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Yang-Mills instantons and dyons on homogeneous G_2-manifolds

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Abstract: We consider Lie$G$-valued Yang-Mills fields on the space $\mathbb{R}\times G/H$, where G/H is a compact nearly Kähler six-dimensional homogeneous space, and the manifold $\mathbb{R}\times G/H$ carries a G_2-structure. After imposing a general G-invariance condition, Yang-Mills theory with torsion on $\mathbb{R}\times G/H$ is reduced to Newtonian mechanics of a particle moving in $\mathbb{R}^6$, $\mathbb{R}^4$ or $\mathbb{R}^2$ under the influence of an inverted double-well-type potential for the cases $G/H =$ SU(3)/U(1)×U(1), Sp(2)/Sp(1)×U(1) or G_2/SU(3), respectively. We analyze all critical points and present analytical and numerical kink- and bounce-type solutions, which yield G-invariant instanton configurations on those cosets. Periodic solutions on $S^1\times G/H$ and dyons on i$\mathbb{R}\times G/H$ are also given.

Keywords: Flux compactifications, Solitons Monopoles and Instantons, Differential and Algebraic Geometry

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Contents

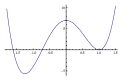
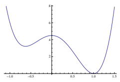
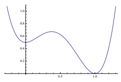
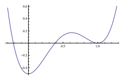
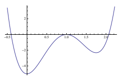
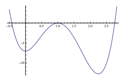
1 Introduction and summary

Interest in Yang-Mills theories in dimensions greater than four grew essentially after the discovery of superstring theory, which contains supersymmetric Yang-Mills in the low-energy limit in the presence of D-branes as well as in the heterotic case. In particular, heterotic strings yield d=10 heterotic supergravity interacting with the $\mathcal{N}$=1 supersymmetric Yang-Mills multiplet [1]. Supersymmetry-preserving compactifications on spacetimes $M_{10-d}\times X^d$ with further reduction to M_{10-d} impose the first-order BPS-type gauge equations which are a generalization of the Yang-Mills anti-self-duality equations in d=4 to higher-dimensional manifolds with special holonomy. Such equations in d>4 dimensions were first introduced in [2] and further considered e.g. in [3–16]. Some of their solutions were found e.g. in [17–24].

Initial choices for the internal manifold X^6 in string theory were Kähler coset spaces and Calabi-Yau manifolds, as well as manifolds with exceptional holonomy group G_2 for d=7 and Spin(7) for d=8. However, it was realized recently that the internal manifold should allow non-trivial p-form fluxes whose back reaction deforms its geometry. In particular, a three-form flux background implies a nonzero torsion whose components are given by the structure constants of the holonomy group, $T^a_{bc} = \varkappa\, f^a_{bc}$, with a real parameter $\varkappa$. String vacua with p-form fields along the extra dimensions ('flux compactifications') have been intensively studied in recent years (see e.g. [25–27] for reviews and references). Flux compactifications have been investigated primarily for type II strings and to a lesser extent in the heterotic theories, despite their long history [28–32]. The number of torsionful geometries that can serve as a background for heterotic string compactifications seems rather limited. Among them there are six-dimensional nilmanifolds, solvmanifolds, nearly Kähler and nearly Calabi-Yau coset spaces. The last two kinds of manifolds carry a natural almost complex structure which is not integrable (for their geometry see e.g. [33–37] and references therein).

In the present paper, we solve the torsionful Yang-Mills equations on G_2-manifolds of topology $\mathbb{R}\times X^6$ with nearly Kähler cosets X^6. The allowed gauge bundle is restricted by the G_2-instanton equations [13, 14]. For each coset $X^6 = G/H$, we parametrize the general G-invariant connection by a set of complex scalars ϕ_i, which depend on the coordinate τ of the $\mathbb{R}$ factor. The Yang-Mills equations then descend to Newton's equations for the coordinates $\phi_i(\tau)$ of a point particle under the influence of an inverted double-well-type potential, whose shape depends on $\varkappa$. For this potential we derive the critical points of zero energy, which correspond to the $\tau\to\pm\infty$ asymptotic configurations of the finite-action Yang-Mills solutions. We then present a variety of zero-energy solutions $\phi_i(\tau)$, of kink and of bounce type, analytically as well as numerically. The kinks translate to instantons for the gauge fields.

Furthermore, by replacing the factor $\mathbb{R}$ with S^1, we obtain periodic solutions with a sphaleron interpretation. Finally, in the Lorentzian case i$\mathbb{R}\times G/H$, the double-well-type potential gets flipped back, and there exist bounce solutions with a dyonic interpretation, some of which have finite action. The different types of finite-action Yang-Mills solutions on $\mathbb{R}\times G/H$ or i$\mathbb{R}\times G/H$ occur in the following ranges of the parameter $\varkappa$:

$\varkappa \in$	$(-\infty,-3)$	$(-3,+1)$	$(+1,+3)$	$(+3,+5)$	$(+5,+9)$	$(+9,+\infty)$
Euclidean	bounces	instantons	instantons	bounces	—	—
Lorentzian	dyons	—	—	—	dyons	dyons
$V_{\mathbb{R}}(\mathrm{Re}\phi)$						

2 Yang-Mills fields on $\mathbb{R} \times G/H$

2.1 Yang-Mills equations with torsion

Instantons [38] play an important role in modern gauge theories [39, 40]. They are non-perturbative BPS configurations in four Euclidean dimensions solving the first-order anti-self-duality equations and forming a subset of solutions to the full Yang-Mills equations. In dimensions higher than four, BPS configurations can still be found as solutions to first-order equations, known as generalized anti-self-duality equations [2–10] or Σ-anti-self-duality [11–14]. These appear in superstring compactifications as conditions of survival of at least one supersymmetry [1]. Various solutions to these first-order equations were found e.g. in [17–24], mostly on flat space $\mathbb{R}^d$ and various cosets.

The BPS-type instanton equations in $d > 4$ dimensions can be introduced as follows. Let Σ be a $(d-4)$-form on a d-dimensional Riemannian manifold M. Consider a complex vector bundle $\mathcal{E}$ over M endowed with a connection $\mathcal{A}$. The Σ-anti-self-dual gauge equations are defined [11, 12] as the first-order equations,

$$* \mathcal{F} \;=\; -\Sigma \wedge \mathcal{F} \,, \tag{2.1}$$

on a connection $\mathcal{A}$ with the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. Here $*$ is the Hodge star operator on M.

Differentiating (2.1), we obtain the Yang-Mills equations with torsion,

$$\mathrm{d} * \mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - *\mathcal{F} \wedge \mathcal{A} + *\mathcal{H} \wedge \mathcal{F} \;=\; 0 \,, \tag{2.2}$$

where the torsion three-form $\mathcal{H}$ is defined by the formula

$$* \mathcal{H} \;:=\; \mathrm{d}\Sigma \qquad \Rightarrow \qquad \mathcal{H} \;=\; (-1)^{3(d-3)} * \mathrm{d}\Sigma \,. \tag{2.3}$$

The torsion term in (2.2) naturally appears in string theory [25–27].[1] If Σ is closed, $\mathcal{H} = 0$ and (2.2) reduce to the standard Yang-Mills equations. The Yang-Mills equations with torsion (2.2) are equations of motion for the action

$$\begin{aligned} S &= \int_M \mathrm{tr}\left(\mathcal{F} \wedge *\mathcal{F} \;+\; (-1)^{d-3} \Sigma \wedge \mathcal{F} \wedge \mathcal{F}\right) \\ &= \int_M \mathrm{tr}\left(\mathcal{F} \wedge *\mathcal{F} \;+\; *\mathcal{H} \wedge \left(\mathrm{d}\mathcal{A} \wedge \mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\right)\right) \;-\; \int_M \mathrm{d}\left(\Sigma \wedge \mathrm{tr}\left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \tfrac{2}{3}\mathcal{A}^3\right)\right) , \end{aligned} \tag{2.4}$$

[1] For a recent discussion of heterotic string theory with torsion see e.g. [41–50] and references therein.

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where the last term is topological. In what follows we consider the equations (2.2) on manifolds $M = \mathbb{R} \times G/H$, where G/H are compact nearly Kähler six-dimensional homogeneous spaces.

2.2 Coset spaces

Consider a compact semisimple Lie group G and a closed subgroup H of G such that G/H is a reductive homogeneous space (coset space). Let $\{I_A\}$ with $A{=}1,\ldots,\dim G$ be the generators of the Lie group G with structure constants f^A_{BC} given by the commutation relations

$$[I_A, I_B] \;=\; f^C_{AB}\, I_C\ . \tag{2.5}$$

We normalize the generators such that the Killing-Cartan metric on the Lie algebra $\mathfrak{g}$ of G coincides with the Kronecker symbol,

$$g_{AB} \;=\; f^C_{AD}\, f^D_{CB} \;=\; \delta_{AB}\ . \tag{2.6}$$

More general left-invariant metrics can be obtained by rescaling the generators.

The Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the orthogonal complement of the Lie algebra $\mathfrak{h}$ of H in $\mathfrak{g}$. Then, the generators of G can be divided into two sets, $\{I_A\} = \{I_a\} \cup \{I_i\}$, where $\{I_i\}$ are the generators of H with $i, j, \ldots = \dim G{-}\dim H{+}1, \ldots, \dim G$, and $\{I_a\}$ span the subspace $\mathfrak{m}$ of $\mathfrak{g}$ with $a, b, \ldots = 1, \ldots, \dim G{-}\dim H$. For reductive homogeneous spaces we have the following commutation relations:

$$[I_i, I_j] \;=\; f^k_{ij}\, I_k\,, \qquad [I_i, I_a] \;=\; f^b_{ia}\, I_b \quad\text{and}\quad [I_a, I_b] \;=\; f^i_{ab}\, I_i + f^c_{ab}\, I_c\ . \tag{2.7}$$

For the metric (2.6) on $\mathfrak{g}$ we have

$$g_{ab} \;=\; 2f^i_{ad}f^d_{ib} + f^c_{ad}f^d_{cb} \;=\; \delta_{ab}\,, \tag{2.8}$$

$$g_{ij} \;=\; f^k_{il}f^l_{kj} + f^b_{ia}f^a_{bj} \;=\; \delta_{ij} \quad\text{and}\quad g_{ia} \;=\; 0\ . \tag{2.9}$$

2.3 Torsionful spin connection on G/H

The metric (2.8) on $\mathfrak{m}$ lifts to a G-invariant metric on G/H. A local expression for this can be obtained by introducing an orthonormal frame as follows. The basis elements I_A of the Lie algebra $\mathfrak{g}$ can be represented by left-invariant vector fields $\hat{E}_A$ on the Lie group G, and the dual basis $\hat{e}^A$ is a set of left-invariant one-forms. The space G/H consists of left cosets gH and the natural projection $g \mapsto gH$ is denoted $\pi : G \to G/H$. Over a small contractible open subset U of G/H, one can choose a map $L : U \to G$ such that $\pi \circ L$ is the identity, i.e. L is a local section of the principal bundle $G \to G/H$. The pull-backs of $\hat{e}^A$ by L are denoted e^A. Among these, the e^a form an orthonormal frame for $T^*(G/H)$ over U, and for the remaining forms we can write $e^i = e^i_a e^a$ with real functions e^i_a. The dual frame for $T(G/H)$ will be denoted E_a. By the group action we can transport e^a and E_a from inside U to everywhere in G/H. The forms e^A obey the Maurer-Cartan equations,

$$\mathrm{d}e^a = -f^a_{ib}\, e^i \wedge e^b - \tfrac{1}{2}\, f^a_{bc}\, e^b \wedge e^c \quad\text{and}\quad \mathrm{d}e^i = -\tfrac{1}{2}\, f^i_{bc}\, e^b \wedge e^c - \tfrac{1}{2}\, f^i_{jk}\, e^j \wedge e^k\ . \tag{2.10}$$

