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# The Weyl double copy in vacuum spacetimes with a cosmological constant

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Abstract: We examine the Weyl double copy relation for vacuum solutions of the Einstein equations with a cosmological constant using the approach we previously described, in which the spin-1/2 massless free-field spinors (Dirac-Weyl fields) are regarded as basic units. Based on the exact non-twisting vacuum type N and vacuum type D solutions, the finding explicitly shows that the single and zeroth copies fulfill conformally invariant field equations in conformally flat spacetime. In addition, irrespective of the presence of a cosmological constant, we demonstrate that the zeroth copy connects Dirac-Weyl fields with the degenerate electromagnetic fields in the curved spacetime in addition to connecting gravity fields with the single copy in conformally flat spacetime. Moreover, the study also demonstrates the critical significance the zeroth copy plays in time-dependent radiation solutions. In particular, for Robinson-Trautman ($\Lambda$) gravitational waves, unlike the single copy, we find that the zeroth copy carries additional information to specify whether the sources of associated gravitational waves are time-like, null, or space-like, at least in the weak field limit.

Keywords: Black Holes, Classical Theories of Gravity, Gauge-Gravity Correspondence, Black Holes in String Theory

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## Contents





## 1 Introduction

The double copy originates from the study of perturbative scattering amplitudes [1–3], which brings forth a fascinating connection between gauge amplitudes and gravity amplitudes. Moreover, this idea has been extended to the classical context. In Kerr-Schild coordinate system, a map between gravity theory and gauge theory was proposed, called Kerr-Schild double copy [4]. A wide array of such classes of spacetimes has been studied [5–19]. Inspired by this, a new type of double copy relation called Weyl double copy is drawing more attention [20–28]. This prescription is represented by

$$\Psi_{ABCD} = \frac{\Phi_{(AB}\Phi_{CD)}}{S}, \tag{1.1}$$

where $\Psi_{ABCD}$ is a Weyl spinor describing vacuum gravity fields, $\Phi_{AB}$ is an electromagnetic spinor referring to a Maxwell field in Minkowski spacetime — the simplest solution of the gauge theory, and $S$ is an auxiliary scalar field satisfying the wave equation in Minkowski spacetime. The last two fields are called single copy and zeroth copy, respectively. Starting from the gravity fields, the Weyl double copy relation leads to a gauge field that is completely independent of the gravity theory. As a result, it is thought that, the Weyl double copy relation could serve as a link between gravity theory and gauge theory.

Luna et al. proposed for the first time the Weyl double copy relation for the case of vacuum type D solutions [20]. Then, in spinor language, this relation was extended to non-twisting vacuum type N solutions by Godazgar et al. [25]. Making use of the peeling property [29, 30] of the Weyl tensor, they further showed that the Weyl double copy relation also holds asymptotically for algebraically general solutions [27]. In addition, at the linearised lever, the Weyl double copy relation was shown to hold for arbitrary Petrov type solutions using the twistor formalism [22, 23]. An extended Weyl double copy prescription was also proposed recently for non-vacuum solutions, whose Weyl spinor is decomposed into a sum of source terms [28]. Very recently, regarding the Dirac-Weyl (DW) spinors (spin-1/2 massless free-field spinors) as the basic units of other higher spin massless free-field spinors, we systematically revisited the Weyl double copy relation for non-twisting vacuum type N and vacuum type D solutions [31]. We further found a map similar to the Weyl double copy prescription for non-twisting vacuum type III spacetimes.

However, the Weyl double copy relation for the exact vacuum solutions with a cosmological constant has not yet been investigated. This is the primary objective of the current effort. In fact, since 1998, by the observations of supernovae of Ia type [32, 33], studies have shown that the expansion of our universe is accelerating, which strongly supports the condition that the cosmological constant $\Lambda$ is nonzero and positive. On the other hand, although Anti-de Sitter (AdS) spacetime does not appear to have direct cosmological applications, it plays a crucial role in AdS/CFT correspondence. Therefore, investigating the Weyl double copy relation in the presence of a cosmological constant would be of interest. Currently, there are two possible research directions: one is to interpret the cosmological constant as a source of the single and zeroth copies in the flat spacetime; the other is to consider the (A)dS spacetime to be the background of the single and zeroth copies. The former idea was proposed for the first time in Kerr-Schild double copy in Taub-NUT spacetime [8] and it would be natural in the direct investigation of the relationship between gravity theory and gauge theory. On the other hand, the latter can be viewed as a precursor to the former. Moreover, it is also advantageous for extending the remit of the Weyl double copy, including cosmological applications and perturbation theory. This has been done in ref. [15] for Kerr-Schild($\Lambda$) double copy, which shows that the single and zeroth copies satisfy different equations for time-dependent and time-independent solutions. These outcomes encourage us to study whether or not the Weyl double copy relation shares this property. In this paper, we shall give an explicit demonstration to show that, different from the Kerr-Schild($\Lambda$) double copy, the single and zeroth copies in the Weyl double copy prescription all satisfy conformal invariant field equations in conformally flat spacetime, both for time-independent solutions and time-dependent solutions. Our finding coincides with the statement of ref. [22] in the twistorial version. Some interesting relations between the zeroth copy and gravitational waves will also be discussed.

The structure of this paper is as follows. In section 2, we will briefly review how to construct electromagnetic spinors in vacuum type N and type D spacetimes by regarding DW spinors as the basic units. Then, we will study the Weyl double copy for exact vacuum solutions with a cosmological constant in section 3. The interpretations of the single copy and the zeroth copy will also be included. Discussion and conclusions are given in section 4. The notation of this paper follows the conventions of ref. [31].

## 2 Massless free-fields in spinor formalism

In this section, we will briefly review how to construct electromagnetic spinors in order to verify the Weyl double copy relation using the methodology of the previous work [31].

In spinor formalism, spin-$k/2$ massless free-field equations have a simple form [34]

$$\nabla^{A_1 A_1'} \mathcal{S}_{A_1 A_2 \ldots A_k} = 0, \tag{2.1}$$

where the spinor $\mathcal{S}_{A_1 A_2 \ldots A_k}$ is totally symmetric.

