $\hat{A}$. We decompose the fermion fields as

$$\begin{aligned}
\Lambda_{\alpha A'} &= \frac{1}{2}\left(\sigma^\mu\right)_{\alpha A'}\Lambda_\mu, & \Lambda_{\alpha\hat{A}} &= \frac{1}{2}\left(\sigma^\mu\right)_{\alpha\hat{A}}\hat{\Lambda}_\mu,\\
\bar{\Lambda}^{A'}_{\dot{\alpha}} &= \frac{1}{2}\delta^{A'}_{\dot{\alpha}}\bar{\Lambda} + \frac{1}{2}\left(\bar{\sigma}^{\mu\nu}\right)^{A'}{}_{\dot{\alpha}}\bar{\Lambda}_{\mu\nu}, & \bar{\Lambda}^{\hat{A}}_{\dot{\alpha}} &= \frac{1}{2}\delta^{\hat{A}}_{\dot{\alpha}}\hat{\bar{\Lambda}} + \frac{1}{2}\left(\bar{\sigma}^{\mu\nu}\right)^{\hat{A}}{}_{\dot{\alpha}}\hat{\bar{\Lambda}}_{\mu\nu}\,.
\end{aligned} \tag{2.15}$$

The scalar fields in (2.8) are further decomposed as

$$\varphi^{A'B'} = \epsilon^{A'B'}\varphi, \qquad \varphi^{\hat{A}\hat{B}} = -\epsilon^{\hat{A}\hat{B}}\bar{\varphi}, \qquad \varphi^{A'\hat{A}} = \frac{1}{2}\epsilon^{A'\hat{A}}\hat{\varphi} + \frac{1}{2}\epsilon^{\hat{A}\hat{B}}\left(\bar{\sigma}^{\mu\nu}\right)^{A'}{}_{\hat{B}}\hat{\varphi}_{\mu\nu}\,. \tag{2.16}$$

The supercharges $Q^A_\alpha$ and $\bar{Q}^{\dot{\alpha}}_A$ are decomposed into the two scalars $\bar{Q}$, $\hat{\bar{Q}}$, the two vectors $Q_\mu$, $\hat{Q}_\mu$ and the two tensor supercharges $\bar{Q}_{\mu\nu}$, $\hat{\bar{Q}}_{\mu\nu}$.

The solution to the parallel spinor and the torsion conditions for preserving $\bar{Q}$ and $\hat{\bar{Q}}$ is

$$\Omega_{\mu\nu a'} = \begin{pmatrix} 0 & \epsilon^1_{a'} & 0 & 0\\ -\epsilon^1_{a'} & 0 & 0 & 0\\ 0 & 0 & 0 & -\epsilon^2_{a'}\\ 0 & 0 & \epsilon^2_{a'} & 0 \end{pmatrix}, \qquad \Omega_{\mu\nu\hat{a}} = \begin{pmatrix} 0 & \epsilon^1_{\hat{a}} & 0 & 0\\ -\epsilon^1_{\hat{a}} & 0 & 0 & 0\\ 0 & 0 & 0 & -\epsilon^2_{\hat{a}}\\ 0 & 0 & \epsilon^2_{\hat{a}} & 0 \end{pmatrix},$$

$$(\mathcal{A}_{a'})^A{}_B = \begin{pmatrix} \frac{1}{4}\left(\epsilon^1_{a'}+\epsilon^2_{a'}\right)\tau^3 & 0\\ 0 & \frac{1}{4}\left(\epsilon^1_{a'}+\epsilon^2_{a'}\right)\tau^3 \end{pmatrix}, \quad (\mathcal{A}_{\hat{a}})^A{}_B = \begin{pmatrix} \frac{1}{4}\left(\epsilon^1_{\hat{a}}+\epsilon^2_{\hat{a}}\right)\tau^3 & 0\\ 0 & \frac{1}{4}\left(\epsilon^1_{\hat{a}}+\epsilon^2_{\hat{a}}\right)\tau^3 \end{pmatrix},$$

$$\left(a' = 1,2,\ \hat{a} = 5,6\right),$$

$$\Omega_{\mu\nu 3} = \Omega_{\mu\nu 4} = (\mathcal{A}_3)^A{}_B = (\mathcal{A}_4)^A{}_B = 0\,. \tag{2.17}$$

It is convenient to rewrite the $\Omega$-background matrices $\Omega_{\mu\nu a}$ as

$$\begin{aligned}
\Omega^{AB}_{\mu\nu} &= \frac{i}{\sqrt{2}}\left(\Sigma_a\right)^{AB}\Omega^a_{\mu\nu}, & \bar{\Omega}^{\mu\nu}_{AB} &= -\frac{i}{\sqrt{2}}\left(\bar{\Sigma}_a\right)_{AB}\Omega^{a\mu\nu},\\
\Omega^{AB}_{\mu} &= \Omega^{AB}_{\mu\nu}x^\nu, & \bar{\Omega}^{\mu}_{AB} &= \bar{\Omega}^{\mu\nu}_{AB}x_\nu\,.
\end{aligned} \tag{2.18}$$

We further decompose these matrices as

$$\Omega^{A'B'}_{\mu\nu} = \epsilon^{A'B'}\Omega_{\mu\nu}, \quad \Omega^{\hat{A}\hat{B}}_{\mu\nu} = -\epsilon^{\hat{A}\hat{B}}\bar{\Omega}_{\mu\nu}, \quad \Omega^{A'\hat{A}}_{\mu\nu} = \frac{1}{2}\epsilon^{A'\hat{A}}\hat{\Omega}_{\mu\nu} + \frac{1}{2}\epsilon^{\hat{A}\hat{B}}\left(\bar{\sigma}^{\rho\sigma}\right)^{A'}{}_{\hat{B}}\hat{\Omega}_{\mu\nu,\rho\sigma}, \tag{2.19}$$



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where $\hat{\Omega}_{\mu\nu,\rho\sigma}$ satisfies the anti-self-dual condition with respect to the last two indices. From the decomposition (2.19), we obtain

$$\hat{\Omega}_{\mu\nu} = \sqrt{2}\begin{pmatrix} 0 & \epsilon_6^1 & 0 & 0 \\ -\epsilon_6^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_6^2 \\ 0 & 0 & \epsilon_6^2 & 0 \end{pmatrix}, \quad \hat{\Omega}_{\mu\nu,12} = -\hat{\Omega}_{\mu\nu,34} = -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \epsilon_5^1 & 0 & 0 \\ -\epsilon_5^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_5^2 \\ 0 & 0 & \epsilon_5^2 & 0 \end{pmatrix}$$
$$\hat{\Omega}_{\mu\nu,\rho\sigma} = 0, \qquad \big((\rho,\sigma) \neq (1,2),(3,4)\big). \tag{2.20}$$

The matrices $\Omega_{\mu\nu}$, $\bar{\Omega}_{\mu\nu}$ are given in (2.11). The $\bar{Q}$-transformations are given by

$$\begin{aligned}
\bar{Q}A_\mu &= \Lambda_\mu, & \bar{Q}\Lambda_\mu &= -2\sqrt{2}\left(D_\mu\varphi - F_{\mu\nu}\Omega^\nu\right), \\
\bar{Q}\varphi &= \Omega^\mu\Lambda_\mu, & & \\
\bar{Q}\bar{\varphi} &= -\sqrt{2}\bar{\Lambda} + \bar{\Omega}^\mu\Lambda_\mu, & \bar{Q}\bar{\Lambda} &= -2i\left([\varphi,\bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right), \\
\bar{Q}\bar{\Lambda}_{\mu\nu} &= -2F^-_{\mu\nu} + i\Big([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\Big) \\
&\quad - \frac{i}{2}\Big([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\Big), \\
\bar{Q}\hat{\Lambda}_\mu &= -\sqrt{2}\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) - 2\sqrt{2}\left(D^\nu\hat{\varphi}_{\mu\nu} - F^{\nu\rho}\hat{\Omega}_{\rho,\mu\nu}\right), \\
\bar{Q}\hat{\varphi} &= -\sqrt{2}\hat{\bar{\Lambda}} + \hat{\Omega}^\mu\Lambda_\mu, & \bar{Q}\hat{\bar{\Lambda}} &= -2i\left([\varphi,\hat{\varphi}] + i\Omega^\mu D_\mu\hat{\varphi} - i\hat{\Omega}^\mu D_\mu\varphi + i\hat{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right), \\
\bar{Q}\hat{\varphi}_{\mu\nu} &= -\sqrt{2}\hat{\bar{\Lambda}}_{\mu\nu} + \hat{\Omega}^{\rho,}{}_{\mu\nu}\Lambda_\rho, & & \\
\bar{Q}\hat{\bar{\Lambda}}_{\mu\nu} &= -2i\left([\varphi,\hat{\varphi}_{\mu\nu}] + i\Omega^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\varphi + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\Omega^\sigma F_{\rho\sigma} - i\Omega_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\Omega_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right).
\end{aligned}$$

Here $\hat{\Omega}^\mu$ and $\hat{\Omega}^{\rho,}{}_{\mu\nu}$ are defined by

$$\hat{\Omega}^\mu = \hat{\Omega}^{\mu\nu}x_\nu, \qquad \hat{\Omega}^{\rho,}{}_{\mu\nu} = \hat{\Omega}^{\rho\sigma,}{}_{\mu\nu}\,x_\sigma, \tag{2.22}$$

and $\hat{\Omega}^-_{\mu\nu,\rho\sigma}$ is the anti-self-dual part of $\hat{\Omega}_{\mu\nu,\rho\sigma}$ with respect to the first two indices. The $\hat{\bar{Q}}$-transformations are also given by

$$\begin{aligned}
\hat{\bar{Q}}A_\mu &= \hat{\Lambda}_\mu, & \hat{\bar{Q}}\hat{\Lambda}_\mu &= 2\sqrt{2}\left(D_\mu\bar{\varphi} - F_{\mu\nu}\bar{\Omega}^\nu\right), \\
\hat{\bar{Q}}\bar{\varphi} &= \bar{\Omega}^\mu\hat{\Lambda}_\mu, & & \\
\hat{\bar{Q}}\varphi &= \sqrt{2}\hat{\bar{\Lambda}} + \Omega^\mu\hat{\Lambda}_\mu, & \hat{\bar{Q}}\hat{\bar{\Lambda}} &= 2i\left([\varphi,\bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right), \\
\hat{\bar{Q}}\hat{\bar{\Lambda}}_{\mu\nu} &= -2F^-_{\mu\nu} - i\Big([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\Big), \\
&\quad - \frac{i}{2}\Big([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\Big), \\
\hat{\bar{Q}}\Lambda_\mu &= -\sqrt{2}\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) + 2\sqrt{2}\left(D^\nu\hat{\varphi}_{\mu\nu} - F^{\nu\rho}\hat{\Omega}_{\rho,\mu\nu}\right), \\
\hat{\bar{Q}}\hat{\varphi} &= \sqrt{2}\bar{\Lambda} + \hat{\Omega}^\mu\hat{\Lambda}_\mu, & \hat{\bar{Q}}\bar{\Lambda} &= -2i\left([\bar{\varphi},\hat{\varphi}] + i\bar{\Omega}^\mu D_\mu\hat{\varphi} - i\hat{\Omega}^\mu D_\mu\bar{\varphi} + i\hat{\Omega}^\mu\bar{\Omega}^\nu F_{\mu\nu}\right), \\
\hat{\bar{Q}}\hat{\varphi}_{\mu\nu} &= \sqrt{2}\bar{\Lambda}_{\mu\nu} + \hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Lambda}_\rho, & & \\
\hat{\bar{Q}}\hat{\bar{\Lambda}}_{\mu\nu} &= -2i\left([\bar{\varphi},\hat{\varphi}_{\mu\nu}] + i\bar{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\bar{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\bar{\Omega}^\sigma F_{\rho\sigma} - i\bar{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\bar{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right).
\end{aligned}$$

