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Predicting the τ strange branching ratios and implications for V_{us}

Mario Antonelli,[a] Vincenzo Cirigliano,[b] Alberto Lusiani[c] and Emilie Passemar[b]

[a] *INFN, Laboratori Nazionali di Frascati, Via E. Fermi 40, I-00044, Italy*
[b] *Theoretical Division, Los Alamos National Laboratory, MS B283, Los Alamos, NM 87545, U.S.A.*
[c] *INFN Sezione di Pisa, Scuola Normale Superiore di Pisa, I-56127 Pisa, Italy*

E-mail: mario.antonelli@lnf.infn.it, cirigliano@lanl.gov, alberto.lusiani@pi.infn.it, passemar@lanl.gov

Abstract: Hadronic τ decays provide several ways to extract the Cabbibo-Kobashi-Maskawa (CKM) matrix element V_{us}. The most precise determination involves using inclusive τ decays and requires as input the total branching ratio into strange final states. Recent results from B-factories have led to a discrepancy of about 3.4σ from the value of V_{us} implied by CKM unitarity and direct determination from Kaon semi-leptonic modes. In this paper we predict the three leading strange τ branching ratios, using dispersive parameterizations of the hadronic form factors and taking as experimental input the measured Kaon decay rates and the $\tau \to K\pi\nu_\tau$ decay spectrum. We then use our results to reevaluate V_{us}, for which we find $|V_{us}| = 0.2207 \pm 0.0027$, in better agreement with CKM unitarity.

Keywords: Quark Masses and SM Parameters, Kaon Physics, QCD, Chiral Lagrangians

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Contents

1 Introduction

Inclusive hadronic decays of the τ lepton provide a unique laboratory to study QCD at low energy [1]. However, predicting exclusive decay rates is a notoriously difficult task, that requires knowing the relevant non-perturbative form factors over a wide kinematical range. While near threshold rigorous chiral perturbation theory (ChPT) methods can be employed, the allowed kinematical region extends well into the resonance domain, where different non-perturbative tools are needed, such as a combination of dispersion relations and data.

Focusing on τ decays into strange hadrons (see table 1, adapted from ref. [2]) one notices that $\Gamma_{10} \equiv \Gamma_{\tau^- \to K^-\nu_\tau}$, $\Gamma_{16} \equiv \Gamma_{\tau^- \to K^-\pi^0\nu_\tau}$ and $\Gamma_{35} \equiv \Gamma_{\tau^- \to \pi^-\bar{K}^0\nu_\tau}$, which represent 68% of the total strange width, are crossed channels from kaon physics. This suggests that, assuming lepton universality, one can predict $\Gamma_{\tau^- \to K^-\nu_\tau}$, $\Gamma_{\tau^- \to K^-\pi^0\nu_\tau}$ and $\Gamma_{\tau^- \to \pi^-\bar{K}^0\nu_\tau}$ using the following ingredients: (i) kaon branching ratios (BRs), precisely measured; (ii) shape of the $K\pi$ form factors determined by a combined fit to the $K_{\ell 3}$ decay distribution

Branching fraction	HFAG Winter 2012 fit
$\Gamma_{10} = K^- \nu_\tau$	$(0.6955 \pm 0.0096) \cdot 10^{-2}$
$\Gamma_{16} = K^- \pi^0 \nu_\tau$	$(0.4322 \pm 0.0149) \cdot 10^{-2}$
$\Gamma_{23} = K^- 2\pi^0 \nu_\tau$ (ex. K^0)	$(0.0630 \pm 0.0222) \cdot 10^{-2}$
$\Gamma_{28} = K^- 3\pi^0 \nu_\tau$ (ex. K^0, η)	$(0.0419 \pm 0.0218) \cdot 10^{-2}$
$\Gamma_{35} = \pi^- \bar{K}^0 \nu_\tau$	$(0.8206 \pm 0.0182) \cdot 10^{-2}$
$\Gamma_{40} = \pi^- \bar{K}^0 \pi^0 \nu_\tau$	$(0.3649 \pm 0.0108) \cdot 10^{-2}$
$\Gamma_{44} = \pi^- \bar{K}^0 \pi^0 \pi^0 \nu_\tau$	$(0.0269 \pm 0.0230) \cdot 10^{-2}$
$\Gamma_{53} = \bar{K}^0 h^- h^- h^+ \nu_\tau$	$(0.0222 \pm 0.0202) \cdot 10^{-2}$
$\Gamma_{128} = K^- \eta \nu_\tau$	$(0.0153 \pm 0.0008) \cdot 10^{-2}$
$\Gamma_{130} = K^- \pi^0 \eta \nu_\tau$	$(0.0048 \pm 0.0012) \cdot 10^{-2}$
$\Gamma_{132} = \pi^- \bar{K}^0 \eta \nu_\tau$	$(0.0094 \pm 0.0015) \cdot 10^{-2}$
$\Gamma_{151} = K^- \omega \nu_\tau$	$(0.0410 \pm 0.0092) \cdot 10^{-2}$
$\Gamma_{801} = K^- \phi \nu_\tau (\phi \to KK)$	$(0.0037 \pm 0.0014) \cdot 10^{-2}$
$\Gamma_{802} = K^- \pi^- \pi^+ \nu_\tau$ (ex. K^0, ω)	$(0.2923 \pm 0.0068) \cdot 10^{-2}$
$\Gamma_{803} = K^- \pi^- \pi^+ \pi^0 \nu_\tau$ (ex. K^0, ω, η)	$(0.0411 \pm 0.0143) \cdot 10^{-2}$
$\Gamma_{110} = X_s^- \nu_\tau$	$(2.8746 \pm 0.0498) \cdot 10^{-2}$

Table 1. HFAG Winter 2012 Tau branching fractions to strange final states [2].

and the $\tau^- \to K\pi\nu_\tau$ invariant mass distribution using a dispersive parametrization for the form factors as presented in refs. [3, 4]; (iii) theoretical input on the electromagnetic and isospin breaking corrections.

The primary purpose of this work is to predict the leading strange τ branching ratios along the lines outlined above. We will then use the predicted BRs to update the extraction of V_{us} from inclusive τ decays [5, 6] and explore how this affects the 3.4σ discrepancy with the extractions of V_{us} based on CKM unitarity and kaon decays [7].

The paper is organized as follows. In section 2 we review the prediction of $\tau \to K\nu_\tau$ from $K_{\mu 2}$. In section 3 we discuss all the ingredients needed to predict $\tau \to K\pi\nu_\tau$ branching ratios in the Standard Model and give our results and error estimates. In section 4 we work out the implications of the new predicted strange BRs on the inclusive extraction of V_{us}, and in section 5 we give our conclusions.

2 $\tau \to K\nu_\tau$ from $K_{\mu 2}$ rate in the Standard Model

Assuming $\tau - \mu$ universality in the charged weak current, the $\tau \to K\nu_\tau$ decay rate can be predicted from the $K \to \mu\nu_\mu$ decay rate:

$$\mathrm{BR}(\tau \to K\nu_\tau) = \frac{m_\tau^3}{2 m_K m_\mu^2} \frac{S_{\mathrm{EW}}^\tau}{S_{\mathrm{EW}}^K} \left(\frac{1 - m_K^2/m_\tau^2}{1 - m_\mu^2/m_K^2} \right)^2 \frac{\tau_\tau}{\tau_K} R_{\mathrm{EM}}^{\tau/K} \, \mathrm{BR}(K_{\mu 2}) \, , \tag{2.1}$$

with $\tau_\tau = 290.6(1.0)$ fs [8] and $\tau_K = 12.384(15)$ ns [7] the charged τ and kaon lifetime respectively. $S_{\mathrm{EW}}^{\tau/K}$ represent the short distance electroweak radiative corrections [9, 10]

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evaluated at the scale $\mu = m_\tau$ and $\mu = m_\rho$, respectively. The long-distance electromagnetic corrections are given by $R_{\rm EM}^{\tau/K} = 1.0090(22)$ [11]. Using eq. (2.1) one finds

$$\mathrm{BR}(\tau \to K\nu_\tau) = (0.713 \pm 0.003) \cdot 10^{-2} \,. \tag{2.2}$$