For spin-2 massless free-fields, the spinor $\mathcal{S}$ refers to the Weyl spinor $\Psi_{ABCD}$ translated from the Weyl tensor $C_{abcd}$

$$C_{abcd} = C_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}. \tag{2.2}$$

It is easy to find that the Weyl spinor $\Psi_{ABCD}$ plays the same role as the Weyl tensor $C_{abcd}$. For a vacuum spacetime (with or without a cosmological constant $\Lambda$), the Einstein field equation is absorbed into the Bianchi identity, which reads

$$\nabla^{AA'}\Psi_{ABCD} = 0. \tag{2.3}$$

This is nothing but a spin-2 massless free-field equation. Notably, the fact that this field equation remains the same regardless of the presence or absence of a cosmological constant motivates us to generalize the original Weyl double copy to the case with a cosmological constant. As is well known, ten independent real components of the Weyl tensor can be reduced to 5 independent complex scalars with the aid of a null tetrad, as defined in ref. [35]. Using the totally symmetric property of the Weyl spinor, we can define them as follows,

$$\begin{aligned}
\psi_0 &= \Psi_{ABCD}o^A o^B o^C o^D = C_{abcd}\ell^a m^b \ell^c m^d,\\
\psi_1 &= \Psi_{ABCD}o^A o^B o^C \iota^D = C_{abcd}\ell^a m^b \ell^c n^d,\\
\psi_2 &= \Psi_{ABCD}o^A o^B \iota^C \iota^D = C_{abcd}\ell^a m^b \bar{m}^c n^d,\\
\psi_3 &= \Psi_{ABCD}o^A \iota^B \iota^C \iota^D = C_{abcd}\ell^a n^b \bar{m}^c n^d,\\
\psi_4 &= \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D = C_{abcd}\bar{m}^a n^b \bar{m}^c n^d,
\end{aligned} \tag{2.4}$$

where the second equations hold based on the definition of the null tetrad in the spinor bases

$$\begin{aligned}
\ell^a &= o^A\bar{o}^{A'}, \quad n^a = \iota^A\bar{\iota}^{A'}, \quad m^a = o^A\bar{\iota}^{A'}, \quad \bar{m}^a = \iota^A\bar{o}^{A'},\\
\ell_a &= o_A\bar{o}_{A'}, \quad n_a = \iota_A\bar{\iota}_{A'}, \quad m_a = o_A\bar{\iota}_{A'}, \quad \bar{m}_a = \iota^A\bar{o}^{A'}.
\end{aligned} \tag{2.5}$$

It is easy to check that the above correspondence indeed defines a null tetrad such that

$$\begin{aligned}
\ell^2 = n^2 = m^2 = \bar{m}^2 = 0,\\
\ell\cdot n = 1, \quad m\cdot\bar{m} = -1, \quad \ell\cdot m = n\cdot m = \ell\cdot\bar{m} = n\cdot\bar{m} = 0.
\end{aligned} \tag{2.6}$$

The spin coefficients are defined the same as in the preceding work [31],

$$\begin{aligned}
\kappa^* &= m^{\mathbf{a}}\ell^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}}, & \pi^* &= n^{\mathbf{a}}\ell^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}, & \epsilon^* &= \frac{1}{2}(n^{\mathbf{a}}\ell^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}} + m^{\mathbf{a}}\ell^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}),\\
\tau^* &= m^{\mathbf{a}}n^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}}, & \nu^* &= n^{\mathbf{a}}n^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}, & \gamma^* &= \frac{1}{2}(n^{\mathbf{a}}n^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}} + m^{\mathbf{a}}n^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}),\\
\sigma^* &= m^{\mathbf{a}}m^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}}, & \mu^* &= n^{\mathbf{a}}m^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}, & \beta^* &= \frac{1}{2}(n^{\mathbf{a}}m^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}} + m^{\mathbf{a}}m^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}),\\
\rho^* &= m^{\mathbf{a}}\bar{m}^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}}, & \lambda^* &= n^{\mathbf{a}}\bar{m}^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}, & \alpha^* &= \frac{1}{2}(n^{\mathbf{a}}\bar{m}^{\mathbf{b}}\nabla_{\mathbf{b}}\ell_{\mathbf{a}} + m^{\mathbf{a}}\bar{m}^{\mathbf{b}}\nabla_{\mathbf{b}}\bar{m}_{\mathbf{a}}).
\end{aligned} \tag{2.7}$$



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For more details, one may refer to refs. [35, 36]. To distinguish from other symbols in this paper, we use $*$ to mark these spin coefficients in the following, such as $\kappa^*$, $\alpha^*$, $\beta^*$, etc. Expanding out the Weyl spinor, the general form reads

$$\begin{aligned}\Psi_{ABCD} = {}& \psi_0 \iota_A \iota_B \iota_C \iota_D - 4\psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + 6\psi_2 o_{(A} o_B \iota_C \iota_{D)} \\ & - 4\psi_3 o_{(A} o_B o_C \iota_{D)} + \psi_4 o_A o_B o_C o_D.\end{aligned} \tag{2.8}$$

For vacuum type N and type D solutions, the Weyl spinors are reduced to

$$\text{type N}: \quad \Psi_{ABCD} = \psi_4 o_A o_B o_C o_D, \tag{2.9}$$

$$\text{type D}: \quad \Psi_{ABCD} = 6\psi_2 o_{(A} o_B \iota_C \iota_{D)}. \tag{2.10}$$

For spin-1/2 massless free-fields, the spinor $\mathcal{S}$ refers to a DW spinor $\xi_A$. Eq. (2.1) in this case represents the DW field equation

$$\nabla^{AA'} \xi_A = 0. \tag{2.11}$$

With the map proposed in the preceding work [31]

$$\Psi_{ABCD} = \frac{\xi_{(A} \eta_B \zeta_C \chi_{D)}}{S_{14}}, \tag{2.12}$$

where the four DW spinors on the right side can be chosen to be the same (depending on which type of spacetime we are focusing on), we are now able to derive the DW spinors in a certain vacuum spacetime with a cosmological constant. Correspondingly, the electromagnetic spinors in curved spacetime will be formulated.

Specifically, for vacuum type N solutions, the map eq. (2.12) reduces to

$$\Psi_{ABCD} = \frac{\xi_{(A} \xi_B \xi_C \xi_{D)}}{S_{14}} = \frac{\xi_{(A} \xi_B \xi_C \xi_{D)}}{(S_{12})^3}. \tag{2.13}$$

From eq. (2.9) one can see that $\xi_A = \xi o_A$. According to ref. [31], we know that $S_{12} = S_{24} = S_{14}^{1/3}$, where $S_{ij}$ is an auxiliary scalar connecting a spin-$i/2$ massless free-field spinor with a spin-$j/2$ massless free-field spinor. The independent dyad components of the Weyl field equation eq. (2.3) then read

$$o_A \nabla^{AA'} \log\ \Psi_4 + 4 o_A \iota^B \nabla^{AA'} o_B - \iota_A o^B \nabla^{AA'} o_B = 0, \tag{2.14}$$

where we define the Weyl scalar $\Psi_4 = \psi_4$. On the other hand, the dyad component of the DW field equation eq. (2.11) is given by

$$o_A \nabla^{AA'} \log\ \xi + o_A \iota^B \nabla^{AA'} o_A - \iota_A o^B \nabla^{AA'} o_B = 0. \tag{2.15}$$

