S-duality and the $\mathcal{N} = 2$ lens space index

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ABSTRACT: We discuss some of the analytic properties of lens space indices for 4d $\mathcal{N} = 2$ theories of class $\mathcal{S}$. The S-duality properties of these theories highly constrain the lens space indices, and imply in particular that they are naturally acted upon by a set of commuting difference operators corresponding to surface defects. We explicitly identify the difference operators to be a matrix-valued generalization of the elliptic Ruijsenaars-Schneider model. In a special limit these difference operators can be expressed naturally in terms of Cherednik operators appearing in the double affine Hecke algebras, with the eigenfunctions given by non-symmetric Macdonald polynomials.

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1 Introduction

The supersymmetric index [1, 2], *a.k.a.* the twisted partition function on $\mathbb{S}^3 \times \mathbb{S}^1$, is a powerful tool to extract quantitative data about strongly coupled superconformal field theories (SCFTs). While the lack of small parameters prevents us from performing direct computations at the IR fixed point, the indices can often be computed in the UV, where we have a known Lagrangian description of the physics. Since the index is independent of gauge couplings [3, 4], we can then identify the supersymmetric indices computed from the UV description with the superconformal indices at the IR fixed point. Following this logic the superconformal indices have been computed for many different theories in various dimensions, and have provided impressive tests of non-perturbative dualities. For example [5], the equality of supersymmetric indices for Seiberg-dual pairs in 4d [6] is due to remarkable identities of special functions appearing in [7] (see also [8, 9]).

In this paper we will be interested in 4d $\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$ [10, 11] which are obtained by compactifying the 6d $(2,0)$ theory of type A_{N-1} on punctured Riemann surfaces $\mathcal{C}$. In most of these examples with $N > 2$ there are no known Lagrangian descriptions of these theories even in the UV. However, the indices of these theories are severely constrained by their symmetries and interrelations. The symmetry which is most useful here is S-duality: the index should be invariant under marginal deformations and thus should be the same when computed in any one of the duality frames.[1] In fact, it has been pointed out in [14] that the assumption of 4d $\mathcal{N} = 2$ S-duality is powerful enough to completely determine the superconformal indices, and leads to manifestly S-duality invariant expressions for them (obtained previously in [15]).

Here we will discuss another interesting twist of this story. We will be interested in studying the lens space index [16], a twisted partition function on $\mathbb{S}^3/\mathbb{Z}_r \times \mathbb{S}^1$. This is a generalization of the ordinary superconformal indices ($r = 1$), and has some new features not present in their $r = 1$ counterparts. The lens index of a gauge theory is determined as a sum over the integer holonomies i.e., discrete Wilson lines parametrized by $(\mathbb{Z}_r)^{N-1}$. Moreover, one can turn on non-trivial holonomies for global symmetries. The lens index is thus a function of fugacities for the global symmetries of the theory, *and* of the discrete holonomies for those symmetries. With this extra structure the lens index contains more refined information about the IR fixed points than the ordinary superconformal indices.[2]

Our goal thus will be to outline some of the features of the lens indices of theories of class $\mathcal{S}$ which can be deduced by exploiting their S-duality properties. Recently, the authors of [31] have verified that, at least in certain limits and for A_1 quivers, the lens space indices of theories of class $\mathcal{S}$ are consistent with S-dualities of these theories. Our approach will be however orthogonal to this: we will *assume* that the lens index of all theories of class $\mathcal{S}$ is independent of the S-duality frame it is computed in, and will discuss what properties of the index follow from this assumption. In this way we thus will be able to say something about lens indices of theories which do not have any known Lagrangian description.

In particular following the technology developed in [14] for the $r = 1$ case we will study the analytical properties of the lens index as a function of certain flavor fugacities. We will show that a class of poles of these indices can be easily deduced. Moreover the residues of these poles are encoded implicitly in certain difference operators. It then will follow that the lens index has simple form when written as a sum of eigenfunctions of these operators. As argued in [14], such residue computations are related to RG flows triggered by turning on space-time dependent VEVs. The residue of the index then describes the index of the IR theory in presence of such a VEV. In general turning on space-time dependent VEVs will result in the IR theory having extended defects which in our setup are surface defects. One

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[1]It follows from this that the index defines a 3-parameter family of 2d TQFTs on $\mathcal{C}$ [12]. This TQFT in a 1-parameter slice coincides with the 2d q-deformed Yang-Mills theory in the zero-area limit [13].

[2]In particular the lens index is sensitive to the global structure of the group, unlike the $r = 1$ index, and can be used to distinguish dualities differing by such global properties [17, 18] : this fact however will not be important to us. Moreover, some of the known exactly-localized partition functions, such as 3d S^3 partition function ([19–21]), 3d lens space partition function ([22–25]), 3d $S^1 \times S^2$ index ([26, 27]), and 2d S^2 partition function ([28, 29]), are believed to be deducible from a suitable reduction of the 4d lens indices [16, 30].

should thus view the difference operators obtained in the procedure of [14] as introducing certain surface defects into the index computation.[3]

The difference operators we will obtain depend on three parameters ($p, q,$ and t), the $\mathcal{N} = 2$ superconformal fugacities, and act non-locally on the lattice $(\mathbb{Z}_r)^{N-1}$ parameterizing the integer holonomies for a global symmetry. Mathematically these operators can be thought of as matrix-valued generalizations of elliptic Ruijsenaars-Schneider operators. S-duality is translated into a number of mathematical properties satisfied by the difference operators, such as the commutativity and self-adjointness under the vector multiplet measure.[4]

In a certain limit of the superconformal fugacities, $p = 0$, similar to the Macdonald limit for the $r = 1$ case [14, 15], we find that our difference operators can be related to a well-studied structure in mathematics. Namely, these difference operators are related (by conjugation) to a symmetric combination of the Cherednik operators of the double affine Hecke algebra (DAHA) [33], and their eigenfunctions are given by *non-symmetric* Macdonald polynomials studied for example in [33, 34].

This paper is organized as follows. After a brief summary of the lens space indices for $\mathcal{N} = 2$ theories in section 2, we discuss the general strategy of computing poles and residues of the lens index in section 3. We then analyze the difference operators, their eigenfunctions and the lens space indices in more detail in section 4. We will also discuss two simplifying limits, namely the $p = 0$ limit (section 5) and $r \to \infty$ limit (section 6). Finally we make some further comments on our results in section 7. Several appendices include technical details and developments.

2 Lens space index

Let us first briefly review the 4d $\mathcal{N} = 2$ lens space index, the supersymmetric partition function on $\mathbb{S}^3/\mathbb{Z}_r \times \mathbb{S}^1$ [16]. The lens space $L(r, 1) = \mathbb{S}^3/\mathbb{Z}_r$ is given by the following discrete identification on S^3

$$(z_1, \, z_2) \sim (e^{\frac{2\pi i}{r}} z_1, \, e^{-\frac{2\pi i}{r}} z_2), \qquad |z_1|^2 + |z_2|^2 = 1. \tag{2.1}$$

This orbifold acts on the Hopf fiber of $\mathbb{S}^3$: $\mathbb{Z}_r \subset \mathrm{U}(1)_1 \subset \mathrm{SU}(2)_1 \subset \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \sim \mathrm{SO}(4)$. For the special case $r = 1$ we recover the round sphere $\mathbb{S}^3$, whereas in the opposite limit $r \to \infty$ the Hopf fiber shrinks and we obtain $L(r \to \infty, 1) \sim \mathbb{S}^2$.

The 4d $\mathcal{N} = 2$ lens index is defined as[5]

$$\mathcal{I}(p, q, t; a) = \mathrm{Tr} \left[(-1)^F \left(\frac{t}{pq} \right)^r p^{j_2 + j_1} \, q^{j_2 - j_1} \, t^R \prod_i a_i^{f_i} \right], \tag{2.2}$$

where the trace is over the Hilbert space on $\mathbb{S}^3/\mathbb{Z}_r$, F the fermion number, j_1, j_2 the Cartans of the rotation group $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \sim \mathrm{SO}(4)$, R the U(1) generator of $\mathrm{SU}(2)_R$

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[3]See [32] for a different, more direct, computation of the indices of such surface defects.

[4]We would like to urge the more mathematically-oriented readers to prove these properties explicitly.

[5]The fugacities p, q, t in our paper are related to $t_{\text{there}}, y_{\text{there}}, v_{\text{there}}$ of [16] as $p = t_{\text{there}}^3 \, y_{\text{there}}$, $\quad q = t_{\text{there}}^3 \, y_{\text{there}}^{-1}$, $\quad t = t_{\text{there}}^4 \, v_{\text{there}}^{-1}$. We have denoted the $\mathbb{Z}_{p_{\text{there}}}$ orbifold action in [16] by $\mathbb{Z}_r$, in order to save the notation p for fugacity.

R-symmetry and r the generator of $\mathrm{U}(1)_R$, and f_i the flavor $\mathrm{U}(1)$ symmetry (if there are any). The index depends on the superconformal fugacities (p, q, t) and the fugacities for flavor symmetries, the a_i's. We assume that the fugacities satisfy the following conditions:

$$|p|, |q|, |t| < 1 \,, \quad |t| > |pq| \,, \quad |a_i| = 1 \,. \tag{2.3}$$

This ensures the convergence of the definition (2.2), and will be important for the residue calculus in the next section. In our residue calculus we will analytically continue the index by taking some of the fugacities a_i to be more general while keeping the rest on the unit circle.

The definition (2.2) of the lens index is similar to the ordinary superconformal index ($r = 1$). However there is one qualitatively new feature which one should consider for $r > 1$: we should (can) turn on non-trivial discrete Wilson lines (holonomies) V for the gauge (flavor) vector fields, since $\pi_1(\mathbb{S}^3/\mathbb{Z}_r) = \mathbb{Z}_r$ and thus is non-trivial. For a simply-connected gauge group this is parameterized by elements in the Cartan of the gauge group G

$$V = \mathrm{diag}(e^{\frac{2\pi i m_1}{r}}, e^{\frac{2\pi i m_2}{r}}, \cdots, e^{\frac{2\pi i m_N}{r}}) \,, \tag{2.4}$$

where the integers m_i's take values in $\mathbb{Z}_r$. In this paper we will be interested in $G = \mathrm{SU}(N)$ and then we also have $\sum_i m_i = 0$ modulo r. The holonomies satisfy $V^r = 1$ since the rth power of this discrete Wilson line is contractible. In presence of the holonomies the gauge group is broken. For $G = \mathrm{SU}(N)$ we have

$$\mathrm{SU}(N) \to S\left[\prod_{i=1}^{N} \mathrm{U}(N_i)\right] \,, \qquad \sum_{i=1}^{N} N_i = N \,, \tag{2.5}$$

where we defined

$$N_i := \#\{1 \le j \le N \,|\, m_j = i\} \,, \tag{2.6}$$

and we defined $\mathrm{U}(0)$ to be the trivial group. The Hilbert space factorizes into sectors with different values of m's, and the lens index is defined as a sum over different holonomy sectors specified by m. We will see that this subtlety modifies the discussion of [14] in an interesting way, and generalizes the mathematical structures behind the usual ($r = 1$) superconformal index.