3 $\tau \to K\pi\nu_\tau$ branching ratios in the Standard Model

3.1 Relating $K \to \pi\ell\bar\nu_\ell$ and $\tau \to \bar K\pi\nu_\tau$ rates

The decays $\tau \to K\pi\nu_\tau$ and $K \to \pi\ell\bar\nu_\ell$ ($\ell = e, \mu$) are generated by the same underlying quark-lepton level operator in the charged current effective Lagrangian (with the replacement $\tau \leftrightarrow \ell$). This is true in the Standard Model (SM) and in any extension that respects lepton universality. Therefore, the hadronic matrix elements for the above two processes are related by crossing. Considering only the SM operator, the $K \to \pi\ell\bar\nu_\ell$ amplitude involves

$$\begin{aligned}\langle \pi(p_\pi)|\bar s\gamma_\mu u|K(p_K)\rangle &= (p_K+p_\pi)_\mu f_+^{K\pi}(t) + (p_K-p_\pi)_\mu f_-^{K\pi}(t)\,, \\ &= \frac{\Delta_{K\pi}}{t}(p_K-p_\pi)_\mu f_0^{K\pi}(t) + \left[(p_K+p_\pi)_\mu - \frac{\Delta_{K\pi}}{t}(p_K+p_\pi)_\mu\right] f_+^{K\pi}(t)\,,\end{aligned} \tag{3.1}$$

where $t = (p_K - p_\pi)^2$ and $\Delta_{K\pi} = m_K^2 - m_\pi^2$. The vector (scalar) form factors $f_+(t)$ ($f_0(t)$) represent the P-wave (S-wave) projection of the crossed channel matrix element $\langle K\pi|\bar s\gamma^\mu u|0\rangle$. The scalar form factor $f_0(t)$ can be expressed in terms of $f_+(t)$ and $f_-(t)$ as $f_0(t) = f_+(t) + t/\Delta_{K\pi} f_-(t)$, and by construction, $f_0(0) = f_+(0)$. The hadronic matrix element relevant for $\tau \to K\pi\nu_\tau$ reads

$$\langle \bar K(p_K)\pi(p_\pi)|\bar s\gamma_\mu u|0\rangle = -\frac{\Delta_{K\pi}}{s}(p_K+p_\pi)_\mu f_0^{K\pi}(s) - \left[(p_K-p_\pi)_\mu - \frac{\Delta_{K\pi}}{s}(p_K+p_\pi)_\mu\right] f_+^{K\pi}(s)\,, \tag{3.2}$$

with in this case $s = (p_K + p_\pi)^2$. The decay rates for $\tau \to K\pi\nu_\tau$ and $K \to \pi\ell\bar\nu_\ell$ involve integrals of the form factors over the appropriate phase space. The overall normalization, common to both modes is controlled by $f_+^{K\pi}(0)$. It is therefore convenient to factor out $f_+^{K^0\pi^-}(0)$, denoted $f_+(0)$ in the following, in the $K_{\ell 3}$ and $\tau \to K\pi\nu_\tau$ decay rates. The phase space integrals depend then on the normalized form factors, defined by

$$\bar f_+(s) = \frac{f_+(s)}{f_+(0)},\ \ \bar f_0(s) = \frac{f_0(s)}{f_+(0)},\ \ \bar f_+(0) = \bar f_0(0) = 1\,. \tag{3.3}$$

With the above definitions for the hadronic form factors, the $K_{\ell 3}$ decay rate reads

$$\Gamma(K \to \pi\ell\bar\nu_\ell[\gamma]) = \frac{G_F^2 m_K^5}{192\pi^3}\, C_K^2\, S_{\rm EW}^K\, (|V_{us}|f_+(0))^2\, I_K^\ell\, \left(1 + \delta_{\rm EM}^{K\ell} + \delta_{\rm SU(2)}^{K\pi}\right)^2 \,. \tag{3.4}$$

Here $S_{\rm EW}^K$ represents the short distance electroweak radiative corrections [9, 10] evaluated at the scale $\mu = m_\rho$, C_K the Clebsch-Gordan coefficients, equal to 1 for K^0 and $1/\sqrt{2}$ for K^-. The quantity $\delta_{\rm EM}^{K\ell}$ encodes the channel dependent long-distance electromagnetic corrections [12, 13], and $\delta_{\rm SU(2)}^{K\pi}$ the correction for strong isospin breaking. It is defined to parameterize the difference between the $K^- \to \pi^0$ and $K^0 \to \pi^-$ form factors, so that $\delta^{K^0\pi^-}_{\rm SU(2)} = 0$ and $\delta^{K^+\pi^0}_{\rm SU(2)} \neq 0$. Finally, the dimensionless phase space integral is given by

$$I_K^\ell = \int_{m_\ell^2}^{t_{\rm max}} dt \, \frac{1}{m_K^8} \, \lambda^{3/2} \left(1 + \frac{m_\ell^2}{2t}\right) \left(1 - \frac{m_\ell^2}{2t}\right)^2 \left(|\bar{f}_+(t)|^2 + \frac{3m_\ell^2 \Delta_{K\pi}^2}{(2t + m_\ell^2)\lambda} \, |\bar{f}_0(t)|^2\right) , \qquad (3.5)$$

with $\lambda = [t - (m_K + m_\pi)^2][t - (m_K - m_\pi)^2]$ and $t_{\rm max} = (m_K - m_\pi)^2$.

The $\tau \to \bar{K}\pi\nu_\tau$ decay rate has a structure similar to $\Gamma(K \to \pi\ell\bar{\nu}_\ell[\gamma])$. Including electromagnetic and strong isospin breaking corrections one has

$$\Gamma(\tau \to \bar{K}\pi\nu_\tau[\gamma]) = \frac{G_F^2 m_\tau^5}{96\pi^3} \; C_K^2 \; S_{\rm EW}^\tau \; \left(|V_{us}| f_+(0)\right)^2 I_K^\tau \left(1 + \delta_{\rm EM}^{K\tau} + \tilde{\delta}_{\rm SU(2)}^{K\pi}\right)^2 . \qquad (3.6)$$

$S_{\rm EW}^\tau$ represents the short distance electroweak radiative corrections [9, 10] evaluated at the scale $\mu = m_\tau$. C_K is the Clebsch-Gordan coefficient defined above. $\delta_{\rm EM}^{K\tau}$ is the channel dependent long-distance electromagnetic correction and $\tilde{\delta}_{\rm SU(2)}^{K\pi}$ the correction for strong isospin breaking. As before, $\tilde{\delta}^{K^0\pi^-}_{\rm SU(2)} = 0$ and $\tilde{\delta}^{K^+\pi^0}_{\rm SU(2)} \neq 0$. Note that $\tilde{\delta}^{K\pi}_{\rm SU(2)} \neq \delta^{K\pi}_{\rm SU(2)}$ because the K and τ decay rates involve integrals of the form factors over very different energy regions. Finally, the dimensionless phase space integral , I_K^τ is given by

$$I_K^\tau = \frac{1}{m_\tau^2} \int_{s_{K\pi}}^{m_\tau^2} \frac{ds}{s\sqrt{s}} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + \frac{2s}{m_\tau^2}\right) q_{K\pi}^3(s) |\bar{f}_+(s)|^2 + \frac{3\Delta_{K\pi}^2}{4s} q_{K\pi}(s) |\bar{f}_0(s)|^2\right] , \qquad (3.7)$$

with $s_{K\pi} = (m_K + m_\pi)^2$ and $q_{K\pi}$ the kaon momentum in the rest frame of the hadronic system:

$$q_{K\pi} = \frac{1}{2\sqrt{s}} \sqrt{(s - s_{K\pi})\,(s - t_{K\pi})} \times \theta\,(s - s_{K\pi})\,, \qquad t_{K\pi} = (m_K - m_\pi)^2 \,. \qquad (3.8)$$

Taking the ratios of eqs. (3.4) and (3.6) and multiplying by the ratio of τ and K lifetimes, one obtains the following relation for $\mathrm{BR}(\tau \to \bar{K}\pi\nu_\tau)$ in terms of the crossed channel branching fraction $\mathrm{BR}(K \to \pi\ell\bar{\nu}_\ell)$:

$$\mathrm{BR}(\tau \to \bar{K}\pi\nu_\tau) = \frac{2m_\tau^5}{m_K^5} \frac{S_{\rm EW}^\tau}{S_{\rm EW}^K} \frac{I_K^\tau}{I_K^\ell} \frac{\left(1 + \delta_{\rm EM}^{K\tau} + \tilde{\delta}_{\rm SU(2)}^{K\pi}\right)^2}{\left(1 + \delta_{\rm EM}^{K\ell} + \delta_{\rm SU(2)}^{K\pi}\right)^2} \frac{\tau_\tau}{\tau_K} \, \mathrm{BR}(K \to \pi e\bar{\nu}_e)\,. \qquad (3.9)$$

We will use the above formula to predict $\mathrm{BR}(\tau \to \bar{K}\pi\nu_\tau)$. All the theoretical and experimental quantities involving $K_{\ell 3}$ decays in eq. (3.9) are very accurately known [7]. The key new ingredients are the phase space integrals I_K^τ, that require knowledge of the form factors over a wide energy range, and the electromagnetic and isospin-breaking corrections relevant to the τ decays, $\delta_{\rm EM}^{K\tau}$ and $\tilde{\delta}_{\rm SU(2)}^{K\pi}$. In what follows, we describe in detail the evaluation of these three input quantities. Before doing that, we make the following general observations about our approach:

- In order to compute I_K^τ (see eq. (3.7)), we determine $\bar{f}_{+,0}^{K^0\pi^-}(s)$ by a combined fit to the $K_{\ell 3}$ decay distribution and the $\tau^- \to K_S\pi^-\nu_\tau$ invariant mass distribution using a dispersive parametrization for the form factors [3, 4].

