

Quantization of integrable systems and a 2d/4d duality

Nick Dorey,^a Sungjay Lee^a and Timothy J. Hollowood^b

^a*DAMTP, Centre for Mathematical Sciences, University of Cambridge,
Wilberforce Road, Cambridge CB3 0WA, U.K.*

^b*Department of Physics, Swansea University,
Swansea SA2 8PP, U.K.*

E-mail: n.dorey@damtp.cam.ac.uk, S.Lee@damtp.cam.ac.uk,
T.Hollowood@swansea.ac.uk

ABSTRACT: We present a new duality between the F-terms of supersymmetric field theories defined in two- and four-dimensions respectively. The duality relates $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions, deformed by an Ω -background in one plane, to $\mathcal{N} = (2, 2)$ gauged linear σ -models in two dimensions. On the four dimensional side, our main example is $\mathcal{N} = 2$ SQCD with gauge group $G = \text{SU}(L)$ and $N_F = 2L$ fundamental flavours. Using ideas of Nekrasov and Shatashvili, we argue that the Coulomb branch of this theory provides a quantization of the classical Heisenberg $\text{SL}(2)$ spin chain. Agreement with the standard quantization via the Algebraic Bethe Ansatz implies the existence of an isomorphism between the chiral ring of the 4d theory and that of a certain two-dimensional theory. The latter can be understood as the worldvolume theory on a surface operator/vortex string probing the Higgs branch of the same 4d theory. We check the proposed duality by explicit calculation at low orders in the instanton expansion. One striking consequence is that the Seiberg-Witten solution of the 4d theory is captured by a one-loop computation in two dimensions. The duality also has interesting connections with the AGT conjecture, matrix models and topological string theory where it corresponds to a refined version of the geometric transition.

KEYWORDS: Supersymmetry and Duality, Supersymmetric gauge theory, Bethe Ansatz

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1 Introduction

Supersymmetric gauge theories in two dimensions exhibit intriguing similarities to their four dimensional counterparts which have been noted many times in the past. Their common features include the existence of protected quantities with holomorphic dependence on F-term couplings and a related spectrum of BPS states which undergo non-trivial monodromies and wall-crossing transitions in the space of couplings/VEVs. In this paper we will propose a precise duality between specific theories in four- and two-dimensions (henceforth denoted Theory I and Theory II respectively). The duality applies to the large class of four dimensional theories with $\mathcal{N} = 2$ supersymmetry which can be realised by the standard quiver construction as in [1]. As our main example we have,

Theory I. Four-dimensional $\mathcal{N} = 2$ SQCD with gauge group $SU(L)$, L hypermultiplets in the fundamental representation with masses $\vec{m}_F = (m_1, \dots, m_L)$ and L hypermultiplets in the anti-fundamental with masses $\vec{m}_{AF} = (\tilde{m}_1, \dots, \tilde{m}_L)$. The theory is conformally invariant in the UV with marginal coupling $\tau = 4\pi i/g^2 + \vartheta/2\pi$.

For some purposes it will also be useful to consider the corresponding $U(L)$ gauge theory. We consider Theory I in the presence of a particular¹ Nekrasov deformation with parameter ϵ which preserves $\mathcal{N} = (2, 2)$ supersymmetry in an $\mathbb{R}^{1,1}$ subspace of four-dimensional spacetime. The resulting effective theory in two dimensions is characterised by a (twisted) superpotential, $\mathcal{W}^{(I)}$ with holomorphic dependence on (twisted) chiral superfields. The superpotential $\mathcal{W}^{(I)}$ receives an infinite series of corrections from perturbation theory and instantons which encode the four-dimensional origin of the theory. It has an L -dimensional lattice of stationary points corresponding to supersymmetric vacua of the deformed theory. These are determined by the F-term equation,

$$\vec{a} = \vec{m}_F - \vec{n}\epsilon \quad \vec{n} = (n_1, \dots, n_L) \in \mathbb{Z}^L$$

where $\vec{a} = (a_1, \dots, a_L)$ are the usual special Kähler coordinates on the Coulomb branch of the four-dimensional theory. A generic point on the Coulomb branch of the undeformed theory can be recovered in an appropriate $\epsilon \rightarrow 0$, $|\vec{n}| \rightarrow \infty$ limit.

We will propose an exact duality of Theory I to a surprisingly simple model defined in two-dimensions which holds for all positive values of the integers $\{n_l\}$ introduced above;

Theory II. Two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric Yang-Mills theory with gauge group $U(N)$ with L chiral multiplets in the fundamental representation with twisted masses $\vec{M}_F = (M_1, \dots, M_L)$ and L chiral multiplets in the anti-fundamental with twisted masses $\vec{M}_{AF} = (\tilde{M}_1, \dots, \tilde{M}_L)$. In addition the theory has a single chiral multiplet in the adjoint representation with mass ϵ . The FI parameter r and 2d vacuum angle θ combine to form a complex marginal coupling $\hat{\tau} = ir + \theta/2\pi$.

Theory II has a twisted effective superpotential $\mathcal{W}^{(II)}$ which is one-loop exact [2]. In both Theory I and Theory II, the superpotential determines the chiral ring of supersymmetric vacuum states.

Claim. The chiral rings of Theory I and Theory II are isomorphic. In particular, there is a 1-1 correspondence between the supersymmetric vacua of the two theories and, with an appropriate identification of complex parameters, the values of the twisted superpotentials coincide in corresponding vacua (up to a vacuum-independent additive constant),

$$\mathcal{W}^{(I)} \stackrel{\text{on-shell}}{\equiv} \mathcal{W}^{(II)}$$

The rank N of the 2d gauge group is identified in terms of the 4d parameters according to $N + L = \sum_{l=1}^L n_l$. Thus, when $|\epsilon|$ is small, low values of N correspond to points near the Higgs branch root of the 4d theory. The deformation parameter ϵ of Theory I is identified with adjoint mass of Theory II. The explicit map between the remaining parameters takes the form,

$$\hat{\tau} = \tau + \frac{1}{2}(N + 1), \quad \vec{M}_F = \vec{m}_F - \frac{3}{2}\vec{\epsilon}, \quad \vec{M}_{AF} = \vec{m}_{AF} + \frac{1}{2}\vec{\epsilon}. \quad (1.1)$$

¹As we explain in section 2.2 below there are a family of inequivalent deformations related to each other by the low-energy electromagnetic duality group of the four-dimensional theory.

where $\vec{\epsilon} = (\epsilon, \epsilon, \dots, \epsilon)$. Further details of the map between the chiral rings of the two theories is given in Subsection 2.5 below.

The initial motivation for this duality comes from the mysterious connection between supersymmetric gauge theories and quantum integrable systems developed in a remarkable series of papers by Nekrasov and Shatashvili (NS) [3–5]. These authors propose a general correspondence in which the space of supersymmetric vacua of a theory with $\mathcal{N} = (2, 2)$ supersymmetry is identified with the Hilbert space of a quantum integrable system. The generators of the chiral ring are mapped to the commuting conserved charges of the integrable system. The twisted superpotential itself corresponds to the so-called Yang-Yang potential which is naturally thought of as a generating function for the conserved charges. The ideas of [5] also extend the well known connection between $\mathcal{N} = 2$ supersymmetric gauge theory in four-dimensions and classical integrable systems [7–10] which is reviewed in section 2 below. In particular they propose that the introduction of a Nekrasov deformation in one plane, breaking four-dimensional supersymmetry to an $\mathcal{N} = (2, 2)$ subalgebra, corresponds to a quantization of the corresponding classical integrable system with the deformation parameter ϵ playing the role of Planck’s constant \hbar .

Our main observation is that the same quantum integrable system arises in two different contexts. In the case $\epsilon = 0$, it has been known for some time [11–13] that Theory I corresponds to a *classical* Heisenberg spin chain with spins whose Poisson brackets provide a representation of $\mathfrak{sl}(2)$ at each site. After imposing appropriate reality conditions, we will adapt the ideas of [5] and argue that introducing non-zero ϵ corresponds to a specific quantization of this system² in which the classical spins at each site are replaced by quantum operators acting in a highest-weight representation of $SL(2, \mathbb{R})$. The resulting quantum chain is integrable and can be diagonalised exactly using the Quantum Inverse Scattering method which leads to a simple set of rational Bethe Ansatz equations. However, precisely these equations also arise as the F-term equations of Theory II and the corresponding twisted superpotential, $\mathcal{W}^{(II)}$, coincides with the Yang-Yang potential of the spin chain [3, 4]. This is not a coincidence: with the identification of parameters proposed above, Theory II can be identified as the worldvolume theory of a vortex string or surface operator probing the Higgs branch of Theory I.³ As we discuss below, this connection suggests a physical explanation of the correspondence along the lines of [17, 18] as well as relations to several other recent developments.

Equivalence between the NS quantization and the standard quantization of the spin chain implies the duality between Theories I and II proposed above. In section 3 below we test the duality by an explicit calculation of $\mathcal{W}^{(I)}$ in each vacuum including classical, perturbative contributions as well as non-perturbative contributions up to second order in the four-dimensional instanton expansion. The corresponding computation of $\mathcal{W}^{(II)}$ involves an iterative solution of the one-loop exact F-term equations in powers of the parameter $q = \exp(2\pi i\tau)$. The calculation yields precise agreement in all vacua when the parameters are identified according to (1.1).

²For other recent work relating $\mathcal{N} = 2$ SQCD and the quantum Heisenberg spin chain see [46–48, 51].

³There are some subtleties to this relation which we discuss in subsection 2.4 below.

One interesting consequence of the proposed duality is that the full Seiberg-Witten solution of Theory I can be recovered by taking an $N \rightarrow \infty$, $\epsilon \rightarrow 0$ limit of the one-loop exact F-term equations of Theory II. In fact this is just the standard semiclassical limit of the non-compact spin chain (see e.g. [43, 45]) where the Bethe roots condense to form branch cuts in the spectral plane. In our context the resulting double cover of the complex plane is the Seiberg-Witten curve. This is very reminiscent of the Dijkgraaf-Vafa matrix model [14–16] approach where the eigenvalues of an $N \times N$ matrix condense to form the branch cuts of the Seiberg-Witten curve. An important difference is that, in the NS limit, the Bethe Ansatz equations provide an exact solution of the system even at finite N (see point 4 below).

In a forthcoming paper [61] we will also sketch a similar correspondence for a larger class of $\mathcal{N} = 2$ quiver theories in four (as well as five and six) dimensions. A common feature is that quantization leads to spin chains based on highest weight representations of Lie algebras where the ferromagnetic ground-state of the spin chain corresponds to the root of the gauge theory Higgs branch. In all cases, the Algebraic Bethe Ansatz leads to simple rational, (trigonometric, elliptic) equations which coincide with the F-term equations of a dual two-dimensional theory. In forthcoming work [61] we will show how the correspondence can be proved by directly relating the Bethe Ansatz equation to the saddle-point equation describing the instanton density in the Nekrasov-Shatashvili limit [5] (see also the recent papers [59, 60]).

There are also several interesting connections to other developments in supersymmetric gauge theory both new and old:

1. As already mentioned, there is a simple physical relation between the two theories which holds with identifications between the parameters described above: Theory II can be understood as the theory on the world-volume of a vortex string or surface operator [34, 35] probing the Higgs branch of the same four-dimensional theory. More precisely, Theory II is a gauged linear σ -model for the moduli space⁴ of N non-abelian vortices of Theory I [19] (See also [36, 37]). The proposed duality relates the world-volume theory of N vortices to the bulk theory (i.e. Theory I) on its Coulomb branch. As usual, Higgs phase vortices carry quantized magnetic flux. Interestingly, the dual Coulomb branch vacua of Theory I also exhibit quantized magnetic fluxes in the presence of the Ω -deformation.
2. The duality provides an analog of the geometric transition of Gopakumar and Vafa [38] for the “refined” topological string [54] with refinement parameters $\epsilon_1 \neq \epsilon_2$ in the Nekrasov-Shatashvili (NS) limit $\epsilon_2 \rightarrow 0$. Theories I and II correspond to the closed and open string sides of the transition respectively.
3. The duality can also be understood in terms of the conjecture [33] relating four dimensional supersymmetric gauge theory with Liouville theory. We will argue that in certain cases the proposed duality is equivalent to the conjecture of Alday *et al* [34] that a surface operator in gauge theory corresponds to a particular degenerate operator in Liouville theory.

⁴There are some subtleties to this relation which are discussed in section 2.4 below.

4. Recent work [39] has advocated a further duality between $\mathcal{N} = 2$ supersymmetric gauge theories in a four dimensional Ω -background and matrix models. In particular, the Nekrasov partition function of Theory I should have a matrix integral representation. Here we identify the dimension of matrix as the rank N of the gauge group of Theory II. We conjecture that the resulting integral over the matrix eigenvalues can be evaluated explicitly by a saddle point in the NS limit *even at finite N* . The saddle-point equations are precisely the Bethe Ansatz equations of the spin chain and the free energy is equal to the prepotential of the gauge theory. This should also have interesting consequences for refined open topological string amplitudes in the limit $\epsilon_2 \rightarrow 0$.

5. Some time ago a duality was proposed [40, 41] relating the BPS spectrum of a two-dimensional theory and that of a corresponding four-dimensional gauge theory at the root of its Higgs branch. The correspondence is further studied in [42] with care given to precise supermultiplet counting together with their wall-crossing behaviour. We show that the present proposal reduces to the earlier one in the limit $\epsilon \rightarrow 0$.

The rest of the paper is organised as follows. In section 2, we review the basic features of Theory I, its relation to the classical Heisenberg spin chain in the case $\epsilon = 0$ and its quantization for $\epsilon \neq 0$. We also introduce Theory II and review its realisation on the worldvolume of a vortex string/surface operator and provide the precise statement of our duality conjecture. Section 3 is devoted to a detailed check of this proposal. Discussion of our results, generalisations and connections to topological strings, matrix models and Liouville theory are presented in section 4.

2 Supersymmetric gauge theory and integrable systems

In this section we will begin reviewing the relevant feature of the four-dimensional theory introduced in section 1. As above, we focus on four-dimensional $\mathcal{N} = 2$ Super QCD with gauge group $G = \text{SU}(L)$ and $N_F = 2L$ hypermultiplets. For an $\text{SU}(L)$ gauge theory, hypermultiplets in the fundamental and anti-fundamental representations of the gauge group are essentially equivalent. It is nevertheless convenient to focus on the case where half of the $2L$ hypermultiplets are in the fundamental representation and the rest in the anti-fundamental representation. The duality discussed below will apply equally to the corresponding $\text{U}(L)$ gauge theory which differs from the $\text{SU}(L)$ theory by an additional $\text{U}(1)$ factor which is IR free. The fundamental and anti-fundamental hypermultiplet masses are denoted by m_l and \tilde{m}_l with $l = 1, 2, \dots, L$ respectively. The $\text{SU}(L)$ theory is conformally invariant in the UV with marginal coupling $\tau = 4\pi i/g^2 + \vartheta/2\pi$.

The exact low-energy solution of the undeformed theory is governed by the corresponding Seiberg-Witten curve,

$$\prod_{l=1}^L (v - \tilde{m}_l)t^2 - 2 \prod_{l=1}^L (v - \phi_l)t - h(h+2) \prod_{l=1}^L (v - m_l) = 0. \quad h(\tau) = -\frac{2q}{q+1}, \quad (2.1)$$

where $q = \exp(2\pi i\tau)$ is the factor associated with a four-dimensional Yang-Mills instanton. Here ϕ_l denote the L classical eigenvalues of the adjoint scalar field φ in the $\mathcal{N} = 2$ vector multiplet. As the gauge group is $\text{SU}(L)$ we impose a traceless condition $\sum_{l=1}^L \phi_l = 0$.