Given these general definitions one can compute the lens index of different multiplets. The 4d $\mathcal{N} = 2$ A_{N-1} theories of class $\mathcal{S}$ are constructed from two basic ingredients: the trinion theory and the vector multiplet associated with cylinders [10]. The trinion theory depends on the types of punctures, and is generically strongly-coupled. For the computations in this paper we only need to know the lens index of bi-fundamental hypermultiplets, i.e. the theory corresponding to a sphere with two full punctures and one simple puncture. This multiplet is in bifundamental representation of $\mathrm{SU}(N)_{\mathbf{b}} \times \mathrm{SU}(N)_{\mathbf{z}}$ flavor symmetries. Moreover the half-hypers are charged (with opposite charges) under a $\mathrm{U}(1)_a$ symmetry.[6]

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[6]In the special case $N = 2$ there is no distinction between full and simple punctures. Since **2** of SU(2) is pseudoreal, the hypermultiplet here can be decomposed into two half-hypermultiplets, and the trinion theory is given by trifundamental half-hypermultiplets under global symmetries $\mathrm{SU}(2)^3$.

The lens index of this theory, in addition to the superconformal fugacities p, q, t, also depends on fugacities $\mathbf{b}$, $\mathbf{z}$, and a for the global symmetry $\mathrm{SU}(N)_{\mathbf{b}} \times \mathrm{SU}(N)_{\mathbf{z}} \times \mathrm{U}(1)_a$. Moreover, we can also turn on non-trivial holonomies for these symmetries. In what follows we will need to consider only holonomies for $\mathrm{SU}(N)_{\mathbf{b}}$ and $\mathrm{SU}(N)_{\mathbf{z}}$ which we will denote by $\widetilde{\mathbf{m}}$ and $\mathbf{m}$ respectively.[7] The lens index for this trinion theory is given by [16]

$$\mathcal{I}_H^{(\mathbf{m},\widetilde{\mathbf{m}})}(a,\mathbf{b},\mathbf{z}) = \mathcal{I}_H^0 \prod_{s=\pm 1} \prod_{i,j=1}^N \Gamma\left(t^{\frac{1}{2}} p^{[\![s(m_i+\widetilde{m}_j)]\!]}(z_i)^s (b_j)^s a^s; pq, p^r \right)$$
$$\times \Gamma\left(t^{\frac{1}{2}} q^{r-[\![s(m_i+\widetilde{m}_j)]\!]}(z_i)^s (b_j)^s a^s; pq, q^r \right) . \qquad (2.7)$$

We have defined the elliptic gamma function $\Gamma(x; p, q)$ by

$$\Gamma(x; p, q) = \prod_{i,j \geq 0} \frac{1 - x^{-1} p^{i+1} q^{j+1}}{1 - x\, p^i q^j} . \qquad (2.8)$$

For an integer m we define $[\![m]\!]$ to be m modulo r, i.e., an integer $0 \leq [\![m]\!] < r$ such that $m \equiv [\![m]\!]$ modulo r. The holonomies for $\mathrm{SU}(N)$ flavor symmetries satisfy $[\![\sum_{i=1}^N \widetilde{m}_i]\!] = [\![\sum_{i=1}^N m_i]\!] = 0$.

We will also need the lens index of the $\mathcal{N}=2$ $\mathrm{SU}(N)$ vector multiplet

$$\mathcal{I}_V^{(\mathbf{m})}(\mathbf{z}) = \mathcal{I}_V^0 \left(\frac{(p^r; p^r)}{\Gamma(t; pq, p^r)} \frac{(q^r; q^r)}{\Gamma(t\, q^r; pq, q^r)} \right)^{N-1} \prod_{1 \leq i < j \leq N\, :\, m_i = m_j} \left(1 - \frac{z_i}{z_j} \right)^{-1} \left(1 - \frac{z_j}{z_i} \right)^{-1}$$
$$\times \prod_{i \neq j} \frac{1}{\Gamma\left(t\, p^{[\![m_i - m_j]\!]} z_i/z_j; pq, p^r \right)} \frac{1}{\Gamma\left(t\, q^{r - [\![m_i - m_j]\!]} z_i/z_j; pq, q^r \right)}$$
$$\times \prod_{i \neq j} \frac{1}{\Gamma\left(p^{[\![m_i - m_j]\!]} z_i/z_j; pq, p^r \right)} \frac{1}{\Gamma\left(q^{r - [\![m_i - m_j]\!]} z_i/z_j; pq, q^r \right)} . \qquad (2.9)$$

In the expressions (2.7), (2.9), $\mathcal{I}_H^0$ and $\mathcal{I}_V^0$ are the zero-point contributions and are given by[8]

$$\mathcal{I}_H^0 = \left(\frac{pq}{t} \right)^{\frac{1}{4}\left[\sum_{s=\pm 1} \sum_{i,j=1}^N \left([\![s(m_i+\widetilde{m}_j)]\!] - \frac{1}{r} [\![s(m_i+\widetilde{m}_j)]\!]^2 \right) \right]} ,$$
$$\mathcal{I}_V^0 = \left(\frac{pq}{t} \right)^{-\frac{1}{2}\left[\sum_{i,j=1}^N \left([\![m^i - m^j]\!] - \frac{1}{r} [\![m^i - m^j]\!]^2 \right) \right]} . \qquad (2.10)$$

When we gauge a global symmetry, we need to include the index of the vector multiplet, $\mathcal{I}_V(z)$, sum over all the possible holonomies $\mathbf{m}$, and integrate over the corresponding fugacity, with a measure given by

$$[d\mathbf{z}]_{\mathbf{m}} = \frac{1}{\prod_{i=1}^N (N_i!)} \prod_{i=1}^{N-1} \frac{dz_i}{2\pi z_i} \prod_{1 \leq i < j \leq N\, :\, m_i = m_j} \left(1 - \frac{z_i}{z_j} \right) \left(1 - \frac{z_j}{z_i} \right) , \qquad (2.11)$$

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[7]In principle, one could also discuss adding a holonomy for the U(1) symmetry under which the hyper-multiplet is charged. However, this adds complexity not needed for our discussion and thus we will refrain from doing so.

[8]Note that the zero-point contribution explicitly depends on the holonomies m, and is trivial in the case $r = 1$, when we obtain the ordinary superconformal index.
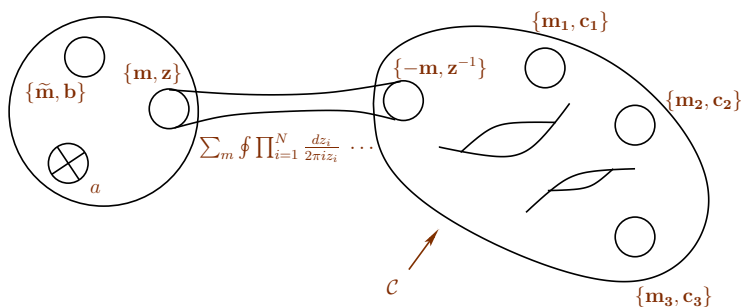
Figure 1. The trinion glued to a general Riemann surface $\mathcal{C}$.

which is the invariant Haar measure of the unbroken gauge group (2.5). Note also that since $\prod_{i=1}^{N} z_i = 1$, only $N-1$ of z_i's are independent. The integral over the z_i's is performed over the contour $|z_i| = 1$.

3 Strategy

A theory of class $\mathcal{S}$ corresponding to a Riemann surface $\mathcal{C}$ admits in general several descriptions. These descriptions correspond to different pair-of-pants decompositions of the underlying Riemann surface. A given description is natural when certain couplings are small (i.e. the corresponding tubes are long). Since the (lens) index is independent of the continuous couplings of the model, the indices computed using different descriptions should agree. This invariance of the index has far-reaching implications for the form of the index. In what follows we will deduce some of these implications. To do so we will follow the general strategy of [14].

Suppose we consider a theory corresponding to a Riemann surface $\mathcal{C}'$ which degenerates into a trinion connected to the rest of the Riemann surface, $\mathcal{C}$, by a cylinder. We also assume that two of the punctures of the trinion are full and one is simple (see figure 1).

Translated into the language of supersymmetric gauge theories, this means that the theory $\mathcal{T}[\mathcal{C}']$ associated with the surface $\mathcal{C}'$ is obtained by gauging the diagonal SU(N) symmetry inside SU(N)^2, one coming from the trinion theory and another from the theory $\mathcal{T}[\mathcal{C}]$ for the surface $\mathcal{C}$. The trinion is given by free N^2 $\mathcal{N} = 2$ hypermultiplets transforming under global symmetries SU(N)$_\mathbf{b} \times$ SU(N)$_\mathbf{z} \times$ U(1)$_a$. Let us turn on holonomies $\mathbf{m}$ for the SU(N)$_z$ global symmetry and $\widetilde{\mathbf{m}}$ for SU(N)$_b$.

The lens index of the theory $\mathcal{T}[\mathcal{C}']$ (denoted by $\mathcal{I}$) is obtained by gluing that of the trinion ($\mathcal{I}_H$) and of the theory $\mathcal{T}[\mathcal{C}]$ ($\widetilde{\mathcal{I}}$) with a measure coming from the vector multiplet

$$\mathcal{I}_{\widetilde{\mathbf{m}}}(a, \mathbf{b}, -) = \sum_{\mathbf{m}} \oint [d\mathbf{z}]_{\mathbf{m}} \, \mathcal{I}_H^{(\mathbf{m}, \widetilde{\mathbf{m}})}(a, \mathbf{b}, \mathbf{z}) \, \mathcal{I}_V^{(\mathbf{m})}(\mathbf{z}) \, \widetilde{\mathcal{I}}_{-\mathbf{m}}(\mathbf{z}^{-1}, -) \,, \tag{3.1}$$

where $-$ inside the arguments of $\widetilde{\mathcal{I}}$ represents fugacities for the global symmetries associated with the remaining punctures of $\mathcal{C}$. It should be emphasized here that the punctures of $\mathcal{C}$ other than those associated with SU(N)$_\mathbf{z}$ are arbitrary, and in particular theory $\mathcal{T}[\mathcal{C}]$

is in general strongly-coupled with no known Lagrangian description; the only requirement for our computation is that $\mathcal{T}[\mathcal{C}']$ contains at least one minimal and one maximal puncture so that we will be able to go to a description with a free bi-fundamental hypermultiplet coupled to $\mathcal{T}[\mathcal{C}]$.

In what follows we will study the poles $a = a^*$ of the expression $\mathcal{I}_{\widetilde{\mathbf{m}}}(a, \mathbf{b}, -)$ with respect to the fugacity a. Physically a pole signifies that for a particular choice of fugacities a flat direction opens up, making the index divergent. We can then trigger an RG flow by giving a VEV to the operator corresponding to the flat direction. This VEV in general will imply a non-trivial spatial profile for the operator and one can argue that in IR the theory will be $\mathcal{T}[\mathcal{C}]$ with a certain surface defect. The residue is identified with the index of the theory $\mathcal{T}[\mathcal{C}]$ with such a surface defect [14].

We will find that the residues of the poles of the index of $\mathcal{T}[\mathcal{C}']$ in the $\mathrm{U}(1)_a$ fugacity are computed by certain difference operators $\mathcal{O}_{a^*}$ acting on the index of $\mathcal{T}[\mathcal{C}]$:

$$\mathrm{Res}_{a \to a^*} \mathcal{I}_{\mathbf{m}}(a, \mathbf{b}, -) = \sum_{\mathbf{n} \in (\mathbb{Z}_r)^{N-1}} \mathcal{O}_{a^*}{}^{\mathbf{n}}{}_{\mathbf{m}} \, \widetilde{\mathcal{I}}_{\mathbf{n}}(\{\alpha_i b_i\}, -) \,, \tag{3.2}$$

where α_i's are monomials in fugacities p, q, and t. This difference operator in general acts non-locally on the discrete periodic $N-1$ dimensional lattice $(\mathbb{Z}_r)^{N-1}$ parametrizing the discrete Wilson lines (which are denoted by $\mathbf{m}, \mathbf{n}$ here).

