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The geometry of decoupling fields

Ibrahima Bah,[a,b] Federico Bonetti,[c] Enoch Leung[a] and Peter Weck[a]

[a] *Department of Physics and Astronomy, Johns Hopkins University, 3400 North Charles Street, Baltimore, MD 21218, U.S.A.*
[b] *Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ 08540, U.S.A.*
[c] *Mathematical Institute, University of Oxford, Woodstock Road, Oxford, OX2 6GG, U.K.*

E-mail: iboubah@jhu.edu, federico.bonetti@maths.ox.ac.uk, yleung5@jhu.edu, pweck1@jhu.edu

Abstract: We consider 4d field theories obtained by reducing the 6d (1,0) SCFT of N M5-branes probing a $\mathbb{C}^2/\mathbb{Z}_k$ singularity on a Riemann surface with fluxes. We follow two different routes. On the one hand, we consider the integration of the anomaly polynomial of the parent 6d SCFT on the Riemann surface. On the other hand, we perform an anomaly inflow analysis directly from eleven dimensions, from a setup with M5-branes probing a resolved $\mathbb{C}^2/\mathbb{Z}_k$ singularity fibered over the Riemann surface. By comparing the 4d anomaly polynomials, we provide a characterization of a class of modes that decouple along the RG flow from six to four dimensions, for generic N, k, and genus. These modes are identified with the flip fields encountered in the Lagrangian descriptions of these 4d models, when they are available. We show that such fields couple to operators originating from M2-branes wrapping the resolution cycles. This provides a geometric origin of flip fields. They interpolate between the 6d theory in the UV, where the M2-brane operators are projected out, and the 4d theory in the IR, where these M2-brane operators are part of the spectrum.

Keywords: Anomalies in Field and String Theories, Field Theories in Higher Dimensions, Global Symmetries, M-Theory

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1 Introduction and summary

The reduction of higher-dimensional superconformal field theories (SCFTs) to lower dimensions has proven to be a powerful framework to construct non-trivial field theories, study their properties, and, more broadly, organize the space of quantum field theories (QFTs) using topological and geometric tools. One of the most prominent realizations of this paradigm is provided by class $\mathcal{S}$ constructions [1, 2] and their generalizations [3–13]. The central idea is to start with a 6d SCFT and reduce it on a Riemann surface, triggering a renormalization group (RG) flow that can yield a non-trivial 4d SCFT in the IR.

In most instances, the parent 6d SCFT admits a realization in string/M/F-theory. Indeed, an atomic classification of 6d (1,0) SCFTs has been proposed in F-theory [14]. When an explicit string theoretic construction of the parent SCFT is available, we have two ways of thinking about the resulting 4d SCFT: it is the IR fixed point of a purely field-theoretic RG flow from six dimensions, and it is also the QFT capturing the low-energy dynamics of a string theory setup with four non-compact dimensions of spacetime. As a simple example, we may consider the 6d (2,0) SCFT of type A_{N-1}, which can be realized by a stack of N M5-branes. This 6d SCFT can be reduced on a smooth Riemann surface preserving 4d $\mathcal{N}=2$ or $\mathcal{N}=1$ supersymmetry, depending on the choice of topological twist [1, 7]. In M-theory, the M5-brane stack is wrapped on the Riemann surface, and the choice of topological twist is mapped to the topology of the normal bundle to the M5-branes.

In this work, we explore a generalization of this circle of ideas to 4d $\mathcal{N}=1$ SCFTs of class $\mathcal{S}_k$ [8]. In the class $\mathcal{S}_k$ program, the parent 6d SCFT is the 6d (1,0) theory realized by a stack of N M5-branes probing a $\mathbb{C}^2/\mathbb{Z}_k$ singularity. The global symmetries of this SCFT for generic N, k include the $\mathrm{SU}(2)_R$ R-symmetry and a $\mathrm{U}(1)\times\mathrm{SU}(k)\times\mathrm{SU}(k)$ flavor symmetry.[1] This 6d (1,0) SCFT is reduced on a Riemann surface with a topological twist that preserves 4d $\mathcal{N}=1$ supersymmetry. The data that specify the construction include the topology of the Riemann surface, possible defects (punctures) that can decorate the Riemann surface, and a choice of fluxes for global symmetries of the 6d (1,0) SCFT.

For simplicity, in this paper we restrict our attention to the case of a smooth Riemann surface without punctures. Even in this simpler class of setups, the RG flow from six to four dimensions can exhibit subtle features, such as a non-trivial pattern of modes that decouple along the flow, as demonstrated in [12] for reductions on tori with flux.

In the construction of Lagrangian models for 4d SCFTs originating from reductions of a 6d SCFT, it is not unusual to encounter flip fields, i.e. gauge singlets ϕ that participate in a superpotential coupling $W_{\text{flip}}=\phi\,\mathcal{O}$, where $\mathcal{O}$ is a gauge invariant operator (for example, a baryonic operator constructed from a bifundamental field in a quiver gauge theory). For a gauge theory of sufficiently large rank, the superpotential coupling $W_{\text{flip}}=\phi\,\mathcal{O}$ is irrelevant: in the deep IR, the flip field ϕ is expected to behave as a free field, and therefore decouple from the interacting SCFT. Flip fields are ubiquitous in the literature on class $\mathcal{S}$ and its generalizations [8, 11, 13, 15–24], and appear in particular in many models studied in [12]. One of the aims of this work is to revisit this class of models, with the aim of shedding light on the origin of flip fields from a geometric M-theory perspective.

The main goal of this paper is to contrast the field-theoretic point of view on $\mathcal{S}_k$ constructions with a point of view based on a direct construction from M-theory, as illustrated in figure 1. Our objective is two-fold:

(i) Identify the M-theory setups that correspond to class $\mathcal{S}_k$ reductions on a smooth Riemann surface with non-zero fluxes for the global $\mathrm{SU}(k)^2$ flavor symmetry of the parent 6d SCFT.

(ii) Exploit these M-theory setups to gain insights on the field theory flow from 6d to 4d.

Let us now proceed to summarize the main results of this paper.

[1] For $k=2$, the U(1) factor enhances to SU(2). For $N=2$ the flavor symmetry enhances to SU(2k). For $k=2$, $N=2$, it enhances to SO(7). For the remainder of this work, we focus on the $\mathrm{U}(1)\times\mathrm{SU}(k)\times\mathrm{SU}(k)$ symmetry of the general case. Moreover, we are cavalier about the global form of the symmetry group, since it does not play a role in our discussion.

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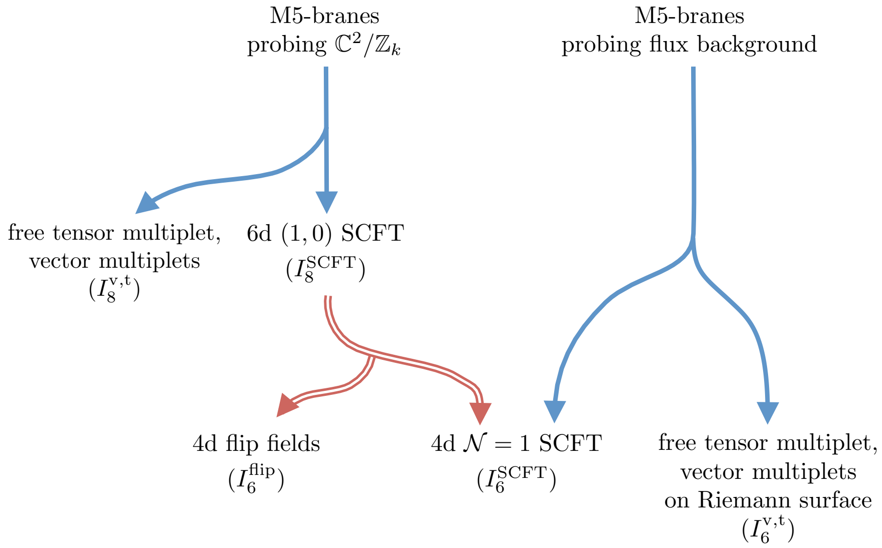

Figure 1. Schematic depiction of the flows across dimensions considered in this work. On the left: a stack of M5-branes with flat 6d worldvolume probes a $\mathbb{C}^2/\mathbb{Z}_k$ singularity, yielding a 6d (1,0) SCFT, plus 6d modes associated to free tensor, vector multiplets. The interacting 6d (1,0) SCFT is reduced on a Riemann surface with fluxes. The outcome is organized into an interacting 4d SCFT, and a collection of 4d free fields, interpreted as flip fields. On the right: a stack of M5-branes probes a resolved $\mathbb{C}^2/\mathbb{Z}_k$ singularity and is wrapped on a Riemann surface. At low energies, this M-theory setup gives the same 4d SCFT, plus other 4d modes coming from free tensor, vector multiplets on the Riemann surface. The blue, solid arrows denote anomaly inflow from the 11d bulk onto the M5-branes. The red, hollow arrows denote the purely field-theoretical reduction of the 6d (1,0) SCFT on the Riemann surface.

Inflow for wrapped M5-branes probing flux backgrounds. As far as objective (i) is concerned, a natural proposal is as follows: in M-theory, we should consider a stack of M5-branes that probes a *resolved* $\mathbb{C}^2/\mathbb{Z}_k$ singularity, further wrapped on a Riemann surface. The 11d background probed by the M5-branes is expected to be a flux background: a non-zero G_4-flux threads non-trivial 4-cycles, obtained combining the Riemann surface with the 2-cycles originating from the resolution of the $\mathbb{C}^2/\mathbb{Z}_k$ singularity.

The above discussion can be made more precise for $k = 2$. Indeed, in this case we can establish a connection to a class of AdS_5 solutions in 11d supergravity, first discussed by Gauntlett, Martelli, Sparks, and Waldram (GMSW) [25]. These solutions take the form of a warped product $\mathrm{AdS}_5 \times_w M_6$, in which the internal space M_6 is a fibration of a 4-manifold M_4 over a Riemann surface Σ_g. The space M_4 can be regarded as a resolution of the orbifold $S^4/\mathbb{Z}_2$: the north and south pole of the S^4 are fixed points of the $\mathbb{Z}_2$ action, yielding locally $\mathbb{R}^4/\mathbb{Z}_2$ singularities; the singularity at each pole is resolved introducing a 2-cycle. The internal G_4-flux configuration is specified by three positive integer flux quanta, which we denote by N, N_{N_1}, N_{S_1}. The flux quantum N measures G_4-flux on the 4-cycle given by the fiber M_4 at a generic point of the base Σ_g. The integer N_{N_1} quantifies instead the G_4-flux on the 4-cycle obtained by combining the Riemann surface with the resolution 2-cycle at the north pole of S^4. Similar remarks apply to N_{S_1}. We propose the following interpretation of this class of solutions: they describe the near-horizon geometry of a stack of N wrapped M5-branes probing a resolved $\mathbb{C}^2/\mathbb{Z}_2$ singularity [26].

As discussed in [27], the topology and G_4-flux configuration of the internal space M_6 for $k = 2$ admit a natural generalization for higher k. In this case, M_6 is still taken to be a fibration of a 4-manifold M_4 over a Riemann surface Σ, but M_4 is identified with the resolution of the orbifold $S^4/\mathbb{Z}_k$. The fixed points of the orbifold action are locally $\mathbb{R}^4/\mathbb{Z}_k$ and can be resolved by blow-up, introducing a collection of $k-1$ resolution 2-cycles near each pole of the S^4. The G_4-flux configuration is described by a total of $2k-1$ flux quanta, N, $\{N_{\mathrm{N}_i}\}_{i=1}^{k-1}$, $\{N_{\mathrm{S}_i}\}_{i=1}^{k-1}$, in direct analogy to the GMSW solution.

Explicit AdS$_5$ solutions in which the internal space M_6 has the topology described in the previous paragraph are not known for $k > 2$. While the relevant BPS system is well-studied [25, 28], the search for such solutions proves to be a challenging task. This is certainly an important problem, but one which we choose to set aside for the purposes of this paper. Our working assumption is that the topology and flux configuration for $k > 2$ can be realized in the near horizon limit of a well-defined M-theory setup.

Crucially, in order to extract the physical consequences of our working assumption we do not need an explicit holographic solution. Building on [29, 30], systematic methods have been developed [31] (see also [32]) to compute the inflow anomaly polynomial I_6^{inflow} for the wrapped M5-branes probing the resolved singularity, using as input the topology and flux configuration of M_6. The quantity I_6^{inflow} is a 6-form characteristic class that captures the anomalous variation of the bulk 11d supergravity action in the presence of the wrapped M5-branes. According to the standard anomaly inflow paradigm, I_6^{inflow} is expected to be canceled by the 't Hooft anomalies of the 4d degrees of freedom living along the non-compact directions of the M5-brane stack. The computation of I_6^{inflow} was performed in [26] for $k = 2$ and in [27] for general k, at cubic order in the flux quanta (which originate from the 2-derivative $C_3G_4G_4$ coupling in the M-theory effective action). In this work we complete the computation of I_6^{inflow} by deriving the terms linear in the flux quanta (which originate from the higher-derivative coupling C_3X_8).

Comparison with the integrated anomaly polynomial in field theory. In order to contrast the field-theoretical and M-theory perspectives on class $\mathcal{S}_k$ theories, it is natural to compare the quantity I_6^{inflow} on the M-theory side with the quantity $\int_{\Sigma_g} I_8^{\text{SCFT}}$ on the field theory side. Here, I_8^{SCFT} denotes the anomaly polynomial of the parent interacting 6d (1,0) SCFT [33], and $\int_{\Sigma_g} I_8^{\text{SCFT}}$ denotes the 6-form characteristic class obtained upon integrating I_8^{SCFT} on the Riemann surface, taking into account both R-symmetry and flavor fluxes [12]. In particular, the quantity $\int_{\Sigma_g} I_8^{\text{SCFT}}$ depends not only on N, k, and the Euler characteristic χ of the Riemann surface, but also on the flavor fluxes for the Cartan subgroups of the $\mathrm{SU}(k)^2$ flavor symmetry, denoted by N_{b_i}, N_{c_i}, $i = 1, \dots, k$ (subject to the constraints $\sum_{i=1}^k N_{b_i} = 0$, $\sum_{i=1}^k N_{c_i} = 0$).

One of the main results of this paper is a precise match between $\int_{\Sigma_g} I_8^{\text{SCFT}}$ and I_6^{inflow}, including terms of order 1 in the flux parameters. The match takes the following form,

$$- I_6^{\text{inflow}} - I_6^{\text{v,t}} = \int_{\Sigma_g} I_8^{\text{SCFT}} - I_6^{\text{flip}} = I_6^{\text{SCFT}} \,. \tag{1.1}$$

The 6-form I_6^{SCFT} is the anomaly polynomial of the *interacting* 4d SCFT of class $\mathcal{S}_k$ that we want to study.[2] The quantity $I_6^{\text{v,t}}$ is the anomaly polynomial of a collection of free 4d fields, which we interpret as the reduction on Σ_g of a free 6d tensor multiplet and free 6d vector multiplets. The quantity I_6^{flip} is also the anomaly polynomial of a collection of free 4d fields, but with a different interpretation: they are flip chiral multiplets, i.e. gauge singlets coupled to 4d operators of the interacting 4d SCFT via irrelevant interactions. They are free fields, but due to their superpotential couplings to non-trivial operators in the interacting SCFT, they give large contributions to the integrated anomaly polynomial $\int_\Sigma I_8^{\text{SCFT}}$ (here "large" refers to the fact that these contributions scale with N and the flux quanta, and are not order one fixed numbers). These contributions have to be suitably accounted for in order to extract the anomaly polynomial I_6^{SCFT} of the interacting 4d SCFT from the integrated polynomial $\int_\Sigma I_8^{\text{SCFT}}$. The role of flip fields in the field-theory flow from 6d to 4d is studied in detail in [12] with several examples where the Riemann surface is a torus.

Our analysis furnishes a general expression for the contribution I_6^{flip} of flip fields, for a Riemann surface of arbitrary genus and for arbitrary values of k and the flux parameters. Our findings match the results of [12] in the case of genus one.

The case $k = 2$ at genus one: D3-branes at the tip of Cone($Y^{p,q}$). In the special case $k = 2$ and genus one, the GMSW solution is related by a chain of string theory dualities to the $\text{AdS}_5 \times Y^{p,q}$ solutions in Type IIB string theory [25], which are holographically dual to the SCFTs engineered by a stack of D3-branes at the tip of the Calabi-Yau cone over $Y^{p,q}$. By analyzing this special case, we find further evidence in favor of (1.1), by verifying the relation

$$k = 2, \text{ genus } g = 1: \qquad -I_6^{\text{inflow}} = I_6^{\text{SCFT}}(Y^{p,q})\,. \tag{1.2}$$

The above equality holds exactly in N, and not only at large N. We confirm the map between the p, q integers of $Y^{p,q}$ and the flux quanta on the M-theory side, established in [34]. Notice that the term $I_6^{\text{v,t}}$ in the general relation (1.1) is absent in (1.2), because it is proportional to the Euler characteristic of the Riemann surface.

Flip fields and M2-brane operators. The anomaly polynomial I_6^{flip} encodes the anomalies of the flip fields we encounter upon reducing the 6d (1,0) SCFT on the Riemann surface. From the data in I_6^{flip} it is straightforward to extract the charges and multiplicities of the operators that get flipped. Such charges and multiplicities can be matched precisely with those of wrapped M2-brane states in the M-theory setup.

In the case $k = 2$, one can resort to the explicit AdS_5 GMSW solution to study supersymmetric M2-brane probes, by identifying the calibrated 2-cycles in the internal space M_6 [34]. Moreover, the charges of the operators originating from these wrapped M2-brane probes can be extracted systematically from the terms in the uplift ansatz for G_4 that are linear in the external gauge fields. These are in turn conveniently extracted from the same 4-form E_4 that we utilize in the anomaly inflow computation.

[2] While we expect that the flows from six to four dimensions studied in this work yield non-trivial SCFTs in the IR, we do not have a general proof. Our analysis of the case of genus one in section 5 and of the central charges in section 6 supports our expectations.

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For $k > 2$ we do not have an explicit holographic solution, nor do we have a solution describing the flux background probed by the M5-brane stack. For these reasons, a direct analysis of the calibration conditions for wrapped M2-brane probes is challenging. Nonetheless, we can identify non-trivial 2-cycles in the internal space M_6. Motivated by the analogy with the $k = 2$ case, we make the working assumption that the relevant non-trivial 2-homology classes in M_6 admit a calibrated representative, so that the associated wrapped M2-brane operators are BPS. We can then proceed to compute their charges from the 4-form E_4 utilized in the anomaly inflow analysis. We obtain a perfect match with the charges of the operators that are flipped by the fields in I_6^{flip}. Moreover, we can also reproduce their multiplicities: they are simply given by the units of G_4 flux threading the relevant 2-cycle (combined with the Riemann surface), by virtue of a standard Landau-level degeneracy argument [35], which we review in section 4.

The identification of charges and multiplicities of flipped operators, and wrapped M2-brane operators, suggests the following physical picture. The wrapped M2-branes operators are associated to blow-up modes for the $\mathbb{C}^2/\mathbb{Z}_k$ singularity. In the 6d (1,0) SCFT, however, such modes are not present [36–44]. In contrast, we expect the M2-brane operators to be part of the spectrum of the 4d theory obtained by reduction on the Riemann surface. Indeed, in Lagrangian models, they are baryonic operators. As a result, a mechanism is needed to interpolate between six and four dimensions; this mechanism is precisely given by the flip fields from the term I_6^{flip}. They act as Lagrange multipliers that project away the wrapped M2-brane operators in the integration of the anomaly polynomial on Σ_g. In the 4d theory, they are expected to be free fields and decouple, thus effectively reintroducing the M2-brane operators.

Central charges. We study a-maximization [45] on the combination $-I_6^{\text{inflow}} - I_6^{\text{v,t}}$, in order to support the interpretation of this quantity as the anomaly polynomial of the interacting SCFT of class $\mathcal{S}_k$. By a combination of analytic and numeric methods, we explore vast regions of the flux parameter space. Our findings provide evidence for the existence of a unique local maximum of the trial a central charge. The resulting a and c are compatible with the Hofman-Maldacena bounds [46].

Organization of the paper. The rest of this paper is organized as follows. In section 2 we review the M-theory flux setups studied in [27], giving a brief account of the isometries and topology of the relevant internal space M_6. In section 3 we discuss the anomaly inflow computation (including the $E_4 X_8$ contribution) and we present in detail the main relation (1.1), giving the explicit expressions for $I_6^{\text{v,t}}$ and I_6^{flip}. Section 4 is devoted to the match between the charges and multiplicities of the flip fields entering I_6^{flip}, and those of operators originating from M2-branes wrapping resolution 2-cycles in M_6. In section 5 we focus on the case in which the Riemann surface is a torus, establishing a precise correspondence with D3-brane theories dual to $\text{AdS}_5 \times Y^{p,q}$, for $k = 2$. We also consider some explicit Lagrangian examples with $k = 2, 3$. In section 6 we use a-maximization to compute conformal and flavor central charges from $-I_6^{\text{inflow}} - I_6^{\text{v,t}}$ and establish various properties of these quantities. We conclude with an outlook in section 7. Several appendices collect useful material and detailed derivations.

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2 Review of the eleven dimensional flux setups

In this section we summarize the basic features of the internal space M_6 in the putative 11d flux backgrounds of relevance for the 4d field theories of $\mathcal{S}_k$. For a more detailed account of the geometry and homology of M_6, we refer the reader to [27]. M_6 is characterized by the fibration

$$M_4 \hookrightarrow M_6 \to \Sigma_g \,, \tag{2.1}$$

where Σ_g is a Riemann surface of genus g, and M_4 is the manifold obtained by resolving the fixed points of the orbifold $S^4/\mathbb{Z}_k$ via a blow-up procedure,

$$M_4 = [S^4/\mathbb{Z}_k]_{\text{resolved}} \,. \tag{2.2}$$

This M_4 is locally a multi-center Gibbons-Hawking space, with $k-1$ 2-cycles separated by k (unit-charge) Kaluza-Klein monopoles aligned along a common axis. It admits two U(1) isometries, and can be expressed in turn as a fibration

$$S^1_\varphi \hookrightarrow M_4 \to S^1_\psi \times M_2 \,, \tag{2.3}$$

where M_2 is a compact 2d space. Schematically, the metric on M_4 can be cast in the form

$$\begin{aligned} ds^2(M_4) &= ds^2(M_2) + R^2_\psi(\eta,\theta) d\psi^2 + R^2_\varphi(\eta,\theta) D\varphi^2 \,, \\ D\varphi &= d\varphi - L(\eta,\theta) d\psi \,, \end{aligned} \tag{2.4}$$

where the angular coordinates η and θ span the 2d space M_2. The S^1_ψ circle shrinks everywhere on the boundary ∂M_2, at $\eta, \theta = 0, \pi$. Before the blow-up, the orbifold fixed points are labeled by $\eta = 0, \pi$, which we refer to here as the north and south poles, respectively. After the blow-up, each pole is replaced by a chain of $k-1$ resolution 2-cycles. The k monopoles in the north carry Kaluza-Klein charge $+1$, while the k monopoles in the south carry charge -1, with the relative sign accounting for the opposite orientations relative to M_2 at $\eta = 0, \pi$. There is a U(1) gauge symmetry associated with each resolution 2-cycle, so there is an overall $\mathrm{U}(1)^{k-1}$ symmetry in both the north and the south. The topology of the $S^4/\mathbb{Z}_k$ and its resolved counterpart M_4 are illustrated in figure 2.