We note that when $\epsilon^1_{\hat{a}} = \epsilon^2_{\hat{a}} = 0$ ($\hat{a} = 5, 6$), the theory reduces to the $\Omega$-deformed $\mathcal{N} = 2^*$ theory with the mass parameters $m = \frac{1}{4}\left(\epsilon^1 + \epsilon^2\right)$ and $\bar{m} = \frac{1}{4}\left(\bar{\epsilon}^1 + \bar{\epsilon}^2\right)$. In other words, we have a supersymmetry enhancement in the $\Omega$-deformed $\mathcal{N} = 2^*$ theory by choosing the mass parameters to the above special values. A similar enhancement of supersymmetry was also discussed in [17, 18].

### 2.3 The Marcus twist

In the case of the Marcus twist, we replace $\mathrm{SU}(2)_L$ by the diagonal subgroup of $\mathrm{SU}(2)_{L'} \times \mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ by the diagonal subgroup of $\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_R$. The new Lorentz group becomes $[\mathrm{SU}(2)_{L'} \times \mathrm{SU}(2)_L]_{\mathrm{diag}} \times [\mathrm{SU}(2)_{R'} \times \mathrm{SU}(2)_R]_{\mathrm{diag}}$, where the indices $\alpha$ and $\hat{A}$, $\dot{\alpha}$ and $A'$ are identified respectively. We decompose the Weyl fermions as

$$\begin{aligned}
\Lambda_\alpha^{A'} &= \frac{1}{2}\epsilon^{A'\dot{\beta}}\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\Lambda_\mu, \qquad & \bar{\Lambda}^{\dot{\alpha}}_{A'} &= -\frac{1}{2}\delta^{\dot{\alpha}}_{A'}\bar{\Lambda} + \frac{1}{2}\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}{}_{A'}\bar{\Lambda}_{\mu\nu}, \\
\bar{\Lambda}^{\dot{\alpha}\hat{A}} &= \frac{1}{2}\left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\hat{A}}\bar{\Lambda}_\mu, & \Lambda^{\hat{A}}_\alpha &= \frac{1}{2}\delta^{\hat{A}}_\alpha\Lambda + \frac{1}{2}\left(\sigma^{\mu\nu}\right)_\alpha{}^{\hat{A}}\Lambda_{\mu\nu}.
\end{aligned} \tag{2.24}$$

We define the field $\varphi_\mu$ as

$$\varphi_\mu = \left(\sigma_\mu\right)_{\hat{B}A'}\varphi^{A'\hat{B}}. \tag{2.25}$$

The supercharges are decomposed into the two scalars $Q$ and $\bar{Q}$, the two vectors $Q_\mu$, $\bar{Q}_\mu$ and the two tensor supercharges $Q_{\mu\nu}$, $\bar{Q}_{\mu\nu}$. The solution to the parallel spinor and the torsion conditions for preserving $Q$ and $\bar{Q}$ is given by

$$\begin{aligned}
\Omega_{\mu\nu a'} &= \begin{pmatrix} 0 & \epsilon^1_{a'} & 0 & 0 \\ -\epsilon^1_{a'} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon^2_{a'} \\ 0 & 0 & \epsilon^2_{a'} & 0 \end{pmatrix}, \quad \left(\mathcal{A}_{a'}\right)^A{}_B = \begin{pmatrix} \frac{1}{4}\left(\epsilon^1_{a'} + \epsilon^2_{a'}\right)\tau^3 & 0 \\ 0 & \frac{1}{4}\left(\epsilon^1_{a'} - \epsilon^2_{a'}\right)\tau^3 \end{pmatrix}, \\
\Omega_{\mu\nu\hat{a}} &= \left(\mathcal{A}_{\hat{a}}\right)^A{}_B = 0, \qquad \left(a' = 1, 2,\ \hat{a} = 3, 4, 5, 6\right).
\end{aligned} \tag{2.26}$$

The supersymmetry transformations generated by $\bar{Q}$ and $Q$ are

$$\begin{aligned}
\bar{Q}A_\mu &= \Lambda_\mu, & \bar{Q}\Lambda_\mu &= -2\sqrt{2}\left(D_\mu\varphi - F_{\mu\nu}\Omega^\nu\right), \\
\bar{Q}\varphi &= \Omega^\mu\Lambda_\mu, & & \\
\bar{Q}\bar{\varphi} &= -\sqrt{2}\bar{\Lambda} + \bar{\Omega}^\mu\Lambda_\mu, & \bar{Q}\bar{\Lambda} &= -2i\left([\varphi, \bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right), \\
\bar{Q}\varphi_\mu &= -\sqrt{2}\bar{\Lambda}_\mu, & \bar{Q}\bar{\Lambda}_\mu &= -2i\left([\varphi, \varphi_\mu] + i\Omega^\nu D_\nu\varphi_\mu - i\Omega_\mu{}^\nu\varphi_\nu\right), \\
\bar{Q}\Lambda &= \sqrt{2}D_\mu\varphi^\mu, & \bar{Q}\Lambda_{\mu\nu} &= \sqrt{2}\left(D_\mu\varphi_\nu - D_\nu\varphi_\mu\right)^+, \\
\bar{Q}\bar{\Lambda}_{\mu\nu} &= -2F^-_{\mu\nu} + i\left[\varphi_\mu, \varphi_\nu\right]^-, & & 
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
QA_\mu &= \bar{\Lambda}_\mu, & Q\bar{\Lambda}_\mu &= -2\sqrt{2}\left(D_\mu\varphi - F_{\mu\nu}\Omega^\nu\right), \\
Q\varphi &= \Omega^\mu\bar{\Lambda}_\mu, & & \\
Q\bar{\varphi} &= \sqrt{2}\Lambda + \bar{\Omega}^\mu\bar{\Lambda}_\mu, & Q\Lambda &= 2i\left([\varphi, \bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right), \\
Q\varphi_\mu &= \sqrt{2}\Lambda_\mu, & Q\Lambda_\mu &= 2i\left([\varphi, \varphi_\mu] + i\Omega^\nu D_\nu\varphi_\mu - i\Omega_\mu{}^\nu\varphi_\nu\right), \\
Q\bar{\Lambda} &= \sqrt{2}D_\mu\varphi^\mu, & Q\Lambda_{\mu\nu} &= -2F^+_{\mu\nu} + i\left[\varphi_\mu, \varphi_\nu\right]^+, \\
Q\bar{\Lambda}_{\mu\nu} &= -\sqrt{2}\left(D_\mu\varphi_\nu - D_\nu\varphi_\mu\right)^-. & &
\end{aligned} \tag{2.28}$$



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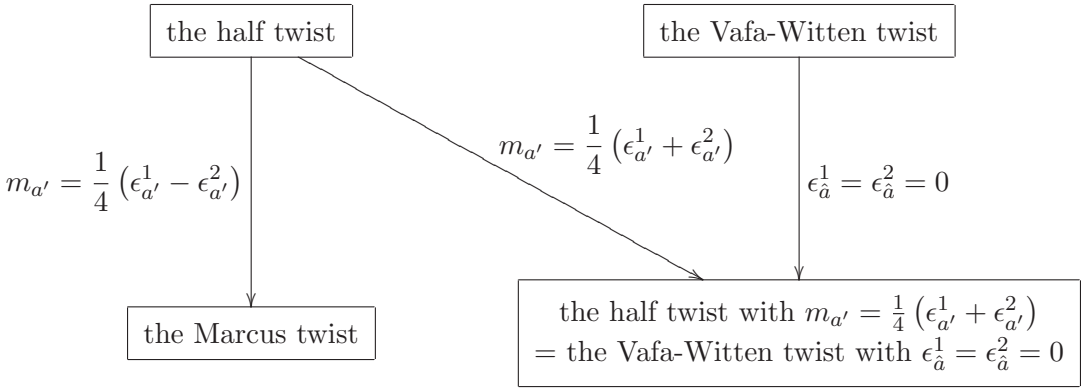


**Figure 1**. The relations among the theories with the different twists.

The Ω-deformed theory corresponding to the Marcus twist can be obtained by choosing the mass parameters $m = \frac{1}{4}\left(\epsilon^1 - \epsilon^2\right)$ and $\bar{m} = \frac{1}{4}\left(\bar{\epsilon}^1 - \bar{\epsilon}^2\right)$ in the Ω-deformed $\mathcal{N} = 2^*$ theory. The relations among three types of the Ω-deformed theories are summarized in figure 1. The Ω-deformed $\mathcal{N} = 2^*$ theory with generic $\epsilon^1$ and $\epsilon^2$ has two special supersymmetry enhancements at the mass parameters $m_{a'} = \frac{1}{4}\left(\epsilon^1_{a'} \pm \epsilon^2_{a'}\right)$.

## 3 Off-shell supersymmetries and exactness of action

In order to compute the partition function and the correlation functions of observables with the help of localization technique, we need to investigate off-shell supersymmetry [17]. In this section, we study off-shell supersymmetry generated by the scalar supercharges in the Ω-deformed $\mathcal{N} = 4$ super Yang-Mills theories by introducing the auxiliary fields associated with the fermions. For the undeformed case, the off-shell supersymmetry has been discussed in [15, 16, 19–21].