- Calculations of $\delta_{\rm EM}^{K\tau}$ and $\tilde{\delta}_{\rm SU(2)}^{K^+\pi^0} \neq 0$ are not as robust as the corresponding quantities for K decays, because a rigorous ChPT analysis can only be performed in a corner of τ decay phase space. However, we will provide in this paper first estimates for these quantities. In order to estimate the electromagnetic effects we will use a point-like description of pions and kaons, neglecting all structure-dependent effects both in loops with virtual photons and Bremsstrahlung amplitudes. For the strong isospin breaking effects, we will obtain a rough estimate by using a parameterization of the s dependence of the form factor based on a simple resonance model. In both cases we will assign conservative uncertainties to the results we obtain.

One important consequence of the above discussion is that we will be able to predict $\mathrm{BR}(\tau^- \to \bar{K}^0\pi^-\nu_\tau)$ more accurately than $\mathrm{BR}(\tau^- \to K^-\pi^0\nu_\tau)$, since the latter involves the poorly known $\tilde{\delta}_{\rm SU(2)}^{K^+\pi^0}$.

3.2 $K\pi$ form factors

3.2.1 Parametrization of the form factors

To compute the phase space integrals, I_K^ℓ, one needs to know the normalized $K\pi$ form factors, $\bar{f}_+(s)$ and $\bar{f}_0(s)$ in the two energy regions $m_\ell^2 < s < (m_K - m_\pi)^2$ (for $K_{\ell 3}$ decays) and $(m_K + m_\pi)^2 < s < m_\tau^2$ (for $\tau \to \bar{K}\pi\nu_\tau$). To this end, a dispersive representation for the form factors has been introduced in ref. [3]. Here we briefly recall the key ingredients of the two parametrizations used. For more details see ref. [4]. For the scalar form factor, a dispersion relation with three subtractions is written for $\ln \bar{f}_0(s)$, one at the Callan-Treiman point and the other two at zero. This leads to the following representation for $\bar{f}_0(s)$

$$\bar{f}_0(s) = \exp\left[\frac{s}{\Delta_{K\pi}}\left(\ln C + (s-\Delta_{K\pi})\left(\frac{\ln C}{\Delta_{K\pi}} - \frac{\lambda_0'}{m_\pi^2}\right)\right.\right. \\ \left.\left. + \frac{\Delta_{K\pi}\, s\, (s-\Delta_{K\pi})}{\pi}\int_{s_{K\pi}}^{\infty} \frac{ds'}{s'^2}\frac{\phi_0(s')}{(s'-\Delta_{K\pi})(s'-s-i\epsilon)}\right)\right] . \tag{3.10}$$

The two subtraction constants a priori unknown, $\ln C \equiv \ln \bar{f}_0(\Delta_{K\pi})$ and λ_0', the slope of the form factor (the third one being fixed since $\bar{f}_0(0) \equiv 1$, see eq. (3.3)), are determined from a fit to the data. $\phi_0(s)$ represents the phase of the form factor. In the low energy region $s \le s_{\rm cut}$ we use the S-wave $I = 1/2$ $K\pi$ scattering phase from ref. [14]. For the high-energy region, see discussion below.

A dispersive representation for the vector form factor $\bar{f}_+(s)$ is built in a similar way [3, 15–17]. In this case the three subtractions are performed at $s = 0$. Hence the dispersive representation for $\bar{f}_+(s)$ reads:

$$\bar{f}_+(s) = \exp\left[\lambda_+'\frac{s}{m_\pi^2} + \frac{1}{2}\left(\lambda_+'' - \lambda_+'^2\right)\left(\frac{s}{m_\pi^2}\right)^2 + \frac{s^3}{\pi}\int_{s_{K\pi}}^{\infty}\frac{ds'}{s'^3}\frac{\phi_+(s')}{(s'-s-i\epsilon)}\right] . \tag{3.11}$$

Use has been made of $\bar{f}_+(0) \equiv 1$ to fix one subtraction constant. λ_+' and λ_+'' are the two other subtractions constants corresponding to the slope and curvature of the form factor. They are determined from a fit to the data. As for the phase of the form factor, $\phi_+(s)$, we parameterize it as $\tan\phi_+(s) = \text{Im}\tilde{f}_+(s)/\text{Re}\tilde{f}_+(s)$ in terms of a model for the form factor $\tilde{f}_+(s)$ that includes two resonances $K^*(892)$ and $K^*(1414)$, with mixing parameter β, see refs. [16–19]:

$$\tilde{f}_+(s) = \frac{\tilde{m}_{K^*}^2 - \kappa_{K^*}\tilde{H}_{K\pi}(0) + \beta s}{D(\tilde{m}_{K^*}, \tilde{\Gamma}_{K^*})} - \frac{\beta s}{D(\tilde{m}_{K^{*\prime}}, \tilde{\Gamma}_{K^{*\prime}})}, \tag{3.12}$$

with

$$D(\tilde{m}_R, \tilde{\Gamma}_R) = \tilde{m}_R^2 - s - \kappa_R \,\text{Re}\, \tilde{H}_{K\pi}(s) - i\tilde{m}_R\tilde{\Gamma}_R(s)\ . \tag{3.13}$$

In this equation, $\tilde{m}_R$ and $\tilde{\Gamma}_R$ are model parameters and $\tilde{\Gamma}_R(s)$ and κ_R are given by:

$$\tilde{\Gamma}_R(s) = \tilde{\Gamma}_R \frac{s}{\tilde{m}_R^2} \frac{\sigma_{K\pi}^3(s)}{\sigma_{K\pi}^3(\tilde{m}_R^2)}, \qquad \kappa_R = \frac{\tilde{\Gamma}_R}{\tilde{m}_R} \frac{192\pi F_K F_\pi}{\sigma_{K\pi}^3(\tilde{m}_R^2)}, \tag{3.14}$$

with $\sigma_{K\pi}(s) = 2q_{K\pi}(s)/\sqrt{s}$. $\tilde{H}_{K\pi}(s)$ is the $K\pi$ loop function in ChPT [18, 19]. We emphasize here that $\tilde{m}_R$ and $\tilde{\Gamma}_R$ are model parameters and do not correspond to the physical resonance mass and width. To find them one has to find the poles of eq. (3.12) or equivalently the zeros of eq. (3.13) on the second Riemann sheet. Note that this model inspired by the Gounaris-Sakurai parametrization [3, 15–22] is built such that the good properties of analyticity, unitarity and perturbative QCD are fulfilled. This model is only valid in the τ decay region. Therefore we will use it for $s \leq s_{\text{cut}} \sim m_\tau^2$. Hence there will be seven parameters to fit from the data: λ_+' and λ_+'' the slope and the curvature of the form factor and the resonance parameters used to model the phase: $\tilde{m}_{K^*}$ and $\tilde{\Gamma}_{K^*}$ the mass and decay width of K*(892) and $\tilde{m}_{K^{*\prime}}$ and $\tilde{\Gamma}_{K^{*\prime}}$ the mass and decay width of K*(1414) and β the mixing parameter between the two resonances.

For the high-energy region of the dispersive integrals eqs. (3.10), (3.11), ($s \geq s_{\text{cut}} \sim m_\tau^2$) the phase is unknown and following refs. [3, 4, 23, 24], we take a conservative interval between 0 and 2π centered at the asymptotic value of the phase which is π. The use of a three time subtracted dispersion relation reduces the impact of our ignorance of the phase at relatively high energies. The price to pay is that the correct asymptotic behaviour of the two form factors is subjected to a set of sum rules derived in [3, 4], which is used to constrain our fit parameters.