The function $L(\eta,\theta)$ is piecewise constant on ∂M_2, with its difference across a given monopole measuring that monopole's Kaluza-Klein charge. Labeling the resolution 2-cycles in the north by $i = 1, \dots, k-1$ and those in the south by $i = k+1, \dots, 2k-1$, we have explicitly

$$\ell_i \equiv L(t_i < t < t_{i+1}) = \begin{cases} i - \dfrac{k}{2} & \text{for} \quad i = 1, \dots, k, \\ \dfrac{3k}{2} - i & \text{for} \quad i = k+1, \dots, 2k \,, \end{cases} \tag{2.5}$$

where t is a periodic coordinate parameterizing the boundary ∂M_2.

In the full internal space M_6, twisting of M_4 over the Riemann surface introduces a U(1) connection over Σ_g to the form $d\psi$. $\mathcal{N}=1$ supersymmetry is preserved specifically by a topological twist [6, 7]

$$d\psi \to D\psi = d\psi - 2\pi\chi A_\Sigma \,. \tag{2.6}$$

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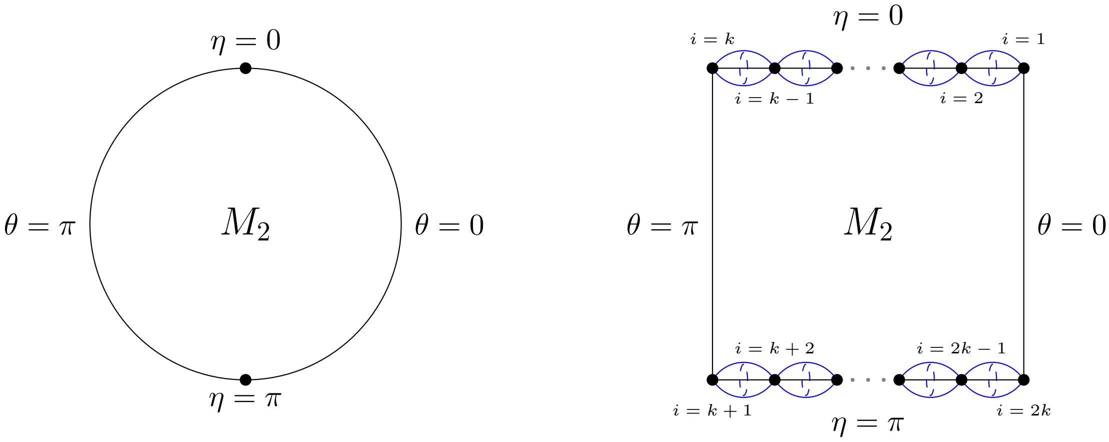

Figure 2. Illustration of the topology of the unresolved $S^4/\mathbb{Z}_k$ space (*left*) and resolved M_4 space (*right*), with the ψ and φ angles suppressed, taken from [27]. The circle S^1_ψ vanishes along the entire boundary ∂M_2, while S^1_φ vanishes only at the $2k$ monopoles, labeled by the index i. The blue bubbles represent the resolution 2-cycles connected by unit-charge Kaluza-Klein monopoles.

The quantity $\chi = 2(1-g)$ is the Euler characteristic of the genus-g Riemann surface, with volume form V_Σ normalized as $\int_{\Sigma_g} V_\Sigma = 1$, and A_Σ is the local antiderivative of V_Σ. This topological twist leads to nontrivial relations in the homology of M_6. Consider first the 2-cycles in M_6. There are two types: the boundary 2-cycles in M_4, and the Riemann surface Σ_g itself, at the position of each of the monopoles. We thus have $4k$ total 2-cycles,

$$\mathcal{C}_2^{\Sigma,i} = \Sigma_g|_{t=t_i}\,, \qquad \mathcal{C}_2^i = [t_i, t_{i+1}] \times S^1_\varphi\,, \tag{2.7}$$

one of each type for every $i = 1, \ldots, 2k$. However, as described in [27], only $2k-1$ of these 2-cycles are independent. The situation is analogous for the 4-cycles. The region M_4 constitutes one 4-cycle in the full M_6 space, as do the $2k$ pairings of the boundary 2-cycles with Σ_g,

$$\mathcal{C}_4^{\mathrm{C}} = M_4\,, \qquad \mathcal{C}_4^i = \mathcal{C}_2^i \times \Sigma_g = [t_i, t_{i+1}] \times S^1_\varphi \times \Sigma_g\,. \tag{2.8}$$

The topological twist (2.6) trivializes certain linear combinations of these 4-cycles, however,

$$\sum_{i=1}^{2k} \mathcal{C}_4^i = \chi \mathcal{C}_4^{\mathrm{C}}\,, \qquad \sum_{i=1}^{2k} \ell_i\, \mathcal{C}_4^i = 0\,, \tag{2.9}$$

leaving $2k-1$ independent 4-cycles. It is convenient to adopt a complete basis of 4-cycles $\mathcal{C}_{4,\alpha}$ consisting of the M_4 bulk 4-cycle, the $k-1$ 4-cycles in the north, and the $k-1$ 4-cycles in the south,

$$\mathcal{C}_{4,\alpha=C} = \mathcal{C}_4^{\mathrm{C}}\,, \quad \mathcal{C}_{4,\alpha=\mathrm{N}_{i=1,\ldots,k-1}} = \mathcal{C}_4^{i=1,\ldots,k-1}\,, \quad \mathcal{C}_{4,\alpha=\mathrm{S}_{i=1,\ldots,k-1}} = \mathcal{C}_4^{i=2k-1,\ldots,k+1}\,. \tag{2.10}$$

The corresponding basis of $2k-1$ 2-cycles $\mathcal{C}_2^\alpha$ can be taken to be Poincaré-dual to these 4-cycles. Quantization of the M-theory 4-form flux G_4 associates each 4-cycle with an integer,

$$N = \int_{\mathcal{C}_4^{\mathrm{C}}} \frac{G_4}{2\pi}\,, \qquad N_i = \int_{\mathcal{C}_4^i} \frac{G_4}{2\pi}\,, \tag{2.11}$$

subject to linear constraints inherited from (2.9), namely,

$$\sum_{i=1}^{2k} N_i = \chi N \,, \qquad \sum_{i=1}^{2k} \ell_i N_i = 0 \,. \tag{2.12}$$

In the basis (2.10), we have $2k-1$ independent flux quanta,

$$N_{\mathrm{N}_{i=1,\dots,k-1}} = N_{i=1,\dots,k-1} \,, \quad N_{\mathrm{S}_{i=1,\dots,k-1}} = N_{i=2k-1,\dots,k+1} \,, \quad N_{\mathrm{C}} = N \,. \tag{2.13}$$

All told, the space M_6 is characterized by the integer parameters k, χ, N, N_{N_i}, N_{S_i}.

3 Anomaly polynomials in class $\mathcal{S}_k$ from inflow

In this section we argue that the inflow anomaly polynomial for wrapped M5-branes probing a resolved $\mathbb{C}^2/\mathbb{Z}_k$ singularity is to be identified with the anomaly polynomial of a class $\mathcal{S}_k$ theory, obtained from reduction of the parent 6d (1,0) SCFT on a smooth Riemann surface with non-trivial $\mathrm{SU}(k)^2$ flavor fluxes. The identification holds up to the contribution of a suitable collection of free fields, which we discuss in detail.

3.1 Integrated anomaly polynomial from six dimensions

Here we review briefly the integration of the 6d 8-form anomaly polynomial I_8^{SCFT} on a smooth genus-g Riemann surface, with a non-trivial topological twist and flavor fluxes [12]. Let us stress that I_8^{SCFT} denotes the anomaly polynonomial of the *interacting* 6d (1,0) SCFT realized by a stack of M5-branes probing a $\mathbb{C}^2/\mathbb{Z}_k$ singularity [33], without the inclusion of decoupled sectors, such as the center-of-mass mode of the stack.

The R-symmetry of the parent 6d (1,0) theory is $\mathrm{SU}(2)_R$. In the reduction to four dimensions, the Chern root of the $\mathrm{SU}(2)_R$ bundle is shifted to implement the topological twist that preserves 4d $\mathcal{N}=1$ supersymmetry,

$$c_2(\mathrm{SU}(2)_R) = -\Big[c_1(R') - \frac{\chi}{2}\, V_\Sigma\Big]^2 \,. \tag{3.1}$$

The label R' on the 4d background Chern class $c_1(R')$ is a reminder that this is a reference R-symmetry, which does not generically coincide with the 4d $\mathcal{N}=1$ superconformal R-symmetry $R_{\mathcal{N}=1}$.

For generic N, k, the 6d SCFT admits a $\mathrm{U}(1)_s \times \mathrm{SU}(k)_b \times \mathrm{SU}(k)_c$ flavor symmetry. The Chern roots of the $\mathrm{SU}(k)_b$, $\mathrm{SU}(k)_c$ bundles are denoted by b_i, c_i ($i=1,\dots,k$), respectively, and they are subject to the constraints $\sum_{i=1}^k b_i = 0 = \sum_{i=1}^k c_i$. The reduction to four dimensions is performed with the following Chern root shifts,

$$c_1(\mathrm{U}(1)_s) = c_1(t) \,, \qquad b_i = c_1(\beta_i) + N_{\beta_i}\, V_\Sigma \,, \qquad c_i = c_1(\gamma_i) + N_{\gamma_i}\, V_\Sigma \,. \tag{3.2}$$

Notice in particular that, for simplicity, in this work we do not turn on a flavor flux for the $\mathrm{U}(1)_s$ symmetry. The quantities $c_1(t)$, $c_1(\beta_i)$, $c_1(\gamma_i)$ are the first Chern classes of background fields for 4d U(1) global symmetries. We observe that $c_1(\beta_i)$, $c_1(\gamma_i)$, as well as the flavor fluxes N_{β_i}, N_{γ_i}, are constrained quantities,

$$\sum_{i=1}^{k} c_1(\beta_i) = \sum_{i=1}^{k} c_1(\gamma_i) = 0\,, \qquad \sum_{i=1}^{k} N_{\beta_i} = \sum_{i=1}^{k} N_{\gamma_i} = 0\,. \tag{3.3}$$

We find it convenient to adopt the following parametrizations of the above constraints,

$$\begin{aligned} N_{\beta_i} &= N_{\widetilde{\beta}_i} - N_{\widetilde{\beta}_{i-1}}\,, \qquad & c_1(\beta_i) &= c_1(\widetilde{\beta}_i) - c_1(\widetilde{\beta}_{i-1})\,, \\ N_{\gamma_i} &= N_{\widetilde{\gamma}_i} - N_{\widetilde{\gamma}_{i-1}}\,, \qquad & c_1(\gamma_i) &= c_1(\widetilde{\gamma}_i) - c_1(\widetilde{\gamma}_{i-1})\,, \qquad i = 1, \dots, k\,, \end{aligned} \tag{3.4}$$

with the conventions $N_{\widetilde{\beta}_0} \coloneqq 0$, $N_{\widetilde{\beta}_k} \coloneqq 0$, $c_1(\widetilde{\beta}_0) \coloneqq 0$, $c_1(\widetilde{\beta}_k) \coloneqq 0$, $N_{\widetilde{\gamma}_0} \coloneqq 0$, $N_{\widetilde{\gamma}_k} \coloneqq 0$, $c_1(\widetilde{\gamma}_0) \coloneqq 0$, $c_1(\widetilde{\gamma}_k) \coloneqq 0$.

We are now in a position to quote the result of the integration of the anomaly polynomial $I_8^{\rm SCFT}$ of the parent 6d (1,0) SCFT on the Riemann surface Σ_g,

$$\begin{aligned} \int_{\Sigma_g} I_8^{\rm SCFT} =& -\frac{[k^2(N^2+N-1)+2]\chi(N-1)}{12}\, c_1^3(R') + \frac{k^2\chi N(N^2-1)}{12}\, c_1(R')\, c_1^2(t) \\ & - \frac{1}{2}[kN(N-1)c_1^2(R') - kN^2 c_1^2(t)] \sum_{i=1}^{k} [N_{\beta_i} c_1(\beta_i) + N_{\gamma_i} c_1(\gamma_i)] \\ & + \frac{k\chi N^2(N-1)}{4}\, c_1(R') \sum_{i=1}^{k} [c_1^2(\beta_i) + c_1^2(\gamma_i)] \\ & + \frac{kN^2}{2} \sum_{i=1}^{k} [N_{\beta_i} c_1(t)\, c_1^2(\beta_i) - N_{\gamma_i} c_1(t)\, c_1^2(\gamma_i)] \\ & + \frac{kN^3}{6} \sum_{i=1}^{k} [N_{\beta_i} c_1^3(\beta_i) + N_{\gamma_i} c_1^3(\gamma_i)] + \frac{N^2(N-1)}{2} \sum_{i=1}^{k} c_1^2(\beta_i) \sum_{j=1}^{k} N_{\beta_j} c_1(\beta_j) \\ & + \frac{N^2(N-1)}{2} \sum_{i=1}^{k} c_1^2(\gamma_i) \sum_{j=1}^{k} N_{\gamma_j} c_1(\gamma_j) \\ & + \frac{N^2}{2} \left[\sum_{i=1}^{k} c_1^2(\beta_i) \sum_{j=1}^{k} N_{\gamma_j} c_1(\gamma_j) + \sum_{i} c_1^2(\gamma_i) \sum_{j=1}^{k} N_{\beta_j} c_1(\beta_j) \right] \\ & - \frac{(k^2-2)\chi(N-1)}{48}\, c_1(R')\, p_1(T_4) - \frac{kN}{24} \sum_{i=1}^{k} [N_{\beta_i} c_1(\beta_i) + N_{\gamma_i} c_1(\gamma_i)]\, p_1(T_4)\,. \end{aligned} \tag{3.5}$$

3.2 Anomaly inflow from eleven dimensions

The inflow anomaly polynomial for a stack of M5-branes probing a background associated with the internal geometry M_6 and background flux configuration $\overline{G}_4$ is given by [31]

$$-\, I_6^{\rm inflow} = \int_{M_6} \left[\frac{1}{6}\, E_4^3 + E_4 \wedge X_8 \right], \qquad X_8 = \frac{1}{192} \left[p_1(TM_{11})^2 - 4\, p_2(TM_{11}) \right]. \tag{3.6}$$

The closed and gauge-invariant 4-form E_4 is constructed from $\overline{G}_4$ by including the external gauge fields associated with the isometries and the non-trivial cohomology classes of M_6.

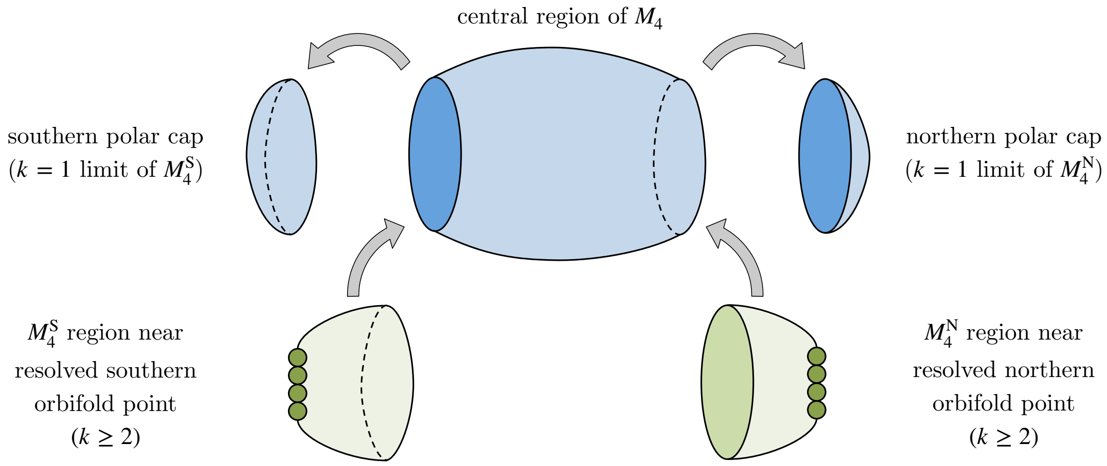

Figure 3. Pictorial depiction of the strategy used to evaluate the E_4X_8 contribution to the inflow anomaly polynomial. In order to obtain the resolved orbifold M_4, we consider the central "cylindrical" region of S^4 away from the poles, and we glue in the resolved orbifolds at each pole. These include resolution 2-cycles, depicted as small green circles in the figure. The central contribution is obtained by taking a full S^4 and removing small polar caps. The contribution of the latter is computed by taking the limit $k = 1$ in the contribution of the resolved orbifolds.

The 8-form characteristic class X_8 is built with the Pontryagin classes $p_i(TM_{11})$ of the tangent bundle of the 11d spacetime.

The computation of the inflow anomaly polynomial for the flux setups reviewed in section 2 is discussed in detail in [27] for general k, where the contribution of the E_4^3 term was studied. The contribution of the $E_4\,X_8$ term for general k is derived in appendix B. We have a completely explicit expression for $-I_6^{\rm inflow}$ in terms of k, N, χ, the flux parameters $N_{{\rm N}_i}$, $N_{{\rm S}_i}$, the external field strengths F_2^{ψ}, F_2^{φ}, $F_2^{{\rm N}_i}$, $F_2^{{\rm S}_i}$, and the first Pontryagin class $p_1(TW_4)$ of the external spacetime. Since this expression is quite complicated, however, we refrain from reproducing it in the main text. The E_4^3 contribution to $-I_6^{\rm inflow}$ is given in appendix A, while the result for the E_4X_8 part can be found in appendix B.

Without going into the technical details of the E_4X_8 computation, it is interesting to comment on the general strategy we follow, which is depicted pictorially in figure 3. The main idea is to organize the contributions to E_4X_8 into a bulk part, originating from the central region of S^4 away from the poles, and the parts associated to the resolved orbifold points at the north and south poles of the S^4. Operationally, we obtain an expression for the resolved orbifolds at the poles for generic k. If we specialize to $k = 1$, i.e. no orbifolding, this contribution is identified with the contribution of the polar caps of S^4, i.e. small neighborhoods of the two poles. If we take the full S^4 contribution, and we subtract these polar caps for $k = 1$, we get the contribution of the central "cylindrical" region. Finally, we glue back in the resolved orbifolds with the appropriate value $k > 1$, in order to get the final desired result.

In concluding, we observe that the E_4^3 contribution was computed in [27] using a different strategy, but is nonetheless compatible with this cut-and-glue picture. Both for E_4X_8, and for E_4^3, the contribution associated to the central region of M_4 is equal to the inflow anomaly polynomial for a 4d $\mathcal{N}=1$ SCFT originating from the 6d (2,0) SCFT of type A_{N-1} with twist parameters p, q satisfying $p=q$ [7].

3.3 Matching the two sides: decoupled modes and flip fields

Let us now discuss the relation between $\int_{\Sigma_g} I_8^{\rm SCFT}$ and $-I_6^{\rm inflow}$. To this end, it is convenient to introduce the following notation,

$$\begin{aligned} I_6^{\rm free}(q_{R'};q_t;q_{\widetilde{\beta}_i};q_{\widetilde{\gamma}_i}) = \frac{1}{6}\left[q_{R'}\,c_1(R') + q_t\,c_1(t) + \sum_{i=1}^{k-1} q_{\widetilde{\beta}_i}\,c_1(\widetilde{\beta}_i) + \sum_{i=1}^{k-1} q_{\widetilde{\gamma}_i}\,c_1(\widetilde{\gamma}_i)\right]^3 \\ - \frac{1}{24}\,p_1(TW_4)\left[q_{R'}\,c_1(R') + q_t\,c_1(t) + \sum_{i=1}^{k-1} q_{\widetilde{\beta}_i}\,c_1(\widetilde{\beta}_i) + \sum_{i=1}^{k-1} q_{\widetilde{\gamma}_i}\,c_1(\widetilde{\gamma}_i)\right]. \end{aligned} \tag{3.7}$$

This quantity is the anomaly polynomial of a free, positive-chirality Weyl fermion in four dimensions, with prescribed charges $q_{R'}$, q_t, $q_{\widetilde{\beta}_i}$, $q_{\widetilde{\gamma}_i}$ under the 4d U(1) symmetries $\mathrm{U}(1)_{R'}$, $\mathrm{U}(1)_t$, $\mathrm{U}(1)_{\widetilde{\beta}_i}$, $\mathrm{U}(1)_{\widetilde{\gamma}_i}$, in the notation of section 3.1. The quantities $c_1(\widetilde{\beta}_i)$, $c_1(\widetilde{\gamma}_i)$ are the unconstrained Chern roots defined by (3.4).

We can write the difference between $\int_{\Sigma_g} I_8^{\rm SCFT}$ and $-I_6^{\rm inflow}$ in terms of a collection of free fermions, with anomaly given by (3.7) for appropriate charges. More precisely, we find

$$-I_6^{\rm inflow} - I_6^{\rm v,t} = \int_{\Sigma_g} I_8^{\rm SCFT} - I_6^{\rm flip}\,, \tag{3.8}$$

where $I_6^{\rm v,t}$ and $I_6^{\rm flip}$ are given by

$$\begin{aligned} I_6^{\rm v,t} &= \frac{\chi}{2}\,(k-2)\,I_6^{\rm free}(q_{R'}=1;\,q_t=0;\,q_{\widetilde{\beta}_i}=0;\,q_{\widetilde{\gamma}_i}=0) \\ &\quad + \frac{\chi}{2}\sum_{1\le a<b\le k} I_6^{\rm free}(q_{R'}=1;\,q_t=0;\,q_{\widetilde{\beta}_i}=N\,L^i_{a,b};\,q_{\widetilde{\gamma}_i}=0) \\ &\quad + \frac{\chi}{2}\sum_{1\le a<b\le k} I_6^{\rm free}(q_{R'}=1;\,q_t=0;\,q_{\widetilde{\beta}_i}=0;\,q_{\widetilde{\gamma}_i}=N\,L^i_{a,b})\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} I_6^{\rm flip} &= \sum_{1\le a<b\le k} m^{(\beta)}_{a,b}\,I_6^{\rm free}(q_{R'}=1;\,q_t=0;\,q_{\widetilde{\beta}_i}=N\,L^i_{a,b};\,q_{\widetilde{\gamma}_i}=0) \\ &\quad + \sum_{1\le a<b\le k} m^{(\gamma)}_{a,b}\,I_6^{\rm free}(q_{R'}=1;\,q_t=0;\,q_{\widetilde{\beta}_i}=0;\,q_{\widetilde{\gamma}_i}=N\,L^i_{a,b})\,. \end{aligned} \tag{3.10}$$

Some remarks on our notation are in order. The quantities $L^i_{a,b}$ can be identified with the components of the positive roots of the Lie algebra $\mathfrak{su}(k)$,

$$L^i_{a,b} = L^i_a - L^i_b\,,\quad L^i_a = \delta_{a,i} - \delta_{a,i+1}\,,\quad 1\le a<b\le k\,,\quad i=1,\dots,k-1\,. \tag{3.11}$$

The multiplicity factors $m^{(\beta)}_{a,b}$, $m^{(\gamma)}_{a,b}$ are given in terms of the unconstrained flavor fluxes $N_{\widetilde{\beta}_i}$, $N_{\widetilde{\gamma}_i}$ introduced in (3.4) by the following expressions,

$$m^{(\beta)}_{a,b} = N_{\widetilde{\beta}_a} - N_{\widetilde{\beta}_{a-1}} - N_{\widetilde{\beta}_b} + N_{\widetilde{\beta}_{b-1}}\,,\quad m^{(\gamma)}_{a,b} = N_{\widetilde{\gamma}_a} - N_{\widetilde{\gamma}_{a-1}} - N_{\widetilde{\gamma}_b} + N_{\widetilde{\gamma}_{b-1}}\,. \tag{3.12}$$

The relation (3.8) holds provided that we make the following identifications among quantities related to anomaly inflow from eleven dimensions, and quantities related to integration from six dimensions,

$$\begin{aligned}
\frac{F_2^{\psi}}{2\pi} &= 2\,c_1(R')\,, & \frac{F_2^{\varphi}}{2\pi} &= -k\,c_1(t)\,, \\
\frac{F_2^{\mathrm{N}_i}}{2\pi} &= c_1(\widetilde\beta_i) + \frac{2}{N\,\chi}\,N_{\widetilde\beta_i}\,c_1(R')\,, & \frac{F_2^{\mathrm{S}_{k-i}}}{2\pi} &= -c_1(\widetilde\gamma_i) - \frac{2}{N\,\chi}\,N_{\widetilde\gamma_i}\,c_1(R')\,, \\
N_{\mathrm{N}_i} &= \sum_{j=1}^{k-1} (\mathsf{A}_{k-1})_{ij}\,N_{\widetilde\beta_j}\,, & N_{\mathrm{S}_{k-i}} &= \sum_{j=1}^{k-1} (\mathsf{A}_{k-1})_{ij}\,N_{\widetilde\gamma_j}\,.
\end{aligned} \tag{3.13}$$

The quantities $(\mathsf{A}_{k-1})_{ij}$ are the entries of the Cartan matrix of $\mathfrak{su}(k)$,

$$(\mathsf{A}_{k-1})_{ij} = 2\,\delta_{i,j} - \delta_{i,j+1} - \delta_{i+1,j}\,, \qquad i,j = 1,\dots,k-1\,. \tag{3.14}$$

Notice the $k-i$ label on southern quantities, as opposed to the i label on northern quantities.