We will discuss the difference operators in more detail in the next section, but let us here first explain the general implications of S-duality for the difference operators.

3.1 The implications of S-duality

As we mentioned in the beginning of this section the fact that different descriptions of a theory of class $\mathcal{S}$ are interconnected by S-dualities implies that the lens index computed in different duality frames should be the same. In particular it should not matter which of the maximal punctures we decouple together with the minimal puncture corresponding to $\mathrm{U}(1)_a$ in the residue computation above. Quantitatively this implies that when acting on the lens index of a theory of class $\mathcal{S}$ the difference operators satisfy

$$\sum_{\mathbf{n}} \mathcal{O}_{a^*}{}^{\mathbf{n}}{}_{\mathbf{m}} \, \widetilde{\mathcal{I}}_{\mathbf{n}, \mathbf{m}', \cdots}(\{\alpha_i b_i\}, \mathbf{b}', \dots) = \sum_{\mathbf{n}} \mathcal{O}_{a^*}{}^{\mathbf{n}}{}_{\mathbf{m}'} \, \widetilde{\mathcal{I}}_{\mathbf{m}, \mathbf{n}, \cdots}(\mathbf{b}, \{\alpha_i b_i'\}, \dots) \,. \tag{3.3}$$

Moreover the invariance of the index under S-duality implies [14] that the operators $\mathcal{O}_{a^*}$ should commute with each other for all the possible choices of a^*, and that $\mathcal{O}_{a^*}$ are self-adjoint with respect to the measure given by $\mathcal{I}_V^{(\mathbf{m})}$ (2.9). One can thus seek for a set of joint eigenfunctions, $\psi_\Lambda(\mathbf{z}; \mathbf{m})$, of all $\mathcal{O}_{a^*}$ which are orthonormal under the natural vector multiplet measure appearing in the problem,

$$\sum_{\mathbf{n}} \oint [d\mathbf{z}] \, \mathcal{I}_V^{(\mathbf{n})}(\mathbf{z}) \, \psi_\Lambda(\mathbf{z}; \mathbf{n}) \, \psi_{\widetilde{\Lambda}}(\mathbf{z}^{-1}; [-\mathbf{n}]) = \delta_{\widetilde{\Lambda}, \Lambda} \,. \tag{3.4}$$

Given a set of such eigenfunctions one can, in principle, write the index of a theory of class $\mathcal{S}$ corresponding to a Riemann surface with genus g and s maximal punctures as

$$\mathcal{I} = \sum_\Lambda (C_\Lambda)^{2g-2+s} \prod_{\ell=1}^s \psi_\Lambda(\mathbf{z}_\ell; \mathbf{n}_\ell) \,. \tag{3.5}$$
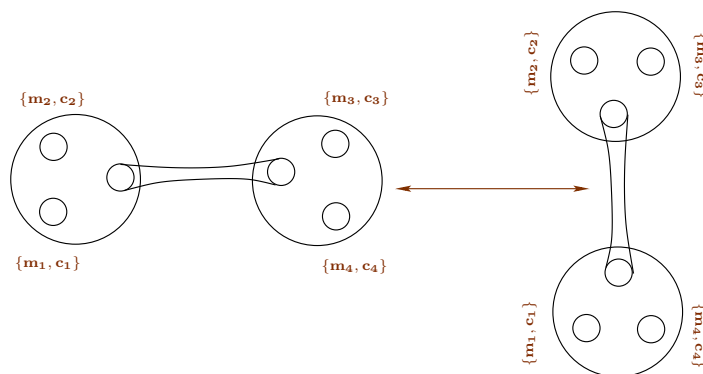
Figure 2. The lens index is expected to be invariant under S-duality.

In writing such an expression one assumes that the spectrum of the eigenvalues is non-degenerate. This assumption is indeed correct in the $r = 1$ case [14]. However as we will see in next sections, it is not true at least in certain limits of the lens index with $r > 1$, and the above "diagonal" form of the index has to be modified to be "block diagonal" (see for example [31]).[9] It is not unlikely that the structure constants C_Λ can be also fixed by residue computations as was done in [14] for $r = 1$ case. Using S-duality one then in principle can translate the physical problem of finding the value of the lens index to the mathematical problem of finding the complete set of orthonormal eigenfunctions for a set of commuting matrix-valued difference operators. To the best of our knowledge this mathematical problem however has not been solved yet for the difference operators at hand.

4 Residues and difference operators

Let us derive next the difference operator (3.2) explicitly. We will make use of the formulas summarized in section 2.

4.1 The poles

The loci of the poles of (3.1) in the $U(1)_a$ fugacity can be deduced as follows. The index $\mathcal{I}_{\widetilde{\mathbf{m}}}(a, \mathbf{b}, -)$ is computed by z_i contour integrals. The integrands of these integrals have numerous poles in z_i with the position of the poles depending on various fugacities. In particular when one varies these fugacities pairs of poles from opposite sides of the integration contour can collide and pinch it: if all the contours are simultaneously pinched the integrals giving $\mathcal{I}_{\widetilde{\mathbf{m}}}(a, \mathbf{b}, -)$ diverge.[10] Thus to find the loci of poles in a one has to understand for which values of a the contour integrals in z_i are simultaneously pinched.

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[9]One can expect that it should be possible to diagonalize also the "blocks": the limits of the parameters in which the eigenfunctions are explicitly known, [31] and section 5 below, the lens index behaves in a somewhat subtle way so this statement was not explicitly checked.

[10]If poles hit the integration contour without pinching it no divergence occurs since in this case the contour can be smoothly deformed away from the poles.

Restricting to the case of $|a| < 1$ we claim that this occurs when[11]

$$a^* = t^{\frac{1}{2}}(pq)^{\frac{n_1}{N}} p^{\frac{rn_2}{N}} q^{\frac{rn_3}{N}}, \quad n_1, n_2, n_3 \geq 0.$$ (4.1)

The poles for the z_i integration which depend on the value of a come only from the index $\mathcal{I}_H^{(\mathbf{m}, \widetilde{\mathbf{m}})}(a, \mathbf{b}, \mathbf{z})$ of the trinion which we decoupled from $\mathcal{T}[\mathcal{C}']$ to obtain $\mathcal{T}[\mathcal{C}]$. This index is given by (2.7). The poles of the integrand of (3.1) coming from the index of the hypermultiplet inside the integration contours are given by (recall (2.3))

$$z_i^{(q)} = \frac{1}{b_{\sigma(i)}\, a}\, t^{\frac{1}{2}}\, q^{r\ell_i + [\![m_i + \widetilde{m}_{\sigma(i)}]\!]}(q\, p)^{s_i}, \qquad \ell_i, s_i \geq 0, \quad i = 1, \cdots, N-1,$$

$$z_i^{(p)} = \frac{1}{b_{\sigma(i)}\, a}\, t^{\frac{1}{2}}\, p^{r\ell_i + r - [\![m_i + \widetilde{m}_{\sigma(i)}]\!]}(q\, p)^{s_i}, \quad \ell_i, s_i \geq 0, \quad i = 1, \cdots, N-1,$$ (4.2)

where we have introduced a permutation $\sigma \in S_N$. Note that $z_i^{(p)}$ and $z_i^{(q)}$ do not coincide for general values of p and q. There are also two interesting sets of poles outside the integration contours which come from terms with z_N in the index of the decoupled trinion,

$$\widetilde{z}_N^{(q)} = \frac{1}{\prod_{i=1}^{N-1} z_i} = \frac{1}{b_{\sigma(N)}\, a}\, t^{\frac{1}{2}}\, q^{r\ell_N + [\![m_N + \widetilde{m}_{\sigma(N)}]\!]}(qp)^{s_N}, \qquad \ell_N, s_N \geq 0,$$

$$\widetilde{z}_N^{(p)} = \frac{1}{\prod_{i=1}^{N-1} z_i} = \frac{1}{b_{\sigma(N)}\, a}\, t^{\frac{1}{2}}\, p^{r\ell_N + r - [\![m_N + \widetilde{m}_{\sigma(N)}]\!]}(qp)^{s_N}, \quad \ell_N, s_N \geq 0.$$ (4.3)

When a pole inside the contour coincides with one of the poles outside, all the integration contours are pinched at once and the whole integral has a pole. Let us look for the poles of the form

$$a = t^{\frac{1}{2}} q^\alpha p^\beta.$$ (4.4)

Then α and β fit into one of the following four possibilities

$$(q, q): \quad N\beta = \sum_{i=1}^{N} s_i, \qquad N\alpha = N\beta + r\sum_{i=1}^{N} \ell_i + \sum_{i=1}^{N}[\![m_i + \widetilde{m}_{\sigma(i)}]\!],$$

$$(p, p): \quad N\alpha = \sum_{i=1}^{N} s_i, \qquad N\beta = N\alpha + r\sum_{i=1}^{N} \ell_i + \sum_{i=1}^{N}\left(r - [\![m_i + \widetilde{m}_{\sigma(i)}]\!]\right),$$

$$(q, p): \quad N\beta = \sum_{i=1}^{N} s_i + r\,\ell_N + \left(r - [\![m_N + \widetilde{m}_{\sigma(N)}]\!]\right),$$

$$N\alpha = \sum_{i=1}^{N} s_i + r\sum_{i=1}^{N-1} \ell_i + \sum_{i=1}^{N-1}[\![m_i + \widetilde{m}_{\sigma(i)}]\!],$$

$$(p, q): \quad N\alpha = \sum_{i=1}^{N} s_i + r\,\ell_N + [\![m_N + \widetilde{m}_{\sigma(N)}]\!],$$

$$N\beta = \sum_{i=1}^{N} s_i + r\sum_{i=1}^{N-1} \ell_i + \sum_{i=1}^{N-1}\left(r - [\![m_i + \widetilde{m}_{\sigma(i)}]\!]\right).$$ (4.5)

---

[11]The restriction $|a| < 1$ corresponds to looking for poles coming from baryons. There are also poles coming from anti-baryons which have $|a| > 1$. Since these do not teach us anything new we do not discuss them.

For example, a pole of the form $z_i^{(q)}$ and a pole $\tilde{z}_N^{(q)}$ coincide when

$$\prod_{i=1}^{N-1} z_i^{(q)} = (\tilde{z}_N^{(q)})^{-1} \,, \tag{4.6}$$

which leads to the case (q, q) in (4.5). Similarly a pole of type $z_i^{(p)}$ coinciding with pole $\tilde{z}_N^{(p)}$ results in case (p, p) in (4.5), and poles coming from $z_i^{(p)}$ ($z_i^{(p)}$) coinciding with $\tilde{z}_N^{(q)}$ ($\tilde{z}_N^{(p)}$) result in case (q, p) ((p, q)) in (4.5). Thus from (4.5) and (4.4) we derive (4.1).

The loci of the poles (4.4) have a simple physical explanation in terms of surface defects. For $r = 1$ case discussed in [14], we have poles located at

$$a^* = t^{\frac{1}{2}} p^{\frac{n_1}{N}} q^{\frac{n_2}{N}}, \qquad n_1, n_2 \geq 0 \,. \tag{4.7}$$

The most basic pole is at $a^* = t^{\frac{1}{2}}$, which corresponds to a VEV for a baryonic operator $B \sim Q^N$ built from the decoupled hypermultiplet. The two towers of poles in (4.7) correspond then to VEVs for derivative operators $\partial_{12}^{n_1} \partial_{34}^{n_2} B$. It was argued in [14] that such VEVs result in the IR theory having a certain surface defect. When we take $\mathbb{Z}_r$ orbifold (2.1) , the surviving states which are not charged under global symmetries for which holonomies are turned on satisfy $2j_1 = 0$ modulo r. This means that the surface defects are allowed only when $n_1 - n_2 = 0$ modulo r. After keeping only such poles from (4.7), we find (4.1).

