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The complete one-loop spin chain of $\mathcal{N} = 1$ SQCD

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Abstract: We evaluate the complete planar one-loop dilation operator of $\mathcal{N} = 1$ Super QCD, at the large N Banks-Zaks fixed point near the upper edge of the superconformal window. The spin-chain Hamiltonian turns out to be entirely fixed by the constraints of superconformal symmetry, as in $\mathcal{N} = 4$ Super Yang-Mills and in $\mathcal{N} = 2$ SuperConformal QCD.

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1 Introduction

Superconformal field theories have come to occupy a central position in theoretical physics. Much has been learnt, especially about the most supersymmetric models in four and three dimensions, but some of the most interesting questions remain unanswered. To mention a glaring example, which is the focus of this paper, no substantial progress has been made over the years on the simplest four-dimensional $\mathcal{N} = 1$ model originally discovered by Seiberg [1]: SU(N_c) super QCD in the conformal window. No weakly-coupled AdS gravity dual description is expected to exist for this class of theories; non-critical string duals (which have an intrinsically strongly-coupled worldsheet description) may well exist (see e.g. [2–6]), but these proposals are not yet well understood. And Seiberg duality remains largely a mystery.[1]

A strategy that has proved very fruitful in the case of $\mathcal{N} = 4$ super Yang-Mills theory is the systematic perturbative evaluation of the planar dilation operator [10–16], which turns out to be described by an integrable spin chain (see e.g. [17–22] for a very partial list of references and [23] for a comprehensive review). One does not *a priori* expect the remarkable integrability properties of $\mathcal{N} = 4$ SYM to fully persist in less symmetric theories, but one cannot really know until one tries.[2] For $\mathcal{N} = 1$ SQCD, a perturbative expansion is meaningful in the large N Veneziano limit [28] of $N_c \to \infty$, $N_f \to \infty$ with N_f/N_c fixed, near the upper edge of the superconformal window $N_f \lesssim 3N_c$. If one defines

$$\frac{N_f}{N_c} = 3 - \epsilon \,, \tag{1.1}$$

the large N theory flows for $\epsilon \ll 1$ to a weakly-coupled Banks-Zaks fixed point [29], with 't Hooft coupling $g^2_{\rm YM} N_c \sim \epsilon$. The one-loop planar dilation operator captures the spectrum of the theory at this isolated fixed point, while higher-loop corrections (reorganized in powers of ϵ) correspond to moving down the conformal window. The dual "magnetic" theory admits a perturbative expansion starting from the lower edge of the conformal window $N_f \gtrsim \frac{3}{2} N_f$, with a Banks-Zaks fixed point that is weakly-coupled for $\tilde{\epsilon} \ll 1$, where

$$\frac{N_f}{N_c} = \frac{3}{2} + \tilde{\epsilon} \,. \tag{1.2}$$

A complete large N solution of SQCD would entail determining the dilation operator of the electric theory to all orders in ϵ, and that of the magnetic theory to all orders in $\tilde{\epsilon}$. The resummations of both expansions should then coincide — in the ultimate triumph of Seiberg duality. Needless to say, this is a tall order, and one cannot hope to fulfill this program unless integrability comes to the rescue.

[1] We should however mention the new precision checks of Seiberg duality for the protected spectrum, obtained with the help of the superconformal index [7–9].

[2] There has been much work on integrability in $4d$ gauge theories with $\mathcal{N} < 4$, see e.g. [24–26] and the recent review [27]. Most investigations to date have focused on theories in the standard 't Hooft limit, where the number of fundamental flavors N_f is zero or anyway kept fixed as the number of colors N_c is sent to infinity. The 't Hooft limit has the drawback that conformal invariance is inevitably broken (except of course in $\mathcal{N} = 4$ SYM). The breaking of conformal invariance does not affect the one-loop dilation operator, but it kicks in at two loops.

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Recently the one-loop dilation operator of $\mathcal{N} = 1$ SQCD in the Veneziano limit has been determined in the scalar subsector [30], and shown to coincide with the Ising spin chain in a transverse magnetic field, one of the best known integrable models. This is a tantalizing hint, well-worth subjecting to more stringent tests. In this paper we determine the *complete* planar one-loop spin chain Hamiltonian of $\mathcal{N} = 1$ SQCD and make a preliminary investigation of its integrability.

We eschew direct Feynman diagram calculations and rely instead on symmetry. The constraints of superconformal invariance are sufficient to completely fix the one-loop planar dilation operator of $\mathcal{N} = 4$ super Yang-Mills [12, 31], and also, somewhat unexpectedly, of $\mathcal{N} = 2$ superconformal QCD, as recently shown in [32]. The calculation for $\mathcal{N} = 1$ SQCD proceeds along similar lines as $\mathcal{N} = 2$ SCQCD, and again we are able to fix the one-loop Hamiltonian from symmetry considerations alone. This is a nice surprise. For $\mathcal{N} = 4$ SYM at each site of the chain sits a single irreducible multiplet, which moreover has a simple tensor product with itself; by contrast in the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ cases each site hosts a handful of irreducible representations, and their tensor products have a more complicated decomposition, leading to a rather intricate mixing problem. Despite these complications, the general structure of the calculation is the same as in $\mathcal{N} = 4$ SYM: the full Hamiltonian can be uplifted from the Hamiltonian in a simple subsector, which is in turn uniquely fixed by a centrally extended SU(1|1) symmetry.

The rest of the paper is organized as follows. We begin in section 2 by introducing $\mathcal{N} = 1$ SQCD and its spin chain. We discuss the decomposition of the single-letter state space into irreducible representations of the $\mathcal{N} = 1$ superconformal algebra and give the tensor products of any two such representations. In section 3 we show how the full planar one-loop Hamiltonian can be uplifted from a closed subsector with $\mathrm{SU}(1,1) \times \mathrm{U}(1|1)$ symmetry, where it is fixed (up to overall normalization) by the centrally extended superalgebra. By this route we obtain an expression for the full Hamiltonian in terms of the superconformal projectors onto the different irreps that appear in the two-site state space. By making contact with the scalar sector results of [30] we fix the overall normalization. Finally we obtain the "harmonic action" form of the full Hamiltonian — a completely explicit oscillator representation that can be easily implemented on any state. In section 4 we present a preliminary investigation of the integrability of the one-loop spin chains for $\mathcal{N} = 2$ SCQCD and $\mathcal{N} = 1$ SQCD. We show that both models admit a "parity" symmetry that commutes with the Hamiltonian to all loops. In $\mathcal{N} = 4$ SYM, the existence of degenerate parity pairs provided early circumstantial evidence for integrability. We diagonalize the spectrum and search for parity pairs, both in the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ spin chains (in the subsectors that were used to uplift the full Hamiltonians, for states of length $L \leq 5$). We find that the appearance of parity pairs is much less systematic than in $\mathcal{N} = 4$ SYM. We conclude in section 5 with a brief discussion. Two appendices contain background material on the $\mathcal{N} = 1$ superconformal algebra and its oscillator representation, and the explicit expressions of the two-site superconformal primaries.

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	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)$	$\mathrm{SU}(N_f)$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_r$
$\mathcal{Q}_\alpha$	**1**	**1**	**1**	0	-1
$\mathcal{S}^\alpha$	**1**	**1**	**1**	0	1
λ_α	Adj	**1**	**1**	0	1
$A_{\alpha\dot\alpha}$	Adj	**1**	**1**	0	0
Q	$\square$	$\square$	**1**	1	$1-\frac{N_c}{N_f}$
ψ_α	$\square$	$\square$	**1**	1	$-\frac{N_c}{N_f}$
$\tilde{Q}$	$\overline{\square}$	**1**	$\overline{\square}$	-1	$1-\frac{N_c}{N_f}$
$\tilde{\psi}_\alpha$	$\overline{\square}$	**1**	$\overline{\square}$	-1	$-\frac{N_c}{N_f}$

Table 1. Field content and symmetries of $\mathcal{N}=1$ SQCD. We use $\alpha=\pm$ and $\dot\alpha=\dot\pm$ for Lorentz spinor indices. $\mathcal{Q}_\alpha$ and $\mathcal{S}^\alpha$ denote respectively the Poincaré and conformal supercharges. Conjugate objects such as $\bar\lambda_{\dot\alpha}$ are not written explicitly.

2 Preliminaries

We consider $\mathcal{N}=1$ SQCD, the $\mathcal{N}=1$ supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(N_c)$ and N_f flavors of fundamental quarks. In table 1 we recall the familiar symmetries of the theory and set our notations. Besides the $\mathcal{N}=1$ vector multiplet $(A_{\alpha\dot\alpha},\,\lambda_\alpha)$, in the adjoint representation of the gauge group, there are two sets of N_f chiral multiplets, (Q,ψ_α) and $(\tilde{Q},\tilde{\psi}_\alpha)$, respectively in the fundamental and antifundamental representations of $\mathrm{SU}(N_c)$. The color and flavor structure is then

$$(A^a_{\ b},\lambda^a_{\ b})\,,\qquad (Q^{ai},\psi^{ai})\,,\qquad (\tilde{Q}_{a\tilde{\imath}},\tilde{\psi}_{a\tilde{\imath}})\,, \tag{2.1}$$

where $a=1,\dots N_c$ are color indices, and $i=1,\dots N_f$ and $\tilde{\imath}=1,\dots N_f$ two independent sets of flavor indices, corresponding to the independent flavor symmetries of the gauge-fundamental and of the gauge-antifundamental chiral multiplets.

In the large N Veneziano limit, the basic flavor-singlet local gauge-invariant operators are "generalized single-traces" [33, 34], of the schematic form

$$\mathrm{Tr}\left(\phi^{k_1}\mathcal{M}^{k_2}\phi^{k_3}\mathcal{M}^{k_4}\dots\right). \tag{2.2}$$

Here ϕ denotes any of the color-adjoint "letters", for example $\phi^a_{\ b}=(\mathcal{D}^n\lambda)^a_{\ b}$, where $\mathcal{D}$ is a gauge-covariant derivative, while $\mathcal{M}^a_{\ b}$ is any of the gauge-adjoint composite objects obtained by the flavor contraction of a fundamental and an antifundamental letter, for example $\mathcal{M}^a_{\ b}=Q^{ai}\bar{Q}_{bi}$ or $\mathcal{M}^a_{\ b}=\bar{\tilde{\psi}}^{a\tilde{\imath}}\tilde{Q}_{b\tilde{\imath}}$. In the Veneziano limit these are the building blocks: a generic gauge-singlet operator factorizes into products of generalized single-trace operators, up to $1/N$ corrections. To leading order in the large N limit (the planar theory) generalized single-trace operators are closed under the action of the dilation operator, which takes the familiar form of a spin chain Hamiltonian; as usual, the locality of the Hamiltonian (nearest neighbor, next-to-nearest-neighbor, ...) is related to the loop order of the planar perturbative expansion.

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One can also consider operators with open flavor indices of the schematic forms

$$\bar{q}_i\,\phi^{k_1}\mathcal{M}^{k_2}\dots q^j\,,\qquad \bar{q}_i\,\phi^{k_1}\mathcal{M}^{k_2}\dots\bar{\tilde{q}}^{\tilde{\jmath}}\,,\qquad \tilde{q}_{\tilde{\imath}}\,\phi^{k_1}\mathcal{M}^{k_2}\dots q^j\,,\qquad \tilde{q}_{\tilde{\imath}}\,\phi^{k_1}\mathcal{M}^{k_2}\dots\bar{\tilde{q}}^{\tilde{\jmath}}\,, \tag{2.3}$$

where q and $\tilde{q}$ stand for any of the (anti)fundamental letters. It is also understood that for operators with index structures ${}_i{}^j$ and ${}_{\tilde{\imath}}{}^{\tilde{\jmath}}$ we are projecting into the adjoint representation of SU(N_f) by removing the flavor-trace term. In the Veneziano limit operators of the form (2.3) are closed under renormalization and they can be viewed as open spin chains.