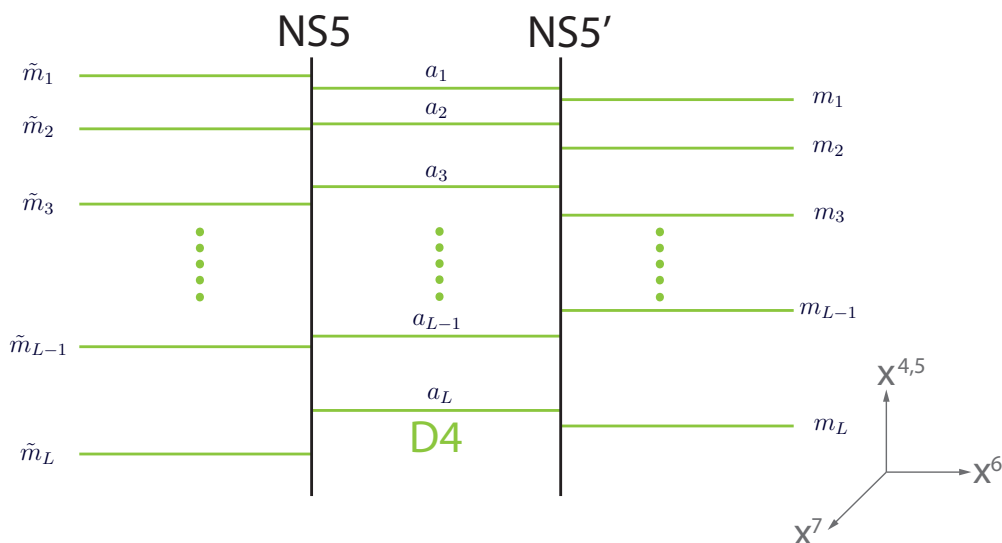


Figure 1. IIA brane construction for a generic point on the Coulomb branch of Theory I.

The masses of BPS states in the four-dimensional theory are determined by a meromorphic differential, $\lambda_{\text{SW}} = v dt/t$ on the curve. A standard basis of A- and B-cycles on the Seiberg-Witten curve, with $A_l \cap B_m = \delta_{lm}$ can be defined in the weak-coupling limit $\tau \rightarrow i\infty$. Electric and magnetic central charges are determined by the periods of λ_{SW} in this basis,

$$\vec{a} = \frac{1}{2\pi i} \oint_{\vec{A}} \lambda_{\text{SW}}, \quad \vec{a}^D = \frac{1}{2\pi i} \frac{\partial \mathcal{F}}{\partial \vec{a}} = \frac{1}{2\pi i} \oint_{\vec{B}} \lambda_{\text{SW}}, \quad (2.2)$$

where $\vec{a} = (a_1, \dots, a_L)$ with similar notation for other L -component vectors. For theories with matter the Seiberg-Witten differential also has simple poles at the points $x = m_l, \tilde{m}_l$ with residues m_l and \tilde{m}_l respectively. It is convenient to introduce additional cycles $\vec{C}_F = (C_1, \dots, C_L)$ and $\vec{C}_{AF} = (\tilde{C}_1, \dots, \tilde{C}_L)$ encircling these poles so that the corresponding periods of λ_{SW} are,

$$\vec{m}_F = \frac{1}{2\pi i} \oint_{\vec{C}_F} \lambda_{\text{SW}}, \quad \vec{m}_{AF} = \frac{1}{2\pi i} \oint_{\vec{C}_{AF}} \lambda_{\text{SW}}.$$

The standard IIA brane construction of Theory I at a generic point on its Coulomb branch is shown in figure 1. Here we follow the conventions of [1]. Each horizontal line corresponds to a D4 brane. In the figure each D4 is labelled by the corresponding value of the complex coordinate $v = x_4 + ix_5$. In the following, the point on the Coulomb branch where it touches the Higgs branch will have a particular significance. The configuration corresponding to this Higgs branch root is shown in figure 2. At the root of baryonic Higgs branch, the $\vec{A} - \vec{C}$ cycles degenerate leading to $L - 1$ additional massless hypermultiplets. This corresponds to a factorization of the Seiberg-Witten curve. More precisely, when ϕ_l 's

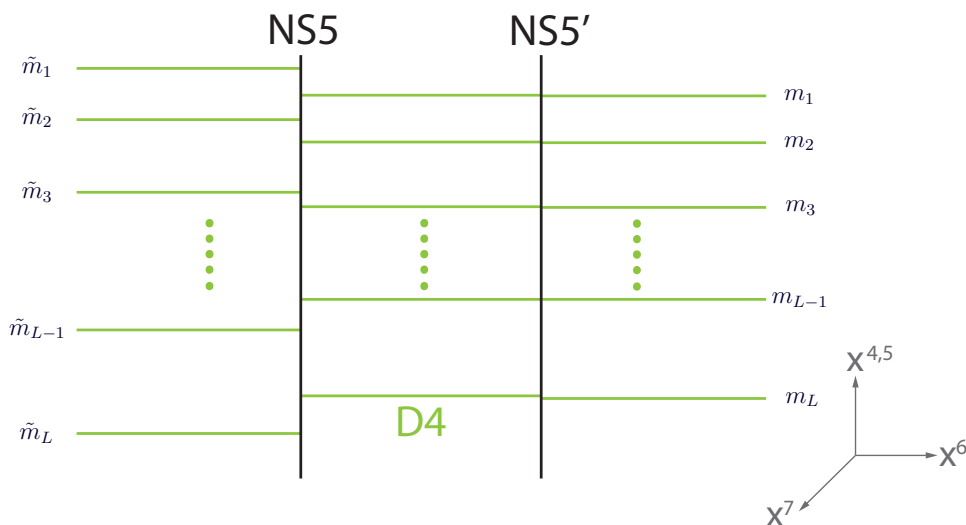


Figure 2. The Higgs branch root $\vec{a} = \vec{m}_F$.

are tuned to satisfy a relation

$$-h \prod_{l=1}^L (v - \tilde{m}_l) + (h + 2) \prod_{l=1}^L (v - m_l) = 2 \prod_{l=1}^L (v - \phi_l), \quad (2.3)$$

the Seiberg-Witten curve becomes degenerate

$$\left[\prod_{l=1}^L (v - \tilde{m}_l)t - (h + 2) \prod_{l=1}^L (v - m_l) \right] \times [t + h] = 0, \quad (2.4)$$

and $\vec{A} = \vec{C}_F$. We will soon explain a correspondence between the root of baryonic Higgs branch and ferromagnetic vacuum of the $SL(2, \mathbb{R})$ integrable model.

2.1 The classical integrable system

We now review the connection between $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions and complex classical integrable systems. We begin by introducing the Heisenberg spin chain.

We will consider a chain of L complex “spins” [43–45] corresponding to classical variables, $\mathcal{L}_l^\pm, \mathcal{L}_l^0$, for $l = 1, 2, \dots, L$ with Poisson brackets:

$$\{\mathcal{L}_l^+, \mathcal{L}_m^-\} = 2i\delta_{lm}\mathcal{L}_m^0 \quad \{\mathcal{L}_l^0, \mathcal{L}_m^\pm\} = \pm i\delta_{lm}\mathcal{L}_m^\pm. \quad (2.5)$$

Here $+$, $-$ and 0 are indices in the Lie algebra $\mathfrak{sl}(2)$. The spins at each site have a fixed value of the quadratic Casimir,

$$\mathcal{L}_l^+ \mathcal{L}_l^- + (\mathcal{L}_l^0)^2 = J_l^2. \quad (2.6)$$

Integrability of the classical spin chain starts from an auxiliary linear problem based on the Lax matrix,

$$\mathbb{L}_l(x) = \begin{pmatrix} x + i\mathcal{L}_l^0 & i\mathcal{L}_l^+ \\ i\mathcal{L}_l^- & x - i\mathcal{L}_l^0 \end{pmatrix}, \quad (2.7)$$

where $x \in \mathbb{C}$ is a spectral parameter. A tower of commuting conserved quantities are obtained by constructing the corresponding monodromy matrix,

$$\mathbb{T}(x) = \mathbb{V} \prod_{l=1}^L \mathbb{L}_l(x - \theta_l),$$

where we have included inhomogeneities θ_l , $l = 1, 2, \dots, L$, at each site and a diagonal “twist” matrix,

$$\mathbb{V} = \begin{pmatrix} -h & 0 \\ 0 & h + 2 \end{pmatrix}.$$

As usual, the trace of the monodromy matrix is the generating function for a tower of conserved charges,

$$\begin{aligned} 2P(x) &= \text{tr}_2 [\mathbb{T}(x)] \\ &= 2x^L + q_1 x^{L-1} + \dots + q_{L-1} x + q_L. \end{aligned} \quad (2.8)$$

One may check starting from the Poisson brackets (2.5) that the conserved charges, q_l , $l = 1, 2, \dots, L$ are in involution: $\{q_l, q_m\} = 0, \forall l, m$, which establishes the Liouville integrability of the chain. The lowest charge q_1 can be set to zero by a linear shift of the spectral parameter and we will do so in the following.

As for any integrable Hamiltonian system, the exact classical trajectories of the spins can be found by a canonical transformation to action-angle variables. Using standard methods, the action variables are identified as the moduli of a spectral curve $\Gamma_L \subset \mathbb{C}^2$ defined by the equation,

$$F(x, y) = \det(y\mathbf{1}_2 - \mathbb{T}(x)) = 0,$$

while the angle variables naturally parameterise the Jacobian variety $\mathcal{J}(\Gamma_L)$. More explicitly the curve takes the form,

$$\Gamma_L : \quad y^2 - 2P(x)y - h(h+2)K(x)\tilde{K}(x) = 0, \quad (2.9)$$

where,

$$K(x) = \prod_{l=1}^L (x - \theta_l - iJ_l) \quad \tilde{K}(x) = \prod_{l=1}^L (x - \theta_l + iJ_l).$$

This curve is an equivalent form of the Seiberg-Witten curve (2.1) for Theory I provided we make the identifications,

$$\vec{m}_F = \vec{\theta} + i\vec{J} \quad \vec{m}_{AF} = \vec{\theta} - i\vec{J},$$

and

$$\begin{aligned}
 2P(x) &= 2x^L + q_2x^{L-2} + \dots + q_{L-1}x + q_L \\
 &= 2 \prod_{l=1}^L (x - \phi_l).
 \end{aligned}$$

Also $h(\tau)$ is identified with the twist parameter of the spin chain denoted by the same letter. The relation between (2.1) and (2.9) corresponds to the holomorphic change of variables, $t = y/\tilde{K}(x)$, $v = x$.

With this identification different values of the moduli of Theory I are associated with different values of the integrals of motion of the complex spin chain. One point of particular interest is the ferromagnetic vacuum of the chain where each spin is in its classical groundstate: $\vec{\mathcal{L}}_l^0 = \vec{J}$, $\vec{\mathcal{L}}^\pm = 0$. It is easy to check that this corresponds to the root of the Higgs branch in the gauge theory where the VEVs take the values $\vec{a} = \vec{m}_F$.

2.2 Nekrasov-Shatashvili quantization

We now turn our attention to the gauge theory in the presence of the so-called Ω -background. This deformation, which breaks four-dimensional Lorentz invariance, is specified by parameters ϵ_1 and ϵ_2 . The $\mathcal{N} = 2$ F-terms of the deformed theory are determined by the Nekrasov partition function [28, 29],

$$\mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2).$$

In this paper we will be mainly concerned with the Nekrasov-Shatashvili limit $\epsilon_2 \rightarrow 0$ with ϵ_1 held fixed, where the deformation is restricted to one plane in \mathbb{R}^4 . We define a quantum prepotential in this limit as,

$$\mathcal{F}(\vec{a}, \epsilon) = \lim_{\epsilon_2 \rightarrow 0} [\epsilon_1 \epsilon_2 \log \mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2)|_{\epsilon_1 = \epsilon}].$$

In the further limit $\epsilon \rightarrow 0$, the quantum prepotential reduces to the familiar prepotential of the undeformed theory: $\mathcal{F}(\vec{a}, \epsilon) \rightarrow \mathcal{F}(\vec{a})$. Following [46–48, 50], the quantum prepotential can be obtained by a suitable deformation of the Seiberg-Witten differential appearing in (2.2),

$$\lambda(\epsilon) = \lambda_{\text{SW}} + \mathcal{O}(\epsilon),$$

with periods,

$$\vec{a}(\epsilon) = \frac{1}{2\pi i} \oint_{\vec{A}} \lambda(\epsilon), \quad \vec{a}^D(\epsilon) = \frac{1}{2\pi i} \oint_{\vec{B}} \lambda(\epsilon), \tag{2.10}$$

such that,

$$\vec{a}^D(\epsilon) = \frac{1}{2\pi i} \frac{\partial}{\partial \vec{a}(\epsilon)} \mathcal{F}(\vec{a}, \epsilon). \tag{2.11}$$

For convenience we will suppress the ϵ dependence of the deformed central charges from now on and denote them simply as \vec{a} and \vec{a}^D .

For $\epsilon \neq 0$, the four-dimensional $\mathcal{N} = 2$ supersymmetry is broken down to $\mathcal{N} = (2, 2)$ supersymmetry two dimensions. The zero modes of the $U(1)^{L-1}$ vector multiplet in the four-dimensional low energy theory give rise to a field strength multiplet in two dimensions. This multiplet includes the gauge field strength \vec{F}_{01} in the undeformed directions and the scalar fields \vec{a} which parametrize the 4d Coulomb branch. Thus \vec{a} is the lowest component of a twisted chiral superfield in $\mathcal{N} = (2, 2)$ superspace. This superfield inherits a twisted superpotential from the partition function of the four-dimensional theory. The resulting twisted superpotential is a multi-valued function on the Coulomb branch,

$$\mathcal{W}^{(I)}(\vec{a}, \epsilon) = \frac{1}{\epsilon} \mathcal{F}(\vec{a}, \epsilon) - 2\pi i \vec{k} \cdot \vec{a} \tag{2.12}$$

where the integer-valued vector⁵ $\vec{k} \in \mathbb{Z}^L$ corresponds to the choice of branch. This choice corresponds to the freedom to shift the 2d vacuum angle associated with each $U(1)$ factor in the low energy gauge group by an integer multiple of 2π . This is equivalent to introducing a constant electric field \vec{F}_{01} in spacetime which is then screened by pair creation [2, 26]. The vector \vec{k} thus specifies the choice of a quantized two-dimensional electric flux in the Cartan subalgebra of $SU(L)$.

The main claim of [5] is that $\mathcal{W}^{(I)}(\vec{a}, \epsilon)$ is the Yang-Yang potential for a quantization of the classical integrable system described above, in which the effective Planck constant \hbar is proportional to the deformation parameter ϵ . This requires further explanation: corresponding to a complex classical integrable system there can be several choices of reality condition which yield inequivalent real integrable systems. Each of these real systems gives rise upon quantization to different quantum integrable systems. As discussed in [5], the twisted superpotential given above gives rise to a particular quantization which they refer to as Type A which we now review.

The F-term equations coming from (2.12) can be written in terms of the deformed magnetic central charge using (2.11) as follows,

$$\frac{\partial \mathcal{W}^{(I)}}{\partial \vec{a}} = 0 \Rightarrow \vec{a}^D \in \epsilon \mathbb{Z}^L \tag{2.13}$$

This corresponds to a quantization condition for the conserved charges of the integrable system, each point of the lattice \mathbb{Z}^L corresponds to a different quantum state. The values of the commuting conserved charges in each state are encoded in the on-shell value of the superpotential $\mathcal{W}^{(I)}$. Each branch of this multi-valued function corresponds to a supersymmetric vacuum and to a state of the quantum integrable system. Differentiating with respect to the parameters yields the vacuum expectation values of chiral operators in the corresponding vacuum. For example,

$$\langle \text{Tr } \varphi^2 \rangle = -\frac{\epsilon}{\pi i} \frac{\partial}{\partial \tau} \mathcal{W}^{(I)} \Big|_{\vec{a}^D \in \epsilon \mathbb{Z}^L} .$$

To extract the VEVs of higher dimension chiral operators $\hat{O}_k = \text{Tr } \varphi^k$ for $k > 2$ we should deform the prepotential of the UV theory with appropriate source terms [25].