4.2 The residues

Next we give an example of how to compute the residues at the poles (4.1). There are three "basic" poles: $a^* = t^{\frac{1}{2}}(p\,q)^{\frac{1}{N}}, a^* = t^{\frac{1}{2}} q^{\frac{r}{N}}$ and $a^* = t^{\frac{1}{2}} q^{\frac{r}{N}}$. All the other poles are located at positions which are given by some product of these three. Let us first quote the results for the residue at the $a^* = t^{\frac{1}{2}}(p\,q)^{\frac{1}{N}}$

$$
\operatorname{Res}_{a \to t^{\frac{1}{2}}(p\,q)^{\frac{1}{N}}} \mathcal{I}_{\mathbf{m}}(a, \mathbf{b}, \dots) = \sum_{\mathbf{n}} \left[\mathcal{O}_{a^* = t^{\frac{1}{2}}(p\,q)^{\frac{1}{N}}} \right]^{\mathbf{n}}_{\mathbf{m}} \widetilde{\mathcal{I}}_{\mathbf{n}}(a, \mathbf{b}, \dots)
$$

$$
= \sum_{I=1}^{N} F_I^{\{m\}}(\mathbf{b})\, \widetilde{\mathcal{I}}_{\mathbf{m}} \left(b_I \to b_I (q\,p)^{\frac{1-N}{N}}, b_{i \neq I} \to (q\,p)^{\frac{1}{N}} b_i \right)
$$

$$
+ \sum_{I \neq J}^{N} G_{I,J}^{\{m\}}(\mathbf{b})\, \widetilde{\mathcal{I}}_{\{m_1, m_2, \dots, m_I+1, \dots, m_J-1, \dots\}} \left(b_I \to q^{\frac{1-N}{N}} p^{\frac{1}{N}} b_I, \right.
$$

$$
\left. b_J \to p^{\frac{1-N}{N}} q^{\frac{1}{N}} b_J, b_{i \neq I,J} \to (q\,p)^{\frac{1}{N}} b_i \right), \tag{4.8}
$$

One can compute explcitly the functions F and G. For example, in the A_1 case with $a^* = t^{\frac{1}{2}}(p\,q)^{\frac{1}{N}}$ we obtain (no restrictions on m)

$$
F_1^m = \left(\mathcal{I}_V^{(m=0)} \right)^{-1} \frac{\Gamma((pq)^{\pm 1}; p^r, q^r)}{\Gamma(t^{\pm 1}; p^r, q^r)} \frac{\theta(q^{2m} \frac{t}{pq} b^{-2}; q^r) \theta(p^{2m} \frac{pq}{t} b^2; p^r)}{\theta(q^{2m} b^{-2}; q^r) \theta(p^{2m} b^2; p^r)} \,,
$$

$$
F_2^m = \left(\mathcal{I}_V^{(m=0)} \right)^{-1} \frac{\Gamma((pq)^{\pm 1}; p^r, q^r)}{\Gamma(t^{\pm 1}; p^r, q^r)} \frac{\theta(q^{2m} \frac{pq}{t} b^{-2}; q^r) \theta(p^{2m} \frac{t}{pq} b^2; p^r)}{\theta(q^{2m} b^{-2}; q^r) \theta(p^{2m} b^2; p^r)} \,,
$$

$$
G_{12}^m = \left(\frac{p\,q}{t} \right)^{\frac{2+4m-r}{r}} \left(\mathcal{I}_V^{(m=0)} \right)^{-1} \frac{\Gamma((pq)^{\pm 1}; p^r, q^r)}{\Gamma(t^{\pm 1}; p^r, q^r)} \frac{\theta(q^{2m} \frac{pq}{t} b^{-2}; q^r) \theta(p^{2m} \frac{pq}{t} b^2; p^r)}{\theta(q^{2m} b^{-2}; q^r) \theta(p^{2m} b^2; p^r)} \,,
$$

$$
G_{21}^m = \left(\frac{p\,q}{t} \right)^{\frac{2-4m+r}{r}} \left(\mathcal{I}_V^{(m=0)} \right)^{-1} \frac{\Gamma((pq)^{\pm 1}; p^r, q^r)}{\Gamma(t^{\pm 1}; p^r, q^r)} \frac{\theta(q^{2m} \frac{t}{pq} b^{-2}; q^r) \theta(p^{2m} \frac{t}{pq} b^2; p^r)}{\theta(q^{2m} b^{-2}; q^r) \theta(p^{2m} b^2; p^r)} \,, \tag{4.9}
$$

where we denoted $(b_1, b_2) = (b, b^{-1})$, $(m_1, m_2) = (m, -m)$ and defined

$$\theta(x; q) := (x; q) \left(\frac{q}{x}; q\right), \qquad (x; q) := \prod_{i=1}^{\infty} (1 - xq^i). \tag{4.10}$$

We also used the short-hand notation that $\pm$ in the argument of an expression represents the product of two instances of the expression with the plus sign and the minus sign in argument. For example $\Gamma((pq)^{\pm 1}; p^r, q^r) = \Gamma((pq)^{+1}; p^r, q^r)\Gamma((pq)^{-1}; p^r, q^r)$. Note that (4.9) are explicitly periodic in $m \sim m + r$. Similar expressions can be obtained for the higher rank cases.

Let us now derive (4.8). Note first that the z-poles for $a^* = t^{\frac{1}{2}}(pq)^{\frac{1}{N}}$ appear in the (q, q) and (q, p) sectors in (4.5). In (q, q) sector we have to set $\ell_i = 0$, $\widetilde{m}_{\sigma(i)} = r - m_i$ and $s_I = 1$ with $s_{i \neq I} = 0$. The poles in z_i which pinch the integration contours are located at

$$z_i = (q\,p)^{\delta_{iI} - \frac{1}{N}} \frac{1}{b_{\sigma(i)}}. \tag{4.11}$$

In (q, p) sector we have to set $\ell_i = 0$, $s_i = 0$ for all i; for $i \neq I, N$ $\widetilde{m}_{\sigma(i)} = r - m_i$, and $\widetilde{m}_{\sigma(I)} = r - m_I + 1$, $\widetilde{m}_{\sigma(N)} = r - m_N - 1$. The relevant poles in z_i are located then at

$$z_{i \neq I, N} = (q\,p)^{-\frac{1}{N}} \frac{1}{b_{\sigma(i)}}, \qquad z_I = q^{\frac{N-1}{N}} p^{-\frac{1}{N}} \frac{1}{b_{\sigma(I)}}, \qquad z_N = p^{\frac{N-1}{N}} q^{-\frac{1}{N}} \frac{1}{b_{\sigma(N)}}. \tag{4.12}$$

These two contributes gives the F and G terms in (4.8), respectively.

The difference operator computing the residue at $a^* = t^{\frac{1}{2}}(p\,q)^{\frac{1}{N}}$ of the lens index with holonomy $\mathbf{m}$ involves "nearest neighbor" points on the $\mathbf{m}$ lattice (figure 1). This is to be contrasted with the difference operator for a generic residue which will involve all points on the $\mathbf{m}$ lattice.

The b-dependent part of (4.9) can be interpreted as counting the 2d degrees of freedom localized on a surface defect. Let us take F_1^m as an example. For $r = 1$ F_1^m involves factors of the form

$$\frac{\theta(\frac{t}{pq}b^{-2}; q)}{\theta(b^{-2}; q)}, \qquad \frac{\theta(\frac{pq}{t}b^2; p)}{\theta(b^2; p)}, \tag{4.13}$$

which have the form of elliptic genera of 2d multiplets [14, 32, 35]. These indices are products and ratios of terms of the form $(1 - b^{\pm 2}t^i p^j q^k)$. Such terms survive the orbifold projection only when $j - k \pm 2m = 0$ modulo r. This projection condition follows from the definition of the lens index (2.2) and the $\mathbb{Z}_r$ action (2.1); when translated along the Hopf fiber of $S^3/\mathbb{Z}_r$ the wavefunction acquires a phase $(e^{\frac{2\pi i}{N}})^{j-k}(e^{\frac{2\pi i}{N}})^{\pm 2m}$, where the first factor comes from the spin j_1 and the second from the gauge field flux along the Hopf fiber (Aharanov-Bohm effect). Keeping only such terms from the product in (4.13), we obtain the combinations which appear in F_1^m

$$\frac{\theta(q^{2m}\frac{t}{pq}b^{-2}; q^r)}{\theta(q^{2m}b^{-2}; q^r)}, \qquad \frac{\theta(p^{2m}\frac{pq}{t}b^2; q^r)}{\theta(p^{2m}b^2; p^r)}. \tag{4.14}$$

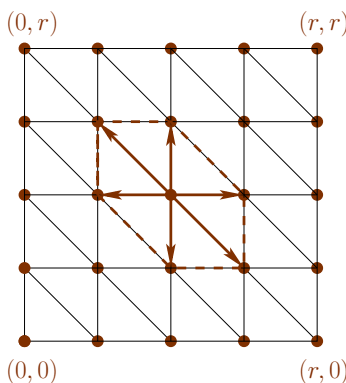In the next sections we will discuss in more detail two simplifying limits of the difference operators.

Figure 3. The (m_1, m_2) lattice of the A_2 case for $r = 4$ (opposite sides are identified). The red arrows represent the nearest neighbor sites for the difference operator computing residues at $a^* = t^{\frac{1}{2}} (p\,q)^{\frac{1}{N}}$.

5 Macdonald limit

Let us discuss the limit of the index when one of the fugacities p or q is vanishing. In what follows for concreteness we will take $p \to 0$. For the $r = 1$ case this limit is called the Macdonald index [15] and we will keep this name also here: this will be justified by the appearance of the non-symmetric Macdonald polynomials. A further limit of $p = 0$, $t = q^r$ (Schur limit) for A_1 quivers was discussed in [31]. Unlike the $r = 1$ case where p always appears in the index with non-negative powers, the lens index for $r > 1$ might contains also negative powers of p: these powers come from the zero point energies (2.10).[12] However, one can show (see [31] and/or appendix A here) that the following quantity

$$\hat{\mathcal{I}}_H(a, \mathbf{b}, \mathbf{z}) \equiv \left(\mathcal{I}_V^0(\mathbf{m}) \right)^{\frac{1}{2}} \left(\mathcal{I}_V^0(\widetilde{\mathbf{m}}) \right)^{\frac{1}{2}} \mathcal{I}_H(a, \mathbf{b}, \mathbf{z}), \tag{5.1}$$

has a well-defined limit as p is taken to vanish. Moreover $\hat{\mathcal{I}}_H$ vanishes in the limit unless there is a permutation $\hat{\sigma} \in S_N$ such that

$$\forall\, i \qquad [\![m_i + \widetilde{m}_{\hat{\sigma}(i)}]\!] = 0 \,. \tag{5.2}$$

In the rest of the section we will have in mind such a rescaled index, and will drop the hat from the notations. Note that after the rescaling (5.1) one does not have to include the zero-point energy in the vector multiplets.