### 3.1 The half twist

We begin with the half twist case. In the background (2.9), the action (2.4) is invariant under the deformed on-shell $\bar{Q}$-transformations (2.12). In order to get the off-shell $\bar{Q}$-supersymmetry, we introduce the auxiliary fields $D_{\mu\nu}$ and $K^{\hat{A}}_{\alpha}$ associated with the fermions $\bar{\Lambda}_{\mu\nu}$ and $\Lambda^{\hat{A}}_{\alpha}$. We add the Gaussian terms of these fields to the action. Then the action becomes

$$S_1 = S + \frac{1}{\kappa g^2}\int d^4x \ \mathrm{Tr}\left[-\frac{1}{2}\left(D_{\mu\nu}\right)^2 + \frac{1}{2}K^{\hat{A}}_{\alpha}K^{\alpha}_{\hat{A}}\right]. \tag{3.1}$$

Now we modify the supersymmetry transformation of the fields $\bar{\Lambda}_{\mu\nu}$ and $\Lambda^{\hat{A}}_{\alpha}$ as

$$\begin{aligned}\bar{Q}\bar{\Lambda}_{\mu\nu} &= 2D_{\mu\nu} - 2F^{-}_{\mu\nu} - i\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\left[\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}\right], \\ \bar{Q}\Lambda^{\hat{A}}_{\alpha} &= 2K^{\hat{A}}_{\alpha} + \sqrt{2}\left(\sigma^{\mu}\right)_{\alpha\dot{\alpha}} D_{\mu}\varphi^{\dot{\alpha}\hat{A}}.\end{aligned} \tag{3.2}$$



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Then the supersymmetry transformations of the fields $D_{\mu\nu}$ and $K_\alpha^{\hat{A}}$ are determined from the condition that the linear term in the auxiliary fields in the $\bar{Q}$-transformation of the Lagrangian vanishes. The result is

$$\begin{aligned}
\bar{Q}D_{\mu\nu} &= (D_\mu\Lambda_\nu - D_\nu\Lambda_\mu)^- - \sqrt{2}i\,(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\left[\bar{\Lambda}^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}\right] \\
&\quad + \sqrt{2}i\left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\rho D_\rho\bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\rho\bar{\Lambda}_{\rho\nu} - \Omega_\nu{}^\rho\bar{\Lambda}_{\rho\mu}\right), \\
\bar{Q}K_\alpha^{\hat{A}} &= (\sigma^\mu)_{\alpha\dot{\alpha}}\,D_\mu\bar{\Lambda}^{\dot{\alpha}\hat{A}} - \frac{i}{\sqrt{2}}(\sigma^\mu)_{\alpha\dot{\alpha}}\left[\Lambda_\mu, \varphi^{\dot{\alpha}\hat{A}}\right] \\
&\quad + \sqrt{2}i\left(\left[\varphi, \Lambda_\alpha^{\hat{A}}\right] + i\Omega^\mu D_\mu\Lambda_\alpha^{\hat{A}} + M^{\hat{A}}{}_{\hat{B}}\Lambda^{\hat{B}}{}_\alpha - \frac{i}{2}\Omega^{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta\Lambda_\beta^{\hat{A}}\right).
\end{aligned} \tag{3.3}$$

Then the action (3.1) is invariant under the $\bar{Q}$-transformation. We redefine the auxiliary fields as

$$\begin{aligned}
H_{\mu\nu} &= D_{\mu\nu} - F_{\mu\nu}^- - \frac{i}{2}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\left[\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}\right], \\
G_\alpha^{\hat{A}} &= K_\alpha^{\hat{A}} + \frac{1}{\sqrt{2}}(\sigma^\mu)_{\alpha\dot{\alpha}}\,D_\mu\varphi^{\dot{\alpha}\hat{A}},
\end{aligned} \tag{3.4}$$

so that (3.2) takes a simple form

$$\bar{Q}\bar{\Lambda}_{\mu\nu} = 2H_{\mu\nu}, \qquad \bar{Q}\Lambda_\alpha^{\hat{A}} = 2G_\alpha^{\hat{A}}. \tag{3.5}$$

Then the transformations for $H_{\mu\nu}$ and $G_\alpha^{\hat{A}}$ become

$$\begin{aligned}
\bar{Q}H_{\mu\nu} &= \sqrt{2}i\left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\rho D_\rho\bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\rho\bar{\Lambda}_{\rho\nu} - \Omega_\nu{}^\rho\bar{\Lambda}_{\rho\mu}\right), \\
\bar{Q}G_\alpha^{\hat{A}} &= \sqrt{2}i\left(\left[\varphi, \Lambda_\alpha^{\hat{A}}\right] + i\Omega^\mu D_\mu\Lambda_\alpha^{\hat{A}} + M^{\hat{A}}{}_{\hat{B}}\Lambda^{\hat{B}}{}_\alpha - \frac{i}{2}\Omega^{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta\Lambda_\beta^{\hat{A}}\right).
\end{aligned} \tag{3.6}$$

We find that the $\bar{Q}^2$ action on a field $\Psi$ results in the form,

$$\bar{Q}^2\Psi = 2\sqrt{2}\big(\,\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\Omega) + \delta_{\text{flavor}}(M)\big)\Psi. \tag{3.7}$$

Here the symbol $\delta_{\text{gauge}}(\varphi)$ stands for the gauge transformation by the parameter $\varphi$, $\delta_{\text{Lorentz}}(\Omega)$ for the Lorentz transformation by the parameter $\Omega_{\mu\nu}$ and $\delta_{\text{flavor}}(M)$ for the $\mathrm{SU}(2)_{L'}$ rotation by the parameter $M^{\hat{A}}{}_{\hat{B}}$ defined in the subsection 2.1. We note that due to the conditions (2.9) for the parameters, the rotations $\delta_{\text{Lorentz}}(\Omega)$ and $\delta_{\text{flavor}}(M)$ are reduced to the ones generated by their Cartan subgroups. The algebra of the symmetry generated by $\bar{Q}$ closes off-shell.

We next examine the $\bar{Q}$-exactness of the action (3.1). This property is important to study the cohomological structure of the theory. Using the transformations (3.5), (3.6) and (2.12) for the other fields, we find that the action (3.1) is written in the $\bar{Q}$-exact form:

$$S_1 = \bar{Q}\Xi_1 + \int d^4x\frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \tag{3.8}$$



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where $\Xi_1$, which is called as the gauge fermion, is defined by

$$\begin{aligned}\Xi_1 = \int d^4x \frac{1}{\kappa g^2}\mathrm{Tr}\Bigg[&-\frac{1}{2}F^-_{\mu\nu}\bar{\Lambda}^{\mu\nu} - \frac{1}{4}H_{\mu\nu}\bar{\Lambda}^{\mu\nu} - \frac{1}{2\sqrt{2}}\Lambda^\mu\left(D_\mu\bar{\varphi} - F_{\mu\nu}\bar{\Omega}^\nu\right)\\ &+\frac{i}{4}\bar{\Lambda}\left([\varphi,\bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right)\\ &+\frac{1}{2}\Lambda^\alpha_{\hat{A}}G^{\hat{A}}_\alpha - \frac{1}{\sqrt{2}}\Lambda^\alpha_{\hat{A}}(\sigma^\mu)_{\alpha\dot{\alpha}}D_\mu\varphi^{\dot{\alpha}\hat{A}}\\ &-\frac{i}{2}\bar{\Lambda}^{\dot{\alpha}\hat{A}}\left(\left[\bar{\varphi},\bar{\varphi}_{\hat{A}\dot{\alpha}}\right] + i\bar{\Omega}^\mu D_\mu\bar{\varphi}_{\hat{A}\dot{\alpha}} - \frac{i}{2}\bar{\Omega}_{\mu\nu}\left(\bar{\sigma}^{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\bar{\varphi}_{\hat{A}\dot{\beta}} + \bar{M}^{\hat{B}}{}_{\hat{A}}\bar{\varphi}_{\hat{B}\dot{\alpha}}\right)\\ &-\frac{i}{4}\bar{\Lambda}^{\mu\nu}\left(\bar{\sigma}_{\mu\nu}\right)^{\dot{\beta}}{}_{\dot{\alpha}}\left[\varphi^{\dot{\alpha}\hat{A}},\bar{\varphi}_{\hat{A}\dot{\beta}}\right]\Bigg].\end{aligned} \tag{3.9}$$

In [6], we have shown that the $\mathcal{N}=2$ super Yang-Mills theory in the $\Omega$-background defined in six dimensions is written in the $\bar{Q}$-exact form. When the $\mathcal{N}=2$ hypermultiplet $(\varphi^{\dot{\alpha}\hat{A}}, \Lambda^\alpha{}_{\hat{A}}, \bar{\Lambda}^{\dot{\alpha}\hat{A}}, K^{\hat{A}}_\alpha)$ is removed, the action (3.8) becomes that of the $\Omega$-deformed $\mathcal{N}=2$ super Yang-Mills theory. Then the gauge fermion (3.9) indeed becomes the one which was found in the $\Omega$-deformed $\mathcal{N}=2$ super Yang-Mills theory.

### 3.2 The Vafa-Witten twist

In the case of the Vafa-Witten twist, we have obtained the two on-shell scalar supercharges $\bar{Q}$ and $\hat{\bar{Q}}$ with the same chirality in subsection 2.2. The on-shell transformations by $\bar{Q}$, $\hat{\bar{Q}}$ are (2.21) and (2.23). For the undeformed case, the off-shell supersymmetry transformations were constructed in [13, 19, 20].