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The local expression for the G-invariant metric then is

$$g_{G/H} \;=\; \delta_{ab} e^a e^b \;. \tag{2.11}$$

Recall that a linear connection is a matrix of one-forms $\Gamma = (\Gamma^a_b) = (\Gamma^a_{cb} e^c)$. The connection is metric compatible if $g_{ac}\Gamma^c_b$ is anti-symmetric, and its torsion is a vector of two-forms T^a determined by the structure equations

$$\mathrm{d}e^a + \Gamma^a_b \wedge e^b \;=\; T^a \;=\; \tfrac12 T^a_{bc}\, e^b \wedge e^c \;. \tag{2.12}$$

We choose the torsion tensor components on G/H proportional to the structure constants f^a_{bc},

$$T^a_{bc} \;=\; \varkappa\, f^a_{bc}\,, \tag{2.13}$$

where $\varkappa$ is an arbitrary real parameter. Then the torsionful spin connection on G/H becomes

$$\Gamma^a_b \;=\; f^a_{ib} e^i + \tfrac12 (\varkappa{+}1)\, f^a_{cb}\, e^c \;=:\; \Gamma^a_{cb} e^c \;. \tag{2.14}$$

2.4 Yang-Mills equations on $\mathbb{R} \times G/H$

Consider the space $\mathbb{R} \times G/H$ with a coordinate τ on $\mathbb{R}$, a one-form $e^0 := \mathrm{d}\tau$ and the Euclidean metric

$$g \;=\; (e^0)^2 \;+\; \delta_{ab}\, e^a e^b \;. \tag{2.15}$$

The torsionful spin connection Γ on $\mathbb{R} \times G/H$ is given by (2.14), with

$$\Gamma^a_{cb} \;=\; e^i_c\, f^a_{ib} + \tfrac12 (\varkappa{+}1)\, f^a_{cb} \quad \text{and} \quad \Gamma^0_{0b} = \Gamma^a_{0b} = \Gamma^0_{cb} = 0 \;. \tag{2.16}$$

For our choice of the metric, $g_{ab} = \delta_{ab}$, we can pull down indices in (2.13) and introduce the three-form

$$\mathcal{H} = \tfrac{1}{3!} T_{abc}\, e^a \wedge e^b \wedge e^c = \tfrac16 \varkappa f_{abc}\, e^a \wedge e^b \wedge e^c \quad \Longrightarrow \quad \mathcal{H}_{abc} = T_{abc} = \varkappa f_{abc} \;. \tag{2.17}$$

Consider the trivial principal bundle $P(\mathbb{R}{\times}G/H, G) = (\mathbb{R}{\times}G/H){\times}G$ over $\mathbb{R}{\times}G/H$ with the structure group G, the associated trivial complex vector bundle $\mathcal{E}$ over $\mathbb{R}{\times}G/H$ and a $\mathfrak{g}$-valued connection one-form $\mathcal{A}$ on $\mathcal{E}$ with the curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. In the basis of one-forms $\{e^0, e^a\}$ on $\mathbb{R}{\times}G/H$, we have

$$\mathcal{A} \;=\; \mathcal{A}_0 e^0 + \mathcal{A}_a e^a \quad \text{and} \quad \mathcal{F} \;=\; \mathcal{F}_{0a}\, e^0 \wedge e^a + \tfrac12 \mathcal{F}_{ab}\, e^a \wedge e^b \;. \tag{2.18}$$

In the following we choose a 'temporal' gauge in which $\mathcal{A}_0 \equiv \mathcal{A}_\tau = 0$.

The Yang-Mills equations with torsion (2.2) on $\mathbb{R}{\times}G/H$ are equivalent to

$$E_a \mathcal{F}^{a0} + \Gamma^a_{ab} \mathcal{F}^{b0} + [\mathcal{A}_a, \mathcal{F}^{a0}] = 0\,, \tag{2.19}$$

$$E_0 \mathcal{F}^{0b} + E_a \mathcal{F}^{ab} + \Gamma^d_{da} \mathcal{F}^{ab} + \Gamma^b_{cd} \mathcal{F}^{cd} + [\mathcal{A}_a, \mathcal{F}^{ab}] = 0\,, \tag{2.20}$$

where we used (2.16) and (2.17) and the gauge $\mathcal{A}_0 = 0$ with $E_0 = \mathrm{d}/\mathrm{d}\tau$. Note that these equations also follow from the action functional (2.4) with $\mathcal{H}$ given in (2.17).

2.5 G-invariant gauge fields

Let us associate our complex vector bundle $\mathcal{E} \to \mathbb{R}\times G/H$ with the adjoint representation $\operatorname{adj}(G)$ of the structure group G. Then the generators of G are realized as $\dim G \times \dim G$ matrices

$$I_i \;=\; \big(I^A_{iB}\big) \;=\; \big(f^A_{iB}\big) \;=\; \big(f^j_{ik}\big)\oplus\big(f^a_{ib}\big) \quad \text{and} \quad I_a \;=\; \big(I^A_{aB}\big) \;=\; \big(f^A_{aB}\big)\;. \tag{2.21}$$

According to [51] (see also [52–55]), G-invariant connections on $\mathcal{E}$ are determined by linear maps $\Lambda : \mathfrak{m} \to \mathfrak{g}$ which commute with the adjoint action of H:

$$\Lambda\big(\mathrm{Ad}(h)Y\big) \;=\; \mathrm{Ad}(h)\,\Lambda(Y) \qquad \forall h\in H \quad \text{and} \quad Y\in\mathfrak{m}\;. \tag{2.22}$$

Such a linear map is represented by a matrix (X^B_a), appearing in

$$X_a \;:=\; \Lambda(I_a) \;=\; X^B_a I_B \;=\; X^i_a I_i + X^b_a I_b\;. \tag{2.23}$$

For the cases we will consider one can always choose $X^i_a = 0$. In local coordinates the connection is written

$$\mathcal{A} \;=\; e^i I_i + e^a X_a \qquad \Leftrightarrow \qquad \mathcal{A}_a \;=\; e^i_a I_i + X_a\,, \tag{2.24}$$

and its G-invariance imposes the condition

$$[I_i, X_a] \;=\; f^b_{ia} X_b \qquad \Leftrightarrow \qquad X^b_a f^c_{bi} \;=\; f^b_{ia} X^c_b\;. \tag{2.25}$$

The curvature $\mathcal{F}$ of the invariant connection (2.24) reads

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} \;=\; \dot{X}_a e^0\wedge e^a - \tfrac{1}{2}\left(f^i_{bc} I_i + f^a_{bc} X_a - [X_b, X_c]\right) e^b\wedge e^c \quad \Leftrightarrow \tag{2.26}$$

$$\mathcal{F}_{0a} = \dot{X}_a \quad \text{and} \quad \mathcal{F}_{bc} \;=\; -\big(f^i_{bc} I_i + f^a_{bc} X_a - [X_b, X_c]\big)\,, \tag{2.27}$$

where dots denote derivatives with respect to τ. For our choice (2.8) and (2.9) of the metric one can pull down all indices in the Yang-Mills equations (2.19) and (2.20) as well as in (2.16). It is now a matter of computation to substitute (2.24) and (2.27) into (2.19) and (2.20), making use of the Jacobi identity for the structure constants. One finds that (2.20) is equivalent to

$$\ddot{X}_a = \big(\tfrac{1}{2}(\varkappa{+}1) f_{acd} f_{bcd} - f_{acj} f_{bcj}\big) X_b \;-\; \tfrac{1}{2}(\varkappa{+}3) f_{abc}[X_b, X_c] \;-\; \big[X_b, [X_b, X_a]\big]\,, \tag{2.28}$$

and (2.19) reduces to the constraint

$$[X_a, \dot{X}_a] \;=\; 0 \qquad (\text{sum over } a) \tag{2.29}$$

on the matrices X_a. Note that the equations (2.28) can also be obtained from the action (2.4) reduced to a matrix-model action after substituting (2.24) and (2.27) into (2.4). The subsidiary relation (2.29) is the Gauss-law constraint following from the gauge fixing $\mathcal{A}_0 = 0$.

3 Invariant gauge fields on homogeneous G_2-manifolds

Here, we choose G/H to be a compact six-dimensional nearly Kähler coset space. Such manifolds are important examples of SU(3)-structure manifolds used in flux compactifications of string theories (see e.g. [35–37, 48–50] and references therein). Their geometry is fairly rigid and features a 3-symmetry, which generalizes the reflection symmetry of symmetric spaces. This allows for a very explicit description of their structure and a complete parametrization of G-invariant Yang-Mills fields, which we present in this section.

3.1 Nearly Kähler six-manifolds

An SU(3)-structure on a six-manifold is by definition a reduction of the structure group of the tangent bundle from SO(6) to SU(3). Manifolds of dimension six with SU(3)-structure admit a set of canonical objects, consisting of an almost complex structure J, a Riemannian metric g, a real two-form ω and a complex three-form Ω. With respect to J, the forms ω and Ω are of type (1,1) and (3,0), respectively, and there is a compatibility condition, $g(J\cdot,\cdot) = \omega(\cdot,\cdot)$. With respect to the volume form V_g of g, the forms ω and Ω are normalized so that

$$\omega\wedge\omega\wedge\omega \;=\; 6V_g \quad \text{and} \quad \Omega\wedge\bar{\Omega} \;=\; -8\mathrm{i}V_g\ . \tag{3.1}$$

Then, a nearly Kähler six-manifold is an SU(3)-structure manifold with the differentials

$$\mathrm{d}\omega \;=\; 3\rho\,\mathrm{Im}\Omega \quad \text{and} \quad \mathrm{d}\Omega \;=\; 2\rho\,\omega\wedge\omega \tag{3.2}$$

for some real non-zero constant ρ (if ρ was zero, the manifold would be Calabi-Yau). More generally, six-manifolds with SU(3)-structure are classified by their intrinsic torsion [56], and nearly Kähler manifolds form one particular intrinsic torsion class.

There are only four known examples of compact nearly Kähler six-manifolds, and they are all coset spaces [33, 34]:

$$\mathrm{SU(3)/U(1){\times}U(1)}\,,\ \ \mathrm{Sp(2)/Sp(1){\times}U(1)}\,,\ \ G_2/\mathrm{SU(3)}=S^6,\ \ \mathrm{SU(2)^3/SU(2)}=S^3{\times}S^3\,. \tag{3.3}$$

Here Sp(1)×U(1) is chosen to be a non-maximal subgroup of Sp(2): if the elements of Sp(2) are written as 2×2 quaternionic matrices, then the elements of Sp(1)×U(1) have the form diag(p,q), with $p\in$Sp(1) and $q\in$U(1). Also, SU(2) is the diagonal subgroup of SU(2)^3. These coset spaces are all 3-symmetric, because the subgroup H is the fixed point set of an automorphism s of G satisfying $s^3=\mathrm{Id}$ [33, 34].