2.1 Superconformal representations

We are going to make crucial use of superconformal symmetry to constraint the form of the one-loop dilation operator (the spin chain Hamiltonian). The letters that occupy each site of the $\mathcal{N}=1$ SQCD chain belong to four distinct irreducible representations of the SU(2, 2|1) superconformal algebra. We denote them by $\mathcal{X}$ (chiral multiplet), $\bar{\mathcal{X}}$ (antichiral multiplet), $\mathcal{V}$ (vector multiplet) and $\bar{\mathcal{V}}$ (conjugate vector multiplet). This is in contrast with $\mathcal{N}=4$ SYM, where the letters belong to a single irreducible representation, but rather similar to $\mathcal{N}=2$ SCQCD, where the single-site state space decomposes in the sum of three different representations [32].

At one loop, the Hamiltonian is of nearest-neighbor form. The Hamiltonian density acts on two adjacent sites and can be written as a sum of projectors onto the irreducible representations that span the two-site state space. Because of the index structure of the spin chain, not all orderings of two single-site representations are allowed in the two-site state space. For example, it is not possible to have two Qs adjacent to each other because there is no way in which to contract the indices, so two adjacent $\mathcal{X}$ representations are not allowed. On the other hand, Q and $\bar{Q}$ can be placed together and, in fact, there are two ways in which this can be done, we can contract either adjacent *gauge* indices or adjacent *flavor* indices. The gauge-contracted combinations are (the order matters):

$$\mathcal{V}\times\mathcal{V}\qquad \bar{\mathcal{V}}\times\mathcal{V}\qquad \bar{\mathcal{V}}\times\bar{\mathcal{V}}\qquad \mathcal{V}\times\bar{\mathcal{V}} \tag{2.4}$$

$$\mathcal{V}\times\mathcal{X}\qquad \tilde{\mathcal{X}}\times\mathcal{V}\qquad \bar{\mathcal{V}}\times\mathcal{X}\qquad \tilde{\mathcal{X}}\times\bar{\mathcal{V}} \tag{2.5}$$

$$\mathcal{V}\times\bar{\tilde{\mathcal{X}}}\qquad \bar{\mathcal{X}}\times\mathcal{V}\qquad \bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}\qquad \bar{\mathcal{X}}\times\bar{\mathcal{V}} \tag{2.6}$$

$$\bar{\mathcal{X}}\times\mathcal{X}\qquad \bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}\qquad \tilde{\mathcal{X}}\times\mathcal{X}\qquad \tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}\,, \tag{2.7}$$

while the flavor-contracted combinations are:

$$\mathcal{X}\times\bar{\mathcal{X}}\qquad\qquad \bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}\,. \tag{2.8}$$

For clarity we have added a "tilde" to distinguish the fundamental from the antifundamental chiral multiplets, though of course this is a distinction that pertains to the color and flavor structure, not the superconformal structure ($\mathcal{X}$ and $\tilde{\mathcal{X}}$ are isomorphic as superconformal representations).

The classification of multiplets of the $\mathcal{N}=1$ superconformal algebra is reviewed in appendix A. We follow the notations of [9], according to which the multiplets that span the single-site state space are given by

$$\mathcal{X}=\tilde{\mathcal{X}}=\bar{\mathcal{D}}_{(0,0)}\,,\qquad \bar{\mathcal{X}}=\bar{\tilde{\mathcal{X}}}=\mathcal{D}_{(0,0)},\qquad \mathcal{V}=\bar{\mathcal{D}}_{(\frac{1}{2},0)}\,,\qquad \bar{\mathcal{V}}=\mathcal{D}_{(0,\frac{1}{2})}\,. \tag{2.9}$$

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Using superconformal characters it is not difficult to decompose the tensor products of any two such multiplets into irreducible representations. We find

$$\tilde{\mathcal{X}} \times \mathcal{X} = \bar{\mathcal{B}}_{\frac{4}{3}(0,0)} \oplus \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q+1}{2},\frac{q}{2}\right)}\,, \tag{2.10}$$

$$\bar{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}} = \mathcal{B}_{-\frac{4}{3}(0,0)} \oplus \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q}{2},\frac{q+1}{2}\right)}\,, \tag{2.11}$$

$$\mathcal{X} \times \bar{\mathcal{X}} = \bar{\mathcal{X}} \times \mathcal{X} = \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q}{2},\frac{q}{2}\right)} = \tilde{\mathcal{X}} \times \bar{\tilde{\mathcal{X}}} = \bar{\tilde{\mathcal{X}}} \times \tilde{\mathcal{X}}\,, \tag{2.12}$$

$$\mathcal{V} \times \mathcal{X} = \bar{\mathcal{B}}_{\frac{5}{3}\left(\frac{1}{2},0\right)} \oplus \bigoplus_{q=1}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q+1}{2},\frac{q-1}{2}\right)} = \tilde{\mathcal{X}} \times \mathcal{V}\,, \tag{2.13}$$

$$\bar{\mathcal{V}} \times \mathcal{X} = \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q}{2},\frac{q+1}{2}\right)} = \tilde{\mathcal{X}} \times \bar{\mathcal{V}}\,, \tag{2.14}$$

$$\bar{\mathcal{X}} \times \mathcal{V} = \bigoplus_{q=0}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q+1}{2},\frac{q}{2}\right)} = \mathcal{V} \times \bar{\tilde{\mathcal{X}}}\,, \tag{2.15}$$

$$\bar{\mathcal{X}} \times \bar{\mathcal{V}} = \mathcal{B}_{-\frac{5}{3}\left(0,\frac{1}{2}\right)} \oplus \bigoplus_{q=1}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q-1}{2},\frac{q+1}{2}\right)} = \bar{\mathcal{V}} \times \bar{\tilde{\mathcal{X}}}\,, \tag{2.16}$$

$$\mathcal{V} \times \mathcal{V} = \bar{\mathcal{B}}_{2(0,0)} \oplus \bar{\mathcal{B}}_{2(1,0)} \oplus \bigoplus_{q=2}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q+1}{2},\frac{q-2}{2}\right)}\,, \tag{2.17}$$

$$\bar{\mathcal{V}} \times \bar{\mathcal{V}} = \mathcal{B}_{-2(0,0)} \oplus \mathcal{B}_{-2(0,1)} \oplus \bigoplus_{q=2}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q-2}{2},\frac{q+1}{2}\right)}\,, \tag{2.18}$$

$$\mathcal{V} \times \bar{\mathcal{V}} = \bigoplus_{q=1}^{\infty} \hat{\mathcal{C}}_{\left(\frac{q}{2},\frac{q}{2}\right)} = \bar{\mathcal{V}} \times \mathcal{V}\,. \tag{2.19}$$

3 Algebraic evaluation of the Hamiltonian

The evaluation of the one-loop Hamiltonian proceeds in three steps. First, we identify a closed $\mathrm{SU}(1,1) \times \mathrm{U}(1|1)$ subsector and determine the Hamiltonian in the subsector by using the constraints of the centrally-extended symmetry. We then uplift the result to the complete theory and obtain the full Hamiltonian as a sum of superconformal projectors. Finally we rewrite the Hamiltonian in an explicit "harmonic action" form.

3.1 The $\mathrm{SU}(1,1) \times \mathrm{U}(1|1)$ subsector

Consider the subsector generated by the letters

$$\lambda_k = \frac{\mathcal{D}^k}{k!}\lambda_+\,, \qquad \bar{\mathcal{F}}_k = \frac{\mathcal{D}^k}{k!}\bar{\mathcal{F}}_{\dot{+}\dot{+}}\,, \tag{3.1}$$

$$Q_k = \frac{\mathcal{D}^k}{k!}Q\,, \qquad \bar{\psi}_k = \frac{\mathcal{D}^k}{k!}\bar{\psi}_{\dot{+}}\,, \tag{3.2}$$

$$\tilde{Q}_k = \frac{\mathcal{D}^k}{k!}\tilde{Q}\,, \qquad \bar{\tilde{\psi}}_k = \frac{\mathcal{D}^k}{k!}\bar{\tilde{\psi}}_{\dot{+}}\,. \tag{3.3}$$

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with $\mathcal{D} \equiv \mathcal{D}_{++\dot{+}}$. By using conservation of the engineering dimension, of the Lorentz spins and of the R-charge it is easy to see that this sector is closed to all loops under the action of the dilation operator. Moreoever, the one-loop Hamiltonian restricted to this subsector can be uplifted to the full one-loop Hamiltonian, as each of the modules appearing on the right hand side of the tensor products (2.10)–(2.19) contains a representative within the subsector. The representatives are primaries of SU(1, 1), and descendants with respect to the full SU(2, 2|1) algebra.

To obtain the Hamiltonian in the subsector one could perform an explicit Feynman diagram calculation. Instead we use a purely algebraic approach that uses the restrictions imposed by superconformal symmetry. The algebraic method was successfully used in [31] and [32] to find the dilation operator of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD respectively.

The subgroup of the superconformal group acting on the sector is SU(1, 1) × U(1|1). The SU(1, 1) generators are

$$\mathcal{J}'_{+}(g) = \mathcal{P}_{++\dot{+}}(g)\,, \tag{3.4}$$

$$\mathcal{J}'_{-}(g) = \mathcal{K}^{++\dot{+}}(g)\,, \tag{3.5}$$

$$\mathcal{J}'_{3}(g) = \frac{1}{2}D_0 + \frac{1}{2}\delta D(g) + \frac{1}{2}\mathcal{L}_{+}{}^{+} + \frac{1}{2}\dot{\mathcal{L}}_{\dot{+}}{}^{\dot{+}}\,. \tag{3.6}$$

The U(1|1) generators are given by

$$L = \mathcal{L}_{-}{}^{-} + \dot{\mathcal{L}}_{\dot{-}}{}^{\dot{-}} + D_0\,, \quad \bar{\mathcal{Q}}(g) = \bar{Q}_{\dot{-}}(g)\,, \quad \bar{\mathcal{S}}(g) = \bar{\mathcal{S}}_{\dot{-}}(g)\,, \quad \delta D(g) \tag{3.7}$$

and can be checked to commute with the SU(1, 1) generators (3.4)–(3.6). Their (anti)commutators are[3]

$$[L, \bar{\mathcal{Q}}(g)] = \bar{\mathcal{Q}}(g)\,, \tag{3.8}$$

$$[L, \bar{\mathcal{S}}(g)] = -\bar{\mathcal{S}}(g)\,, \tag{3.9}$$

$$\{\bar{\mathcal{S}}(g), \bar{\mathcal{Q}}(g)\} = \frac{1}{2}\delta D(g)\,. \tag{3.10}$$

The generator L can be identified with the length operator, this implies that $\bar{\mathcal{Q}}$ increases the length of a chain while $\bar{\mathcal{S}}$ decreases it.

3.2 First order expressions for $\mathcal{Q}(g)$ and $\mathcal{S}(g)$

The procedure is now very similar to the one followed in [32] for $\mathcal{N} = 2$ SCQCD and we shall be brief. Writing $\mathcal{Q}(g) = g\mathcal{Q} + O(g^2)$, we formulate an ansatz for the action of the supercharges on the states of the sector compatible with the quantum numbers of the fields and impose invariance under the SU(1, 1) algebra, $[\mathcal{J}', \bar{\mathcal{Q}}] = 0$ and $[\mathcal{J}', \bar{\mathcal{S}}] = 0$. In fact strict invariance is too restrictive and one needs only to impose vanishing of these commutators up to local gauge transformations on the chain. The only gauge transformation that preserves

[3]The bar in $\bar{\mathcal{Q}}$ and $\bar{\mathcal{S}}$ does not denote complex conjugation, we are going to impose the appropriate hermiticity condition below.