We first consider the deformed off-shell supersymmetry generated by $\bar{Q}$. Following [19, 20], we introduce the auxiliary fields $D_{\mu\nu}$, $K_\mu$. Then we add the quadratic terms of the auxiliary fields to the action (2.4) as

$$S_2 = S + \int d^4x \frac{1}{\kappa g^2}\mathrm{Tr}\left[-\frac{1}{2}\left(D_{\mu\nu}\right)^2 - \frac{1}{2}\left(K_\mu\right)^2\right]. \tag{3.10}$$

We modify the $\bar{Q}$-transformations of the fields $\bar{\Lambda}_{\mu\nu}$, $\hat{\Lambda}_\mu$ as

$$\begin{aligned}\bar{Q}\bar{\Lambda}_{\mu\nu} &= 2D_{\mu\nu} - 2F^-_{\mu\nu}\\ &\quad + i\left([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right)\\ &\quad - \frac{i}{2}\left([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right),\\ \bar{Q}\hat{\Lambda}_\mu &= 2K_\mu - \sqrt{2}\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) - 2\sqrt{2}\left(D^\nu\hat{\varphi}_{\mu\nu} - F^{\nu\rho}\hat{\Omega}_{\rho,\mu\nu}\right).\end{aligned} \tag{3.11}$$

As in the half twist case, we find that the supersymmetry transformations of the auxiliary



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fields are given by

$$
\begin{aligned}
\bar{Q}D_{\mu\nu} &= (D_\mu\Lambda_\nu - D_\nu\Lambda_\mu)^- - \frac{i}{2\sqrt{2}}\left[\hat{\varphi}_{\mu\rho}, \hat{\bar{\Lambda}}_\nu{}^\rho\right] + \frac{i}{2\sqrt{2}}\left[\hat{\varphi}_{\nu\rho}, \hat{\bar{\Lambda}}_\mu{}^\rho\right] \\
&\quad + \frac{1}{2\sqrt{2}}\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\bar{\Lambda}}_\nu{}^\sigma - \frac{1}{2\sqrt{2}}\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\bar{\Lambda}}_\mu{}^\sigma - \frac{i}{2\sqrt{2}}\hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\bar{\Lambda}}^{\rho\sigma} + \frac{i}{2\sqrt{2}}\hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\bar{\Lambda}}_{\mu\nu} \\
&\quad + \frac{i}{\sqrt{2}}\left[\hat{\varphi}, \hat{\bar{\Lambda}}_{\mu\nu}\right] - \frac{i}{\sqrt{2}}\hat{\Omega}^\rho D_\rho\hat{\bar{\Lambda}}_{\mu\nu} + \frac{i}{\sqrt{2}}\left(\hat{\Omega}_\mu{}^\rho\hat{\bar{\Lambda}}_{\rho\nu} - \hat{\Omega}_\nu{}^\rho\hat{\bar{\Lambda}}_{\rho\mu}\right) \\
&\quad + \sqrt{2}i\left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\rho D_\rho\bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\rho\bar{\Lambda}_{\rho\nu} - \Omega_\nu{}^\rho\bar{\Lambda}_{\rho\mu}\right) \\
&\quad - \frac{i}{\sqrt{2}}\left[\hat{\varphi}_{\mu\nu}, \hat{\bar{\Lambda}}\right] + \frac{1}{\sqrt{2}}\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\bar{\Lambda}}, \\
\bar{Q}K_\mu &= -D_\mu\hat{\bar{\Lambda}} - 2D^\nu\hat{\bar{\Lambda}}_{\mu\nu} - \sqrt{2}i\left[\hat{\varphi}_{\mu\nu}, \Lambda^\nu\right] + \sqrt{2}\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\Lambda^\nu + \sqrt{2}\hat{\Omega}^{\rho\nu,}{}_{\mu\nu}\Lambda_\rho \\
&\quad - \frac{i}{\sqrt{2}}\left[\hat{\varphi}, \Lambda_\mu\right] - \frac{1}{\sqrt{2}}\hat{\Omega}^\rho D_\rho\Lambda_\mu + \frac{1}{\sqrt{2}}\hat{\Omega}_\mu{}^\nu\Lambda_\nu \\
&\quad + \sqrt{2}i\left[\varphi, \hat{\Lambda}_\mu\right] - \sqrt{2}\Omega^\rho D_\rho\hat{\Lambda}_\mu + \sqrt{2}\Omega_\mu{}^\nu\hat{\Lambda}_\nu\,.
\end{aligned}
\tag{3.12}
$$

We redefine the auxiliary fields as

$$
\begin{aligned}
H_{\mu\nu} &= D_{\mu\nu} - F^-_{\mu\nu} \\
&\quad + \frac{i}{2}\left([\hat{\varphi}, \hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right) \\
&\quad - \frac{i}{4}\left([\hat{\varphi}_{\mu\rho}, \hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right), \\
G_\mu &= K_\mu - \frac{1}{\sqrt{2}}\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) - \sqrt{2}\left(D^\nu\hat{\varphi}_{\mu\nu} - F^{\nu\rho}\hat{\Omega}_{\rho,\mu\nu}\right),
\end{aligned}
\tag{3.13}
$$

so that (3.12) is rewritten in a simple form as

$$
\bar{Q}\bar{\Lambda}_{\mu\nu} = 2H_{\mu\nu}, \qquad \bar{Q}\hat{\bar{\Lambda}}_\mu = 2G_\mu\,. \tag{3.14}
$$

Then the transformations of $H_{\mu\nu}$ and $G_\mu$ become

$$
\begin{aligned}
\bar{Q}H_{\mu\nu} &= \sqrt{2}i\left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\rho D_\rho\bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\rho\bar{\Lambda}_{\rho\nu} - \Omega_\nu{}^\rho\bar{\Lambda}_{\rho\mu}\right), \\
\bar{Q}G_\mu &= \sqrt{2}i\left[\varphi, \hat{\bar{\Lambda}}_\mu\right] - \sqrt{2}\Omega^\rho D_\rho\hat{\bar{\Lambda}}_\mu + \sqrt{2}\Omega_\mu{}^\nu\hat{\bar{\Lambda}}_\nu.
\end{aligned}
\tag{3.15}
$$

Using the transformations (3.14), (3.15), and (2.21) for the other fields, we find that the



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action (3.10) is written in the $\bar{Q}$-exact form up to the topological term:

$$
\begin{aligned}
S_2 &= \bar{Q}\Xi_2 + \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \qquad (3.16)\\
\Xi_2 &= \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\Bigg[-\frac{1}{2}F^-_{\mu\nu}\bar{\Lambda}^{\mu\nu} - \frac{1}{4}H_{\mu\nu}\bar{\Lambda}^{\mu\nu} - \frac{1}{2\sqrt{2}}\Lambda^\mu(D_\mu\bar{\varphi} - F_{\mu\nu}\bar{\Omega}^\nu)\\
&\quad + \frac{i}{4}\bar{\Lambda}\left([\varphi,\bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right)\\
&\quad - \frac{1}{4}G_\mu\hat{\Lambda}^\mu - \frac{1}{2\sqrt{2}}\hat{\Lambda}^\mu\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) + \frac{1}{\sqrt{2}}\hat{\Lambda}_\nu\left(D_\mu\hat{\varphi}^{\mu\nu} - F_{\mu\rho}\hat{\Omega}^{\rho,\mu\nu}\right)\\
&\quad + \frac{i}{4}\hat{\bar{\Lambda}}\left([\bar{\varphi},\hat{\varphi}] + i\bar{\Omega}^\mu D_\mu\hat{\varphi} - i\hat{\Omega}^\mu D_\mu\bar{\varphi} + i\hat{\Omega}^\mu\bar{\Omega}^\nu F_{\mu\nu}\right)\\
&\quad - \frac{i}{4}\hat{\bar{\Lambda}}^{\mu\nu}\left([\bar{\varphi},\hat{\varphi}_{\mu\nu}] + i\bar{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\bar{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\bar{\Omega}^\sigma F_{\rho\sigma} - i\bar{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\bar{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right)\\
&\quad + \frac{i}{4}\bar{\Lambda}^{\mu\nu}\left([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right)\\
&\quad - \frac{i}{8}\bar{\Lambda}^{\mu\nu}\left([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right)\Bigg].
\end{aligned}
$$

We next study the off-shell transformations generated by $\hat{\bar{Q}}$. We modify the $\hat{\bar{Q}}$-transformations of $\hat{\bar{\Lambda}}_{\mu\nu}$, $\Lambda_\mu$ as

$$
\begin{aligned}
\hat{\bar{Q}}\hat{\bar{\Lambda}}_{\mu\nu} &= 2D_{\mu\nu} - 2F^-_{\mu\nu}\\
&\quad - i\left([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right)\\
&\quad - \frac{i}{2}\left([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right),\\
\hat{\bar{Q}}\Lambda_\mu &= -2K_\mu - \sqrt{2}\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) + 2\sqrt{2}\left(D^\nu\hat{\varphi}_{\mu\nu} - F^{\nu\rho}\hat{\Omega}_{\rho,\mu\nu}\right). \qquad (3.17)
\end{aligned}
$$

The $\hat{\bar{Q}}$-transformations of the auxiliary fields are given by

$$
\begin{aligned}
\hat{\bar{Q}}D_{\mu\nu} &= \left(D_\mu\hat{\Lambda}_\nu - D_\nu\hat{\Lambda}_\mu\right)^- + \frac{i}{2\sqrt{2}}\left[\hat{\varphi}_{\mu\rho},\bar{\Lambda}_\nu{}^\rho\right] - \frac{i}{2\sqrt{2}}\left[\hat{\varphi}_{\nu\rho},\bar{\Lambda}_\mu{}^\rho\right]\\
&\quad - \frac{1}{2\sqrt{2}}\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\bar{\Lambda}_\nu{}^\sigma + \frac{1}{2\sqrt{2}}\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\bar{\Lambda}_\mu{}^\sigma + \frac{i}{2\sqrt{2}}\hat{\Omega}^-_{\mu\nu,\rho\sigma}\bar{\Lambda}^{\rho\sigma} - \frac{i}{2\sqrt{2}}\hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\bar{\Lambda}_{\mu\nu}\\
&\quad + \frac{i}{\sqrt{2}}\left[\hat{\varphi},\bar{\Lambda}_{\mu\nu}\right] - \frac{i}{\sqrt{2}}\hat{\Omega}^\rho D_\rho\bar{\Lambda}_{\mu\nu} + \frac{i}{\sqrt{2}}\left(\hat{\Omega}_\mu{}^\rho\bar{\Lambda}_{\rho\nu} - \hat{\Omega}_\nu{}^\rho\bar{\Lambda}_{\rho\mu}\right)\\
&\quad - \sqrt{2}i\left[\bar{\varphi},\hat{\bar{\Lambda}}_{\mu\nu}\right] + \sqrt{2}\bar{\Omega}^\rho D_\rho\hat{\bar{\Lambda}}_{\mu\nu} - \sqrt{2}\left(\bar{\Omega}_\mu{}^\rho\hat{\bar{\Lambda}}_{\rho\nu} - \bar{\Omega}_\nu{}^\rho\hat{\bar{\Lambda}}_{\rho\mu}\right)\\
&\quad + \frac{i}{\sqrt{2}}\left[\hat{\varphi}_{\mu\nu},\bar{\Lambda}\right] - \frac{1}{\sqrt{2}}\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\bar{\Lambda},\\
\hat{\bar{Q}}K_\mu &= D_\mu\bar{\Lambda} + 2D^\nu\bar{\Lambda}_{\mu\nu} - \sqrt{2}i\left[\hat{\varphi}_{\mu\nu},\hat{\Lambda}^\nu\right] + \sqrt{2}\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\Lambda}^\nu + \sqrt{2}\hat{\Omega}^{\rho\nu,}{}_{\mu\nu}\hat{\Lambda}_\rho\\
&\quad + \frac{i}{\sqrt{2}}\left[\hat{\varphi},\hat{\Lambda}_\mu\right] - \frac{1}{\sqrt{2}}\hat{\Omega}^\nu D_\nu\hat{\Lambda}_\mu + \frac{1}{\sqrt{2}}\hat{\Omega}_\mu{}^\nu\hat{\Lambda}_\nu\\
&\quad + \sqrt{2}i\left[\bar{\varphi},\Lambda_\mu\right] - \sqrt{2}\bar{\Omega}^\nu D_\nu\Lambda_\mu + \sqrt{2}\bar{\Omega}_\mu{}^\nu\Lambda_\nu. \qquad (3.18)
\end{aligned}
$$