Interpretation of $I_6^{\mathrm{v,t}}$. The free-field contributions in $I_6^{\mathrm{v,t}}$ are interpreted as originating from the reduction on Σ_g of free fields in six dimensions, as suggested by the fact that they are proportional to the Euler characteristic χ. The 8-form anomaly polynomials of a free 6d (1,0) tensor multiplet, and a free 6d (1,0) vector multiplet, are readily computed and integrated on the Riemann surface, with the result

$$I_8^{(1,0)\ \mathrm{tens}} = -I_8^{(1,0)\ \mathrm{vec}} = -\frac{\chi}{2}\,I_6^{\mathrm{free}}(q_{R'} = 1;\ q_t = 0;\ q_{\widetilde\beta_i} = 0;\ q_{\widetilde\gamma_i} = 0)\,. \tag{3.15}$$

This observation suggests us to interpret the first line of $I_6^{\mathrm{v,t}}$ in (3.9) as a contribution of one tensor and $k-1$ vectors. The former is associated with the center of mass of the M5-brane stack, while the latter is thought of as the Cartan generators of $\mathrm{SU}(k)$. By a similar token, we interpret the other terms in (3.9) as coming from the reduction of 6d W-bosons of $\mathrm{SU}(k)$, whose charges are indeed given by the roots of $\mathfrak{su}(k)$.

Interpretation of I_6^{flip}. While the free-field contributions in $I_6^{\mathrm{v,t}}$ have a 6d interpretation, those in I_6^{flip} are interpreted in four-dimensional terms. More precisely, we identify I_6^{flip} with the anomaly polynomial of a collection of flip fields. Here, by flip field we mean a 4d $\mathcal{N}=1$ chiral multiplet ϕ that couples to an operator $\mathcal{O}$ of the interacting 4d SCFT with a superpotential coupling of the form

$$W_{\mathrm{flip}} = \phi\,\mathcal{O}\,. \tag{3.16}$$

The field ϕ has canonical kinetic terms and, if it were not for (3.16), would be completely decoupled from the 4d SCFT. Since the superpotential has R-charge 2 and is neutral under other global symmetries, the coupling (3.16) implies

$$q_{R'}[\Psi_\phi] = (2 - q_{R'}[\mathcal{O}]) - 1\,, \qquad q_X[\Psi_\phi] = -q_X[\mathcal{O}]\,, \tag{3.17}$$

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where Ψ_ϕ denotes the fermion in the chiral multiplet ϕ, and q_X is a shorthand notation for q_t, $q_{\widetilde{\beta}_i}$, $q_{\widetilde{\gamma}_i}$. The charges $q_{R'}[\Psi_\phi]$, $q_X[\Psi_\phi]$ are the charges given $I_6^{\rm flip}$ in (3.10). They are readily translated into charges of the operators $\mathcal{O}$ that get flipped,

$$\begin{aligned} &\text{flipped op.'s } \mathcal{O}^{(\beta)}_{a,b}: \quad q_{R'}[\mathcal{O}^{(\beta)}_{a,b}] = 0\,, \quad q_{\widetilde{\beta}_i}[\mathcal{O}^{(\beta)}_{a,b}] = -N\,L^i_{a,b}\,, \quad q_{\widetilde{\gamma}_i}[\mathcal{O}^{(\beta)}_{a,b}] = 0\,, \\ &\text{flipped op.'s } \mathcal{O}^{(\gamma)}_{a,b}: \quad q_{R'}[\mathcal{O}^{(\gamma)}_{a,b}] = 0\,, \quad q_{\widetilde{\beta}_i}[\mathcal{O}^{(\gamma)}_{a,b}] = 0\,, \quad q_{\widetilde{\gamma}_i}[\mathcal{O}^{(\gamma)}_{a,b}] = -N\,L^i_{a,b}\,. \end{aligned} \tag{3.18}$$

Notice how all flipped operators have zero charge under $\mathrm{U}(1)_{R'}$.

In section 4 we match the charges of the flipped operators (3.18) with charges of wrapped M2-brane states, and we comment further on the physical mechanism underlying the flipping of these operators.

Flavor fluxes versus resolution fluxes: a geometric picture. Let us motivate the identification (3.13) by considering the geometric interpretation of the flux quanta $N_{\tilde{\beta}_i}, N_{\tilde{\gamma}_i}$ in the 6d setup. Prior to compactification over the Riemann surface, the internal space of the 6d theories is $S^4/\mathbb{Z}_k$, and has two orbifold fixed points which can be resolved by the set of 2-cycles $\mathcal{C}_2^i$ defined in (2.7). Associated with each 2-cycle is a U(1) flavor symmetry. The only 4-cycle in such a setup is the (unresolved) $S^4/\mathbb{Z}_k$ that is analogous to $\mathcal{C}_4^{\rm C}$; there is no analogue of the 4-cycles $\mathcal{C}_4^i$ defined in (2.8). As a consequence, the flux quanta $N_{\tilde{\beta}_i}, N_{\tilde{\gamma}_i}$ assigned to the $\mathrm{U}(1)_{\tilde{\beta}_i}, \mathrm{U}(1)_{\tilde{\gamma}_i}$ flavor symmetries in the compactification are naturally associated with $\mathcal{C}_2^i$ but not $\mathcal{C}_4^i$. From the perspective of the expansion

$$E_4 \supset N_\alpha (\Omega_4^\alpha)^{\rm eq} + N\,\frac{F_2^\alpha}{2\pi} \wedge (\omega_{2,\alpha})^{\rm eq}\,, \tag{3.19}$$

flux quanta are intrinsically paired to 4-cycles [27]. So we conjecture that $N_{\tilde{\beta}_i}, N_{\tilde{\gamma}_i}$ are really flux quanta with respect to the 4-cycles Poincaré-dual to the resolution 2-cycles $\mathcal{C}_2^i$ (after reduction on the Riemann surface). In contrast, the flux quanta $N_{\mathrm{N}_i}, N_{\mathrm{S}_i}$ were defined with respect to the 4-cycles $\mathcal{C}_{4,\mathrm{N_i}}, \mathcal{C}_{4,\mathrm{S_i}}$ described in (2.10). Direct comparison between $\int_{\Sigma_g} I_8^{\rm SCFT}$ and $-I_6^{\rm inflow}$ therefore requires that we find the transformation matrices relating these two distinct bases of homology classes. As worked out in appendix D, these transformation matrices turn out to be block diagonal, with each of the two nontrivial blocks proportional to A_{k-1}, the Cartan matrix of $\mathfrak{su}(k)$. One may heuristically interpret this as a remnant of the enhanced $\mathrm{SU}(k)$ symmetry present at each orbifold fixed point before being resolved into $k-1$ 2-cycles with U(1) symmetries. Indeed, the identification in (3.13) is precisely the change of basis we have described, with an additional sign change for the southern flux quanta to preserve their positivity.

Next, we argue that the factor of $1/k$ in the identification (3.13) of the field strength F_2^φ with the Chern root $c_1(t)$ can be attributed to the different periodicities of t and φ. More specifically, the periodicity of the angular variable t in the 6d theories is reduced from 2π to $2\pi/k$ by the $\mathbb{Z}_k$ quotient, but the periodicity of φ in the inflow computation directly from 11d is simply 2π by construction. In the former case, we gauge the $\mathrm{U}(1)_t$ isometry as $(Dt)^{\rm g} = Dt - A_1^t$, while in the latter case we gauge the $\mathrm{U}(1)_\varphi$ isometry as $(D\varphi)^{\rm g} = D\varphi + A_1^\varphi$. This motivates the identifications,

$$t = \frac{1}{k}\,\varphi\,, \quad A_1^t = -\frac{1}{k}\,A_1^\varphi\,, \quad F_2^t = -\frac{1}{k}\,F_2^\varphi\,. \tag{3.20}$$

Lastly, the factor of 2 appearing in the identification (3.13) between $F_2^\psi/2\pi$ and $c_1(R')$ is needed to ensure an appropriate R-charge normalization [34].

4 Wrapped M2-branes and flip fields

Operators from M2-branes wrapping calibrated 2-cycles. The calibration conditions for probe M2-branes wrapping 2-cycles in the internal space M_6 were derived in [34], where they were also analyzed for the $k = 2$ GMSW solutions. The homology classes of the resolution 2-cycles of M_4 (at a generic point on the Riemann surface) admit a calibrated representative. In the notation of section 2, these homology classes are the $\mathcal{C}_2^i$ in (2.7) with labels $i = 1$ and $i = 3$. Wrapping M2-brane probes on such 2-cycles yields BPS particle states in the external spacetime.

A direct analysis of the calibration conditions for $k \geq 3$ is challenging. Indeed, we do not have an explicit AdS_5 solution in which the internal space has the topology and flux quanta of M_6. Solutions describing the flux background probed by the M5-brane stack are not known, either. For these reasons, we refrain from studying the calibration conditions for M2-brane probes for $k \geq 3$, and we make the following working assumption: the homology classes of the resolution 2-cycles of M_4 admit calibrated representative 2-cycles. We wrap probe M2-branes on such cycles, getting BPS states in the external spacetime.

In what follows, we study for generic $k \geq 2$ the charges and multiplicities of such states. Crucially, these data do not depend on the (putative, for $k > 2$) explicit calibrated representative, but only on its homology class.

Charges of M2-branes operators. The method for the computation of the charges of wrapped M2-brane operators is explained in [34]. The key point is to use the standard coupling of the 11d 3-form C_3 to the worldvolume of the M2-brane probe. The desired charges are extracted by integrating this 3d coupling along the compact directions of the 2-cycle wrapped by the M2-brane, thus extracting terms linear in the external gauge fields.

In order to determine how the external gauge fields enter the 11d 3-form C_3, we resort to the 4-form E_4 used in the anomaly inflow computation. More precisely, we proceed as follows: extract the terms in E_4 that are linear in the external 2-form field strengths F_2^ψ, F_2^φ, $F_2^{\mathrm{N}_i}$, $F_2^{\mathrm{S}_i}$, and cast these terms as a total derivative of a 3-form δC_3 that is linear in the corresponding external gauge fields A_1^ψ, A_1^φ, $A_1^{\mathrm{N}_i}$, $A_1^{\mathrm{S}_i}$. All the relevant charges of interest are then obtained by integrating δC_3 on the 2-cycle wrapped by the M2-brane.

After these general preliminaries, we are in a position to outline the results we obtain for the setups of interest in this work. The 3-form δC_3 derived from E_4 takes the form

$$\delta C_3 \supset (\omega_{2,\beta})^{\mathrm{g}} \wedge A_1^\beta + (\Omega_{2,I})^{\mathrm{g}} \wedge A_1^I \,, \tag{4.1}$$

where the label β runs over all independent 2-cycles in M_6, while I is a collective label for the isometry directions ψ, φ. The 2-forms $(\omega_{2,\beta})^{\mathrm{g}}$ are obtained from the harmonic 2-forms on M_6 by means of the replacements $d\psi \to d\psi + A_1^\psi$, $d\varphi \to d\varphi + A_1^\varphi$. The 2-forms $(\Omega_{2,I})^{\mathrm{g}}$ are derived in the process of constructing the equivariant completion of the harmonic

4-forms on M_6. We refer the interested reader to [27], where the explicit expressions of $(\omega_{2,\beta})^{\mathrm{g}}$ and $(\Omega_{2,I})^{\mathrm{g}}$ can be found.

The charges under the U(1) field strengths for flavor symmetries and isometries, respectively F_2^β and F_2^I, can then by computed from the integrals

$$(Q_\beta)^i = N \int_{\mathcal{C}_2^i} (\omega_{2,\beta})^{\mathrm{g}}, \qquad (Q_I)^i = \int_{\mathcal{C}_2^i} (\Omega_{2,I})^{\mathrm{g}}. \tag{4.2}$$

The results can be summarized in the following table, up to an overall orientation choice,

	Q_ψ	Q_φ	Q_{N_j}	Q_{S_j}	mult.
$i = 1, \dots, k-1$	$\chi^{-1} N_{\mathrm{N}_i}$	0	$-N\,(\mathsf{A}_{k-1})_{ij}$	0	N_{N_i}
$i = 2k-1, \dots, k+1$	$\chi^{-1} N_{\mathrm{S}_i}$	0	0	$+N\,(\mathsf{A}_{k-1})_{ij}$	N_{S_i}

(4.3)

where we have written the flavor charges in terms of the entries of the Cartan matrix of $\mathfrak{su}(k)$. Notice that the charges are given in a basis that corresponds to the external gauge fields in the anomaly inflow computation. Making use of (3.13), they are readily translated to the basis used in the integration of the anomaly polynomial of the 6d SCFT,

	$Q_{R'}$	Q_t	$Q_{\widetilde{\beta}_j}$	$Q_{\widetilde{\gamma}_j}$	mult.
$i = 1, \dots, k-1$	0	0	$-N\,(\mathsf{A}_{k-1})_{ij}$	0	N_{N_i}
$i = 2k-1, \dots, k+1$	0	0	0	$-N\,(\mathsf{A}_{k-1})_{ij}$	N_{S_i}

(4.4)

In the above tables we have also reported the multiplicity of the M2-brane operators, which is derived as explained below.

Multiplicities of M2-branes operators. The multiplicities, or degeneracies, of the states originating from an M2-brane probe wrapping a 2-cycle can be derived using a Landau-level argument [35]. The probe M2-branes of interest in this work wrap a resolution 2-cycle $\mathcal{C}_2^i$ in M_4, and sit at a point on Σ_g. Moreover, a non-trivial G_4-flux is turned on along the 4-cycle that results from combining the resolution 2-cycle $\mathcal{C}_2^i$ and the Riemann surface Σ_g. As a result, the M2-brane behaves like a point particle on Σ_g, in the presence of a non-zero magnetic field. This quantum-mechanical system exhibits a well-known Landau degeneracy of states, which is simply given by the total magnetic flux, measured in units of the minimal magnetic flux that can be turned on. This quantized magnetic flux is indeed identified with the quantized G_4-flux through the 4-cycle obtained combining Σ_g and $\mathcal{C}_2^i$. In conclusion, the expected degeneracies of the wrapped M2-brane operators of interest are simply given by the values of the corresponding G_4-flux.

Comparison to flip fields. The anomaly polynomial I_6^{flip} is interpreted as a sum over *flip* fields. The charges of the corresponding *flipped* operators are collected in (3.18), for a generic pair of labels $a < b$, $a, b = 1, \dots, k$. These pairs correspond to all positive roots of $\mathfrak{su}(k)$. Those pairs $a < b$ with $b = a + 1$ correspond to the *simple* roots of $\mathfrak{su}(k)$. Based on standard intuition regarding M-theory on a $\mathbb{Z}_k$ singularity, we expect the M2-branes wrapping the $(k-1)$ resolution 2-cycles at the north and south poles to correspond to the

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simple roots of $\mathfrak{su}(k)_{\rm N}$, $\mathfrak{su}(k)_{\rm S}$. (The full set of positive roots corresponds to BPS bound states of the M2-brane states corresponding to the simple roots.) Now, the identity

$$L^i_{a,a+1} = (\mathsf{A}_{k-1})_{ai}\,, \qquad a, i = 1, \dots, k-1\,, \tag{4.5}$$

shows a match between the charges of the flipped operators in (3.18), and the charges of M2-branes wrapping resolution 2-cycles in (4.4).

The multiplicities of the flip fields for generic $a < b$ are reported in (3.12). Let us specialize to $b = a + 1$, and combine (3.12) with the relations (3.13) between the flavor fluxes and the resolution fluxes. We obtain

$$m^{(\beta)}_{a,a+1} = N_{{\rm N}_a}\,, \qquad m^{(\gamma)}_{a,a+1} = N_{{\rm S}_a}\,, \qquad a = 1, \dots, k-1\,. \tag{4.6}$$

We thus verify that, for pairs with $b = a + 1$, corresponding to simple roots, the multiplicities that enter $I_6^{\rm flip}$ coincide with the degeneracies given by the Landau-level argument discussed above.

M-theory origin of the flipping mechanism. Recall that the flip fields enter the 4d theory via the schematic superpotential coupling

$$W_{\rm flip} = \phi\, \mathcal{O}\,. \tag{4.7}$$

Based on the previous analysis, we identify the flipped operators $\mathcal{O}$ with the operators originating from M2-branes wrapping the resolution 2-cycles in the internal space. For generic N, the coupling (4.7) is irrelevant. It is therefore important in the UV, where its effect is to project out the operators $\mathcal{O}$. This fits with the fact that the 6d parent SCFT admits no blow-up modes for the $\mathbb{C}^2/\mathbb{Z}_k$ singularity [36–44]. In contrast, (4.7) is irrelevant in the deep IR, where the flip fields ϕ become free fields and decouple. The operators $\mathcal{O}$ are thus effectively reintroduced in the 4d theory.

5 The case of genus one

In this section, we consider several explicit examples at genus one in order to gather evidence for a series of connected claims:

- Since $I_6^{\rm v,t}$ drops out of (3.8) when $\chi = 0$, in this case the quantity $-I_6^{\rm inflow}$ provides direct access to the anomaly polynomial of the corresponding 4d SCFT,

$$-\,I_6^{\rm inflow} = I_6^{\rm SCFT}. \tag{5.1}$$

- The quantity we have called $I_6^{\rm flip}$ in (3.8) is indeed the anomaly polynomial of the flip fields that appear in the field-theoretic construction on tori with fluxes.

- The difference between $\int_{\Sigma_g} I_8^{\rm SCFT}$ and $I_6^{\rm flip}$ as given by our formula (3.10) is indeed equal to the anomaly polynomial of the *interacting* SCFT in four dimensions,

$$\int_{\Sigma_g} I_8^{\rm SCFT} - I_6^{\rm flip} = I_6^{\rm SCFT}\,. \tag{5.2}$$

Note that both the E^3 contribution to the inflow anomaly polynomial recorded in appendix A and the E_4X_8 contribution derived in appendix B are expressed in a form which only applies for higher-genus Riemann surfaces, i.e. $\chi < 0$. The genus-one result can be obtained either by first using (3.13) and subsequently fixing $\chi = 0$, or by repeating an analogous anomaly inflow computation as in [27] with cohomology class representatives chosen consistently with $\chi = 0$ from the beginning (which we describe in appendix C). We have verified that both paths produce the same result,

$$\begin{aligned}
-I_6^{\text{inflow}}(\chi=0) = &-\frac{1}{2}[kN(N-1)c_1^2(R') - kN^2c_1^2(t)]\sum_{i=1}^{k}[N_{\beta_i}c_1(\beta_i) + N_{\gamma_i}c_1(\gamma_i)] \\
&+ \frac{kN^2}{2}\sum_{i=1}^{k}[N_{\beta_i}c_1(t)\,c_1^2(\beta_i) - N_{\gamma_i}c_1(t)\,c_1^2(\gamma_i)] \\
&+ \frac{N^2(N-1)}{2}\sum_{i=1}^{k}c_1^2(\beta_i)\sum_{j=1}^{k}N_{\beta_j}c_1(\beta_j) \\
&+ \frac{N^2(N-1)}{2}\sum_{i=1}^{k}c_1^2(\gamma_i)\sum_{j=1}^{k}N_{\gamma_j}c_1(\gamma_j) \\
&+ \frac{N^2}{2}\left[\sum_{i=1}^{k}c_1^2(\beta_i)\sum_{j=1}^{k}N_{\gamma_j}c_1(\gamma_j) + \sum_{i}c_1^2(\gamma_i)\sum_{j=1}^{k}N_{\beta_j}c_1(\beta_j)\right] \\
&+ \frac{kN^3}{6}\sum_{i=1}^{k}[N_{\beta_i}c_1^3(\beta_i) + N_{\gamma_i}c_1^3(\gamma_i)] \\
&- \frac{kN}{24}\sum_{i=1}^{k}[N_{\beta_i}c_1(\beta_i) + N_{\gamma_i}c_1(\gamma_i)]\,p_1(T_4) \\
&- \sum_{1\le a<b\le k} m_{a,b}^{(\beta)}\,I_6^{\text{free}}(q_{R'}=1;\ q_t=0;\ q_{\widetilde{\beta}_i}=N\,L_{a,b}^i;\ q_{\widetilde{\gamma}_i}=0) \\
&- \sum_{1\le a<b\le k} m_{a,b}^{(\gamma)}\,I_6^{\text{free}}(q_{R'}=1;\ q_t=0;\ q_{\widetilde{\beta}_i}=0;\ q_{\widetilde{\gamma}_i}=N\,L_{a,b}^i)\,. \qquad (5.3)
\end{aligned}$$

We explore how the expression (5.3) can be used to verify the claims highlighted above in various examples.

5.1 The $Y^{p,q}$ quiver theories from inflow

Consider the case $k = 2$ for genus one. Here we find that the equation (5.3) reproduces the anomaly polynomial of the $Y^{p,q}$ quiver gauge theories, which can be engineered on a stack of D3-branes at the tip of the Calabi-Yau cone over $Y^{p,q}$ [47].

The $Y^{p,q}$ are an infinite family of Sasaki-Einstein manifolds labeled by positive integers p and q with $0 \le q \le p$ [25]. The holographic duals of the corresponding $\text{AdS}_5 \times Y^{p,q}$ solutions in Type IIB string theory were constructed in [47], using an iterative procedure on the quiver for $Y^{p,p}$. The field content of this family of quiver gauge theories is summarized in table 1. The quiver associated with $Y^{p,q}$ has $2p$ gauge groups, represented diagrammatically by $2p$ nodes. All fields are in either a spin-0 or spin-1/2 representation of a global SU(2) symmetry. There are two additional global U(1)'s, labeled here as $\text{U}(1)_B$ and $\text{U}(1)_F$.

	Degeneracy	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_F$	$\mathrm{U}(1)_\varphi$	$\mathrm{U}(1)_{R_0}$
Y singlets	$p+q$	$p-q$	-1	0	1
Z singlets	$p-q$	$p+q$	1	0	0
U doublets	p	$-p$	0	$\pm 1/2$	$1/2$
V doublets	q	q	1	$\pm 1/2$	$1/2$

Table 1. Field content for the infinite family of $Y^{p,q}$ quiver gauge theories. See, e.g. [47].