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the sector and is consistent with conservation of quantum numbers is the one with gauge parameter $\alpha\lambda_+$, where α is an arbitrary coefficient. Using the compact notation defined above we have,

$$[\bar{\mathcal{Q}}, \mathcal{J}'_+]\lambda_n = \alpha\left(\lambda_n\lambda + \lambda\lambda_n\right), \qquad [\bar{\mathcal{Q}}, \mathcal{J}'_+]\bar{\mathcal{F}}_n = \alpha\left(-\bar{\mathcal{F}}_n\lambda + \lambda\bar{\mathcal{F}}_n\right), \tag{3.11}$$

$$[\bar{\mathcal{Q}}, \mathcal{J}'_+]Q_n = \alpha\lambda Q_n, \qquad [\bar{\mathcal{Q}}, \mathcal{J}'_+]\bar{\psi}_n = \alpha\bar{\psi}_n\lambda, \tag{3.12}$$

$$[\bar{\mathcal{Q}}, \mathcal{J}'_+]\tilde{Q}_n = -\alpha\tilde{Q}_n\lambda, \qquad [\bar{\mathcal{Q}}, \mathcal{J}'_+]\bar{\tilde{\psi}}_n = \alpha\lambda\bar{\tilde{\psi}}_n. \tag{3.13}$$

The action of $\bar{\mathcal{Q}}$ is easily obtained from these expressions. First, we consider an ansatz consistent with conservation of quantum numbers and the fact that $\bar{\mathcal{Q}}$ increases the length by one. This ansatz has a set of unknown coefficients that can be uniquely fixed recursively (see section 3.2 of [31] for a similar calculation). The final answer is given by

$$\bar{\mathcal{Q}}\lambda_n = \alpha\sum_{k'=0}^{n-1}\frac{n+1}{(k'+1)(n-k')}\lambda_{k'}\lambda_{n-k'-1}, \tag{3.14}$$

$$\begin{aligned}\bar{\mathcal{Q}}\bar{\mathcal{F}}_n = {}& \alpha\sum_{k'=0}^{n-1}\left(-\frac{1}{n-k'}\bar{\mathcal{F}}_{k'}\lambda_{n-k'-1} + \frac{1}{k'+1}\lambda_{k'}\bar{\mathcal{F}}_{n-k'+1}\right)\\ & +\alpha'\sum_{k'=0}^{n} Q^i_{k'}\bar{\psi}_{n-k'\,i} + \alpha''\sum_{k'=0}^{n}\bar{\tilde{\psi}}^{\tilde{\imath}}_{k'}\tilde{Q}_{n-k'\,\tilde{\imath}},\end{aligned} \tag{3.15}$$

$$\bar{\mathcal{Q}}Q_n = \alpha\sum_{k'=0}^{n-1}\frac{1}{k'+1}\lambda_{k'}Q_{n-k'-1}, \tag{3.16}$$

$$\bar{\mathcal{Q}}\bar{\psi}_n = \alpha\sum_{k'=0}^{n-1}\frac{1}{n-k'}\bar{\psi}_{k'}\lambda_{n-k'-1}, \tag{3.17}$$

$$\bar{\mathcal{Q}}\tilde{Q}_n = -\alpha\sum_{k'=0}^{n-1}\frac{1}{n-k'}\tilde{Q}_{k'}\lambda_{n-k'+1}, \tag{3.18}$$

$$\bar{\mathcal{Q}}\bar{\tilde{\psi}}_n = \alpha\sum_{k'=0}^{n-1}\frac{1}{k'+1}\lambda_{k'}\bar{\tilde{\psi}}_{n-k'+1}. \tag{3.19}$$

The terms with α' and α'' are invariant on their own and that's why we assigned them independent gauge parameters. Similarly, the gauge transformations associated with the $\bar{\mathcal{S}}$ supercharge are

$$[\bar{\mathcal{S}}, \mathcal{J}'_-]\lambda_k\lambda_{n-k} = \beta\left(\delta_{k=0} + \delta_{n=k}\right)\lambda_n, \tag{3.20}$$

$$[\bar{\mathcal{S}}, \mathcal{J}'_-]\lambda_k\bar{\mathcal{F}}_{n-k} = \beta\delta_{k=0}\bar{\mathcal{F}}_n, \qquad [\bar{\mathcal{S}}, \mathcal{J}'_-]\bar{\mathcal{F}}_k\lambda_{n-k} = -\beta\delta_{n=k}\bar{\mathcal{F}}_n, \tag{3.21}$$

$$[\bar{\mathcal{S}}, \mathcal{J}'_-]\lambda_k Q_{n-k} = \beta\delta_{k=0}Q_n, \qquad [\bar{\mathcal{S}}, \mathcal{J}'_-]\tilde{Q}_k\lambda_{n-k} = -\beta\delta_{n=k}\tilde{Q}_n, \tag{3.22}$$

$$[\bar{\mathcal{S}}, \mathcal{J}'_-]\bar{\psi}_k\lambda_{n-k} = \beta\delta_{n=k}\bar{\psi}_n, \qquad [\bar{\mathcal{S}}, \mathcal{J}'_-]\lambda_k\bar{\tilde{\psi}}_{n-k} = \beta\delta_{k=0}\bar{\tilde{\psi}}_n, \tag{3.23}$$

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and the action of $\bar{\mathcal{S}}$ consistent with them is

$$\bar{\mathcal{S}}\lambda_k\lambda_{n-k} = \beta\lambda_{n+1}\,, \tag{3.24}$$

$$\mathcal{S}\lambda_k\bar{\mathcal{F}}_{n-k} = \beta\frac{(n-k+2)(n-k+1)}{(n+3)(n+2)}\bar{\mathcal{F}}_{n+1}\,, \tag{3.25}$$

$$\mathcal{S}\bar{\mathcal{F}}_k\lambda_{n-k} = -\beta\frac{(k+2)(k+1)}{(n+3)(n+2)}\bar{\mathcal{F}}_{n+1}\,, \tag{3.26}$$

$$\bar{\mathcal{S}}\lambda_k Q_{n-k} = \beta Q_{n+1}\,, \tag{3.27}$$

$$\bar{\mathcal{S}}\tilde{Q}_k\lambda_{n-k} = -\beta\tilde{Q}_{n+1}\,, \tag{3.28}$$

$$\bar{\mathcal{S}}\bar{\psi}_k\lambda_{n-k} = \beta\frac{k+1}{n+2}\bar{\psi}_{n+1}\,, \tag{3.29}$$

$$\bar{\mathcal{S}}\lambda_k\bar{\tilde{\psi}}_{n-k} = \beta\frac{n-k+1}{n+2}\bar{\tilde{\psi}}_{n+1}\,, \tag{3.30}$$

$$\mathcal{S}Q^i_k\bar{\psi}_{n-k\,i} = \beta'\frac{n-k+1}{(n+2)(n+1)}\bar{\mathcal{F}}_n\,, \tag{3.31}$$

$$\mathcal{S}\bar{\tilde{\psi}}^{\tilde{\imath}}_k\tilde{Q}_{n-k\,\tilde{\imath}} = \beta''\frac{k+1}{(n+2)(n+1)}\bar{\mathcal{F}}_n\,. \tag{3.32}$$

As before, the terms with β' and β'' are invariant on their own. Note that the action of $\bar{\mathcal{S}}$ on $Q_k\bar{\psi}_{n-k}$ and $\bar{\tilde{\psi}}_k\tilde{Q}_{n-k}$ is non-zero only for the flavor-contracted combinations. Indeed the action on the gauge-contracted combinations would have to give a single letter with open flavor indices which is impossible. Now we impose the hermiticity condition

$$\bar{\mathcal{Q}}^\dagger = \bar{\mathcal{S}}\,, \tag{3.33}$$

which implies the following reality constraints for the undetermined coefficients:

$$\alpha = \beta^*\,, \quad \alpha' = \beta'^*\,, \quad \alpha'' = \beta''^*\,. \tag{3.34}$$

3.3 The Hamiltonian as a sum of projectors

Having determined the $O(g)$ action of $\bar{\mathcal{Q}}(g)$ and $\bar{\mathcal{S}}(g)$, the one-loop Hamiltonian is easily obtained from

$$H' = 2\{\bar{\mathcal{S}}, \bar{\mathcal{Q}}\}\,. \tag{3.35}$$

This result suffers from a certain gauge ambiguity, as we may redefine

$$H'_{\ell,\ell+1} \to H'_{\ell,\ell+1} - K_\ell + K_{\ell+1}\,, \tag{3.36}$$

where $H'_{\ell,\ell+1}$ is the Hamiltonian density at sites $(\ell, \ell+1)$ and K_ℓ an arbitrary local operator at site ℓ. The total Hamiltonian $H' = \sum_\ell H'_{\ell,\ell+1}$ is guaranteed to be invariant under SU(1, 1). As in [32], we fix the gauge ambiguity (3.36) by demanding the stronger condition that the Hamiltonian density be SU(1, 1) invariant as well. We can then write $H'_{\ell,\ell+1}$ as a sum of projectors onto the SU(1, 1) irreps of the two-site state space, with coefficients determined by explicit evaluation on the primary of each module. The uplifting procedure is straightforward: one writes the full Hamiltonian as a sum over SU(2, 2|2) projectors and fixes the coefficients by comparison with the SU(1, 1) subsector, as each SU(1, 1) primary is also a SU(2, 2|2) descendant. We simply quote the results in the various subspaces.

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3.3.1 $\mathcal{V}\times\mathcal{V}$ and $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$

For $\mathcal{V}\times\mathcal{V}$ and $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$ we find,[4]

$$H_{12} = 0\times\mathcal{P}_{\bar{\mathcal{B}}_{2(0,0)}} + 2|\alpha|^2\sum_{q=1}^{\infty}2h(q)\mathcal{P}_{\left(\frac{q+1}{2},\frac{q-2}{2}\right)}\,, \tag{3.37}$$

$$H_{12} = 0\times\mathcal{P}_{\mathcal{B}_{-2(0,0)}} + 2\left(-2|\alpha|^2+\frac{|\alpha'|^2}{2}+\frac{|\alpha''|^2}{2}\right)\sum_{q=1}^{\infty}2h(q)\mathcal{P}_{\left(\frac{q-2}{2},\frac{q+1}{2}\right)}\,, \tag{3.38}$$

where $h(q)$ are the harmonic numbers, $h(q)=\sum_{j=1}^{q}\frac{1}{j}$ and $h(0)\equiv 0$. CPT invariance implies that these expressions should be identical, which imposes an extra restriction on $|\alpha'|^2$ and $|\alpha''|^2$ namely,

$$|\alpha'|^2+|\alpha''|^2 = 6|\alpha|^2\,. \tag{3.39}$$

Now, α' and α'' are parameters associated with $(\mathcal{X},\bar{\mathcal{X}})$ and $(\tilde{\mathcal{X}},\bar{\tilde{\mathcal{X}}})$ repectively. Parity (which is a symmetry of the theory, see section 4) interchanges the two, in order to have a parity invariant Hamiltonian we need to set

$$\alpha' = \sqrt{3}e^{i\theta}\alpha\,, \tag{3.40}$$

$$\alpha'' = -\sqrt{3}e^{i\theta}\alpha\,, \tag{3.41}$$

where θ is an arbitrary phase, which can be set to zero by a similarity transformation.

In the next subsections we present our results for the different two-site combinations.[5]

3.3.2 $\mathcal{V}\times\mathcal{X}$, $\bar{\mathcal{X}}\times\bar{\mathcal{V}}$, $\bar{\mathcal{V}}\times\mathcal{X}$ and $\bar{\mathcal{X}}\times\mathcal{V}$

$$\mathcal{V}\times\mathcal{X} = 2|\alpha|^2\sum_{q=0}^{\infty}\left(h(q+1)+h(q)-\frac{1}{2}\right)\mathcal{P}_{\left(\frac{q+1}{2},\frac{q-1}{2}\right)} = \tilde{\mathcal{X}}\times\mathcal{V}\,, \tag{3.42}$$

$$\bar{\mathcal{X}}\times\bar{\mathcal{V}} = 2|\alpha|^2\sum_{q=0}^{\infty}\left(h(q+1)+h(q)-\frac{1}{2}\right)\mathcal{P}_{\left(\frac{q-1}{2},\frac{q+1}{2}\right)} = \bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}\,, \tag{3.43}$$

$$\bar{\mathcal{V}}\times\mathcal{X} = 2|\alpha|^2\sum_{q=0}^{\infty}\left(h(q+2)+h(q)-\frac{1}{2}\right)\mathcal{P}_{\left(\frac{q}{2},\frac{q+1}{2}\right)} = \tilde{\mathcal{X}}\times\bar{\mathcal{V}}\,, \tag{3.44}$$

$$\bar{\mathcal{X}}\times\mathcal{V} = 2|\alpha|^2\sum_{q=0}^{\infty}\left(h(q+2)+h(q)-\frac{1}{2}\right)\mathcal{P}_{\left(\frac{q+1}{2},\frac{q}{2}\right)} = \mathcal{V}\times\bar{\tilde{\mathcal{X}}}\,. \tag{3.45}$$

[4] In this section we use as extra input the fact that $\bar{\mathcal{B}}_{2(0,0)}$ and $\mathcal{B}_{-2(0,0)}$ are protected multiplets.