⁵For an $SU(L)$ theory we should also impose $\sum_l k_l = 0$.

Theories with $\mathcal{N} = 2$ supersymmetry in four dimensions exhibit two distinct manifestations of electromagnetic duality. First there is the low-energy electromagnetic duality which provides an alternative description of the low-energy effective theory at any point on the Coulomb branch in terms of a dual field \vec{a}^D with dual prepotential $\mathcal{F}_D(\vec{a}_D, \epsilon)$, which is related to the original prepotential by Legendre transform.⁶

$$\mathcal{F}_D(\vec{a}^D) = \mathcal{F}(\vec{a}) - 2\pi i \vec{a} \cdot \vec{a}^D . \tag{2.14}$$

In an $SU(L)$ gauge theory, the full group of low-energy duality transformation includes a copy of $Sp(2L, \mathbb{Z})$ which acts linearly on the central charges (\vec{a}, \vec{a}^D) . For Theory I, there are also additional additive transformations involving shifts of the central charges by integer multiples of the mass parameters \vec{m}_A, \vec{m}_{AF} [27].

In [5], the authors proposed that these different formulations of the low-energy theory give rise upon deformation to non-zero ϵ , to different quantizations of the same complex classical integrable system. In particular, performing the basic \mathbb{Z}_2 electric-magnetic duality transformation we obtain a dual superpotential,

$$\mathcal{W}_D^{(I)}(\vec{a}_D) = \mathcal{W}^{(I)}(\vec{a}) - \frac{2\pi i}{\epsilon} \vec{a} \cdot \vec{a}^D \tag{2.15}$$

whose F-term equations give rise to dual quantization conditions, denoted Type B in [5],

$$\frac{\partial \mathcal{W}_D^{(I)}}{\partial \vec{a}^D} = 0 \Rightarrow \vec{a} \in \epsilon \mathbb{Z}^L . \tag{2.16}$$

The discrete choice of vacuum now corresponds to a choice of magnetic flux \vec{F}_{23} in the Cartan subalgebra of the gauge group. As a consequence of (2.15), we note that the two superpotentials $\mathcal{W}^{(I)}$ and $\mathcal{W}_D^{(I)}$ are actually equal as multi-valued functions when evaluated on-shell. This is true for both quantization conditions (2.13) and (2.16).

Summarising the above, the Type A and B quantization conditions can be written as,

$$\frac{1}{2\pi i} \oint_{\vec{A}} \lambda(\epsilon) \in \epsilon \mathbb{Z}^L \quad \text{and} \quad \frac{1}{2\pi i} \oint_{\vec{B}} \lambda(\epsilon) \in \epsilon \mathbb{Z}^L ,$$

respectively. Other quantizations corresponding to other transformations in the duality group naturally correspond to period conditions for different choices of basis cycles.

The $\mathcal{N} = 2$ theory with gauge group $SU(L)$ and $2L$ hypermultiplets exhibits another form of electric-magnetic duality. The exact S-duality of the theory relates electric and magnetic observables of the theory at different values of the marginal coupling τ . In the present context it implies a non-trivial duality between Type A and Type B quantization at different values of τ .

⁶Although the electro-magnetic duality transformation is modified and becomes a kind of Fourier transform [29] for general deformation parameters ϵ_1, ϵ_2 , the standard transformation properties are regained in the NS limit.

2.3 Quantization via the Bethe Ansatz

In this section we will review the standard approach to quantizing the Heisenberg spin chain (see e.g. [30, 31, 43, 45]). Starting from the Poisson brackets (2.5) we make the usual replacement of the classical variables $\mathcal{L}_l^\pm, \mathcal{L}_l^0$ at each site by operators $\hat{L}_l^\pm, \hat{L}_l^0$ obeying commutation relations,

$$[\hat{L}_l^+, \hat{L}_m^-] = -2\hbar\delta_{lm}\hat{L}_m^0 \quad [\hat{L}_l^0, \hat{L}_m^\pm] = \mp\hbar\delta_{lm}\hat{L}_m^\pm, \quad (2.17)$$

for $l, m = 1, 2, \dots, L$. The spins at each site each commute with the Casimir operator,

$$\hat{L}_l^2 = \frac{1}{2}(\hat{L}_l^+\hat{L}_l^- + \hat{L}_l^-\hat{L}_l^+) + (\hat{L}_l^0)^2 = s_l(s_l - 1)\hbar^2.$$

Depending on the value of s_l , these operators can act on representations of either SU(2) or SL(2, \mathbb{R}). We will focus on the latter case. If we choose $s \in \mathbb{R}^+$ the spins can be chosen to act in the principal discrete series representation⁷ of SL(2, \mathbb{R}),

$$\mathcal{D}_s^+ = \{|s, \mu\rangle, \mu = s, s + 1, s + 2, \dots\}$$

These are highest weight representations of SL(2, \mathbb{R}) and the resulting spin chain admits a tower of commuting charges which can be simultaneously diagonalised. By restricting to a representation \mathcal{D}_s^+ with $s = s_l > 0$ at the l 'th site we are defining a *real* quantum integrable system. An important feature of this non-compact spin chain is that it has a semiclassical limit where each spin is highly excited. As we discuss below, the relation to the *complex* integrable system we discussed in subsection 2.1 becomes clear in this limit.

As we are dealing with a spin chain based on highest weight representations the Algebraic Bethe Ansatz [30] is applicable and can be used to find the exact spectrum of the model [31]. We now review the solution of the spin chain using an alternative approach based on the Baxter equation. We start by defining a quantum version of the classical Lax matrix (2.7) which takes the form,

$$\hat{\mathbb{L}}_l(x) = \begin{pmatrix} x + i\hat{L}_l^0 & i\hat{L}_l^+ \\ i\hat{L}_l^- & x - i\hat{L}_l^0 \end{pmatrix} \quad (2.18)$$

and we define the quantum transfer matrix for an inhomogeneous chain with twisted boundary conditions by,

$$\begin{aligned} \hat{T}(x) &= \text{tr}_2 \left[\mathbb{V} \prod_{l=1}^{\rightarrow} \hat{\mathbb{L}}_l(x - \theta_l) \right] \\ &= 2x^L + \hat{q}_2 x^{L-2} + \dots + \hat{q}_{L-1} x + \hat{q}_L, \end{aligned}$$

where $\{\hat{q}_l\}$, $l = 1, 2, \dots, L$, are a set of mutually commuting conserved charges which are the quantized versions of the corresponding classical charges $\{q_l\}$.

⁷More precisely, when s is not equal to a half-integer, these are representations of the universal cover of SL(2, \mathbb{R}).

The standard problem for a quantum integrable system is to find the eigenstates of the transfer matrix,

$$\hat{T}(x) |\Psi\rangle = t(x) |\Psi\rangle ,$$

where the eigenvalue, $t(x)$, is a polynomial of degree L in the spectral parameter x by construction. This is accomplished by allowing the transfer matrix to act on a “wavefunction” $Q(x)$ which leads to the Baxter equation,

$$-ha(x)Q(x+i\hbar) + (h+2)d(x)Q(x-i\hbar) = t(x)Q(x) , \tag{2.19}$$

where,

$$a(x) = \prod_{l=1}^L (x - \theta_l + is_l\hbar) \quad d(x) = \prod_{l=1}^L (x - \theta_l - is_l\hbar)$$

Looking for solutions where $Q(x)$ is a polynomial of degree N ,

$$Q(x) = \prod_{j=1}^N (x - x_j)$$

we impose the polynomiality of $t(x)$ to obtain N equations for the N zeros x_j of $Q(x)$,

$$\prod_{l=1}^L \left(\frac{x_j - \theta_l - is_l\hbar}{x_j - \theta_l + is_l\hbar} \right) = q \prod_{k \neq j}^N \left(\frac{x_j - x_k + i\hbar}{x_j - x_k - i\hbar} \right) , \tag{2.20}$$

for $j = 1, 2, \dots, N$ where $q = -h/(h+2)$. These are the Bethe Ansatz Equations (BAE). These equations are very well studied in the case of an untwisted homogeneous $SL(2, \mathbb{R})$ chain with $\theta_l = 0$, $q = 1$ and $s_l = s > 0$. In particular, for this case, it is known that all the Bethe roots $\{x_j\}$ lie on the real axis [43]. The number N of Bethe roots corresponds to the number of magnons in the corresponding state. The ferromagnetic vacuum of the spin chain is defined as an empty state $N = 0$, i.e., $Q(x) = \text{constant}$. It implies from the Baxter equation (2.19) that

$$-ha(x) + (h+2)d(x) = t(x) , \tag{2.21}$$

which is nothing but the condition at the root of baryonic Higgs branch (2.3). The ferromagnetic vacuum of the spin chain can therefore be identified as the Higgs branch root. The eigenvalues of the transfer matrix can easily be written in terms of the solutions of the BAE using the Baxter equation (2.19).

In the following it will also be important that the BAE corresponds to stationary points of an action function,

$$\begin{aligned} \mathbb{Y}(x) = & 2\pi i\tau \sum_{j=1}^N x_j + \sum_{j=1}^N \sum_{l=1}^L \left[f(x_j - \theta_l + is_l\hbar) - f(x_j - \theta_l - is_l\hbar) \right] \\ & + \sum_{j,k=1}^N f(x_j - x_k + i\hbar) + i\pi(N+1) \sum_{j=1}^N x_j , \end{aligned}$$

where $f(x) = x(\log x - 1)$ and $q = \exp(2\pi i\tau)$.

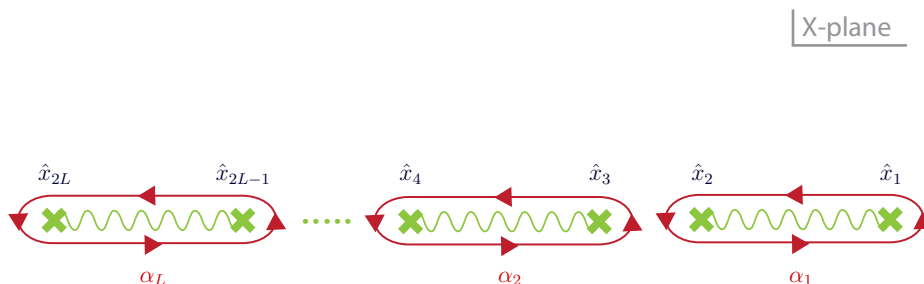


Figure 3. The cut x -plane corresponding to the curve Γ_L .

To understand the relation to the classical system of subsection 2.1, it will be useful to consider the semiclassical limit $\hbar \rightarrow 0$ of the quantum chain in which the excitation numbers at each site become large. The quantum numbers s_l which determine the spin representation at each site scale like \hbar^{-1} . They are related to the classical Casimirs as $J_l \simeq s_l \hbar + O(\hbar)$. Alternatively, if s_l are held fixed as $\hbar \rightarrow 0$ one ends up with the classical Heisenberg magnet of spin $\vec{J} = 0$. In either limit the quantum charges \hat{q}_l go over to Poisson commuting classical quantities q_l .

The key feature of the semiclassical limit is that the Bethe roots $\{x_j\}$ condense to form cuts in the complex plane. In this limit the resolvent $L(x) = d \log Q(x)/dx$ is naturally defined on a double cover of the x -plane with two sheets joined along these cuts. The resulting Riemann surface is exactly the spectral curve Γ_L and the meromorphic differential $L(x)dx$ can be identified with the Seiberg-Witten differential λ_{SW} . For the case of the homogeneous untwisted chain of spin zero, the semiclassical limit is described in detail in section 2.2 of [43]. In this case the Bethe roots condense to form branch cuts on the real axis. Working in the vicinity of the ferromagnetic vacuum, the curve can be represented as a double cover of the x plane with $2L$ real branch points⁸ at $x = \hat{x}_1 \geq \hat{x}_2 \geq \dots \geq \hat{x}_{2L}$ as shown in figure 3. We also define one-cycles α_l , $l = 1, \dots, L$ surrounding each branch cut. In the semiclassical limit, the quantum $\text{SL}(2, \mathbb{R})$ spin chain gives rise to a particular real slice of the complex classical spin chain considered above. The reality conditions select a middle-dimensional subspace of the original complex phase space. Allowing generic complex values of the moduli corresponds to working with a complexification of the spin chain in which $\text{SL}(2, \mathbb{R})$ is replaced by $\text{SL}(2, \mathbb{C})$.

At the classical level, the moduli of the curve vary continuously. The leading semiclassical approximation the quantum spectrum arises from imposing appropriate Bohr-Sommerfeld quantisation conditions which are formulated in terms of the periods of the meromorphic differential $L(x)dx$ on Γ_L which coincides with the Seiberg-Witten differential λ_{SW} ,

$$\frac{1}{2\pi} \oint_{\alpha_l} \lambda_{\text{SW}} = \hbar \hat{n}_l, \tag{2.22}$$

⁸Strictly speaking this picture is correct with a real twist parameter slightly different from unity. In the special case $q = 1$, one cut degenerates and the genus of the curve drops to $L - 2$.

for $l = 1, \dots, L$ where \hat{n}_l are non-negative integers. In terms of the Bethe Ansatz, this is just the condition that each cut contains an integral number of Bethe roots. From this point of view it is obvious that the quantization conditions are unchanged by continuous variations of the parameters including the introduction of inhomogeneities and a non-trivial twist. In particular, they also apply in the weak coupling regime $q \rightarrow 0$ where the standard basis cycles of the Seiberg-Witten curve are defined. It will be useful to express the cycles $\vec{\alpha}$ appearing in the semiclassical quantization condition in this basis. The key point, mentioned above, is that the ferromagnetic vacuum of the spin chain corresponds to the Higgs branch root $\vec{a} = \vec{m}_F$. In terms of the basis cycles defined above this corresponds to the point in moduli space where the cycles $\vec{A} - \vec{C}_F$ vanish. Thus we have $\vec{\alpha} = \vec{A} - \vec{C}_F$.

2.4 Two-dimensional gauge theory and vortex strings

The next observation, following [3, 4], is that the BAE for the Heisenberg spin chain themselves arise as the F-term equations of a certain two-dimensional gauge theory with $\mathcal{N} = (2, 2)$ supersymmetry which we will call Theory II. As above Theory II is a $U(N)$ gauge theory with L fundamental chiral multiplets Q_l with twisted masses M_l and L anti-fundamental chiral multiplets \tilde{Q}^l with twisted masses \tilde{M}_l . The theory also contains an adjoint chiral multiplet Z with twisted mass ϵ and has a marginal complex coupling $\hat{\tau} = ir + \theta/2\pi$ which corresponds to a background twisted chiral superfield.

We begin by focussing on the case $\epsilon = 0$. In this case the model can be realised on the worldvolume of N D2 branes probing a configuration of intersecting NS5 and D4 branes in Type IIA string theory [19, 57] as shown⁹ in figure 4. Since the brane-configuration is invariant under the rotations in {2,3}-, {4,5}- and {8,9}-planes, the Theory II has, at least classically, global symmetry groups $U(1)_{23} \times U(1)_{45} \times U(1)_{89}$ as well as flavor symmetry groups $SU(L) \times SU(L)$. Here the FI parameter r is proportional to Δx^6 . Classical vacua are determined by solving the D-term equations,

$$\sum_{l=1}^L (Q_l Q^{l\dagger} - \tilde{Q}_l^\dagger \tilde{Q}^l) - [Z, Z^\dagger] = r, \tag{2.23}$$

and

$$\sum_{l=1}^L \left| \lambda Q_l - Q_l M_l \right|^2 + \sum_{l=1}^L \left| -\tilde{Q}^l \lambda + \tilde{Q}^l \tilde{M}_l \right|^2 = 0, \tag{2.24}$$

where λ denotes the adjoint scalar field in the vector multiplet.

For $r = 0$, $Q_l = \tilde{Q}_l = 0$ and Theory II has a classical Coulomb branch parametrized by the eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ of the adjoint scalar field in the $U(N)$ vector multiplet. In the figure this corresponds to the special case where each D2 is suspended between NS5 and NS5' and can move independently in the x_4 and x_5 directions. On the other hand, the eigenvalues of Z parameterise the position of D2-branes in the {2,3}-plane.