3.2.2 Determination of the $K\pi$ form factors from $\tau \to K\pi\nu_\tau$ Belle data and $K_{\ell 3}$ analyses

We perform a combined fit to the Belle data [25] as well as the $K_{\ell 3}$ data [7], along the lines described in ref. [3]. We minimize the following quantity:

$$\chi^2 = \sum_i \left(\frac{N_i^{\text{theo}} - N_i^{\text{exp}}}{\sigma_{N_i^{\text{exp}}}}\right)^2 + \begin{pmatrix} \ln C - \ln C^{K_{\ell 3}} \\ \lambda_+' - \lambda_+'^{K_{\ell 3}} \end{pmatrix}^T V^{-1} \begin{pmatrix} \ln C - \ln C^{K_{\ell 3}} \\ \lambda_+' - \lambda_+'^{K_{\ell 3}} \end{pmatrix} + \left(\frac{\alpha_{2s} - \alpha_{2s}^{\text{sr}}}{\sigma_{\alpha_{2s}^{\text{sr}}}}\right)^2 + \left(\frac{\alpha_{2v} - \alpha_{2v}^{\text{sr}}}{\sigma_{\alpha_{2v}^{\text{sr}}}}\right)^2, \tag{3.15}$$

where N_i^{exp} and $\sigma_{N_i^{\text{exp}}}$ are respectively, the experimental number of events and the corresponding uncertainty in the i^{th} bin. The theoretical number of events in a given i

bin is [18, 19]

$$N_i^{\rm theo} = N_{\rm tot} b_w \frac{1}{\Gamma_{\tau\to K\pi\nu}} \frac{d\Gamma_{\tau\to K\pi\nu}}{d\sqrt{s}}(s_i)\,, \tag{3.16}$$

with $N_{\rm tot}$, the total number of events, b_w the bin width and $\Gamma_{\tau\to K\pi\nu}$ the total decay rate given in eq. (3.6). We fit the first 76 points from threshold $s_{K\pi}$ to $s_{\rm fit} \sim 1.51\,{\rm GeV}^2$ where our parametrization is expected to be reliable. Note that following refs. [16, 17, 19] we exclude from the fit the points 5, 6 and 7 that exhibit a bump which is not present in the preliminary BaBar data [26]. We have tested that including these points in the fit amounts to increase the χ^2 from 60/68 to 78/71 without any significant changes in the values of the parameters, which remain within the error bars. The second term of eq. (3.15) encodes the constraints coming from $K_{\ell 3}$ analyses where a dispersive parametrization has been used for the form factors [23, 24]. We are using $\ln C^{K_{\ell 3}} = 0.2004 \pm 0.0091$, $\lambda_+^{\prime K_{\ell 3}} = (25.66 \pm 0.41)\times 10^{-3}$ and $\rho(\ln C, \lambda'_+) = -0.33$ from ref. [7]. V represents the covariance matrix. In the minimization we also impose the constraints given by the sum rules eqs. (15) and (18) of ref. [3, 4][1] with $\alpha_{2s} \equiv \frac{\ln C}{\Delta_{K\pi}} - \frac{\lambda'_0}{m_\pi^2}$, $\alpha_{2v} \equiv \lambda''_+ - \lambda'^2_+$ and

$$\alpha_{2s}^{\rm sr} \equiv \frac{\Delta_{K\pi}}{\pi} \int_{s_{K\pi}}^{\infty} \frac{ds'}{s'^2} \frac{\phi_0(s')}{(s' - \Delta_{K\pi})}\,, \tag{3.17}$$

$$\alpha_{2v}^{\rm sr} \equiv \frac{2 m_\pi^4}{\pi} \int_{s_{K\pi}}^{\infty} ds' \frac{\phi_+(s')}{s'^3}\,. \tag{3.18}$$

The results of the fit are presented on figure 1 and in table 2 with the correlations between the parameters in table 3. Table 2 displays the results for the fit to real data [25] and also to projected data from a super-B factory, obtained by keeping the same central values of current Belle data [25] and rescaling the errors according to the expected sensitivity of a second generation B factory assuming an integrated luminosity of 40 ab^{-1}, see e.g. ref. [27]. Using these results we can compute the phase space integrals eqs. (3.5), (3.7) given in tables 4 and 5.

3.3 Electromagnetic effects in $\tau \to K\pi\nu_\tau$

While the electromagnetic corrections are known for $K_{\ell 3}$ to order $(e^2 p^2)$ in ChPT [12, 13, 28], they have never been computed in the case of $\tau \to K\pi\nu_\tau$. In this case there are no rigorous methods to compute electromagnetic effects over the entire phase space, because the kinematics of τ decays allows the hadronic invariant mass squared $s = (p_K + p_\pi)^2$ to extend well beyond the chiral regime, all the way to $s = m_\tau^2$. While resonance-model calculations are possible [29, 30], here we will give a first estimate of the long-distance electromagnetic corrections to $\tau \to K\pi\nu_\tau$ based on point-like mesons and leading Low bremsstrahlung contributions, i.e. neglecting structure dependent effects. With these approximations we provide the corrections to both differential and total rate for the processes $\tau \to K\pi\nu_\tau$.

The leading $O(\alpha)$ long-distance EM corrections arise from one-loop corrections to the decay amplitudes and real photon emission. Only the one-photon-inclusive decay

[1]The constraints from the other two sum-rules, see refs. [3, 4] are not imposed in the fit, see eq. (3.15), since they are automatically satisfied due to the large band taken for $\phi_{0,\rm as}$ and $\phi_{+,\rm as}$.

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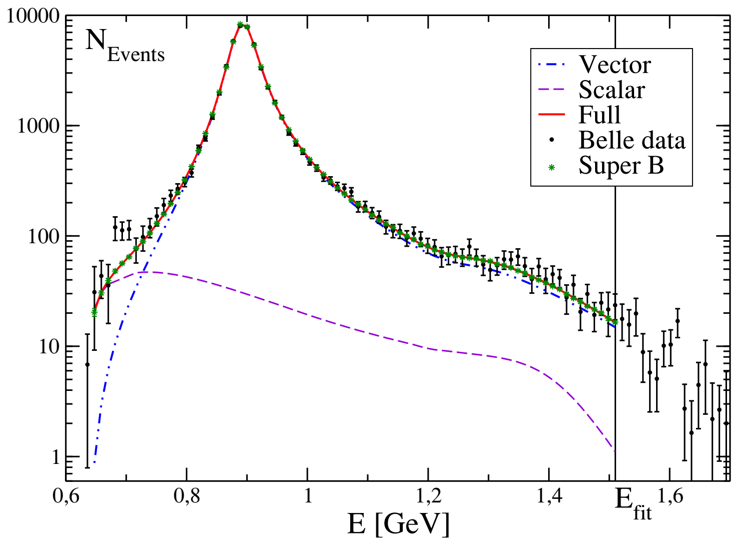

Figure 1. Fit result for the spectrum of $\tau \to K\pi\nu_\tau$. The data in black are from Belle Collaboration [25]. The points in green are projected data for a second generation B factory with integrated luminosity of 40 ab^{-1} with the same central values of current Belle data and rescaling errors according to the expected sensitivity. The dashed violet line represents the scalar form factor contribution. The dot-dashed blue line is the vector form factor contribution and the solid red line gives the full result.

	$\tau \to K\pi\nu_\tau$ & $K_{\ell 3}$ Belle	$\tau \to K\pi\nu_\tau$ & $K_{\ell 3}$ 2$^{\text{nd}}$ generation B factory (projected)
$\ln C$	0.20352 ± 0.00890	0.19880 ± 0.00498
$\lambda'_0 \times 10^3$	13.824 ± 0.824	13.703 ± 0.521
$\tilde{m}_{K^*}$[MeV]	943.59 ± 0.58	943.76 ± 0.06
$\tilde{\Gamma}_{K^*}$[MeV]	67.064 ± 0.846	67.290 ± 0.088
$\tilde{m}_{K^{*\prime}}$[MeV]	1392.2 ± 57.6	1361.7 ± 6.3
$\tilde{\Gamma}_{K^{*\prime}}$[MeV]	296.67 ± 160.28	254.62 ± 17.45
β	-0.0404 ± 0.0206	-0.0338 ± 0.0023
$\lambda'_+ \times 10^3$	25.621 ± 0.405	25.601 ± 0.277
$\lambda''_+ \times 10^3$	1.2221 ± 0.0183	1.2150 ± 0.0090
$\chi^2/d.o.f$	60.2/68	28.1/71