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We redefine the auxiliary fields as

$$\begin{aligned}\hat{H}_{\mu\nu} &= D_{\mu\nu} - F^-_{\mu\nu} \\ &\quad - \frac{i}{2}\left([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right) \\ &\quad - \frac{i}{4}\left([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}^-_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right), \\ \hat{G}_\mu &= K_\mu + \frac{1}{\sqrt{2}}\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) - \sqrt{2}\left(D^\nu\hat{\varphi}_{\mu\nu} - F^{\nu\rho}\hat{\Omega}_{\rho,\mu\nu}\right), \end{aligned} \tag{3.19}$$

such that (3.18) is simply rewritten as

$$\hat{\bar{Q}}\hat{\bar{\Lambda}}_{\mu\nu} = 2\hat{H}_{\mu\nu}, \qquad \hat{\bar{Q}}\Lambda_\mu = -2\hat{G}_\mu. \tag{3.20}$$

Then the transformations for $\hat{H}_{\mu\nu}$ and $\hat{G}_\mu$ are

$$\begin{aligned}\hat{\bar{Q}}\hat{H}_{\mu\nu} &= -\sqrt{2}i\left[\bar{\varphi},\hat{\bar{\Lambda}}_{\mu\nu}\right] + \sqrt{2}\bar{\Omega}^\rho D_\rho\hat{\bar{\Lambda}}_{\mu\nu} - \sqrt{2}\left(\bar{\Omega}_\mu{}^\rho\hat{\bar{\Lambda}}_{\rho\nu} - \bar{\Omega}_\nu{}^\rho\hat{\bar{\Lambda}}_{\rho\mu}\right), \\ \hat{\bar{Q}}\hat{G}_\mu &= \sqrt{2}i\left[\bar{\varphi},\Lambda_\mu\right] - \sqrt{2}\bar{\Omega}^\nu D_\nu\Lambda_\mu + \sqrt{2}\bar{\Omega}_\mu{}^\nu\Lambda_\nu. \end{aligned} \tag{3.21}$$

Using the transformations (3.17), (3.21) and (2.23) for the other fields, we find that the action is written in the $\hat{\bar{Q}}$-exact form up to the topological term:

$$S_2 = \hat{\bar{Q}}\Xi_2' + \int d^4x\,\frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \tag{3.22}$$

$$\begin{aligned}\Xi_2' = \int d^4x\,\frac{1}{\kappa g^2}\mathrm{Tr}\Bigg[&-\frac{1}{2}F^-_{\mu\nu}\hat{\bar{\Lambda}}^{\mu\nu} - \frac{1}{4}\hat{H}_{\mu\nu}\hat{\bar{\Lambda}}^{\mu\nu} + \frac{1}{2\sqrt{2}}\hat{\Lambda}^\mu\left(D_\mu\varphi - F_{\mu\nu}\Omega^\nu\right) \\ &- \frac{i}{4}\hat{\bar{\Lambda}}\left([\varphi,\bar{\varphi}] + i\Omega^\mu D_\mu\bar{\varphi} - i\bar{\Omega}^\mu D_\mu\varphi + i\bar{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right) \\ &- \frac{1}{4}\hat{G}_\mu\Lambda^\mu + \frac{1}{2\sqrt{2}}\Lambda^\mu\left(D_\mu\hat{\varphi} - F_{\mu\nu}\hat{\Omega}^\nu\right) - \frac{1}{\sqrt{2}}\Lambda_\nu\left(D_\mu\hat{\varphi}^{\mu\nu} - F_{\mu\rho}\hat{\Omega}^{\rho,\mu\nu}\right) \\ &+ \frac{i}{4}\bar{\Lambda}\left([\varphi,\hat{\varphi}] + i\Omega^\mu D_\mu\hat{\varphi} - i\hat{\Omega}^\mu D_\mu\varphi + i\hat{\Omega}^\mu\Omega^\nu F_{\mu\nu}\right) \\ &- \frac{i}{4}\bar{\Lambda}^{\mu\nu}\left([\varphi,\hat{\varphi}_{\mu\nu}] + i\Omega^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\varphi + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\Omega^\sigma F_{\rho\sigma} - i\Omega_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\Omega_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right) \\ &- \frac{i}{4}\hat{\bar{\Lambda}}^{\mu\nu}\left([\hat{\varphi},\hat{\varphi}_{\mu\nu}] + i\hat{\Omega}^\rho D_\rho\hat{\varphi}_{\mu\nu} - i\hat{\Omega}^{\rho,}{}_{\mu\nu}D_\rho\hat{\varphi} + i\hat{\Omega}^{\rho,}{}_{\mu\nu}\hat{\Omega}^\sigma F_{\rho\sigma} - i\hat{\Omega}_\mu{}^\rho\hat{\varphi}_{\rho\nu} + i\hat{\Omega}_\nu{}^\rho\hat{\varphi}_{\rho\mu}\right) \\ &+ \frac{i}{8}\hat{\bar{\Lambda}}^{\mu\nu}\left([\hat{\varphi}_{\mu\rho},\hat{\varphi}_\nu{}^\rho] + i\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - i\hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma + \hat{\Omega}_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} - \hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right)\Bigg]. \end{aligned}$$

The transformations of a field $\Psi$ by $\bar{Q}$ and $\hat{\bar{Q}}$ satisfy the following off-shell algebra:

$$\begin{aligned}\bar{Q}^2\Psi &= 2\sqrt{2}\big(\,\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\Omega)\big)\Psi, \\ \hat{\bar{Q}}^2\Psi &= -2\sqrt{2}\big(\,\delta_{\text{gauge}}(\bar{\varphi}) + \delta_{\text{Lorentz}}(\bar{\Omega})\big)\Psi, \\ \{\bar{Q},\hat{\bar{Q}}\}\Psi &= 2\sqrt{2}\big(\,\delta_{\text{gauge}}(\hat{\varphi}) + \delta_{\text{Lorentz}}(\hat{\Omega})\big)\Psi. \end{aligned} \tag{3.23}$$

In the undeformed case, the action is written in the exact form by the two scalar supercharges simultaneously [13, 14, 22]. We find that this is also true in the deformed



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theory. The action is expressed as

$$S_2 = \bar{Q}\hat{\bar{Q}}\mathcal{F} + \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \tag{3.24}$$

where

$$\begin{aligned}\mathcal{F} = \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\Bigg[&-\frac{1}{2\sqrt{2}}\hat{\varphi}^{\mu\nu}F^-_{\mu\nu} + \frac{1}{8}\bar{\Lambda}^{\mu\nu}\hat{\bar{\Lambda}}_{\mu\nu} + \frac{1}{8}\Lambda^\mu\Lambda_\mu - \frac{1}{8}\bar{\Lambda}\hat{\bar{\Lambda}} + \frac{i}{24\sqrt{2}}\hat{\varphi}^{\mu\nu}\left[\hat{\varphi}_\mu{}^\lambda, \hat{\varphi}_{\lambda\nu}\right] \\ &+ \frac{1}{16\sqrt{2}}\hat{\varphi}^{\mu\nu}\left(\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_\rho\hat{\varphi}_\nu{}^\sigma - \hat{\Omega}^{\rho,}{}_{\nu\sigma}D_\rho\hat{\varphi}_\mu{}^\sigma - i\hat{\Omega}_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} + i\hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu}\right) \\ &+ \frac{3}{2\sqrt{2}}\hat{\Omega}^{[\rho,\mu\nu]}\left(A_{[\mu}F_{\nu\rho]} - \frac{i}{3}A_{[\mu}A_\nu A_{\rho]}\right)\Bigg].\end{aligned} \tag{3.25}$$

Here the three indices in the square bracket are totally antisymmetrized with the normalization 1/3!. Note that $\mathcal{F}$ is gauge invariant. This is because the gauge transformation of $\mathcal{F}$ by the gauge parameter $\alpha$ is computed as

$$\delta_{\text{gauge}}(\alpha)\mathcal{F} = \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\left[-\frac{1}{2\sqrt{2}}\left(\hat{\Omega}_\rho{}^{\mu,\nu\rho} - \hat{\Omega}_\rho{}^{\nu,\mu\rho}\right)F_{\mu\nu}\alpha\right], \tag{3.26}$$

and $\hat{\Omega}_\rho{}^{\mu,\nu\rho}$ is symmetric with respect to $\mu$ and $\nu$ from (2.20).

### 3.3 The Marcus twist

In the case of the Marcus twist, there are two scalar supercharges $Q$ and $\bar{Q}$, which have the opposite chirality. As studied in [15, 16], in the undeformed case, one cannot make both $Q$ and $\bar{Q}$ off-shell but can make only their linear combination off-shell. This charge plays an important role for studying the generalized Langlands duality of $\mathcal{N} = 4$ theory compactified on a Riemann surface [16]. Now we will examine whether this off-shell supersymmetry structure is kept under the $\Omega$-deformation.