Combining eq. (2.13), eq. (2.14) and eq. (2.15), the auxiliary scalar $S_{12}$ and the DW scalar $\xi$ will be identified by solving

$$\ell \cdot \nabla \log\ S_{12} - \rho^* = 0, \quad m \cdot \nabla \log\ S_{12} - \tau^* = 0. \tag{2.16}$$



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Since there is only one type of DW spinor $\xi_A = \xi o_A$, correspondingly, only one type of electromagnetic spinor can exist — the degenerate electromagnetic spinor

$$\Phi_{AB} = \frac{\xi_A \xi_B}{S_{12}} = \frac{\xi^2}{S_{12}} o_A o_B = \phi_2 o_A o_B. \tag{2.17}$$

Furthermore, the electromagnetic tensor $F_{ab} = F_{AA'BB'} = \Phi_{AB}\varepsilon_{A'B'} + \bar{\Phi}_{A'B'}\varepsilon_{AB}$, where $\varepsilon_{AB} = 2o_{[A}\iota_{B]}$, in the null tetrad we have

$$F_{ab} = 2\phi_2 \ell_{[a} m_{b]} + 2\bar{\phi}_2 \ell_{[a}\bar{m}_{b]}. \tag{2.18}$$

For vacuum type D solutions, many of spacetimes that we are familiar with belong to this class, such as Kerr (A)dS black holes, Reissner-Nordström (A)dS black holes, NUT solutions, $C$-metric, etc. In this case, the map eq. (2.12) reduces to

$$\Psi_{ABCD} = \frac{\xi_{(A}\xi_B \eta_C \eta_{D)}}{S_{14}}, \tag{2.19}$$

where we choose two DW spinors with the same coefficient, in other words,

$$\xi_A = \xi o_A, \qquad \eta_A = \xi \iota_A. \tag{2.20}$$

The dyad components of gravity field equation eq. (2.3) are then given by

$$o_A \nabla^{AA'} \log(\Psi_2) - 3\iota_A o^B \nabla^{AA'} o_B = 0, \tag{2.21}$$
$$\iota_A \nabla^{AA'} \log(\Psi_2) + 3o_A \iota^B \nabla^{AA'} \iota_B = 0, \tag{2.22}$$

where we let the Weyl scalar $\Psi_2 = 6\psi_2$. Two dyad components of the DW field equations read

$$o_A \nabla^{AA'} \log\ \xi - \iota_A o^B \nabla^{AA'} o_B + o_A \iota^B \nabla^{AA'} o_B = 0, \tag{2.23}$$
$$\iota_A \nabla^{AA'} \log\ \xi + o_A \iota^B \nabla^{AA'} \iota_B - \iota_A o^B \nabla^{AA'} \iota_B = 0. \tag{2.24}$$

By making use of the map eq. (2.19), the auxiliary scalar $S_{14}$ and the DW scalar will be identified by solving

$$\begin{aligned} &\ell \cdot \nabla \log S_{14} + 4\epsilon^* - \rho^* = 0, \qquad && m \cdot \nabla \log S_{14} + 4\beta^* - \tau^* = 0, \\ &\bar{m} \cdot \nabla \log S_{14} - 4\alpha^* + \pi^* = 0, \qquad && n \cdot \nabla \log S_{14} - 4\gamma^* + \mu^* = 0. \end{aligned} \tag{2.25}$$

Different from the type N case, since there are two different types of DW spinors, we thereby have two different types of electromagnetic spinors. Apart from the degenerate electromagnetic spinor we discussed above, the other type is a non-degenerate electromagnetic spinor,

$$\Phi^{(1)}_{AB} = \phi_1 o_{(A}\iota_{B)} = \frac{\xi^2 o_{(A}\iota_{B)}}{S^{(1)}_{12}}. \tag{2.26}$$

In order to distinguish two different types of electromagnetic spinors, we use a upper index (1) to refer to non-degenerate ones and (0), (2) to refer to degenerate ones $\Phi^{(0)}_{AB} = \phi_0 \iota_A \iota_B$ and

$\Phi^{(2)}_{AB} = \phi_2 o_A o_B$, repectively.[1] The dyad components of the non-degenerate electromagnetic field equation are given by

$$o_A \nabla^{AA'} \log \phi_1 - 2\iota_A o^B \nabla^{AA'} o_B = 0, \tag{2.27}$$
$$\iota_A \nabla^{AA'} \log \phi_1 + 2 o_A \iota^B \nabla^{AA'} \iota_B = 0. \tag{2.28}$$

Substitution of the map eq. (2.26) into the above equations and multiplying $\bar{o}_{A'}$ and $\bar{\iota}_{A'}$ respectively yield

$$\begin{aligned} \ell \cdot \nabla \log S^{(1)}_{12} + 2\epsilon^* &= 0, \qquad m \cdot \nabla \log S^{(1)}_{12} + 2\beta^* = 0, \\ \bar{m} \cdot \nabla \log S^{(1)}_{12} - 2\alpha^* &= 0, \qquad n \cdot \nabla \log S^{(1)}_{12} - 2\gamma^* = 0. \end{aligned} \tag{2.29}$$

Solving the above equations, we are able to obtain the auxiliary scalar field $S^{(1)}_{12}$. The electromagnetic scalar $\phi_1$ will then be determined from eq. (2.26). In analogy to eq. (2.18), the non-degenerate electromagnetic tensor in the null tetrad reads

$$F_{ab} = 2\phi_1 \left(\ell_{[a} n_{b]} + \bar{m}_{[a} m_{b]}\right) + 2\bar{\phi}_1 \left(\ell_{[a} n_{b]} + m_{[a} \bar{m}_{b]}\right). \tag{2.30}$$

Correspondingly, the auxiliary scalar field connecting the Weyl field and the non-degenerate electromagnetic field is denoted by $S^{(1,1)}_{24}$, which satisfies

$$\Psi_2 = \frac{(\phi_1)^2}{S^{(1,1)}_{24}}, \tag{2.31}$$

where superscript $(1,1)$ corresponds to the product of two electromagnetic scalars $\phi_1$.

In general, both for type N solutions and type D solutions, once DW spinors are identified, all electromagnetic fields (or other higher spin massless free-fields) in the curved spacetime principally can be formulated with the aid of an auxiliary scalar field. To verify the Weyl double copy relation, it is only necessary to locate a specific set of electromagnetic fields, which are independent of the source parameters or structure functions that determine how the spacetime deviates from the (A)dS background. If such electromagnetic fields do exist, they will also satisfy the field equation in (A)dS spacetime. These fields are nothing but the single copy, and the associated auxiliary scalar fields are the zeroth copy.