The 3-symmetry actually plays a fundamental role in defining the canonical structures on the coset spaces. The automorphism s induces an automorphism S of the Lie algebra $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ of G which acts trivially on $\mathfrak{h}$ and non-trivially on $\mathfrak{m}$; one can define a map

$$J:\mathfrak{m}\to\mathfrak{m} \qquad \text{by} \qquad S|_{\mathfrak{m}} \;=\; -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}J \;=\; \exp\left(\tfrac{2\pi}{3}\,J\right)\ . \tag{3.4}$$

The map J satisfies $J^2=-1$ and provides the almost complex structure on G/H. The components J^a_b of the almost complex structure J are defined via $J(I_b)=J^a_bI_a$. Local

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expressions for the G-invariant metric, almost complex structure, and the two-form ω on a nearly Kähler space G/H in an orthonormal frame $\{e^a\}$ are

$$g \;=\; \delta_{ab} e^a e^b\,, \qquad J \;=\; J_a^b e^a E_b \quad\text{and}\quad \omega \;=\; \tfrac{1}{2} J_{ab} e^a \wedge e^b\,. \tag{3.5}$$

One can also obtain a local expression for (3,0)-form Ω by using (3.2) and the Maurer-Cartan equations. From (2.10) one can compute $\mathrm{d}\omega$ and hence $*\mathrm{d}\omega$:

$$\mathrm{d}\omega \;=\; -\tfrac{1}{2}\,\tilde{f}_{abc}\, e^a \wedge e^b \wedge e^c \quad\text{and}\quad *\,\mathrm{d}\omega \;=\; \tfrac{1}{2}\, f_{abc}\, e^a \wedge e^b \wedge e^c\,, \tag{3.6}$$

where

$$\tilde{f}_{abc} \;:=\; f_{abd} J_{dc} \tag{3.7}$$

are the components of a totally antisymmetric tensor on a nearly Kähler six-manifold in the list (3.3). The structure constants on nearly Kähler cosets obey the identities

$$f_{aci} f_{bci} \;=\; f_{acd} f_{bcd} = \tfrac{1}{3}\,\delta_{ab}\,, \tag{3.8}$$

$$J_{cd} f_{adi} \;=\; J_{ad} f_{cdi} \quad\text{and}\quad J_{ab} f_{abi} = 0\,. \tag{3.9}$$

From the normalization (3.1) and (3.8) we compute that

$$||\omega||^2 \;:=\; \omega_{ab}\omega_{ab} \;=\; 3 \quad\text{and}\quad ||\mathrm{Im}\,\Omega||^2 \;:=\; (\mathrm{Im}\,\Omega)_{abc} (\mathrm{Im}\,\Omega)_{abc} \;=\; 4\,. \tag{3.10}$$

So it must be that

$$\mathrm{Im}\Omega = -\tfrac{1}{\sqrt{3}}\,\tilde{f}_{abc}\, e^a \wedge e^b \wedge e^c\,, \quad \mathrm{Re}\Omega = -\tfrac{1}{\sqrt{3}}\, f_{abc}\, e^a \wedge e^b \wedge e^c \quad\text{and}\quad \rho = \tfrac{1}{2\sqrt{3}}\,. \tag{3.11}$$

Note that on all four nearly Kähler coset spaces (3.3) one can choose the non-vanishing structure constants such that

$$\{f_{abc}\}: \quad f_{135} = f_{425} = f_{416} = f_{326} = -\tfrac{1}{2\sqrt{3}} \tag{3.12}$$

and therefore

$$\{\tilde{f}_{abc}\}: \quad \tilde{f}_{136} = \tilde{f}_{426} = \tilde{f}_{145} = \tilde{f}_{235} = -\tfrac{1}{2\sqrt{3}} \tag{3.13}$$

for J such that

$$\omega \;=\; \tfrac{1}{2} J_{ab}\, e^a \wedge e^b \;=\; e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6\,. \tag{3.14}$$

Then we have

$$\Omega = \mathrm{Re}\,\Omega + \mathrm{i}\,\mathrm{Im}\,\Omega = e^{135} + e^{425} + e^{416} + e^{326} + \mathrm{i}(e^{136} + e^{426} + e^{145} + e^{235}) =: \Theta^1 \wedge \Theta^2 \wedge \Theta^3\,, \tag{3.15}$$

where $e^{abc} \equiv e^a \wedge e^b \wedge e^c$ and

$$\Theta^1 := e^1 + \mathrm{i}e^2\,, \qquad \Theta^2 := e^3 + \mathrm{i}e^4 \quad\text{and}\quad \Theta^3 := e^5 + \mathrm{i}e^6 \tag{3.16}$$

are forms of type (1,0) with respect to J.

3.2 Yang-Mills equations and action functional

In the previous subsection we described the geometry of nearly Kähler six-manifolds. Now we would like to consider the Yang-Mills theory on seven-manifolds $\mathbb{R}\times G/H$, where G/H is a nearly Kähler coset space. Note that on such manifolds

$$M \;=\; \mathbb{R}\times G/H \tag{3.17}$$

one can introduce three-forms

$$\Sigma \;=\; e^0\wedge\omega \;+\; \mathrm{Im}\,\Omega\,, \tag{3.18}$$

and

$$\Sigma' \;=\; e^0\wedge\omega \;+\; \mathrm{Re}\,\Omega\,. \tag{3.19}$$

Each of the two, Σ as well as Σ', defines a G_2-structure on $\mathbb{R}\times G/H$, i.e. a reduction of the holonomy group SO(7) to a subgroup $G_2\subset\mathrm{SO}(7)$. From (3.18) and (3.19) one sees that both G_2-structures are induced from the SU(3)-structure on G/H.

On the seven-manifold (3.17), the matrix equations (2.28) and (2.29) simplify to

$$\ddot{X}_a = \tfrac{1}{6}(\varkappa-1)X_a \;-\; \tfrac{1}{2}(\varkappa+3)f_{abc}[X_b,X_c] \;-\; \big[X_b,[X_b,X_a]\big]\,, \tag{3.20}$$

$$[X_a,\dot{X}_a] = 0 \qquad \text{(sum over } a) \tag{3.21}$$

after using the identities (3.8). We notice that the equations (3.20) and (3.21) are the equation of motion and the Gauss constraint for the action

$$S \;=\; -\tfrac{1}{4}\int_{\mathbb{R}\times G/H}\mathrm{tr}\left(\mathcal{F}\wedge *\mathcal{F} \;+\; \frac{\varkappa}{3}\,e^0\wedge\omega\wedge\mathcal{F}\wedge\mathcal{F}\right)\,. \tag{3.22}$$

Substituting (2.24) and (2.27) into (3.22) and imposing the gauge $\mathcal{A}_0=0$, we obtain

$$\begin{aligned} S \;=\; -\tfrac{1}{4}\,\mathrm{Vol}(G/H)\textstyle\int\!\mathrm{d}\tau\;\mathrm{tr}\Big(\dot{X}_a\dot{X}_a \;&-\; \tfrac{1}{6}(\varkappa-3)f_{iab}f_{jab}I_iI_j \;+\; \tfrac{1}{6}(\varkappa-1)X_aX_a \\ &-\; \tfrac{1}{3}(\varkappa+3)f_{abc}X_a[X_b,X_c] \;+\; \tfrac{1}{2}[X_b,X_c][X_b,X_c]\Big)\,. \end{aligned} \tag{3.23}$$

The Euler-Lagrange equations for this matrix-model action are (3.20).

3.3 Solution of the G-invariance condition

The G-invariance condition (2.25),

$$[I_i,X_a] \;=\; f^b_{ia}X_b \qquad \text{for} \qquad X_a \;=\; X^b_aI_b \;\in\mathrm{Lie}(G){-}\mathrm{Lie}(H)\,, \tag{3.24}$$

says that the X_a must transform in the six-dimensional representation $\mathcal{R}$ of H which arises in the decomposition (2.21),

$$\mathrm{adj}(G)\big|_H \;=\; \mathrm{adj}(H)\oplus\mathcal{R}\,, \tag{3.25}$$

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of the adjoint of G restricted to H, i.e. $(\mathcal{R}(I_i))_a^b = f_{ia}^b$. It is real but reducible and decomposes into complex irreducible parts as

$$\mathcal{R} \;=\; \sum_{p=1}^{q} \mathcal{R}_p \;\oplus\; \sum_{p=1}^{q} \overline{\mathcal{R}}_p\,, \tag{3.26}$$

with $\sum_{p=1}^{q} \dim \mathcal{R}_p = 3$. This is the same H-representation as furnished by the I_a. Hence, for each irrep $\mathcal{R}_p$ one can find complex linear combinations $I_{\alpha_p}^{(p)}$ of the I_a, with $\alpha_p = 1, \dots, \dim \mathcal{R}_p$, such that

$$[I_i\,,\,I_{\alpha_p}^{(p)}] \;=\; f_{i\,\alpha_p}^{\beta_p}\, I_{\beta_p}^{(p)} \tag{3.27}$$

close among themselves for each p. In the absence of a condition on $[X_a, X_b]$, the X_a appear linearly and thus may always be multiplied by a common factor ϕ_p inside each irrep $\mathcal{R}_p$. By Schur's lemma this is in fact the only freedom, i.e.

$$X_{\alpha_p}^{(p)} \;=\; \phi_p\, I_{\alpha_p}^{(p)} \qquad \text{with} \quad \phi_p \in \mathbb{C} \quad \text{and} \quad \alpha_p = 1, \dots, \dim \mathcal{R}_p \tag{3.28}$$

is the unique solution to the G-invariance condition inside $\mathcal{R}_p$. The six antihermitian matrices X_a are then easily reconstructed via

$$\{X_a\} \;=\; \Big\{ \tfrac{1}{2}\big(X_{\alpha_p}^{(p)} - \overline{X}_{\alpha_p}^{(p)}\big)\,,\, \tfrac{1}{2\mathrm{i}}\big(X_{\alpha_p}^{(p)} + \overline{X}_{\alpha_p}^{(p)}\big) \Big\} \tag{3.29}$$

and will depend on q complex functions $\phi_p(\tau)$. The same holds for any smaller G-representation $\mathcal{D}$ instead of $\mathrm{adj}(G)$.

For computations, we choose a basis in $\mathfrak{g}$ such that the first $\dim(\mathcal{R}_1)$ generators I_{α_1} span $\mathcal{R}_1$, the next $\dim(\mathcal{R}_2)$ generators I_{α_2} span $\mathcal{R}_2$ etc., and the last $\dim(H)$ generators span $\mathfrak{h}$. Such a basis decomposes $\mathcal{R}$ into the said blocks. Fusing all irreducible blocks and $\mathrm{adj}(H)$ together again, we obtain a realization of I_i, I_a and X_a as matrices in $\mathrm{adj}(G)$. Since G is the gauge group, these matrices enter in the action (3.23). However, for calculations it is more convenient to take a smaller G-representation $\mathcal{D}$. This affects only the normalization of the trace,

$$\mathrm{tr}_{\mathcal{D}}(I_A I_B) \;=\; -\chi_{\mathcal{D}}\, \delta_{AB}\,, \tag{3.30}$$

where the (2nd-order) Dynkin index $\chi_{\mathcal{D}}$ depends on the representation used. We normalize our generators such that $\chi_{\mathrm{adj}(G)} = 1$, and choose $\mathcal{D}$ in all cases (see below) such that $\chi_{\mathcal{D}} = \frac{1}{6}$. With this, the constant term in the action (3.23) computes to

$$-\tfrac{1}{6}(\varkappa{-}3) f_{iab} f_{jab}\, \mathrm{tr}_{\mathcal{D}}(I_i I_j) \;=\; \tfrac{1}{36}(\varkappa{-}3) f_{iab} f_{iab} \;=\; \tfrac{1}{18}(\varkappa{-}3)\;. \tag{3.31}$$

4 Yang-Mills fields on $\mathbb{R}\times$ SU(3)/U(1)×U(1)

4.1 Explicit form of X_a matrices

The structure constants for SU(3) which conform with the nearly Kähler structure (3.12)–(3.16) are

$$f_{135} = f_{425} = f_{416} = f_{326} = -\frac{1}{2\sqrt{3}}\,, \tag{4.1}$$
$$f_{127} = f_{347} = \frac{1}{2\sqrt{3}}\,, \quad f_{128} = -f_{348} = -\frac{1}{2} \quad \text{and} \quad f_{567} = -\frac{1}{\sqrt{3}}\;.$$

The adjoint of SU(3), restricted to U(1)×U(1), decomposes as

$$\mathbf{8}\,(\text{of SU}(3)) = ((0,0)+(0,0))_{\text{adj}}+(3,1)+(-3,-1)+(3,-1)+(-3,1)+(0,2)+(0,-2)\,, \quad (4.2)$$

where the $\mathcal{R}_p$ are labelled by the charges (r,s) under U(1)×U(1). Obviously, we have $q{=}3$ complex parameters. We employ the fundamental representation $\mathcal{D} = \mathbf{3}$ of SU(3). It is easy to check that indeed $\chi_{\mathbf{3}}/\chi_{\mathbf{8}} = 1/6$.