[5] Some of the modules with low q are not present in the subsector so the corresponding coefficients are not determined by the algebraic constraints. However, these coefficients can be fixed by invoking CPT.

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3.3.3 $\bar{\mathcal{X}}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$ and $\bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$ (gauge contracted)

$$\bar{\mathcal{X}}\times\mathcal{X}=2|\alpha|^2\sum_{q=0}^{\infty}(h(q+1)+h(q)-1)\mathcal{P}_{\left(\frac{q}{2},\frac{q}{2}\right)}=\tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}\,, \tag{3.46}$$

$$\tilde{\mathcal{X}}\times\mathcal{X}=2|\alpha|^2\sum_{q=-1}^{\infty}(2h(q+1)-1)\mathcal{P}_{\left(\frac{q+1}{2},\frac{q}{2}\right)}\,, \tag{3.47}$$

$$\bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}=2|\alpha|^2\sum_{q=-1}^{\infty}(2h(q+1)-1)\mathcal{P}_{\left(\frac{q}{2},\frac{q+1}{2}\right)}\,. \tag{3.48}$$

3.3.4 $\mathcal{X}\times\bar{\mathcal{X}}$ (flavor contracted), $\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}$ (flavor contracted), $\mathcal{V}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\mathcal{V}$

This is the most involved subspace since we have mixing between different copies of the same superconformal multiplets. The Hamiltonian acting on this subspace is a 4×4 matrix,

$$H_{12}=3|\alpha|^2\left(\begin{array}{c|cccc} & \mathcal{X}\bar{\mathcal{X}} & \bar{\tilde{\mathcal{X}}}\tilde{\mathcal{X}} & \mathcal{V}\bar{\mathcal{V}} & \bar{\mathcal{V}}\mathcal{V}\\ \hline \bar{\mathcal{X}}\mathcal{X} & 1 & -1 & 0 & 0\\ \tilde{\mathcal{X}}\bar{\tilde{\mathcal{X}}} & -1 & 1 & 0 & 0\\ \bar{\mathcal{V}}\mathcal{V} & 0 & 0 & 0 & 0\\ \mathcal{V}\bar{\mathcal{V}} & 0 & 0 & 0 & 0 \end{array}\right)\mathcal{P}_{(0,0)} \tag{3.49}$$

$$+\,2|\alpha|^2\sum_{q=1}^{\infty}\left(\begin{array}{c|cccc} & \mathcal{X}\bar{\mathcal{X}} & \bar{\tilde{\mathcal{X}}}\tilde{\mathcal{X}} & \mathcal{V}\bar{\mathcal{V}} & \bar{\mathcal{V}}\mathcal{V}\\ \hline \bar{\mathcal{X}}\mathcal{X} & 0 & 0 & \sqrt{3}e^{-i\theta} & -\sqrt{3}\frac{e^{-i\theta}}{q+1}\\ \tilde{\mathcal{X}}\bar{\tilde{\mathcal{X}}} & 0 & 0 & -\sqrt{3}\frac{e^{-i\theta}}{q+1} & \sqrt{3}e^{-i\theta}\\ \bar{\mathcal{V}}\mathcal{V} & \sqrt{3}\frac{e^{i\theta}}{q(q+2)} & -\sqrt{3}\frac{e^{i\theta}}{q(q+1)(q+2)} & h(q+2)+h(q-1) & \frac{2}{q(q+1)(q+2)}\\ \mathcal{V}\bar{\mathcal{V}} & -\sqrt{3}\frac{e^{i\theta}}{q(q+1)(q+2)} & \sqrt{3}\frac{e^{i\theta}}{q(q+2)} & \frac{2}{q(q+1)(q+2)} & h(q+2)+h(q-1) \end{array}\right)\mathcal{P}_{\left(\frac{q}{2},\frac{q}{2}\right)}\,.$$

3.4 Scalar sector

Let us compare our results with the scalar sector computation of [30]. Apart from providing a check of our procedure, this comparison allows us to fix the overall normalization of the Hamiltonian in terms of the gauge theory 't Hooft coupling. The action of the Hamiltonian for the gauge contracted Q and $\tilde{Q}$ pairs can be obtained from (3.46) at $q=0$ and (3.47), (3.48) at $q=-1$,

$$H_{12}=2|\alpha|^2\left(\begin{array}{c|cccc} & \bar{Q}Q & \bar{Q}\bar{\tilde{Q}} & \tilde{Q}Q & \tilde{Q}\bar{\tilde{Q}}\\ \hline Q\bar{Q} & 0 & & & \\ Q\tilde{Q} & & -1 & & \\ \bar{\tilde{Q}}\bar{Q} & & & -1 & \\ \bar{\tilde{Q}}\tilde{Q} & & & & 0 \end{array}\right)\,, \tag{3.50}$$

in perfect agreement with equation (3.6) of [30] provided we identify $2|\alpha|^2=\lambda$. For the flavor contracted pairs we obtain from the first matrix in (3.49),

$$H_{12}=3|\alpha|^2\left(\begin{array}{c|cc} & Q\bar{Q} & \bar{\tilde{Q}}\tilde{Q}\\ \hline \bar{Q}Q & 1 & -1\\ \tilde{Q}\bar{\tilde{Q}} & -1 & 1 \end{array}\right)\,, \tag{3.51}$$

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again in agreement with [30]. It is interesting that the value of the transverse magnetic field for the Ising spin chain in the scalar sector, namely $h_{\text{Ising}} = N_f/N_c = 3$ [30], turns out to be determined by superconformal symmetry alone.

3.5 Open chain

The extension of the previous results to an open chain with adjoint or bi-fundamental flavor indices is straightforward. To obtain the full Hamiltonian we start considering the open chain states of the SU(1, 1) subsector,

$$\bar{\psi}_{ai}\dots Q^{aj}\,,\quad \bar{\psi}_{ai}\dots\bar{\tilde{\psi}}^{a\tilde{j}}\,,\quad \tilde{Q}_{a\tilde{i}}\dots Q^{aj}\,,\quad \tilde{Q}_{a\tilde{i}}\dots\bar{\tilde{\psi}}^{a\tilde{j}}\,, \tag{3.52}$$

where ${}_i{}^j$, ${}_{\tilde{i}}{}^{\tilde{j}}$ are projected to its adjoint irreducible component (the singlet is subtracted). One easily check that the gauge transformations (3.11)–(3.13) and (3.24)–(3.32) leave invariant open chains of this form. The uplift to the full Hamiltonian works just as for the closed chain. The upshot is that the Hamiltonian density derived for the closed chain applies with no modification to the open chains.

3.6 The harmonic action

In this section we present an explicit oscillator form of the Hamiltonian analogous to Beisert's harmonic action [12] for $\mathcal{N} = 4$ SYM.

3.6.1 $\mathcal{V}\times\mathcal{V}$ and $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$

For states in these two subspaces the action of the Hamiltonian is identical with that of $\mathcal{N} = 4$ SYM and of $\mathcal{N} = 2$ SCQCD. General states in $\mathcal{V}\times\mathcal{V}$ and $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$ can be written as

$$|s_1,\dots,s_n;A\rangle_{\mathcal{V}\times\mathcal{V}} = A^\dagger_{s_1A_1}\dots A^\dagger_{s_nA_n}|0\rangle\otimes|0\rangle\,, \tag{3.53}$$

$$|s_1,\dots,s_n;A\rangle_{\bar{\mathcal{V}}\times\bar{\mathcal{V}}} = A^\dagger_{s_1A_1}\dots A^\dagger_{s_nA_n}|\mathbf{d}_3\rangle\otimes|\mathbf{d}_3\rangle\,, \tag{3.54}$$

where $A^\dagger_A = (\mathbf{a}^\dagger_\alpha, \mathbf{b}^\dagger_{\dot\alpha}, \mathbf{c}^\dagger)$ and $s_i = 1, 2$ indicates in which site the oscillator sits. The action of the Hamiltonian on this state does not change the number of oscillators but merely shifts them from site 1 to site 2 (or vice versa) in all possible combinations. This can be written as

$$H_{12}|s_1,\dots,s_n;A\rangle_{\mathcal{V}\times\mathcal{V}} = \sum_{s'_1,\dots,s'_n} c_{n,n_{12},n_{21}}\delta_{C_1,0}\delta_{C_2,0}|s'_1,\dots,s'_n;A\rangle_{\mathcal{V}\times\mathcal{V}}\,, \tag{3.55}$$

$$H_{12}|s_1,\dots,s_n;A\rangle_{\bar{\mathcal{V}}\times\bar{\mathcal{V}}} = \sum_{s'_1,\dots,s'_n} c_{n,n_{12},n_{21}}\delta_{C_1,0}\delta_{C_2,0}|s'_1,\dots,s'_n;A\rangle_{\bar{\mathcal{V}}\times\bar{\mathcal{V}}}\,, \tag{3.56}$$

where the Kronecker deltas project onto states with zero central charge and n_{ij} counts the number of oscillators moving from site i to site j. The explicit formula for the function $c_{n,n_{12},n_{21}}$ is

$$c_{n,n_{12},n_{21}} = (-1)^{1+n_{12}n_{21}}\frac{\Gamma\left(\frac{1}{2}(n_{12}+n_{21})\right)\Gamma\left(1+\frac{1}{2}(n-n_{12}-n_{21})\right)}{\Gamma\left(1+\frac{1}{2}n\right)}\,, \tag{3.57}$$

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with $c_{n,0,0} = h(\frac{n}{2})$. In [12] it was proven that this function is a superconformal invariant and that it has the appropriate eigenvalues when acting on the $\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})}$ and $\hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})}$ modules, namely

$$H_{12}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})} = 2h(q)\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})}\,, \tag{3.58}$$

$$H_{12}\hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})} = 2h(q)\hat{\mathcal{C}}_{(\frac{q-2}{2},\frac{q+1}{2})}\,. \tag{3.59}$$

3.6.2 $\mathcal{V}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\mathcal{V}$, $\bar{\mathcal{X}}\times\bar{\mathcal{V}}$ and $\bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}$

General states in these four subspaces can be written as

$$|s_1,\ldots,s_n;A\rangle_{\mathcal{V}\times\mathcal{X}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|0\rangle\otimes|\mathbf{d}_1\rangle\,, \tag{3.60}$$

$$|s_1,\ldots,s_n;A\rangle_{\tilde{\mathcal{X}}\times\mathcal{V}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|\tilde{\mathbf{d}}_1\rangle\otimes|0\rangle\,, \tag{3.61}$$

$$|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{X}}\times\bar{\mathcal{V}}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|\mathbf{d}_1\mathbf{d}_2\rangle\otimes|\mathbf{d}_3\rangle\,, \tag{3.62}$$

$$|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|\mathbf{d}_3\rangle\otimes|\tilde{\mathbf{d}}_1\tilde{\mathbf{d}}_2\rangle\,, \tag{3.63}$$

where $|\mathbf{d}_i\rangle = \mathbf{d}_i^\dagger|0\rangle$.[6] The action of H_{12} for all these four subspaces is given by[7]

$$H_{12}|s_1,\ldots,s_n;A\rangle = \sum_{s'_1,\ldots,s'_n} c_{n+1,n_{12},n_{21}}|s_1,\ldots,s_n;A\rangle - \frac{1}{2}|s_1,\ldots,s_n;A\rangle\,. \tag{3.64}$$

Invariance under the superconformal group is guaranteed because we are using the $c_{n,n_{12},n_{21}}$ constants. We need only to check that this expression has the correct eigenvalues when acting on the corresponding $\mathcal{N}=1$ primaries (see appendix B.3), which can be easily done using an algebra software like `Mathematica`.