Table 2. Results for the $K\pi$ form factors parameters from a combined fit to $\tau \to K\pi\nu_\tau$ and $K_{\ell 3}$. Note that $\tilde{m}_R$ and $\tilde{\Gamma}_R$ are model parameters and do not correspond to the physical resonance mass and width.

rate is infrared (IR) finite to $O(\alpha)$. Our approach here relies on the analysis of EM corrections to $K \to \pi\ell\bar{\nu}_\ell$ and $\tau \to \pi\pi\nu_\tau$ presented in refs. [12, 13] and [29, 30], respectively. Adapting the arguments presented in refs. [12, 13] we find that long distance EM effects

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Parameter	$\ln C$	λ'_0	$\tilde{m}_{K^*}$	$\tilde{\Gamma}_{K^*}$	$\tilde{m}_{K^{*\prime}}$	$\tilde{\Gamma}_{K^{*\prime}}$	β	λ'_+	λ''_+
$\ln C$	1	0.943	-0.093	-0.117	0.047	0.005	-0.003	0.342	0.135
λ'_0	–	1	-0.066	-0.068	0.040	0.027	-0.067	0.318	0.266
$\tilde{m}_{K^*}$	–	–	1	0.951	0.196	0.240	-0.345	0.001	-0.250
$\tilde{\Gamma}_{K^*}$	–	–	–	1	0.145	0.179	-0.273	0.017	-0.160
$\tilde{m}_{K^{*\prime}}$	–	–	–	–	1	0.926	-0.842	0.088	0.030
$\tilde{\Gamma}_{K^{*\prime}}$	–	–	–	–	–	1	-0.917	0.088	0.030
β	–	–	–	–	–	–	1	-0.128	-0.018
λ'_+	–	–	–	–	–	–	–	1	0.735

Table 3. Correlations between the parameters of the fit to Belle and $K_{\ell 3}$ data presented in table 2.

Integral	result	error	exp	theo
$I^{\tau}_{K^0}$	0.50418	0.01762	0.01689	0.00501
$I^{e}_{K^0}$	0.15472	0.00022	0.00022	0.00000
$I^{\tau}_{K^0}/I^{e}_{K^0}$	3.25864	0.11115	0.10634	0.03235
$I^{\tau}_{K^+}$	0.52387	0.01958	0.01889	0.00515
$I^{e}_{K^+}$	0.15909	0.00025	0.00025	0.00000
$I^{\tau}_{K^+}/I^{e}_{K^+}$	3.29282	0.12032	0.11589	0.03235

Table 4. Phase space integrals for the charged and neutral modes of $\tau \to K\pi\nu$ and K_{e3} as well as their ratio using the results of the fit to Belle and $K_{\ell 3}$ data, see table 2. The experimental uncertainty comes from the uncertainties from the fit parameters and the theoretical uncertainty comes from the uncertainty of the phase of the form factors in the inelastic region, where a large band of 2π has been taken, see section 2.3.1. The two uncertainties have been summed in quadrature to give the final one.

Integral	result	error	exp	theo
$I^{\tau}_{K^0}$	0.49590	0.00820	0.00662	0.00484
$I^{e}_{K^0}$	0.15471	0.00015	0.00015	0.00000
$I^{\tau}_{K^0}/I^{e}_{K^0}$	3.20545	0.05060	0.03562	0.03130
$I^{\tau}_{K^+}$	0.51536	0.00858	0.00631	0.00498
$I^{e}_{K^+}$	0.15908	0.00017	0.00017	0.00000
$I^{\tau}_{K^+}/I^{e}_{K^+}$	3.23973	0.05114	0.03635	0.03132

Table 5. Phase space integrals for the charged and neutral modes of $\tau \to K\pi\nu$ and K_{e3} as well as their ratio using the results of the fit to the projected 2^{nd} generation of B-factories and $K_{\ell 3}$ data, see table 2.

in $\tau \to K\pi\nu_\tau$ induce:[2]

(i) An overall correction $g_{\rm rad}(s,u)$ to the differential decay rate, that combines the effect of soft real photon emission and the universal soft part of one-loop diagrams. The virtual- and real-photon corrections are IR divergent and depend on the IR regulator M_γ, while their sum is finite:

$$g_{\rm rad}(s,u) \equiv \frac{\alpha}{2\pi}\Gamma_C(u,m_\tau^2,m_1^2,M_\gamma^2) + g_{\rm brems}(s,u,m_1^2,m_2^2,M_\gamma^2)\ . \tag{3.19}$$

The expression for $\Gamma_C(u,m_\tau^2,m_1^2,M_\gamma^2)$ can be found in refs. [12, 13] and is reported for completeness in appendix B. $g_{\rm brems}(s,u,m_1^2,m_2^2,M_\gamma^2)$ encodes the Bremsstrahlung effects in the leading Low approximation and its expression can be found in refs. [29, 30] and appendix C.

(ii) Shifts to the form factors: $\bar f^{K\pi}_{\pm,0}(s) \to \bar f^{K\pi}_{\pm,0}(s) + \delta \bar f^{K\pi}_{\pm,0}(s,u)$. These shifts arise already when treating K and π as point-like as soon as one uses momentum-dependent vertices for the weak hadronic current. $\delta \bar f_\pm(u)$ are given by

$$\delta \bar f_\pm^{K^-\pi^0}(u) = \frac{\alpha}{4\pi}\frac{1}{f_+(0)}\left[\Gamma_1(u,m_\tau^2,m_K^2) \pm \Gamma_2(u,m_\tau^2,m_K^2)\right] + \dots\,, \tag{3.20}$$

$$\delta \bar f_\pm^{\bar K^0\pi^-}(u) = \frac{\alpha}{4\pi}\frac{1}{f_+(0)}\left[\Gamma_2(u,m_\tau^2,m_\pi^2) \pm \Gamma_1(u,m_\tau^2,m_\pi^2)\right] + \dots\,, \tag{3.21}$$

The dots denote structure-dependent corrections that are hard to estimate over all the phase space. Near threshold, the ChPT expressions in terms of low-energy constants can be found in refs. [12, 13]. The loop functions $\Gamma_{1,2}(u,m_\tau^2,m_1^2)$ can be found in refs. [12, 13] and in appendix B. Finally, in terms of the shifts $\delta \bar f_\pm^{K\pi}(u)$, the corrections to the scalar form factor reads $\delta \bar f_0^{K\pi}(s,u) \equiv \delta \bar f_+^{K\pi}(u) + s/\Delta_{K\pi}\ \delta \bar f_-^{K\pi}(u)$.

With the above prescriptions, and linearizing in the corrections to the form factors, we obtain the following expression for the photon-inclusive double differential rate $\tau \to K\pi\nu_\tau[\gamma]$ decay:

$$\begin{aligned}\frac{d\Gamma_{\tau\to K\pi\nu[\gamma]}}{ds\ du} = \frac{G_F^2 C_K^{\tau 2} S_{\rm EW}^\tau |f_+(0)V_{us}|^2}{128\pi^3 m_\tau^3}&\Bigg[D_+^{\bar K\pi}(s,u)\Big(|\bar f_+(s)|^2 + 2{\rm Re}\left[\bar f_+(s)\delta \bar f_+^*(u)\right]\Big)\\ &+D_0^{\bar K\pi}(s,u)\Big(|\bar f_0(s)|^2 + 2{\rm Re}\left[\bar f_0(s)\delta \bar f_0^*(s,u)\right]\Big)\\ &+D_{+0}^{\bar K\pi}(s,u){\rm Re}\Big[\bar f_+(s)\bar f_0^*(s) + \bar f_+(s)\delta \bar f_0^*(s,u) + \delta \bar f_+(u)\bar f_0^*(s)\Big]\Bigg]\times\Big[1+g_{\rm rad}(s,u)\Big].\end{aligned} \tag{3.22}$$

The expression for the Dalitz plot kinematic densities $D_{+,0,+0}(s,u)$ can be found in appendix A. Integrating over the u variable we obtain the EM-corrected distribution in the

[2] For the two decay modes we adopt this conventions for the particle four-momenta: $\tau^-(p_\tau) \to \pi^-(p_1)K^0(p_2)\nu_\tau(q)$ and $\tau^-(p_\tau) \to K^-(p_1)\pi^0(p_2)\nu_\tau(q)$. The EM corrections involve the Mandelstam variable $u = (p_\tau - p_1)^2$, where p_τ and p_1 denote the four-momentum of the τ and the charged meson (K or π) in the final state. Moreover, $m_1^2 = p_1^2$ denotes the mass squared of the charged meson.