For the generators $I_{7,8}$ of the subgroup U(1)×U(1) of SU(3) chosen in the form

$$I_7 \;=\; -\frac{\mathrm{i}}{2\sqrt{3}}\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix} \quad\text{and}\quad I_8 \;=\; \frac{\mathrm{i}}{6}\begin{pmatrix}2&0&0\\0&-1&0\\0&0&-1\end{pmatrix}, \tag{4.3}$$

the solution to the SU(3)-invariance equation (3.24) then reads

$$\begin{aligned}
X_1 &= \frac{1}{2\sqrt{3}}\begin{pmatrix}0&0&-\phi_1\\0&0&0\\\bar{\phi}_1&0&0\end{pmatrix}, \quad
X_3 = \frac{1}{2\sqrt{3}}\begin{pmatrix}0&-\bar{\phi}_2&0\\\phi_2&0&0\\0&0&0\end{pmatrix}, \quad
X_5 = \frac{1}{2\sqrt{3}}\begin{pmatrix}0&0&0\\0&0&-\bar{\phi}_3\\0&\phi_3&0\end{pmatrix},\\
X_2 &= \frac{1}{2\sqrt{3}}\begin{pmatrix}0&0&\mathrm{i}\phi_1\\0&0&0\\\mathrm{i}\bar{\phi}_1&0&0\end{pmatrix}, \quad
X_4 = \frac{-1}{2\sqrt{3}}\begin{pmatrix}0&\mathrm{i}\bar{\phi}_2&0\\\mathrm{i}\phi_2&0&0\\0&0&0\end{pmatrix}, \quad
X_6 = \frac{-1}{2\sqrt{3}}\begin{pmatrix}0&0&0\\0&0&\mathrm{i}\bar{\phi}_3\\0&\mathrm{i}\phi_3&0\end{pmatrix},
\end{aligned} \tag{4.4}$$

where ϕ_1, ϕ_2, ϕ_3 are complex-valued functions of τ. Note that for $\phi_1 = \phi_2 = \phi_3 = 1$ from (4.4) one obtains the normalized basis for $\mathfrak{m}$ which yields the nearly Kähler structure on SU(3)/U(1)×U(1) in the standard form (3.2), (3.5) and (3.12)–(3.16).

4.2 Equations of motion

Substituting (4.4) into the action (3.23), we obtain the Lagrangian

$$\begin{aligned}
18\,\mathcal{L} = {}& 6\left(|\dot{\phi}_1|^2+|\dot{\phi}_2|^2+|\dot{\phi}_3|^2\right) \;-\; (\varkappa{-}3) \;+\; (\varkappa{-}1)\left(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2\right)\\
&-(\varkappa{+}3)\left(\phi_1\phi_2\phi_3+\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3\right) \;+\; |\phi_1\phi_2|^2+|\phi_2\phi_3|^2+|\phi_3\phi_1|^2+|\phi_1|^4+|\phi_2|^4+|\phi_3|^4\,,
\end{aligned} \tag{4.5}$$

whose quartic terms may be rewritten as

$$\tfrac{1}{2}\left(|\phi_1|^4+|\phi_2|^4+|\phi_3|^4\right) \;+\; \tfrac{1}{2}\left(|\phi_1|^2+|\phi_2|^2+|\phi_3|^2\right)^2\,. \tag{4.6}$$

The equations of motion for the gauge fields on $\mathbb{R}\times$ SU(3)/U(1)×U(1) can be obtained by plugging (4.4) in (3.20) and (3.21). We get

$$\begin{aligned}
6\,\ddot{\phi}_1 &= (\varkappa{-}1)\,\phi_1 \;-\; (\varkappa{+}3)\,\bar{\phi}_2\bar{\phi}_3 \;+\; \left(2|\phi_1|^2+|\phi_2|^2+|\phi_3|^2\right)\phi_1\,,\\
6\,\ddot{\phi}_2 &= (\varkappa{-}1)\,\phi_2 \;-\; (\varkappa{+}3)\,\bar{\phi}_1\bar{\phi}_3 \;+\; \left(|\phi_1|^2+2|\phi_2|^2+|\phi_3|^2\right)\phi_2\,,\\
6\,\ddot{\phi}_3 &= (\varkappa{-}1)\,\phi_3 \;-\; (\varkappa{+}3)\,\bar{\phi}_1\bar{\phi}_2 \;+\; \left(|\phi_1|^2+|\phi_2|^2+2|\phi_3|^2\right)\phi_3\,,
\end{aligned} \tag{4.7}$$

as well as

$$\phi_1\dot{\bar{\phi}}_1-\dot{\phi}_1\bar{\phi}_1 \;=\; \phi_2\dot{\bar{\phi}}_2-\dot{\phi}_2\bar{\phi}_2 \;=\; \phi_3\dot{\bar{\phi}}_3-\dot{\phi}_3\bar{\phi}_3\,. \tag{4.8}$$

The equations (4.7) are the Euler-Lagrange equations for the Lagrangian (4.5) obtained from (3.22) after fixing the gauge $\mathcal{A}_0 = 0$.

4.3 Zero-energy critical points

Writing the equations of motion (4.7) as

$$6\,\ddot{\phi}_i \;=\; \frac{\partial V}{\partial\bar{\phi}_i}\,, \tag{4.9}$$

we see that they describe the motion of a particle on $\mathbb{C}^3$ under the influence of the inverted quartic potential $-V$, where

$$\begin{aligned} V \;=\; &-(\varkappa{-}3) \;+\; (\varkappa{-}1)\big(|\phi_1|^2{+}\,|\phi_2|^2{+}\,|\phi_3|^2\big) \;+\; \big(|\phi_1|^4{+}\,|\phi_2|^4{+}\,|\phi_3|^4\big) \\ &-(\varkappa{+}3)\big(\phi_1\phi_2\phi_3+\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3\big) \;+\; |\phi_1\phi_2|^2+|\phi_2\phi_3|^2+|\phi_3\phi_1|^2\,, \end{aligned} \tag{4.10}$$

or, alternatively, the dynamics of three identical particles on the complex plane, with an external potential given by the (negative of) the first line in (4.10) and two- and three-body interactions in the second line.

The potential (4.10) is invariant under permutations of the ϕ_i as well as under the $\mathrm{U}(1){\times}\mathrm{U}(1)$ transformations

$$\big(\phi_1\,,\,\phi_2\,,\,\phi_3\big) \;\mapsto\; \big(\mathrm{e}^{\mathrm{i}\delta_1}\phi_1\,,\,\mathrm{e}^{\mathrm{i}\delta_2}\phi_2\,,\,\mathrm{e}^{\mathrm{i}\delta_3}\phi_3\big) \qquad\text{with}\quad \delta_1+\delta_2+\delta_3=0 \quad\text{mod } 2\pi\,, \tag{4.11}$$

which include the 3-symmetry, $\phi_i \mapsto \mathrm{e}^{2\pi\mathrm{i}/3}\phi_i$. Such a transformation may be used to align the phases of the ϕ_i, i.e. $\arg(\phi_1)=\arg(\phi_2)=\arg(\phi_3)$. These phases only enter in the cubic term of the potential, which is proportional to $\cos(\sum_i \arg\phi_i)$. Therefore, the extrema of V are attained at $\sum_i \arg\phi_i = 0$ or π, and so, employing (4.11), we may take $\phi_i\in\mathbb{R}$ in our search for them.[2] Furthermore, the Noether charges of the $\mathrm{U}(1){\times}\mathrm{U}(1)$ symmetry (4.11) are just the differences $\ell_i-\ell_j$ of the 'angular momenta'

$$\ell_i \;:=\; \phi_i\dot{\bar{\phi}}_i - \dot{\phi}_i\bar{\phi}_i\;. \tag{4.12}$$

Hence, the constraints (4.8) may be interpreted as putting these charges to zero. Note, however, that the individual angular momenta are not conserved, since

$$\dot{\ell}_i \;=\; -\tfrac{1}{6}(\varkappa{+}3)\,\big(\phi_1\phi_2\phi_3-\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3\big)\;. \tag{4.13}$$

Finite-action solutions $\phi_i(\tau)$ must interpolate between critical points with zero potential,

$$\lim_{\tau\to\pm\infty}\phi_i(\tau) =: \phi_i^{\pm} \quad\text{and}\quad (\phi_1^{\pm},\phi_2^{\pm},\phi_3^{\pm})\in\{\widehat{\phi}\} \quad\text{with}\quad V(\widehat{\phi})=0=\mathrm{d}V(\widehat{\phi})\;. \tag{4.14}$$

Modulo the symmetry (4.11) and permutations, the complete list of such critical points reads: where $\gamma_\pm = -(1{+}\sqrt{3})\pm 2\sqrt{2(\sqrt{3}{-}1)}$ takes the numerical values of -0.31 and -5.15. The zero modes of V'' are enforced by the symmetries; their number indicates the dimension of the critical manifold in $\mathbb{C}^3$. A critical point is marginally stable only when V'' has no positive eigenvalues. At the critical points $\dot{\ell}_i=0$ is guaranteed, hence the product $\widehat{\phi}_1\widehat{\phi}_2\widehat{\phi}_3$ has to be real unless $\varkappa=-3$. The latter value is special because all phase dependence disappears, and the symmetry (4.11) is enhanced to $\mathrm{U}(1)^3$. We will not consider this special situation (type A') further. Appendix A proves that the list below is complete.

[2] We thank N. Dragon for this remark.

type	$\widehat{\phi}_1$	$\widehat{\phi}_2$	$\widehat{\phi}_3$	$\varkappa$	eigenvalues of V''					
A	1	1	1	any	0	0	$3(\varkappa{+}3)$	$2(\varkappa{+}4)$	$2(\varkappa{+}4)$	$5{-}\varkappa$
A'	$\mathrm{e}^{\mathrm{i}\alpha}$	$\mathrm{e}^{\mathrm{i}\alpha}$	$\mathrm{e}^{\mathrm{i}\alpha}$	-3	0	0	0	2	2	8
B	0	0	0	$+3$	2	2	2	2	2	2
C	0	0	$\sqrt{1{+}\sqrt{3}}$	$-1{-}2\sqrt{3}$	0	γ_-	γ_-	γ_+	γ_+	$4(1{+}\sqrt{3})$

4.4 Some solutions

Finite-action trajectories $\phi_i(\tau)$ require the conserved Newtonian energy to vanish,

$$E \;:=\; 6\left(|\dot\phi_1|^2{+}|\dot\phi_2|^2{+}|\dot\phi_3|^2\right) \;-\; V(\phi_1,\phi_2,\phi_3) \;\stackrel{!}{=}\; 0. \tag{4.15}$$

They can be of two types: Either $\phi_i^+ \neq \phi_i^-$ (kink), or $\phi_i^+ = \phi_i^-$ (bounce). Since this choice occurs for each value of $i=1,2,3$, mixed solutions are possible. We now present some special cases.