The poles which survive in the limit are located at

$$a = t^{\frac{1}{2}} q^{\frac{r}{N} n} \,, \qquad n \geq 0 \,. \tag{5.3}$$

The residue is computed to be of the following form (for simplicity we assume here that all m_i are different)

$$\mathrm{Res}_{a \to t^{\frac{1}{2}} q^{\frac{r}{N}}} \mathcal{I}_{\mathbf{m}}(a,\, \mathbf{z}) = \sum_{\mathbf{n}} \left[\mathcal{O}_{a^* = t^{\frac{1}{2}} q^{\frac{r}{N}}} \right]_{\mathbf{m}}^{\mathbf{n}} \widetilde{\mathcal{I}}_{\mathbf{n}}(a,\, \mathbf{z})$$

---

[12]In the $r = 1$ case the power of p couples to $\delta = \{\mathcal{Q}_{1+},\, \mathcal{Q}_{1+}^{\dagger}\}$ and thus is non-negative. However when $r > 1$ the supercharge $\mathcal{Q}_{1+}$ does not correspond to a symmetry of the theory anymore due to the orbifold projection.

$$
= \sum_{I=1}^{N} F_I(\mathbf{z}) \; \widetilde{\mathcal{I}}_{\mathbf{m}}(z_I \to q^{\frac{r(1-N)}{N}} z_{\hat{\sigma}(I)}, \, z_{i \neq I} \to q^{\frac{r}{N}} z_{\hat{\sigma}(i)})
$$

$$
+ \sum_{I<J} G_{(I\,J)}(\mathbf{z}) \; \widetilde{\mathcal{I}}_{\mathbf{m}}(z_I \to q^{\frac{r}{N}-m_I+m_J} z_{\hat{\sigma}(J)}, \, z_J \to q^{\frac{r(1-N)}{N}-m_J+m_I} z_{\hat{\sigma}(I)}, \, z_{i \neq I,\,J} \to q^{\frac{r}{N}} z_{\hat{\sigma}(i)})
$$

$$
+ \sum_{I<J<K} G_{(I\,J\,K)}(\mathbf{z}) \; \widetilde{\mathcal{I}}_{\mathbf{m}}(z_I \to q^{\frac{r}{N}-m_I+m_J} z_{\hat{\sigma}(J)}, \, z_J \to q^{\frac{r}{N}-m_J+m_K} z_{\hat{\sigma}(K)},
$$

$$
z_K \to q^{\frac{r(1-N)}{N}-m_K+m_I} z_{\hat{\sigma}(I)}, \, z_{i \neq I,\,J} \to q^{\frac{r}{N}} z_{\hat{\sigma}(i)})
$$

$$
+ \cdots . \tag{5.4}
$$

It is straightforward to evaluate the functions F_I and $G_{(I\,J\,\dots)}$: we will quote the answer for A_1 case momentarily (and for A_2 in appendix B). Note that in the Macdonald limit the difference operators are local on the lattice defined by $\mathbf{m}$ (i.e. the residue computed by this difference operator for the index with holonomy $\mathbf{m}$ act only on the index with the same value of $\mathbf{m}$).

To derive (5.4), note that the pole (5.3) come only from the (q, q) sector in (4.5):

$$
a^* = t^{\frac{1}{2}} q^{\alpha}, \qquad \alpha = \frac{r}{N} \sum_{i=1}^{N} \ell_i + \frac{1}{N} \sum_{i=1}^{N} [\![m_i + \widetilde{m}_{\sigma(i)}]\!] , \tag{5.5}
$$

where $\sum_{i=1}^{N} [\![m_i + \widetilde{m}_{\sigma(i)}]\!]$ is always divisible by r. The permutation σ introduced in section 3 is in general different from $\hat{\sigma}$ introduced above. We will assume the generic scenario in which all m_i are different and we order them such that $m_i > m_j$ if $i > j$. We will comment shortly on the case when this assumption does not hold. From (5.2) we deduce that $\widetilde{m}_{\hat{\sigma}(i)} = r - m_i$. Let us evaluate the residues and the associated difference operators for the simplest case of $n = 1$ in (5.3). Here $\alpha = \frac{r}{N}$ which can be achieved either by setting $\{\ell_I = 1, \ell_{i \neq I} = 0\}$ and $\sigma = \hat{\sigma}$, or by setting $\ell_i = 0$ and choosing $\sigma \in S_N$ such that only for one value of i $m_{\hat{\sigma}(i)} < m_{\sigma(i)}$. This can happen if σ and $\hat{\sigma}$ differ by a single cycle of the form $(I_1 I_2 \cdots I_k)$ with $I_1 < I_2 < \cdots < I_k$. The positions of the z_i poles are thus given by

$$
\{ z_I = q^{\frac{r(N-1)}{N}} b_{\hat{\sigma}(I)}^{-1}, \quad z_{i \neq I} = q^{-\frac{r}{N}} b_{\hat{\sigma}(i)}^{-1} \},
$$

$$
\{ z_I = q^{-\frac{r}{N}+m_I-m_J} b_{\hat{\sigma}(J)}^{-1}, \quad z_J = q^{\frac{r(N-1)}{N}+m_J-m_I} b_{\hat{\sigma}(I)}^{-1}, \quad z_{i \neq I,\,J} = q^{-\frac{r}{N}} b_{\hat{\sigma}(i)}^{-1} \},
$$

$$
\{ z_I = q^{-\frac{r}{N}+m_I-m_J} b_{\hat{\sigma}(J)}^{-1}, \quad z_J = q^{-\frac{r}{N}+m_J-m_K} b_{\hat{\sigma}(K)}^{-1},
$$

$$
z_K = q^{\frac{r(N-1)}{N}+m_K-m_I} b_{\hat{\sigma}(I)}^{-1}, \, z_{i \neq I,J,K} = q^{-\frac{r}{N}} b_{\hat{\sigma}(i)}^{-1} \},
$$

$$
\cdots . \tag{5.6}
$$

Here we assumed without loss of generality that $I < J < K < \cdots$. Collecting these contributions, we obtain (5.4). Finally, when not all m_i are different the terms $G_{(I_1 \cdots I_k)}$ corresponding to cycles permuting equal masses are absent from the difference operator.

5.1 A_1

Let us consider the A_1 quivers in more detail. The A_1 case is special since $\mathrm{U}(1)_a$ symmetry enhances to $\mathrm{SU}(2)_a$, and the trinion theory is a tri-fundamental half-hypermultiplet under

$SU(2)^3$, i.e. there is no distinction here between minimal and maximal punctures. To take advantage of this symmetry let us use the notation $\vec{z} = (z^1, z^2, z^3) = (a, b, z)$ and $\vec{m} = (m^1, m^2, m^3) = (m_a, m_b, m_c)$ where we also turned on a non-zero holonomy for the $SU(2)_a$ symmetry.[13] In this notation, the Macdonald index of the trinion is given by

$$\mathcal{I}_H(z^1, z^2, z^3) = \prod_{\vec{s}=\pm 1} \frac{1}{\left(t^{\frac{1}{2}} q^{[\![-\vec{m}\cdot\vec{s}]\!]} (z^1)^{s^1}(z^2)^{s^2}(z^3)^{s^3}; q^r\right)}. \tag{5.7}$$

The residue at $t^{\frac{1}{2}}q^{\frac{1}{2}}$ (5.4) in the A_1 case evaluates explicitly to

$$\underset{a\to t^{\frac{1}{2}}q^{\frac{1}{2}r}}{\mathrm{Res}}\mathcal{I}_m(a, b) = \frac{1-t}{2\left(1-q^{-r}\right)(q^r; q^r)(t; q^r)} \, \mathcal{H}_b \cdot \widetilde{\mathcal{I}}_m(b) = \frac{(1-t)^2}{2\left(q^r; q^r\right)(t; q^r)} \times$$
$$\times \left(\frac{1}{(1-t)(1-q^{-r})}\left[\frac{1-t\,q^{2m-r}\,b^{-2}}{1-q^{r-2m}\,b^2}\widetilde{\mathcal{I}}_m(q^{\frac{1}{2}r}\,b) + \frac{1-t\,q^{-2m}\,b^2}{1-q^{2m}\,b^{-2}}\widetilde{\mathcal{I}}_m(q^{-\frac{1}{2}r}\,b)\right] + \right.$$
$$\left. + \frac{1}{(1-q^{2m-r}\,b^{-2})(1-q^{-2m}\,b^2)}\widetilde{\mathcal{I}}_m(q^{2m-\frac{1}{2}r}\,b^{-1})\right). \tag{5.8}$$

The two terms on the second line come from $F_{1,2}$ and the term on the third line is $G_{(1\,2)}$. We have defined $\mathcal{H}_b$ as the difference operator computing the residue in this case. A priori from the derivation of the previous sub-section this operator computes the residue when $m \neq 0, \frac{r}{2}$ since then the two m_i are different. However it is easy to show that the operator one obtains when $m = 0, \frac{r}{2}$ is equivalent to $\mathcal{H}_b$ when the latter acts on symmetric functions: i.e. functions which are symmetric under the action of the Weyl group which here is $f(b) = f(b^{-1})$. Indeed, when $m = 0$ or $\frac{r}{2}$ the flavor group enhances from $S(U(1) \times U(1))$ to $SU(2)$ and the index is invariant under the Weyl group of $SU(2)$.

As we discussed in section 3 a consequence of the invariance of the lens index under S-duality is that

$$\mathcal{H}_{z^1}\widetilde{\mathcal{I}}_{m^1, m^2, \ldots}(z^1, z^2, \ldots) = \mathcal{H}_{z^2}\widetilde{\mathcal{I}}_{m^1, m^2, \ldots}(z^1, z^2, \ldots). \tag{5.9}$$

Here $\widetilde{\mathcal{I}}_{m^1, m^2, \ldots}(z^1, z^2, \ldots)$ is the lens index of a general A_1 theory of class $\mathcal{S}$. We can check explicitly for the trinion (5.7) that this property holds,

$$\mathcal{H}_{z^1}\prod_{s_i=\pm 1} \frac{1}{\left(t^{\frac{1}{2}} q^{[\![-\vec{m}\cdot\vec{s}]\!]} (z^1)^{s^1}(z^2)^{s^2}(z^3)^{s^3}; q^r\right)} \propto$$
$$\propto \prod_{s_i=\pm 1} \frac{1}{\left(\left(\frac{t}{q^r}\right)^{1/2} q^{[\![-\vec{m}\cdot\vec{s}]\!]} (z^1)^{s^1}(z^2)^{s^2}(z^3)^{s^3}; q^r\right)} \times$$
$$\times \left[1 - 3\,t - 3q^{-r}\,t^2 + q^{-r}\,t^3 - t\,(1+t)\sum_{i=1}^{3}((z^i)^2 q^{-2m^i} + q^{-r}(z^i)^{-2}q^{2m^i})\right.$$

$$+2\,t^{\frac{3}{2}}\,q^{-\frac{r}{2}}\sum_{s^i=\pm1}\prod_{i=1}^{3}(z^i q^{-m^i})^{s^i}+2\,t^{\frac{3}{2}}(q^{\frac{r}{2}}-q^{-\frac{r}{2}})z^1 z^2 z^3 q^{-m^1-m^2-m^3}+$$

$$+2\,t^{\frac{3}{2}}(q^{-\frac{3r}{2}}-q^{-\frac{r}{2}})(z^1 z^2 z^3)^{-1}q^{m^1+m^2+m^3}\bigg]\,. \tag{5.10}$$

Here we assumed for simplicity that the triplet (m^1, m^2, m^3) satisfies strict triangle inequality, all m_i are different and satisfy $0 < m_i \ll r$. The right-hand-side is explicitly symmetric in the three punctures although the operator acted only on the first one. This fact can be viewed either as a non-trivial check of S-duality of A_1 quivers, or if one takes S-duality for granted as a check of our technical procedure. Note that the holonomies m^i can be absorbed into z^i by redefining $\hat{z}^i = z^i\,q^{-m_i}$. The only information about holonomies affecting the index is whether or not they satisfy certain exclusions. The three holonomies have to satisfy

$$|m^1 - m^2| \le m^3 \le \min\{m^1 + m^2,\ r - m^1 - m^2\}\,, \tag{5.11}$$

because of the zero point energy factors as discussed in [31]. This condition, when applied to (5.10), changes the factors $[\![m^i - m^j - m^k]\!] = 0,\ r + m^i - m^j - m^k$ depending on whether $m^i - m^j - m^k$ is zero or not, and $[\![m^1 + m^2 + m^3]\!] = 0,\ m^1 + m^2 + m^3$ depending whether $m^1 + m^2 + m^3 = r$ or not.