⁹More precisely Theory II arises in a particular decoupling limit of the brane configuration. All conventions relating to the intersecting brane configuration shown in figure 4 are the same as in [41].

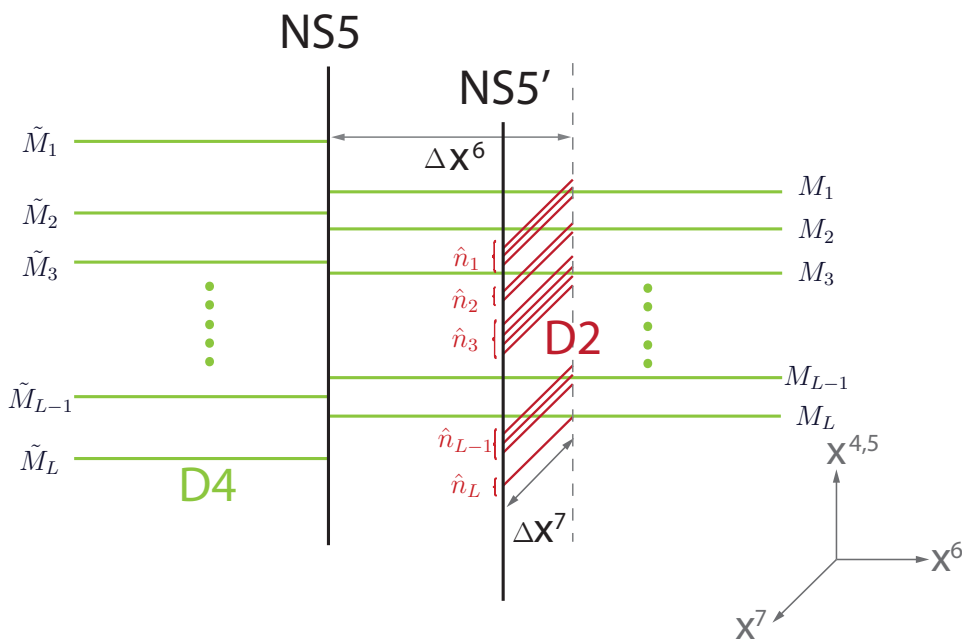


Figure 4. A IIA brane construction for Theory II with $\epsilon = 0$.

For $r > 0$, the theory is on a Higgs branch with $Q \neq 0$, $\tilde{Q} = 0$. The vector multiplet VEVs are fixed by the second D-term condition (2.24). Solutions are labelled by the number of ways of distributing the N scalars $\{\lambda_j\}$ between the L values $\{M_l\}$. Thus we specify a vacuum by choosing L non-negative integers $\{\hat{n}_l\}$ with $\sum_{l=1}^L \hat{n}_l = N$. In the brane construction these correspond to the number of D2 branes ending on each D4 brane as shown in the figure.

The brane construction also reveals an interesting physical relation between Theory I and Theory II [18]. In the absence of the D2 branes, the worldvolume theory on the intersection of the remaining branes is precisely $\mathcal{N} = 2$ SQCD with gauge group $SU(L)$ and $N_F = 2L$ hypermultiplets with masses $m_l = M_l$ and $\tilde{m}_l = \tilde{M}_l$ and complex gauge coupling τ . In other words, it is Theory I in the undeformed case $\epsilon = 0$. To understand the relation, compare the brane configuration shown in figure 1 with the configuration in figure 4 (in the absence of D2 branes). The former configuration represents a generic point on the Coulomb branch of Theory I. To pass to the configuration shown in figure 4, we reconnect the D4 branes on NS5' and then move NS5' away from the D4 branes in the x_7 direction. The first step corresponds to moving on the Coulomb branch to the Higgs branch root root (see figure 2), the second to moving out along the Higgs branch. The theory on the Higgs branch admits vortex strings which corresponds to the D2 branes in the figure 4. Thus we identify Theory II as the worldvolume theory on N vortex strings probing the Higgs branch of Theory I. For a review of such non-abelian vortex string see [20, 21].

At this point there are several subtleties. First, to have BPS vortices with a supersymmetric worldvolume theory we must consider a four-dimensional theory with gauge group $U(L)$ rather than $SU(L)$. Thus, although the F-term duality described in this paper works

the same for both theories, the interpretation in terms of vortex strings is only available in the $U(L)$ case. Next, the gauge theory designated as Theory II above cannot be directly identified as the worldvolume theory of the vortex. Rather, Theory II is an $\mathcal{N} = (2, 2)$ gauged linear σ -model which flows in the IR to a non-linear σ -model (with twisted mass terms) whose target space is a certain Kähler manifold which is closely related to the moduli space of N vortices in Theory I [19]. In the present case, where the four-dimensional theory is conformal, the target space has zero first Chern class and can therefore be expected to admit a unique Ricci-flat metric which provides a natural IR fixed-point for the worldsheet theory. In fact, the actual Kähler metric on the classical vortex moduli space differs from this target space metric. Even worse, as we are discussing semi-local vortices, some elements of the true classical moduli space metric diverge due to the non-normalisability of some of the vortex zero modes (see eg [22, 23]). Although this has not been analysed in detail, we expect that this problem is cured once the Ω -background is reintroduced.¹⁰ Indeed, as above, a Nekrasov deformation in one plane renders even the vacuum moduli of the four-dimensional gauge theory normalisable as fields in a two-dimensional effective action. Correspondingly we expect that, in the present context, the true world-volume theory of N vortices flows to the same IR fixed point as Theory II. However, what we actually need here is much weaker: that the two theories agree at the level of $\mathcal{N} = (2, 2)$ F-terms and we will assume this is the case.

With the identification described above, the numbers $\{\hat{n}_l\}$ labelling the classical vacua of the worldvolume theory determine the number of units of magnetic flux in each $U(1)$ of the Cartan subalgebra of $U(L)$. The tension of the vortex strings is controlled by the Higgs branch VEV v of the four-dimensional theory, proportional to Δx^7 . In the limit, $v \rightarrow \infty$, the tension diverges and the vortex strings become static surface operators of the type considered in [34].

Next we restore the Nekrasov deformation to Theory I. From a two-dimensional perspective this corresponds to a twisted mass term for fields charged under rotations in the x_2 - x_3 plane. The $U(N)$ adjoint chiral multiplet has charge +1 under this symmetry and this field acquires mass ϵ . Thus the adjoint chiral mass in Theory II is identified with the Nekrasov deformation parameter ϵ in Theory I. More generally, we can consider a twisted mass corresponding to 2-3 rotations mixed with the other $U(1)$ global symmetries which act on the chiral multiplets Q and \tilde{Q} . In fact the relevant symmetry for the duality we consider turns out to be one under which Q has charge $-3/2$ and \tilde{Q} has charge $+1/2$. The corresponding dictionary between the 2d and 4d masses then becomes,

$$\vec{M}_F = \vec{m}_F - \frac{3}{2}\vec{\epsilon}, \quad \vec{M}_{AF} = \vec{m}_{AF} + \frac{1}{2}\vec{\epsilon}. \tag{2.25}$$

where $\vec{\epsilon} = (\epsilon, \epsilon, \dots, \epsilon)$.

Having discussed the classical behaviour of Theory I, we now turn our attention to the quantum theory. Quantum effects lift the classical Coulomb and Higgs branches described above leaving only isolated supersymmetric vacua corresponding to stationary points of the twisted superpotential. Upon integrating out the matter multiplets we obtain an effective

¹⁰An alternative way of regulating this divergence is proposed in [23].

twisted superpotential on the Coulomb branch of the form,

$$\begin{aligned} \mathcal{W}^{(II)}(\lambda) = & 2\pi i \hat{\tau} \sum_{j=1}^N \lambda_j - \epsilon \sum_{j=1}^N \sum_{l=1}^L f\left(\frac{\lambda_j - M_l}{\epsilon}\right) \\ & + \epsilon \sum_{j=1}^N \sum_{l=1}^L f\left(\frac{\lambda_j - \tilde{M}_l}{\epsilon}\right) + \epsilon \sum_{j,k=1}^N f\left(\frac{\lambda_j - \lambda_k - \epsilon}{\epsilon}\right), \end{aligned} \quad (2.26)$$

where $f(x) = x(\log x - 1)$ and $\hat{\tau} = ir + \frac{\theta}{2\pi}$ denotes the two-dimensional holomorphic coupling constant. It is known that this twisted superpotential is one-loop exact. Defining

$$q = \exp(2\pi i \tau) \equiv (-1)^{N+1} \exp(2\pi i \hat{\tau}),$$

the resulting F-term equations can be expressed as below

$$\prod_{l=1}^L \left(\frac{\lambda_j - M_l}{\lambda_j - \tilde{M}_l} \right) = q \prod_{k \neq j} \left(\frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon} \right) \quad (2.27)$$

with $j = 1, 2, \dots, N$, coincide with the BAE (2.20) with the identification of the variables $\{\lambda_j\}$ with $\{x_j\}$ and setting,

$$\vec{M}_F = \vec{\theta} + i\vec{s}\hbar \quad \vec{M}_{AF} = \vec{\theta} - i\vec{s}\hbar$$

and $\epsilon = -i\hbar$. The marginal coupling τ is identified with the corresponding parameter in the Yang-Yang functional.

The space of supersymmetric ground states of Theory II is now identified with the Hilbert space of the quantum $SL(2, \mathbb{R})$ spin chain. Strictly speaking this correspondence holds when the complex parameters of Theory II satisfy the reality conditions implied by the above identifications. More generally one must consider an analytic continuation of the quantum spin chain to complex values of its parameters. The rank N of the 2d gauge group corresponds to the number of magnons in the spin chain state. Pleasingly the total number of states of the spin chain containing N magnons is the number of partitions of N into L non-negative integers which is the same as the number of classical SUSY vacua of the theory. It is important to note that these vacua, like the states of the spin chain, correspond to *non-degenerate* solutions of (2.27) where $\lambda_i \neq \lambda_j$ for all $i \neq j$. Degenerate solutions are points where the Coulomb branch effective description breaks down and any corresponding vacuum states would have unbroken non-abelian gauge symmetry. The absence of such vacua is consistent with the results of [24].

To summarise the above discussion, Theory II corresponds to the worldvolume theory of N vortex strings probing the Higgs branch of Theory I. The tension of these strings is controlled by the Higgs branch VEV of the four-dimensional theory which corresponds to a D-term in $\mathcal{N} = (2, 2)$ superspace [18]. The F-term equations of motion of Theory II coincide with the BAE for the $SL(2, \mathbb{R})$ Heisenberg spin chain. This conclusion is independent of D-term couplings and therefore hold equally well in the limit of infinite tension where the vortex string becomes a surface operator.

2.5 The duality proposal

As mentioned above starting from a complex classical integrable system there can be many inequivalent reality conditions and many different quantizations. In subsections 2.1 and 2.3 we have discussed the particular reality conditions which lead to the $SL(2, \mathbb{R})$ Heisenberg spin chain and the standard quantization which leads to an integrable quantum spin chain. On the other hand we have reviewed the Nekrasov-Shatashvili quantization which produces a family of reality conditions and quantizations related by electric-magnetic duality transformations. It is natural to ask whether one of this family corresponds to the standard quantization of the $SL(2, \mathbb{R})$ spin chain. To start with we can address this question in the semiclassical regime of small ϵ . In this case the quantization conditions of the NS proposal take the generic form,

$$\oint_{\vec{\mathcal{A}}} \lambda_{\text{SW}} \in \epsilon \mathbb{Z},$$

for some choice of L one-cycles $\vec{\mathcal{A}} = (\mathcal{A}_1, \dots, \mathcal{A}_L)$ on Γ_L which are the image under the low-energy duality group of the basis cycles \vec{B} appearing in the Type A quantization condition. We obtain agreement with the semiclassical limit of the $SL(2, \mathbb{R})$ chain if $\epsilon = -i\hbar$ and the basis cycles are chosen as $\vec{\mathcal{A}} = \vec{\alpha} = \vec{A} - \vec{C}_F$.

With the choices outlined above, the NS quantization condition coincides with the standard one at leading semiclassical order. In a generic quantum system it is easy to imagine different quantizations which agree at leading order in \hbar . However, for a real classical integrable system with a given symplectic form, quantizations which preserve integrability are very special and we do not know of an example where two distinct quantizations coincide at leading semiclassical order. For this reason, we conjecture that the agreement between the NS quantization and the standard one persists to all orders.

The Hilbert space resulting from the NS quantization corresponds to the space of SUSY vacua determined by the superpotential,

$$\hat{\mathcal{W}}_D^{(I)}(\vec{a}_D) = \mathcal{W}^{(I)}(\vec{a}) - \frac{2\pi i}{\epsilon} (\vec{a} - \vec{m}_F) \cdot \vec{a}^D, \tag{2.28}$$

which is obtained by applying a duality transformation to the superpotential $\mathcal{W}^{(I)}$ given in (2.12). In addition to the standard electric-magnetic duality, the transformation includes the shift $\vec{a} \rightarrow \vec{a} - \vec{m}_F$, $\vec{a}_D \rightarrow \vec{a}_D$ which is also part of the low-energy duality group [27]. The dual superpotential leads to the F-term equations,

$$\vec{a} - \vec{m}_F \in \epsilon \mathbb{Z}^L,$$

while the Hilbert space of the standard quantization coincides with the space of SUSY vacua with superpotential (2.26) whose F-term equations coincide with the BAE. Equivalence between the two quantization schemes leads to the following duality conjecture,

Proposal. The twisted chiral rings of Theory I and Theory II are isomorphic. This means in particular that the stationary points of the twisted superpotentials (2.28) and (2.26) are

in one-to-one correspondence and, with appropriate identifications between the complex parameters, these superpotentials take the same value in corresponding vacua;

$$\mathcal{W}^{(I)} \stackrel{\text{on-shell}}{\equiv} \mathcal{W}^{(II)}$$

where equality holds up to an additive vacuum-independent constant. Here, we use the fact that $\mathcal{W}^{(I)}$ and $\hat{\mathcal{W}}_D^{(I)}$ are equal on-shell as multi-valued functions. In the next section we will test this proposal by explicit calculation on both sides.

To complete the map between the chiral rings of the two theories we should also give formulae for the VEVs of the tower of chiral operators¹¹ $\hat{O}_k = \text{Tr } \varphi^k = \sum_{l=1}^L \phi_l^k$ in each vacuum state in terms of the corresponding set of Bethe roots $\{\lambda_j\}$. Given the correspondence between these operators and the conserved charges of the classical spin chain which holds for $\epsilon = 0$, it is natural to conjecture that, in the deformed theory, they are related to the corresponding conserved charges of the quantum spin chain;

$$\left\langle \{\lambda_j\} \left| \prod_{l=1}^L (\lambda - \phi_l) \right| \{\lambda_j\} \right\rangle = \frac{1}{2} t(\lambda - \lambda_0) \tag{2.29}$$

with,

$$t(\lambda) = \frac{-h \prod_{l=1}^L (\lambda - \tilde{M}_l) \prod_{j=1}^N (\lambda - \lambda_j - \epsilon) + (h + 2) \prod_{l=1}^L (\lambda - M_l) \prod_{j=1}^N (\lambda - \lambda_j + \epsilon)}{\prod_{j=1}^N (\lambda - \lambda_j)}$$

Here the shift parameter λ_0 is chosen to impose the $SU(L)$ condition $\sum_{l=1}^L \phi_l = 0$. In the weak-coupling limit $\tau \rightarrow \infty$, one may check the formula holds with $\lambda_0 = \epsilon/2$ using the results of the next section. These additional predictions will be discussed further in [61].

Finally we note that other aspects of the relation between $\mathcal{N} = 2$ SQCD and the quantum spin chain have been checked very recently in [51]. Specifically, in this paper, it is verified that the A- and B-type quantization conditions match the quantisation conditions for periodicity of a wave-function obeying the Baxter equation of the spin chain for general complex values of the parameters. A similar agreement to the quantization condition of the Gaudin model was obtained earlier in [52].