3.6.3 $\bar{\mathcal{V}}\times\mathcal{X}$, $\tilde{\mathcal{X}}\times\bar{\mathcal{V}}$, $\bar{\mathcal{X}}\times\mathcal{V}$ and $\mathcal{V}\times\bar{\tilde{\mathcal{X}}}$

In this case we have

$$|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{X}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|\mathbf{d}_3\rangle\otimes|\mathbf{d}_1\rangle\,, \tag{3.65}$$

$$|s_1,\ldots,s_n;A\rangle_{\tilde{\mathcal{X}}\times\bar{\mathcal{V}}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|\tilde{\mathbf{d}}_1\rangle\otimes|\mathbf{d}_3\rangle\,, \tag{3.66}$$

$$|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{X}}\times\mathcal{V}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|\mathbf{d}_1\mathbf{d}_2\rangle\otimes|0\rangle\,, \tag{3.67}$$

$$|s_1,\ldots,s_n;A\rangle_{\mathcal{V}\times\bar{\tilde{\mathcal{X}}}} = A^\dagger_{s_1A_1}\ldots A^\dagger_{s_nA_n}|0\rangle\otimes|\tilde{\mathbf{d}}_1\tilde{\mathbf{d}}_2\rangle\,, \tag{3.68}$$

and the action of H_{12} reads

$$H_{12}|s_1,\ldots,s_n;A\rangle = \sum_{s'_1,\ldots,s'_n} c_{n+2,n_{12},n_{21}}|s_1,\ldots,s_n;A\rangle - \frac{1}{2}|s_1,\ldots,s_n;A\rangle\,. \tag{3.69}$$

[6] The tilde in some of the $\mathbf{d}$ oscillators is just a reminder that we are looking at the $\tilde{\mathcal{X}}$ multiplet or its conjugate.

[7] To simplify the notation we will omit the Kronecker deltas $\delta_{C_1,0}\delta_{C_2,0}$.

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3.6.4 $\bar{\mathcal{X}} \times \mathcal{X}$ and $\tilde{\bar{\mathcal{X}}} \times \tilde{\mathcal{X}}$ (gauge contracted)

The states are given by

$$|s_1, \ldots, s_n; A\rangle_{\bar{\mathcal{X}} \times \mathcal{X}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\mathbf{d}_1 \mathbf{d}_2\rangle \otimes |\mathbf{d}_1\rangle \,, \tag{3.70}$$

$$|s_1, \ldots, s_n; A\rangle_{\tilde{\mathcal{X}} \times \tilde{\bar{\mathcal{X}}}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\tilde{\mathbf{d}}_1\rangle \otimes |\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2\rangle \,, \tag{3.71}$$

and the action of H_{12} is

$$H_{12}|s_1, \ldots, s_n; A\rangle = \sum_{s'_1, \ldots, s'_n} c_{n+1, n_{12}, n_{21}} |s_1, \ldots, s_n; A\rangle - |s_1, \ldots, s_n; A\rangle \,. \tag{3.72}$$

3.6.5 $\tilde{\mathcal{X}} \times \mathcal{X}$ and $\bar{\mathcal{X}} \times \tilde{\bar{\mathcal{X}}}$

The states are given by

$$|s_1, \ldots, s_n; A\rangle_{\tilde{\mathcal{X}} \times \mathcal{X}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\tilde{\mathbf{d}}_1\rangle \otimes |\mathbf{d}_1\rangle \,, \tag{3.73}$$

$$|s_1, \ldots, s_n; A\rangle_{\bar{\mathcal{X}} \times \tilde{\bar{\mathcal{X}}}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\mathbf{d}_1 \mathbf{d}_2\rangle \otimes |\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2\rangle \,, \tag{3.74}$$

and the action of H_{12} is

$$H_{12}|s_1, \ldots, s_n; A\rangle = \sum_{s'_1, \ldots, s'_n} \left(c_{n+2, n_{12}, n_{21}} - c_{n+2, n_{12}+1, n_{21}+1}\right) |s_1, \ldots, s_n; A\rangle - |s_1, \ldots, s_n; A\rangle \,. \tag{3.75}$$

3.6.6 $\mathcal{X} \times \bar{\mathcal{X}}$ (flavor contracted), $\tilde{\bar{\mathcal{X}}} \times \tilde{\mathcal{X}}$ (flavor contracted), $\mathcal{V} \times \bar{\mathcal{V}}$ and $\bar{\mathcal{V}} \times \mathcal{V}$

The states are

$$|s_1, \ldots, s_n; A\rangle_{\mathcal{X} \times \bar{\mathcal{X}}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\mathbf{d}_1\rangle \otimes |\mathbf{d}_1 \mathbf{d}_2\rangle \,, \tag{3.76}$$

$$|s_1, \ldots, s_n; A\rangle_{\tilde{\bar{\mathcal{X}}} \times \tilde{\mathcal{X}}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2\rangle \otimes |\tilde{\mathbf{d}}_1\rangle \,, \tag{3.77}$$

$$|s_1, \ldots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |0\rangle \otimes |\mathbf{d}_3\rangle \,, \tag{3.78}$$

$$|s_1, \ldots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}} = A^\dagger_{s_1 A_1} \ldots A^\dagger_{s_n A_n} |\mathbf{d}_3\rangle \otimes |0\rangle \,. \tag{3.79}$$

The action of H_{12} is given by

$$\begin{aligned} H_{12}|s_1, \ldots, s_n; A\rangle_{\mathcal{X} \times \bar{\mathcal{X}}} = & \sum_{s'_1, \ldots, s'_n} c_{n+3, n_{12}+1, n_{21}+1} |s_1, \ldots, s_n; A\rangle_{\mathcal{X} \times \bar{\mathcal{X}}} \\ & + 3 \sum_{s'_1, \ldots, s'_n} c_{n+3, n_{12}+1, n_{21}+2} |s_1, \ldots, s_n; A\rangle_{\tilde{\bar{\mathcal{X}}} \times \tilde{\mathcal{X}}} \\ & + \sqrt{3} e^{-i\theta} \sum_{s'_1, \ldots, s'_n} c_{n+3, n_{12}, n_{21}+1} |s_1, \ldots, s_n; A\rangle_{\mathcal{V} \times \bar{\mathcal{V}}} \\ & + \sqrt{3} e^{-i\theta} \sum_{s'_1, \ldots, s'_n} c_{n+3, n_{12}+1, n_{21}+1} |s_1, \ldots, s_n; A\rangle_{\bar{\mathcal{V}} \times \mathcal{V}} \,, \end{aligned} \tag{3.80}$$

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$$
\begin{aligned}
H_{12}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{X}}\times\tilde{\mathcal{X}}} = & \sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+1,n_{21}+1}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{X}}\times\tilde{\mathcal{X}}} \\
& -3\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+1,n_{21}+2}|s_1,\ldots,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} \\
& +\sqrt{3}e^{-i\theta}\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+1,n_{21}+1}|s_1,\ldots,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
& +\sqrt{3}e^{-i\theta}\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+1,n_{21}}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}}\,, \qquad (3.81)\\
H_{12}|s_1,\ldots,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} = & \sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12},n_{21}}|s_1,\ldots,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
& +\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+2,n_{21}+1}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \\
& +\sqrt{3}e^{i\theta}\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+1,n_{21}}|s_1,\ldots,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} \\
& -\sqrt{3}e^{i\theta}\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12},n_{21}+2}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{X}}\times\tilde{\mathcal{X}}}\,, \qquad (3.82)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{12}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} = & \sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12},n_{21}}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{V}}\times\mathcal{V}} \\
& -\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+2,n_{21}+1}|s_1,\ldots,s_n;A\rangle_{\mathcal{V}\times\bar{\mathcal{V}}} \\
& -\sqrt{3}e^{i\theta}\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12}+2,n_{21}}|s_1,\ldots,s_n;A\rangle_{\mathcal{X}\times\bar{\mathcal{X}}} \\
& +\sqrt{3}e^{i\theta}\sum_{s'_1,\ldots,s'_n} c_{n+3,n_{12},n_{21}+1}|s_1,\ldots,s_n;A\rangle_{\bar{\mathcal{X}}\times\tilde{\mathcal{X}}}\,. \qquad (3.83)
\end{aligned}
$$

4 Spectral analysis

Spectral studies in planar $\mathcal{N}=4$ SYM have shown the systematic presence of degenerate pairs of states of opposite "parity", where parity is the $\mathbb{Z}_2$ symmetry associated with complex conjugation of the SU(N) gauge group [11, 13, 14, 31]. These degeneracies persist at higher loops, but are lifted by non-planar corrections. This phenomenon is naturally explained by the integrable structures of planar $\mathcal{N}=4$ SYM: the theory admits higher conserved charges that are parity-odd and map the degenerate eigenstates. In some models it is even possible to prove that the existence of parity pairs is a sufficient condition for integrability [31].

The upshot is that in $\mathcal{N}=4$ SYM the existence of parity pairs is one of the many pieces of evidence for the complete integrability of the theory. With this precedent in mind, we can look forward to a similar analysis in $\mathcal{N}=2$ SCQCD and in $\mathcal{N}=1$ SQCD. In this section we determine the low-lying spectrum of the one-loop dilation operator of both theories, in the closed non-compact subsectors that were used to uplift the full Hamiltonian.

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4.1 $\mathcal{N}=2$ SCQCD

We start our analysis with $\mathcal{N}=2$ SCQCD and with the more general quiver theory that interpolates between the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SCQCD. For background material we refer the reader to [32, 34, 35] where this model was studied extensively.

4.1.1 Parity

The first thing we need to do is define a meaningful parity operation. We will take $\mathcal{N}=4$ SYM as our starting point where parity amounts to conjugation of the SU(N) gauge group. Under parity, the Lie algebra generators transform as

$$T^a{}_b \to -(T^a{}_b)^* = -T^b{}_a\,, \tag{4.1}$$

where we have used hermiticity to trade conjugation by transposition.

Now, as reviewed in [33, 34] $\mathcal{N}=2$ SCQCD can be thought of as a limit of a two-parameter $(g,\check{g})$ quiver theory with gauge group $\mathrm{SU}(N_c)\times\mathrm{SU}(N_{\check{c}})$ (with $N_{\check{c}}\equiv N_c$): one has $\mathcal{N}=2$ SCQCD at $\check{g}=0$ and the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM at $g=\check{g}$. Starting from $\mathcal{N}=4$ SYM with gauge group SU($2N_c$) the $\mathbb{Z}_2$ orbifold theory is obtained by the projection

$$A_{\alpha\dot\alpha} = \begin{pmatrix} A^a_{\alpha\dot\alpha\, b} & 0 \\ 0 & \check{A}^{\check{a}}_{\alpha\dot\alpha\,\check{b}} \end{pmatrix}, \quad Z = \begin{pmatrix} \phi^a{}_b & 0 \\ 0 & \check{\phi}^{\check{a}}{}_{\check{b}} \end{pmatrix}, \tag{4.2}$$

$$\lambda_{\mathcal{I}} = \begin{pmatrix} \lambda^a_{\mathcal{I}b} & 0 \\ 0 & \check{\lambda}^{\check{a}}_{\mathcal{I}\check{b}} \end{pmatrix}, \qquad \lambda_{\hat{\mathcal{I}}} = \begin{pmatrix} 0 & \epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\psi^{a\hat{\mathcal{J}}}_{\check{a}} \\ \epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\tilde{\psi}^{\check{b}\hat{\mathcal{J}}}_{b} & 0 \end{pmatrix}, \tag{4.3}$$

$$\mathcal{X}_{\mathcal{I}\hat{\mathcal{I}}} = \begin{pmatrix} 0 & Q^{\ a}_{\mathcal{I}\hat{\mathcal{I}}\check{a}} \\ -\epsilon_{\mathcal{I}\mathcal{J}}\epsilon_{\hat{\mathcal{I}}\hat{\mathcal{J}}}\bar{Q}^{\check{b}\hat{\mathcal{J}}\mathcal{J}}_{\ \ b} & 0 \end{pmatrix}, \tag{4.4}$$

where $\mathcal{I},\hat{\mathcal{I}}=1,2$. The parity operation described above implies the following transformations. For the fields in the vector multiplets,

$$\begin{aligned} &A^a_{\alpha\dot\alpha\, b} \leftrightarrow -A^b_{\alpha\dot\alpha\, a} \qquad \lambda^a_{\mathcal{I}b} \leftrightarrow -\lambda^b_{\mathcal{I}a} \qquad \phi^a{}_b \leftrightarrow -\phi^b{}_a\,, \\ &\check{A}^{\check{a}}_{\alpha\dot\alpha\,\check{b}} \leftrightarrow -\check{A}^{\check{b}}_{\alpha\dot\alpha\,\check{a}} \qquad \check{\lambda}^{\check{a}}_{\mathcal{I}\check{b}} \leftrightarrow -\check{\lambda}^{\check{b}}_{\mathcal{I}\check{a}} \qquad \check{\phi}^{\check{a}}{}_{\check{b}} \leftrightarrow -\check{\phi}^{\check{b}}{}_{\check{a}}\,, \end{aligned} \tag{4.5}$$

and analogous expressions for the conjugate fields. For the fields in the hypermultiplets,

$$\psi^a_{\hat{\mathcal{I}}\check{b}} \leftrightarrow -\tilde{\psi}^{\check{b}}_{\hat{\mathcal{I}}a} \quad Q^a_{\mathcal{I}\hat{\mathcal{I}}\check{b}} \leftrightarrow \bar{Q}^{\check{b}}_{\mathcal{I}\hat{\mathcal{I}}a}\,, \tag{4.6}$$

and analogous expressions for the conjugate fields. These transformations remain a symmetry also away from the orbifold point (that is, for arbitrary $(g,\check{g})$), as can be easily checked by inspection of the Lagrangian (see e.g. [34] for the explicit expression of the Lagrangian). This implies that the parity operation commutes with the dilation operator to all loops. Its action on single-trace states is then given by

$$P|A_1\dots A_L\rangle = (-1)^{L+k(k+1)/2}|A_L\dots A_1\rangle\,, \tag{4.7}$$

where k is the number of fermions and we replace $\psi\leftrightarrow\tilde{\psi}$, $\bar{\psi}\leftrightarrow\bar{\tilde{\psi}}$, $Q\leftrightarrow-\bar{Q}$.