It is convenient to define a new operator $\hat{\mathcal{H}}_z$ related to $\mathcal{H}_z$ by a conjugation,

$$\hat{\mathcal{H}}_z = K^{-1}\,\mathcal{H}_z\,K\,, \qquad K = \frac{1}{\prod_{\vec{s}=\pm1}\left((z^1)^{s^1}(z^2)^{s^2}(z^3)^{s^3}\,t\,q^{[\![-2\vec{m}\cdot\vec{s}]\!]}; q^r\right)}\,. \tag{5.12}$$

This operator takes the following form

$$\hat{\mathcal{H}}_z\,F(\{m, z\}) =$$

$$\times\left[\frac{1 - t\,q^{r-2m}\,z^2}{1 - q^{r-2m}\,z^2}\,F(\{m, q^{\frac{1}{2}r}\,z\}) + \frac{1 - t\,q^{2m}\,z^{-2}}{1 - q^{2m}\,z^{-2}}\,F(\{m, q^{-\frac{1}{2}r}\,z\})\right] +$$

$$+\frac{(1-t)(1-q^{-r})}{(1 - q^{2m-r}\,z^{-2})(1 - q^{-2m}\,z^2)}\,F(\{m, q^{2m-\frac{1}{2}r}\,z^{-1}\})\,. \tag{5.13}$$

As a matter of fact $\hat{\mathcal{H}}_z$ is a well-known object in mathematical literature, as we will discuss in the next sub-section. The eigenfunctions of this operator are neatly given in terms of Macdonald polynomials[14]

$$\psi_\ell^1(\{m, z\}) = P_\ell(q^{-m}\,z; q^r, t)\,,$$
$$\hat{\mathcal{H}}_z\,\psi_\ell^1(\{m, z\}) = q^{-\frac{\ell}{2}r}(1 + t\,q^{\ell r})\,\psi_\ell^1(\{m, z\})\,,$$
$$\psi_{\ell>0}^2(\{m, z\}) = S_\ell(q^{-m}\,z; q, t) \equiv (q^{-m}z - t\,q^m z^{-1})\,P_{\ell-1}(q^{-m}\,z; q^r, q^r\,t)\,,$$
$$\psi_{l=0}^2(\{m, z\}) = 0\,,$$
$$\hat{\mathcal{H}}_z\,\psi_\ell^2(\{m, z\}) = q^{-\frac{\ell}{2}r}(1 + t\,q^{\ell r})\,\psi_\ell^2(\{m, z\})\,, \tag{5.14}$$

where the polynomial $P_\ell(z; q, t)$ is the usual (i.e. symmetric) Macdonald polynomial [36]. For A_1 case this polynomial is symmetric, i.e. $P_\ell(z; q, t) = P_\ell(z^{-1}; q, t)$, and is given by

$$P_\ell(z; q, t) = \frac{(q; q)_\ell}{(t; q)_\ell}\sum_{i=0}^{\ell}\frac{(t; q)_i}{(q; q)_i}\frac{(t; q)_{\ell-i}}{(q; q)_{\ell-i}}\,z^{\ell-2i}\,. \tag{5.15}$$

---

[14]Note that the eigenfunction satisfy $\phi_\ell^i(\{m, q^k\,z\}) = \phi_\ell^i(\{m - k, z\})$.

The polynomial $S_\ell(z; q, t)$ is called the t-antisymmetric Macdonald Polynomial [37], and as the name suggests is not a symmetric function. The operators $\hat{\mathcal{H}}_z$ are self-adjoint under the natural measure of our problem: the measure with which we glue Riemann surfaces together, i.e. the vector multiplet measure,

$$\Delta(\mathbf{z}) = \prod_{i<j} \frac{\theta(q^{m_j - m_i} z_i/z_j; q^r)}{\theta(t\, q^{m_j - m_i} z_i/z_j; q^r)}\,. \tag{5.16}$$

Here we write the measure for general A_{N-1} case and assume as before that for $i > j$ $m_i > m_j$. This is precisely the measure under which the non-symmetric Macdonald polynomials are orthogonal, see e.g. [37].

Note the degeneracy of the spectrum: the two eigenfunctions ψ_ℓ^i have the same eigenvalue under the operator $\hat{\mathcal{H}}_z$ and they also are independent of m. In particular the discussion up to this point implies that we can write the lens index of a generic theory of class $\mathcal{S}$ of type A_1 (genus g and s punctures) in the following form

$$\mathcal{I}(\{m^i, z^i\}) = \sum_{\ell=0}^\infty \sum_{\gamma_i=1}^2 C_\ell^{(g)\,\gamma_1 \cdots \gamma_s; m^1 \cdots m^s} \prod_{j=1}^s \phi_\ell^{\gamma_j}(\{m^j, z^j\})\,, \qquad \phi = K\,\psi\,. \tag{5.17}$$

This structure was derived in the further Schur limit, $t = q^r$, in [31] by explicitly studying the trinion theory. We did not use all the constraints following from S-duality in writing (5.17). For the $r = 1$ case one can fully exploit such constraints to completely fix the structure constants $C_\ell^{(g)\,\gamma_1 \cdots \gamma_s; m^1 \cdots m^s}$ [14]. We leave the problem of figuring out whether this is also the case for $r > 1$ for future work.

5.2 A_{N-1} — Cherednik operators

We can also evaluate the difference operators for the A_{N-1} cases with $N > 2$; the explicit expression for the A_2 case can be found in appendix B. Surprisingly, it turns out that the difference operators we obtain in the Macdonald limit are related to a well studied object in mathematics, the *(double) affine Hecke algebra* ((D)AHA) and especially the Cherednik operators [33].

Let us thus make a brief interlude to define this mathematical structure. To define AHA, one introduces the following operators acting on functions $f(\hat{z}_1, \cdots, \hat{z}_N)$ of N variables

$$
\begin{aligned}
\sigma_i\, f(\cdots, \hat{z}_i, \hat{z}_{i+1}, \cdots) &= f(\cdots, \hat{z}_{i+1}, \hat{z}_i, \cdots)\,, \\
\tau_i\, f(\hat{z}_1, \cdots, \hat{z}_N) &= f(\hat{z}_1, \cdots, q\,\hat{z}_i, \cdots \hat{z}_N)\,, \\
T_i &= t + \frac{t\,\hat{z}_i - \hat{z}_{i+1}}{\hat{z}_i - \hat{z}_{i+1}}(\sigma_i - 1)\,, & i &= 1, \cdots, N-1\,, \\
T_0 &= t + \frac{q\,t\,\hat{z}_N - \hat{z}_1}{q\,\hat{z}_N - \hat{z}_1}(\sigma_N\,\tau_1\tau_N^{-1} - 1)\,, \\
\omega &= \sigma_{N-1}\,\sigma_{N-2} \cdots \sigma_2\,\sigma_1\,\tau_1\,.
\end{aligned} \tag{5.18}
$$

The operator T_i, sometimes called the Demazure-Lusztig operator, is a deformation of the permutation σ_i by two parameters q, t. It follows from this definition that the operators satisfy a set of nice relations,

$$(T_i - t)(T_i + 1) = 0, \qquad\qquad \text{i.e.} \quad T_i^{-1} = t^{-1} - 1 + t^{-1},$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad\qquad \omega T_i = T_{i-1} \omega,$$
$$[T_i, T_j] = 0, \qquad\qquad |i - j| \geq 2. \qquad (5.19)$$

These are the defining relations of the AHA.[15] Let us define the Cherednik operators Y_i by

$$Y_i = t^{i-N} T_i T_{i+1} \cdots T_{N-1} \omega T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1}, \qquad i = 1, \cdots, N. \qquad (5.20)$$

We can show from the relations (5.19) that all these operators commute with each other[16]

$$[Y_i, Y_j] = 0. \qquad (5.21)$$

This mathematical structure is then related to the lens index in the following way. After redefinition

$$\hat{z}_{N-i} = q^{-m_i + \frac{1}{N} \sum_{\ell=1}^N m_\ell} z_i, \qquad (5.22)$$

and the identification $(t \to t, q^r \to q)$ the operator of (5.4) computing the residue at $a^* = t^{\frac{1}{2}} q^{\frac{1}{N}}$ for A_{N-1} quiver theories is given by,

$$K^{-1} \mathcal{H} K \sim \sum_{\ell=1}^N Y_\ell, \qquad (5.23)$$

where

$$K = \prod_{i \neq j}^N \frac{1}{(t\, q^{[\![m_j - m_i]\!]}\, z_i/z_j; q^r)}, \qquad (5.24)$$

and $\sim$ in (5.23) represents the equality up to an overall multiplicative factor which depends only on q and t. We have checked this for A_2 case, whose explicit difference operator can be found in appendix B.

The lens index in the Macdonald limit of general A_{N-1} quiver is then naturally given in terms of eigenfunctions of $\mathcal{H}$ which are K times the non-symmetric Macdonald polynomials. Since the spectrum of non-symmetric Macdonald polynomials is degenerate the index will not be completely diagonal in this basis as we already discussed in the A_1 case. We expect that the Cherednik operators will play a prominent role if one will be interested in higher residues and in difference operators which compute indices in presence of surface defects labeled by general representations of A_{N-1}. We briefly discuss this issue in appendix C.

---

[15] AHA is defined by T_i's and ω satisfying (5.18), and equivalently can be defined in terms of T_i's and Y_i's satisfying certain defining relations. We can also define an operator X_i which acts as a multiplication by $\hat{z}_i$, and it turns out that T_i's and X_i's also define another AHA. We can combine all of T_i, X_i, Y_i containing two AHAs, and this is known as the DAHA. See [33] for details.

[16] We also have $\prod_i Y_i = \omega^N$, where $\omega^N f(\hat{z}_1, \cdots, \hat{z}_N) = f(q\hat{z}_1, \cdots, q\hat{z}_N)$, and hence only $N-1$ of the Y_i's are in practice sufficient.

6 Large r limit

Let us comment on the lens index in the limit of large r. In this limit the size of the Hopf fiber of S^3 shrinks to zero length and we are left with S^2. One can think thus of the lens index in $r \to \infty$ limit as the $S^2 \times S^1$, *a.k.a.* the index, of the $3d$ theory with same matter content and the same global symmetry as the $4d$ theory. The lattice $(\mathbb{Z}_r)^{N-1}$ on which the index is defined becomes non-compact $(\mathbb{Z}^{N-1})$.