3 A weak-coupling test

We present in this section a detailed comparison at the weak coupling limit between the twisted superpotentials of both theories in the proposed duality. As discussed below, a careful analysis shows perfect agreement.

In the following we will find a correspondence where the numbers \hat{n}_l of D2 branes in Theory II are identified with the integers appearing in the quantization condition $a_l =$

¹¹In general, the definition of these quantities is afflicted by the usual ambiguities in parametrising the Coulomb branch [49]. For fixed L , the ambiguity corresponds to a finite number of *vacuum independent* coefficients.

$m_l - n_l \epsilon$ for Theory I according to $\hat{n}_l = n_l - 1$. Suggestively both sets of integers correspond to quantized magnetic fluxes in the Cartan subalgebra of the gauge group. As \hat{n}_l are by definition non-negative n_l must be strictly positive. The trace condition for the $SU(L)$ gauge group also implies the additional constraint on the sum of the hypermultiplet masses; $\sum_{l=1}^L m_l = (N + L)\epsilon$.

3.1 Theory I

The Nekrasov partition function is composed of three factors corresponding to the classical, the one-loop perturbative, and the instanton contributions:

$$\mathcal{Z}_{\text{Nek}}(\vec{a}, \tau, \epsilon_1, \epsilon_2) = \mathcal{Z}_{\text{cl}} \times \mathcal{Z}_{1\text{-loop}} \times \mathcal{Z}_{\text{inst}},$$

which leads to a corresponding decomposition of the twisted superpotential of Theory I

$$\mathcal{W}^{(I)}(\vec{a}, \tau, \epsilon) = \mathcal{W}_{\text{cl}} + \mathcal{W}_{1\text{-loop}} + \mathcal{W}_{\text{inst}}. \tag{3.1}$$

We will evaluate the on-shell value of these terms in order. More precisely, we will compute a difference of the twisted superpotential value at a vacuum $\vec{a} = \vec{m} - \vec{n}\epsilon$ and at the root of the baryonic Higgs branch¹² $\vec{a} = \vec{m} - \epsilon$ which corresponds to the ferromagnetic ground state of the spin chain. This subtraction corresponds to a vacuum independent (i.e. \vec{n} -independent) constant shift of the superpotential.

$$\mathcal{G} = \mathcal{W}^{(I)} \Big|_{\vec{a}=\vec{m}-\vec{n}\epsilon} - \mathcal{W}^{(I)} \Big|_{\vec{a}=\vec{m}-\epsilon}. \tag{3.2}$$

Strictly speaking, the standard results of [28, 29] apply to the partition function of the $U(L)$ theory which differs from the Nekrasov partition function of the $SU(L)$ theory by a multiplicative factor which does not depend on the Coulomb branch moduli \vec{a} [33]. The resulting difference between the superpotentials of the $SU(L)$ and $U(L)$ theories is therefore vacuum-independent additive constant which is not visible in our analysis.

In order to avoid degenerate cases, we suppose that $n_l, l = 1, 2, \dots, L$, satisfy the following relations

$$n_1 \geq n_2 \geq \dots \geq n_L, \quad m_l - n_l \epsilon \geq m_n - n_n \epsilon \quad \text{if } l < n. \tag{3.3}$$

Classical part. The classical part \mathcal{Z}_{cl} of the partition function is given by

$$\mathcal{Z}_{\text{cl}} = \exp \left[-\frac{2\pi i}{\epsilon_1 \epsilon_2} \sum_{l=1}^L \frac{\tau_0}{2} a_l^2 \right],$$

where as above τ_0 denotes the bare marginal coupling constant of the $SU(L)$ gauge group. This coupling is slightly different to the the coupling τ in the Seiberg-Witten curve (2.1),

¹²Note the shift of $-\epsilon$ in the location of the Higgs branch root which follows from the conventions for the Ω background used in [29, 33]. The location of the Higgs branch root is determined as the point on the Coulomb branch where additional hypermultiplets become massless.

due to some ambiguities in the perturbative computation. We will discuss it further below. The classical part of the function \mathcal{G} is therefore given by

$$\begin{aligned} \mathcal{G}_{\text{cl}} &= -\frac{2\pi i\tau_0}{\epsilon} \sum_{l=1}^L \frac{1}{2} \left((m_l - n_l\epsilon)^2 - (m_l - \epsilon)^2 \right) \\ &= 2\pi i\tau_0 \sum_{l=1}^L \left((n_l - 1)(m_l - \epsilon) - \frac{1}{2}(n_l - 1)^2\epsilon \right). \end{aligned} \quad (3.4)$$

1-loop part. The one-loop contribution $\mathcal{Z}_{1\text{-loop}}$ consists of three factors coming from the vector multiplet and L fundamental/anti-fundamental hypermultiplets. As mentioned in [28] and above, the one-loop contribution has an ambiguity in fixing the quadratic terms. It describes the finite renormalization of the classical part of Z . Following the convention of [29],

$$\mathcal{Z}_{1\text{-loop}} = z_{\text{vec}} \times z_{\text{fund}} \times z_{\text{anti-fund}},$$

with

$$\begin{aligned} z_{\text{vec}} &= \prod_{l,m=1}^L \frac{1}{\text{Exp}[\gamma_{\epsilon_1,\epsilon_2}(a_{lm})]}, \\ z_{\text{fund}} &= \prod_{l,m=1}^L \text{Exp}[\gamma_{\epsilon_1,\epsilon_2}(a_l - m_m)], \\ z_{\text{anti-fund}} &= \prod_{l,m=1}^L \text{Exp}[\gamma_{\epsilon_1,\epsilon_2}(a_l - \tilde{m}_m - \epsilon_1 - \epsilon_2)]. \end{aligned} \quad (3.5)$$

where $a_{lm} = a_l - a_m$. Here, the function $\gamma_{\epsilon_1,\epsilon_2}(x)$ is the logarithm of Barnes' double gamma function whose properties are summarized in [29]. In particular, the function $\gamma_{\epsilon_1,\epsilon_2}(x)$ obeys a difference equation below

$$\gamma_{\epsilon_1,\epsilon_2}(x) + \gamma_{\epsilon_1,\epsilon_2}(x - \epsilon_1 - \epsilon_2) - \gamma_{\epsilon_1,\epsilon_2}(x - \epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(x - \epsilon_2) = \log \frac{\Lambda}{x}. \quad (3.6)$$

Here, we are primarily interested in its limiting form as $\epsilon_2 \rightarrow 0$. Defining a new function $\omega_\epsilon(x)$ with $\Lambda = \epsilon$ by

$$\omega_\epsilon(x) = \lim_{\epsilon_2 \rightarrow 0} [\epsilon_2 \gamma_{\epsilon_1=\epsilon,\epsilon_2}(x)], \quad (3.7)$$

one can then verify from (3.6) the following relation

$$\omega'_\epsilon(x) - \omega'_\epsilon(x - \epsilon) = \log \frac{x}{\epsilon}, \quad \omega'_\epsilon(x) = \frac{d}{dx} \omega_\epsilon(x).$$

As in [5], it immediately implies that

$$\omega'_\epsilon(x) = -\log \Gamma(1 + x/\epsilon). \quad (3.8)$$

For later convenience, we present a relation for $\omega_\epsilon(x)$ with positive n

$$\omega_\epsilon(x + n\epsilon) - \omega_\epsilon(x) = -\epsilon \sum_{j=1}^n f\left(\frac{x}{\epsilon} + j\right) - \frac{n\epsilon}{2} \log 2\pi, \quad f(x) = x(\log x - 1), \quad (3.9)$$

which will be frequently used in what follows. The branch of the multi-valued function $f(x)$ is chosen here that, for non-negative x ,

$$f(x) = -f(-x) + i\pi x.$$

The one-loop contribution to the twisted superpotential of Nekrasov and Shatashvili can therefore be expressed as

$$\mathcal{W}_{1\text{-loop}}^{(I)} = \varpi_{\text{vec}} + \varpi_{\text{fund}} + \varpi_{\text{anti-fund}}$$

with

$$\begin{aligned} \varpi_{\text{vec}} &= -\sum_{l,m} \omega_\epsilon(a_{lm}), \\ \varpi_{\text{fund}} &= \sum_{l,m} \omega_\epsilon(a_l - m_m), \\ \varpi_{\text{anti-fund}} &= \sum_{l,m} \omega_\epsilon(a_l - \tilde{m}_m - \epsilon). \end{aligned} \quad (3.10)$$

Using the relation (3.9), one can show that

$$\begin{aligned} \mathcal{G}_{1\text{-loop}}^{\text{fund}} &= \epsilon \sum_{l,m} \left[\sum_{k=1-n_l}^{-1} f\left(\frac{m_{lm}}{\epsilon} + k\right) + \frac{n_l - 1}{2} \log 2\pi \right] \\ \mathcal{G}_{1\text{-loop}}^{\text{anti-fund}} &= \epsilon \sum_{l,m} \left[\sum_{k=0}^{n_l-2} f\left(\frac{m_l - \tilde{m}_m - 2\epsilon}{\epsilon} - k\right) + \frac{n_l - 1}{2} \log 2\pi \right], \\ \mathcal{G}_{1\text{-loop}}^{\text{vec}} &= -\epsilon \sum_l \sum_{m>l} \left[\sum_{k=1+n_m-n_l}^0 + \sum_{k=n_m-n_l}^{-1} \right] f\left(\frac{m_{lm}}{\epsilon} + k\right) + i\pi\epsilon \sum_{l,m>l} \sum_{k=n_m-n_l}^{-1} \left(\frac{m_{lm}}{\epsilon} + k\right). \end{aligned} \quad (3.11)$$

After some algebra whose details are discussed in appendix A, one can finally obtain

$$\mathcal{G}_{1\text{-loop}} = -\epsilon \sum_{l,m=1}^L \sum_{k=1+n_m-n_l}^{n_m-1} f\left(\frac{m_{lm}}{\epsilon} + k\right) + i\pi L \sum_l \left[(n_l - 1)m_l - \frac{(n_l - 1)^2}{2}\epsilon \right], \quad (3.12)$$

up to some constants which do not depend on the choice of vacuum.

Collecting the results so far, the perturbative contribution $\mathcal{G}_{\text{pert}} = \mathcal{G}_{\text{cl}} + \mathcal{G}_{1\text{-loop}}$ takes the following form

$$\mathcal{G}_{\text{pert}} = 2\pi i\tau \sum_{l=1}^L \left((n_l - 1)(m_l - \epsilon) - \frac{1}{2}(n_l - 1)^2\epsilon \right) - \epsilon \sum_{l,m=1}^L \sum_{k=1+n_m-n_l}^{n_m-1} f\left(\frac{m_{lm}}{\epsilon} + k\right), \quad (3.13)$$

where $\tau = \tau_0 + \frac{L}{2}$ denotes the marginal coupling constant in the curve.

Instanton contribution. The instanton contribution to the partition function is written as a sum over coloured partitions

$$\begin{aligned} \mathcal{Z}_{\text{inst}} &= 1 + \sum_{k=1}^{\infty} \mathcal{Z}_{\text{inst}}^{(k)} ((-1)^L q)^k \\ &= \sum_{\vec{Y}} \mathcal{Z}_{\vec{Y}} (-1)^{L|\vec{Y}|} q^{|\vec{Y}|}, \end{aligned}$$

where the factors of $(-1)^L$ are included so that definition of $q = e^{2\pi i \tau}$ agrees with the coupling that appears in the Seiberg-Witten curve. For $U(L)$ gauge group, the coloured partitions $\vec{Y} = (Y_1, Y_2, \dots, Y_L)$ are labelled by N Young diagrams Y_i . The total number of boxes $|\vec{Y}| = \sum_{i=1}^N |Y_i|$ is the instanton number k . The explicit form of $\mathcal{Z}_{\vec{Y}}$ can be found in [33].

Denoting the unique partition of unity as $\{\mathbf{1}\}$ and the two inequivalent partitions of two as $\{\mathbf{2}\}$ and $\{\mathbf{1}, \mathbf{1}\}$, the following coloured partitions can contribute:

- In the one-instanton sector $k = 1$, we have partitions $\vec{Y}_{\{\mathbf{1}\}}^{(l)}$ with components $(Y_{\{\mathbf{1}\}}^{(l)})_m = \delta_{lm} \{\mathbf{1}\}$ for $l, m = 1, 2, \dots, L$. The one-instanton contribution to the partition function therefore becomes

$$\mathcal{Z}_{\text{inst}}^{(1)} = \mathcal{Z}_{\{\mathbf{1}\}} = - \frac{(-1)^L}{\epsilon_1 \epsilon_2} \sum_{l=1}^L R_l(a_l, \epsilon_1 + \epsilon_2), \quad (3.14)$$

where

$$R_l(x, \epsilon) = \frac{\prod_{m=1}^L (x - m_m + \epsilon)(x - \tilde{m}_m)}{\prod_{m \neq l} (x - a_m + \epsilon)(x - a_m)}. \quad (3.15)$$

- In the two-instanton sector $k = 2$, we have coloured partitions $\vec{Y}_{\{\mathbf{2}\}}^{(l)}$, $\vec{Y}_{\{\mathbf{1}, \mathbf{1}\}}^{(l)}$ ($l = 1, 2, \dots, L$), and $\vec{Y}_{\{\mathbf{1}\}, \{\mathbf{1}\}}^{(l, m)}$ ($l, m = 1, 2, \dots, L$ and $l \neq m$) with components

$$\delta_{ln} \{\mathbf{2}\}, \quad \delta_{ln} \{\mathbf{1}, \mathbf{1}\}, \quad \text{and} \quad \delta_{ln} \{\mathbf{1}\} + \delta_{mn} \{\mathbf{1}\},$$

respectively. The corresponding contributions to the partition function are given by

$$\begin{aligned} \mathcal{Z}_{\{\mathbf{2}\}} &= \frac{1}{2\epsilon_1 \epsilon_2^2 (\epsilon_1 - \epsilon_2)} \sum_{l=1}^L R_l(a_l, \epsilon_1 + \epsilon_2) R_l(a_l + \epsilon_2, \epsilon_1 + \epsilon_2), \\ \mathcal{Z}_{\{\mathbf{1}, \mathbf{1}\}} &= \frac{1}{2\epsilon_2 \epsilon_1^2 (\epsilon_2 - \epsilon_1)} \sum_{l=1}^L R_l(a_l, \epsilon_1 + \epsilon_2) R_l(a_l + \epsilon_1, \epsilon_1 + \epsilon_2), \\ \mathcal{Z}_{\{\mathbf{1}\}, \{\mathbf{1}\}} &= \frac{1}{\epsilon_1^2 \epsilon_2^2} \sum_{l \neq m} R_l(a_l, \epsilon_1 + \epsilon_2) R_m(a_m, \epsilon_1 + \epsilon_2) \frac{a_{lm}^2 (a_{lm}^2 - (\epsilon_1 + \epsilon_2)^2)}{(a_{lm}^2 - \epsilon_1^2) (a_{lm}^2 - \epsilon_2^2)}. \end{aligned} \quad (3.16)$$

The two-instanton contribution to the partition function takes the following form

$$\mathcal{Z}_{\text{inst}}^{(2)} = \mathcal{Z}_{\{\mathbf{2}\}} + \mathcal{Z}_{\{\mathbf{1}, \mathbf{1}\}} + \mathcal{Z}_{\{\mathbf{1}\}, \{\mathbf{1}\}}. \quad (3.17)$$