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L	r	Δ_0	$\delta\Delta^P\,[2g^2_{\rm YM}N/\pi^2]$
3	$\frac{3}{2}$	7.5	$\frac{5}{4}^{\pm}$
		9.5	$\frac{133}{96}^{\pm}$
		10.5	$\frac{761}{480}^{\pm}$, $\frac{761}{560}^{-}$
		11.5	$\frac{179}{120}^{\pm}$
4	2	8	$\frac{5}{4}^{\pm}$
		9	$\frac{1}{48}\left(73\pm\sqrt{37}\right)^{-}$
		10	$\frac{19}{12}^{\pm}$, $\frac{133}{96}^{\pm}$
		11	$P_3(x)^{-}$, $\frac{761}{480}^{\pm}$
5	$\frac{5}{2}$	9.5	$\frac{1}{48}\left(73\pm\sqrt{37}\right)^{+}$
		10.5	$\frac{7}{4}^{\pm}$, $\frac{19}{12}^{\pm}$
		11.5	$P_3(x)^{-}$, $\frac{1}{24}\left(43\pm\sqrt{5}\right)^{\pm}$
Paired eigenvalues: ~ 69 %			

Table 2. SU(1, 1) primaries with maximal r-charge $\left(r=\frac{L}{2}\right)$ in the SU(1, 1) × SU(1|1) × SU(1|1) × U(1) sector of $\mathcal{N}=2$ SCQCD. We have omitted the one-dimensional subspaces where there is no room for a parity pair.

4.1.2 Diagonalization

We consider the SU(1, 1) × SU(1|1) × SU(1|1) × U(1) subsector defined in [32]. We focus on SU(1, 1) primaries (since descendants have the same anomalous dimensions) and take with no loss of generalities $r \geq 0$. Table 2 corresponds to states with maximal r-charge. This subsector is made exclusively out of $\{\mathcal{D}^k\bar{\lambda}\}$ and is therefore identical to the SU(1, 1) subsector used in [31] to obtain the complete dilation operator of $\mathcal{N}=4$ SYM. Being a subsector of $\mathcal{N}=4$, it is integrable and, indeed, our results indicate that degenerate states with opposite parity show up consistently at each stage of the diagonalization. The notation $P_n(x)$ denotes the roots of a polynomial of order n, we will not write the polynomial explicitly because we are really interested in the amount of parity pairs and not in the actual values of the energies. We will denote by $P_n(x)$ all the polynomials of order n we encounter, even if they are different from each other.

Being identical to the analogous $\mathcal{N}=4$ SYM sector, we cannot use the results of table 2 as a test for integrability. The true dynamics of $\mathcal{N}=2$ SCQCD is encoded in subspaces where the r-charge is not maximal. For this we need states with Q and $\bar{Q}$. Our results are presented in table 3. As opposed to the results of table 2 the presence of parity pairs here is less systematic.

More insight is obtained if we also look at the $\mathbb{Z}_2$ orbifold ($\breve{g}=g$). For the the orbifold theory (and for the whole interpolating theory with general $\breve{g}$, g) we have an $SU(2)_L$ symmetry not present in $\mathcal{N}=2$ SCQCD, so to make the analysis more transparent we restrict the diagonalization to $SU(2)_L$ singlets. Our results for the $\mathbb{Z}_2$ orbifold are shown

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L	r	Δ_0	$\delta\Delta^P\,[2g^2_{\rm YM}N/\pi^2]$
3	$\frac{1}{2}$	4.5	$\frac{3}{4}^-$, $\frac{3}{4}^-$, $\frac{3}{8}^+$
		5.5	$\frac{15}{16}^-$, $\frac{1}{24}\left(16\pm\sqrt{31}\right)^-$, $\frac{1}{32}\left(21\pm\sqrt{57}\right)^+$
		6.5	$\frac{25}{24}^-$, $\frac{25}{24}^-$, $\frac{25}{48}^+$
			$\frac{1}{96}\left(81\pm\sqrt{561}\right)^-$, $\frac{1}{96}\left(83\pm\sqrt{409}\right)^+$
4	0	5	$\frac{3}{4}^\pm$, $\frac{3}{8}^-$
		6	$1^\pm$, $\frac{15}{16}^+$, $\frac{15}{16}^+$
			$\frac{1}{32}\left(21\pm\sqrt{57}\right)^-$, $\frac{1}{32}\left(21\pm\sqrt{57}\right)^-$
			$\frac{1}{24}\left(16\pm\sqrt{31}\right)^+$, $\frac{5}{8}$, 0^+
	1	6	1^-, $\frac{15}{16}^+$, $\frac{1}{32}\left(21\pm\sqrt{57}\right)^-$
		7	$\frac{5}{4}^\pm$, $\frac{9}{8}^-$, $\frac{25}{24}^+$
			$\frac{1}{16}\left(16\pm\sqrt{6}\right)^+$, $\frac{1}{96}\left(81\pm\sqrt{561}\right)^+$
			$\frac{1}{96}\left(83\pm\sqrt{409}\right)^-$
5	$\frac{1}{2}$	6.5	$1^\pm$, 1^+, $\frac{15}{16}^-$
			$\frac{1}{32}\left(21\pm\sqrt{57}\right)^+$
Paired eigenvalues: 16 %			

Table 3. SU(1, 1) primaries with $0 \le r < \frac{L}{2}$ in the SU(1, 1) × SU(1|1) × SU(1|1) × U(1) sector of $\mathcal{N} = 2$ SCQCD.

in table 4. As in the case with maximal r-charge, parity pairs show up consistently. This is again expected because this theory is known to be integrable [36].

Finally we look at how some sample parity pairs of the orbifold theory evolve when we move away from the orbifold point. Our results are shown in table 5. We see that for arbitrary values of $\kappa \equiv \check{g}/g$ the pairs are lifted and they are not in general recovered in the SCQCD limit $\kappa \to 0$. (Note that not all $SU(2)_L$ gauge singlets evolve to legitimate states of $\mathcal{N} = 2$ SCQCD, which must obey the stronger condition of being $SU(N_f)$ singlets. In the last column of table 5 we indicate whether the states belong or not to $\mathcal{N} = 2$ SCQCD.)

4.2 $\mathcal{N} = 1$ SQCD

We now repeat the same analysis for $\mathcal{N} = 1$ SQCD. Inspired by the transformation used in the $\mathcal{N} = 2$ theory, we define the following parity operation. For the fields in the vector multiplet,

$$A^a_{\mu b} \leftrightarrow -A^b_{\mu a} \qquad \lambda^a{}_b \leftrightarrow -\lambda^b{}_a\,, \tag{4.8}$$

and analogous expressions for the conjugate fields. For the chiral multipets,

$$Q^{ai} \leftrightarrow -\tilde{Q}_{\tilde{\imath}a}, \qquad \psi^{ai} \leftrightarrow -\tilde{\psi}_{\tilde{\imath}a}\,, \tag{4.9}$$

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L	r	Δ_0	$\delta\Delta^P\,[2g^2_{\rm YM}N/\pi^2]$
3	$\frac{1}{2}$	4.5	$\frac{1}{2}^+,\ \frac{3}{4}^-,\ \frac{3}{4}^-,\ \frac{3}{4}^-$
			$\frac{3}{4}^\pm$
		5.5	$\frac{3}{4}^-,\ \frac{7}{8}^-,\ \frac{25}{24}^-,\ \frac{1}{2}^+$
			$\frac{15}{16}^\pm,\ \frac{1}{32}\left(27\pm\sqrt{57}\right)^\pm$
		6.5	$\frac{3}{4}^+,\ \frac{25}{24}^-,\ \frac{25}{24}^-,\ \frac{25}{24}^-$
			$\frac{5}{4}^\pm,\ \frac{15}{16}^\pm,\ \frac{25}{24}^\pm$
			$\frac{1}{96}\left(93\pm\sqrt{249}\right)^\pm$
4	0	5	$\frac{1}{2}^-,\ \frac{1}{2}^-,\ \frac{1}{8}\left(5\pm\sqrt{13}\right)^-$
			$\frac{3}{4}^\pm,\ \frac{3}{4}^\pm$
		6	$\frac{7}{8}^+,\ \frac{25}{24}^+,\ \frac{1}{2}^-,\ \frac{1}{2}^-$
			$\frac{1}{8}\left(5\pm\sqrt{5}\right)^+$
			$\frac{3}{4}^\pm,\ \frac{5}{4}^\pm,\ \frac{5}{8}^\pm,\ \frac{7}{8}^\pm$
			$\frac{15}{16}^\pm,\ \frac{15}{16}^\pm,\ \frac{1}{4}\left(3\pm\sqrt{2}\right)^\pm$
			$\frac{1}{32}\left(27\pm\sqrt{57}\right)^\pm,\ \frac{1}{32}\left(27\pm\sqrt{57}\right)^\pm$
Paired eigenvalues: $\sim 68\ \%$			

Table 4. SU(1, 1) primaries with $0 \le r < \frac{L}{2}$ in the SU(1, 1) × SU(1|1) × SU(1|1) × U(1) sector of the orbifold theory ($\breve{g} = g$). We have restricted the diagonalization to $\mathrm{SU}(2)_L$ singlets.

and analogous expressions for the conjugate fields. Again, these transformations are a symmetry of the Lagrangian and therefore commute with the dilation operator to all loops. The action on single-trace states is given by

$$P|A_1 \dots A_L\rangle = (-1)^{L+k(k+1)/2}|A_L \dots A_1\rangle\,, \tag{4.10}$$

where k is the number of fermions and we make the replacements $\psi \leftrightarrow \tilde{\psi}$, $\bar{\psi} \leftrightarrow \bar{\tilde{\psi}}$, $Q \leftrightarrow \tilde{Q}$, $\bar{Q} \leftrightarrow \bar{\tilde{Q}}$. Our results for the diagonalization of generalized single-trace operators of length $L \le 5$ are shown in table 6. We restrict to states with r-charge $0 < r < L$. (We omit the sectors with $r = L$ and $r = 0$, which are spanned respectively by $\{\lambda_k\}$ and $\{\bar{\mathcal{F}}_k\}$. These sectors are isomorphic to the analogous sectors in $\mathcal{N} = 4$ SYM and thus inherit their integrability.) The results are qualitatively similar to the ones for $\mathcal{N} = 2$ SCQCD: there are a few parity pairs, but their presence is not as striking and systematic as in $\mathcal{N} = 4$ SYM.