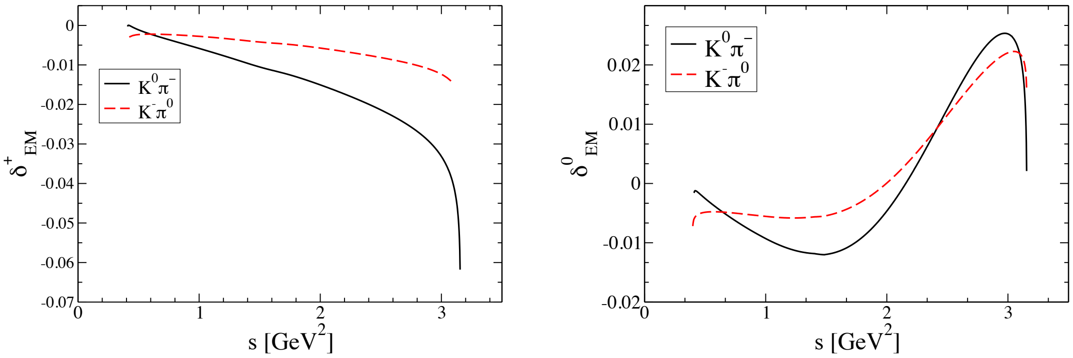

Figure 2. Correction factors $\delta^+_{\rm EM}(s)$ (left panel) and $\delta^0_{\rm EM}(s)$ (right panel) to the vector and scalar contribution to the differential decay rates of both $\tau \to K\pi\nu_\tau$ modes.

$K\pi$ invariant mass:

$$\frac{d\Gamma_{K\pi[\gamma]}}{ds} = \frac{G_F^2 C_K^2 S_{\rm EW} |f_+(0)V_{us}|^2 m_\tau^3}{96\pi^3 s\sqrt{s}} \left[\left(1-\frac{s}{m_\tau^2}\right)^2 \left(\left(1+\frac{2s}{m_\tau^2}\right) q_{K\pi}^3(s)|\bar{f}_+(s)|^2 \left[1+\delta^+_{\rm EM}(s)\right] \right.\right.$$
$$\left.\left. + \frac{3\Delta^2_{K\pi}}{4s} q_{K\pi}(s)\,|\bar{f}_0(s)|^2 \left[1+\delta^0_{\rm EM}(s)\right]\right) + {\rm Re}\left[\bar{f}_+(s)\bar{f}_0^*(s)\right] \delta^{+0}_{\rm EM}(s)\right] , \qquad (3.23)$$

with

$$\delta^+_{\rm EM}(s) \equiv \frac{\int_{u_{\rm min}(s)}^{u_{\rm max}(s)} du\; D_+(s,u) \left(|\bar{f}_+(s)|^2 g_{\rm rad}(s,u) + 2{\rm Re}\left[\bar{f}_+(s)\delta\bar{f}_+^*(u)\right]\right)}{\int_{u_{\rm min}(s)}^{u_{\rm max}(s)} du\; D_+(s,u)|\bar{f}_+(s)|^2} , \qquad (3.24)$$

$$\delta^0_{\rm EM}(s) \equiv \frac{\int_{u_{\rm min}(s)}^{u_{\rm max}(s)} du\; D_0(s,u) \left(|\bar{f}_0(s)|^2 g_{\rm rad}(s,u) + 2{\rm Re}\left[\bar{f}_0(s)\delta\bar{f}_0^*(s,u)\right]\right)}{\int_{u_{\rm min}(s)}^{u_{\rm max}(s)} du\; D_0(s,u)|\bar{f}_0(s)|^2} , \qquad (3.25)$$

$$\delta^{+0}_{\rm EM}(s) \equiv \frac{3s\sqrt{s}}{4m_\tau^6} \int_{u_{\rm min}(s)}^{u_{\rm max}(s)} du\; D_{+0}(s,u) \Big({\rm Re}\left[\bar{f}_+(s)\bar{f}_0^*(s)\right] g_{\rm rad}(s,u)$$
$$+ {\rm Re}\left[\bar{f}_+(s)\delta\bar{f}_0^*(s,u) + \delta\bar{f}_+(u)\bar{f}_0^*(s)\right] \Big) . \qquad (3.26)$$

$u_{\rm min,max}(s)$ can be found in the appendix. The functions $\delta^{+,0}_{\rm EM}(s)$ are shown on figure 2. Further integrating the distribution (3.23) over s with and without electromagnetic corrections, and taking the ratio, we get $\delta^{K\tau}_{\rm EM}$. Assigning an uncertainty of $\sim \alpha/\pi$ to the unknown structure-dependent corrections, we get:

$$\delta^{K^-\tau}_{\rm EM} = -(0.2 \pm 0.2)\% , \qquad \delta^{\bar{K}^0\tau}_{\rm EM} = -(0.15 \pm 0.2)\% . \qquad (3.27)$$

Note that a comparison between the leading Low approximation and the full calculation is performed in refs. [29, 30], and it shows that it leads to a comparable correction to the decay rate. Hence we expect this calculation to give a reasonable estimate for the electromagnetic corrections to the $\tau \to K\pi\nu_\tau$ total decay rates. We have introduced the EM correction factors $\delta^{+,0,+0}_{\rm EM}(s)$ in the fitting procedure and we have found that these corrections do not affect the determination of the form factors at the current level of precision.

3.4 Isospin breaking corrections in $\tau \to K^-\pi^0\nu_\tau$

In order to estimate the strong isospin breaking effects, we focus on the dominant vector form factor. We adopt a simple parameterization of the ratio $f_+^{K^-\pi^0}(s)/f_+^{\bar{K}^0\pi^-}(s)$ based on a single vector meson resonance exchange. The ratio $f_+^{K^-\pi^0}(s)/f_+^{\bar{K}^0\pi^-}(s)$ differs from unity because of (i) $\pi^0-\eta$ mixing and (ii) possible isospin-breaking effects in the coupling of K^{*-} to $K\pi$. To leading order in isospin-breaking, the first effect is independent of s, and completely controlled by the $\pi^0-\eta$ mixing angle $\epsilon = \frac{\sqrt{3}}{4}\frac{m_d-m_u}{m_s-1/2(m_u+m_d)} = 1.16(13)\times 10^{-2}$ [31]. The second effect can be estimated by using couplings of vector mesons to Goldstone Bosons that involve insertions of quark mass matrices, such as those introduced in ref. [32]. Requiring that the form factors in the isospin-symmetric limit fall off as $1/s$, single vector meson resonance exchange implies the parameterization:

$$f_+^{K^-\pi^0}(s)/f_+^{\bar{K}^0\pi^-}(s) = \left(1+\sqrt{3}\,\epsilon\right)\left(1+\tilde{g}\frac{m_K^2}{(4\pi F_\pi)^2}\frac{s}{m_{K^*}^2}\,\epsilon\right). \tag{3.28}$$

The only unknown parameter in the above expression is the coupling $\tilde{g}\sim O(1)$, which we vary between -2 and $+2$. This gives a first rough estimate of the effect of s-dependent isospin breaking effect, namely $\tilde{\delta}_{\mathrm{SU(2)}}^{K^-\pi^0} = \pm 0.5\%$. On the other hand, the constant part due to $\pi^0-\eta$ mixing is better known and is 100% correlated with the analogous $K_{\ell 3}$ quantity. Putting the two ingredients together, this procedure leads to $\tilde{\delta}_{\mathrm{SU(2)}}^{K\pi} = (2.9\pm 0.4_{\mathrm{mixing}}\pm 0.5)\%$. We emphasize that this is a far-from-complete estimate of strong isospin breaking effects, and it is only meant to provide a rough estimate of the central value and uncertainty associated with these effects.

3.5 Branching ratios

Using eq. (3.9) we predict $\mathrm{Br}(\tau^-\to K^-\pi^0\nu_\tau)$ and $\mathrm{Br}(\tau^-\to\bar{K}^0\pi^-\nu_\tau)$. In table 6 we summarize the input values used for the predictions. We find for the branching ratios

$$\mathrm{BR}(\tau^-\to\bar{K}^0\pi^-\nu_\tau) = (0.857\pm 0.030)\cdot 10^{-2}\,, \tag{3.29}$$

$$\mathrm{BR}(\tau^-\to K^-\pi^0\nu_\tau) = (0.471\pm 0.018)\cdot 10^{-2}\,, \tag{3.30}$$

with a 100% correlation. The error comes exclusively from the uncertainty on the τ phase space integrals. In table 7 results for the 2$^{\mathrm{nd}}$ generation of flavour factory with the error budget can be found. One sees that the uncertainty coming from the evaluation of the phase space integrals can be reduced by a factor of three. Then the uncertainties coming from EM corrections start to matter.