Transverse kinks at $-3{<}\varkappa{<}{+}3$. The two-dimensional type A critical manifold exists for any value of $\varkappa$, so one may try to find trajectories connecting two critical points of type A. As a particularly symmetric choice we wish to interpolate

$$(\phi_i^-) = (1\,,\mathrm{e}^{2\pi\mathrm{i}/3},\mathrm{e}^{-2\pi\mathrm{i}/3}) \qquad\longrightarrow\qquad (\phi_i^+) = (\mathrm{e}^{2\pi\mathrm{i}/3},\mathrm{e}^{-2\pi\mathrm{i}/3},1)\;. \tag{4.16}$$

The three independent conserved quantities $(E,\ell_i{-}\ell_j)$ do not suffice to integrate the equations of motion (4.7), so generically one has to resort to numerical methods. With a little effort, zero-energy 'transverse' kinks can be found in the range $\varkappa\in(-3,+3)$. We display the trajectory $(\phi_i(\tau))\in\mathbb{C}^3$ as three curves $\phi_i(\tau)\in\mathbb{C}$ in figure 1 for $\varkappa=-2,-1,0,+1,+2$. Apparently, the 3-symmetry effects a permutation since $\phi_2(\tau)=\mathrm{e}^{2\pi\mathrm{i}/3}\phi_1(\tau)=\mathrm{e}^{-2\pi\mathrm{i}/3}\phi_3(\tau)$. This relation takes care of the constraint (4.8). Of course, acting with the transformations (4.11) generates a two-parameter family of such 'transverse' kinks.

At the magical value of $\varkappa{=}{-1}$ the trajectories become straight, and the solution analytic:

$$\begin{aligned}
\phi_1(\tau) &= \big(\tfrac14{+}\mathrm{i}\tfrac{\sqrt3}{4}\big) + \big(-\tfrac34{+}\mathrm{i}\tfrac{\sqrt3}{4}\big)\tanh\big(\tfrac{\tau-\tau_0}{2}\big)\,,\\
\phi_2(\tau) &= -\tfrac12 - \mathrm{i}\tfrac{\sqrt3}{2}\tanh\big(\tfrac{\tau-\tau_0}{2}\big)\,,\\
\phi_3(\tau) &= \big(\tfrac14{-}\mathrm{i}\tfrac{\sqrt3}{4}\big) + \big(\tfrac34{+}\mathrm{i}\tfrac{\sqrt3}{4}\big)\tanh\big(\tfrac{\tau-\tau_0}{2}\big)\;.
\end{aligned} \tag{4.17}$$

Radial kinks at $\varkappa=3$. For this value of $\varkappa$ the critial point at the origin is degenerate with $(1,1,1)$ and its symmetry orbits. Therefore, we can connect any type A critical point to the unique type B point via 'radial kinks', such as

$$\begin{aligned}
\phi_1(\tau) &= \tfrac12\Big(1+\tanh\big(\tfrac{\tau-\tau_0}{2\sqrt3}\big)\Big)\,,\\
\phi_2(\tau) &= \big(-\tfrac14{+}\mathrm{i}\tfrac{\sqrt3}{4}\big)\Big(1+\tanh\big(\tfrac{\tau-\tau_0}{2\sqrt3}\big)\Big)\,,\\
\phi_3(\tau) &= \big(-\tfrac14{-}\mathrm{i}\tfrac{\sqrt3}{4}\big)\Big(1+\tanh\big(\tfrac{\tau-\tau_0}{2\sqrt3}\big)\Big)\,,
\end{aligned} \tag{4.18}$$

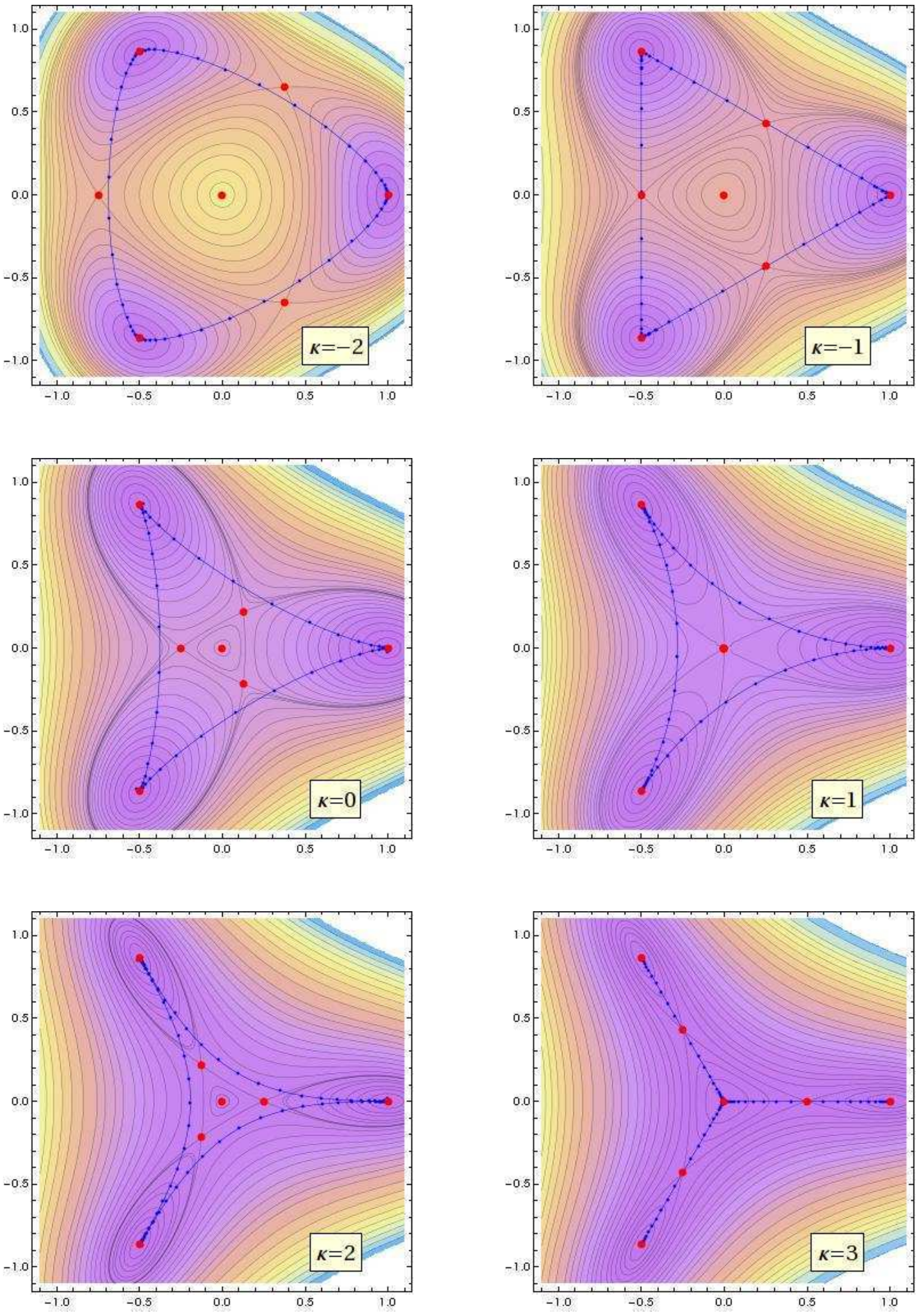

Figure 1. Contour plots of $V(\phi_1{=}\phi_2{=}\phi_3)$, with critical points and zero-energy kink trajectories.

which connects

$$(0\,,0\,,0) \qquad \longrightarrow \qquad (1\,,\mathrm{e}^{2\pi\mathrm{i}/3},\mathrm{e}^{-2\pi\mathrm{i}/3}) \tag{4.19}$$

in a 3-symmetric fashion and is also marked in the lower right plot of figure 1. It is the limiting case of the transverse kinks for $\varkappa \to +3$. In the other limit, $\varkappa \to -3$, the particles move infinitely slowly on the degenerate unit circle, $|\phi| = 1$.

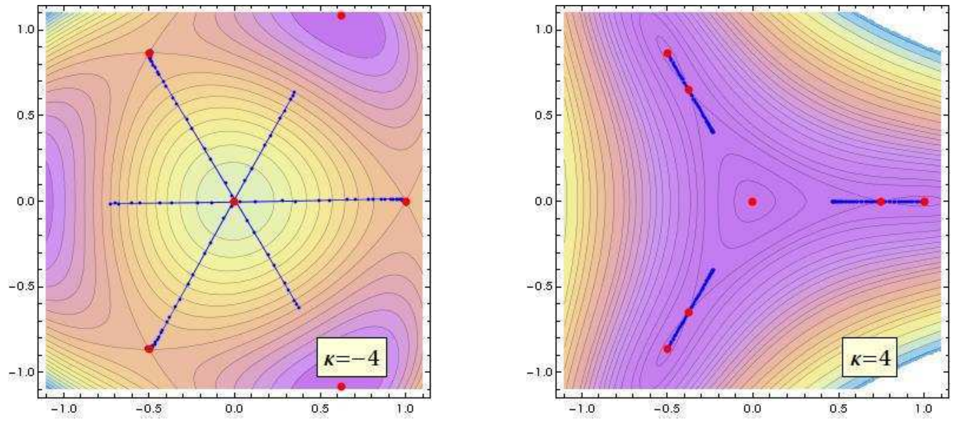

Figure 2. Contour plots of $V(\phi_1{=}\phi_2{=}\phi_3)$, with critical points and zero-energy bounce trajectories.

Bounces at $\varkappa{<}{-}3$ and $+3{<}\varkappa{<}{+}5$. In the range $\varkappa \in (-\infty,-3) \cup (+3,+5)$ finite-action bounce solutions must exist, in the form

$$\phi_k(\tau) = \mathrm{e}^{2\pi\mathrm{i}(k-1)/3}\, f_\varkappa(\tau) \quad \text{with} \quad f_\varkappa(\pm\infty) = 1 \quad \text{and} \quad f_\varkappa(0) = \tfrac{1}{6}\big(\varkappa{-}3{+}\sqrt{\varkappa^2{-}9}\big)\,, \tag{4.20}$$

where $f_\varkappa(\tau)$ is a real function, so the trajectories are straight. It is easy to find it numerically. Figure 2 shows the trajectories for $\varkappa = -4$ and $\varkappa = +4$.

Radial bounce/kink at $\varkappa = -1{-}2\sqrt{3}$. If we put $\phi_1(\tau) = \phi_2(\tau) \equiv 0$ at this $\varkappa$ value, the remaining function is governed by the rotationally symmetric potential

$$V(0,0,\phi_3) \;=\; 2(2{+}\sqrt{3}) \;-\; (1{+}\sqrt{3})|\phi_3|^2 \;+\; |\phi_3|^4\,, \tag{4.21}$$

admitting the kink solution

$$\phi_3(\tau) \;=\; \mathrm{e}^{\mathrm{i}\alpha}\sqrt{1{+}\sqrt{3}}\,\tanh\left\{\sqrt{\tfrac{1+\sqrt{3}}{6}}\,\tau\right\} \qquad \text{while} \quad \phi_1(\tau) = \phi_2(\tau) \equiv 0\,, \tag{4.22}$$

which interpolates between antipodal type C critical points via point B,

$$(0\,,0\,,-\mathrm{e}^{\mathrm{i}\alpha}\sqrt{1{+}\sqrt{3}}) \qquad \longrightarrow \qquad (0\,,0\,,+\mathrm{e}^{\mathrm{i}\alpha}\sqrt{1{+}\sqrt{3}})\ . \tag{4.23}$$

5 Yang-Mills fields on $\mathbb{R}\times$ Sp(2)/Sp(1)×U(1)

5.1 Explicit form of X_a matrices

The adjoint of Sp(2), restricted to Sp(1)×U(1), decomposes as

$$\mathbf{10}\ (\text{of Sp}(2)) \;=\; (\mathbf{3}_0 + \mathbf{1}_0)_{\text{adj}} + \mathbf{2}_{+1} + \mathbf{2}_{-1} + \mathbf{1}_{+2} + \mathbf{1}_{-2}\,, \tag{5.1}$$

where the subscript denotes the U(1) charge. Clearly, one has $q{=}2$ complex parameters. As a convenient representation, let us take the fundamental $\mathcal{D} = \mathbf{4}$ of Sp(2) ⊂ U(4). Again, it turns out that $\chi_{\mathbf{4}}/\chi_{\mathbf{10}} = 1/6$.