## 3 The Weyl double copy in curved spacetimes

Regarding DW spinors as basic units, electromagnetic spinors living in a certain curved spacetime are constructed according to the maps eq. (2.17) and eq. (2.26). Then, they are converted to tensor form and expanded in terms of the products of the null tetrad bases, such as eq. (2.18) and eq. (2.30). As we will see later, except for the electromagnetic scalars, the products of the null tetrad bases of the degenerate electromagnetic tensors in non-twisting vacuum type N spacetimes are independent of the structure functions and

[1] The complete expression of the degenerate electromagnetic spinors in eq. (2.17) should be $\Phi^{(2)}_{AB}$, since there is only one type of the field in the type N case, for simplicity, we omit the superscript (2) there.

source parameters. The same is true for non-degenerate electromagnetic tensors in vacuum type D spacetimes. For the sake of brevity, structure functions and source parameters will be referred to as deviation-information in the following. Once we have demonstrated that the electromagnetic scalars are independent of deviation-information, the same will be true of electromagnetic fields. Therefore, they should satisfy the field equations in conformally flat spacetime. Surprisingly, one will find that the associated scalar fields automatically satisfy their conformally invariant field equations in conformally flat spacetime. An explicit demonstration of the Weyl double copy for non-twisting vacuum type N and vacuum type D solutions is given in the following. The signature of the spacetime metric is chosen as $(+,-,-,-)$ in this work.

### 3.1 The case of non-twisting vacuum type N solutions

As the solutions of gravitational waves, non-twisting vacuum type N solutions $(\Lambda)$ are composed of two classes [37, 38], one is the non-expanding Kundt$(\Lambda)$ class, and the other is the expanding Robinson-Trautman$(\Lambda)$ class.

#### 3.1.1 The Kundt$(\boldsymbol{\Lambda})$ class

The metric in this case reads

$$\mathrm{d}s^2 = -F\mathrm{d}u^2 + 2\frac{q^2}{p^2}\mathrm{d}u\mathrm{d}v - 2\frac{1}{p^2}\mathrm{d}z\mathrm{d}\bar{z}, \tag{3.1}$$

with

$$\begin{aligned} p &= 1 + \frac{\Lambda}{6}z\bar{z}, & q &= \left(1 - \frac{\Lambda}{6}z\bar{z}\right)\alpha + \bar{\beta}z + \beta\bar{z}, \\ F &= \kappa\frac{q^2}{p^2}v^2 - \frac{(q^2)_{,u}}{p^2}v - \frac{q}{p}H, & \kappa &= \frac{\Lambda}{3}\alpha^2 + 2\beta\bar{\beta}, \\ H &= H(u,z,\bar{z}) = \left(f_{,z} + \bar{f}_{,\bar{z}}\right) - \frac{\Lambda}{3p}\left(\bar{z}f + z\bar{f}\right). \end{aligned} \tag{3.2}$$

where $f$ is an arbitrary complex function of $u$ and $z$, analytic in $z$. Further more, $\alpha$ and $\beta$ are two arbitrary real and complex functions of $u$, respectively. In fact, according to ref. [37], one can see that the parameter $\kappa$ is sign invariant. For the case $\Lambda = 0$, there are two classes of solutions — generalised pp-waves $(\kappa = 0)$ and generalised Kundt waves $(\kappa > 0)$. If our universe admits a positive cosmological constant, namely $\Lambda > 0$, there is no limit on $\alpha$ and $\beta$, and there is only one kind of solution — generalised Kundt waves. For the case $\Lambda < 0$, the values of parameters $\alpha$ and $\beta$ classify the metric into three types of solutions — generalised Kundt waves $(\kappa > 0)$, generalised Siklos waves $(\kappa = 0)$, and generalised pp-waves $(\kappa < 0)$. We will soon see that the zeroth copy inherits this property to classify the gravity solutions.

Choosing the null tetrad

$$\ell = \mathrm{d}u, \quad n = -\frac{F}{2}\mathrm{d}u + \frac{q^2}{p^2}\mathrm{d}v, \quad m = \frac{1}{p}\mathrm{d}\bar{z}, \tag{3.3}$$

we have

$$\rho^* = 0, \qquad \tau^* = -\frac{2\Lambda\bar{z}\alpha + \Lambda\bar{z}^2\beta - 6\bar{\beta}}{(6-\Lambda z\bar{z})\alpha + 6(z\bar{\beta}+\bar{z}\beta)}, \tag{3.4}$$

$$\Psi_4 = \frac{1}{72}\left(\Lambda z\bar{z}+6\right)\left[(\Lambda z\bar{z}-6)\alpha - 6(\bar{z}\beta + z\bar{\beta})\right]\partial^3_{\bar{z}}\bar{f}. \tag{3.5}$$

Recalling eq. (2.18), it is easy to check

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[a}\bar{m}_{b]} = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

where $I = \frac{6}{6+\Lambda z\bar{z}}$. Both matrices do not depend on the deviation-information, so the electromagnetic scalar will decide whether this kind of electromagnetic field is dependent of the deviation-information or not. From eq. (2.16), the auxiliary scalar $S_{12}$ is solved by

$$S_{12} = \mathcal{C}(u,\bar{z})\frac{\Lambda z\bar{z}+6}{(\Lambda z\bar{z}-6)\,\alpha - 6\left(z\bar{\beta}+\bar{z}\beta\right)}, \tag{3.7}$$

where $\mathcal{C}(u,\bar{z})$ is an arbitrary function of $u$ and $\bar{z}$. Clearly $S_{12}$ itself is independent of the deviation-information. According to eq. (2.13) and eq. (2.17), the DW scalar $\xi$ and the electromagnetic scalar $\phi_2$ are solved by

$$\xi^4 = \frac{(6+\Lambda z\bar{z})^4\mathcal{C}(u,\bar{z})^3\partial^3_{\bar{z}}\bar{f}}{72\left[(\Lambda z\bar{z}-6)\alpha - 6(\bar{z}\beta+z\bar{\beta})\right]^2}, \tag{3.8}$$

$$\phi_2 = \frac{(6+\Lambda z\bar{z})}{6\sqrt{2}}\sqrt{\mathcal{C}(u,\bar{z})\partial^3_{\bar{z}}\bar{f}}. \tag{3.9}$$

The structure function $\bar{f}$, which measures the value of $\Psi_4$, is absorbed by an arbitrary function $\mathcal{C}(u,\bar{z})$. The electromagnetic scalar thus does not depend on $\partial^3_{\bar{z}}\bar{f}(u,\bar{z})$. So we obtain a particular degenerate electromagnetic field which is independent of the deviation-information. It is easy to check that this type of electromagnetic field satisfies its field equation even for the case $\partial^3_{\bar{z}}\bar{f} = 0$. Namely,

$$\tilde{\nabla}_a F^{ab} = 0, \tag{3.10}$$

where the symbol tilde denotes that the background is (A)dS spacetimes — conformally flat spacetimes — where we just need to let $f = 1$ in the original metric. In fact, there is a freedom to choose a polynomial function $f = c_0(u) + c_1(u)\bar{z} + c_2(u)\bar{z}^2$ as long as $\partial^3_{\bar{z}}\bar{f}(u,\bar{z}) = 0$, where $c_i(u)$ are expanding parameters of $\bar{z}$. Furthermore, with the fact that the Ricci scalar $R = -4\Lambda$, it is easy to verify that the auxiliary scalar field $S_{24}(= S_{12})$ satisfies the conformally invariant scalar field equation not only in the curved spacetime but also in conformally flat spacetime. So we have

$$\tilde{\nabla}^a\tilde{\nabla}_a S_{24} - \frac{1}{6}\tilde{R}S_{24} = 0. \tag{3.11}$$

When $\Lambda \to 0$, the result reduces to the Kundt ($\Lambda = 0$) class, the single copy and the zeroth copy satisfy their field equations in Minkowski spacetime.