The anomaly polynomial for general $Y^{p,q}$ quiver gauge theories can be computed directly from the field content and associated degeneracies, and is given by

$$\begin{aligned}
I_6^{\mathrm{SCFT}}(Y_{pq}) = &\frac{N^2}{8}(p+q)\,c_1(R_0)^3 - \frac{1}{12}\left[4c_1(R_0)^2 - p\,p_1(TW_4)\right]c_1(R_0) \\
&+ \frac{N^2}{4}(2p-q)\,c_1(R_0)^2 c_1(F) + \frac{N^2}{4}(p^2-q^2)\,c_1(R_0)^2 c_1(B) \\
&- \frac{N^2}{8}(p+q)\,c_1(\varphi)^2 c_1(R_0) + \frac{N^2 q}{4}\,c_1(\varphi)^2 c_1(F) - \frac{N^2}{4}(p^2-q^2)\,c_1(\varphi)^2 c_1(B) \\
&- \frac{N^2 p}{2}\left[c_1(F) + p\,c_1(B)\right]^2 c_1(R_0) - \frac{N^2 p}{2}(p^2+pq-q^2)\,c_1(B)^2 c_1(R_0) \\
&+ N^2 p^2\left[c_1(F) + q\,c_1(B)\right] c_1(B) c_1(F)\,.
\end{aligned} \tag{5.4}$$

In virtue of a chain of dualities connecting the GMSW solution in 11d supergravity to the $\mathrm{AdS}_5 \times Y^{p,q}$ solutions in Type IIB, the internal manifold M_6 in the corresponding inflow setup is defined by $k=2$ and $\chi=0$. The quantity $-I_6^{\mathrm{inflow}}$ can be obtained for example from (5.3),

$$\begin{aligned}
&-I_6^{\mathrm{inflow}}(k=2,\chi=0) = \\
&\quad 2N^2\left[N_{\widetilde\beta_1} c_1(\widetilde\beta_1) + N_{\widetilde\gamma_1} c_1(\widetilde\gamma_1)\right]\left(c_1(t)^2 - c_1(R')^2\right) \\
&\quad - 4N^2\left[N_{\widetilde\beta_1} c_1(\widetilde\beta_1)^2 + N_{\widetilde\gamma_1} c_1(\widetilde\gamma_1)^2\right] c_1(R') + 2N^2\left[N_{\widetilde\beta_1} c_1(\widetilde\beta_1) c_1(\widetilde\gamma_1)^2 + N_{\widetilde\gamma_1} c_1(\widetilde\gamma_1) c_1(\widetilde\beta_1)^2\right] \\
&\quad - 2N^2\left[N_{\widetilde\beta_1} c_1(\widetilde\beta_1)^3 + N_{\widetilde\gamma_1} c_1(\widetilde\gamma_1)^3\right] - \frac{1}{12}(N_{\widetilde\beta_1} + N_{\widetilde\gamma_1})\left[4c_1(R')^2 - p_1(TW_4)\right] c_1(R').
\end{aligned} \tag{5.5}$$

Under the field strength redefinitions,

$$\begin{aligned}
c_1(\widetilde\beta_1) &= \frac{1}{2}\,c_1(F) - \frac{1}{2}(p-q)\,c_1(B) - \frac{1}{2}\,c_1(R_0)\,, & c_1(\widetilde\gamma_1) &= -\frac{1}{2}\,c_1(F) - \frac{1}{2}(p+q)\,c_1(B)\,, \\
c_1(R') &= c_1(R_0)\,, & c_1(t) &= -\frac{1}{k}\,c_1(\varphi)\,,
\end{aligned}$$

and the identifications

$$N_{\widetilde\beta_1} = 2(p+q)\,, \qquad N_{\widetilde\gamma_1} = 2(p-q) \tag{5.6}$$

between the integers p, q and the resolution flux quanta $N_{\widetilde\beta_1}$, $N_{\widetilde\gamma_1}$, we verify an exact match, $I_6^{\mathrm{SCFT}}(Y_{pq}) = -I_6^{\mathrm{inflow}}$. This match supports our claim (5.1) that the topological and geometric data of M_6 fully characterize the anomaly polynomial of the corresponding (genus-one) 4d SCFT.

5.2 More quiver theories and flip fields at genus one

Next we revisit some explicit examples at genus one reported in [12] in order to provide further evidence for the interpretation of I_6^{flip} in (3.8) and the equality (5.2). In these examples, we consider N to be generic, but make the implicit assumption that N is large enough to ensure that all the couplings between flip fields and baryons are irrelevant. It would be interesting to consider in greater detail low values of N, but we refrain from such analysis in this work.

Example no. 1. Consider the gauge theory with the quiver depicted in figure 4 of [12], reproduced for convenience as quiver (a) in figure 4. This theory corresponds to $k = 2$. In our notation, the flavor fluxes are

$$N_{\widetilde{\beta}_1} = 1\,, \qquad N_{\widetilde{\gamma}_1} = 0\,. \tag{5.7}$$

From the quiver one extracts the charges of all fields. These charges are: the charge $\widehat{q}_R$ under a convenient reference R-symmetry (given in the caption of figure 4); the charges $\widehat{q}_t$, $\widehat{q}_\beta$, $\widehat{q}_\gamma$ extracted from the exponents of the t, β, γ fugacities in the quiver. In order to compare the quiver data with our formulae, we need the map between the charges $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_\beta; \widehat{q}_\gamma)$ of the quiver description, and the charges $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\gamma}_1})$ used in this work,

$$\widehat{q}_R = q_{R'} - \frac{1}{3}\, q_{\widetilde{\beta}_1}\,, \qquad \widehat{q}_t = q_t\,, \qquad \widehat{q}_\beta = -q_{\widetilde{\beta}_1}\,, \qquad \widehat{q}_\gamma = -q_{\widetilde{\gamma}_1}\,. \tag{5.8}$$

The last two relations simply state how to relate our conventions for the flavor charges to those of [12]. The first relation encodes how the reference R-symmetry in the quiver compares to the reference R-symmetry R' from six dimensions.

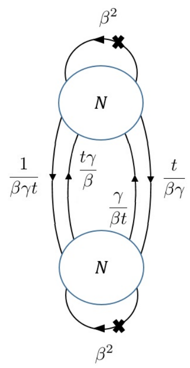
According to our general formula (3.10), we have one species of flip field with charges $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\gamma}_1}) = (1; 0; 2N; 0)$ and multiplicity 2, as computed from (3.12) using (5.7) (with labels $a = 1$, $b = 2$). Making use of the dictionary (5.8), the charges in the notation of the quiver are $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_\beta; \widehat{q}_\gamma) = (1 - \frac{2}{3}N; 0; -2N; 0)$. These are indeed the correct charges for the (fermions in the chiral) fields that flip the baryons of the adjoints that carry an "X" in the quiver diagram.

Finally, using again (5.7) and (5.8), one can verify (5.2), where the anomaly of the SCFT is extracted from the quiver, simply ignoring the flip fields. (The gauge singlets associated with the arrows that connect a node to itself are kept, because they participate in relevant superpotential couplings.) Note that due to the restriction (5.7) on the flavor fluxes, there is a symmetry enhancement of the $\text{U}(1)_{\widetilde{\gamma}_1}$ to an $\text{SU}(2)$. This can be seen at the level of the anomaly polynomial in that $c_1(\widetilde{\gamma}_1)$ enters only quadratically, via the term

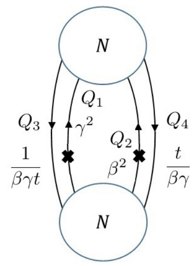
$$I_6^{\text{SCFT}} \supset 2N^2 c_1(\widetilde{\beta}_1) c_1(\widetilde{\gamma}_1)^2. \tag{5.9}$$

Example no. 2. This example is the gauge theory with quiver depicted in figure 5 of [12], reproduced for convenience as quiver (b) in figure 4. This theory also corresponds to $k = 2$. In our notation, the flavor fluxes are

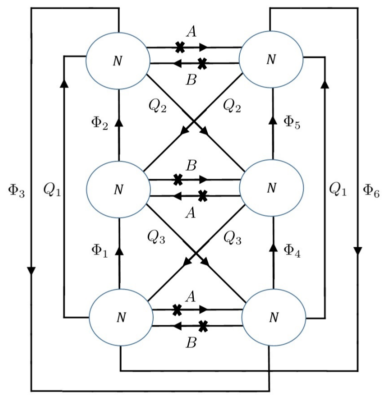
$$N_{\widetilde{\beta}_1} = \frac{1}{2}\,, \qquad N_{\widetilde{\gamma}_1} = \frac{1}{2}\,. \tag{5.10}$$



(a)

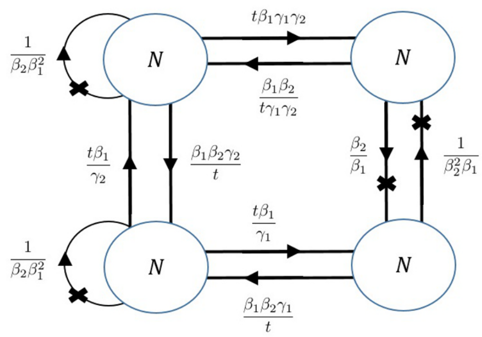


(b)



A	$\frac{\beta_2}{\beta_1}$
B	$\frac{1}{\beta_2\beta_1^2}$
Φ_1	$\frac{\beta_1\beta_2\gamma_2}{t}$
Φ_2	$\frac{\gamma_1}{t\beta_2}$
Φ_3	$\frac{\beta_1\beta_2}{t\gamma_1\gamma_2}$
Φ_4	$\frac{\gamma_2}{t\beta_2}$
Φ_5	$\frac{\beta_1\beta_2\gamma_1}{t}$
Φ_6	$\frac{1}{t\beta_2\gamma_1\gamma_2}$
Q_1	$t\beta_1\gamma_1\gamma_2$
Q_2	$\frac{t\beta_1}{\gamma_1}$
Q_3	$\frac{t\beta_1}{\gamma_2}$

(c)



(d)

Figure 4. Explicit examples of quiver gauge theories of class $\mathcal{S}_k$ realized on a torus with fluxes, taken from [12]. The gauge nodes are $\mathrm{SU}(N)$ groups. An arrow connecting two distinct nodes is a bifundamental chiral multiplet. An arrow connecting a node to itself represents an adjoint-plus-singlet chiral multiplet. An "X" on an arrow signals that the baryon associated with that field is flipped. The charges under the global, non-R-symmetries are given in terms of fugacities. The reference R-symmetry charges for the various quivers are as follows. In quiver (a), all fields (except the flip fields) have R-charge 2/3. In quiver (b), all fields (except the flip fields) have R-charge 1/2. In quiver (c), all fields (except the flip fields) have R-charge 2/3. In quiver (d), the bifundamentals have R-charge 1/2 and the adjoint-plus-singlet's have R-charge 1. For a description of the superpotential of these models, we refer the reader to [12].

Once again, we need the dictionary between the charges $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_\beta; \widehat{q}_\gamma)$ of the quiver description, and the charges $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\gamma}_1})$ used in this work,

$$\widehat{q}_R = q_{R'} - \frac{1}{4} q_{\widetilde{\beta}_1} - \frac{1}{4} q_{\widetilde{\beta}_2} \,, \qquad \widehat{q}_t = q_t \,, \qquad \widehat{q}_\beta = -q_{\widetilde{\beta}_1} \,, \qquad \widehat{q}_\gamma = -q_{\widetilde{\gamma}_1} \,. \tag{5.11}$$

Our formula (3.10) predicts one species of flip fields with charges $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\gamma}_1}) = (1; 0; 2N; 0)$, and one species with charges $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\gamma}_1}) = (1; 0; 0; 2N)$. According to (3.12) and (5.10), both species have multiplicity 1. In the notation of the quiver, the charges are $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_\beta; \widehat{q}_\gamma) = (1 - \frac{1}{2}N; 0; -2N; 0)$ and $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_\beta; \widehat{q}_\gamma) = (1 - \frac{1}{2}N; 0; 0; -2N)$, which indeed match with the charges of the (fermions in the chiral) fields that flip the baryons of the fields marked with an "X" in the quiver. We can verify (5.2) in this example as well, making use of (5.10) and (5.11).

Example no. 3. This example is the gauge theory with quiver depicted in figure 6 of [12], reproduced for convenience as quiver (c) in figure 4. This theory has $k = 3$. In our notation, the flavor fluxes are

$$N_{\widetilde{\beta}_1} = 2\,, \qquad N_{\widetilde{\beta}_2} = 1\,, \qquad N_{\widetilde{\gamma}_1} = 0\,, \qquad N_{\widetilde{\gamma}_2} = 0\,. \tag{5.12}$$

Let $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_{\beta_1}; \widehat{q}_{\beta_2}; \widehat{q}_{\gamma_1}; \widehat{q}_{\gamma_2})$ denote the charges extracted from the quiver (and its adjacent table). They are related to the charges $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\beta}_2}; q_{\widetilde{\gamma}_1}; q_{\widetilde{\gamma}_2})$ used in this work via

$$\widehat{q}_R = q_{R'} - \frac{4}{9}\,q_{\widetilde{\beta}_1} - \frac{2}{9}\,q_{\widetilde{\beta}_2} + \frac{1}{9}\,q_t\,, \quad \widehat{q}_t = q_t\,, \quad \begin{array}{l} \widehat{q}_{\beta_1} = q_{\widetilde{\beta}_1} + q_{\widetilde{\beta}_2} \\ \widehat{q}_{\beta_2} = q_{\widetilde{\beta}_2} \end{array}, \quad \begin{array}{l} \widehat{q}_{\gamma_1} = q_{\widetilde{\gamma}_1} + q_{\widetilde{\gamma}_2} \\ \widehat{q}_{\gamma_2} = q_{\widetilde{\gamma}_2} \end{array}. \tag{5.13}$$

As in the previous examples, using (5.12) and (5.13) we can verify that the charges and multiplicities of flip fields given by our formulae match with the quiver data. We can also verify (5.2), where the SCFT anomaly is computed from the quiver, simply ignoring all flip fields.

Example no. 4. This example is the gauge theory with quiver depicted in figure 7 of [12], reproduced for convenience as quiver (d) in figure 4. This theory has $k = 3$. In our notation, the flavor fluxes are

$$N_{\widetilde{\beta}_1} = 1\,, \qquad N_{\widetilde{\beta}_2} = 1\,, \qquad N_{\widetilde{\gamma}_1} = 0\,, \qquad N_{\widetilde{\gamma}_2} = 0\,. \tag{5.14}$$

The dictionary between $(\widehat{q}_R; \widehat{q}_t; \widehat{q}_{\beta_1}; \widehat{q}_{\beta_2}; \widehat{q}_{\gamma_1}; \widehat{q}_{\gamma_2})$ and $(q_{R'}; q_t; q_{\widetilde{\beta}_1}; q_{\widetilde{\beta}_2}; q_{\widetilde{\gamma}_1}; q_{\widetilde{\gamma}_2})$ in this example is

$$\widehat{q}_R = q_{R'} - \frac{1}{2}\,q_{\widetilde{\beta}_1} - \frac{1}{2}\,q_{\widetilde{\beta}_2}\,, \quad \widehat{q}_t = q_t\,, \quad \begin{array}{l} \widehat{q}_{\beta_1} = q_{\widetilde{\beta}_1} + q_{\widetilde{\beta}_2} \\ \widehat{q}_{\beta_2} = q_{\widetilde{\beta}_2} \end{array}, \quad \begin{array}{l} \widehat{q}_{\gamma_1} = q_{\widetilde{\gamma}_1} + q_{\widetilde{\gamma}_2} \\ \widehat{q}_{\gamma_2} = q_{\widetilde{\gamma}_2} \end{array}. \tag{5.15}$$

Once again, we have a match of charges and multiplicities of flip fields, and (5.2) can be verified.

6 Central charges

We continue our study of the putative 4d $\mathcal{N} = 1$ SCFTs of interest in this paper by computing their central charges using a-maximization [45]. In this section we will first describe the computational complexity of this a-maximization problem. We then study several important properties of the resulting central charges. Some exact results are presented for a special family of G_4-flux configurations with sufficiently few independent parameters that the maximization problem can be solved analytically. Finally, we treat the general case with a perturbative analysis in the regime where the ratios $N_{\mathrm{N}_i}/|\chi|N$ and $N_{\mathrm{S}_i}/|\chi|N$ are small. Most of the discussion in this section implicitly assumes that we are working with a $g \geq 2$ Riemann surface, unless otherwise specified.

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6.1 Computational setup

The anomaly polynomial of a 4d $\mathcal{N}=1$ SCFT contains

$$I_6^{\rm SCFT} \supset \frac{{\rm tr}R^3}{6}\left(\frac{F_2^R}{2\pi}\right)^3 - \frac{{\rm tr}R}{24}\frac{F_2^R}{2\pi}p_1(TW_4)\,, \tag{6.1}$$

where R is the generator of the superconformal R-symmetry and F_2^R its background field strength, and $p_1(TW_4)$ is the first Pontryagin class of the 4d worldvolume W_4. Its central charges are [48]

$$a = \frac{3}{32}(3{\rm tr}R^3 - {\rm tr}R)\,, \quad c = \frac{1}{32}(9{\rm tr}R^3 - 5{\rm tr}R)\,. \tag{6.2}$$

In the presence of additional flavor symmetries, one may also compute the associated flavor central charges [49],

$$b_G\delta^{ab} = -6{\rm tr}(R\,T_G^a\,T_G^b)\,, \tag{6.3}$$

where T_G^a are generators of the flavor symmetry G, with the normalization ${\rm tr}(T_G^a\,T_G^b) = \delta^{ab}/2$ in the fundamental representation.

As described before, we can access $I_6^{\rm SCFT}$ through either of the equalities in (1.1), repeated here for convenience,

$$I_6^{\rm SCFT} = -I_6^{\rm inflow} - I_6^{\rm v,t} = \int_{\Sigma_g} I_8^{\rm SCFT} - I_6^{\rm flip}\,. \tag{6.4}$$

For concreteness, suppose we restrict our attention to the first equality above, we can define a trial R-symmetry as

$$R_{\rm trial} = 2T_\psi + q^\varphi T_\varphi + \sum_{i=1}^{k-1}(q^{{\rm N}_i}T_{{\rm N}_i} + q^{{\rm S}_i}T_{{\rm S}_i})\,. \tag{6.5}$$

The various T_G are the generators of the $\mathrm{U}(1)_G$ flavor symmetries, and the factor of 2 in front of T_ψ is inserted to ensure it has the appropriately normalized R-charge [34]. This is equivalent to carrying out the following replacements in the anomaly polynomial(s),

$$F_2^\psi \to 2F_2^R\,, \quad F_2^\varphi \to F_2^\varphi + q^\varphi F_2^R\,, \quad F_2^{{\rm N}_i} \to F_2^{{\rm N}_i} + q^{{\rm N}_i}F_2^R\,, \quad F_2^{{\rm S}_i} \to F_2^{{\rm S}_i} + q^{{\rm S}_i}F_2^R\,, \tag{6.6}$$

from which we can construct a trial central charge,

$$a_{\rm trial}(\{q^G\}) = \frac{27}{16}\mathcal{A}_{RRR} + \frac{9}{4}\mathcal{A}_R\,, \tag{6.7}$$

where $\mathcal{A}_{RRR}$ and $\mathcal{A}_R$ are respectively the coefficients of $(F_2^R/2\pi)^3$ and $F_2^R/2\pi$ in the polynomial $I_6^{\rm SCFT}(\{q^G\})$. The central charge a corresponds to the local maximum of $a_{\rm trial}(\{q^G\})$ with respect to the $2k-1$ real parameters q^G. Such a solution is hereafter denoted by $\{q^G_{\rm max}\}$. Up to an overall normalization, the flavor central charge b_G for a given flavor symmetry G can be extracted by plugging $\{q^G_{\rm max}\}$ into $I_6^{\rm SCFT}(\{q^G\})$, and then reading off the coefficient $\mathcal{A}_{RGG}$ in front of $(F_2^R/2\pi)(F_2^G/2\pi)^2$. We can also similarly find c.

Analytically searching for a local maximum of $a_{\rm trial}(\{q^G\})$ amounts to performing a two-step process: first solving the quadratic equations $\partial a_{\rm trial}/\partial q^G = 0$ simultaneously

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for all G, and then identifying a solution $\{q^G_{\max}\}$ that yields a negative-definite Hessian matrix $[\partial^2 a_{\text{trial}}/\partial q^{G_i}\partial q^{G_j}]_{\{q^G\}=\{q^G_{\max}\}}$. As far as the former is concerned, Bezout's theorem asserts that $a_{\text{trial}}(\{q^G\})$ has at most 2^{2k-1} (or infinitely many) critical points. Hence, the computational complexity of the problem scales roughly as 4^k at large k. This presents a formidable challenge even for state-of-the-art algebraic solvers, in which case we have to resort to numerical methods. In fact, the only scenario where we are able to analytically solve the a-maximization problem for generic combinations of χ, N, N_{N_i}, N_{S_i} for $i = 1, \ldots, k-1$ is $k = 2$, in which we are able to fully reproduce the result of [26]. As we will discuss later in this section, there exists a family of theories with certain special flux configurations $\{N_{\text{N}_i}, N_{\text{S}_i}\}$ that significantly reduce the number of independent extremization parameters q^G, thus rendering the a-maximization problem analytically solvable for $k > 2$ as well.

6.2 Salient properties of the central charge

The central charge a thus obtained for the SCFTs via a-maximization exhibits a number of crucial properties which we collect in this subsection.

Uniqueness. For a given choice of k, χ, N, N_{N_i}, N_{S_i}, if the central charge a exists then it is necessarily unique. The proof of this statement proceeds as follows. Given the replacements made in (6.6), the trial central charge $a_{\text{trial}}(\{q^G\})$ defined in (6.7) is a cubic polynomial of the $2k-1$ variables q^G where $G = \varphi, \text{N}_i, \text{S}_i$ for $i = 1, \ldots, k-1$. Suppose a local maximum of a_{trial} exists at some point $\{q^G_{\max}\} \in \mathbb{R}^{2k-1}$. We can project the $(2k-1)$-dimensional vector $\vec{q} = (q^{G_1}, \ldots, q^{G_{2k-1}})$ onto an arbitrary line $\mathbb{R}_p \subset \mathbb{R}^{2k-1}$ passing through $\{q^G_{\max}\}$. Restricted to this line, the trial central charge becomes a univariate cubic polynomial $a_{\text{trial}}(q_p) = a_{\text{trial}}(\{q^G\})|_{\mathbb{R}_p}$ where $\vec{q}_p = \text{proj}_{\mathbb{R}_p}\vec{q}$.

A univariate cubic polynomial admits at most one local maximum, so by construction $\{q^G_{\max}\}$ is the unique local maximum of $a_{\text{trial}}(q_p)$ along any $\mathbb{R}_p$. Since $\mathbb{R}_p$ can be chosen to connect $\{q^G_{\max}\}$ to any other point in the parameter space, along which $\{q^G_{\max}\}$ is always the unique local maximum, we conclude that a_{trial} has at most one local maximum at $\{q^G\} = \{q^G_{\max}\}$.[3]

Large-N scaling relation. Recall that the central charges of the SCFT are encoded by the anomaly polynomial $I_6^{\text{SCFT}} = -I_6^{\text{inflow}} - I_6^{\text{v,t}}$. The former can be further decomposed into an $\mathcal{O}(N^3, N^3_{\text{N}_i,\text{S}_i})$ contribution $I_6^{\text{inflow},E_4^3}$ and an $\mathcal{O}(N, N_{\text{N}_i,\text{S}_i})$ contribution $I_6^{\text{inflow},E_4 X_8}$. Although $I_6^{\text{v,t}}$ is of $\mathcal{O}(N^3)$ according to (3.9), we have checked that it always contribute only at $\mathcal{O}(N)$ to the central charge a. Hence in the large-$N, N_{\text{N}_i,\text{S}_i}$ limit (or loosely, the large-N limit), a can be effectively determined by performing a-maximization directly on $-I_6^{\text{inflow},E_4^3}$.