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We choose the generators of the subgroup Sp(1)×U(1) of Sp(2) in the form

$$I_{7,8,9} \;=\; \frac{\mathrm{i}}{2\sqrt{3}}\begin{pmatrix}\sigma_{1,2,3} & \mathbf{0}_2\\ \mathbf{0}_2 & \mathbf{0}_2\end{pmatrix} \quad\text{and}\quad I_{10} \;=\; \frac{\mathrm{i}}{2\sqrt{3}}\begin{pmatrix}\mathbf{0}_2 & \mathbf{0}_2\\ \mathbf{0}_2 & \sigma_3\end{pmatrix}\;. \tag{5.2}$$

Then solutions of the Sp(2)-invariance conditions (2.25) are given by matrices

$$\begin{aligned}
X_1 &= \frac{1}{2\sqrt{6}}\begin{pmatrix}0&0&0&-\varphi\\0&0&-\bar\varphi&0\\0&\varphi&0&0\\\bar\varphi&0&0&0\end{pmatrix}, \quad X_2 = \frac{1}{2\sqrt{6}}\begin{pmatrix}0&0&0&\mathrm{i}\varphi\\0&0&-\mathrm{i}\bar\varphi&0\\0&-\mathrm{i}\varphi&0&0\\\mathrm{i}\bar\varphi&0&0&0\end{pmatrix},\\
X_3 &= \frac{1}{2\sqrt{6}}\begin{pmatrix}0&0&-\bar\varphi&0\\0&0&0&\varphi\\\varphi&0&0&0\\0&-\bar\varphi&0&0\end{pmatrix}, \quad X_4 = \frac{-1}{2\sqrt{6}}\begin{pmatrix}0&0&\mathrm{i}\bar\varphi&0\\0&0&0&\mathrm{i}\varphi\\\mathrm{i}\varphi&0&0&0\\0&\mathrm{i}\bar\varphi&0&0\end{pmatrix},\\
X_5 &= \frac{1}{2\sqrt{3}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\bar\chi\\0&0&-\chi&0\end{pmatrix}, \quad X_6 = \frac{1}{2\sqrt{3}}\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\mathrm{i}\bar\chi\\0&0&\mathrm{i}\chi&0\end{pmatrix},
\end{aligned} \tag{5.3}$$

where φ and χ are complex-valued functions of τ. Note that the generators $\{I_a\}$ of the group Sp(2) are obtained from (5.3) if one put $\varphi = 1 = \chi$. The choice (5.2) and (5.3) agrees with the standard form (3.2), (3.5) and (3.12)–(3.16) of the nearly Kähler structure on the manifold Sp(2)/Sp(1)×U(1).

5.2 Equations of motion

The equations of motion for Sp(2)-invariant gauge fields on $\mathbb{R}$×Sp(2)/Sp(1)×U(1) are obtained by plugging (5.3) into (3.20) and (3.21). After tedious calculations we get

$$\begin{aligned}
6\,\ddot\varphi &= (\varkappa{-}1)\,\varphi \;-\; (\varkappa{+}3)\,\bar\varphi\bar\chi \;+\; (3|\varphi|^2 + \;|\chi|^2)\,\varphi\,,\\
6\,\ddot\chi &= (\varkappa{-}1)\,\chi \;-\; (\varkappa{+}3)\,\bar\varphi^2 \;\;+\; (2|\varphi|^2 + 2|\chi|^2)\,\chi\,,
\end{aligned} \tag{5.4}$$

and

$$\varphi\dot{\bar\varphi} - \dot\varphi\bar\varphi \;=\; \chi\dot{\bar\chi} - \dot\chi\bar\chi \tag{5.5}$$

Notice that these equations follow from (4.7), (4.8) after identification

$$\phi_1 = \phi_2 =: \varphi \quad\text{and}\quad \phi_3 =: \chi\;. \tag{5.6}$$

Furthermore, substituting (5.3) into the action functional (3.23), we obtain the Lagrangian

$$18\,\mathcal{L} = 12|\dot\varphi|^2{+}6|\dot\chi|^2{-}(\varkappa{-}3){+}(\varkappa{-}1)\big(2|\varphi|^2{+}|\chi|^2\big){-}(\varkappa{+}3)\big(\varphi^2\chi{+}\bar\varphi^2\bar\chi\big){+}3|\varphi|^4{+}2|\varphi\chi|^2{+}|\chi|^4\,, \tag{5.7}$$

which also follows from (4.5) after identification (5.6). The equations (5.4) are the Euler-Lagrange equations for the Lagrangian (5.7),

$$12\,\ddot\varphi \;=\; \frac{\partial V}{\partial\bar\varphi} \quad\text{and}\quad 6\,\ddot\chi \;=\; \frac{\partial V}{\partial\bar\chi}\,, \tag{5.8}$$

and the constraint (5.5) derives from the U(1) symmetry

$$\big(\varphi\,,\chi\big)\ \mapsto\ \big(\mathrm{e}^{\mathrm{i}\delta}\varphi\,,\,\mathrm{e}^{-2\mathrm{i}\delta}\chi\big) \tag{5.9}$$

of the potential

$$V=-(\varkappa{-}3)\,+\,(\varkappa{-}1)\big(2|\varphi|^2{+}|\chi|^2\big)\,-\,(\varkappa{+}3)\big(\varphi^2\chi{+}\bar\varphi^2\bar\chi\big)\,+\,3|\varphi|^4{+}2|\varphi\chi|^2{+}|\chi|^4\,. \tag{5.10}$$

5.3 Some solutions

Clearly, the solutions to (5.4) and (5.5) form a subset of the solutions to (4.7) and (4.8), namely those where two functions coincide. Since in all examples of the previous section this can be arranged by applying a U(1)×U(1) transformation (4.11), one gets $\varphi(\tau)=\chi(\tau)$ equal to any of the functions appearing on the right-hand sides of (4.17) and (4.18) or depicted in figure 1, after dialling the corresponding $\varkappa$ value. In addition, (4.22) translates to a solution with $\varphi\equiv 0$ and a kink χ.

5.4 Specialization to S^6 and flow equations

By further identification

$$\phi_1=\phi_2=\phi_3=:\phi \tag{5.11}$$

we resolve the constraint equations (4.8) and reduce (4.7) to the equation

$$6\,\ddot\phi\ =\ (\varkappa{-}1)\,\phi\ -\ (\varkappa{+}3)\,\bar\phi^2\ +\ 4|\phi|^2\phi\ =\ \frac{1}{3}\frac{\partial V}{\partial\bar\phi} \tag{5.12}$$

with

$$V\ =\ -(\varkappa{-}3)\ +\ 3(\varkappa{-}1)\,|\phi|^2\ -\ (\varkappa{+}3)\big(\phi^3{+}\bar\phi^3\big)\ +\ 6\,|\phi|^4\,. \tag{5.13}$$

The U(1) symmetry (5.9) is broken to the discrete 3-symmetry. Clearly, the Lagrangian (4.5) maps to

$$18\,\mathcal{L}\ =\ 18\,|\dot\phi|^2\ +\ V(\phi)\,, \tag{5.14}$$

which describes G_2-invariant gauge fields on $\mathbb{R}\times S^6$, where $S^6=G_2/\mathrm{SU}(3)$ [24]. All is consistent with the decomposition

$$\mathbf{14}\ (\text{of } G_2)\ =\ \mathbf{8}_{\text{adj}}+\mathbf{3}+\bar{\mathbf{3}}\ (\text{of SU(3)})\ . \tag{5.15}$$

Obviously, any function on the right-hand sides of (4.17) and (4.18) or shown in figure 1 is a zero-energy solution $\phi(\tau)$, as was already noticed in [24]. Vice versa, any solution of (5.12) gives a special solution to the equations (5.4), (5.5) and (4.7), (4.8).

Let us for a moment investigate the possibility of straight-trajectory solutions $\phi(\tau)\in\mathbb{C}$ to (5.12). With a 3-symmetry transformation, any such solution can be brought into a form where either $\mathrm{Re}\phi(\tau)=\text{const}$ or $\mathrm{Im}\phi(\tau)=\text{const}$. Then, the vanishing of the left-hand side of Re (5.12) yields two conditions on $\mathrm{Re}\phi$ and $\varkappa$, whose solutions follow a Hamiltonian flow [24]:

$$\begin{aligned}
\varkappa=-1\quad&\text{and}\quad \mathrm{Re}\phi=-\tfrac12 &&\Rightarrow\quad \sqrt3\,\mathrm{Im}\dot\phi=\tfrac34-(\mathrm{Im}\phi)^2 &&\Leftrightarrow\quad \sqrt3\,\dot\phi=\mathrm{i}\,(\bar\phi^2-\phi)\,,\\
\varkappa=-3\quad&\text{and}\quad \mathrm{Re}\phi=0 &&\Rightarrow\quad \sqrt3\,\mathrm{Im}\dot\phi=1-(\mathrm{Im}\phi)^2 &&\Leftrightarrow\quad \sqrt3\,\dot\phi=\tfrac{\phi}{|\phi|}\,(1-|\phi|^2)\,,\\
\varkappa=-7\quad&\text{and}\quad \mathrm{Re}\phi=1 &&\Rightarrow\quad \sqrt3\,\mathrm{Im}\dot\phi=3-(\mathrm{Im}\phi)^2 &&\Leftrightarrow\quad \sqrt3\,\dot\phi=\mathrm{i}\,(\bar\phi^2+2\phi)\,.
\end{aligned} \tag{5.16}$$

On the other hand, for $\mathrm{Im}\ddot\phi=0$ one finds

$$\text{any } \varkappa \quad \text{and} \quad \mathrm{Im}\phi=0 \quad \Rightarrow \quad 6\,\mathrm{Re}\ddot\phi=(\varkappa{-}1)\mathrm{Re}\phi-(\varkappa{+}3)(\mathrm{Re}\phi)^2+4(\mathrm{Re}\phi)^3=\frac{1}{3}\frac{\partial V_{\mathbb{R}}}{\partial \mathrm{Re}\phi}\,, \tag{5.17}$$

with

$$V_{\mathbb{R}} \;=\; \big(\mathrm{Re}\phi-1\big)^2\,\big(6(\mathrm{Re}\phi)^2-(\varkappa{-}3)(2\mathrm{Re}\phi+1)\big)\,. \tag{5.18}$$

This includes the gradient-flow situations [24]

$$\begin{aligned} \varkappa=+3 \quad \text{and} \quad \mathrm{Im}\phi=0 \;&\Rightarrow\; \sqrt{3}\,\mathrm{Re}\dot\phi=(\mathrm{Re}\phi)^2-\mathrm{Re}\phi \quad \Leftrightarrow\quad \sqrt{3}\,\dot\phi=\bar\phi^2-\phi\,,\\ \varkappa=+9 \quad \text{and} \quad \mathrm{Im}\phi=0 \;&\Rightarrow\; \sqrt{3}\,\mathrm{Re}\dot\phi=(\mathrm{Re}\phi)^2-2\,\mathrm{Re}\phi \quad \Leftrightarrow\quad \sqrt{3}\,\dot\phi=\bar\phi^2-2\phi\,. \end{aligned} \tag{5.19}$$

All kink solutions to (5.16) and (5.19) were given in [24]. They have zero energy and thus finite action only for $\varkappa=-3$, -1 and $+3$. The latter two cases are also displayed in (4.17) and (4.18), respectively. In addition, for $\varkappa{<}{-3}$ and $+3{<}\varkappa{<}{+5}$ one can also numerically construct finite-action bounce solutions to (5.17).

Remark. Note that a nearly Kähler structure exists also on the space $S^3\times S^3$. However, we do not consider the Yang-Mills equations on $\mathbb{R}\times S^3\times S^3$ since this was already done in [21].