More interestingly, one can find that the single copy only confines the structure function. For example, it does not depend on the parameters $\alpha$, $\beta$, and $\Lambda$; the function $f$ in Maxwell scalar only needs to be a function of coordinates $u$ and $z$, and there are no other restrictions. In contrast, the zeroth copy is closely associated with $\alpha$, $\beta$, and $\Lambda$. With a negative cosmological constant and in conjunction with the introduction in the first paragraph of this section, one can see that for different $\kappa$, it is the zeroth copy that specifies the sort of curved spacetimes they map.

### 3.1.2 The Robinson-Trautman($\Lambda$) class

One of the familiar form of the metric for Robinson-Trautman ($\Lambda$) solutions is given by García Díaz and Plebański [37, 39]

$$\begin{aligned} &\mathrm{d}s^2 = -2(A\bar{A} + \psi B)\mathrm{d}u^2 - 2\psi \mathrm{d}u\mathrm{d}v - 2v\bar{A}\mathrm{d}u\mathrm{d}z - 2vA\mathrm{d}u\mathrm{d}\bar{z} - 2v^2\mathrm{d}z\mathrm{d}\bar{z}, \\ &A = \epsilon z - vf, \quad B = -\epsilon + \frac{v}{2}(f_{,z} + \bar{f}_{,\bar{z}}) + \frac{\Lambda}{6}v^2\psi, \quad \psi = 1 + \epsilon z\bar{z}, \end{aligned} \tag{3.12}$$

where $\epsilon = +1, 0, -1$ corresponds to the source of the transverse gravitational waves being time-like, null, or space-like, respectively, at least in the weak field limit. This is consistent with the case that $\Lambda = 0$ [40]. One can also refer to ref. [41] for more details on the interpretation of the Robinson-Trautman solutions. In addition, this metric only depends linearly on an arbitrary structure function $f(u, z)$, which will help facilitate the following discussions.

Choosing the null tetrad

$$\ell = du, \quad n = -(A\bar{A} + \psi B)du - \psi dv - \bar{A}vdz - vAd\bar{z}, \quad m = vd\bar{z}, \tag{3.13}$$

we have

$$\rho^* = \frac{1}{v(1 + \epsilon z\bar{z})}, \quad \tau^* = \frac{\bar{f}}{1 + \epsilon z\bar{z}}, \tag{3.14}$$

$$\Psi_4 = \frac{(1 + \epsilon z\bar{z})\partial_{\bar{z}}^3 \bar{f}}{2v}. \tag{3.15}$$

In this case, the Weyl scalar does not even depend on the cosmological constant. Recalling eq. (2.18), one observes

$$2\ell_{[a}m_{b]} = \begin{pmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad 2\ell_{[a}\bar{m}_{b]} = \begin{pmatrix} 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 \\ -v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

both terms are independent of the structure function $f(u, \xi)$. Solving eq. (2.16), the auxiliary scalar field $S_{12}$ is given by

$$S_{12} = \frac{C(u, \bar{z})}{v(1 + \epsilon z\bar{z})}, \tag{3.17}$$



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where the function $C(u,\bar{z})$ is arbitrary. Following eq. (2.13) and eq. (2.17), the DW scalar and the electromagnetic scalar are solved by

$$\xi^4 = \frac{C(u,\bar{z})^3 \partial_{\bar{z}}^3 \bar{f}}{2v^4(1+\epsilon z\bar{z})^2}, \tag{3.18}$$

$$\phi_2 = \sqrt{\frac{C(u,\bar{z})\partial_{\bar{z}}^3 \bar{f}}{2}}\frac{1}{v}. \tag{3.19}$$

Clearly, the function $C(u,\bar{z})$ lets $\phi_2$ be independent of the structure function $f$. Thus, the electromagnetic field also satisfies the field equation in conformally flat spacetime. We can further check that the auxiliary scalar filed $S_{24}(=S_{12})$ satisfies eq. (3.11) both in the curved spacetime and in conformally flat spacetime.

It is worth noting that the single copy does not depend on the parameter $\epsilon$. It is the zero copy that decides what kind of sources of gravitational waves they are mapping, at least in the weak field limit. For example, given the same electromagnetic field in conformally flat spacetime, following the map eq. (1.1) the scalar field $S_{12}$ with $\epsilon = 1$ will lead to a class of transverse gravitational waves whose source is time-like. On the other hand, a scalar field $S_{12}$ with $\epsilon = 0$ will lead to another class of transverse gravitational waves whose source is null.

So far, we only consider the time-dependent vacuum solutions. Next, we will investigate time-independent vacuum solutions by focusing on type D spacetimes. More interpretations about the single copy and the zeroth copy will be discussed later.

## 3.2 The case of vacuum type D solutions

### 3.2.1 Kerr-(A)dS black holes

As we know, rotating black holes are believed to be the most typical astrophysical black holes in the universe. It is necessary to take the case of Kerr-(A)dS black holes as a specific example to study the double copy relation before going to the most general vacuum type D solutions.

The metric of Kerr-(A)dS black holes in the Boyer-Lindquist coordinates reads [42–44]

$$\begin{aligned} \mathrm{d}s^2 = \frac{\mathcal{R}}{\rho^2}\left(\mathrm{d}t - \frac{a}{\Sigma}\sin^2\theta \mathrm{d}\phi\right)^2 - \frac{\rho^2}{\mathcal{R}}\mathrm{d}r^2 - \frac{\rho^2}{\Theta}\mathrm{d}\theta^2 \\ - \frac{\Theta}{\rho^2}\sin^2\theta\left(a\mathrm{d}t - \frac{r^2+a^2}{\Sigma}\mathrm{d}\phi\right)^2, \end{aligned} \tag{3.20}$$

where

$$\mathcal{R} = (r^2+a^2)\left(1+\frac{r^2}{l^2}\right) - 2Mr, \qquad \Theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \tag{3.21}$$

$$\Sigma = 1 - \frac{a^2}{l^2}, \quad \rho^2 = r^2 + a^2\cos^2\theta, \quad l^2 = -\frac{3}{\Lambda}, \tag{3.22}$$

with mass $M/\Sigma^2$ and angular momentum $J = aM/\Sigma^2$. Clearly, $M$ and $a$ can be regarded as mass parameter and angular momentum parameter, respectively.