We observe that the large-N inflow anomaly polynomial $I_6^{\text{inflow},E_4^3}$, constructed in [27] and reviewed in appendix A, satisfies an interesting identity. Suppose we have two distinct setups with generally different Euler characteristics, χ_A, χ_B, of the $(g \geq 2)$ Riemann

[3] To the best of our knowledge, this result concerning the number of local maxima/minima of a multivariate cubic polynomial was first explicitly proven by [50].

surfaces, and they are wrapped by different numbers, N_A, N_B, of M5-branes, then the corresponding large-N inflow anomaly polynomials follow

$$\begin{aligned}&I_6^{\text{inflow},E_4^3}(k,\chi_A,N_A,\{N_{\mathrm{N}_i}\},\{N_{\mathrm{S}_i}\},F_2^\psi,F_2^\varphi,\{F_2^{\mathrm{N}_i}\},\{F_2^{\mathrm{S}_i}\})\\&=\frac{\chi_A N_A^3}{\chi_B N_B^3}I_6^{\text{inflow},E_4^3}\left(k,\chi_B,N_B,\left\{\frac{\chi_B N_B}{\chi_A N_A}N_{\mathrm{N}_i}\right\},\left\{\frac{\chi_B N_B}{\chi_A N_A}N_{\mathrm{S}_i}\right\},F_2^\psi,F_2^\varphi,\{F_2^{\mathrm{N}_i}\},\{F_2^{\mathrm{S}_i}\}\right).\end{aligned} \tag{6.8}$$

Consequently, we have an analogous scaling relation for the trial central charge,

$$\begin{aligned}&a_{\text{trial}}(k,\chi_A,N_A,\{N_{\mathrm{N}_i}\},\{N_{\mathrm{S}_i}\},q^\varphi,\{q^{\mathrm{N}_i}\},\{q^{\mathrm{S}_i}\})\\&=\frac{\chi_A N_A^3}{\chi_B N_B^3}a_{\text{trial}}\left(k,\chi_B,N_B,\left\{\frac{\chi_B N_B}{\chi_A N_A}N_{\mathrm{N}_i}\right\},\left\{\frac{\chi_B N_B}{\chi_A N_A}N_{\mathrm{S}_i}\right\},q^\varphi,\{q^{\mathrm{N}_i}\},\{q^{\mathrm{S}_i}\}\right).\end{aligned} \tag{6.9}$$

This motivates the definitions of "reduced flux quanta,"

$$n_{\mathrm{N}_i}=\frac{N_{\mathrm{N}_i}}{|\chi|N}\,,\qquad n_{\mathrm{S}_i}=\frac{N_{\mathrm{S}_i}}{|\chi|N}\,, \tag{6.10}$$

such that the central charge, which is the (unique) local maximum of a_{trial}, obeys

$$\frac{1}{\chi_A N_A^3}a(k,\chi_A,N_A,\{n_{\mathrm{N}_i}\},\{n_{\mathrm{S}_i}\})=\frac{1}{\chi_B N_B^3}a(k,\chi_B,N_B,\{n_{\mathrm{N}_i}\},\{n_{\mathrm{S}_i}\}) \tag{6.11}$$

in the large-N limit. It follows from (6.8) that the same scaling relation applies to the other central charges c and b_G as well.

Existence and flux positivity. As alluded to earlier, a central charge does not necessarily exist for an arbitrary combination of k, χ, N, N_{N_i}, N_{S_i}. This is possible because while Bezout's theorem states that there can be at most 2^{2k-1} critical points of $a_{\text{trial}}(\{q^G\})$, they may be all saddle points (possibly with a local minimum swapped in). Moreover, given a choice of orientation for the internal space M_6, we define the charge (or more precisely the number of M5-branes) N to be positive under this orientation. Similarly, we can define the rest of the flux quanta using the same choice of orientation for the relevant cycles.[4] For a supersymmetric theory, we expect all of these flux quanta to have the same sign as N. It is therefore important for us to examine the range of parameters within which our construction admits a central charge.

In figure 5 we illustrate the range of existence of the central charge a for a variety of specific configurations of N_{N_i} and N_{S_i}, given fixed k, χ, N. For $k=2$ there is only one trivial pair of axes, i.e. N_{N_1} and N_{S_1}, but for $k=3$ there are four independent flux quanta, so we show here several representative 2d cross-sections in the (discrete) space of flux quanta. In all cases considered we observe a clear-cut boundary separating the regions of the flux lattice with or without a central charge. We also note that a always exists when the flux quanta are strictly nonnegative; the red "exclusion region" always lies in the

[4] Alternatively, flipping the orientation for M_6 amounts to flipping the signs of all the flux quanta, including N.

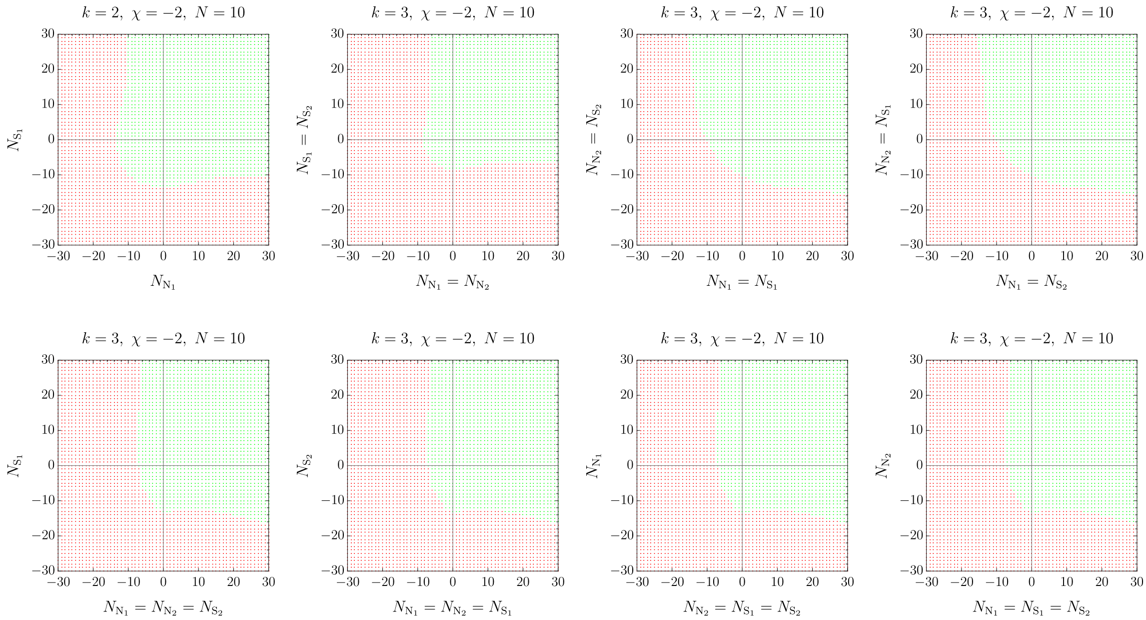

Figure 5. 2d exclusion plots visualizing the existence/nonexistence of the central charge a for $k = 2$ and $k = 3$. The choices of k, χ, N are labeled on top of each plot, whereas the specific configuration of the (integral) flux quanta, N_{N_i}, N_{S_i}, can be read off from the coordinates of a given dot. A dot is green if a exists for the corresponding combination of flux quanta; it is red if a does not exist. Note that the upper four plots are symmetric with respect to the exchange of axes as a consequence of the D_2 symmetry of M_4.

three quadrants with at least one negative flux quantum. In fact, we find that the same qualitative feature applies to general k: setups characterized by strictly nonnegative flux configurations have (unique) central charges.

Furthermore, we can deduce from the large-N scaling relation (6.11) that as we decrease $|\chi|$ or N, the boundary of the exclusion region retreats towards the two positive axes but never crosses them. This is because the ratio $\chi_B N_B/\chi_A N_A$ cannot change sign as long as we keep $g \geq 2$. In other words, the central charge is guaranteed to exist everywhere in the first quadrant (where all flux quanta are positive) once the signs of χ and N are fixed, thus conforming to our expectation that all fluxes in SCFTs should be of uniform sign.

For the case of $k = 2$ studied in [26], we can see that the relative signs of fluxes are fixed directly in the holographic dual, namely, the GMSW solution [25]. If explicit holographic duals are found for $k > 2$ in the future, we anticipate the same uniform-flux-sign condition to hold.

Dihedral symmetry. It was noted in [27] that the inflow anomaly polynomial is invariant (up to a sign) under the two parity transformations of M_4. Unsurprisingly, this nice property carries over to the central charge. Let us first consider the action of a north-south involution, the trial central charge $a_{\text{trial}}(\{N_{\mathrm{N}_i}\}, \{N_{\mathrm{S}_i}\}, q^\varphi, \{q^{\mathrm{N}_i}\}, \{q^{\mathrm{S}_i}\})$ is invariant under

$$N_{\mathrm{N}_i} \leftrightarrow N_{\mathrm{S}_i}\,, \quad q^\varphi \to q^\varphi\,, \quad q^{\mathrm{N}_i} \leftrightarrow -q^{\mathrm{S}_i} \tag{6.12}$$

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for $i = 1, \ldots, k-1$. Similarly, if we consider the action of an east-west involution, a_{trial} is invariant under

$$N_{\mathrm{N}_i} \leftrightarrow N_{\mathrm{N}_{k-i}}\,, \quad N_{\mathrm{S}_i} \leftrightarrow N_{\mathrm{S}_{k-i}}\,, \quad q^{\varphi} \to -q^{\varphi}\,, \quad q^{\mathrm{N}_i} \leftrightarrow q^{\mathrm{N}_{k-i}}\,, \quad q^{\mathrm{S}_i} \leftrightarrow q^{\mathrm{S}_{k-i}}\,. \tag{6.13}$$

Omitting the explicit functional dependence on k, χ, and N, we see that the central charge exhibits a dihedral symmetry,

$$a(\{N_{\mathrm{N}_i}\},\{N_{\mathrm{S}_i}\}) = a(\{N_{\mathrm{S}_i}\},\{N_{\mathrm{N}_i}\}) = a(\{N_{\mathrm{N}_{k-i}}\},\{N_{\mathrm{S}_{k-i}}\}) = a(\{N_{\mathrm{S}_{k-i}}\},\{N_{\mathrm{N}_{k-i}}\})\,, \tag{6.14}$$

where the arguments should be understood to be ordered.

When fluxes are configured symmetrically, we can further use this symmetry of a_{trial} to place powerful constraints on the values of the parameters q^G_{max} that determine the exact R-symmetry. For example, consider a flux configuration with $N_{\mathrm{N}_i} = N_{\mathrm{S}_i}$ for all i, (6.12) implies that local maxima of a_{trial} should exist in pairs distributed symmetrically across the fixed locus of this involution in the $(2k-1)$-dimensional parameter space, i.e. if a local maximum is located at some point $(q^{\varphi}_{\text{max}}, \{q^{\mathrm{N}_i}_{\text{max}}\}, \{q^{\mathrm{S}_i}_{\text{max}}\})$, then there must also be another local maximum at $(q^{\varphi}_{\text{max}}, \{-q^{\mathrm{S}_i}_{\text{max}}\}, \{-q^{\mathrm{N}_i}_{\text{max}}\})$. However, since a_{trial} has at most one local maximum, all (pairs of) local maxima must coincide and lie on the fixed locus $q^{\mathrm{N}_i} = -q^{\mathrm{S}_i}$ of the involution (6.12). Applying the same reasoning to (6.13), we conclude that for any choice of k, χ, N,

1. $N_{\mathrm{N}_i} = N_{\mathrm{S}_i} \Rightarrow q^{\mathrm{N}_i}_{\text{max}} = -q^{\mathrm{S}_i}_{\text{max}}\,;$
2. $N_{\mathrm{N}_i} = N_{\mathrm{N}_{k-i}}\,,\, N_{\mathrm{S}_i} = N_{\mathrm{S}_{k-i}} \Rightarrow q^{\varphi}_{\text{max}} = 0\,,\, q^{\mathrm{N}_i}_{\text{max}} = q^{\mathrm{N}_{k-i}}_{\text{max}}\,,\, q^{\mathrm{S}_i}_{\text{max}} = q^{\mathrm{S}_{k-i}}_{\text{max}}\,;$
3. $N_{\mathrm{N}_i} = N_{\mathrm{N}_{k-i}} = N_{\mathrm{S}_i} = N_{\mathrm{S}_{k-i}} \Rightarrow q^{\varphi}_{\text{max}} = 0\,,\, q^{\mathrm{N}_i}_{\text{max}} = q^{\mathrm{N}_{k-i}}_{\text{max}} = -q^{\mathrm{S}_i}_{\text{max}} = -q^{\mathrm{S}_{k-i}}_{\text{max}}\,.$

Note that $k = 2$ setups are described by just two resolution flux quanta N_{N_1} and N_{S_1}, so they automatically fall under the second family. The invariance of a_{trial} under (6.13) then fixes $q^{\varphi}_{\text{max}} = 0$. Indeed, this is consistent with the fact that for $k = 2$ the $\mathrm{U}(1)_{\varphi}$ flavor symmetry is enhanced to $\mathrm{SU}(2)_{\varphi}$, whose non-abelian nature prohibits it from mixing with the R-symmetry, as argued in [45]. For general k, the last family of flux configurations listed above is of the most interest because the central charge can be much more efficiently determined through a modified a-maximization problem. Instead of (6.5) the trial R-symmetry for such flux configurations can be expressed as

$$R_{\text{trial}} = 2T_{\psi} + \sum_{i=1}^{\lceil (k-1)/2 \rceil} q^{\mathrm{N}_i} (T_{\mathrm{N}_i} + T_{\mathrm{N}_{k-i}} - T_{\mathrm{S}_i} - T_{\mathrm{S}_{k-i}})\,. \tag{6.15}$$

In this way, the dimension of the parameter space is reduced from $2k-1$ to $\lceil (k-1)/2 \rceil$, i.e. there are roughly a factor of four fewer degrees of freedom. We refer to such setups as bisymmetric flux configurations. In the next subsection we will study a special case of bisymmetric flux configurations in which all flux quanta N_{N_i} and N_{S_i} are equal.

6.3 Exact results for uniform flux configurations

The a-maximization problem is much more tractable for uniform flux configurations, that is, $N_{\mathrm{N}_i} = N_{\mathrm{S}_i} := N_{\mathrm{N}}$ for $i = 1, 2, \ldots, k-1$, than for arbitrary configurations. It is indeed exactly solvable for sufficiently small k. As described earlier, finding the central charges for $k = 2$ and $k = 3$ with uniform flux configurations is effectively a 1d maximization problem, and in both cases the various central charges can be written as reasonably compact closed-form expressions. For $k = 2$, we get

$$a = \frac{\left[\chi^2(9N^2-8)-18N_{\mathrm{N}}(\chi N-2N_{\mathrm{N}})\right]^{3/2}-9\chi^2 N_{\mathrm{N}}(9N^2-6)+54N_{\mathrm{N}}^2(3\chi N-4N_{\mathrm{N}})}{48\chi^2}, \tag{6.16}$$

$$\begin{aligned} c = &\frac{\left[\chi^2(9N^2-8)-18N_{\mathrm{N}}(\chi N-2N_{\mathrm{N}})\right]^{3/2}-9\chi^2 N_{\mathrm{N}}(9N^2-6)+54N_{\mathrm{N}}^2(3\chi N-4N_{\mathrm{N}})}{48\chi^2} \\ &+\frac{\sqrt{\chi^2(9N^2-8)-18N_{\mathrm{N}}(\chi N-2N_{\mathrm{N}})}}{48}, \end{aligned} \tag{6.17}$$

$$\mathcal{A}_{R\varphi\varphi} = \frac{\chi^2 N(N^2-1)+NN_{\mathrm{N}}\left[3(\chi N-2N_{\mathrm{N}})+\sqrt{\chi^2(9N^2-8)-18N_{\mathrm{N}}(\chi N-2N_{\mathrm{N}})}\right]}{12\chi}, \tag{6.18}$$

$$\mathcal{A}_{Rii} = \frac{N(\chi N-N_{\mathrm{N}})\sqrt{\chi^2(9N^2-8)-18N_{\mathrm{N}}(\chi N-2N_{\mathrm{N}})}-3NN_{\mathrm{N}}(\chi N-2N_{\mathrm{N}})}{3\chi}, \tag{6.19}$$

whereas for $k = 3$, we have

$$a = \frac{9\chi^2(N-1)^2(27N^2+27N-14)-6\chi N_{\mathrm{N}}(135N^3-253N+118)+12N_{\mathrm{N}}^2(81N^2-80)}{64\left[10N_{\mathrm{N}}-3\chi(N-1)\right]}, \tag{6.20}$$

$$c = \frac{3\chi^2(N-1)^2(81N^2+81N-28)-2\chi N_{\mathrm{N}}(405N^3-641N+236)+N_{\mathrm{N}}^2(972N^2-640)}{64\left[10N_{\mathrm{N}}-3\chi(N-1)\right]}, \tag{6.21}$$

$$\mathcal{A}_{R\varphi\varphi} = \frac{3\chi^2 N(N-1)(N^2-1)-10\chi NN_{\mathrm{N}}(N^2-1)+24N^2N_{\mathrm{N}}^2}{120N_{\mathrm{N}}-36\chi(N-1)}, \tag{6.22}$$

$$\mathcal{A}_{Rii} = \frac{9\chi^2N^2(N-1)^2-51\chi N^2N_{\mathrm{N}}(N-1)+64N^2N_{\mathrm{N}}^2}{20N_{\mathrm{N}}-6\chi(N-1)}. \tag{6.23}$$

The divergence of the expressions above when $N = 1$ and $N_{\mathrm{N}} = 0$ shall not worry us as long as we are working in the large-N limit. Moreover, it can be easily checked that a and c are the same at leading order. Interestingly, we observe that $\mathcal{A}_{R\mathrm{N}_1\mathrm{N}_1} = \mathcal{A}_{R\mathrm{S}_1\mathrm{S}_1}$ for $k = 2$, and $\mathcal{A}_{R\mathrm{N}_1\mathrm{N}_1} = \mathcal{A}_{R\mathrm{N}_2\mathrm{N}_2} = \mathcal{A}_{R\mathrm{S}_1\mathrm{S}_1} = \mathcal{A}_{R\mathrm{S}_2\mathrm{S}_2}$ for $k = 3$. This pattern has a natural generalization for higher k, as we will soon see.

While being fully analytic, the central charges we find for $k \geq 4$ cannot be reduced to similarly compact forms. Nevertheless, figure 6 illustrates the functional dependence of a, a/c, $\mathcal{A}_{R\varphi\varphi}$, $\mathcal{A}_{Rii}$ on N_{N} for a range of k. Note that the scaling relation (6.8) implies up to $\mathcal{O}(N^3, N_{\mathrm{N}}^3)$, changing χ and N amounts to rescaling the axes of the plots of a, $\mathcal{A}_{R\varphi\varphi}$, $\mathcal{A}_{Rii}$ without altering their qualitative behavior. It can be seen that all the central

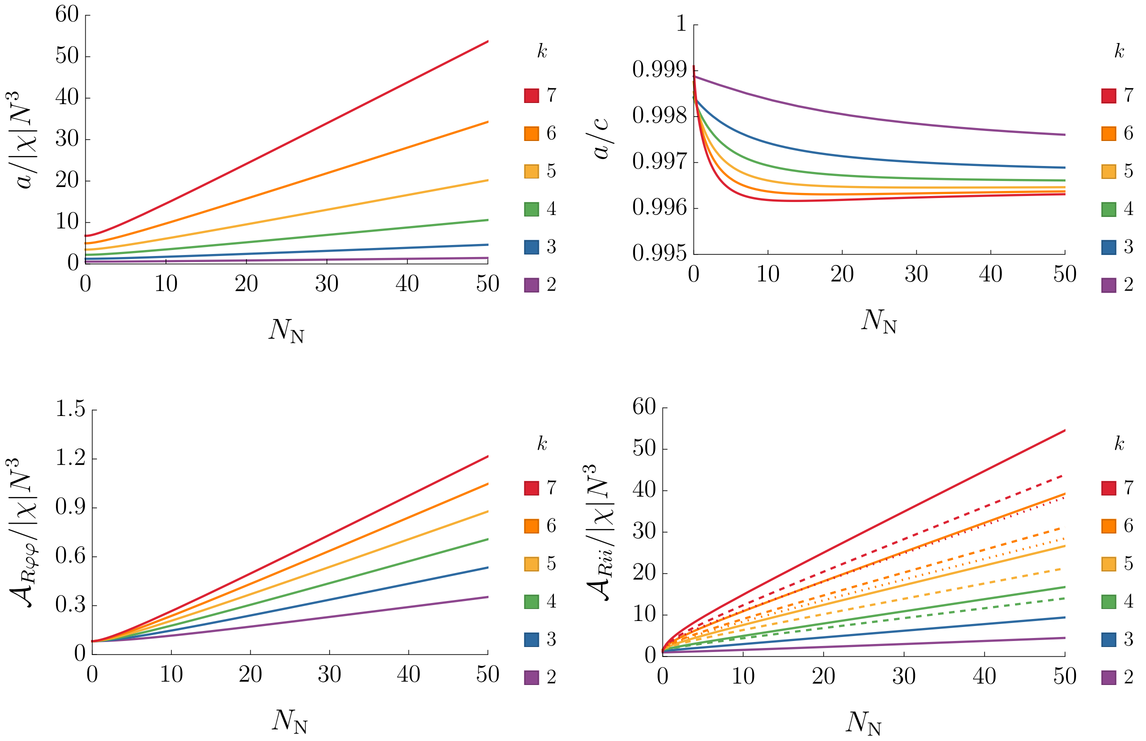

Figure 6. Plots of a, a/c, $\mathcal{A}_{R\varphi\varphi}$, $\mathcal{A}_{Rii}$ for $k = 2$ to $k = 7$ with identical flux quanta $N_{\mathrm{N}_i} = N_{\mathrm{S}_i} := N_{\mathrm{N}}$ for all $i = 1, \ldots, k-1$. N_{N} is treated as a continuous real parameter for visualization purposes. All of the plots are evaluated with $\chi = -2$ and $N = 10$. For a given k, there are $\lceil (k-1)/2 \rceil$ independent $\mathcal{A}_{Rii}$ anomaly coefficients, which are proportional to the flavor central charges b_i associated with the resolution cycles of M_4. We use solid lines for $i = 1$, dashed lines for $i = 2$, and dotted lines for $i = 3$ where applicable.

charges are monotonic in k and N_{N}. We also note that the ratio a/c is well within the Hofman-Maldacena bounds [46] on $\mathcal{N} = 1$ SCFTs,

$$\frac{1}{2} \leq \frac{a}{c} \leq \frac{3}{2} \,. \tag{6.24}$$

Let us briefly comment on the flavor central charge $b_i \propto \mathcal{A}_{Rii}$. In general, because of the D_2 symmetry of M_4, there are $\lceil (k-1)/2 \rceil$ independent $\mathcal{A}_{Rii}$ when all the resolution flux quanta are equal, i.e.

$$\mathcal{A}_{R\,\mathrm{N}_i\mathrm{N}_i} = \mathcal{A}_{R\,\mathrm{N}_{k-i}\mathrm{N}_{k-i}} = \mathcal{A}_{R\,\mathrm{S}_i\mathrm{S}_i} = \mathcal{A}_{R\,\mathrm{S}_{k-i}\mathrm{S}_{k-i}} \tag{6.25}$$

for $i = 1, 2, \ldots, \lceil (k-1)/2 \rceil$, hence the notation $\mathcal{A}_{Rii} := \mathcal{A}_{R\,\mathrm{N}_i\mathrm{N}_i} = \mathcal{A}_{R\,\mathrm{S}_i\mathrm{S}_i}$. Specifically, there is one independent $\mathcal{A}_{Rii}$ for $k = 2, 3$, two for $k = 4, 5$, and three for $k = 6, 7$. It is evident from the separation between lines of like color in figure 6 that

$$\mathcal{A}_{Rii} \geq \mathcal{A}_{R(i+1)(i+1)} \geq \cdots \geq \mathcal{A}_{R\lceil (k-1)/2 \rceil \lceil (k-1)/2 \rceil} \,. \tag{6.26}$$

The inequalities are simultaneously saturated when $N_{\mathrm{N}} = 0$.