4 Implications for the inclusive determination of V_{us}

The most precise determination of $|V_{us}|$ from τ decays comes from the measurements of inclusive $|\Delta S|=0$ and $|\Delta S|=1$ tau decay widths. Indeed one can build the theoretical quantity

$$\delta R_{\tau,\mathrm{th}} = \frac{R_{\tau,NS}}{|V_{ud}|^2} - \frac{R_{\tau,S}}{|V_{us}|^2}\,, \tag{4.1}$$

Parameter	Value	ref.	Parameter	Value	ref.
$\mathrm{BR}(K^{\pm}_{e3})$	0.05078(31)	[7]	$\mathrm{BR}(K_{Le3})$	0.4056(9)	[7]
$\tau_{K^\pm}$	(12.384 ± 0.015) ns	[7]	τ_{K_L}	(51.16 ± 0.21) ns	[7]
τ_τ	(290.6 ± 1.0) fs	[8]	τ_τ	(290.6 ± 1.0) fs	[8]
S^K_{EW}	1.0232 ± 0.0003	[9]	S^K_{EW}	1.0232 ± 0.0003	[9]
S^τ_{EW}	1.0201 ± 0.0003	[10]	S^τ_{EW}	1.0201 ± 0.0003	[10]
$\delta^{K\ell}_{\mathrm{EM}}(\%)$	0.050 ± 0.125	[28]	$\delta^{K\ell}_{\mathrm{EM}}(\%)$	0.495 ± 0.110	[28]
$\delta^{K^-\tau}_{\mathrm{EM}}$	-(0.2 ± 0.2) %	section 3.3	$\delta^{\bar{K}^0\tau}_{\mathrm{EM}}$	-(0.15 ± 0.2) %	section 3.3
$\delta^{K\pi}_{\mathrm{SU(2)}}$	0.029 ± 0.004	[7, 33]	$\delta^{K\pi}_{\mathrm{SU(2)}}$	0	[7, 33]
$\tilde{\delta}^{K\pi}_{\mathrm{SU(2)}}$	0.029 ± 0.006	section 3.4	$\tilde{\delta}^{K\pi}_{\mathrm{SU(2)}}$	0	section 3.4
I^τ_{K+}/I^e_{K+}	10.32059 ± 0.48240	table 4	I^τ_{K+}/I^e_{K+}	10.21432 ± 0.43058	table 4

Table 6. Inputs used to compute $\tau^- \to K^-\pi^0\nu_\tau$ and $\tau^- \to \bar{K}^0\pi^-\nu_\tau$ branching ratios.

Mode	σ(BR)	% err	$\mathrm{BR}(K_{e3})$	τ_K	τ_τ	I^τ_K/I^e_K	EM	SU(2)
$\tau^- \to \bar{K}^0\pi^-\nu_\tau$	$\pm$ 0.0122	1.45	0.22	0.41	0.34	1.24	0.46	0
$\tau^- \to K^-\pi^0\nu_\tau$	$\pm$ 0.0079	1.71	0.06	0.12	0.34	1.25	0.47	1.00

Table 7. Prediction for the uncertainty of the $\tau \to K\pi\nu_\tau$ branching fraction in % from the K_{e3} branching ratio using the 2$^{\mathrm{nd}}$ generation of B factory projected results for the phase space integrals, see table 5. The different sources of uncertainty are given. They have been summed in quadrature to give the total one.

where R_τ is defined as

$$R_\tau = \frac{\Gamma[\tau \to \text{hadrons } \nu_\tau]}{\Gamma[\tau \to \bar{\nu}_e e \nu_\tau]} \ . \tag{4.2}$$

This quantity vanishes in the SU(3) limit and can be precisely determined within QCD combining perturbative QCD and low energy data [5, 6, 34]. Hence, we can extract V_{us} from eq. (4.1) using the theoretical estimate of $\delta R_{\tau,th}$ and the precise measurements of non-strange ($R_{\tau,NS}$) and strange ($R_{\tau,S}$) inclusive decays, and $|V_{ud}|$. Following ref. [2], we take $\delta R_{\tau,th} = 0.240 \pm 0.032$, with a systematic error on $|V_{us}|$ that lies between the two more recent estimates [35, 36]. We use $|V_{ud}| = 0.97425 \pm 0.00022$ from the superallowed $0^+ \to 0^+$ nuclear β decays [37]. Using the HFAG Early 2012 averages from the τ branching fractions reported in table 1 together with their reported statistical correlations and replacing the $\tau \to K\nu_\tau$ and $\tau \to K\pi\nu_\tau$ results by our prediction in eq. (2.2) and eqs. (3.29), (3.30), we obtain [2]

$$\mathrm{BR}_{\tau,S} \equiv \mathrm{BR}(\tau \to X_s^- \nu_\tau) = (2.965 \pm 0.066) \cdot 10^{-2} \ . \tag{4.3}$$

With this estimate and $\mathrm{BR}_{\tau,NS} = (61.85 \pm 0.11)\%$ and $\mathrm{BR}_{\tau,e} = (17.839 \pm 0.028)\%$ [2], we get

$$|V_{us}| = 0.2207 \pm 0.0027 \ . \tag{4.4}$$

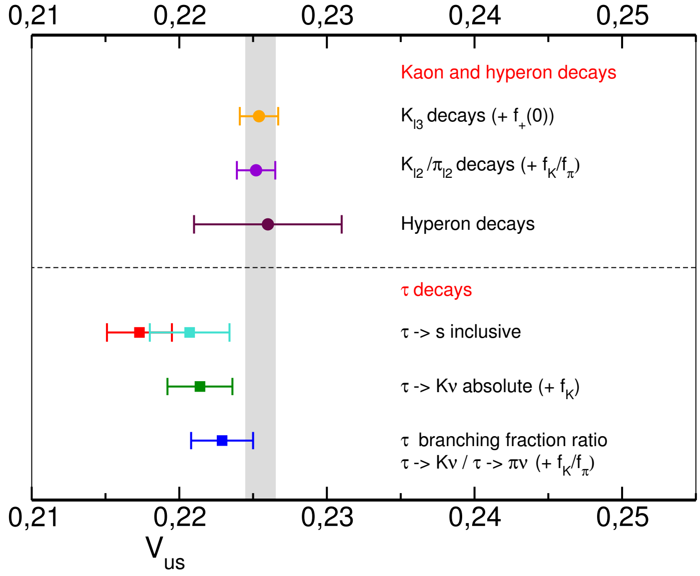

Figure 3. Determination of $|V_{us}|$ from semileptonic, leptonic kaon decays [7], hyperon decays [38] and inclusive and exclusive τ decays [2]. The errors bars correspond to the determination from exclusive τ decays (blue), the inclusive hadronic τ decays (red), and our prediction (cyan). The grey band displays the value of $|V_{us}|$ assuming unitarity of the first row of the CKM matrix.

We summarize in figure 3 the different extractions of $|V_{us}|$ from semileptonic and leptonic kaon decays, hyperon and τ decays. Our prediction shifts the inclusive determination of $|V_{us}|$ towards the exclusive one by $\sim 1.5\sigma$.

5 Conclusion

The experimental precision of data on leptonic and semileptonic kaon decays matched by sub-percent theoretical calculations allowed the most accurate determination of V_{us} [7]. Assuming lepton universality, we use the same data in combination with the measurement of the $K\pi$ invariant mass distribution in the $\tau \to K\pi\nu$ decay and a dispersive parametrization for the form factors [3, 4] to obtain a precise prediction for about 68% of the total strange width. A first evaluation of electromagnetic and SU(2) breaking effects has been derived to this purpose. We find:

$$\begin{aligned}
\mathrm{BR}(\tau^- \to K^-\nu_\tau) &= (0.713 \pm 0.003) \cdot 10^{-2}\,, \\
\mathrm{BR}(\tau^- \to \bar{K}^0\pi^-\nu_\tau) &= (0.857 \pm 0.030) \cdot 10^{-2}\,, \\
\mathrm{BR}(\tau^- \to K^-\pi^0\nu_\tau) &= (0.471 \pm 0.018) \cdot 10^{-2}\,, \\
B_3 \equiv \mathrm{BR}(\tau \to K\nu_\tau) + \mathrm{BR}(\tau^- \to \bar{K}^0\pi^-\nu_\tau) + \mathrm{BR}(\tau^- \to K^-\pi^0\nu_\tau) &= (2.040 \pm 0.048) \cdot 10^{-2}\,.
\end{aligned}$$

B_3 is 1.7σ higher with respect to the world average measured value. In addition we obtain a determination of V_{us} from inclusive tau decays using the above prediction for the branching ratios and the world average values for the rest of tau branching fractions. We find:

$$|V_{us}| = 0.2207 \pm 0.0027\,,$$

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and for the unitarity of the CKM quark mixing matrix as applied to the first row, we obtain:

$$1 - |V_{us}|^2 - |V_{ud}|^2 = 0.0021 \pm 0.0013 \; (-1.6\sigma) \; .$$

Finally, we have shown that measurements of the $K\pi$ invariant mass distribution at a second generation B factory with integrated luminosity of 40 ab^{-1} would reduce the uncertainty in the $\tau \to K\pi\nu_\tau$ BRs by a factor of three, and therefore further reduce the error on V_{us}.