We first study the off-shell supersymmetry generated by $\bar{Q}$. In order to construct off-shell supersymmetry, we introduce the auxiliary fields $K$, $K_{\mu\nu}$, $D_{\mu\nu}$. Then we add the quadratic terms of the auxiliary fields to the action (2.4) as

$$S_3 = S + \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\left[-\frac{1}{2}\left(D_{\mu\nu}\right)^2 - \frac{1}{2}\left(K_{\mu\nu}\right)^2 - \frac{1}{2}K^2\right]. \tag{3.27}$$

We modify the transformations of the fields $\Lambda$, $\Lambda_{\mu\nu}$, $\bar{\Lambda}_{\mu\nu}$ as

$$\begin{aligned}\bar{Q}\Lambda &= 2K + \sqrt{2}D_\mu\varphi^\mu, \\ \bar{Q}\Lambda_{\mu\nu} &= 2K_{\mu\nu} + \sqrt{2}\left(D_\mu\varphi_\nu - D_\nu\varphi_\mu\right)^+, \\ \bar{Q}\bar{\Lambda}_{\mu\nu} &= 2D_{\mu\nu} - 2F^-_{\mu\nu} + i\left[\varphi_\mu, \varphi_\nu\right]^-.\end{aligned} \tag{3.28}$$



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The transformations of the auxiliary fields are determined as

$$\begin{aligned}
\bar{Q}K &= D_\mu \bar{\Lambda}^\mu - \frac{i}{\sqrt{2}}\left[\Lambda_\mu, \varphi^\mu\right] + \sqrt{2}i\left([\varphi, \Lambda] + i\Omega^\mu D_\mu \Lambda\right), \\
\bar{Q}K_{\mu\nu} &= \left(D_\mu \bar{\Lambda}_\nu - D_\nu \bar{\Lambda}_\mu\right)^+ + \frac{i}{\sqrt{2}}\left([\varphi_\mu, \Lambda_\nu] - [\varphi_\nu, \Lambda_\mu]\right)^+ \\
&\quad + \sqrt{2}i\left[\varphi, \Lambda_{\mu\nu}\right] - \sqrt{2}\Omega^\lambda D_\lambda \Lambda_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\lambda \Lambda_{\lambda\nu} - \Omega_\nu{}^\lambda \Lambda_{\lambda\mu}\right), \\
\bar{Q}D_{\mu\nu} &= \left(D_\mu \Lambda_\nu - D_\nu \Lambda_\mu\right)^- - \frac{i}{\sqrt{2}}\left(\left[\varphi_\mu, \bar{\Lambda}_\nu\right] - \left[\varphi_\nu, \bar{\Lambda}_\mu\right]\right)^- \\
&\quad + \sqrt{2}i\left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\lambda D_\lambda \bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\lambda \bar{\Lambda}_{\lambda\nu} - \Omega_\nu{}^\lambda \bar{\Lambda}_{\lambda\mu}\right).
\end{aligned} \tag{3.29}$$

We redefine the auxiliary fields as

$$\begin{aligned}
G =& K + \frac{1}{\sqrt{2}} D_\mu \varphi^\mu, \\
G_{\mu\nu} =& K_{\mu\nu} + \frac{1}{\sqrt{2}}(D_\mu \varphi_\nu - D_\nu \varphi_\mu)^+, \\
H_{\mu\nu} =& D_{\mu\nu} - F^-_{\mu\nu} + \frac{i}{2}[\varphi_\mu, \varphi_\nu]^-,
\end{aligned} \tag{3.30}$$

such that (3.29) takes a simple form as

$$\bar{Q}\Lambda = 2G, \qquad \bar{Q}\Lambda_{\mu\nu} = 2G_{\mu\nu}, \qquad \bar{Q}\bar{\Lambda}_{\mu\nu} = 2H_{\mu\nu}. \tag{3.31}$$

Then the transformations for $G$, $G_{\mu\nu}$ and $H_{\mu\nu}$ are given by

$$\begin{aligned}
\bar{Q}G &= \sqrt{2}i([\varphi, \Lambda] + i\Omega^\mu D_\mu \Lambda), \\
\bar{Q}G_{\mu\nu} &= \sqrt{2}i[\varphi, \Lambda_{\mu\nu}] - \sqrt{2}\Omega^\lambda D_\lambda \Lambda_{\mu\nu} + \sqrt{2}(\Omega_\mu{}^\lambda \Lambda_{\lambda\nu} - \Omega_\nu{}^\lambda \Lambda_{\lambda\mu}), \\
\bar{Q}H_{\mu\nu} &= \sqrt{2}i[\varphi, \bar{\Lambda}_{\mu\nu}] - \sqrt{2}\Omega^\lambda D_\lambda \bar{\Lambda}_{\mu\nu} + \sqrt{2}(\Omega_\mu{}^\lambda \bar{\Lambda}_{\lambda\nu} - \Omega_\nu{}^\lambda \bar{\Lambda}_{\lambda\mu}).
\end{aligned} \tag{3.32}$$

Using the transformations, (3.31), (3.32) and (2.27) for the other fields, we find that the action (3.27) is written in the $\bar{Q}$-exact form up to the topological term:

$$S_3 = \bar{Q}\Xi_3 + \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\left[\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}\right], \tag{3.33}$$

where

$$\begin{aligned}
\Xi_3 = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\Big[ & -\frac{1}{2} F^-_{\mu\nu} \bar{\Lambda}^{\mu\nu} + \frac{i}{4} \bar{\Lambda}^{\mu\nu} \left[\varphi_\mu, \varphi_\nu\right]^- - \frac{1}{4} H_{\mu\nu} \bar{\Lambda}^{\mu\nu} \\
& - \frac{1}{4} \Lambda_{\mu\nu} G^{\mu\nu} + \frac{1}{2\sqrt{2}} \Lambda^{\mu\nu} \left(D_\mu \varphi_\nu - D_\nu \varphi_\mu\right)^+ \\
& - \frac{1}{4} \Lambda G + \frac{i}{4} \bar{\Lambda} \left([\varphi, \bar{\varphi}] + i\Omega^\mu D_\mu \bar{\varphi} - i\bar{\Omega}^\mu D_\mu \varphi + i\bar{\Omega}^\mu \Omega^\nu F_{\mu\nu}\right) \\
& + \frac{1}{2\sqrt{2}} \Lambda D_\mu \varphi^\mu - \frac{1}{2\sqrt{2}} \Lambda^\mu \left(D_\mu \bar{\varphi} - F_{\mu\nu} \bar{\Omega}^\nu\right) \\
& - \frac{i}{4} \bar{\Lambda}^\mu \left([\bar{\varphi}, \varphi_\mu] + i\bar{\Omega}^\nu D_\nu \varphi_\mu - i\bar{\Omega}_\mu{}^\nu \varphi_\nu\right)\Big].
\end{aligned} \tag{3.34}$$



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We next study the transformations generated by $Q$. We modify the transformations of the fields $\bar{\Lambda}$, $\Lambda_{\mu\nu}$ and $\bar{\Lambda}_{\mu\nu}$ as

$$\begin{aligned} Q\bar{\Lambda} =& 2K + \sqrt{2} D_\mu \varphi^\mu, \\ Q\Lambda_{\mu\nu} =& 2K_{\mu\nu} - 2F^+_{\mu\nu} + i\left[\varphi_\mu, \varphi_\nu\right]^+, \\ Q\bar{\Lambda}_{\mu\nu} =& 2D_{\mu\nu} - \sqrt{2}\left(D_\mu \varphi_\nu - D_\nu \varphi_\mu\right)^-. \end{aligned} \tag{3.35}$$

We find that the transformations of the auxiliary fields are

$$\begin{aligned} QK =& -D_\mu \Lambda^\mu - \frac{i}{\sqrt{2}}\left[\bar{\Lambda}_\mu, \varphi^\mu\right] + \sqrt{2} i \left(\left[\varphi, \bar{\Lambda}\right] + i\Omega^\mu D_\mu \bar{\Lambda}\right), \\ QK_{\mu\nu} =& \left(D_\mu \bar{\Lambda}_\nu - D_\nu \bar{\Lambda}_\mu\right)^+ + \frac{i}{\sqrt{2}}\left(\left[\varphi_\mu, \Lambda_\nu\right] - \left[\varphi_\nu, \Lambda_\mu\right]\right)^+ \\ & + \sqrt{2} i \left[\varphi, \Lambda_{\mu\nu}\right] - \sqrt{2}\Omega^\lambda D_\lambda \Lambda_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\lambda \Lambda_{\lambda\nu} - \Omega_\nu{}^\lambda \Lambda_{\lambda\mu}\right), \\ QD_{\mu\nu} =& \left(D_\mu \Lambda_\nu - D_\nu \Lambda_\mu\right)^- - \frac{i}{\sqrt{2}}\left(\left[\varphi_\mu, \bar{\Lambda}_\nu\right] - \left[\varphi_\nu, \bar{\Lambda}_\mu\right]\right)^- \\ & + \sqrt{2} i \left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\lambda D_\lambda \bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\lambda \bar{\Lambda}_{\lambda\nu} - \Omega_\nu{}^\lambda \bar{\Lambda}_{\lambda\mu}\right). \end{aligned} \tag{3.36}$$

We redefine the auxiliary fields such that (3.35) takes a simple form as

$$\begin{aligned} G'_{\mu\nu} =& K_{\mu\nu} - F^+_{\mu\nu} + \frac{i}{2}\left[\varphi_\mu, \varphi_\nu\right]^+, \\ H'_{\mu\nu} =& D_{\mu\nu} - \frac{1}{\sqrt{2}}\left(D_\mu \varphi_\nu - D_\nu \varphi_\mu\right)^-. \end{aligned} \tag{3.37}$$

Then (3.35) becomes

$$Q\bar{\Lambda} = 2G, \qquad Q\Lambda_{\mu\nu} = 2G'_{\mu\nu}, \qquad Q\bar{\Lambda}_{\mu\nu} = 2H'_{\mu\nu}. \tag{3.38}$$

The transformations for $G$, $G'_{\mu\nu}$ and $H'_{\mu\nu}$ are

$$\begin{aligned} QG =& \sqrt{2} i \left(\left[\varphi, \bar{\Lambda}\right] + i\Omega^\mu D_\mu \bar{\Lambda}\right), \\ QG'_{\mu\nu} =& \sqrt{2} i \left[\varphi, \Lambda_{\mu\nu}\right] - \sqrt{2}\Omega^\lambda D_\lambda \Lambda_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\lambda \Lambda_{\lambda\nu} - \Omega_\nu{}^\lambda \Lambda_{\lambda\mu}\right), \\ QH'_{\mu\nu} =& \sqrt{2} i \left[\varphi, \bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^\lambda D_\lambda \bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_\mu{}^\lambda \bar{\Lambda}_{\lambda\nu} - \Omega_\nu{}^\lambda \bar{\Lambda}_{\lambda\mu}\right). \end{aligned} \tag{3.39}$$

Again, we find that the action (3.27) is written in the $Q$-exact form up to the topological term:

$$S_3 = Q\bar{\Xi}_3 - \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\left[\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}\right], \tag{3.40}$$



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where

$$\begin{aligned}\bar{\Xi}_3 = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr} \Big[ & -\frac{1}{2} F^+_{\mu\nu} \Lambda^{\mu\nu} + \frac{i}{4} \Lambda^{\mu\nu} \left[\varphi_\mu, \varphi_\nu\right]^+ - \frac{1}{4} G'_{\mu\nu} \Lambda^{\mu\nu} \\ & - \frac{1}{4} \bar{\Lambda}^{\mu\nu} H'_{\mu\nu} - \frac{1}{2\sqrt{2}} \bar{\Lambda}^{\mu\nu} \left(D_\mu \varphi_\nu - D_\nu \varphi_\mu\right)^- \\ & - \frac{1}{4} \bar{\Lambda} G - \frac{i}{4} \Lambda \left( [\varphi, \bar{\varphi}] + i \Omega^\mu D_\mu \bar{\varphi} - i \bar{\Omega}^\mu D_\mu \varphi + i \bar{\Omega}^\mu \Omega^\nu F_{\mu\nu} \right) \\ & + \frac{1}{2\sqrt{2}} \bar{\Lambda} D_\mu \varphi^\mu - \frac{1}{2\sqrt{2}} \bar{\Lambda}^\mu \left( D_\mu \bar{\varphi} - F_{\mu\nu} \bar{\Omega}^\nu \right) \\ & + \frac{i}{4} \bar{\Lambda}^\mu \left( [\bar{\varphi}, \varphi_\mu] + i \bar{\Omega}^\nu D_\nu \varphi_\mu - i \bar{\Omega}_\mu{}^\nu \varphi_\nu \right) \Big]. \end{aligned} \tag{3.41}$$

The supercharges $Q$ and $\bar{Q}$ satisfy the following on-shell relations on a field $\Psi$

$$\bar{Q}^2 \Psi = Q^2 \Psi = 2\sqrt{2} \big( \delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\Omega) \big) \Psi, \tag{3.42}$$

$$\{Q, \bar{Q}\} \Psi = 0. \tag{3.43}$$

We find that (3.42) holds off-shell for all the fields but (3.43) does not hold off-shell on the fields $\Lambda_{\mu\nu}$, $\bar{\Lambda}_{\mu\nu}$, $K_{\mu\nu}$ and $D_{\mu\nu}$. Therefore the algebra of symmetry generated by two supercharges $Q$ and $\bar{Q}$ does not close off-shell.