6.4 Perturbative analysis

Even though it is exceptionally challenging to analytically determine the central charge through a-maximization for arbitrary combinations of k, χ, N, N_{N_i}, N_{S_i}, we can use perturbation theory to solve the equations $\partial a_{\text{trial}}/\partial q^G = 0$ order by order in the regime where

$$\frac{N_{\mathrm{N}_i}^3}{\chi^2}\,,\ \frac{N_{\mathrm{S}_i}^3}{\chi^2} \gg |\chi|N \gg N_{\mathrm{N}_i}, N_{\mathrm{S}_i} \gg 1\,. \tag{6.27}$$

The first and the last inequalities are required to ensure that the $\mathcal{O}(N)$ contributions from $I_6^{\text{inflow},E_4X_8}$ and $I_6^{\text{v,t}}$ are negligible.[5] We find that the perturbative expansions of the central charges $a = c$ and the anomaly coefficient $\mathcal{A}_{R\varphi\varphi}$ can be written as

$$a = c = -\frac{9k^2\chi N^3}{64} - \frac{27N}{64\chi}\sum_{i,j=1}^{k-1} i(k-j)(N_{\mathrm{N}_i}N_{\mathrm{N}_j} + N_{\mathrm{S}_i}N_{\mathrm{S}_j}) + \mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i})\,, \tag{6.28}$$

$$\mathcal{A}_{R\varphi\varphi} = -\frac{\chi N^3}{12} - \frac{N}{2k^2\chi}\sum_{i,j=1}^{k-1} i(k-j)(N_{\mathrm{N}_i}N_{\mathrm{N}_j} + N_{\mathrm{S}_i}N_{\mathrm{S}_j}) + \mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i})\,. \tag{6.29}$$

For uniform flux configurations, these perturbative expansions have been checked to be consistent with the previously shown exact expressions (6.16), (6.18), (6.20), (6.22) derived for $k = 2$ and $k = 3$.

The symmetry (6.25) between flavor central charges $b_{\mathrm{N}_i,\mathrm{S}_i} \propto \mathcal{A}_{R(\mathrm{N}_i,\mathrm{S}_i)(\mathrm{N}_i,\mathrm{S}_i)}$ no longer holds for nonuniform flux configurations. We list below the perturbative expansions of various flavor central charges for $k = 2$ and $k = 3$, so the reader can compare them to their uniform-flux analogs (6.19) and (6.23). For $k = 2$, we obtain

$$\mathcal{A}_{R\mathrm{N}_1\mathrm{N}_1} = -\chi N^3 + N^2 N_{\mathrm{N}_1} - \frac{NN_{\mathrm{S}_1}^2}{2\chi} + \mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i})\,, \tag{6.30}$$

$$\mathcal{A}_{R\mathrm{S}_1\mathrm{S}_1} = -\chi N^3 + N^2 N_{\mathrm{S}_1} - \frac{NN_{\mathrm{N}_1}^2}{2\chi} + \mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i})\,, \tag{6.31}$$

whereas for $k = 3$, we obtain

$$\begin{aligned}\mathcal{A}_{R\mathrm{N}_1\mathrm{N}_1} = &-\frac{3\chi N^3}{2} + \frac{N^2(5N_{\mathrm{N}_1}+2N_{\mathrm{N}_2})}{4}\\ &-\frac{N[5N_{\mathrm{N}_1}^2+26N_{\mathrm{N}_1}N_{\mathrm{N}_2}+2N_{\mathrm{N}_2}^2+16(N_{\mathrm{S}_1}^2+N_{\mathrm{S}_1}N_{\mathrm{S}_2}+N_{\mathrm{S}_2}^2)]}{24\chi}+\mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i})\,,\end{aligned} \tag{6.32}$$

$$\begin{aligned}\mathcal{A}_{R\mathrm{N}_2\mathrm{N}_2} = &-\frac{3\chi N^3}{2} + \frac{N^2(2N_{\mathrm{N}_1}+5N_{\mathrm{N}_2})}{4}\\ &-\frac{N[2N_{\mathrm{N}_1}^2+26N_{\mathrm{N}_1}N_{\mathrm{N}_2}+5N_{\mathrm{N}_2}^2+16(N_{\mathrm{S}_1}^2+N_{\mathrm{S}_1}N_{\mathrm{S}_2}+N_{\mathrm{S}_2}^2)]}{24\chi}+\mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i})\,,\end{aligned} \tag{6.33}$$

[5] Appropriate powers of χ are inserted in the inequalities here based on the facts that $N_{\mathrm{N}_i}/|\chi|N$, $N_{\mathrm{S}_i}/|\chi|N \sim 1$ are the "characteristic scales" and that $I_6^{\text{inflow},E_4^3}$ scales as χN^3.

$$\mathcal{A}_{R\mathrm{S}_1\mathrm{S}_1} = -\frac{3\chi N^3}{2} + \frac{N^2(5N_{\mathrm{S}_1}+2N_{\mathrm{S}_2})}{4} - \frac{N[5N_{\mathrm{S}_1}^2+26N_{\mathrm{S}_1}N_{\mathrm{S}_2}+2N_{\mathrm{S}_2}^2+16(N_{\mathrm{N}_1}^2+N_{\mathrm{N}_1}N_{\mathrm{N}_2}+N_{\mathrm{N}_2}^2)]}{24\chi} + \mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i}), \quad (6.34)$$

$$\mathcal{A}_{R\mathrm{S}_2\mathrm{S}_2} = -\frac{3\chi N^3}{2} + \frac{N^2(2N_{\mathrm{S}_1}+5N_{\mathrm{S}_2})}{4} - \frac{N[2N_{\mathrm{S}_1}^2+26N_{\mathrm{S}_1}N_{\mathrm{S}_2}+5N_{\mathrm{S}_2}^2+16(N_{\mathrm{N}_1}^2+N_{\mathrm{N}_1}N_{\mathrm{N}_2}+N_{\mathrm{N}_2}^2)]}{24\chi} + \mathcal{O}(N^3_{\mathrm{N}_i,\mathrm{S}_i}). \quad (6.35)$$

6.5 Genus-one cases

The expressions reported earlier in this section are not applicable to the cases where the Riemann surface is a torus, although we shall remark that our previous arguments leading to the uniqueness theorem of central charges continue to hold here. To determine the central charges in such cases, we can perform a-maximization on the expression (5.3) obtained from re-writing the inflow anomaly polynomial using (3.13). Note that $I_6^{\mathrm{v,t}}$ vanishes in this limit, so that the anomalies of the interacting 4d SCFT can be read off directly, as stated in (5.1).

Let us first focus on the case of $k=2$. We find that the various central charges admit qualitatively different expressions depending on whether the two independent flux quanta, N_{N_1} and N_{S_1}, are the same or different. Specifically, if $N_{\mathrm{N}_1} \neq N_{\mathrm{S}_1}$, we recover the central charges of the $Y^{p,q}$ theories obtained by [47] for $q>0$, using the identifications (5.6). On the other hand, if $N_{\mathrm{N}_1} = N_{\mathrm{S}_1} := N_{\mathrm{N}}$, or equivalently, $q=0$, we instead have

$$a = \frac{3(9N^2-8)N_{\mathrm{N}}}{64}, \qquad c = \frac{(27N^2-16)N_{\mathrm{N}}}{64}, \qquad \mathcal{A}_{R\varphi\varphi} = \frac{N^2 N_{\mathrm{N}}}{8}, \qquad \mathcal{A}_{R\mathrm{N}_1\mathrm{N}_1} = \mathcal{A}_{R\mathrm{S}_1\mathrm{S}_1} = \frac{3N^2N_{\mathrm{N}}}{2}. \quad (6.36)$$

Similarly to the higher-genus cases, it is technically challenging to analytically solve the a-maximization problem corresponding to generic flux configurations for $k>2$, except when all the resolution flux quanta are equal. For these uniform flux configurations, the central charges admit rather simple functional forms as follows,

$$a = \mathsf{a}_1(k)N^2N_{\mathrm{N}} - \mathsf{a}_2(k)N_{\mathrm{N}}, \quad c = \mathsf{c}_1(k)N^2N_{\mathrm{N}} - \mathsf{c}_2(k)N_{\mathrm{N}}, \quad \mathcal{A}_{R\varphi\varphi} = \mathsf{b}(k)N^2N_{\mathrm{N}}. \quad (6.37)$$

We record in table 2 the values of these coefficients for $k=2$ to $k=7$. Note that $\mathsf{a}_1(k) = \mathsf{c}_1(k)$ as expected.

7 Conclusion and outlook

In this work, we have identified a geometric origin of flip fields in 4d $\mathcal{N}=1$ SCFTs of class $\mathcal{S}_k$, by adopting an 11d perspective on these models. The comparison between anomaly inflow from M-theory, and integration of the 6d anomaly polynomial, led us to the relation (1.1), which is central to our analysis. The charges and multiplicities in I_6^{flip} are then interpreted in terms of M2-brane operators, associated to blow-up modes of the $\mathbb{C}^2/\mathbb{Z}_k$ singularity. We thus get a physical picture of the role of flip fields: they are necessary to interpolate

k	2	3	4	5	6	7
a_1	27/64	243/160	81/22	14175/1952	120285/9536	17199/856
a_2	3/8	3/2	15/4	15/2	105/8	21
c_1	27/64	243/160	81/22	14175/1952	120285/9536	17199/856
c_2	1/4	1	5/2	5	35/4	14
b	1/8	1/5	3/11	21/61	495/1192	52/107

Table 2. Central charge coefficients (with genus one) for various k.

between six dimensions, where such blow-up modes are not present, and four dimensions, where they are part of the SCFT.

The results of this paper suggest several directions for future research. Firstly, it would be interesting to find explicit AdS_5 solutions in 11d supergravity, which generalize the GMSW solutions from $k = 2$ to higher values of k. This work shows that the topology and flux configuration of M_6 give rise, via inflow, to the anomaly polynomial of a 4d SCFT of class $\mathcal{S}_k$ with fluxes for the $\mathrm{SU}(k)_b$, $\mathrm{SU}(k)_c$ flavor symmetries. This observation is a strong hint that AdS_5 solutions should exist, whose internal space has the topology and G_4-flux quanta of M_6.

The special case in which the Riemann surface is a torus also deserves further investigation. For $k = 2$, we have obtained a precise match between the M-theory inflow anomaly polynomial, and the anomaly polynomial of the SCFT realized by N D3-branes at the tip of the cone over the Sasaki-Einstein space $Y^{p,q}$ (with p, q determined by the flavor fluxes in the M-theory construction). It is natural to study generalizations to higher values of k, for instance exploring possible connections to other families of explicit Sasaki-Einstein metrics, such as [51, 52].

Another natural direction for further study is to consider 4d theories of class $\mathcal{S}_\Gamma$, i.e. theories obtained from reduction of the 6d (1,0) SCFT realized by N M5-branes probing the singularity $\mathbb{C}^2/\Gamma$, with Γ an ADE subgroup of SU(2). Based on our results, we conjecture that the pattern of charges of flip fields for these models should be given in terms of the roots and Cartan matrix of $\mathfrak{g}_\Gamma$, the ADE Lie algebra associated to Γ. It would be interesting to perform explicit checks of this conjecture, for instance against the Lagrangian models of [53].

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A Review of the E_4^3 contribution to anomaly inflow

In this appendix, we record the E_4^3 contribution to $-I_6^{\text{inflow}}$, derived in [27] for $\chi < 0$,

$$\begin{aligned}
-I_6^{\text{inflow},E_4^3} &= \\
&-\frac{2}{3\chi^2}\sum_{i=1}^{2k} N_i\left[\ell_i^2 N_i^2 + 3U_iU_{i+1}\right]\left(\frac{F_2^\psi}{2\pi}\right)^3 \\
&+\frac{3}{2\chi}\sum_{i=1}^{2k}\left[\frac{2\ell_i}{3\chi}N_i^3 + N_i(U_iY_{i+1}^\psi + U_{i+1}Y_i^\psi) + \frac{N}{k}(U_iU_{i+1} + \ell_i^2N_i^2)(\delta_i^{2k} - \delta_i^k)\right]\left(\frac{F_2^\psi}{2\pi}\right)^2\frac{F_2^\varphi}{2\pi} \\
&-\sum_{i=1}^{2k}\left[\frac{N_i^3}{3\chi^2} + N_iY_i^\psi Y_{i+1}^\psi - \frac{NN_i}{2k\chi}(\ell_{i+1}-\ell_i)(U_i + U_{i+1})\right. \\
&\qquad\left. + \frac{N}{k\chi}(\delta_i^{2k} - \delta_i^k)\left(N_iU_i + \chi U_iY_i^\psi + \chi U_{i+1}Y_{i+1}^\psi\right)\right]\frac{F_2^\psi}{2\pi}\left(\frac{F_2^\varphi}{2\pi}\right)^2 \\
&+\frac{3N}{2\chi}\sum_{i=1}^{2k} N_i(U_i + U_{i+1})w_\alpha^i\left(\frac{F_2^\psi}{2\pi}\right)^2\frac{F_2^\alpha}{2\pi} - \frac{\chi N^2}{2k}\sum_{i=1}^{2k} Y_{i+1}^\psi(\ell_{i+1}-\ell_i)(w_\alpha^{i+1} - w_\alpha^i)\left(\frac{F_2^\varphi}{2\pi}\right)^2\frac{F_2^\alpha}{2\pi} \\
&-2N\sum_{i=1}^{2k}\left[N_i\left(\frac{N_i}{2\chi} + Y_{i+1}^\psi\right)w_\alpha^i - \frac{N}{k\chi}(\delta_i^{2k} - \delta_i^k)\delta_\alpha^{\mathrm{C}}U_i\right]\frac{F_2^\psi}{2\pi}\frac{F_2^\varphi}{2\pi}\frac{F_2^\alpha}{2\pi} \\
&-N^2\sum_{i=1}^{2k}\left[N_iw_\alpha^iw_\beta^i - (\ell_{i+1}-\ell_i)(w_\alpha^{i+1} - w_\alpha^i)(w_\beta^{i+1} - w_\beta^i)U_{i+1}\right]\frac{F_2^\psi}{2\pi}\frac{F_2^\alpha}{2\pi}\frac{F_2^\beta}{2\pi} \\
&-\frac{\chi N^2}{2}\sum_{i=1}^{2k}(\ell_{i+1}-\ell_i)(w_\alpha^{i+1} - w_\alpha^i)(w_\beta^{i+1} - w_\beta^i)Y_{i+1}^\psi\frac{F_2^\varphi}{2\pi}\frac{F_2^\alpha}{2\pi}\frac{F_2^\beta}{2\pi} \\
&-\frac{\chi N^3}{6}\sum_{i=1}^{2k}(w_\alpha^{i+1} - w_\alpha^i)(w_\beta^{i+1} - w_\beta^i)\left[\ell_{i+1}w_\gamma^i - \ell_iw_\gamma^{i+1} + (\ell_{i+1}-\ell_i)(w_\gamma^{i+1} + w_\gamma^i)\right]\frac{F_2^\alpha}{2\pi}\frac{F_2^\beta}{2\pi}\frac{F_2^\gamma}{2\pi},
\end{aligned} \tag{A.1}$$

where $U_i \equiv N_\alpha U_0^\alpha(t_i)$, $Y_i^\psi \equiv N_\alpha Y_{0,\psi}^\alpha(t_i)$ are values of functions parameterizing basis forms in cohomology at the positions of the monopoles, while ℓ_i, w_α^i are constants on the intervals composing ∂M_2. Explicitly, in the basis introduced in (2.10), we have

$$\begin{aligned}
&U_i = U_1 - \sum_{j=1}^{i-1}\ell_jN_j\,, \qquad Y_{i=1,\cdots,k}^\psi = Y_1^\psi - \sum_{j=1}^{i-1}\frac{N_j}{\chi}\,, \qquad Y_{i=k+1,\cdots,2k}^\psi = Y_1^\psi + \frac{N}{2} - \sum_{j=1}^{i-1}\frac{N_j}{\chi}\,, \\
&w_{i=1,\cdots,k}^\alpha = -\delta_{\mathrm{N}_i}^\alpha - \frac{\delta_C^\alpha}{\chi}\,, \qquad w_{i=k+1,\cdots,2k-1}^\alpha = -\delta_{\mathrm{S}_{2k-i}}^\alpha - \frac{\delta_C^\alpha}{\chi}\,, \qquad w_{i=k}^\alpha = w_{i=2k}^\alpha = -\frac{\delta_C^\alpha}{\chi}\,, \\
&\ell_{i=1,\cdots,k} = i - \frac{k}{2}\,, \qquad \ell_{i=k+1,\cdots,2k} = \frac{3k}{2} - i\,.
\end{aligned} \tag{A.2}$$

To study perturbative anomalies for continuous symmetries, topological mass terms from the 11d effective action must be integrated out. As discussed in [27], this can be accomplished by imposing the condition

$$\sum_\alpha N_\alpha F_2^\alpha = 0 \tag{A.3}$$

on the original set of $2k-1$ U(1) field strengths associated with the non-trivial cohomology classes of M_6.

B Computation of the $E_4 X_8$ contribution to anomaly inflow

Recall that the internal space M_6 is a fibration of M_4 over the Riemann surface Σ_g, where M_4 is the resolved orbifold $S^4/\mathbb{Z}_k$. In order to evaluate the $E_4 X_8$ contribution to the inflow anomaly polynomial, it is convenient to regard M_4 as consisting of three regions: the region near the north pole, the region near the south pole, and the central region. In this appendix we present a computation of the contribution from each region, using the same parameterization of E_4 as employed in [27]. Note that this parameterization assumes $\chi < 0$. After the field redefinitions (3.13), however, the final result can be extended to the case $\chi = 0$.

B.1 Contribution from the resolved orbifold singularities

The contributions to X_8 originating from the polar regions can be evaluated recalling that M_4 is an S^1_φ fibration over a 3d space parametrized by the ψ circle and the 2d space M_2. The S^1_φ fibration has monopole sources located along the boundary of M_2, grouped into a collection of k monopoles with charge $+1$ in the region near the north pole, and a collection of k monopoles with charge -1 in the region near the south pole. In the vicinity of each monopole, M_4 is locally approximated by single-center ALF Taub-NUT metric. This 4d space has self-dual curvature, and correspondingly it has only one independent Chern root. If we denote the Chern roots as λ_1, λ_2, we have

$$\lambda_1 + \lambda_2 = 0\,, \qquad \lambda_1 - \lambda_2 = 2\,\lambda_1 \coloneqq 2\,\lambda\,. \tag{B.1}$$

The independent Chern root λ can be identified with the first Chern class of the S^1_φ bundle, which is effectively localized at the center of the Taub-NUT space. The relations (B.1) also apply if we consider a multicenter Taub-NUT metric to model the union of the northern and southern regions. In this case, λ is supported at the locations of the various monopoles.

It is useful to recall that, for a single-center ALF Taub-NUT space TN_n with monopole charge n, we have [54]

$$p_1(\mathrm{TN}_n) = \lambda_1^2 + \lambda_2^2 = 2\,\lambda^2\,, \qquad \int_{\mathrm{TN}_n} p_1(\mathrm{TN}_n) = 2\,n\,. \tag{B.2}$$

If we consider a multi-center Taub-NUT, we can write

$$2\,\lambda^2 = \sum_{i=1}^{2k} p_1(\mathrm{TN}_{n_i})\,, \tag{B.3}$$

where the charge n_i is $+1$ for $1 \le i \le k$ (northern region) and -1 for $k+1 \le i \le 2k$ (sourthern region). We observe that the 4-form λ^2 has legs along M_4 only, and is supported at the locations of the monopoles.

Upon twisting M_4 over the Riemann surface, and activating the external gauge fields, the Chern roots λ_1, λ_2 get shifted: the sum $\lambda_1 + \lambda_2$, which is associated with the angle ψ

in the base of the Taub-NUT S^1_φ fibration, is shifted by the total connection for the angle ψ, consisting of a contribution along the Riemann surface (implementing the topological twist), and a contribution along the external spacetime,

$$\lambda_1 + \lambda_2 \to \lambda_1 + \lambda_2 - \chi\, V_2^\Sigma + \frac{F_2^\psi}{2\pi}\,. \tag{B.4}$$

The difference $\lambda_1 - \lambda_2$, which is instead associated with the S^1_φ fiber, is shifted by the external gauge field for the φ isometry,

$$\lambda_1 - \lambda_2 \to \lambda_1 - \lambda_2 + \frac{F_2^\varphi}{2\pi}\,. \tag{B.5}$$

Recalling (B.1), the Chern roots of (the polar regions of) M_4 after twisting and gauging take the form

$$\lambda_{1,2} = \pm\lambda - \frac{\chi}{2}\, V_2^\Sigma + \frac{1}{2}\,\frac{F_2^\psi}{2\pi} \pm \frac{1}{2}\,\frac{F_2^\varphi}{2\pi}\,. \tag{B.6}$$

After these preliminaries, we can proceed with the computation of X_8 that captures the northern and southern caps of M_4. To compute the Pontryagin classes of the total 11d spacetime, we can apply the splitting principle, with reference to the schematic decomposition

$$TM_{11} \to TW_4 \oplus T\Sigma_g \oplus TM_4\,, \tag{B.7}$$

using (B.6) for the Chern roots of the last summand. We obtain

$$p_1(TM_{11}) = p_1(TW_4) + \lambda_1^2 + \lambda_2^2\,, \qquad p_2(TM_{11}) = p_1(TW_4)\,(\lambda_1^2 + \lambda_2^2) + \lambda_1^2\,\lambda_2^2\,, \tag{B.8}$$

and hence (neglecting terms with more than six legs in the external spacetime)

$$X_8 = \frac{1}{192}\Big[p_1(TM_{11})^2 - 4\,p_2(TM_{11})\Big] = \frac{1}{192}\,(\lambda_1^2 - \lambda_2^2)^2 - \frac{1}{96}\,p_1(TW_4)\,(\lambda_1^2 + \lambda_2^2)\,. \tag{B.9}$$

As noted above, λ is supported at the locations of the monopoles. Upon expanding (B.9), we encounter terms without λ, terms linear in λ, and terms with λ^2. Higher powers of λ vanish, because they have too many legs along M_4. Next, we observe that the terms that are linear in λ do not contribute to $E_4\,X_8$. This is due to the fact that, in our construction of E_4, we have imposed that E_4 be regular as we approach the locations of the monopoles. As a result, E_4 localized at a monopole cannot provide the additional M_4 legs that would be necessary (together with λ) to saturate the integral in the M_4 directions. Furthermore, we can drop all terms in X_8 that have purely external legs. Taking these considerations into account, we see that the relevant terms in X_8 are given by

$$X_8 = \frac{\chi}{96}\,V_2^\Sigma\,\frac{F_2^\psi}{2\pi}\left[p_1(TW_4) - \left(\frac{F_2^\varphi}{2\pi}\right)^2\right] + \frac{1}{48}\,\lambda^2\left[\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right] - \frac{\chi}{24}\,\lambda^2\,V_2^\Sigma\,\frac{F_2^\psi}{2\pi}\,. \tag{B.10}$$

We may now consider the parametrization of E_4 given by equation (4.8) of [27], and reported here for convenience,

$$\begin{aligned} E_4 = {}& N_\alpha\,(\Omega_4^\alpha)^{\rm g} + N_\alpha\,(\Omega_{2,I}^\alpha)^{\rm g}\,\frac{F_2^I}{2\pi} + N_\alpha\,\Omega_{0,IJ}^\alpha\,\frac{F_2^I}{2\pi}\,\frac{F_2^J}{2\pi} \\ & + N\,\frac{F_2^\alpha}{2\pi}\,(\omega_{2,\alpha})^{\rm g} + N\,\frac{F_2^\alpha}{2\pi}\,\omega_{0,\alpha I}\,\frac{F_2^I}{2\pi} + N\,\frac{\gamma_4}{2\pi}\,. \end{aligned} \tag{B.11}$$

For the explicit parameterization and properties of the various forms appearing in this expansion we refer the reader to appendix B of [27]. Upon expanding $E_4\,X_8$ and collecting the terms that can saturate the M_6 integration, we arrive at

$$\begin{aligned} E_4 X_8 \supset {}& -\frac{\chi}{96}\,N_\alpha\,\Omega_4^\alpha\,V_2^\Sigma\left[\left(\frac{F_2^\varphi}{2\pi}\right)^2 - p_1(TW_4)\right]\frac{F_2^\psi}{2\pi} \\ & + \frac{1}{48}\,\lambda^2\left[N_\alpha\,(\Omega_{2,I}^\alpha)^{\rm g}\,\frac{F_2^I}{2\pi} + N\,\frac{F_2^\alpha}{2\pi}\,(\omega_{2,\alpha})^{\rm g}\right]\left[\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right] \\ & - \frac{\chi}{24}\,\lambda^2\,V_2^\Sigma\left(N\,\frac{F_2^\alpha}{2\pi}\,\omega_{0,\alpha I}\,\frac{F_2^I}{2\pi} + N_\alpha\,\Omega_{0,IJ}^\alpha\,\frac{F_2^I}{2\pi}\,\frac{F_2^J}{2\pi}\right)\frac{F_2^\psi}{2\pi}\,. \end{aligned} \tag{B.12}$$

The terms with γ_4 are omitted, as it can be easily verified that they drop from the computation, given our prescription to compute integrals of λ^2 described below.