For simplicity we focus on the A_1 case, where the lattice is simply given by the line of integers $\mathbb{Z}$. We define

$$x = \sqrt{pq}, \qquad y = \sqrt{q/p}, \qquad \mathcal{I}_m(b) = \mathcal{J}_m((p/q)^{m/2} b), \qquad (p/q)^{m/2} b = \beta. \qquad (6.1)$$

Taking $r \to \infty$ the only residues are at $a^* = t^{\frac{1}{2}}(pq)^{\frac{r}{2}}$. The basic difference operator for $a^* = (p\,q\,t)^{\frac{1}{2}}$ becomes

$$\mathcal{O}_{(p\,q\,t)^{\frac{1}{2}}} \cdot \mathcal{J} = F_1^m \, \mathcal{J}_m(x\,\beta) + F_2^m \, \mathcal{J}_m(x^{-1}\,\beta) + G_1^m \, \mathcal{J}_{m+1}(\beta) + G_2^m \, \mathcal{J}_{m-1}(\beta), \qquad (6.2)$$

where we have for $-\infty < m < \infty$ (recall (4.9))

$$F_1^m = \left(\mathcal{I}_V^{(m=0)}\right)^{-1} \frac{1-t^{\pm 1}}{1-x^{\pm 2}} \frac{(1-x^{2m}\frac{t}{x^2}\beta^{-2})(1-x^{2m}\frac{x^2}{t}\beta^2)}{(1-x^{2m}\beta^{-2})(1-x^{2m}\beta^2)},$$

$$F_2^m = \left(\mathcal{I}_V^{(m=0)}\right)^{-1} \frac{1-t^{\pm 1}}{1-x^{\pm 2}} \frac{(1-x^{2m}\frac{x^2}{t}\beta^{-2})(1-x^{2m}\frac{t}{x^2}\beta^2)}{(1-x^{2m}\beta^{-2})(1-x^{2m}\beta^2)},$$

$$G_1^m = \left(\mathcal{I}_V^{(m=0)}\right)^{-1} \frac{t}{x^2} \frac{1-t^{\pm 1}}{1-x^{\pm 2}} \frac{(1-x^{2m}\frac{x^2}{t}\beta^{-2})(1-x^{2m}\frac{x^2}{t}\beta^2)}{(1-x^{2m}\beta^{-2})(1-x^{2m}\beta^2)},$$

$$G_2^m = \left(\mathcal{I}_V^{(m=0)}\right)^{-1} \frac{x^2}{t} \frac{1-t^{\pm 1}}{1-x^{\pm 2}} \frac{(1-x^{2m}\frac{t}{x^2}\beta^{-2})(1-x^{2m}\frac{t}{x^2}\beta^2)}{(1-x^{2m}\beta^{-2})(1-x^{2m}\beta^2)}. \qquad (6.3)$$

Note that the dependence on y drops out completely. This is to be expected since the fugacity y couples to the momentum along the Hopf fiber (recall (2.2)) which shrinks in the large r limit.

One can check the consistency of S-duality following similar discussion of the previous section. For example, the index of the trinion in the large r limit is given by

$$\mathcal{J}_{H,\{m_i\}}(a,b,c) =$$
$$\times \left(\frac{x^2}{t}\right)^{\alpha(m^1,m^2,m^3)} \frac{(t^{\frac{1}{2}}\frac{x^2}{t}x^{|m^1+m^2-m^3|}(\frac{ab}{c})^{\pm 1}; x^2)(t^{\frac{1}{2}}\frac{x^2}{t}x^{|m^1+m^3-m^2|}(\frac{ac}{b})^{\pm 1}; x^2)}{(t^{\frac{1}{2}}x^{|m^1+m^2-m^3|}(\frac{ab}{c})^{\pm 1}; x^2)(t^{\frac{1}{2}}x^{|m^1+m^3-m^2|}(\frac{ac}{b})^{\pm 1}; x^2)}$$
$$\times \frac{(t^{\frac{1}{2}}\frac{x^2}{t}x^{|m^3+m^2-m^1|}(\frac{cb}{a})^{\pm 1}; x^2)(t^{\frac{1}{2}}\frac{x^2}{t}x^{|m^1+m^2+m^3|}(abc)^{\pm 1}; x^2)}{(t^{\frac{1}{2}}x^{|m^3+m^2-m^1|}(\frac{cb}{a})^{\pm 1}; x^2)(t^{\frac{1}{2}}x^{|m^1+m^2+m^3|}(abc)^{\pm 1}; x^2)}, \qquad (6.4)$$

where $\alpha(m^1, m^2, m^3)$ represents the zero-point contribution (2.10).[17] Acting on the trinion with the operator $\mathcal{O}_{(p\,q\,t)^{\frac{1}{2}}}$ on any one of the three SU(2) flavor fugacities one obtains the same result, as is expected from S-duality (cf. (5.9)).

---

[17]Note that the zero point contribution $\alpha(m^1, m^2, m^3)$ here includes also the holonomy m_a [16, 31]. For example when m_i satisfy triangle inequality and are positive $\alpha(m^1, m^2, m^3) = m^1 + m^2 + m^3$.

One curious observation is that $\mathcal{O}_{(p\,q\,t)^{\frac{1}{2}}}$ simplifies tremendously if we make the following Ansatz

$$\mathcal{J}_m(\beta) = \mathcal{J}(x^{-m}\beta)\,, \tag{6.5}$$

and then

$$\mathcal{J}_{m+1}(\beta) = \mathcal{J}_m(x^{-1}\beta)\,, \qquad \mathcal{J}_{m-1}(\beta) = \mathcal{J}_m(x\beta)\,. \tag{6.6}$$

The difference operator (6.2) becomes

$$\mathcal{O}_{(p\,q\,t)^{\frac{1}{2}}} \cdot \mathcal{J} = \frac{(1-t)^2(x^2+t)}{t\,(1-x^2)^2} \left[\frac{1-\frac{x^2}{t}\,\alpha^2}{1-\alpha^2}\,\mathcal{J}(x\,\alpha) + \frac{1-\frac{x^2}{t}\,\alpha^{-2}}{1-\alpha^{-2}}\,\mathcal{J}(x^{-1}\,\alpha) \right]\,, \tag{6.7}$$

where $\alpha = x^{-m}\beta$. This is precisely proportional to A_1 Macdonald operator $T_{\hat{q},\hat{t}}$ with parameters $\hat{t} = \frac{x^2}{t}$ and $\hat{q} = x^2$. Thus the operator $\mathcal{O}_{(p\,q\,t)^{\frac{1}{2}}}$ can be thought of as yet another generalization of the Macdonald operator. Note however that the index of the trinion (6.4) is not of the form (6.5) and thus this Ansatz does not hold in our case.

7 Final remarks

In this paper we have discussed an explicit procedure to obtain a set of difference operators which act naturally on the lens space index of theories of class $\mathcal{S}$. In particular finding the set of orthogonal eigenfunctions of these difference operators reduces the problem of fixing the lens index of theories of class $\mathcal{S}$ to a much simpler problem of finding a discrete set of structure constants. It will be interesting to see whether the structure constants can be also fixed only by assuming dualities.

We found surprisingly that the difference operators computing the resiudes in U(1) fugacities of the lens index in the Macdonald limit discussed in section 4 can be nicely written in terms of the Cherednik operators appearing in (D)AHA. It would be interesting to see if the (D)AHA for other root systems are relevant for the study of lens indices for Gaiotto theories for D_N and $E_{6,7,8}$ (cf. [38, 39]).

(D)AHA has been discussed in a number of different contexts in mathematical physics. In particular it has recently been used in the construction of knot invariants (see [40] and subsequent works). For torus knots using (D)AHA these papers give knot invariants identical to those coming from the refined Chern-Simons theory of [41]. Since the latter appears to be closely related to the superconformal index (see e.g. [15]), the appearance of (D)AHA in the two different contexts is probably not a coincidence. It would be interesting to explore this point further.

The structure of our lens index, with full fugacities p, q, t, suggests that there is even richer mathematical structure when we incorporate the parameter p. This should be associated with some *elliptic* generalization of DAHA. The situation is more complicated in this case since there are three different types of basic difference operators $\mathcal{O}_{t^{\frac{1}{2}}(pq)^{1/N}}, \mathcal{O}_{t^{\frac{1}{2}}p^{r/N}}, \mathcal{O}_{t^{\frac{1}{2}}q^{r/N}}$. These operators can be viewed as a matrix valued generalization of the "hamiltonians" of the Ruijsenaars-Schneider integrable models.

Finally, it would be interesting to see if similar techniques could be applied to 4d $\mathcal{N} = 1$ theories, such as the theories in [42] and those in [43, 44]. The lens space indices of the latter theories will be discussed in [45] in connection with integrable models.

Acknowledgments

A A technical proof

In this appendix we show that the rescaled index (5.1) is well-defined in the Macdonald limit $p \to 0$, i.e., the rescaled index has only non-negative powers of p and the only remaining contribution is (5.2).

The power of the zero-point contribution of (5.1) is given by

$$\frac{2}{4r} \left[\sum_{i,j} f(m_i + \widetilde{m}_j) - \sum_{i<j} f(m_i - m_j) - \sum_{i<j} f(\widetilde{m}_i - \widetilde{m}_j) \right] , \qquad (A.1)$$

with $f(x) := [\![x]\!](r - [\![x]\!])$. Note that we have $f(0) = f(r) = 0$ and $f(x) = f(-x)$. Since the expression (A.1) is invariant under the permutation of m_i's and also of $\widetilde{m}_i$'s we can assume without generality that

$$0 \le m_N \le m_{N-1} \le \cdots \le m_2 \le x_1 < r, \quad 1 \le \bar{m}_N \le \bar{m}_{N-1} \le \cdots \le \bar{m}_2 \le \bar{m}_1 < r, \quad (A.2)$$

where we defined $\bar{m}_i := r - \widetilde{m}_i$. The expression (A.1) then becomes

$$\frac{2}{4r} \left[\sum_{i,j} g(|m_i - \bar{m}_j|) - \sum_{i<j} g(m_i - m_j) - \sum_{i<j} g(\bar{m}_i - \bar{m}_j) \right] , \qquad (A.3)$$

with $g(x) := x(r - x)$. The expression inside the bracket of (A.3) is non-negative:

$$r \left(\sum_{i,j} |m_i - \bar{m}_j| - \sum_{i<j} (m_i - m_j + \bar{m}_i - \bar{m}_j) \right)$$

$$+ \left(\sum_{i,j} (m_i - \bar{m}_j)^2 - \sum_{i<j} (m_i - m_j)^2 - \sum_{i<j} (\bar{m}_i - \bar{m}_j)^2 \right)$$

$$= r\left(\sum_{i<j}\left(|m_i - \bar{m}_j| - (m_i - \bar{m}_j) + |m_j - \bar{m}_i| + (m_j - \bar{m}_i)\right) + \sum_i |m_i - \bar{m}_i|\right)$$

$$+ \left(\sum_i (m_i - \bar{m}_i)\right)^2 \geq 0\,, \tag{A.4}$$

where the equality holds only when $m_i = \bar{m}_i$ for all i. This gives (5.2), after lifting the condition (A.2).