We can choose the linear combination of the two supercharges

$$\mathcal{Q} = uQ + v\bar{Q}, \quad u, v \in \mathbb{C}, \tag{3.44}$$

such that $\mathcal{Q}$ becomes off-shell. In the undeformed case, when $u^2 + v^2 \neq 0$, the action is shown to be the $\mathcal{Q}$-exact form up to the topological term [16]. When $u^2 + v^2 = 0$, the action is not written in the $\mathcal{Q}$-exact form but it is $\mathcal{Q}$-closed [15]. In the following, we show that this property also holds in the deformed theory. Since the two supercharges $Q$ and $\bar{Q}$ satisfy the relations (3.42), (3.43) on-shell, $\mathcal{Q}$ satisfies the on-shell transformation

$$\mathcal{Q}^2 \Psi = 2\sqrt{2}(u^2 + v^2)(\delta_{\text{gauge}}(\varphi) + \delta_{\text{Lorentz}}(\Omega))\Psi. \tag{3.45}$$

In the following, we study the off-shell generalization of the supersymmetry generated by $\mathcal{Q}$ and examine the $\mathcal{Q}$-exactness of the action $S_3$ in the cases where $u^2 + v^2 \neq 0$ and $u^2 + v^2 = 0$.

**$u^2 + v^2 \neq 0$ case.** Since the algebra of $Q$ and $\bar{Q}$ does not close on the fields $\Lambda_{\mu\nu}$, $\bar{\Lambda}_{\mu\nu}$, $K_{\mu\nu}$ and $D_{\mu\nu}$ off-shell, we need to re-examine the $\mathcal{Q}$-transformations of these fields. The on-shell $\mathcal{Q}$-transformations of $\Lambda_{\mu\nu}$, $\bar{\Lambda}_{\mu\nu}$ are

$$\mathcal{Q} \Lambda_{\mu\nu} = 2U_{\mu\nu}, \qquad \mathcal{Q} \bar{\Lambda}_{\mu\nu} = 2V_{\mu\nu}, \tag{3.46}$$

where we have defined

$$\begin{aligned} U_{\mu\nu} &\equiv -u F^+_{\mu\nu} + \frac{i}{2} u \left[\varphi_\mu, \varphi_\nu\right]^+ + \frac{1}{\sqrt{2}} v \left(D_\mu \varphi_\nu - D_\nu \varphi_\mu\right)^+, \\ V_{\mu\nu} &\equiv -v F^-_{\mu\nu} + \frac{i}{2} v \left[\varphi_\mu, \varphi_\nu\right]^- - \frac{1}{\sqrt{2}} u \left(D_\mu \varphi_\nu - D_\nu \varphi_\mu\right)^-. \end{aligned} \tag{3.47}$$



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We modify the transformation (3.46) as

$$\begin{aligned}\mathcal{Q}\Lambda_{\mu\nu} &= 2\sqrt{u^2+v^2}K_{\mu\nu} + 2U_{\mu\nu},\\ \mathcal{Q}\bar{\Lambda}_{\mu\nu} &= 2\sqrt{u^2+v^2}D_{\mu\nu} + 2V_{\mu\nu}\,.\end{aligned} \tag{3.48}$$

The transformations of $K_{\mu\nu}$ and $D_{\mu\nu}$ are determined as in the $Q$- and $\bar{Q}$-transformations. We redefine the auxiliary fields as

$$\begin{aligned}\mathcal{G}_{\mu\nu} &= \sqrt{u^2+v^2}\,K_{\mu\nu} + U_{\mu\nu},\\ \mathcal{H}_{\mu\nu} &= \sqrt{u^2+v^2}\,D_{\mu\nu} + V_{\mu\nu}\,,\end{aligned} \tag{3.49}$$

so that (3.48) becomes a simple form as

$$\mathcal{Q}\Lambda_{\mu\nu} = 2\mathcal{G}_{\mu\nu}, \qquad \mathcal{Q}\bar{\Lambda}_{\mu\nu} = 2\mathcal{H}_{\mu\nu}. \tag{3.50}$$

Then the transformations of $\mathcal{G}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$ are

$$\begin{aligned}\mathcal{Q}\mathcal{G}_{\mu\nu} &= \left(u^2+v^2\right)\left(\sqrt{2}i\left[\varphi,\Lambda_{\mu\nu}\right] - \sqrt{2}\Omega^{\lambda}D_{\lambda}\Lambda_{\mu\nu} + \sqrt{2}\left(\Omega_{\mu}{}^{\lambda}\Lambda_{\lambda\nu} - \Omega_{\nu}{}^{\lambda}\Lambda_{\lambda\mu}\right)\right),\\ \mathcal{Q}\mathcal{H}_{\mu\nu} &= \left(u^2+v^2\right)\left(\sqrt{2}i\left[\varphi,\bar{\Lambda}_{\mu\nu}\right] - \sqrt{2}\Omega^{\lambda}D_{\lambda}\bar{\Lambda}_{\mu\nu} + \sqrt{2}\left(\Omega_{\mu}{}^{\lambda}\bar{\Lambda}_{\lambda\nu} - \Omega_{\nu}{}^{\lambda}\bar{\Lambda}_{\lambda\mu}\right)\right).\end{aligned} \tag{3.51}$$

The $\mathcal{Q}$-transformations of the other fields are obtained from (3.44).

Now we construct the gauge fermion $\widehat{\Xi}$ which satisfies $S_3 = \mathcal{Q}\widehat{\Xi}$. Since we have changed the transformations of $\Lambda_{\mu\nu}$, $\bar{\Lambda}_{\mu\nu}$, $K_{\mu\nu}$ and $D_{\mu\nu}$, we decompose the gauge fermion as $\widehat{\Xi} = \widehat{\Xi}^{(1)} + \widehat{\Xi}^{(2)}$, where $\widehat{\Xi}^{(1)}$ is the linear terms in $\Lambda_{\mu\nu}$ and $\bar{\Lambda}_{\mu\nu}$ and $\widehat{\Xi}^{(2)}$ does not contain these fields. Using the transformations (3.48), (3.51) and the $\mathcal{Q}$-transformations of the other fields, we find

$$\widehat{\Xi}^{(1)} = \frac{1}{u^2+v^2}\int d^4x\,\frac{1}{\kappa g^2}\mathrm{Tr}\left[\left(\frac{1}{2}U_{\mu\nu}\Lambda^{\mu\nu} - \frac{1}{2}\mathcal{G}_{\mu\nu}\Lambda^{\mu\nu}\right) + \left(\frac{1}{2}V_{\mu\nu}\bar{\Lambda}^{\mu\nu} - \frac{1}{2}\mathcal{H}_{\mu\nu}\bar{\Lambda}^{\mu\nu}\right)\right]. \tag{3.52}$$

In order to find $\widehat{\Xi}^{(2)}$ we take the following ansatz

$$\widehat{\Xi}^{(2)} = a\bar{\Xi}'_3 + b\Xi'_3. \tag{3.53}$$

Here $a$, $b$ are constants and $\Xi'_3$, $\bar{\Xi}'_3$ are terms that do not contain $\Lambda_{\mu\nu}, \bar{\Lambda}_{\mu\nu}$ in $\Xi_3$ and $\bar{\Xi}_3$ respectively. Using the supersymmetry transformations (2.27), (2.28), we can show that $\Xi'_3$ and $\bar{\Xi}'_3$ are the exact forms as

$$\Xi'_3 = QV, \qquad \bar{\Xi}'_3 = -\bar{Q}V, \tag{3.54}$$

where $V$ is given by

$$V = \int d^4x\,\frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{8}\Lambda\bar{\Lambda} - \frac{1}{4}\varphi^{\mu}\left(D_{\mu}\bar{\varphi} - F_{\mu\nu}\bar{\Omega}^{\nu}\right)\right]. \tag{3.55}$$

We can find the constants $a$, $b$ such that

$$\widehat{\Xi}^{(2)} = \frac{1}{u^2+v^2}\left(-u\bar{Q} + vQ\right)V, \tag{3.56}$$



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and the action is written in the $\mathcal{Q}$-exact form. We find that the action can be written in the $\mathcal{Q}$-exact form up to the topological term:

$$S_3 = \mathcal{Q}\left(\widehat{\Xi}^{(1)} + \widehat{\Xi}^{(2)}\right) + \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\left[\frac{u^2 - v^2}{4\left(u^2 + v^2\right)} F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{3.57}$$

The dependence on $u$ and $v$ of the topological term is the same as the undeformed case [16].