5 Discussion

We have used superconformal invariance to constrain the planar one-loop dilation operator for $\mathcal{N} = 1$ SQCD. The structure of the calculation is similar to the one in $\mathcal{N} = 2$ SCQCD [32], and leads to the same surprising result: the one-loop Hamiltonian is uniquely fixed by symmetry. We have worked in the "electric" description of the theory, at the

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L	r	Δ_0	$\kappa = 1$	$\kappa = 0.7$	$\kappa = 0.3$	$\kappa = 0$	SCQCD
3	$\frac{1}{2}$	5.5	$\frac{1}{32}\left(27+\sqrt{57}\right)$	0.97	0.94	$\frac{15}{16}$	Yes
			$\frac{1}{32}\left(27+\sqrt{57}\right)$	0.95	0.90	$\frac{1}{32}\left(27+\sqrt{57}\right)$	Yes
3	$\frac{1}{2}$	5.5	$\frac{1}{32}\left(27-\sqrt{57}\right)$	0.39	0.08	0	No
			$\frac{1}{32}\left(27-\sqrt{57}\right)$	0.47	0.30	$\frac{1}{4}$	No
3	$\frac{1}{2}$	6.5	$\frac{5}{4}$	1.12	1.09	$\frac{1}{96}\left(81+\sqrt{561}\right)$	Yes
			$\frac{5}{4}$	1.11	1.08	$\frac{1}{96}\left(83+\sqrt{409}\right)$	Yes
3	$\frac{1}{2}$	6.5	$\frac{1}{96}\left(93+\sqrt{249}\right)$	0.81	0.63	$\frac{1}{96}\left(81-\sqrt{561}\right)$	Yes
			$\frac{1}{96}\left(93+\sqrt{249}\right)$	0.82	0.68	$\frac{1}{96}\left(83-\sqrt{409}\right)$	Yes
3	$\frac{1}{2}$	6.5	$\frac{1}{96}\left(93-\sqrt{249}\right)$	0.48	0.09	0	No
			$\frac{1}{96}\left(93-\sqrt{249}\right)$	0.57	0.31	$\frac{1}{4}$	No
4	0	6	$\frac{5}{4}$	1.08	1.01	1	Yes
			$\frac{5}{4}$	1.06	1.01	1	Yes
4	0	6	$\frac{5}{8}$	0.45	0.29	$\frac{1}{4}$	No
			$\frac{5}{8}$	0.49	0.28	$\frac{1}{4}$	No
4	0	6	$\frac{1}{4}\left(3+\sqrt{2}\right)$	0.74	0.54	$\frac{1}{2}$	No
			$\frac{1}{4}\left(3+\sqrt{2}\right)$	0.81	0.66	$\frac{5}{8}$	Yes
4	0	6	$\frac{1}{4}\left(3-\sqrt{2}\right)$	0.26	0.06	0	No
			$\frac{1}{4}\left(3-\sqrt{2}\right)$	0.30	0.08	0	No

Table 5. Examples of evolution of $\mathbb{Z}_2$ orbifold pairs for different values of the parameter $\kappa = \frac{\tilde{g}}{g}$.

Banks-Zaks fixed point near the upper edge of the conformal window. It would be interesting to apply the same strategy to the dual magnetic theory, at the Banks-Zaks fixed point in the lower edge of the conformal window.

The recent discovery [30] that in the scalar sector of $\mathcal{N} = 1$ SQCD the planar one-loop dilation operator is captured by the integrable Ising chain in transverse magnetic field was one of the motivations of our work. A skeptic may point out that this identification was kinematically foreordained, since no other structure is allowed (though the precise value of the transverse magnetic field does require dynamical input). The same could be said about the SU(2) scalar sector of $\mathcal{N} = 4$ SYM, which is *a priori* guaranteed to take the form of the Heisenberg spin chain, but not (say) for the SO(6) sector considered in the seminal paper [10]. More convincing tests of one-loop integrability require the complete Hamiltonian, which we have obtained in this paper. Our analysis indicates that the presence of degenerate parity pairs, a hallmark of integrability, is not as systematic in $\mathcal{N} = 1$ SQCD (or in $\mathcal{N} = 2$ SCQCD) as it is in $\mathcal{N} = 4$ SYM. These preliminary results are not particularly encouraging, but should of course be taken *cum grano salis*.

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L	r	Δ_0	$\delta\Delta^P\,[g^2_{\rm YM}N/\pi^2]$
3	1	4.5	$\frac{3}{4}^-$, $\frac{3}{8}^+$
		5.5	$\frac{1}{96}\left(67\pm\sqrt{457}\right)^-$, $P_3(x)^+$
		6.5	$\frac{25}{24}^-$, $\frac{1}{96}\left(81\pm\sqrt{473}\right)^-$, $P_3(x)^+$
			$\frac{25}{48}^\pm$
	2	4	$\frac{3}{16}^-$, $\frac{9}{16}^+$
		5	$\frac{7}{16}^-$, $\frac{7}{24}^-$, $\frac{13}{16}^+$, $\frac{1}{24}\left(13\pm\sqrt{39}\right)^+$
4	1	6.5	$\frac{3}{4}^+$, $\frac{9}{8}^-$
		7.5	$P_3(x)^-$, $P_4(x)^+$
	2	6	$\frac{1}{16}\left(9\pm\sqrt{37}\right)^+$, $\frac{1}{96}\left(67\pm\sqrt{457}\right)^+$, $P_3(x)^-$
			$\frac{9}{16}^\pm$, $0^\pm$
		7	$\frac{1}{96}\left(81\pm\sqrt{473}\right)^+$, $P_3(x)^-$, $P_7(x)^+$, $P_8(x)^-$
			$\frac{25}{48}^\pm$
	3	5.5	$\frac{7}{16}^-$, $\frac{13}{16}^+$
		6.5	$\frac{11}{16}^+$, $\frac{17}{16}^-$, $\frac{1}{96}\left(71\pm\sqrt{553}\right)^+$, $P_3(x)^-$
5	3	6.5	$\frac{1}{16}\left(9\pm\sqrt{37}\right)^-$, $\frac{9}{16}^\pm$
Paired eigenvalues: ~ 13 %			

Table 6. SU(1, 1) primaries with $0 < r < L$ in the SU(1, 1) × U(1|1) sector of $\mathcal{N} = 1$ SQCD.

If the complete theory turns out to be non-integrable, we can still ask whether there is scope for integrability in some closed subsectors. The question is really about all-loop integrability, since at one loop some simple sectors may be "accidentally" integrable. This may well be the case for the scalar sector of [30], which is only closed to lowest order: at higher orders the scalars mix with every other state, so either the whole theory turns out to be integrable or the integrability of the scalar spin chain is a one-loop accident. More promising is the situation in the sectors that happen to coincide at one-loop with analogous sectors of $\mathcal{N} = 4$ SYM, thus inheriting its integrability properties, but that remain closed to all loops. The largest such sector is the SU(2, 1|1) sector spanned by the letters $\{\mathcal{D}^k_{+\dot\alpha}\,\lambda_+\,,\ \mathcal{D}^k_{+\dot\alpha}\,\mathcal{F}_{++}\}$. While at one loop its Hamiltonian coincides with that of $\mathcal{N} = 4$ SYM, it will start differing from it at sufficiently high order. It will be very interesting to investigate whether integrability is preserved.[8] The SU(2, 1|1) sector exists

[8]The same can be asked for the SU(2, 1|2) sector of the $\mathcal{N} = 2$ interpolating theory (in particular of $\mathcal{N} = 2$ SCQCD) consisting of the letters $\{\mathcal{D}^k_{+\dot\alpha}\,\phi\,, \mathcal{D}^k_{+\dot\alpha}\,\lambda_{\mathcal{I}+}\,,\ \mathcal{D}^k_{+\dot\alpha}\,\mathcal{F}_{++}\}$. Here the question is even sharper since on the ϕ vacuum the two-body magnon S-matrix is completely fixed by symmetry (up to the overall dynamical phase) to be the SU(2|2) S-matrix of [20], which automatically satisfies the Yang-Baxter equation. Still, integrability is by no means obvious, since factorization of the n-body S-matrix is a stronger condition than Yang-Baxter. If this sector turns out to be integrable to all loops, its difference from the analogous sector of $\mathcal{N} = 4$ SYM would be fully encoded in the expression of the dynamical phase and of the magnon dispersion relation. The difference with $\mathcal{N} = 4$ SYM should start appearing at three loops [37].

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Shortening Conditions				Multiplet
$\mathcal{B}$	$\mathcal{Q}_\alpha \lvert r\rangle^{h.w.} = 0$	$j_1 = 0$	$\Delta = -\frac{3}{2}r$	$\mathcal{B}_{r(0,j_2)}$
$\bar{\mathcal{B}}$	$\bar{\mathcal{Q}}_{\dot\alpha} \lvert r\rangle^{h.w.} = 0$	$j_2 = 0$	$\Delta = \frac{3}{2}r$	$\bar{\mathcal{B}}_{r(j_1,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B} \cap \bar{\mathcal{B}}$	$j_1, j_2, r = 0$	$\Delta = 0$	$\hat{\mathcal{B}}$
$\mathcal{C}$	$\epsilon^{\alpha\beta}\mathcal{Q}_\beta \lvert r\rangle^{h.w.}_\alpha = 0$		$\Delta = 2 + 2j_1 - \frac{3}{2}r$	$\mathcal{C}_{r(j_1,j_2)}$
	$(\mathcal{Q})^2 \lvert r\rangle^{h.w.} = 0$ for $j_1 = 0$		$\Delta = 2 - \frac{3}{2}r$	$\mathcal{C}_{r(0,j_2)}$
$\bar{\mathcal{C}}$	$\epsilon^{\dot\alpha\dot\beta}\bar{\mathcal{Q}}_{\dot\beta} \lvert r\rangle^{h.w.}_{\dot\alpha} = 0$		$\Delta = 2 + 2j_2 + \frac{3}{2}r$	$\bar{\mathcal{C}}_{r(j_1,j_2)}$
	$(\bar{\mathcal{Q}})^2 \lvert r\rangle^{h.w.} = 0$ for $j_2 = 0$		$\Delta = 2 + \frac{3}{2}r$	$\bar{\mathcal{C}}_{r(j_1,0)}$
$\hat{\mathcal{C}}$	$\mathcal{C} \cap \bar{\mathcal{C}}$	$\frac{3}{2}r = (j_1 - j_2)$	$\Delta = 2 + j_1 + j_2$	$\hat{\mathcal{C}}_{(j_1,j_2)}$
$\mathcal{D}$	$\mathcal{B} \cap \bar{\mathcal{C}}$	$j_1 = 0, -\frac{3}{2}r = j_2 + 1$	$\Delta = -\frac{3}{2}r = 1 + j_2$	$\mathcal{D}_{(0,j_2)}$
$\bar{\mathcal{D}}$	$\bar{\mathcal{B}} \cap \mathcal{C}$	$j_2 = 0, \frac{3}{2}r = j_1 + 1$	$\Delta = \frac{3}{2}r = 1 + j_1$	$\bar{\mathcal{D}}_{(j_1,0)}$

Table 7. Possible shortening conditions for the $\mathcal{N} = 1$ superconformal algebra.

of course also in the dual magnetic theory, so its integrability may allow to interpolate across the whole conformal window. We look forward to future investigations of this scenario.

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A $\mathcal{N} = 1$ superconformal multiplets

In this appendix we summarize some basic facts about $\mathcal{N} = 1$ superconformal representation theory [9]. A generic long multiplet $\mathcal{A}^{\Delta}_{r(j_1,j_2)}$ is generated by the action of 4 Poincaré supercharges $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot\alpha}$ on a superconformal primary which by definition is annihilated by all the conformal supercharges $\mathcal{S}$. In table 7 we have summarized the possible shortening and semi-shortening conditions.