A Kinematic densities

The differential decay rate eq. (3.22) involves the kinematic functions

$$\begin{aligned} D_+^{\bar{K}\pi}(s,u) &= \frac{m_\tau^2}{2}(m_\tau^2 - s) + 2m_1^2 m_2^2 - 2u\,(m_\tau^2 - s + m_1^2 + m_2^2) + 2u^2 \\ &\quad - \frac{\Delta_{21}}{s} m_\tau^2 (2u + s - m_\tau^2 - 2m_2^2) + \frac{\Delta_{21}^2}{s^2}\frac{m_\tau^2}{2}(m_\tau^2 - s)\,, \end{aligned} \tag{A.1}$$

$$D_0^{\bar{K}\pi}(s,u) = \frac{\Delta_{21}^2 m_\tau^4}{2s^2}\left(1 - \frac{s}{m_\tau^2}\right)\,, \tag{A.2}$$

$$D_{+0}^{\bar{K}\pi}(s,u) = \frac{\Delta_{21} m_\tau^2}{s}\left(2u + s - m_\tau^2 - 2m_2^2 - \frac{\Delta_{21}}{s}\left(m_\tau^2 - s\right)\right)\,, \tag{A.3}$$

with $\Delta_{21} = m_2^2 - m_1^2$. The above expressions are valid for both decay modes, with the following conventions for the particle four-momenta: $\tau^-(p_\tau) \to \pi^-(p_1)K^0(p_2)\nu_\tau(q)$ and $\tau^-(p_\tau) \to K^-(p_1)\pi^0(p_2)\nu_\tau(q)$. The Mandelstam variable $u = (p_\tau - p_1)^2$, where p_τ and p_1 denote the four-momentum of the τ and the charged meson (K or π) in the final state. Moreover, $m_1^2 = p_1^2$ denotes the mass squared of the charged meson.

B Loop functions

We now give expressions for the loop functions characterizing virtual photon corrections. We denote by M the charged meson mass, by $m_\ell \to m_\tau$ the charged lepton mass, and by M_γ the photon mass used as infrared regulator. In order to express the loop functions $\Gamma_{1,2,C}$ in a compact way, it is useful to define the following intermediate variables:

$$R = \frac{m_\ell^2}{M^2}, \quad Y = 1 + R - \frac{v}{M^2}, \quad X = \frac{Y - \sqrt{Y^2 - 4R}}{2\sqrt{R}}. \tag{B.1}$$

In terms of such variables, of the dilogarithm

$$\mathrm{Li}_2(x) = -\int_0^1 \frac{dt}{t}\ln(1 - xt)\;, \tag{B.2}$$

and the auxiliary functions

$$\begin{aligned} \mathcal{C}(v, m_\ell^2, M^2) &= \frac{1}{m_\ell M}\frac{X}{1 - X^2}\Big[-\frac{1}{2}\ln^2 X + 2\ln X\ln(1 - X^2) - \frac{\pi^2}{6} + \frac{1}{8}\ln^2 R \\ &\quad + \mathrm{Li}_2(X^2) + \mathrm{Li}_2\left(1 - \frac{X}{\sqrt{R}}\right) + \mathrm{Li}_2(1 - X\sqrt{R})\Big]\;, \end{aligned} \tag{B.3}$$

$$\mathcal{F}(v, m_\ell^2, M^2) = \frac{2}{\sqrt{R}}\frac{X}{1 - X^2}\ln X\;, \tag{B.4}$$

we have:

$$\Gamma_C(v, m_\ell^2, M^2; M_\gamma^2) = 2M^2 Y\, \mathcal{C}(v, m_\ell^2, M^2) + 2\ln\frac{M m_\ell}{M_\gamma^2}\left(1 + \frac{XY \ln X}{\sqrt{R}(1-X^2)}\right), \qquad \text{(B.5)}$$

and

$$\Gamma_1(v, m_\ell^2, M^2) = \frac{1}{2}\Big[-\ln R + (4-3Y)\mathcal{F}(v, m_\ell^2, M^2)\Big], \qquad \text{(B.6)}$$

$$\Gamma_2(v, m_\ell^2, M^2) = \frac{1}{2}\left(1 - \frac{m_\ell^2}{v}\right)\Big[-\mathcal{F}(v, m_\ell^2, M^2)(1-R) + \ln R\Big] - \frac{1}{2}(3-Y)\mathcal{F}(v, m_\ell^2, M^2).$$

C Real photon emission

In refs. [29, 30] the function $g_{\text{brems}}(s, u, m_1^2, m_2^2, M_\gamma^2)$ is denoted by $g_{\text{brems}}(s, u, M_\gamma^2)$, omitting the dependence on the meson masses $m_{1,2}^2$. We report here the full expressions for completeness. The Bremsstrahlung function is given by:

$$g_{\text{brems}}(s, u, M_\gamma) = \frac{\alpha}{\pi}[J_{11}(s, u, M_\gamma) + J_{20}(s, u, M_\gamma) + J_{02}(s, u, M_\gamma)]\,. \qquad \text{(C.1)}$$

$$J_{11}(s, u, M_\gamma) = \ln\left(\frac{2x_+(s,u)\bar{\gamma}}{M_\gamma}\right)\frac{1}{\bar{\beta}}\ln\left(\frac{1+\bar{\beta}}{1-\bar{\beta}}\right) \qquad \text{(C.2)}$$

$$+\frac{1}{\bar{\beta}}\left\{\text{Li}_2(1/Y_2) - \text{Li}_2(Y_1) + \ln^2(-1/Y_2)/4 - \ln^2(-1/Y_1)/4\right\}, \qquad \text{(C.3)}$$

$$J_{20}(s, u, M_\gamma) = \ln\left(\frac{M_\gamma(m_\tau^2 - s)}{m_\tau x_+(s,u)}\right), \qquad \text{(C.4)}$$

$$J_{02}(s, u, M_\gamma) = \ln\left(\frac{M_\gamma(m_\tau^2 + m_2^2 - s - u)}{m_1 x_+(s,u)}\right). \qquad \text{(C.5)}$$

The auxiliary variables are:

$$x_\pm(s,u) = \frac{1}{2\,m_1^2}\Big[2\,m_1^2\,(m_\tau^2 + s) - (s + m_1^2 - M_2^2)\,(m_\tau^2 + m_1^2 - u)$$

$$\pm\sqrt{\lambda(s, m_1^2, m_2^2)\,\lambda(u, m_1^2, m_\tau^2)}\Big], \qquad \text{(C.6)}$$

$$Y_{1,2} = \frac{1 - 2\bar{\alpha} \pm \sqrt{(1-2\bar{\alpha})^2 - (1-\bar{\beta}^2)}}{1+\bar{\beta}}, \qquad \text{(C.7)}$$

$$\bar{\alpha} = \frac{(m_\tau^2 - s)(m_\tau^2 + m_2^2 - s - u)}{(m_1^2 + m_\tau^2 - u)} \cdot \frac{\lambda(u, m_1^2, m_\tau^2)}{2\,\bar{\delta}}, \qquad \text{(C.8)}$$

$$\bar{\beta} = -\frac{\sqrt{\lambda(u, m_1^2, m_\tau^2)}}{m_1^2 + m_\tau^2 - u}, \qquad \text{(C.9)}$$

$$\bar{\gamma} = \frac{\sqrt{\lambda(u, m_1^2, m_\tau^2)}}{2\sqrt{\bar{\delta}}}, \qquad \text{(C.10)}$$

$$\bar{\delta} = -m_2^4 m_\tau^2 + m_1^2(m_\tau^2 - s)(m_2^2 - u) - su(-m_\tau^2 + s + u)$$

$$+m_2^2(-m_\tau^4 + su + m_\tau^2 s + m_\tau^2 u), \qquad \text{(C.11)}$$

with $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2\,(xy + xz + yz)$.

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