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Since the metric has already been written in the orthogonal tetrad $\{e^i\}$ $(i = 1, 2, 3, 4)$ such that $ds^2 = (e^1)^2 - (e^2)^2 - (e^3)^2 - (e^4)^2$, the null tetrad $\{e'^i\}$ then is easily given under the transformation

$$\begin{aligned} e'^1 =& \frac{1}{\sqrt{2}}(e^1 + e^2), \qquad & e'^2 = \frac{1}{\sqrt{2}}(e^1 - e^2), \\ e'^3 =& \frac{1}{\sqrt{2}}(e^3 + ie^4), \qquad & e'^4 = \frac{1}{\sqrt{2}}(e^3 - ie^4). \end{aligned} \tag{3.23}$$

Thus we have

$$e'^1 = \ell = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mathcal{R}}{\rho^2}}\mathrm{d}t + \sqrt{\frac{\rho^2}{\mathcal{R}}}\mathrm{d}r - \sqrt{\frac{\mathcal{R}}{\rho^2}}\frac{a}{\Sigma}\mathrm{d}\phi\right), \tag{3.24}$$

$$e'^2 = n = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\mathcal{R}}{\rho^2}}\mathrm{d}t + \sqrt{\frac{\rho^2}{\mathcal{R}}}\mathrm{d}r - \sqrt{\frac{\mathcal{R}}{\rho^2}}\frac{a}{\Sigma}\mathrm{d}\phi\right), \tag{3.25}$$

$$e'^3 = m = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\Theta}{\rho^2}}a\sin\theta\mathrm{d}t + i\sqrt{\frac{\rho^2}{\Theta}}\mathrm{d}\theta - \sqrt{\frac{\Theta}{\rho^2}}\frac{(r^2+a^2)}{\Sigma}\sin\theta\mathrm{d}\phi\right), \tag{3.26}$$

$$e'^4 = \bar{m} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\Theta}{\rho^2}}a\sin\theta\mathrm{d}t - i\sqrt{\frac{\rho^2}{\Theta}}\mathrm{d}\theta - \sqrt{\frac{\Theta}{\rho^2}}\frac{(r^2+a^2)}{\Sigma}\sin\theta\mathrm{d}\phi\right). \tag{3.27}$$

We obtain the Weyl scalar

$$\Psi_2 = 6\psi_2 = \frac{6M}{(r + ia\cos\theta)^3}, \tag{3.28}$$

and the spin coefficients

$$\begin{aligned} \rho^* &= \mu^* = -\frac{i}{\sqrt{2}(a\cos\theta - ir)}\sqrt{\frac{\mathcal{R}}{\rho^2}}, \\ \tau^* &= \pi^* = -\frac{ia\sin\theta}{\sqrt{2}(a\cos\theta - ir)}\sqrt{\frac{\Theta}{\rho^2}}, \\ \epsilon^* &= \gamma^* = -\frac{a\cos\theta[l^2(r-M) + r(a^2+2r^2)] + i(a^2l^2 - r^4 - l^2Mr)}{2\sqrt{2}l^2(a\cos\theta - ir)}\frac{1}{\sqrt{\rho^2\mathcal{R}}}, \\ \alpha^* &= \beta^* = \frac{r\cos\theta(a^2\cos 2\theta - l^2) + ia(a^2\cos^4\theta - l^2)}{2\sqrt{2}l^2\sin\theta(a\cos\theta - ir)}\frac{1}{\sqrt{\rho^2\Theta}}. \end{aligned} \tag{3.29}$$

According to eq. (2.19), to identify the DW scalar we need to solve the auxiliary scalar field $S_{14}$. Using eq. (2.25) and the identity

$$\arctan(z) = -\frac{i}{2}\log\left(\frac{i-z}{i+z}\right) \qquad z \in \mathbb{C}, \tag{3.30}$$

it is not hard to obtain that

$$S_{14} = \mathcal{K}_1\frac{\csc^2\theta(r + ia\cos\theta)}{l^4\mathcal{R}\Theta}, \tag{3.31}$$

where all of the constant coefficients have been absorbed by a constant of integration $\mathcal{K}_1$. The DW scalar can then be solved by

$$\xi^2 = \sqrt{\Psi_2 S_{14}} = \frac{\sqrt{6\mathcal{K}_1 M}\csc\theta}{(r + ia\cos\theta)l^2\sqrt{\mathcal{R}\Theta}}. \tag{3.32}$$

Note this is also the coefficient of DW tensor in the null tetrad [31]. We are not going to talk about DW tensor in more detail in this paper, the only reason we show this is to construct an electromagnetic scalar. With the help of the auxiliary scalar $S_{12}^{(1)}$, which is solved from eq. (2.29)

$$S_{12}^{(1)} = \mathcal{K}_2 \frac{\csc\theta(r + ia\cos\theta)}{l^2\sqrt{\mathcal{R}\Theta}}, \tag{3.33}$$

we obtain

$$\phi_1 = \frac{\xi^2}{S_{12}^{(1)}} = \frac{\sqrt{6\mathcal{K}_1 M}}{\mathcal{K}_2}\frac{1}{(r + ia\cos\theta)^2} \sim \frac{1}{(r + ia\cos\theta)^2}, \tag{3.34}$$

where $\mathcal{K}_2$, similar to $\mathcal{K}_1$, is an arbitrary constant of integration. One can observe that the mass parameter $M$ is absorbed by a constant of integration. So $\phi_1$ actually does not depend on the source. In addition, it is easy to verify that $(l_{[a}n_{b]} + \bar{m}_{[a}m_{b]})$ and $(l_{[a}n_{b]} + m_{[a}\bar{m}_{b]})$ are all independent of the mass parameter $M$. Hence, from eq. (2.30), we conclude that the non-degenerate electromagnetic field we construct is independent of the source. It even satisfies the conformally invariant field equation in conformally flat spacetime, where we just need to let $M = 0$ in the metric. What about the auxiliary scalar field $S_{24}^{(1,1)}$ associated to this electromagnetic field? Using the formula eq. (1.1), one observes that

$$S_{24}^{(1,1)} = \frac{(\phi_1)^2}{\Psi_2} = \frac{\mathcal{K}_1}{(\mathcal{K}_2)^2}\frac{1}{(r + ia\cos\theta)} \sim \frac{1}{r + ia\cos\theta}. \tag{3.35}$$

As we expect, this satisfies the conformally invariant scalar field equation

$$\tilde{\nabla}^a\tilde{\nabla}_a S_{24}^{(1,1)} - \frac{1}{6}\tilde{R}S_{24}^{(1,1)} = 0. \tag{3.36}$$

When $\Lambda \to 0$, it describes the wave equation on Minkowski background. Thus, we have shown that the single copy and the zeroth copy of Kerr-AdS spacetimes satisfy their conformally invariant field equations in conformally flat spacetime.