Since the logarithm of the instanton partition function can be expanded as below

$$\begin{aligned} \log \mathcal{Z}_{\text{inst}} &= (-1)^L q \mathcal{Z}_{\text{inst}}^{(1)} + q^2 \left[\mathcal{Z}_{\text{inst}}^{(2)} - \frac{1}{2} (\mathcal{Z}_{\text{inst}}^{(1)})^2 \right] + \mathcal{O}(q^3) \\ &= (-1)^L q \mathcal{Z}_{\{1\}} + q^2 \left[\mathcal{Z}_{\{2\}} + \mathcal{Z}_{\{1,1\}} + \mathcal{Z}_{\{1\},\{1\}} - \frac{1}{2} (\mathcal{Z}_{\{1\}})^2 \right] + \mathcal{O}(q^3), \end{aligned}$$

the instanton contribution to the twisted superpotential becomes

$$\mathcal{W}_{\text{inst}}^{(I)} = \lim_{\epsilon_2 \rightarrow 0} [\epsilon_2 \log \mathcal{Z}_{\text{inst}}(\epsilon_1 = \epsilon, \epsilon_2)] = \sum_{k=1}^{\infty} \mathcal{W}_{\text{inst}}^{(k)} q^k, \quad (3.18)$$

where

$$\begin{aligned} \mathcal{W}_{\text{inst}}^{(1)} &= -\frac{1}{\epsilon} \sum_{l=1}^L R_l(a_l, \epsilon) \\ \mathcal{W}_{\text{inst}}^{(2)} &= +\frac{1}{2\epsilon^3} \sum_{l=1}^L [R_l(a_l, \epsilon)^2 - R_l(a_l, \epsilon)R_l(a_l + \epsilon, \epsilon) + \epsilon R_l(a_l, \epsilon)R'_l(a_l, \epsilon)] \\ &\quad - \frac{1}{\epsilon} \sum_{l \neq m} \frac{1}{a_{lm}^2 - \epsilon^2} R_l(a_l, \epsilon)R_m(a_m, \epsilon). \end{aligned} \quad (3.19)$$

Due to the fact that there arise additional zero modes at the root of baryonic branch, the instanton contribution to the partition function should vanish at $\vec{a} = \vec{m} - \epsilon$. Indeed one can show that

$$R_l(m_l - \epsilon) = 0.$$

The above relation simplifies the evaluation of the instanton contribution to the function $\mathcal{G}_{\text{inst}}$ of our interest, and finally we evaluate the superpotential on-shell to obtain,

$$\mathcal{G}_{\text{inst}} = \sum_{k=1}^{\infty} \mathcal{G}_{\text{inst}}^{(k)} q^k$$

with

$$\begin{aligned} \mathcal{G}_{\text{inst}}^{(1)} &= \mathcal{W}_{\text{inst}}^{(1)} \Big|_{a_l = m_l - n_l \epsilon} = -\frac{1}{\epsilon} \sum_{l=1}^L R_l(m_l - n_l \epsilon), \\ \mathcal{G}_{\text{inst}}^{(2)} &= \mathcal{W}_{\text{inst}}^{(2)} \Big|_{a_l = m_l - n_l \epsilon} \\ &= +\frac{1}{2\epsilon^3} \sum_{l=1}^L \left[R_l(m_l - n_l \epsilon, \epsilon)^2 - R_l(m_l - n_l \epsilon, \epsilon)R_l(m_l - (n_l - 1)\epsilon, \epsilon) \right. \\ &\quad \left. + R_l(m_l - n_l \epsilon, \epsilon)R'_l(m_l - n_l \epsilon, \epsilon) \right] \\ &\quad - \frac{1}{\epsilon} \sum_{l \neq m} \frac{1}{(m_{lm} - (n_l - n_m)\epsilon)^2 - \epsilon^2} R_l(m_l - n_l \epsilon, \epsilon)R_m(m_m - n_m \epsilon, \epsilon). \end{aligned} \quad (3.20)$$

3.2 Theory II

Let us now consider the Theory II as introduced in subsection 2.4. The one-loop exact twisted superpotential $\mathcal{W}^{(II)}$ on the Coulomb branch takes the following form

$$\mathcal{W}^{(II)} = \mathcal{W}_{\text{tree}} + \mathcal{W}_{\text{fund}} + \mathcal{W}_{\text{anti-fund}} + \mathcal{W}_{\text{adj}}, \quad (3.21)$$

where

$$\begin{aligned} \mathcal{W}_{\text{tree}} &= 2\pi i\tau \sum_{i=1}^N \lambda_i - i\pi(N+1) \sum_{i=1}^N \lambda_i \\ \mathcal{W}_{\text{fund}} &= -\epsilon \sum_{i=1}^N \sum_{l=1}^L f\left(\frac{\lambda_i - M_l}{\epsilon}\right) \\ \mathcal{W}_{\text{anti-fund}} &= \epsilon \sum_{i=1}^N \sum_{l=1}^L f\left(\frac{\lambda_i - \tilde{M}_l}{\epsilon}\right) \\ \mathcal{W}_{\text{adj}} &= \epsilon \sum_{i,j=1}^N f\left(\frac{\lambda_i - \lambda_j}{\epsilon} - 1\right) \end{aligned} \quad (3.22)$$

with $f(x) = x(\log x - 1)$. Here τ is related to the two-dimensional holomorphic coupling constants $\hat{\tau} = ir + \theta/2\pi$ as $\hat{\tau} = \tau - \frac{N+1}{2}$.

Vacuum solution. As discussed before, it turns out that the F-term vacuum equation $\exp(\partial\mathcal{W}^{(II)}/\partial\lambda_i) = 1$ indeed leads us to an algebraic BAEs

$$\prod_{l=1}^L (\lambda_i - M_l) \prod_{j \neq i} (\lambda_i - \lambda_j + \epsilon) = q \prod_{l=1}^L (\lambda_i - \tilde{M}_l) \prod_{j \neq i} (\lambda_i - \lambda_j - \epsilon), \quad (3.23)$$

where $q = e^{2\pi i\tau}$. One can solve the above BAE in an iterative way by expanding the lowest component of twisted chiral superfield λ_i in the instanton factor q

$$\lambda_i = \sum_{k=1}^{\infty} \lambda_i^{(k)} q^k. \quad (3.24)$$

At leading order, we have

$$\prod_{l=1}^L (\lambda_i^{(0)} - M_l) \prod_{j \neq i} (\lambda_i^{(0)} - \lambda_j^{(0)} + \epsilon) = 0. \quad (3.25)$$

O obeying the non-degeneracy condition $\lambda_i^{(0)} \neq \lambda_j^{(0)}$, one can show that the solution at leading order can be characterized by partition of N with L parts, i.e.,

$$\{\hat{n}_1, \hat{n}_2, \dots, \hat{n}_L\}, \quad \sum_{l=1}^L \hat{n}_l = N, \quad (3.26)$$

and takes the following form

$$\lambda_i^{(0)} = \lambda_{(l,s_l)}^{(0)} = M_l - (s_l - 1)\epsilon, \quad (3.27)$$

where s_l runs over $1, 2, \dots, \hat{n}_l$. In the language of the spin chain these solutions take the form of Bethe strings. Here the strings arise in an unfamiliar limit where the twist parameter $h \simeq -2q$ goes to zero. Such strings are more usually associated with the thermodynamic limit of the spin chain where they represent magnon bound states.

Let us now in turn discuss the solution at the next order $\lambda_i^{(1)}$. Expanding the equation (3.23) to the next order, one can show that $\lambda_{(l,s_l)}^{(1)}$ always vanish except $s_l = \hat{n}_l$. One can thus set

$$\lambda_{(l,s_l)}^{(1)} = \delta_{s_l, \hat{n}_l} \lambda_l^{(1)}, \quad (3.28)$$

where $\lambda_l^{(1)}$ satisfies the relation below

$$\begin{aligned} & \left[\prod_{m=1}^L (M_l - M_m - (\hat{n}_l - 1)\epsilon) \prod_{(m,s_m) \neq (l, \hat{n}_l), (l, \hat{n}_l - 1)} (M_l - M_m - (\hat{n}_l - s_m - 1)\epsilon) \right] \cdot \lambda_l^{(1)} \\ &= \prod_{m=1}^L (M_l - \tilde{M}_m - (\hat{n}_l - 1)\epsilon) \prod_{(m,s_m) \neq (l, \hat{n}_l)} (M_l - M_m - (\hat{n}_l - s_m + 1)\epsilon). \end{aligned} \quad (3.29)$$

After some algebra, it leads to

$$\lambda_l^{(1)} = -\frac{1}{\epsilon} \cdot \frac{\prod_m (M_{lm} - \hat{n}_l \epsilon) \cdot (M_l - \tilde{M}_m - (\hat{n}_l - 1)\epsilon)}{\prod_{m \neq l} (M_{lm} - (\hat{n}_l - \hat{n}_m)\epsilon) \cdot (M_{lm} - (\hat{n}_l - \hat{n}_m - 1)\epsilon)}. \quad (3.30)$$

Expanding (3.23) to the next order we discover that $\lambda_{l,s_l}^{(2)}$ again vanishes unless $s_l = \hat{n}_l$ or $\hat{n}_l - 1$. The expansion (3.24) therefore takes the following form,

$$\lambda_{l,s_l} = \lambda_{l,s_l}^{(0)} + \delta_{s_l, \hat{n}_l} \lambda_l^{(1)} q + \left(\delta_{s_l, \hat{n}_l} \lambda_l^{(2)} + \delta_{s_l, \hat{n}_l - 1} \tilde{\lambda}_l^{(2)} \right) q^2 + \dots$$

As explained later, we will only need an explicit expression for $\tilde{\lambda}_l^{(2)}$ and not $\lambda_l^{(2)}$. Setting $s_l = \hat{n}_l - 1$ in (3.23) and keeping all terms of order q^2 , one can read the resulting equation that $\tilde{\lambda}_l^{(2)}$ should satisfy

$$\begin{aligned} & \left[\prod_{m=1}^L (M_{lm} - (\hat{n}_l - 2)\epsilon) \prod_{(m,s_m) \neq (l, \hat{n}_l - 1), (l, \hat{n}_l - 2)} (M_{lm} - (\hat{n}_l - s_m - 2)\epsilon) \right] \cdot q^2 \tilde{\lambda}_l^{(2)} \\ &= q \left[\prod_{m=1}^L (M_l - \tilde{M}_m - (\hat{n}_l - 2)\epsilon) \prod_{(m,s_m) \neq (l, \hat{n}_l - 1), (l, \hat{n}_l)} (M_{lm} - (\hat{n}_l - s_m)\epsilon) \right] \cdot (-q \lambda_l^{(1)}), \end{aligned} \quad (3.31)$$

which yields

$$\tilde{\lambda}_l^{(2)} = \frac{\lambda_l^{(1)}}{2\epsilon^2} \cdot \frac{\prod_{m=1}^L (M_{lm} - (\hat{n}_l - 1)\epsilon) (M_l - \tilde{M}_m - (\hat{n}_l - 2)\epsilon)}{\prod_{m \neq l} (M_{lm} - (\hat{n}_l - \hat{n}_m - 1)\epsilon) (M_{lm} - (\hat{n}_l - \hat{n}_m - 2)\epsilon)}. \quad (3.32)$$

We are now ready to evaluate the on-shell value of the twisted superpotential $\mathcal{W}^{(II)}$ up to the lowest three orders in the instanton factor q

$$\mathcal{W}^{(II)} = \mathcal{W}^{(0)} + \mathcal{W}^{(1)} q + \mathcal{W}^{(2)} q^2 + \mathcal{O}(q^3). \quad (3.33)$$

Leading order, $\mathcal{W}^{(0)}$. Substituting the solution at leading order $\lambda_m^{(0)}$ (3.27) to (3.22), one can have

$$\mathcal{W}^{(0)} = \mathcal{W}_{\text{tree}}^{(0)} + \mathcal{W}_{\text{fund}}^{(0)} + \mathcal{W}_{\text{anti-fund}}^{(0)} + \mathcal{W}_{\text{adj}}^{(0)},$$

with

$$\begin{aligned} \mathcal{W}_{\text{tree}}^{(0)} &= (2\pi i\tau - i\pi(N+1)) \sum_{l=1}^L \sum_{s_l=1}^{\hat{n}_l} (M_l - (s_l - 1)\epsilon) \\ &= (2\pi i\tau - i\pi(N+1)) \sum_{l=1}^L \left(\hat{n}_l \left(M_l + \frac{\epsilon}{2} \right) - \frac{\hat{n}_l^2}{2} \epsilon \right), \end{aligned} \quad (3.34)$$

and

$$\begin{aligned} \mathcal{W}_{\text{fund}}^{(0)} &= -\epsilon \sum_{l,m=1}^L \sum_{k=0}^{\hat{n}_l-1} f\left(\frac{M_{lm}}{\epsilon} - k\right), \\ \mathcal{W}_{\text{anti-fund}}^{(0)} &= +\epsilon \sum_{l,m=1}^L \sum_{k=0}^{\hat{n}_l-1} f\left(\frac{M_l - \tilde{M}_m}{\epsilon} - k\right), \\ \mathcal{W}_{\text{adj}}^{(0)} &= +\epsilon \sum_{l,m=1}^L \sum_{s_l=1}^{\hat{n}_l} \sum_{s_m=1}^{\hat{n}_m} f\left(\frac{M_{lm}}{\epsilon} - s_l + s_m - 1\right). \end{aligned} \quad (3.35)$$

One can show that, after some algebra presented in appendix A, the twisted superpotential at leading order $\mathcal{W}^{(0)}$ can be simplified into the following form

$$\mathcal{W}^{(0)} = 2\pi i\tau \sum_{l=1}^L \left(\hat{n}_l \left(M_l + \frac{\epsilon}{2} \right) - \frac{\hat{n}_l^2}{2} \epsilon \right) - \epsilon \sum_{l,m=1}^L \sum_{k=\hat{n}_m-\hat{n}_l+1}^{\hat{n}_m} f\left(\frac{M_{lm}}{\epsilon} + k\right), \quad (3.36)$$

again up to some constant independent of the vacuum choice. We also used properties of the multi-valued function $f(x)$ extensively.

Next two leading orders, $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$. Using the vacuum equation, it is straightforward to obtain the next two leading contribution to the twisted superpotential as below

$$\mathcal{W}^{(1)} = \sum_{l=1}^L \lambda_l^{(1)} \quad (3.37)$$

and

$$\begin{aligned} \mathcal{W}^{(2)} &= \sum_{l=1}^L \tilde{\lambda}_l^{(2)} + \frac{1}{2\epsilon} (\lambda_l^{(1)})^2 + \sum_{l=1}^L (\lambda_l^{(1)})^2 \left[\sum_{m=1}^L \left(\frac{1}{M_{lm} - \hat{n}_m \epsilon} + \frac{1}{M_l - \tilde{M}_m - \hat{n}_l \epsilon + \epsilon} \right) \right. \\ &\quad \left. - \sum_{m \neq l} \left(\frac{1}{M_{lm} - (\hat{n}_l - \hat{n}_m) \epsilon} + \frac{1}{M_{lm} - (\hat{n}_l - \hat{n}_m - 1) \epsilon} \right) \right] \\ &\quad - \epsilon \sum_{m \neq l} \frac{\lambda_l^{(1)} \lambda_m^{(1)}}{(M_{lm} - (\hat{n}_l - \hat{n}_m) \epsilon)^2 - \epsilon^2}. \end{aligned} \quad (3.38)$$

Here $\lambda_l^{(1)}$ and $\lambda_l^{(2)}$ are explicitly given by (3.30) and (3.32), respectively.