For the first line of (B.12) we just need the integral

$$N_\alpha \int_{M_6} \Omega_4^\alpha\,V_2^\Sigma = N_\alpha \int_{M_2} dT_1^\alpha = N_\alpha\,a_{\rm C}^\alpha = N\,. \tag{B.13}$$

The terms with λ^2 are handled recalling (B.2) and (B.3), which imply the prescription $\lambda^2\,\mathcal{Z} \to \sum_{i=1}^{2k} n_i\,\mathcal{Z}(t_i)$. Here $\mathcal{Z}$ stands for an arbitrary quantity on M_4, $\mathcal{Z}(t_i)$ denotes $\mathcal{Z}$ evaluated at the i-th monopole, and $n_i = +1$ for $i = 1, \cdots, k$ and $n_i = -1$ for $i = k+1, \cdots, 2k$ are the monopole charges. We also need to recall that

$$\begin{aligned} & N_\alpha\,\Omega_{2,\psi}^\alpha = 2\,N_\alpha\,U_0^\alpha\,V_2^\Sigma + \dots\,, \qquad N_\alpha\,\Omega_{2,\varphi}^\alpha = -\chi\,N_\alpha\,Y_{0,\psi}^\alpha\,V_2^\Sigma + \dots\,, \\ & \omega_{2,\alpha} = -\chi\,W_{0,\alpha}^\psi\,V_2^\Sigma + \dots\,, \qquad \Omega_{0,\psi\psi}^\alpha = -\frac{1}{\chi}\,U_0^\alpha\,, \quad 2\,\Omega_{0,\psi\varphi}^\alpha = Y_{0,\psi}^\alpha\,, \quad \Omega_{0,\varphi\varphi}^\alpha = Y_{0,\varphi}^\alpha\,, \end{aligned} \tag{B.14}$$

where the omitted terms are not relevant for the M_6 integration. We also need

$$\omega_{0,\alpha I} = W_{0,\alpha I}\,, \tag{B.15}$$

where the W's are the 0-forms that enter the parametrization of the harmonic 2-forms $\omega_{2,\alpha}$.

We find it convenient to introduce the shorthand notation

$$\begin{aligned} & U_i := N_\alpha\,U_0^\alpha(t_i)\,, \qquad Y_i^\psi := N_\alpha\,Y_{0,\psi}^\alpha(t_i)\,, \quad Y_i^\varphi := N_\alpha\,Y_{0,\varphi}^\alpha(t_i)\,, \\ & (W_\alpha^\psi)_i := W_{0,\alpha\psi}(t_i)\,, \quad (W_\alpha^\varphi)_i := W_{0,\alpha\varphi}(t_i)\,. \end{aligned} \tag{B.16}$$

Collecting the various contributions to the integral of (B.12), we obtain

$$
\begin{aligned}
\int_{M_6} E_4 X_8 \supset & -\frac{\chi N}{96}\left[\left(\frac{F_2^\varphi}{2\pi}\right)^2 - p_1(TW_4)\right]\frac{F_2^\psi}{2\pi} \\
& + \frac{1}{48}\left[\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right]\sum_{i=1}^{k}\left[2\,\frac{F_2^\psi}{2\pi}(U_i - U_{k+i}) - \chi\,\frac{F_2^\varphi}{2\pi}(Y_i^\psi - Y_{k+i}^\psi)\right] \\
& - \frac{\chi N}{48}\left[\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right]\frac{F_2^\alpha}{2\pi}\sum_{i=1}^{k}\left[(W_\alpha^\psi)_i - (W_\alpha^\psi)_{k+i}\right] \\
& - \frac{\chi}{24}\,\frac{F_2^\psi}{2\pi}\sum_{i=1}^{k}\left[-\frac{1}{\chi}\left(\frac{F_2^\psi}{2\pi}\right)^2(U_i - U_{k+i}) + \frac{F_2^\psi}{2\pi}\,\frac{F_2^\varphi}{2\pi}(Y_i^\psi - Y_{k+i}^\psi) + \left(\frac{F_2^\varphi}{2\pi}\right)^2\left(Y_i^\varphi - Y_{k+i}^\varphi\right)\right] \\
& - \frac{\chi N}{24}\left(\frac{F_2^\psi}{2\pi}\right)^2\frac{F_2^\alpha}{2\pi}\sum_{i=1}^{k}\left[(W_\alpha^\psi)_i - (W_\alpha^\psi)_{k+i}\right] - \frac{\chi N}{24}\,\frac{F_2^\psi}{2\pi}\,\frac{F_2^\varphi}{2\pi}\,\frac{F_2^\alpha}{2\pi}\sum_{i=1}^{k}[(W_\alpha^\varphi)_i - (W_\alpha^\varphi)_{k+i}]\,.
\end{aligned} \tag{B.17}
$$

To evaluate the above sums, we need to recall some relations regarding the quantities U, Y, W for our choice of basis of 4- and 2-cycles in the parameterization for E_4,

$$
\begin{aligned}
& (Y_\varphi^\alpha)_{i=1,\cdots,k} = \frac{a_{\rm C}^\alpha}{2k}\,, \qquad (Y_\varphi^\alpha)_{i=k+1,\cdots,2k} = -\frac{a_{\rm C}^\alpha}{2k}\,, \\
& (Y_\psi^\alpha)_{i=1,\cdots,k} = (Y_\psi^\alpha)_1 - \sum_{j=1}^{i-1}\frac{a_j^\alpha}{\chi}\,, \quad (Y_\psi^\alpha)_{i=k+1,\cdots,2k} = (Y_\psi^\alpha)_1 + \frac{a_{\rm C}^\alpha}{2} - \sum_{j=1}^{i-1}\frac{a_j^\alpha}{\chi}\,, \\
& U_i^\alpha = U_1^\alpha - \sum_{j=1}^{i-1}\ell_j a_j^\alpha\,, \quad (W_\alpha^\varphi)_i = \frac{w_\alpha^i - w_\alpha^{i-1}}{\ell_i - \ell_{i-1}}\,, \qquad (W_\alpha^\psi)_i = \frac{\ell_i w_\alpha^{i-1} - \ell_{i-1} w_\alpha^i}{\ell_i - \ell_{i-1}}\,.
\end{aligned} \tag{B.18}
$$

Note also that $\ell_i - \ell_{i-1} = -(\ell_{k+i} - \ell_{k+i-1}) = +1$ for $i = 1, \cdots, k$. The quantities w_α^i are in turn given by

$$
w_{i=1,\cdots,k}^\alpha = -\delta_{{\rm N}_i}^\alpha - \frac{\delta_{\rm C}^\alpha}{\chi}\,, \quad w_{i=k+1,\cdots,2k-1}^\alpha = -\delta_{{\rm S}_{2k-i}}^\alpha - \frac{\delta_{\rm C}^\alpha}{\chi}\,, \quad w_{i=k}^\alpha = w_{i=2k}^\alpha = -\frac{\delta_{\rm C}^\alpha}{\chi}\,. \tag{B.19}
$$

The above relations imply in particular

$$
\begin{aligned}
& \sum_{i=1}^{k}(Y_i^\varphi - Y_{k+i}^\varphi) = N\,, \quad \sum_{i=1}^{k}(Y_i^\psi - Y_{k+i}^\psi) = 0\,, \quad \sum_{i=1}^{k}(U_i - U_{k+i}) = \sum_{i=1}^{k}\sum_{j=i}^{k+i-1}\ell_j N_j\,, \\
& \sum_{i=1}^{k}[(W_{\alpha,\varphi})_i - (W_{\alpha,\varphi})_{k+i}] = 0\,, \quad \sum_{i=1}^{k}\left[(W_\alpha^\psi)_i - (W_\alpha^\psi)_{k+i}\right] = \sum_{i=1}^{k} 2\left(\delta_{{\rm S}_{k-i}}^\alpha - \delta_{{\rm N}_i}^\alpha\right)\,.
\end{aligned} \tag{B.20}
$$

Furthermore, we need the following identity,

$$
\begin{aligned}
\sum_{i=1}^{k}\sum_{j=i}^{i+k-1}\ell_j N_j &= \frac{k^2}{4}\,\chi N - 2\sum_{i,j=1}^{k-1}(\mathsf{A}_{k-1}^{-1})_{ij}(N_{{\rm N}_i} + N_{{\rm S}_{k-j}}) \\
&= \frac{k^2}{4}\,\chi N - 2\sum_{i,j=1}^{k-1}(N_{\tilde{\beta}_i} + N_{\tilde{\gamma}_i})\,,
\end{aligned} \tag{B.21}
$$

where A_{k-1} is the Cartan matrix of $\mathfrak{su}(k)$ and in the second step we have used the dictionary (3.13) between the flux quanta in the M-theory setup, and the flavor fluxes of the class $\mathcal{S}_k$ reduction.

It follows that the contribution from $E_4 X_8$ to $-I_6^{\text{inflow},E_4X_8}$ capturing the polar caps of the resolved orbifold M_4 can be written as

$$\begin{aligned}(-I_6^{\text{inflow},E_4X_8})^{\text{polar caps}} = &-\frac{\chi N}{96}\left[\left(\frac{F_2^\varphi}{2\pi}\right)^2 - p_1(TW_4)\right]\frac{F_2^\psi}{2\pi} - \frac{\chi N}{24}\left(\frac{F_2^\varphi}{2\pi}\right)^2\frac{F_2^\psi}{2\pi}\\ &+\frac{1}{24}\left[2\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right]\frac{F_2^\psi}{2\pi}\left[\frac{k^2\chi N}{4} - 2\sum_{i,j=1}^{k-1}(N_{\tilde\beta_i}+N_{\tilde\gamma_i})\right]\\ &+\frac{\chi N}{24}\left[3\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right]\sum_{i=1}^{k-1}\left(\frac{F_2^{\mathrm{N}_i}}{2\pi} - \frac{F_2^{\mathrm{S}_i}}{2\pi}\right). \end{aligned} \tag{B.22}$$

B.2 Other contributions and final result

The remaining contribution to consider is associated with the central region in M_4, between the two polar caps. We may extract this contribution as follows,

$$(-I_6^{\text{inflow},E_4X_8})^{\text{central region}} = (-I_6^{\text{inflow},E_4X_8})^{S^4} - (-I_6^{\text{inflow},E_4X_8})^{\text{polar caps}}_{k=1}\,. \tag{B.23}$$

The quantity $(-I_6^{\text{inflow},E_4X_8})^{\text{polar caps}}_{k=1}$ is simply (B.22) evaluated for $k=1$, with the convention that the sums over i with range 1 to $k-1$ be simply dropped. The quantity $(-I_6^{\text{inflow},E_4X_8})^{S^4}$ is the E_4X_8 contribution to anomaly inflow for an (unorbifolded) S^4 fibered over the Riemann surface. The rationale behind (B.23) is that the central region can be obtained starting from S^4 and removing the polar caps without orbifold, which are captured as the special case $k=1$ of the computation of the previous subsection.

The quantity $(-I_6^{\text{inflow},E_4X_8})^{S^4}$ corresponds to a special case of the Bah-Beem-Bobev-Wecht (BBBW) setup [7], which is in general parameterized by two integer twist parameters q_1, q_2, subject to the constraint $q_1+q_2=-\chi/2$. In this work, we are turning off the flavor twist parameter ζ (governing a twist of the φ isometry along the Riemann surface), which corresponds to the case $q_1=q_2=-\chi/2$. Both the E_4^3 and the E_4X_8 contributions to anomaly inflow were studied for general q_1, q_2 in [31], with the results

$$\begin{aligned}(-I_6^{\text{inflow, }E_4^3})^{S^4} &= \frac{2}{3}N^3\,(q_1\,n_1\,n_2^2 + q_2\,n_2\,n_1^2)\,,\\ (-I_6^{\text{inflow, }E_4X_8})^{S^4} &= -\frac{N}{24}\,(q_1\,n_1+q_2\,n_2)\,p_1(TW_4) + \frac{N}{6}\,(q_1\,n_1^3+q_2\,n_2^3)\\ &\quad-\frac{N}{6}\,(q_1\,n_1\,n_2^2+q_2\,n_2\,n_1^2)\,. \end{aligned} \tag{B.24}$$

The quantities n_1, n_2 are the first Chern classes of the external gauge fields associated with the $\mathrm{U}(1)^2$ isometry of the S^4 fibration over Σ_g that is visible for generic q_1, q_2. The relation to the background fields F_2^ψ, F_2^φ considered in this work is

$$n_1 = \frac{1}{4}\frac{F_2^\psi}{2\pi} + \frac{1}{2}\frac{F_2^\varphi}{2\pi}\,,\qquad n_2 = \frac{1}{4}\frac{F_2^\psi}{2\pi} - \frac{1}{2}\frac{F_2^\varphi}{2\pi}\,. \tag{B.25}$$

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notation of [26]		notation of this work and [27]
$N_{\rm N}$	$=$	$-N_{\rm S_1}$
$N_{\rm S}$	$=$	$+N_{\rm N_1}$
$c_1^{\rm N}$	$=$	$-\frac{F_2^{\rm S_1}}{2\pi} + \frac{(N\chi - N_{\rm S_1})(N\chi - N_{\rm N_1} - N_{\rm S_1})}{2N\chi(2N\chi - N_{\rm N_1} - N_{\rm S_1})} \frac{F_2^\psi}{2\pi}$
$c_1^{\rm S}$	$=$	$+\frac{F_2^{\rm N_1}}{2\pi} + \frac{(N\chi - N_{\rm N_1})(N\chi - N_{\rm N_1} - N_{\rm S_1})}{2N\chi(2N\chi - N_{\rm N_1} - N_{\rm S_1})} \frac{F_2^\psi}{2\pi}$
c_1^ψ	$=$	$\frac{1}{2}\frac{F^\psi}{2\pi}$
p_1^φ	$=$	$\left(\frac{F_2^\varphi}{2\pi}\right)^2$

Table 3. Flux parameter and external gauge field strength redefinitions required to match with the results of [26].

We are now in a position to write down the final answer for the E_4X_8 contribution to anomaly inflow. This quantity is the sum of (B.22) and (B.23), with $(-I_6^{\text{inflow},E_4X_8})^{S^4}$ extracted from (B.24) with the identifications (B.25) and $q_1 = q_2 = -\chi/2$. In conclusion, we arrive at

$$\begin{aligned}
-I_6^{\text{inflow, } E_4X_8} = &-\frac{N\chi}{24}\frac{F_2^\psi}{2\pi}\left(\frac{F_2^\varphi}{2\pi}\right)^2 + \frac{N\chi}{24}\left[3\left(\frac{F_2^\psi}{2\pi}\right)^2 - p_1(TW_4)\right]\sum_{i=1}^{k-1}\left(\frac{F_2^{\rm N_i}}{2\pi} - \frac{F_2^{\rm S_i}}{2\pi}\right) \\
&+ \frac{F_2^\psi}{2\pi} p_1(TW_4)\left[-\frac{N\chi}{96}(k^2-2) + \frac{1}{12}\sum_{i=1}^{k-1}\left(N_{\widetilde{\beta}_i} + N_{\widetilde{\gamma}_i}\right)\right] \\
&+ \left(\frac{F_2^\psi}{2\pi}\right)^3\left[\frac{N\chi}{48}(k^2-1) - \frac{1}{6}\sum_{i=1}^{k-1}\left(N_{\widetilde{\beta}_i} + N_{\widetilde{\gamma}_i}\right)\right] .
\end{aligned} \tag{B.26}$$

B.3 Comparison with [26]

The full inflow anomaly polynomial $-I_6^{\text{inflow}}$ for general k, including the E_4X_8 contribution evaluated above, agrees with the results of [26] for $k = 2$. To verify this, one needs to perform a redefinition of flux parameters and external gauge fields. The quantities in [26] are related to those in this paper and [27] as in table 3.

C Anomaly inflow computation for the torus case

We derive in this appendix the E_4^3 contribution to the inflow anomaly polynomial, $I_6^{\text{inflow},E_4^3}$, for the case of genus one, following the general philosophy adopted by [27] for the construction of the higher-genus inflow anomaly polynomial. We are going to describe only the essential elements which distinguish this computation from the original one, and we refer the reader to [27] for a detailed discussion of the formalism.

The Euler characteristic of the torus is $\chi = 2 - 2g = 0$, so it makes the topological twist of the $\mathrm{U}(1)_\psi$ bundle trivial while preserving $\mathcal{N} = 1$ supersymmetry [28, 55]. Specifically, the global angular forms associated with the isometries reduce respectively to

$$(D\psi)^{\mathrm{g}} = d\psi + A_1^{\psi}\,, \qquad (D\varphi)^{\mathrm{g}} = d\varphi - L\,(d\psi + A_1^{\psi}) + A_1^{\varphi}\,. \tag{C.1}$$

Cohomology class representatives of $H^2(M_6)$ and $H^4(M_6)$ can be written as

$$\omega_{2,\alpha} = (dW_{0,\alpha}^{\psi} + L\,dW_{0,\alpha}^{\varphi}) \wedge \frac{d\psi}{2\pi} + dW_{0,\alpha}^{\varphi} \wedge \frac{D\varphi}{2\pi} + W_{0,\alpha}^{\Sigma} V_2^{\Sigma}\,, \tag{C.2}$$

$$\Omega_4^{\alpha} = dT_1^{\alpha} \wedge \frac{d\psi}{2\pi} \wedge \frac{D\varphi}{2\pi} + (dU_{0,\psi}^{\alpha} + L\,dU_{0,\varphi}^{\alpha}) \wedge \frac{d\psi}{2\pi} \wedge V_2^{\Sigma} + dU_{0,\varphi}^{\alpha} \wedge \frac{D\varphi}{2\pi} \wedge V_2^{\Sigma}\,, \tag{C.3}$$

where $W_{0,\alpha}^{\Sigma}$ is a constant such that $\omega_{2,\alpha}$ is closed. Note especially that the expression for Ω_4^{α} is not the same as the naïve $\chi = 0$ limit of (B.17) in [27], otherwise the flux quanta $N_i = \int_{\mathcal{C}_{4,i}} N_\alpha \Omega_4^{\alpha}$ for $i = 1, 2, \ldots, 2k$ are ill-defined. This is also necessary so as to recover the sum rules,

$$\sum_{i=1}^{2k} a_i^{\alpha} = 0\,, \qquad \sum_{i=1}^{2k} \ell_i a_i^{\alpha} = 0\,, \tag{C.4}$$

that are consistent with their higher-genus cousins. As a reminder, we choose to parameterize a given basis of (co)homology classes using the expansion coefficients,

$$\mathcal{C}_{4,\mathrm{C}} = a_{\mathrm{C}}^{\alpha}\,\mathcal{C}_{4,\alpha}\,, \quad \mathcal{C}_{4,i} = a_i^{\alpha}\,\mathcal{C}_{4,\alpha}\,, \quad \mathcal{C}_2^{\Sigma,i} = b_{\alpha}^{\Sigma,i}\,\mathcal{C}_2^{\alpha}\,, \quad \mathcal{C}_2^{i} = b_{\alpha}^{i}\,\mathcal{C}_2^{\alpha}\,, \tag{C.5}$$

where the various cycles are introduced in section 2. These coefficients can be expressed in terms of various auxiliary differential forms defined earlier,

$$a_{\mathrm{C}}^{\alpha} = \int_{\partial M_2} T_1^{\alpha}\,, \quad a_i^{\alpha} = U_{0,\varphi}^{\alpha}(t_{i+1}) - U_{0,\varphi}^{\alpha}(t_i)\,, \quad b_{\alpha}^{\Sigma,i} = W_{0,\alpha}^{\Sigma}\,, \quad b_{\alpha}^{i} = W_{0,\alpha}^{\varphi}(t_{i+1}) - W_{0,\alpha}^{\varphi}(t_i)\,. \tag{C.6}$$

Recall that the four-form flux E_4 (restricting only to continuous zero-form symmetries) can be expanded as follows,

$$E_4 = N_\alpha (\Omega_4^{\alpha})^{\mathrm{g}} + N_\alpha (\Omega_{2,I}^{\alpha})^{\mathrm{g}} \frac{F_2^I}{2\pi} + N (\omega_{2,\alpha})^{\mathrm{g}} \frac{F_2^{\alpha}}{2\pi} + N_\alpha \Omega_{0,IJ}^{\alpha} \frac{F_2^I}{2\pi} \frac{F_2^J}{2\pi} + N \omega_{0,\alpha I} \frac{F_2^I}{2\pi} \frac{F_2^{\alpha}}{2\pi}\,. \tag{C.7}$$

One can show that the following choice of forms,

$$\begin{aligned} \Omega_{2,\psi}^{\alpha} &= (dX_0^{\alpha} - LT_1^{\alpha}) \wedge \frac{d\psi}{2\pi} - T_1^{\alpha} \wedge \frac{D\varphi}{2\pi} + U_{0,\psi}^{\alpha} V_2^{\Sigma}\,, \\ \Omega_{2,\varphi}^{\alpha} &= (dU_{0,\varphi}^{\alpha} + L\,dY_0^{\alpha} + T_1^{\alpha}) \wedge \frac{d\psi}{2\pi} + dY_0^{\alpha} \wedge \frac{D\varphi}{2\pi} + U_{0,\varphi}^{\alpha} V_2^{\Sigma}\,, \\ \Omega_{0,\psi\psi}^{\alpha} &= X_0^{\alpha}\,, \qquad \Omega_{0,\psi\varphi}^{\alpha} = \frac{1}{2}\,U_{0,\varphi}^{\alpha}\,, \qquad \Omega_{0,\varphi\varphi}^{\alpha} = Y_0^{\alpha}\,, \end{aligned} \tag{C.8}$$

are compatible with the closure and regularity constraints on E_4. Applying the convention

$$N_\alpha N_\beta \mathcal{J}_I^{\alpha\beta} = 0 \tag{C.9}$$

for each $I \in \{\psi, \varphi\}$, we can uniquely fix the reference values,

$$\tilde{U}_{0,\psi}(t_1) = \frac{1}{N} \sum_{i=2}^{2k} \Big[N_i + \ell_i (\tilde{Y}_{0,\varphi}(t_{i+1}) - \tilde{Y}_{0,\varphi}(t_i)) \Big] \sum_{j=1}^{i-1} (\ell_i - \ell_j) N_j\,, \tag{C.10}$$

$$\tilde{U}_{0,\varphi}(t_1) = \frac{1}{N} \left[\sum_{i=1}^{2k} \ell_i N_i \tilde{Y}_{0,\varphi}(t_{i+1}) + \sum_{i=2}^{2k} \ell_i (\tilde{Y}_{0,\varphi}(t_{i+1}) - \tilde{Y}_{0,\varphi}(t_i)) \sum_{j=1}^{i-1} N_j \right]\,, \tag{C.11}$$

where we used the shorthand notation $N_i = N_\alpha a_i^\alpha$, $\tilde{U}_{0,\psi} = N_\alpha U_{0,\psi}^\alpha$, $\tilde{U}_{0,\varphi} = N_\alpha U_{0,\varphi}^\alpha$, $\tilde{Y}_0 = N_\alpha Y_0^\alpha$.