6 Instanton-anti-instanton chains and dyons

If we replace $\mathbb{R}\times G/H$ with $S^1\times G/H$, the time interval will be of finite length, namely the circle circumference L, and we are after solutions periodic in τ. In this case, the action is always finite, and the $E{=}0$ requirement gets replaced by $\phi_i(\tau{+}L)=\phi_i(\tau)$. The physical interpretation of such configurations is one of instanton-anti-instanton chains.

6.1 Periodic solutions

As the simplest case we take $G/H=G_2/\mathrm{SU}(3)$ and consider the magical $\varkappa$ values which admit analytic solutions for $\phi(\tau)\in\mathbb{C}$. Switching from $\tau\in\mathbb{R}$ to $\tau\in S^1$, we must impose the periodicity conditions

$$\phi(\tau{+}L) \;=\; \phi(\tau) \tag{6.1}$$

not on the flow equations (5.16) and (5.19) but on the corresponding second-order equations,

$$\begin{aligned} \varkappa=-1 \quad&\text{and}\quad \mathrm{Re}\phi=-\tfrac12 &&\Rightarrow\quad \tfrac32\,\mathrm{Im}\ddot\phi \;=\; \mathrm{Im}\phi\,\big(\mathrm{Im}\phi^2-\tfrac34\big)\,,\\ \varkappa=-3 \quad&\text{and}\quad \mathrm{Re}\phi=0 &&\Rightarrow\quad \tfrac32\,\mathrm{Im}\ddot\phi \;=\; \mathrm{Im}\phi\,(\mathrm{Im}\phi^2-1)\,,\\ \varkappa=-7 \quad&\text{and}\quad \mathrm{Re}\phi=1 &&\Rightarrow\quad \tfrac32\,\mathrm{Im}\ddot\phi \;=\; \mathrm{Im}\phi\,(\mathrm{Im}\phi^2-3)\,,\\ \varkappa=+3 \quad&\text{and}\quad \mathrm{Im}\phi=0 &&\Rightarrow\quad \tfrac32\,\mathrm{Re}\ddot\phi \;=\; \mathrm{Re}\phi\,\big(\mathrm{Re}\phi-\tfrac12\big)\,(\mathrm{Re}\phi-1)\,,\\ \varkappa=+9 \quad&\text{and}\quad \mathrm{Im}\phi=0 &&\Rightarrow\quad \tfrac32\,\mathrm{Re}\ddot\phi \;=\; \mathrm{Re}\phi\,(\mathrm{Re}\phi-1)\,(\mathrm{Re}\phi-2)\,. \end{aligned} \tag{6.2}$$

At finite L, we obtain a different kind of solution (sphalerons), namely

$$\begin{aligned} \phi(\tau)=\beta\pm\mathrm{i}\sqrt{3}\,\gamma\,k\,b(k)\;\mathrm{sn}[b(k)\gamma\tau;k]\text{with}\quad &(\varkappa;\beta,\gamma)=\big(-1;-\tfrac12,1\big),\ \big(-3;0,\tfrac{2}{\sqrt3}\big),\ (-7;1,2)\,,\\ \phi(\tau)=\beta\pm\ \sqrt{3}\,\gamma\,k\,b(k)\;\mathrm{sn}[b(k)\gamma\tau;k]\text{with}\quad &(\varkappa;\beta,\gamma)=\big(+3;\tfrac12,\tfrac{1}{\sqrt3}\big),\ \big(+9;1,\tfrac{2}{\sqrt3}\big)\,. \end{aligned} \tag{6.3}$$

Here $b(k) = (2+2k^2)^{-1/2}$ and $0 \leq k \leq 1$. Since the Jacobi elliptic function $\mathrm{sn}[u;k]$ has a period of $4K(k)$ (see appendix B), the condition (6.1) is satisfied if

$$\gamma\, b(k)\, L \;=\; 4K(k)\, n \qquad \text{for} \quad n \in \mathbb{N}\,, \tag{6.4}$$

which fixes $k = k(L,n)$ so that $\phi(\tau; k(L,n)) =: \phi^{(n)}(\tau)$. Solutions (6.3) exist if $L \geq 2\pi\sqrt{2}\, n$ [57–59].

By virtue of the periodic boundary conditions (6.1), the topological charge of the sphaleron $\phi^{(n)}$ is zero. In fact, the configuration is interpreted as a chain of n kinks and n antikinks, alternating and equally spaced around the circle [40, 57–59]. Interpreted as a static configuration on $S^1 \times G/H$, the energy of the sphaleron is

$$\mathcal{E} \;=\; \int_0^L \mathrm{d}\tau \left\{ |\dot{\phi}|^2 + V(\phi) \right\} \tag{6.5}$$

and e.g. for the case of $\varkappa = -3$ in (6.3) we obtain

$$\mathcal{E}[\phi^{(n)}] \;=\; \frac{2n}{3\sqrt{2}} \Big[8(1+k^2)\, E(k) \;-\; (1-k^2)(5+3k^2)\, K(k) \Big]\,, \tag{6.6}$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind, respectively [57–59].

The non-BPS solutions (6.3) can be embedded into the other cosets G/H, where they are special solutions, with $\varphi = \chi$ or $\phi_1 = \phi_2 = \phi_3$, respectively. Their degeneracy may be lifted by applying a symmetry transformation (5.9) or (4.11), respectively. Substituting our non-BPS solutions into (4.4) or (5.3) and then into (2.24), we obtain a finite-action Yang-Mills configuration which is interpreted as a chain of n instanton-anti-instanton pairs sitting on $S^1 \times G/H$ with six-dimensional nearly Kähler coset space G/H. Away from the magical $\varkappa$ values, such chains are to be found numerically.

6.2 Dyonic solutions

Let us finally change the signature of the metric on $\mathbb{R}\times G/H$ from Euclidean to Lorentzian by choosing on $\mathbb{R}$ a coordinate $t = -\mathrm{i}\tau$ so that $\tilde{e}^0 = \mathrm{d}t = -\mathrm{i}\mathrm{d}\tau$. Then as metric on $\mathbb{R}\times G/H$ we have

$$\mathrm{d}s^2 \;=\; -(\tilde{e}^0)^2 + \delta_{ab} e^a e^b\ . \tag{6.7}$$

The G-invariant solutions (4.4) and (5.3) for the matrices X_a are not changed. After substituting them into the Yang-Mills equations on $\mathbb{R}\times G/H$, we arrive at the same second-order differential equations as in the Euclidean case, except for the replacement

$$\ddot{\phi}_i \qquad \longrightarrow \qquad -\tfrac{\mathrm{d}^2\phi_i}{\mathrm{d}t^2}\ . \tag{6.8}$$

In particular, this implies a sign change of the left-hand side relative to the right-hand side in (4.7), (5.4) and (5.12). Thus, in the Lagrangians we effectively have a sign flip of the potential V, so that the analog Newtonian dynamics for $(\phi_i(t))$ is based on $+V$.

Let us again for simplicity look at the case of $G/H = G_2/\mathrm{SU}(3)$. Although the Lorentzian variant of (5.12),

$$6\,\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} \;=\; -(\varkappa{-}1)\,\phi \;+\; (\varkappa{+}3)\,\bar{\phi}^2 \;-\; 4|\phi|^2\phi \;=\; -\frac{1}{3}\frac{\partial V}{\partial\bar{\phi}} \tag{6.9}$$

with V from (5.13), does not follow from first-order equations for any of the magical values $\varkappa = -1,\,-3,\,-7,\,+3$ or $+9$, it can still be explicitly integrated in those cases,

$$\begin{aligned} \phi(t) \;&=\; \beta \pm \mathrm{i}\sqrt{\tfrac{3}{2}}\,\gamma\,\cosh^{-1}\tfrac{\gamma\,t}{\sqrt{2}} \quad \text{with} \quad (\varkappa;\beta,\gamma) = \big(-1;-\tfrac{1}{2},1\big),\ \big(-3;0,\tfrac{2}{\sqrt{3}}\big),\ (-7;1,2)\,, \\ \phi(t) \;&=\; \beta \pm \sqrt{\tfrac{3}{2}}\,\gamma\,\cosh^{-1}\tfrac{\gamma\,t}{\sqrt{2}} \quad \text{with} \quad (\varkappa;\beta,\gamma) = \big(+3;\tfrac{1}{2},\tfrac{1}{\sqrt{3}}\big),\ \big(+9;1,\tfrac{2}{\sqrt{3}}\big)\,. \end{aligned} \tag{6.10}$$

The 3-symmetry action maps these solutions to rotated ones. Any such configuration is a bounce in our double-well-type potential, which most of the time hovers around a saddle point. For other values of $\varkappa$, such bounce solutions may be found numerically.

Inserting (6.10) into the gauge potential, we arrive at dyon-type configurations with smooth nonvanishing 'electric' and 'magnetic' field strength $\mathcal{F}_{0a}$ and $\mathcal{F}_{ab}$, respectively. The total energy

$$-\,\mathrm{tr}\,(2\mathcal{F}_{0a}\mathcal{F}_{0a} + \mathcal{F}_{ab}\mathcal{F}_{ab}) \times \mathrm{Vol}(G/H) \tag{6.11}$$

for these configurations is finite, but their action diverges unless $\phi(\pm\infty) = \mathrm{e}^{2\pi\mathrm{i}k/3}$. These are saddle points for $\varkappa < -3$ and $\varkappa > +5$. Thus, for $|\varkappa{-}1| > 4$ the potential (5.13) admits pairs $\phi_\pm(t)$ of finite-action dyons, with

$$\phi_\pm(\pm\infty) \;=\; 1 \quad \text{and} \quad \phi_\pm(0) \;=\; \tfrac{1}{6}\big(\varkappa{-}3 \pm \sqrt{\varkappa^2{-}9}\big) \qquad \text{for} \quad \varkappa > +5 \tag{6.12}$$

and a more complex behavior for $\varkappa < -3$. The $\varkappa{=}{-7}$ and $\varkappa{=}{+9}$ straight-line solutions in (6.10) are among these. Numerical trajectories for some intermediate values are shown in the plots of figure 3.

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A Zero-energy critical points

Here, we prove that the table in subsection 4.3 lists all zero-energy critical points $(\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3)$ of the potential (4.10), modulo permutations of the $\widehat{\phi}_i$ and actions of the $\mathrm{U}(1){\times}\mathrm{U}(1)$ symmetry (4.11).