Moreover, in analogy to eq. (2.16) of the type N case, the auxiliary scalars associated with the degenerate electromagnetic fields are given by

$$S_{12}^{(2)} = S_{12}^{(0)} \sim \frac{1}{r + ia\cos\theta}. \tag{3.37}$$

Combining with eq. (3.35), one can see that $S_{12}^{(2)}$ and $S_{12}^{(0)}$ are equivalent to $S_{24}^{(1,1)}$ up to a constant. Therefore, the zeroth copy connects not only gravity fields with the single copy living in the conformally flat spacetime, but also DW fields with those degenerate electromagnetic fields residing in the curved spacetime. This property is consistent with the discovery of the preceding work [31] in the absence of the cosmological constant $\Lambda$.

#### 3.2.2 The most general vacuum type D solutions

Now, we shall investigate the Weyl double copy relation for the most general vacuum type D solutions with a cosmological constant. The metric has been given by Plebanski and



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Demianski[2] [46],

$$ds^2 = \frac{1}{(p+q)^2}\left(-\frac{1+(pq)^2}{\mathcal{P}}\mathrm{d}p^2 - \frac{\mathcal{P}}{1+(pq)^2}(\mathrm{d}\sigma + q^2\mathrm{d}\tau)^2\right. \tag{3.38}$$

$$\left. - \frac{1+(pq)^2}{\mathcal{L}}\mathrm{d}q^2 + \frac{\mathcal{L}}{1+(pq)^2}(-p^2\mathrm{d}\sigma + \mathrm{d}\tau)^2\right), \tag{3.39}$$

where the structure functions read

$$\mathcal{P} = \left(-\frac{\Lambda}{6} + \gamma\right) + 2np - \epsilon p^2 + 2mp^3 + \left(-\frac{\Lambda}{6} - \gamma\right)p^4, \tag{3.40}$$

$$\mathcal{L} = \left(-\frac{\Lambda}{6} - \gamma\right) + 2nq + \epsilon q^2 + 2mq^3 + \left(-\frac{\Lambda}{6} + \gamma\right)q^4, \tag{3.41}$$

$m$ and $n$ are dynamical parameters measuring the curvature, $\epsilon$ and $\gamma$ are called kinematical parameters which will affect the properties of the solutions.

By choosing null tetrad

$$\ell = \frac{1}{\sqrt{2}(p+q)}\left(\sqrt{\frac{1+(pq)^2}{\mathcal{L}}}dq - p^2\sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\sigma + \sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\tau\right), \tag{3.42}$$

$$n = \frac{1}{\sqrt{2}(p+q)}\left(-\sqrt{\frac{1+(pq)^2}{\mathcal{L}}}dq - p^2\sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\sigma + \sqrt{\frac{\mathcal{L}}{1+(pq)^2}}d\tau\right), \tag{3.43}$$

$$m = \frac{1}{\sqrt{2}(p+q)}\left(\sqrt{\frac{1+(pq)^2}{\mathcal{P}}}dp + i\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\sigma + iq^2\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\tau\right), \tag{3.44}$$

$$\bar{m} = \frac{1}{\sqrt{2}(p+q)}\left(\sqrt{\frac{1+(pq)^2}{\mathcal{P}}}dp - i\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\sigma - iq^2\sqrt{\frac{\mathcal{P}}{1+(pq)^2}}d\tau\right), \tag{3.45}$$

we obtain

$$\Psi_2 = 6\psi_2 = 6(m+in)\left(\frac{p+q}{1-ipq}\right)^3. \tag{3.46}$$

Obviously, the cosmological constant does not affect the Weyl scalar according to the above result. Some spin coefficients are given by

$$\begin{aligned}
\rho^* &= \mu^* = \frac{(p^2-i)\sqrt{\mathcal{L}(q)}}{\sqrt{2}(pq+i)\sqrt{1+p^2q^2}},\\
\tau^* &= -\pi^* = \frac{(q^2-i)\sqrt{\mathcal{P}(p)}}{\sqrt{2}(pq+i)\sqrt{1+p^2q^2}},\\
\epsilon^* &= \gamma^* = \frac{2(p^2+2pq+i)\mathcal{L}(q) - (p+q)(pq+i)\mathcal{L}'(q)}{4\sqrt{2}(pq+i)\sqrt{1+p^2q^2}\sqrt{\mathcal{L}(q)}},\\
\beta^* &= -\alpha^* = \frac{2(q^2+2pq+i)\mathcal{P}(p) - (p+q)(pq+i)\mathcal{P}'(q)}{4\sqrt{2}(pq+i)\sqrt{(1+p^2q^2)\mathcal{P}(p)}}.
\end{aligned} \tag{3.47}$$

[2] By doing a coordinate transformation $q \to -1/q$ and some rescalings following in eq. (3) of ref. [45], we will go back to the modified form of the metric applied in the preceding work [31].

Solving eq. (2.25) with the help of eq. (3.30), the auxiliary scalar field $S_{14}$ is given by

$$S_{14} = \mathcal{D}_1 \frac{(p+q)^3(1-ipq)}{\mathcal{P}(p)\mathcal{L}(q)}, \tag{3.48}$$

where $\mathcal{D}_1$ is a constant of integration. Then from eq. (2.19) we have

$$\xi^2 = \frac{\sqrt{6\mathcal{D}_1(m+in)}}{\sqrt{\mathcal{P}(p)\mathcal{L}(q)}} \frac{(p+q)^3}{1-ipq}. \tag{3.49}$$