B Oscillator representation

In this appendix we describe the oscillator representation of the $\mathcal{N} = 1$ superconformal algebra SU(2, 2|1). We introduce two sets of bosonic oscillators $(\mathbf{a}^\alpha, \mathbf{a}^\dagger_\alpha)$, $(\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\alpha})$ and one fermionic oscillator $(\mathbf{c}, \mathbf{c}^\dagger)$, where $(\alpha, \dot\alpha)$ are Lorentz indices. In addition we will need three more "auxiliary" fermionic operators $(\mathbf{d}_i, \mathbf{d}^\dagger_i)$, $i = 1, 2, 3$. The non-zero (anti)commutation

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relations are

$$[\mathbf{a}^\alpha, \mathbf{a}^\dagger_\beta] = \delta^\alpha_\beta \,, \tag{B.1}$$

$$[\mathbf{b}^{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta^{\dot\alpha}_{\dot\beta} \,, \tag{B.2}$$

$$\{\mathbf{c}, \mathbf{c}^\dagger\} = 1 \,, \tag{B.3}$$

$$\{\mathbf{d}_i, \mathbf{d}^\dagger_j\} = \delta_{ij} \,. \tag{B.4}$$

In this oscillator representation the generators of SU(2, 2|1) are given by

$$\mathcal{L}_\beta{}^\alpha = \mathbf{a}^\dagger_\beta \mathbf{a}^\alpha - \frac{1}{2}\delta^\alpha_\beta \mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma \,, \tag{B.5}$$

$$\dot{\mathcal{L}}_{\dot\beta}{}^{\dot\alpha} = \mathbf{b}^\dagger_{\dot\beta} \mathbf{b}^{\dot\alpha} - \frac{1}{2}\delta^{\dot\alpha}_{\dot\beta} \mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} \,, \tag{B.6}$$

$$r = \mathbf{c}^\dagger \mathbf{c} - \frac{1}{3}\mathbf{d}^\dagger_1 \mathbf{d}_1 - \frac{1}{3}\mathbf{d}^\dagger_2 \mathbf{d}_2 - \mathbf{d}^\dagger_3 \mathbf{d}_3 \,, \tag{B.7}$$

$$D = 1 + \frac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} \,, \tag{B.8}$$

$$C = 1 - \frac{1}{2}\mathbf{a}^\dagger_\gamma \mathbf{a}^\gamma + \frac{1}{2}\mathbf{b}^\dagger_{\dot\gamma} \mathbf{b}^{\dot\gamma} - \frac{1}{2}\mathbf{c}^\dagger \mathbf{c} - \frac{1}{2}\mathbf{d}^\dagger_1 \mathbf{d}_1 - \frac{1}{2}\mathbf{d}^\dagger_2 \mathbf{d}_2 - \frac{3}{2}\mathbf{d}^\dagger_3 \mathbf{d}_3 \,, \tag{B.9}$$

$$\mathcal{Q}_\alpha = \mathbf{a}^\dagger_\alpha \mathbf{c} \,, \qquad \bar{\mathcal{Q}}_{\dot\alpha} = \mathbf{b}^\dagger_{\dot\alpha} \mathbf{c}^\dagger \,, \tag{B.10}$$

$$\mathcal{S}^\alpha = \mathbf{c}^\dagger \mathbf{a}^\alpha \,, \qquad \bar{\mathcal{S}}^{\dot\alpha} = \mathbf{b}^{\dot\alpha} \mathbf{c} \,, \tag{B.11}$$

$$\mathcal{P}_{\alpha\dot\beta} = \mathbf{a}^\dagger_\alpha \mathbf{b}^\dagger_{\dot\beta} \,, \qquad \mathcal{K}^{\alpha\dot\beta} = \mathbf{a}^\alpha \mathbf{b}^{\dot\beta} \,, \tag{B.12}$$

Here C is a central charge that must kill any physical state. It could be eliminated from the algebra by redefining $r \to r + \frac{2}{3}C$, but it is useful for implementing the harmonic action so we keep it.

B.1 Vector multiplets $\mathcal{V}$ and $\bar{\mathcal{V}}$

We define a vacuum state $|0\rangle$ annihilated by all the lowering operators. Then we identify

$$\mathcal{D}^k \mathcal{F} = (\mathbf{a}^\dagger)^{k+2} (\mathbf{b}^\dagger)^k |0\rangle \,, \tag{B.13}$$

$$\mathcal{D}^k \lambda = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \mathbf{c}^\dagger |0\rangle \,, \tag{B.14}$$

and

$$\mathcal{D}^k \bar{\mathcal{F}} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+2} \mathbf{c}^\dagger \mathbf{d}^\dagger_3 |0\rangle \,, \tag{B.15}$$

$$\mathcal{D}^k \bar{\lambda} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+1} \mathbf{d}^\dagger_3 |0\rangle \,. \tag{B.16}$$

B.2 Chiral multiplets $\mathcal{X}$ and $\bar{\mathcal{X}}$

Similarly, for the chiral multiplets we identify

$$\mathcal{D}^k Q = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^k \mathbf{c}^\dagger \mathbf{d}^\dagger_1 |0\rangle \,, \tag{B.17}$$

$$\mathcal{D}^k \psi = (\mathbf{a}^\dagger)^{k+1} (\mathbf{b}^\dagger)^k \mathbf{d}^\dagger_1 |0\rangle \,, \tag{B.18}$$

and

$$\mathcal{D}^k \bar{Q} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^k \mathbf{d}^\dagger_1 \mathbf{d}^\dagger_2 |0\rangle \,, \tag{B.19}$$

$$\mathcal{D}^k \bar{\psi} = (\mathbf{a}^\dagger)^k (\mathbf{b}^\dagger)^{k+1} \mathbf{c}^\dagger \mathbf{d}^\dagger_1 \mathbf{d}^\dagger_2 |0\rangle \,. \tag{B.20}$$

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B.3 Two-letter superconformal primaries

By demanding that they are annihilated by all the conformal supercharges and by the appropriate combinations of Poincaré supercharges, we have worked out the expressions for the superconformal primaries of the irreducible modules that appear on the right hand side of the tensor products (2.10)–(2.19). The grassmannOps.m oscillator package by Jeremy Michelson and Matthew Headrick was extremely useful for this task. We simply quote the results:

$\mathcal{V}\times\mathcal{V}$**:**

$$\bar{\mathcal{B}}_{2(0,0)} = \lambda_+\lambda_- - \lambda_-\lambda_+\,, \tag{B.21}$$

$$\bar{\mathcal{B}}_{2(1,0)} = \lambda_+\lambda_+\,, \tag{B.22}$$

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-2}{2})} &= \sum_{k=0}^{q-2}\frac{(-1)^k}{q(q+1)}\binom{q+1}{k+2}\binom{q-2}{k}\mathcal{D}^{q-k-2}\lambda_+\mathcal{D}^k\mathcal{F}_{++}\\ &\quad+\frac{1}{q}\sum_{k=0}^{q-2}\frac{(-1)^{q-k}}{k+2}\binom{q-2}{k}\binom{q}{k+1}\mathcal{D}^k\mathcal{F}_{++}\mathcal{D}^{q-k-2}\lambda_+\,.\end{aligned} \tag{B.23}$$

For $\bar{\mathcal{V}}\times\bar{\mathcal{V}}$ the expressions are identical with $(\lambda,\mathcal{F})$ replaced by $(\bar\lambda,\bar{\mathcal{F}})$.

$\mathcal{V}\times\mathcal{X}$**:**

$$\bar{\mathcal{B}}_{\frac{5}{3}(\frac{1}{2},0)} = \lambda_+ Q\,, \tag{B.24}$$

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q-1}{2})} &= \sum_{k=0}^{q-1}(-1)^k\binom{q-1}{k}\binom{q+1}{k}\mathcal{D}^{q-k-1}\mathcal{F}_{++}\mathcal{D}^k Q\\ &\quad+(q+1)\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\lambda_+\mathcal{D}^k\psi_+\,.\end{aligned} \tag{B.25}$$

For the $\tilde{\mathcal{X}}\times\mathcal{V}$ primary we replace (Q,ψ) by $(\tilde Q,\tilde\psi)$ and interchange the order of the fields (taking into account fermionic minus signs).

$\bar{\mathcal{V}}\times\mathcal{X}$**:**

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q+1}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k}\mathcal{D}^{q-k}\bar\lambda_{\dot+}\mathcal{D}^k Q\\ &\quad-q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q+1}{k}\mathcal{D}^{q-k-1}\bar{\mathcal{F}}_{\dot+\dot+}\mathcal{D}^k\psi_+\,.\end{aligned} \tag{B.26}$$

For the $\tilde{\mathcal{X}}\times\bar{\mathcal{V}}$ primary we replace (Q,ψ) by $(\tilde Q,\tilde\psi)$ and interchange the order of the fields.

$\bar{\mathcal{X}}\times\mathcal{V}$**:**

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k}\bar Q\mathcal{D}^k\lambda_+\\ &\quad+q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+2}\binom{q-1}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k-1}\bar\psi_{\dot+}\mathcal{D}^k\mathcal{F}_{++}\,.\end{aligned} \tag{B.27}$$

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For the $\mathcal{V}\times\bar{\tilde{\mathcal{X}}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$ and interchange the order of the fields.

$\bar{\mathcal{X}}\times\bar{\mathcal{V}}$**:**

$$\mathcal{B}_{-\frac{5}{3}(0,\frac{1}{2})} = \bar{Q}\bar{\lambda}_+\,, \tag{B.28}$$

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q-1}{2},\frac{q+1}{2})} &= \sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^k\bar{\psi}_{\dot{+}}\mathcal{D}^{q-k-1}\bar{\lambda}_{\dot{+}} \\ &\quad+\sum_{k=0}^{q-1}\frac{(-1)^{q-k}}{q+1}\binom{q-1}{k}\binom{q+1}{k+2}\mathcal{D}^{q-k-1}\bar{Q}\mathcal{D}^k\bar{\mathcal{F}}_{\dot{+}\dot{+}}\,.\end{aligned} \tag{B.29}$$

For the $\bar{\mathcal{V}}\times\bar{\tilde{\mathcal{X}}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$ and interchange the order of the fields.

$\bar{\mathcal{X}}\times\mathcal{X}$**:**

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q}{k}\mathcal{D}^{q-k}\bar{Q}\mathcal{D}^kQ \\ &\quad+q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\bar{\psi}_{\dot{+}}\mathcal{D}^k\psi_+\,.\end{aligned} \tag{B.30}$$

This primary is *gauge* contracted. For the *flavor* contracted $\bar{\tilde{\mathcal{X}}}\times\tilde{\mathcal{X}}$ primary we replace (Q,ψ) by $(\tilde{Q},\tilde{\psi})$.

$\tilde{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$**:**

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q}{k}\mathcal{D}^{q-k}\tilde{Q}\mathcal{D}^k\bar{\tilde{Q}} \\ &\quad+q\sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\tilde{\psi}_+\mathcal{D}^k\bar{\tilde{\psi}}_{\dot{+}}\,.\end{aligned} \tag{B.31}$$

This primary is *gauge* contracted. For the *flavor* contracted $\mathcal{X}\times\bar{\tilde{\mathcal{X}}}$ primary we replace $(\tilde{Q},\tilde{\psi})$ by (Q,ψ).

$\tilde{\mathcal{X}}\times\mathcal{X}$**:**

$$\bar{\mathcal{B}}_{\frac{4}{3}(0,0)} = \tilde{Q}Q \tag{B.32}$$

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q+1}{2},\frac{q}{2})} &= \sum_{k=0}^{q}(-1)^k\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^{q-k}\tilde{Q}\mathcal{D}^k\psi_+ \\ &\quad-\sum_{k=0}^{q}(-1)^{q-k}\binom{q}{k}\binom{q+1}{k+1}\mathcal{D}^k\tilde{\psi}_+\mathcal{D}^{q-k}Q\,.\end{aligned} \tag{B.33}$$

For the $\bar{\mathcal{X}}\times\bar{\tilde{\mathcal{X}}}$ primary we interchange $(\tilde{Q},\psi)$ by $(\bar{Q},\bar{\tilde{\psi}})$ (also for the conjugates).

$\mathcal{V}\times\bar{\mathcal{V}}$**:**

$$\begin{aligned}\hat{\mathcal{C}}_{(\frac{q}{2},\frac{q}{2})} &= \sum_{k=0}^{q-1}\frac{(-1)^k}{k+1}\binom{q-1}{k}\binom{q}{k}\mathcal{D}^{q-k-1}\lambda_+\mathcal{D}^k\bar{\lambda}_{\dot{+}} \\ &\quad-\sum_{k=0}^{q-2}\frac{(-1)^{q-k}}{q}\binom{q1}{k}\binom{q}{k+2}\mathcal{D}^k\mathcal{F}_{++}\mathcal{D}^{q-k-2}\bar{\mathcal{F}}_{\dot{+}\dot{+}}\,.\end{aligned} \tag{B.34}$$

For the $\bar{\mathcal{V}}\times\mathcal{V}$ primary we replace $(\lambda,\mathcal{F})$ by$(\bar{\lambda},\bar{\mathcal{F}})$.

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