3.3 Comparison

In order to compare the two theories, we first identify parameters in both theories as follows

$$\hat{n}_l = n_l - 1, \quad M_l = m_l - \frac{3}{2}\epsilon, \quad \tilde{M}_l = \tilde{m}_l + \frac{1}{2}\epsilon, \quad (3.39)$$

Upon the above identification, one can compare the on-shell twisted superpotentials of both theories order by order in instanton factor q . One can find the agreement at leading order between (3.13) and (3.36), i.e.,

$$\mathcal{G}_{\text{pert}} = \mathcal{W}^{(0)}. \quad (3.40)$$

Noting that

$$\begin{aligned} \lambda_l^{(1)} &= -\frac{1}{\epsilon} R_l(m_l - n_l \epsilon), \\ \tilde{\lambda}_l^{(2)} &= -\frac{1}{2\epsilon^3} R_i(m_l - n_l \epsilon) R_l(m_l - (n_l - 1)\epsilon), \end{aligned} \quad (3.41)$$

one can show the agreement further up to the two-instanton sector

$$\mathcal{G}_{\text{inst}}^{(1)} = \mathcal{W}^{(1)}, \quad \mathcal{G}_{\text{inst}}^{(2)} = \mathcal{W}^{(2)}. \quad (3.42)$$

4 Discussion and relation to other developments

The duality proposed above also has several points of contact with other recent developments in the study of supersymmetric gauge theory. In particular we can interpret our results in the context of the AGT conjecture which relates the Nekrasov instanton partition function of $\mathcal{N} = 2$ supersymmetric $G = \text{SU}(2)$ gauge theory in four dimensions to the conformal blocks of Liouville theory. For $L = 2$, Theory I defined above is an $\text{SU}(2)$ gauge theory with $N_F = 4$ fundamental hypermultiplets. The electric Coulomb branch parameters take the form $\vec{a} = (a, -a)$. According to the AGT conjecture the Nekrasov instanton partition function $\mathcal{Z}(\vec{a}, \epsilon_1, \epsilon_2)$ is related to the conformal block for the correlation function of four primary operators $V_{\alpha_i}(z) = \exp[2\alpha_i\phi(z)]$, inserted at points $z = z_i$ ($i = 1, 2, 3, 4$) on the sphere. The marginal coupling constant of the four dimensional gauge theory $q = e^{2\pi i\tau}$ can be identified with conformally invariant cross-ratio of these four points, the single modulus of a sphere with four punctures. Fixing three positions of vertex operators by 0, 1 and ∞ as usual, the left-over one therefore parametrizes the coupling q . The Liouville coupling and effective Planck constant¹³ are given as,

$$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}, \quad \hbar = \sqrt{\epsilon_1 \epsilon_2}$$

and the Liouville background charge Q can be expressed as $Q = b + 1/b$. The conformal block in question corresponds to the factorisation of the four-point function in which the operator $V_\alpha(z) = \exp[2\alpha\phi(z)]$ appears as the intermediate state in the S-channel as shown in figure 5. The external Liouville momenta are related to the gauge theory mass

¹³This should not be confused with the Planck constant of the integrable system discussed in section 2.

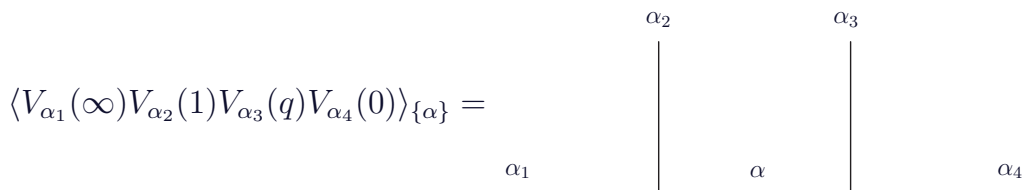


Figure 5. Four-point Liouville conformal block on the sphere.

parameters as,

$$\begin{aligned} \alpha_1 &= \frac{Q}{2} + \frac{(\tilde{m}_1 - \tilde{m}_2)}{2\hbar} & \alpha_2 &= \frac{(\tilde{m}_1 + \tilde{m}_2)}{2\hbar} \\ \alpha_4 &= \frac{Q}{2} + \frac{(m_1 - m_2)}{2\hbar} & \alpha_3 &= \frac{(m_1 + m_2)}{2\hbar} \end{aligned}$$

and the internal momentum is given in terms of the Coulomb branch modulus as $\alpha = Q/2 + a/\hbar$. The conformal dimensions of external and intermediate states are given by

$$\Delta_i = \alpha_i(Q - \alpha_i) \quad \Delta = \alpha(Q - \alpha) . \quad (4.1)$$

Note that the conformal block is invariant under individual flips of Liouville momenta α by $Q - \alpha$, which can be understood as the Weyl group of $SU(2)$ gauge symmetry as well as flavour symmetries.

The Nekrasov-Shatashvili limit is one in which $\epsilon_2 \rightarrow 0$ with $\epsilon_1 = \epsilon$ held fixed. Thus $\hbar \rightarrow 0$ and $b \rightarrow \infty$, while keeping $b\hbar = \epsilon$ fixed. The Liouville background charge $Q \simeq b = \epsilon/\hbar$. The Higgs branch root of Theory I is specified by the following condition $\vec{a} = \vec{m} - \epsilon$ or,

$$a = m_1 - \epsilon \quad -a = m_2 - \epsilon$$

which yield $\alpha_3 = \epsilon/\hbar$ and $\Delta_3 = 0$. Thus, in this limit, the Higgs branch root corresponds to a special case in which $V_{\alpha_3}(q)$ has zero dimension and therefore corresponds to the identity operator. Now suppose we introduce a single $D2$ brane setting $\hat{n}_1 = 1$ and $\hat{n}_2 = 0$. According to our dictionary, $\hat{n}_l = n_l - 1$ this is dual to a point on the Coulomb branch specified as,

$$a = m_1 - 2\epsilon \quad -a = m_2 - \epsilon$$

which yield $\alpha_3 = 3\epsilon/2\hbar$ and $\Delta_3 = -3\epsilon^2/4\hbar^2$. This is precisely the same dimension as the degenerate operator,

$$\Phi_{2,1}(z) = \exp[-b\phi(z)]$$

Interestingly, it is conjectured in [34] that an insertion $\Phi_{2,1}(z)$ is dual to the insertion of a surface operator in the gauge theory. This is precisely consistent with our identification of Theory II, with $\hat{n}_1 = 1$, $\hat{n}_2 = 0$ as the worldvolume theory on a single D2 brane. More general values of \hat{n}_l can therefore correspond to the insertion of multiple surface operators.

The duality proposed in this paper relates the world-volume theory on a surface operator probing the Higgs branch of a four dimensional gauge theory with a corresponding bulk theory (ie the same four dimensional gauge theory without surface operator on its Coulomb branch). As such it is reminiscent of the AdS/CFT correspondence and other large- N dualities. This observation can be made precise in the context of geometric engineering where the Nekrasov partition function of four-dimensional theory is computed by the closed topological string on a suitable local geometry. More precisely we should consider the closed string partition function computed using the refined topological vertex of [54]. On the other hand, the partition function for gauge theory in the presence of a surface operator corresponds to an open topological string partition function [55, 56]. The proposed duality therefore asserts the equality of certain open and closed topological string partition functions and it is natural to ask if it is related to the geometric transition of Gopakumar and Vafa [38]. Strictly speaking the latter is defined only in the unrefined case corresponding to $\epsilon_1 = -\epsilon_2 = g_s$ while our duality proposal applies only to the NS limit $\epsilon_2 \rightarrow 0$. Nevertheless there are strong similarities which suggest that a “refined” geometric transition should exist and should be equivalent in the NS limit to the duality proposed in this paper (see also [35]).

To understand the connection to the Geometric transition it is useful to compare the IIA brane constructions of Theory II and Theory I shown in figures 4 and 1 respectively. Specifically we will focus on the region of these figures corresponding to a single U(1) vector multiplet in the low-energy theory coming from a single D4 brane and a single charged hypermultiplet which arises when the D4 brane breaks on a single NS 5-brane. The corresponding region of figures 4 and 1 are shown in figures 6(a) and 6(b) respectively. This configuration is related to the conifold singularity by the standard sequence of dualities which relate the intersecting brane and geometrical engineering approaches to SUSY gauge theory. In this context, the Coulomb branch of the gauge theory corresponds to the small resolution of the conifold where an S^2 is blown up. The area of the sphere is related to the Coulomb branch modulus a . The Higgs branch of the gauge theory corresponds to the deformation of the conifold where an S^3 of non-zero size appears. Gauge theory vortex strings are realised as D4 branes wrapped on S^3 and extended in two dimensions of four dimensional spacetime. Thus the duality discussed in this paper corresponds to a transition between the deformed conifold with \hat{n} wrapped branes and the resolved conifold without branes. The resulting two sphere has size which is linearly related to the number \hat{n} of branes. Thus the duality is of the same form as the geometric transition. This also seems to be consistent with the relation between the conjecture of [34] and the geometric transition suggested in [35, 56].

Another interesting connection is the one between supersymmetric gauge theory and “holomorphic” matrix models proposed in [39]. In general terms, we expect the Nekrasov partition function of Theory I to be captured by an appropriate matrix model. Here we will

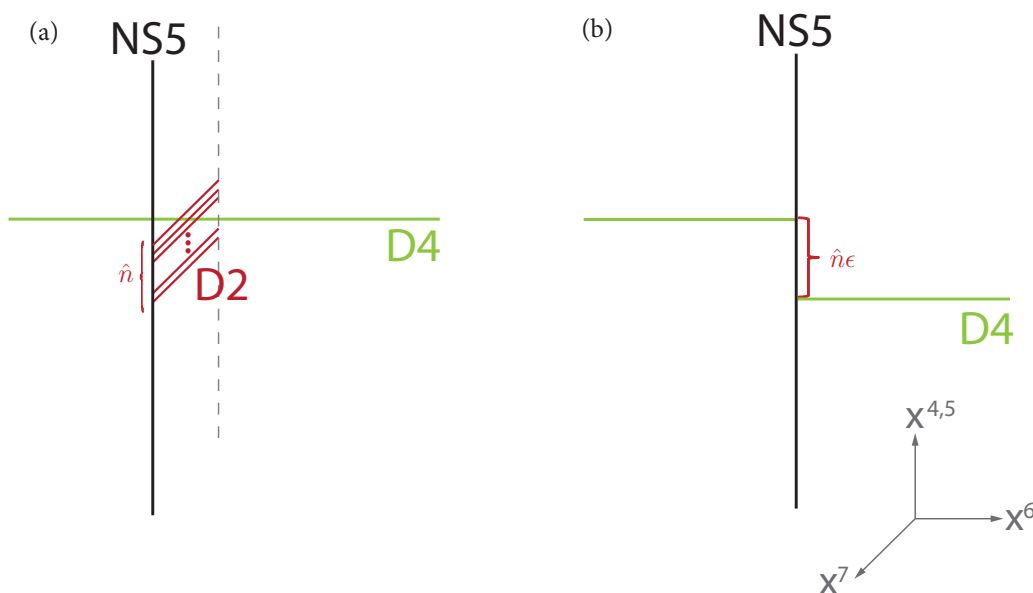


Figure 6. (a) Theory II: \hat{n} D2 branes suspended between a D4 and an NS5. (b) Theory I: D4 brane breaks on NS5.

make a concrete proposal for the form of the matrix model in the NS limit. In particular we will consider an integral over an $N \times N$ complex matrix Φ , but interpreted in the holomorphic sense of [39],

$$\mathcal{Z}_\Phi = \int \mathcal{D}_{\epsilon_1, \epsilon_2} \Phi \exp(-\text{Tr}_N S(\Phi, \epsilon_1, \epsilon_2)) ,$$

where the measure is $\text{SL}(N, \mathbb{C})$ invariant and can be written in terms of the eigenvalues $\{x_1, x_2, \dots, x_N\}$ of Φ . We propose that the leading behaviour of the resulting integral as $\epsilon_2 \rightarrow 0$ has the form,

$$\mathcal{Z}_\Phi = \int \prod_{j=1}^N dx_j \prod_{j < k} \mathcal{M}(x_j - x_k) \exp\left(-\frac{1}{\epsilon_2} \sum_{j=1}^N \mathcal{V}(x_j)\right) ,$$

where the measure is a deformation of the Vandermonde determinant specified as,

$$\mathcal{M}(x) = \exp\left(-\frac{1}{\epsilon_2} [f(x + \epsilon) - f(x - \epsilon)]\right) ,$$

with $f(x) = x(\log x - 1)$ as above and potential,

$$\mathcal{V}(x) = -2\pi i \hat{\tau} x + \sum_{l=1}^L \left[f(x - M_l) - f(x - \tilde{M}_l) \right] ,$$

where each of the above relations is corrected at higher orders in ϵ_2 . On one hand we can reorganise the exponent to write the resulting matrix integral as,

$$\mathcal{Z}_\Phi = \int \prod_{j=1}^N dx_j \exp\left(\frac{1}{\epsilon_2} \mathcal{W}^{(II)}(x)\right) ,$$

where \mathcal{W}_{II} is the superpotential of Theory II as given in (2.26) which is equivalent to the Yang potential of the Bethe Ansatz. As $\epsilon_2 \rightarrow 0$ the matrix integral is dominated by its saddle-points, *even for finite* N . The saddle-point condition coincides with the F-term equations of Theory II which are precisely the BAE of the spin chain. According to the duality proposed above the resulting saddle point value of $\log \mathcal{Z}_\Phi$ is a multivalued function whose branches coincide (up to a vacuum-independent shift) with the quantum prepotential $\mathcal{F}(\vec{a}, \epsilon)$ of Theory I evaluated at the lattice of points $\vec{a} - \vec{m} \in \epsilon \mathbb{Z}^L$. The value of the classical prepotential $\mathcal{F}(\vec{a})$ at any point on the Coulomb branch can be obtained by an appropriate $N \rightarrow \infty$ limit with $\epsilon \rightarrow 0$ and the product $N\epsilon$ fixed. In this limit, the above matrix model is related to the proposal of [39] since the deformed Vandermonde determinant simplifies and the matrix integral becomes

$$\mathcal{Z}_\Phi = \int \prod_{j=1}^N dx_j \prod_{j < k} (x_j - x_k)^{2\beta} \exp \left(-\frac{1}{\epsilon_2} \sum_{j=1}^N \mathcal{V}(x_j) \right)$$

with $\beta = -\epsilon_1/\epsilon_2$. This is a β -ensemble with a Penner-like logarithmic potential, although involving $x \log x$ rather than $\log x$, whose resolvent is precisely the Seiberg-Witten 1-form,

$$L(x)dx = \sum_j \frac{dx}{x - x_j} \equiv \epsilon_1^{-1} \lambda(\epsilon) . \tag{4.2}$$

It follows that in the limit $\epsilon_1 \rightarrow 0$, with $\epsilon_1 \hat{n}_\ell$ fixed, we have

$$a_l - m_l = \lim_{\epsilon_1 \rightarrow 0} \oint_{\alpha_l} \frac{dx}{2\pi i} \epsilon_1 L(x) . \tag{4.3}$$

The free energy in this limit is the prepotential:

$$\mathcal{F}(\vec{a}) \sim \lim_{\epsilon_1, \epsilon_2 \rightarrow 0} \epsilon_1 \epsilon_2 \log \mathcal{Z}_\Phi . \tag{4.4}$$

The resulting matrix model is precisely of the sort discussed in [39]. This proposal also looks superficially similar to the matrix model reformulation of the Nekrasov partition function derived in [53], except that in that formulation the dependence on the $\{a_l\}$ is through the potential whereas in our matrix model it is through the filling fractions.

It is also interesting to relate the duality proposed here to an earlier proposal for a 2d/4d duality made by two of the present authors in [41] (see also [40]). There the massive BPS spectrum of the four-dimensional Theory I at the root of its Higgs branch was related to the BPS spectrum of a U(1) gauge theory in two dimensions closely related to Theory II considered above. In contrast, in the present case, the relation between the superpotentials of Theories I and II takes the form,

$$\mathcal{W}^{(II)} = \mathcal{W}^{(I)} \Big|_{\vec{a}=\vec{m}-\vec{n}\epsilon} - \mathcal{W}^{(I)} \Big|_{\vec{a}=\vec{m}-\epsilon}$$

where as above have fixed the additive constant in the superpotential to by subtracting the value at the root of the Higgs branch. Taking the limit $\epsilon \rightarrow 0$ using (2.12) and (2.11) we get,

$$\lim_{\epsilon \rightarrow 0} \mathcal{W}^{(II)} = - \sum_{l=1}^N \hat{n}_l \vec{a}_l^D \Big|_{\vec{a}=\vec{m}}$$

Thus, in this limit, the on-shell value of the superpotential for the two-dimensional Theory II is related to the magnetic central charge of the four-dimensional theory evaluated at the root of the Higgs branch. In the abelian case where $N = \sum_l \hat{n}_l = 1$, the resulting relation has exactly the same form as that proposed in [40, 41].