**$u^2+v^2=0$ case.** In this case, we can choose $(u, v) = (1, i)$. The supercharge $\mathcal{Q} = Q+i\bar{Q}$ is strictly nilpotent without using the gauge transformation and the Lorentz rotation. To see this, we introduce the following linear combinations of the fields [15]:

$$\begin{aligned}
\mathcal{V}_\mu &= A_\mu + \frac{i}{\sqrt{2}}\varphi_\mu, & \bar{\mathcal{V}}_\mu &= A_\mu - \frac{i}{\sqrt{2}}\varphi_\mu, \\
\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu + i\left[\mathcal{V}_\mu, \mathcal{V}_\nu\right], & \bar{\mathcal{F}}_{\mu\nu} &= \partial_\mu \bar{\mathcal{V}}_\nu - \partial_\nu \bar{\mathcal{V}}_\mu + i\left[\bar{\mathcal{V}}_\mu, \bar{\mathcal{V}}_\nu\right], \\
\psi_\mu &= \Lambda_\mu - i\bar{\Lambda}_\mu, & \bar{\psi}_\mu &= \Lambda_\mu + i\bar{\Lambda}_\mu, \\
\eta &= \Lambda - i\bar{\Lambda}, & \bar{\eta} &= \Lambda + i\bar{\Lambda}, \\
\chi_{\mu\nu} &= \Lambda_{\mu\nu} - i\bar{\Lambda}_{\mu\nu}, & G^+ &= G + [\varphi, \bar{\varphi}], \\
\mathcal{I}_{\mu\nu} &= \mathcal{G}_{\mu\nu} - i\mathcal{H}_{\mu\nu}, & \phi &= \varphi - \frac{i}{\sqrt{2}}\Omega^\mu \varphi_\mu.
\end{aligned} \tag{3.58}$$

We note that $\bar{\chi}_{\mu\nu} = \Lambda_{\mu\nu}+i\bar{\Lambda}_{\mu\nu}$ and $\bar{\mathcal{I}}_{\mu\nu} = \mathcal{G}_{\mu\nu}+i\mathcal{H}_{\mu\nu}$ are equal to $\tilde{\chi}_{\mu\nu}$ and $\tilde{\mathcal{I}}_{\mu\nu}$, respectively. From (2.27), (2.28), (3.50) and (3.51), the off-shell $\mathcal{Q}$-transformations of these fields become

$$\begin{aligned}
\mathcal{Q}\mathcal{V}_\mu &= 2i\psi_\mu, & \mathcal{Q}\psi_\mu &= 0, \\
\mathcal{Q}\bar{\mathcal{V}}_\mu &= 0, & & \\
\mathcal{Q}\bar{\psi}_\mu &= -4\sqrt{2}i\left(\bar{\mathcal{D}}_\mu \phi - \bar{\mathcal{F}}_{\mu\nu}\Omega^\nu\right), & \mathcal{Q}\phi &= 0, \\
\mathcal{Q}\bar{\varphi} &= \sqrt{2}\eta, & \mathcal{Q}\eta &= 0, \\
\mathcal{Q}\bar{\eta} &= 4iG^+, & \mathcal{Q}G^+ &= 0, \\
\mathcal{Q}\chi_{\mu\nu} &= 2\mathcal{I}_{\mu\nu}, & \mathcal{Q}\mathcal{I}_{\mu\nu} &= 0,
\end{aligned} \tag{3.59}$$

where we have defined the gauge covariant derivatives with respect to the gauge fields $\mathcal{V}_\mu$, $\bar{\mathcal{V}}_\mu$ as follows,

$$\mathcal{D}_\mu * = \partial_\mu * + i\left[\mathcal{V}_\mu, *\right], \quad \bar{\mathcal{D}}_\mu * = \partial_\mu * + i\left[\bar{\mathcal{V}}_\mu, *\right]. \tag{3.60}$$

Now we examine the gauge fermion $\widehat{\Xi}$, which is decomposed into the sum of $\widehat{\Xi}^{(1)}$ and $\widehat{\Xi}^{(2)}$. Starting from the ansatz (3.53), we have $\widehat{\Xi}^{(2)} = -\frac{i}{2}\left(Q - i\bar{Q}\right)V$, where $V$ is given by (3.55). However we cannot construct $\widehat{\Xi}^{(1)}$ in a similar way as (3.52) since $\mathcal{Q}\mathcal{I}_{\mu\nu} = 0$. So, instead of using the transformations (3.59), we change the transformation of $\chi_{\mu\nu}$ by eliminating $\mathcal{I}_{\mu\nu}$ using its equation of motion. We define

$$\mathcal{Q}\chi_{\mu\nu} = -2\bar{\mathcal{F}}_{\mu\nu}. \tag{3.61}$$

The new $\mathcal{Q}$-transformations are also nilpotent off-shell because $\mathcal{Q}\bar{\mathcal{F}}_{\mu\nu} = 0$. With respect to $\mathcal{Q}$ we can take $\widehat{\Xi}^{(1)}$ as

$$\widehat{\Xi}^{(1)} = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\left[-\frac{1}{8}\chi^{\mu\nu}\mathcal{F}_{\mu\nu}\right]. \tag{3.62}$$



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Then the action $S_3$ is written as the sum of the $\mathcal{Q}$-exact term and the other part:

$$S_3 = \mathcal{Q}\left(\widehat{\Xi}^{(1)} + \widehat{\Xi}^{(2)}\right) + S_3', \tag{3.63}$$

$$S_3' = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\Big[ -\frac{i}{4}\tilde{\chi}^{\mu\nu}\left(\bar{\mathcal{D}}_\mu \bar{\psi}_\nu - \bar{\mathcal{D}}_\nu \bar{\psi}_\mu\right) + \frac{i}{2\sqrt{2}}\tilde{\chi}^{\mu\nu}\left[\phi, \chi_{\mu\nu}\right] \\ - \frac{1}{2\sqrt{2}}\Omega^\lambda \tilde{\chi}^{\mu\nu}\bar{\mathcal{D}}_\lambda \chi_{\mu\nu} + \frac{1}{2\sqrt{2}}\tilde{\chi}^{\mu\nu}\left(\Omega_\mu{}^\lambda \chi_{\lambda\nu} - \Omega_\nu{}^\lambda \chi_{\lambda\mu}\right)\Big]. \tag{3.64}$$

Here $S_3'$ is not $\mathcal{Q}$-exact but $\mathcal{Q}$-closed. The deformed terms in $S_3'$ are obtained from the undeformed one by using (2.5). Although the undeformed part is independent of the metric [15], the deformed part depends on the metric through $\Omega_{\mu\nu}$.

## 4 Conclusion and discussions

In this paper, we have constructed the off-shell scalar supersymmetry associated with the three different topological twists in the $\Omega$-deformed $\mathcal{N} = 4$ super Yang-Mills theory. The scalar supercharges form the closed algebra up to the gauge transformation, the Lorentz rotation associated with the $\Omega$-vector fields, and the flavor rotation. We have shown that the $\Omega$-deformed action is written in the exact form with respect to the scalar supercharges up to topological terms except the case of the Marcus twist with $u^2 + v^2 = 0$. The twisted $\mathcal{N} = 4$ super Yang-Mills theories can be naturally deformed in the $\Omega$-background.

It would be important to study the quantum aspects of the deformed theory since the fixed point equations for the scalar supersymmetry are deformed by the $\Omega$-background, which could change the partition function. For the half twist, the partition function is indeed the same as the $\mathcal{N} = 2^*$ deformation [4]. For the Vafa-Witten twist, this would be a generalization of [23]. Furthermore it would be an interesting problem to study the S-duality of the $\Omega$-deformed $\mathcal{N} = 4$ theory.

In [7] we showed that the $\Omega$-deformed $\mathcal{N} = 4$ theory has other on-shell supersymmetry associated with the tensor supercharges. It would be interesting to study the off-shell structure of the supersymmetry and its realization in dimensionally reduced theory [24, 25]. We also studied the deformed supersymmetries in the Nekrasov-Shatashvili limit [26]. In this limit, on-shell supersymmetry is enhanced to $\mathcal{N} = (2, 2)$ supersymmetry in the case of the half twist, $\mathcal{N} = (4, 4)$ supersymmetry in the Vafa-Witten and the Marcus twists, where by the notation $\mathcal{N} = (m, n)$ we mean that the theory has $m$ chiral and $n$ anti-chiral supercharges. It is interesting to study the off-shell transformations of these supersymmetries and their BPS states [9–11]. In particular, one can study the BPS equations in the Nekrasov-Shatashivili limit, which has been investigated in the $\mathcal{N} = 2$ case [27, 28].


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## A Dirac matrices in four and six dimensions

The four-dimensional sigma matrices $\sigma^\mu$, $\bar{\sigma}^\mu$ are defined by $\sigma^\mu = (i\tau_1, i\tau_2, i\tau_3, \mathbf{1}_2)$, $\bar{\sigma}^\mu = (-i\tau_1, -i\tau_2, -i\tau_3, \mathbf{1}_2)$ where $\tau_{\tilde{c}}$ ($\tilde{c} = 1, 2, 3$) are the Pauli matrices. The four-dimensional Lorentz generators are defined by $\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$, $\bar{\sigma}^{\mu\nu} = \frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$.

The Dirac matrices $(\Sigma_a)^{AB}$ and $(\bar{\Sigma}_a)_{AB}$ in six dimensions are defined by

$$
\begin{aligned}
&\Sigma_1 = \begin{pmatrix} i\tau^2 & 0 \\ 0 & i\tau^2 \end{pmatrix}, \qquad \Sigma_2 = \begin{pmatrix} \tau^2 & 0 \\ 0 & -\tau^2 \end{pmatrix}, \qquad \Sigma_3 = \begin{pmatrix} 0 & -\tau^3 \\ \tau^3 & 0 \end{pmatrix}, \\
&\Sigma_4 = \begin{pmatrix} 0 & i\mathbf{1}_2 \\ -i\mathbf{1}_2 & 0 \end{pmatrix}, \qquad \Sigma_5 = \begin{pmatrix} 0 & -\tau^1 \\ \tau^1 & 0 \end{pmatrix}, \qquad \Sigma_6 = \begin{pmatrix} 0 & \tau^2 \\ \tau^2 & 0 \end{pmatrix}, \\
&\bar{\Sigma}_1 = \begin{pmatrix} -i\tau^2 & 0 \\ 0 & -i\tau^2 \end{pmatrix}, \qquad \bar{\Sigma}_2 = \begin{pmatrix} \tau^2 & 0 \\ 0 & -\tau^2 \end{pmatrix}, \qquad \bar{\Sigma}_3 = \begin{pmatrix} 0 & \tau^3 \\ -\tau^3 & 0 \end{pmatrix}, \\
&\bar{\Sigma}_4 = \begin{pmatrix} 0 & i\mathbf{1}_2 \\ -i\mathbf{1}_2 & 0 \end{pmatrix}, \qquad \bar{\Sigma}_5 = \begin{pmatrix} 0 & \tau^1 \\ -\tau^1 & 0 \end{pmatrix}, \qquad \bar{\Sigma}_6 = \begin{pmatrix} 0 & \tau^2 \\ \tau^2 & 0 \end{pmatrix}.
\end{aligned} \tag{A.1}
$$

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