Recalling eq. (2.30), one observes that

$$2\left(\ell_{[a}n_{b]} + \bar{m}_{[a}m_{b]}\right) = \begin{pmatrix} 0 & 0 & iA & ip^2A \\ 0 & 0 & q^2A & -A \\ -iA & q^2A & 0 & 0 \\ -ip^2A & A & 0 & 0 \end{pmatrix}, \tag{3.50}$$

and

$$2\left(\ell_{[a}n_{b]} + \bar{m}_{[a}m_{b]}\right) = \begin{pmatrix} 0 & 0 & -iA & -ip^2A \\ 0 & 0 & q^2A & -A \\ iA & q^2A & 0 & 0 \\ ip^2A & A & 0 & 0 \end{pmatrix}, \tag{3.51}$$

where $A = \frac{(p+q)^2}{1+p^2q^2}$. Clearly, they are independent of the dynamical parameters. Consequently, if the electromagnetic scalar is independent of the deviation-information, the electromagnetic field we construct will be independent of the deviation-information as well and the field equation will hold even in conformally flat spacetime. From eq. (2.29), one obtains

$$S_{12}^{(1)} = \mathcal{D}_2 \frac{(p+q)(1-ipq)}{\sqrt{\mathcal{P}(p)\mathcal{L}(q)}}, \tag{3.52}$$

where $\mathcal{D}_2$ is a constant of integration. Following eq. (2.26), the non-degenerate electromagnetic scalar $\phi_1$ is given by

$$\phi_1 = \sqrt{\frac{6\mathcal{D}_1(m+in)}{\mathcal{D}_2}} \frac{(p+q)^2}{(1-ipq)^2} \sim \frac{(p+q)^2}{(1-ipq)^2}. \tag{3.53}$$

One can see that the dynamical parameters which measure the curvature are absorbed by the constants of integration, so $\phi_1$ is independent of the dynamical parameters. Thus, we have discovered a particular non-degenerate electromagnetic field which is independent of the deviation-information. Correspondingly, the auxiliary scalar field $S_{24}^{(1,1)}$ is given by

$$S_{24}^{(1,1)} = \frac{(\phi_1)^2}{\Psi_2} = \frac{\mathcal{D}_1}{(\mathcal{D}_2)^2} \frac{p+q}{1-ipq} \sim \frac{p+q}{1-ipq}. \tag{3.54}$$

It is easy to check that $S_{24}^{(1,1)}$ satisfies the conformal invariant field equation eq. (3.11) in conformally flat spacetime, where we just need to set $m = n = 0$.

Therefore, for vacuum type D solutions with a cosmological constant, the single copy and the zeroth copies satisfy their conformal invariant field equations in conformally flat



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spacetime. When $\Lambda \to 0$, the background goes back to Minkowski spacetime and the situation is consistent with the previous result [20].

In addition, similar to the case of Kerr-AdS spacetime, for the general vacuum type D solutions with or without a cosmological constant, we find that

$$S_{12}^{(0)} = S_{12}^{(2)} \sim \frac{p+q}{1-ipq} \sim S_{24}^{(1,1)}. \tag{3.55}$$

Thus, not only does the zeroth copy connect gravity fields to the single copy, but it also links DW fields to degenerate electromagnetic fields living in curved spacetime. Recalling the previous section, it is evident that this property also applies to non-twisting type N solutions. While, distinct from the type N cases, the zeroth copy now does not possess any extra information about the source. This is mirrored clearly by the double copy scalar relation $(\Psi_2)^{1/3} = (\phi_2)^{1/2} = S_{24}^{(1,1)}$. Therefore, we find that only for the time-dependent solutions, the zeroth copy carries extra information about the source. This provides support for constructing other exact time-dependent radiation solutions in future work.

## 4 Discussion and conclusions

In this paper, using DW spinors (massless spin-1/2 spinors) as basic units, we constructed a particular set of electromagnetic fields in 4-dimensional non-twisting vacuum type N and vacuum type D spacetimes in the presence of a cosmological constant $\Lambda$. These electromagnetic fields are independent of the deviation-information for a given curved metric. Thus they also satisfy the field equation in conformally flat spacetime. Regarding these electromagnetic fields as the single copies in the curved (A)dS spacetime, we verified the Weyl double copy prescription. We found that the single and zeroth copies satisfy the conformally invariant field equations in conformally flat spacetime both for the time-dependent solutions (type N cases) and time-independent solutions (type D cases). When $\Lambda \to 0$, the result reduces to the original case. Namely, they satisfy the field equations in Minkowski spacetime. This is an intriguing outcome. For Kerr-Schild ($\Lambda$) double copy prescription [15], the single and zeroth copies satisfy different equations for time-independent solutions and time-dependent solutions. Specifically, in time-independent cases, the zeroth and single copies satisfy the conformally invariant field equations in conformally flat spacetime; whereas, in time-dependent cases, the zeroth copy satisfies the wave equation and does not admit good conformal transformation properties anymore. Moreover, the single copy does not satisfy the conformally invariant Maxwell's field equation because of an extra term proportional to the Ricci scalar appearing in the equation of motion. Therefore, from this point of view, the Weyl double copy prescription appears as a more fundamental map between gravity theory and gauge theory. This is also consistent with the fact that the Kerr-Schild double copy prescription is linear, whereas the Weyl double copy prescription is essentially more general.

Apart from the above results, we found that the preceding finding [31] also holds in the presence of $\Lambda$. Not only does the zeroth copy connect gravity fields with the single copy in the conformally flat spacetime, but it also connects DW fields with degenerate



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electromagnetic fields in the curved spacetime, both for non-twisting vacuum type N solutions and vacuum type D solutions. More interestingly, we found that the zeroth copy plays a more important role than expected for time-dependent radiation solutions (type N cases). Unlike the single copy, which only restricts the form of the structure function, the zeroth copy carries additional information characterizing the curved spacetimes it is mapping. Specifially for the Robinson-Trautman ($\Lambda$) class, we discovered that it is the zeroth copy that determines whether the sources of associated gravitational waves are time-like, null, or space-like, at least in the weak field limit. This result is reminiscent of previous research on the fluid/gravity duality [21], which showed that all of the information about the fluid is encoded in the zeroth copy for the type N case. Their work further supports our result that the zeroth copy can indeed carry additional information compared with the single copy. We hope this discovery will contribute to constructing other exact time-dependent radiation solutions.

All in all, we have shown explicitly that the single copy and the zeroth copy satisfy conformally invariant equations in conformally flat spacetime by concentrating on non-twisting vacuum type N and vacuum type D solutions. Several novel interpretations of the Weyl double copy prescription are provided, particularly with regard to the zeroth copy. Next, it would be intriguing to check whether the generalized Weyl double copy holds asymptotically for the algebraically general case with a cosmological constant. It is also significant to investigate the applications of the Weyl double copy on astrophysical observations, such as the specific correspondence between the source of gravitational waves and the Weyl double copy. In addition, a natural progression of this work is to analyse the classical Weyl double copy in the flat spacetime instead of the (A)dS background. That would be essential for establishing a bridge between gravity theory and gauge theory. The cosmological constant, in this case, should be considered as the source of the single and zeroth copies. Further studies which treat the Minkovski spacetime as the background of the Weyl double copy prescription in the presence of a cosmological constant will need to be undertaken in the future. Since all of the discussion in this paper is limited to 4-dimensional spacetimes, it would also be worthwhile to extend the study to high-dimensional spacetimes.

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