With the help of this symmetry, we can remove the phases of $\widehat{\phi}_1$ and $\widehat{\phi}_2$. Since it was already argued that extremality implies $\sum_i \arg\widehat{\phi}_i = 0$ or π, also $\widehat{\phi}_3$ must be real. Hence, we may take

$$\widehat{\phi}_1\,,\widehat{\phi}_2 \;\in\; \mathbb{R}_+ \quad \text{and} \quad \widehat{\phi}_3 \;\in \mathbb{R} \tag{A.1}$$

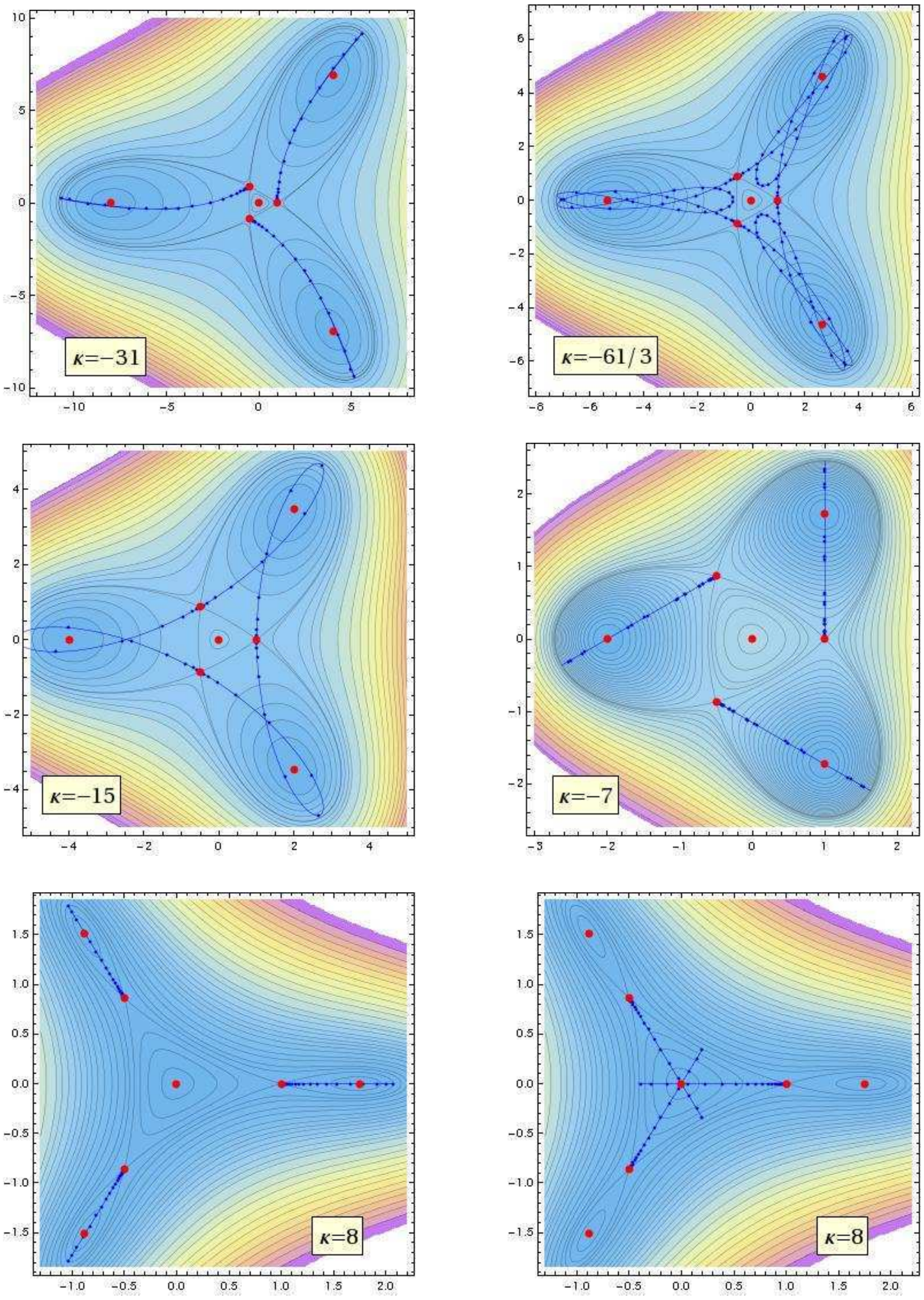

Figure 3. Contour plots of $V(\phi_1{=}\phi_2{=}\phi_3)$, with critical points and finite-action dyon trajectories.

and investigate the solution space of $\mathrm{d}V{=}0{=}V$, i.e.

$$(\varkappa{-}1)\,\widehat{\phi}_i - (\varkappa{+}3)\widehat{\phi}_j\widehat{\phi}_k + (2\widehat{\phi}_i^2 + \widehat{\phi}_j^2 + \widehat{\phi}_k^2)\,\widehat{\phi}_i = 0 \quad \text{for} \quad i{\neq}j{\neq}k \in \{1,2,3\} \tag{A.2}$$

$$\text{and} \quad (\varkappa{-}1)\textstyle\sum_i\widehat{\phi}_i^2 - 2(\varkappa{+}3)\,\widehat{\phi}_1\widehat{\phi}_2\widehat{\phi}_3 + \sum_i\widehat{\phi}_i^4 + \sum_{i<j}\widehat{\phi}_i^2\widehat{\phi}_j^2 = \varkappa{-}3\ . \tag{A.3}$$

Let us first look at the exceptional cases where one of the $\widehat{\phi}_i$ vanishes. From (A.2) it follows that $\hat{\phi}_i = 0$ implies $\hat{\phi}_j\hat{\phi}_k = 0$. The trivial solution is

$$\hat{\phi}_1 = \hat{\phi}_2 = \hat{\phi}_3 = 0 \quad \overset{(A.3)}{\Rightarrow} \quad \varkappa = 3 \tag{A.4}$$

and is labelled as type B in the table. Generically, however, we have

$$\hat{\phi}_1 = \hat{\phi}_2 = 0 \quad \text{and} \quad \hat{\phi}_3 \neq 0 \quad \overset{(A.2)}{\Rightarrow} \quad \varkappa{-}1 + 2\,\hat{\phi}_3^2 = 0 \quad \overset{(A.3)}{\Rightarrow} \quad \varkappa = -1 \pm 2\sqrt{3} \tag{A.5}$$

and reproduce type C in the table.[3]

It remains to study the situation where all $\hat{\phi}_i$ are nonzero. Multiplying (A.2) with $\hat{\phi}_i$ and taking the difference of any two of the resulting three equations, we obtain the three conditions

$$\left(\varkappa{-}1 + 2\hat{\phi}_i^2 + 2\hat{\phi}_j^2 + \hat{\phi}_k^2\right)\left(\hat{\phi}_i^2 - \hat{\phi}_j^2\right) \;=\; 0\;. \tag{A.6}$$

Likewise, multiplying (A.2) with $\hat{\phi}_j\hat{\phi}_k$ and taking the difference of any two of those three equations, we find three more conditions,

$$\left((\varkappa{+}3)\,\hat{\phi}_k^2 + \hat{\phi}_1\hat{\phi}_2\hat{\phi}_3\right)\left(\hat{\phi}_i^2 - \hat{\phi}_j^2\right) \;=\; 0\;. \tag{A.7}$$

A little thought reveals that there are only two options. The first one is

$$\hat{\phi}_1^2 = \hat{\phi}_2^2 = \hat{\phi}_3^2 \quad \Rightarrow \quad \hat{\phi}_1 = \hat{\phi}_2 = \pm\hat{\phi}_3 \;=:\; \hat{\phi} \in \mathbb{R}_+\;. \tag{A.8}$$

The potential on this subspace becomes

$$V(\hat{\phi},\hat{\phi},\pm\hat{\phi}) \;=\; \left(6\,\hat{\phi}^2 \mp (\varkappa{-}3)(2\hat{\phi}-1)\right)\left(\hat{\phi}\mp 1\right)^2, \tag{A.9}$$

and its critical zeros on the positive real axis are

$$(\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3;\;\varkappa) \;=\; (+1,+1,+1;\;\text{any}) \quad \text{and} \quad (+1,+1,-1;\;-3) \tag{A.10}$$

for the two sign choices, respectively. We have recovered types A and A' of our table.

The second option for fulfilling (A.6) and (A.7) is, modulo permutation,

$$\hat{\phi}_1^2 = \hat{\phi}_2^2 \neq \hat{\phi}_3^2 \quad \Rightarrow \quad \hat{\phi}_1 = \hat{\phi}_2 \;=:\; \hat{\varphi} \in \mathbb{R}_+ \quad \text{and} \quad \hat{\phi}_3 \;=:\; \hat{\chi} \in \mathbb{R}\,, \tag{A.11}$$

with the simultaneous requirements

$$\varkappa{-}1 + 3\hat{\varphi}^2 + 2\hat{\chi}^2 \;=\; 0 \quad \text{and} \quad \varkappa{+}3 + \hat{\chi} \;=\; 0 \tag{A.12}$$

from (A.6) and (A.7), respectively. The solution

$$\hat{\varphi} \;=\; \sqrt{-\tfrac{2}{3}\varkappa^2 - \tfrac{13}{3}\varkappa - \tfrac{17}{3}} \quad \text{and} \quad \hat{\chi} \;=\; -\varkappa - 3 \tag{A.13}$$

restricts $-13{-}\sqrt{33} < 4\varkappa < -13{+}\sqrt{33}$, but one finds that

$$V(\hat{\varphi},\hat{\varphi},\hat{\chi}) \;=\; -\tfrac{1}{3}\,(\varkappa{+}1)\,(\varkappa{+}4)^3\,, \tag{A.14}$$

which leaves only

$$\varkappa = -4 \quad \Rightarrow \quad \hat{\varphi} = \hat{\chi} = 1\,, \tag{A.15}$$

falling back to type A. Thus, the list of critical zeros presented in subsection 4.3 is exhaustive.

[3] Only one of the two values for $\varkappa$ leads to a real $\hat{\phi}_3$.

B Jacobi elliptic functions

The Jacobi elliptic functions arise from the inversion of the elliptic integral of the first kind,

$$u = F(\xi, k) = \int_0^\xi \frac{\mathrm{d}x}{\sqrt{1-k^2\sin x}}\,, \qquad 0 \le k^2 < 1\,, \tag{B.1}$$

where $k = \mathrm{mod}\, u$ is the elliptic modulus and $\xi = \mathrm{am}(u,k) = \mathrm{am}(u)$ is the Jacobi amplitude, giving

$$\xi = F^{-1}(u,k) = \mathrm{am}(u,k)\ . \tag{B.2}$$

Then the three basic functions sn, cn and dn are defined by

$$\mathrm{sn}[u;k] = \sin(\mathrm{am}(u,k)) = \sin\xi\,, \tag{B.3}$$

$$\mathrm{cn}[u;k] = \cos(\mathrm{am}(u,k)) = \cos\xi\,, \tag{B.4}$$

$$\mathrm{dn}[u;k]^2 = 1 - k^2\sin^2(\mathrm{am}(u,k)) = 1 - k^2\sin^2\xi\ . \tag{B.5}$$

These functions are periodic in $K(k)$ and $\tilde{K}(k)$,

$$\mathrm{sn}[u{+}2mK{+}2n\mathrm{i}\tilde{K};k] = (-1)^m\mathrm{sn}[u;k]\,, \tag{B.6}$$

$$\mathrm{cn}[u{+}2mK{+}2n\mathrm{i}\tilde{K};k] = (-1)^{m+n}\mathrm{cn}[u;k]\,, \tag{B.7}$$

$$\mathrm{dn}[u{+}2mK{+}2n\mathrm{i}\tilde{K};k] = (-1)^n\mathrm{dn}[u;k]\,, \tag{B.8}$$

where $K(k)$ is the complete elliptic integral of the first kind,

$$K(k) := F(\tfrac{\pi}{2},k) \quad \text{and} \quad \tilde{K}(k) := K(\sqrt{1{-}k^2}) = F(\tfrac{\pi}{2},\sqrt{1{-}k^2})\ . \tag{B.9}$$

In the following we sometimes drop the parameter k, i.e. write $\mathrm{sn}[u;k] = \mathrm{sn}(u)$ etc.

The Jacobi elliptic functions generalize the trigomonetric functions and satisfy analogous identities, including

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1\,, \tag{B.10}$$

$$k^2\mathrm{sn}^2u + \mathrm{dn}^2u = 1\,, \tag{B.11}$$

$$\mathrm{cn}^2u + \sqrt{1{-}k^2}\,\mathrm{sn}^2u = 1 \tag{B.12}$$

as well as

$$\mathrm{sn}[u;0] = \sin u\,, \tag{B.13}$$

$$\mathrm{cn}[u;0] = \cos u\,, \tag{B.14}$$

$$\mathrm{dn}[u;0] = 1\ . \tag{B.15}$$

One may also define cn, dn and sn as solutions $y(x)$ to the respective differential equations

$$y'' = (2{-}k)^2y + y^3\,, \tag{B.16}$$

$$y'' = -(1{-}2k^2)y + 2k^2y^3\,, \tag{B.17}$$

$$y'' = -(1{+}k^2)y + 2k^2y^3\ . \tag{B.18}$$

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