B A_2 Macdonald limit

In this appendix we quote the difference operator computing the basic residue in the Macdonal limit for the A_2 case. The basic difference operator, associated with the pole $a^* = t^{\frac{1}{2}} q^{\frac{r}{3}}$, can be written in terms of seven different functions F_I, $G_{(IJ)}$ and $G_{(1\,3\,2)}$ in (5.4):

$$K^{-1}\,\mathcal{H}\,K\,f(z_1, z_2, z_3) \sim$$
$$\sim \widetilde{F}_1\,f(q^{-\frac{2}{3}}z_1, q^{\frac{1}{3}}z_2, q^{\frac{1}{3}}z_3) + \widetilde{F}_2\,f(q^{\frac{1}{3}}z_1, q^{-\frac{2}{3}}z_2, q^{\frac{1}{3}}z_3) + \widetilde{F}_3\,f(q^{\frac{1}{3}}z_1, q^{\frac{1}{3}}z_2, q^{-\frac{2}{3}}z_3)$$
$$+\widetilde{G}_{(1\,2)}\,f(q^{\frac{1}{3}+m_2-m_1}z_2, q^{-\frac{2}{3}+m_1-m_3}z_1, q^{\frac{1}{3}}z_3)$$
$$+\widetilde{G}_{(1\,3)}\,f(q^{\frac{1}{3}+m_3-m_1}z_3, q^{\frac{1}{3}}z_2, q^{-\frac{2}{3}+m_1-m_3}z_1)$$
$$+\widetilde{G}_{(2\,3)}\,f(q^{\frac{1}{3}}z_1, q^{\frac{1}{3}+m_3-m_2}z_3, q^{-\frac{2}{3}+m_2-m_3}z_2)$$
$$+\widetilde{G}_{(1\,2\,3)}\,f(q^{\frac{1}{3}+m_2-m_1}z_2, q^{\frac{1}{3}+m_3-m_2}z_3, q^{-\frac{2}{3}+m_1-m_3}z_1)\,. \tag{B.1}$$

Assuming for concreteness that $r > m_1 > m_2 > m_3 > 0$ and $m_1 + m_2 + m_3 = r$, we find

$$\widetilde{F}_1: \qquad \frac{1-t}{1-q^{-r}}\,\frac{1-t\,q^r\,q^{m_2-m_1}\frac{z_1}{z_2}}{1-q^r\,q^{m_2-m_1}\frac{z_1}{z_2}}\,\frac{1-t\,q^r\,q^{m_3-m_1}\frac{z_1}{z_3}}{1-q^r\,q^{m_3-m_1}\frac{z_1}{z_3}}\,,$$

$$\widetilde{F}_2: \qquad \frac{1-t}{1-q^{-r}}\,\frac{1-t\,q^{m_1-m_2}\frac{z_2}{z_1}}{1-q^{m_1-m_2}\frac{z_2}{z_1}}\,\frac{1-t\,q^r\,q^{m_3-m_2}\frac{z_2}{z_3}}{1-q^r\,q^{m_3-m_2}\frac{z_2}{z_3}}\,,$$

$$\widetilde{F}_3: \qquad \frac{1-t}{1-q^{-r}}\,\frac{1-t\,q^{m_1-m_3}\frac{z_3}{z_1}}{1-q^{m_1-m_3}\frac{z_3}{z_1}}\,\frac{1-t\,q^{m_2-m_3}\frac{z_3}{z_2}}{1-q^{m_2-m_3}\frac{z_3}{z_2}}\,, \tag{B.2}$$

$$\widetilde{G}_{(1\,2)}: \qquad (1-t)^2\,\frac{1-t\,q^r\,q^{m_3-m_1}\frac{z_1}{z_3}}{(1-q^{m_2-m_1}\frac{z_1}{z_2})(1-q^r q^{m_3-m_1}\frac{z_1}{z_3})(1-q^{-r}q^{m_1-m_2}\frac{z_2}{z_1})}\,,$$

$$\widetilde{G}_{(1\,3)}: \qquad (1-t)^2\,\frac{1-t\,q^{m_2-m_3}\frac{z_3}{z_2}}{(1-q^{m_2-m_3}\frac{z_3}{z_2})(1-q^{-r}q^{m_1-m_3}\frac{z_3}{z_1})(1-q^{m_3-m_1}\frac{z_1}{z_3})}\,,$$

$$\widetilde{G}_{(2\,3)}: \qquad (1-t)^2\,\frac{1-t\,q^{m_1-m_2}\frac{z_2}{z_1}}{(1-q^{m_1-m_2}\frac{z_2}{z_1})(1-q^{-r}q^{m_2-m_3}\frac{z_3}{z_2})(1-q^{m_3-m_2}\frac{z_2}{z_3})}\,,$$

$$\widetilde{G}_{(1\,2\,3)}: \qquad (1-t)^3\,\frac{1}{(1-q^{m_2-m_1}\frac{z_1}{z_2})(1-q^{-r}q^{m_1-m_3}\frac{z_3}{z_1})(1-q^{m_3-m_2}\frac{z_2}{z_3})}\,. \tag{B.3}$$

As explained in the main text, we can explicitly verify that this difference operator is conjugate to the sum of A_2 Cherednik operators $Y_{1,2,3}$:

$$K^{-1}\,\mathcal{H}\,K \sim Y_1 + Y_2 + Y_3\,, \tag{B.4}$$

where we used the parameter identification (5.22), K is given in (5.24), and Y_i can be computed from the definition (5.18), (5.20). Note that we do not have a symmetry interchanging the three indices; for example $\tilde{F}_1$ is different from $\tilde{F}_2$ even after the exchange of z_i's and m_i's. In the index computation this follows from a particular ordering of the holonomies m_i and in the affine Hecke algebra from the non-symmetric definition of Y_i (5.20).

C More comments on difference operators in the Macdonald limit

The discussion of this paper can be generalized to include difference operators associated to general irreducible representations of A_{N-1}. In the $r = 1$ case it was argued in [14] that the difference operators computing residues at $a^* = t^{\frac{1}{2}} q^{\frac{n}{N}}$ for $n = 1, \cdots, N-1$ correspond to introducing certain surface defects to the index computation which are associated to the nth symmetric representation of A_{N-1}. One can then discuss difference operators corresponding to introducing surface operators assocaited to more general representations. Such a generalization was recently elaborated upon in [46] for the ordinary superconformal index $r = 1$ in the Schur limit $p = 0, q = t$. Here we will comment on the lens version of this generalization.

The finite irreducible representations of A_{N-1} are in one to one corerspondence with Young diagrams. For a surface defect represented by a Young diagram (*a.k.a* partition) λ, we propose that the associated difference operator $\mathcal{H}_\lambda$ satisfies

$$K^{-1} \mathcal{H}_\lambda K \sim s_\lambda(Y) \,, \tag{C.1}$$

where K is the same operator (5.24) defined previously and s_λ is the Schur polynomial associated with λ and we have again neglected the overall constant multiplicative factor. Note that the order of Y's does not matter in $s_\lambda(Y)$ since Y's commute with each other (5.21).

The proposal (C.1) passes several non-trivial tests. First for a fundamental representation we have $s_\lambda(Y) = \sum_{i=1}^N Y_i$, so (C.1) reduces to (5.23). Second, we learn immediately from (C.1) and (5.21) that the operators $\mathcal{H}_\lambda$'s commute:

$$[\mathcal{H}_\lambda, \mathcal{H}_\mu] = 0 \,. \tag{C.2}$$

This is consistent with the expectations from the S-duality, see the similar discussion in section 3. Third, (C.1) is consistent with the decomposition of the tensor product of representations, $R_\lambda \otimes R_\mu = \sum_\nu N_{\lambda,\mu}^\nu R_\nu$, where $N_{\lambda,\mu}^\nu$ is the Littlewood-Richardson coefficient. The Schur functions, being the character of irreducible representations, satisfy the corresponding statement

$$s_\lambda(Y)s_\mu(Y) = \sum_\nu N_{\lambda,\mu}^\nu s_\nu(Y) \,, \tag{C.3}$$

This represents operator product expansions of the surface operators, and is similar to the decomposition of degenerate fields in the Liouville theory.

Finally, (C.1) is consistent with the known results for the $r = 1$ index. When restricted to the $r = 1$ case the operators act on the symmetric functions, since there no non-trivial discrete Wilson lines and hence the Weyl group of the gauge group is always unbroken.

It is known that (5.23) acting on symmetric functions coincide with the corresponding Macdonald operators. More formally, there is a $\mathbb{Q}(q,t)$-algebra isomorphism [47]

$$\mathbb{Q}(q,t)[Y_1^{\pm 1}, \cdots, Y_N^{\pm 1}]^{S_N} \simeq \mathbb{Q}(q,t)[D_1, \cdots, D_{N-1}, D_N^{\pm 1}], \qquad (\text{C.4})$$

where S_N is the Weyl group of A_{N-1}, i.e., we have symmetric polynomials of Y_i on the left hand side. On the right we have k-th Macdonald operators

$$D_k := t^{-r(n-r)} \sum_{I \subset \{1,\cdots,N\}: |I|=r} \prod_{i \in I, j \notin I} \frac{tz_i - z_j}{z_i - z_j} \prod_{i \in I} \tau_i, \qquad k = 1, \cdots, N, \qquad (\text{C.5})$$

where τ_i are defined to be the q-multiplication of one of the arguments as in (5.18). When λ is the k-th antisymmetric representation we have $s_\lambda(Y) = \sum_{i=1}^N Y_i^k$, and the image of the map under the isomorphism (C.4) is given by D_k (up to powers of t), as can be checked by explicit computations [48]. The simplest case is the A_1 theory, where we can easily show that the operator (5.13) acting on a symmetric function coincides with the operator D_1.

The eigenfunctions of the operator (C.1) are known as non-symmetric Macdonald polynomials, generalizing our explicit observation for the A_1 case. Given a composition $\eta = (\eta_1, \eta_2, \cdots)$ of non-negative integers, there is an associated non-symmetric Macdonald polynomial $E_\eta(z; q, t)$. This is the simultaneous eigenfunction of the Cherednik operators Y_i (see [33, 34, 37]):

$$Y_i E_\eta(x; q, t) = q^{\eta_i} t^{-l'_\eta(i)} E_\eta(x; q, t), \qquad i = 1, \cdots, N, \qquad (\text{C.6})$$

where we defined the leg co-length $l'_\eta(i)$ by

$$l'_\eta(i) := \#\{k \,|\, k < i, \eta_k \geq \eta_i\} + \#\{k \,|\, k > i, \eta_k > \eta_i\}. \qquad (\text{C.7})$$

For each composition η we can permute the elements such that the resulting expression is a partition. We denote this partition by η^+. It follows from (C.6) that $E_\eta(z; q, t)$ is the eigenfunction of $s_\lambda(Y)$ with eigenvalue $s_\lambda(q^{\eta_i} t^{-l'_\eta(i)})$, which is determined solely in term of the associated partition η^+. In other words $E_\eta(z; q, t)$ have the same eigenvalues under the operator $s_\lambda(Y)$ as long as the associated partitions η^+ are the same. This means that we have degeneracies of the spectrum, as we have already observed for the A_1 case.[18]

In mathematics many of the properties of the ordinary Macdonald polynomials $P_\lambda(x; q, t)$ are more transparent when we consider more general non-symmetric Macdonald polynomials $E_\eta(x; q, t)$. This is mirrored in our physics discussion, where the properties of

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[18]The A_N analogues of symmetric and t-antisymmetric polynomials are given by

$$P_{\eta^+}(z; q, t) = \frac{1}{\gamma_\eta^+(q,t)} U^+ E_\eta(z; q, t), \qquad S_{\eta^+}(z; q, t) = \frac{1}{\gamma_\eta^-(q,t)} U^- E_\eta(z; q, t), \qquad (\text{C.8})$$

where $\gamma_\eta^\pm(q,t)$ are normalization constants and $U^\pm$ are the operators representing t-symmetrization and t-antisymmetrization:

$$U^+ = \sum_{\sigma \in S_N} T_\sigma \qquad U^- = \sum_{\sigma \in S_N} (-t)^{-l(\sigma)} T_\sigma \qquad (\text{C.9})$$

where $l(\sigma)$ is the sign of the permutation ($l(\sigma) := \#\{(i,j) | i < j, \sigma_i > \sigma_j\}$) and T_σ is the product of T_i's when σ is decomposed into a product of adjacent transpositions: $T_\sigma = T_{i_1} T_{i_2} \cdots$ when $s = s_{i_1} s_{i_2} \cdots$.

the superconformal index and the difference operators acting on them are more transparent in the general lens indices. For example, the decomposition of the product of difference operators in (C.3) follows trivially from the commutativity of Cherednik operators, whereas similar statements in terms of Macdonald operators D_k and symmetric functions are more involved.

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