Ideally one would also like to provide a physical explanation of the duality along the lines of [17, 18]. Why should the worldvolume theory of a vortex string have any relation to the bulk theory in the presence of the Nekrasov deformation? Here we note that, in the presence of the specific deformation we consider, magnetic flux is quantized in the resulting supersymmetric vacuum states even on the Coulomb branch. Indeed, the quantum numbers $\{n_l\}$ which describe the number of quanta of magnetic flux in each low energy U(1) are directly related by our conjecture to the magnetic fluxes $\{\hat{n}_l\}$ of the vortices on the Higgs branch. This suggests the possibility of a smooth interpolation between corresponding vacua of Theory I and Theory II corresponding to a variation of $\mathcal{N} = (2, 2)$ D-term couplings, thereby explaining the equality of F-terms between the two theories.

Finally, we have represented the relation between the theories as a conjecture. However, the result can be proved by formulating the BAE as an integral equation, along the lines of the Destri-de Vega equation [58]. The resulting non-linear equation is then essentially identical to the saddle-point equation that describes the instanton partition function in the Nekrasov-Shatashvili limit [5]. The recent paper [60] provides a nice description of the instanton saddle-point equation and there is a direct link between the solution of the BAE and the instanton density function: the pairs $(\lambda_j, \lambda_j^{(0)})$ are precisely the end-points of the intervals — continued into the complex plane — along which the instanton density is non-vanishing. These issues will be described in detail in a companion paper [61].

The formalism that we have developed can also be extended in various ways: to quiver gauge theories, including the elliptic models, and to compactified five and six dimensional $\mathcal{N} = 2$ theories. These generalizations will also be reported elsewhere.

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A Computational details

A.1 Theory I

We present in this section the details in computation of the on-shell twisted superpotential of Theory I, four-dimensional $N = 2$ gauge theories in Ω -background with $\epsilon_2 = 0$. Before presenting the computation, let us first set

$$N = \sum_{l=1}^L (n_l - 1), \tag{A.1}$$

where N represents the number of D2-branes in Theory II.

One can rewrite one-loop contributions from fundamental hyper- and vector-multiplets to the function $\mathcal{G}_{1\text{-loop}}$ into the following forms

$$\begin{aligned} \mathcal{G}_{1\text{-loop}}^{\text{fund}} = & \epsilon \sum_{l,m>l} \left[\sum_{k=1-n_l}^{-1} - \sum_{k=1}^{n_m-1} \right] f\left(\frac{m_{lm}}{\epsilon} + k\right) + \epsilon \sum_l \sum_{k=1}^{n_l-1} f(-k) \\ & + i\pi\epsilon \sum_{l,m>l} \sum_{k=1}^{n_m-1} \left(\frac{m_{lm}}{\epsilon} + k\right) + \frac{LN\epsilon}{2} \log 2\pi, \end{aligned} \quad (\text{A.2})$$

and

$$\mathcal{G}_{1\text{-loop}}^{\text{vec}} = -\epsilon \sum_{l,m>l} \left[\sum_{k=1+n_m-n_l}^0 + \sum_{k=n_m-n_l}^{-1} \right] f\left(\frac{m_{lm}}{\epsilon} + k\right) + i\pi\epsilon \sum_{l,m>l} \sum_{k=n_m-n_l}^{-1} \left(\frac{m_{lm}}{\epsilon} + k\right). \quad (\text{A.3})$$

Summing up the above results, one can obtain

$$\begin{aligned} \mathcal{G}_{1\text{-loop}}^{\text{vec}} + \mathcal{G}_{1\text{-loop}}^{\text{fund}} = & \epsilon \sum_{l,m>l} \left[\sum_{k=1-n_l}^{n_m-n_l-1} - \sum_{k=1+n_m-n_l}^{n_m-1} \right] f\left(\frac{m_{lm}}{\epsilon} + k\right) \\ & - \epsilon \sum_l \sum_{k=1}^{n_l-1} f(k) + i\pi\epsilon \sum_l \frac{n_l(n_l-1)}{2} \\ & + i\pi\epsilon \sum_{l,m>l} \left[\sum_{k=1}^{n_m-1} + \sum_{k=n_m-n_l}^{-1} \right] \left(\frac{m_{lm}}{\epsilon} + k\right) + \frac{LN\epsilon}{2} \log 2\pi, \\ \simeq & -\epsilon \sum_{l,m} \sum_{k=1+n_m-n_l}^{n_m-1} f\left(\frac{m_{lm}}{\epsilon} + k\right) \\ & + i\pi\epsilon \sum_{l,m>l} \left[\sum_{k=1-n_l}^{n_m-n_l-1} - \sum_{k=1}^{n_m-1} + \sum_{k=n_m-n_l}^{-1} \right] \left(\frac{m_{lm}}{\epsilon} + k\right) \\ & + i\pi\epsilon \sum_l \frac{n_l(n_l-1)}{2} + \frac{LN\epsilon}{2} \log 2\pi, \end{aligned} \quad (\text{A.4})$$

where we used for the last equality a property of the multi-valued function $f(x)$. One can further simplify the above expression into

$$\begin{aligned} \mathcal{G}_{1\text{-loop}}^{\text{vec}} + \mathcal{G}_{1\text{-loop}}^{\text{fund}} = & -\epsilon \sum_{l,m} \sum_{k=1+n_m-n_l}^{n_m-1} f\left(\frac{m_{lm}}{\epsilon} + k\right) + i\pi \sum_{l,m>l} m_{lm}(n_l - n_m) - i\pi\epsilon \sum_{l,m} \sum_{k=1}^{n_m-1} k \\ & + i\pi\epsilon \sum_l n_l(n_l-1) + \frac{LN\epsilon}{2} \log 2\pi \\ \simeq & -\epsilon \sum_{l,m} \sum_{k=1+n_m-n_l}^{n_m-1} f\left(\frac{m_{lm}}{\epsilon} + k\right) + i\pi L \sum_l \left[(n_l-1)m_l - \frac{n_l(n_l-1)}{2}\epsilon \right] \\ & - i\pi N \sum_l m_l + \frac{LN\epsilon}{2} \log 2\pi. \end{aligned} \quad (\text{A.5})$$

Demanding the traceless condition of $SU(L)$,

$$\sum_{l=1}^L (m_l - n_l \epsilon) = 0,$$

one can replace the first term in the last line of (A.5) as $\sum_l m_l = (N + L)\epsilon$. In the $U(L)$ theory, the additional non-vanishing term corresponds to a constant vacuum-independent shift of the superpotential.

A.2 Theory II

We now in turn discuss some computational details in simplifying the twisted superpotential at leading order for Theory II. The contributions from the fundamental and adjoint chiral multiplets will be focused in what follows.

One can massage the twisted superpotential at leading order as follows:

$$\begin{aligned} \mathcal{W}_{\text{fund}}^{(0)} &= -\epsilon \sum_{l,m} \sum_{k=0}^{\hat{n}_l-1} f\left(\frac{M_{lm}}{\epsilon} - k\right) \\ &= -\epsilon \sum_{l,m>l} \left[\sum_{k=1-\hat{n}_l}^0 f\left(\frac{M_{lm}}{\epsilon} + k\right) + \sum_{k=0}^{\hat{n}_m-1} f\left(-\frac{M_{lm}}{\epsilon} - k\right) \right] - \epsilon \sum_l \sum_{k=1-\hat{n}_l}^0 f(k) \\ &= -\epsilon \sum_{l,m>l} \left[\sum_{k=1-\hat{n}_l}^0 - \sum_{k=0}^{\hat{n}_m-1} \right] f\left(\frac{M_{lm}}{\epsilon} + k\right) - i\pi\epsilon \sum_{l,m>l} \sum_{k=0}^{\hat{n}_m-1} \left(\frac{M_{lm}}{\epsilon} + k\right) - \epsilon \sum_l \sum_{k=1-\hat{n}_l}^0 f(k). \end{aligned} \quad (\text{A.6})$$

One can also show

$$\begin{aligned} \mathcal{W}_{\text{adj}}^{(0)} &= \epsilon \sum_{l,m} \sum_{s_l=1}^{\hat{n}_l} \sum_{s_m=1}^{\hat{n}_m} f\left(\frac{M_{lm}}{\epsilon} - s_l + s_m - 1\right) \\ &= \epsilon \sum_{l,m>l} \sum_{s_l,s_m} \left[f\left(\frac{M_{lm}}{\epsilon} - s_l + s_m - 1\right) - f\left(\frac{M_{lm}}{\epsilon} - s_l + s_m + 1\right) \right. \\ &\quad \left. + i\pi \left(\frac{M_{lm}}{\epsilon} - s_l + s_m + 1\right) \right] + \epsilon \sum_l \sum_{s_l,t_l} f(-s_l + t_l - 1) \\ &= \epsilon \sum_{l,m>l} \left[\sum_{k=-\hat{n}_l}^{-1} + \sum_{k=1-\hat{n}_l}^0 - \sum_{k=\hat{n}_m-\hat{n}_l}^{\hat{n}_m-1} - \sum_{k=\hat{n}_m-\hat{n}_l+1}^{\hat{n}_m} \right] f\left(\frac{M_{lm}}{\epsilon} + k\right) \\ &\quad + i\pi\epsilon \sum_{l,m>l} \sum_{s_l,s_m} \left(\frac{M_{lm}}{\epsilon} - s_l + s_m + 1\right) + \epsilon \sum_l \sum_{s_l,t_l} f(-s_l + t_l - 1). \end{aligned} \quad (\text{A.7})$$

Here the second and the last terms in the last equality can be simplified into

$$\begin{aligned} &i\pi\epsilon \sum_{l,m>l} \sum_{s_l} \left[\hat{n}_m \left(\frac{M_{lm}}{\epsilon} - s_l + 1\right) + \frac{\hat{n}_m(\hat{n}_m + 1)}{2} \right] \\ &= i\pi \sum_{l,m>l} \hat{n}_l \hat{n}_m M_{lm} + i\pi\epsilon \sum_{l,m>l} \hat{n}_l \hat{n}_m \frac{2 - \hat{n}_l + \hat{n}_m}{2}, \end{aligned} \quad (\text{A.8})$$

and

$$\begin{aligned}
 & \epsilon \sum_l \sum_{t_l > s_l} f(t_l - s_l - 1) - f(t_l - s_l + 1) + i\pi\epsilon \sum_l \sum_{t_l > s_l} (t_l - s_l + 1) + \epsilon N f(-1) \\
 & = \epsilon \sum_l \sum_{s_l} \left[f(0) + f(1) - f(n_l - s_l) - f(n_l - s_l + 1) \right] + \frac{i\pi\epsilon}{6} \sum_l \hat{n}_l (\hat{n}_l - 1) (\hat{n}_l + 4) \\
 & = \epsilon N (f(1) + f(-1)) - \epsilon \sum_l \left[\sum_{k=0}^{\hat{n}_l - 1} + \sum_{k=1}^{\hat{n}_l} \right] f(k) + \frac{i\pi\epsilon}{6} \sum_l \hat{n}_l (\hat{n}_l - 1) (\hat{n}_l + 4). \quad (\text{A.9})
 \end{aligned}$$

It therefore implies that

$$\begin{aligned}
 \mathcal{W}_{\text{adj}}^{(0)} & = \epsilon \sum_{l,m > l} \left[\sum_{k=-\hat{n}_l}^{-1} + \sum_{k=1-\hat{n}_l}^0 - \sum_{k=\hat{n}_m-\hat{n}_l}^{\hat{n}_m-1} - \sum_{k=\hat{n}_m-\hat{n}_l+1}^{\hat{n}_m} \right] f\left(\frac{M_{lm}}{\epsilon} + k\right) \\
 & + i\pi \sum_{l,m > l} \hat{n}_l \hat{n}_m M_{lm} + i\pi\epsilon \sum_{l,m > l} \hat{n}_l \hat{n}_m \frac{2 - \hat{n}_l + \hat{n}_m}{2} \\
 & + i\pi\epsilon N - \epsilon \sum_l \left[\sum_{k=0}^{\hat{n}_l - 1} + \sum_{k=1}^{\hat{n}_l} \right] f(k) + \frac{i\pi\epsilon}{6} \sum_l \hat{n}_l (\hat{n}_l - 1) (\hat{n}_l + 4). \quad (\text{A.10})
 \end{aligned}$$

Collecting all the results, one can obtain

$$\begin{aligned}
 \mathcal{W}_{\text{fund}}^{(0)} + \mathcal{W}_{\text{adj}}^{(0)} & = \epsilon \sum_{l,m > l} \left[\sum_{k=-\hat{n}_l}^{\hat{n}_m - \hat{n}_l - 1} - \sum_{k=\hat{n}_m - \hat{n}_l + 1}^{\hat{n}_m} \right] f\left(\frac{M_{lm}}{\epsilon} + k\right) - i\pi\epsilon \sum_{l,m > l} \sum_{k=0}^{\hat{n}_m - 1} \left(\frac{M_{lm}}{\epsilon} + k\right) \\
 & + i\pi \sum_{l,m > l} \hat{n}_l \hat{n}_m M_{lm} + i\pi\epsilon \sum_{l,m > l} \hat{n}_l \hat{n}_m \frac{2 - \hat{n}_l + \hat{n}_m}{2} \\
 & + i\pi\epsilon N - i\pi\epsilon \sum_l \frac{\hat{n}_l (\hat{n}_l - 1)}{2} + \frac{i\pi\epsilon}{6} \sum_l \hat{n}_l (\hat{n}_l - 1) (\hat{n}_l + 4) - \epsilon \sum_l \sum_{k=1}^{\hat{n}_l} f(k) \\
 & = -\epsilon \sum_{l,m} \sum_{k=\hat{n}_m - \hat{n}_l + 1}^{\hat{n}_m} f\left(\frac{M_{lm}}{\epsilon} + k\right) + i\pi \sum_{l,m > l} \hat{n}_l \hat{n}_m M_{lm} + i\pi\epsilon \sum_{l,m > l} \hat{n}_l \hat{n}_m \frac{\hat{n}_m - \hat{n}_l}{2} \\
 & + \frac{i\pi\epsilon}{6} \sum_l \hat{n}_l (\hat{n}_l - 1) (\hat{n}_l + 1) + i\pi\epsilon N. \quad (\text{A.11})
 \end{aligned}$$

It can be further simplified into

$$\begin{aligned}
 \mathcal{W}_{\text{fund}}^{(0)} + \mathcal{W}_{\text{adj}}^{(0)} & = -\epsilon \sum_{l,m} \sum_{k=\hat{n}_m - \hat{n}_l + 1}^{\hat{n}_m} f\left(\frac{M_{lm}}{\epsilon} + k\right) + i\pi (N + 1) \sum_l \sum_{s_l} (M_l - (s_l - 1)\epsilon) \\
 & - i\pi \sum_l (\hat{n}_l + 1) \hat{n}_l M_l + \frac{2i\pi\epsilon}{3} \sum_l \hat{n}_l (\hat{n}_l - 1) (\hat{n}_l + 1) \\
 & - 2\pi i \sum_l \sum_{s_l} (M_l - (s_l - 1)\epsilon) \sum_{m < l} \hat{n}_m + i\pi\epsilon N \quad (\text{A.12}) \\
 & \simeq -\epsilon \sum_{l,m} \sum_{k=\hat{n}_m - \hat{n}_l + 1}^{\hat{n}_m} f\left(\frac{M_{lm}}{\epsilon} + k\right) + i\pi (N + 1) \sum_l \sum_{s_l} (M_l - (s_l - 1)\epsilon) + i\pi\epsilon N
 \end{aligned}$$

Again, we used for the last equality a property of multi-valued function $f(x)$.

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