The resulting inflow anomaly polynomial can be written as

$$
\begin{aligned}
I_6^{\mathrm{inflow},E_4^3} = \\
&\sum_{i=1}^{2k} \tilde{u}_i \Big[(\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0) \Big[X_0 + \frac{1}{2}\,\ell_i (\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0) \Big] \Big]_{t_i}^{t_{i+1}} \left(\frac{F_2^\psi}{2\pi} \right)^3 \\
&+ \sum_{i=1}^{2k} \Big[\frac{1}{2}\,\ell_i \tilde{U}_{0,\varphi}^3 + \tilde{u}_i \Big(\tilde{X}_0 + \ell_i \tilde{U}_{0,\varphi} + \frac{1}{2}\,\ell_i^2 \tilde{Y}_0 \Big) \tilde{Y}_0 \Big]_{t_i}^{t_{i+1}} \left(\frac{F_2^\psi}{2\pi} \right)^2 \frac{F_2^\varphi}{2\pi} \\
&+ \sum_{i=1}^{2k} \ell_i \Big[\Big(\tilde{U}_{0,\varphi}^2 + \frac{1}{2}\,\tilde{u}_i \tilde{Y}_0 \Big) \tilde{Y}_0 \Big]_{t_i}^{t_{i+1}} \frac{F_2^\psi}{2\pi} \left(\frac{F_2^\varphi}{2\pi} \right)^2 \\
&+ \sum_{i=1}^{2k} \frac{1}{2} \Big[\tilde{U}_{0,\varphi} \tilde{Y}_0 (\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0) \Big]_{t_i}^{t_{i+1}} \left(\frac{F_2^\varphi}{2\pi} \right)^3 \\
&+ \sum_{i=1}^{2k} N \Big[W_{0,\alpha}^\psi \tilde{U}_{0,\varphi} \tilde{X}_0 + \tilde{u}_i \Big[W_{0,\alpha}^\varphi \tilde{X}_0 + w_{\alpha,i} (\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0) \Big] \\
&\qquad + b_\alpha^\Sigma \ell_i \Big[\tilde{X}_0 \tilde{Y}_0 + \frac{1}{2} (\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0)^2 \Big] \Big]_{t_i}^{t_{i+1}} \left(\frac{F_2^\psi}{2\pi} \right)^2 \frac{F_2^\alpha}{2\pi} \\
&+ \sum_{i=1}^{2k} N \Big[\frac{1}{2} (w_{\alpha,i} + \ell_i W_{0,\alpha}^\varphi) \tilde{U}_{0,\varphi}^2 + \ell_i w_{\alpha,i} \tilde{u}_i Y_0 + \frac{1}{2}\,b_\alpha^\Sigma (\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0)^2 \Big]_{t_i}^{t_{i+1}} \frac{F_2^\psi}{2\pi} \frac{F_2^\varphi}{2\pi} \frac{F_2^\alpha}{2\pi} \\
&+ \sum_{i=1}^{2k} N \Big[\ell_i W_{0,\alpha}^\varphi \tilde{U}_{0,\varphi} \tilde{Y}_0 + b_\alpha^\Sigma \Big(\tilde{U}_{0,\varphi} + \frac{1}{2}\,\ell_i \tilde{Y}_0 \Big) \tilde{Y}_0 \Big]_{t_i}^{t_{i+1}} \left(\frac{F_2^\varphi}{2\pi} \right)^2 \frac{F_2^\alpha}{2\pi} \\
&+ \sum_{i=1}^{2k} N^2 \Big[\tilde{u}_i \Big(W_{0,\alpha}^\psi + \frac{1}{2}\,\ell_i W_{0,\alpha}^\varphi \Big) W_{0,\beta}^\varphi + b_\alpha^\Sigma w_{\beta,i} (\tilde{U}_{0,\varphi} + \ell_i \tilde{Y}_0) \Big]_{t_i}^{t_{i+1}} \frac{F_2^\psi}{2\pi} \frac{F_2^\alpha}{2\pi} \frac{F_2^\beta}{2\pi} \\
&+ \sum_{i=1}^{2k} N^2 \ell_i \Big[\frac{1}{2}\, W_{0,\alpha}^\varphi W_{0,\beta}^\varphi \tilde{U}_{0,\varphi} + b_\alpha^\Sigma W_{0,\beta}^\varphi \tilde{Y}_0 \Big]_{t_i}^{t_{i+1}} \frac{F_2^\varphi}{2\pi} \frac{F_2^\alpha}{2\pi} \frac{F_2^\beta}{2\pi} \\
&+ \sum_{i=1}^{2k} \frac{N^3}{2}\, b_\alpha^\Sigma \ell_i \Big[W_{0,\beta}^\varphi W_{0,\gamma}^\varphi \Big]_{t_i}^{t_{i+1}} \frac{F_2^\alpha}{2\pi} \frac{F_2^\beta}{2\pi} \frac{F_2^\gamma}{2\pi} \,,
\end{aligned}
\tag{C.12}
$$

subject to the condition that $N_\alpha F_2^\alpha = 0$. The values of the various auxiliary functions evaluated at any t_i are given by

$$
\ell_i = \begin{cases} i - \dfrac{k}{2} & \text{if } 1 \le i \le k \,, \\[2ex] \dfrac{3k}{2} - i & \text{if } k+1 \le i \le 2k \,, \end{cases} \tag{C.13}
$$

$$
\tilde{u}_i = \tilde{U}_{0,\psi}(t_1) + \ell_i \tilde{U}_{0,\varphi}(t_1) + \sum_{j=1}^{i-1} (\ell_i - \ell_j) N_j \,, \tag{C.14}
$$

$$
\tilde{U}_{0,\psi}(t_i) = \tilde{U}_{0,\psi}(t_1) - \sum_{j=1}^{i-1} \ell_j N_j \,, \tag{C.15}
$$

$$\tilde{U}_{0,\varphi}(t_i) = \tilde{U}_{0,\varphi}(t_1) + \sum_{j=1}^{i-1} N_j\,, \tag{C.16}$$

$$\tilde{Y}_0(t_i) = \begin{cases} \dfrac{N}{2k} & \text{if } 1 \leq i \leq k\,, \\ -\dfrac{N}{2k} & \text{if } k+1 \leq i \leq 2k\,, \end{cases} \tag{C.17}$$

$$\tilde{X}_0(t_i) = -\sum_{j=1}^{i-1} \Big[\ell_j N_j + \ell_j^2 (\tilde{Y}_0(t_{j+1}) - \tilde{Y}_0(t_j))\Big]\,, \tag{C.18}$$

$$W^{\psi}_{0,\alpha}(t_i) = \frac{\ell_i w_{\alpha,i-1} - \ell_{i-1} w_{\alpha,i}}{\ell_i - \ell_{i-1}}\,, \tag{C.19}$$

$$W^{\varphi}_{0,\alpha}(t_i) = \frac{w_{\alpha,i} - w_{\alpha,i-1}}{\ell_i - \ell_{i-1}}\,, \tag{C.20}$$

and in a Poincaré-dual basis of (co)homology classes as introduced in section 2, we have

$$w_{\alpha,i=1,\dots,k-1} = -\delta_{\alpha,\mathrm{N}_i}\,, \quad w_{\alpha,i=k+1,\dots,2k-1} = -\delta_{\alpha,\mathrm{S}_{2k-i}}\,, \quad w_{\alpha,k} = w_{\alpha,2k} = 0\,, \quad b^{\Sigma}_{\alpha} = \delta_{\alpha,\mathrm{C}}\,. \tag{C.21}$$

A prescription (3.13) is provided in the main text to obtain an inflow anomaly polynomial that has a well-defined $\chi = 0$ limit. It turns out that we can exactly reproduce this anomaly polynomial by taking (C.12) and carrying out the replacements,

$$\begin{aligned} \frac{F_2^{\psi}}{2\pi} &= 2c_1(R')\,, & \frac{F_2^{\varphi}}{2\pi} &= -k c_1(t)\,, \\ \frac{F_2^{\mathrm{N}_i}}{2\pi} &= c_1(\widetilde{\beta}_i) - 2N_{\widetilde{\beta}_i} c_1(R')\,, & \frac{F_2^{\mathrm{S}_{k-i}}}{2\pi} &= -c_1(\widetilde{\gamma}_i) + 2N_{\widetilde{\gamma}_i} c_1(R')\,, \\ N_{\mathrm{N}_i} &= \sum_{j=1}^{k-1} (\mathsf{A}_{k-1})_{ij}\, N_{\widetilde{\beta}_j}\,, & N_{\mathrm{S}_{k-i}} &= \sum_{j=1}^{k-1} (\mathsf{A}_{k-1})_{ij}\, N_{\widetilde{\gamma}_j}\,. \end{aligned} \tag{C.22}$$

To conclude, this independent derivation for genus one from first principles confirms the validity of using (3.13) to acquire an inflow anomaly polynomial for arbitrary $g \geq 1$, including the case $\chi = 0$.

D Change of basis between flavor and resolution flux quanta

From the perspective of an 11d flux background probed by an M5-brane stack, the flux quanta N_α appearing in the expansion of E_4 represent the amount of flux threading the 4-cycles $\mathcal{C}_{4,\alpha}$. Therefore, it is natural to express these flux quanta with respect to a (co)homology basis determined by an intuitive choice of 4-cycles, namely,

$$\mathcal{C}_{4,\alpha=\mathrm{N}_1,\dots,\mathrm{N}_{k-1}} = \mathcal{C}_4^{i=1,\dots,k-1}\,, \quad \mathcal{C}_{4,\alpha=C} = \mathcal{C}_4^{\mathrm{C}}\,, \quad \mathcal{C}_{4,\alpha=\mathrm{S}_1,\dots,\mathrm{S}_{k-1}} = \mathcal{C}_4^{i=2k-1,\dots,k+1}\,, \tag{D.1}$$

as in (2.10). One can then proceed to define the "natural" basis 2-cycles $\mathcal{C}_2^{\alpha}$ to be those that are Poincaré-dual to these basis 4-cycles. For simplicity, the basis of (co)homology classes described above will hereafter be referred to as the "resolution flux basis." On the other hand, as explained in section 3.3, the starting point for the basis implicitly used in the compactification of the 6d $\mathcal{N}=(1,0)$ theories is instead an intuitive set of basis 2-cycles,

$$(\mathcal{C}_2^{\alpha=\mathrm{N}_1,\ldots,\mathrm{N}_{k-1}})' = \mathcal{C}_2^{i=1,\ldots,k-1}\,, \quad (\mathcal{C}_2^{\alpha=C})' = \mathcal{C}_2^{\mathrm{C}}\,, \quad (\mathcal{C}_2^{\alpha=\mathrm{S}_1,\ldots,\mathrm{S}_{k-1}})' = \mathcal{C}_2^{i=2k-1,\ldots,k+1}\,, \tag{D.2}$$

where $\mathcal{C}_2^{\mathrm{C}}$ is defined to be the Poincaré dual of $\mathcal{C}_{4,\mathrm{C}}=M_4$. The rest of the basis 4-cycles $(\mathcal{C}_{4,\alpha})'$ can be similarly defined to be the Poincaré duals of $(\mathcal{C}_2^\alpha)'$. We will hereafter refer to this basis of (co)homology classes as the "flavor flux basis."

In this appendix we find the transformation matrix that relates the resolution flux basis and the flavor flux basis. Using the relations following from Poincaré duality and regularity of the basis two-forms in appendix C of [27], we can express the basis 2-cycles $\mathcal{C}_2^\alpha$ in the resolution flux basis in terms of familiar 2-cycles. Let us start with the case of $k=2$. One finds that

$$\begin{aligned} &\mathcal{C}_2^{\Sigma,1} = \mathcal{C}_2^{\Sigma,2} = \chi\,\mathcal{C}_2^{\mathrm{N}_1} + \mathcal{C}_2^{\mathrm{C}}\,, \qquad \mathcal{C}_2^{\Sigma,3} = \mathcal{C}_2^{\Sigma,4} = \chi\,\mathcal{C}_2^{\mathrm{S}_1} + \mathcal{C}_2^{C}\,, \\ &\mathcal{C}_2^{i=1} = 2\,\mathcal{C}_2^{\mathrm{N}_1}\,, \qquad \mathcal{C}_2^{i=2} = \mathcal{C}_2^{i=4} = -\mathcal{C}_2^{\mathrm{N}_1} + \mathcal{C}_2^{\mathrm{S}_1}\,, \qquad \mathcal{C}_2^{i=3} = -2\,\mathcal{C}_2^{\mathrm{S}_1}\,, \end{aligned} \tag{D.3}$$

which can be inverted to give

$$\begin{aligned} \mathcal{C}_2^{\mathrm{N}_1} &= \frac{1}{2}\,\mathcal{C}_2^{i=1}\,, \\ \mathcal{C}_2^{\mathrm{C}} &= \mathcal{C}_2^{\Sigma,1} - \frac{\chi}{2}\,\mathcal{C}_2^{i=1} = \mathcal{C}_2^{\Sigma,3} + \frac{\chi}{2}\,\mathcal{C}_2^{i=3}\,, \\ \mathcal{C}_2^{\mathrm{S}_1} &= -\frac{1}{2}\,\mathcal{C}_2^{i=3}\,. \end{aligned} \tag{D.4}$$

Meanwhile, the basis 2-cycles in the flavor basis are

$$\begin{aligned} (\mathcal{C}_2^{\mathrm{N}_1})' &= \mathcal{C}_2^{i=1} = 2\,\mathcal{C}_2^{\mathrm{N}_1}\,, \\ (\mathcal{C}_2^{\mathrm{C}})' &= \mathcal{C}_2^{\mathrm{C}}\,, \\ (\mathcal{C}_2^{\mathrm{S}_1})' &= \mathcal{C}_2^{i=3} = -2\,\mathcal{C}_2^{\mathrm{S}_1}\,, \end{aligned} \tag{D.5}$$

so we can express the basis transformation compactly as $(\mathcal{C}_2^\alpha)' = (\mathcal{R}_2^{-1})^\alpha_\beta\,\mathcal{C}_2^\beta$ where

$$\mathcal{R}_2^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,. \tag{D.6}$$

The orthonormal pairing between cycles and cohomology representatives is preserved if $(\omega_{2,\beta})' = (\mathcal{R}_2)^\alpha_\beta\,\omega_{2,\alpha}$, such that

$$\int_{(\mathcal{C}_2^\alpha)'} (\omega_{2,\beta})' = \int_{(\mathcal{R}_2^{-1})^\alpha_\gamma\,\mathcal{C}_2^\gamma} (\mathcal{R}_2)^\delta_\beta\,\omega_{2,\delta} = (\mathcal{R}_2^{-1})^\alpha_\gamma\,(\mathcal{R}_2)^\delta_\beta\,\delta^\gamma_\delta = \delta^\alpha_\beta\,. \tag{D.7}$$

Demanding Poincaré duality between the 2-cycles $(\mathcal{C}_2^\alpha)'$ and some set of 4-cycles $(\mathcal{C}_{4,\alpha})'$ amounts to requiring

$$\delta^\alpha_\beta = \int_{M_6} (\Omega_4^\alpha)' \wedge (\omega_{2,\beta})' = \int_{M_6} (\mathcal{R}_4)^\alpha_\gamma\,\Omega_4^\gamma \wedge (\mathcal{R}_2)^\delta_\beta\,\omega_{2,\delta} = (\mathcal{R}_4\mathcal{R}_2)^\alpha_\beta\,, \tag{D.8}$$

which means that the 4-cycles that are Poincaré-dual to the basis 2-cycles in the flavor basis can be expressed as $(\mathcal{C}_{4,\alpha})' = (\mathcal{R}_4^{-1})^\alpha_\beta\, \mathcal{C}_{4,\beta}$ with

$$\mathcal{R}_4^{-1} = \mathcal{R}_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} . \tag{D.9}$$

As discussed in appendix D of [27], the flux quanta associated with the new and old bases of 4-cycles are related by $N'_\alpha = (\mathcal{R}_4^{-1})^\beta_\alpha\, N_\beta$ to make the inflow anomaly polynomial invariant, and hence we get[6]

$$N'_{\mathrm{N}_1} = \frac{1}{2}\, N_{\mathrm{N}_1}\,, \qquad N'_{\mathrm{C}} = N_{\mathrm{C}} = N\,, \qquad N'_{\mathrm{S}_1} = -\frac{1}{2}\, N_{\mathrm{S}_1}\,. \tag{D.10}$$

As noted previously, in order to preserve positivity of the flux quanta an additional overall sign flip must be included in the basis transformation on southern quantities.

We now move on to the case of $k = 3$. By repeating the same exercise as before, we derive that in the resolution flux basis,

$$\begin{aligned}
\mathcal{C}_2^{\mathrm{N}_1} &= \frac{1}{3}\big(2\,\mathcal{C}_2^{i=1} + \mathcal{C}_2^{i=2}\big)\,, \\
\mathcal{C}_2^{\mathrm{N}_2} &= \frac{1}{3}\big(\mathcal{C}_2^{i=1} + 2\,\mathcal{C}_2^{i=2}\big)\,, \\
\mathcal{C}_2^{\mathrm{C}} &= \mathcal{C}_2^{\Sigma,1} - \frac{\chi}{2}\big(2\,\mathcal{C}_2^{i=1} + \mathcal{C}_2^{i=2}\big)\,, \\
\mathcal{C}_2^{\mathrm{S}_2} &= -\frac{1}{3}\big(2\,\mathcal{C}_2^{i=4} + \mathcal{C}_2^{i=5}\big)\,, \\
\mathcal{C}_2^{\mathrm{S}_1} &= -\frac{1}{3}\big(\mathcal{C}_2^{i=4} + 2\,\mathcal{C}_2^{i=5}\big)\,.
\end{aligned} \tag{D.11}$$

On the other hand, the basis 2-cycles in the flavor flux basis are

$$\begin{aligned}
(\mathcal{C}_2^{\mathrm{N}_1})' &= \mathcal{C}_2^{i=1} = 2\,\mathcal{C}_2^{\mathrm{N}_1} - \mathcal{C}_2^{\mathrm{N}_1}\,, \\
(\mathcal{C}_2^{\mathrm{N}_2})' &= \mathcal{C}_2^{i=2} = -\mathcal{C}_2^{\mathrm{N}_1} + 2\,\mathcal{C}_2^{\mathrm{N}_2}\,, \\
(\mathcal{C}_2^{\mathrm{C}})' &= \mathcal{C}_2^{\mathrm{C}}\,, \\
(\mathcal{C}_2^{\mathrm{S}_2})' &= \mathcal{C}_2^{i=4} = -2\,\mathcal{C}_2^{\mathrm{S}_2} + \mathcal{C}_2^{\mathrm{S}_1}\,, \\
(\mathcal{C}_2^{\mathrm{S}_1})' &= \mathcal{C}_2^{i=5} = \mathcal{C}_2^{\mathrm{S}_2} - 2\,\mathcal{C}_2^{\mathrm{S}_1}\,,
\end{aligned} \tag{D.12}$$

[6]Note that the requirement of having no topological mass term associated with γ_4 in the inflow anomaly polynomial is preserved, i.e. $N'_\alpha (F_2^\alpha)' = (\mathcal{R}_4^{-1})^\gamma_\alpha N_\gamma\, (\mathcal{R}_2^{-1})^\alpha_\delta F_2^\delta = (\mathcal{R}_2\mathcal{R}_2^{-1})^\gamma_\delta N_\gamma F_2^\delta = N_\alpha F_2^\alpha = 0$, assuming that we only restrict to the zero-form symmetries and also follow the convention that $N_\alpha N_\beta \mathcal{J}_I^{\alpha\beta} = 0$.

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so the corresponding basis transformation matrix is

$$\mathcal{R}_2^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} . \tag{D.13}$$

The flux quanta in the two bases are then related by

$$\begin{aligned} N'_{\mathrm{N}_1} &= \frac{1}{3}(2N_{\mathrm{N}_1} + N_{\mathrm{N}_2}) , \quad N'_{\mathrm{N}_2} = \frac{1}{3}(N_{\mathrm{N}_1} + 2N_{\mathrm{N}_2}) , \quad N'_{\mathrm{C}} = N , \\ N'_{\mathrm{S}_2} &= -\frac{1}{3}(2N_{\mathrm{S}_2} + N_{\mathrm{S}_1}) , \quad N'_{\mathrm{S}_1} = -\frac{1}{3}(N_{\mathrm{S}_2} + 2N_{\mathrm{S}_1}) . \end{aligned} \tag{D.14}$$

Following the analogous exercise for $k \geq 4$, one can inductively find that the basis transformation matrix $\mathcal{R}_2^{-1}$ is given by

$$\mathcal{R}_2^{-1} = \begin{pmatrix} \mathsf{A}_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mathsf{A}_{k-1} \end{pmatrix} , \tag{D.15}$$

where A_{k-1} is the Cartan matrix of $\mathfrak{su}(k)$. Note that this formula applies also to $k = 2$ and $k = 3$.

To summarize, the flux quanta $N_{\mathrm{N}_i}, N_{\mathrm{S}_i}$ in the resolution flux basis can be expressed in terms of the flux quanta $N'_{\mathrm{N}_i}, N'_{\mathrm{S}_i}$ in the flavor flux basis as

$$N_{\mathrm{N}_i} = \sum_{i=1}^{k-1} (\mathsf{A}_{k-1})_{ij} \, N'_{\mathrm{N}_j} , \qquad N_{\mathrm{S}_i} = -\sum_{i=1}^{k-1} (\mathsf{A}_{k-1})_{ij} \, N'_{\mathrm{S}_j} . \tag{D.16}$$

Alternately, we have

$$N'_{\mathrm{N}_i} = \sum_{i=1}^{k-1} (\mathsf{A}_{k-1}^{-1})_{ij} \, N_{\mathrm{N}_j} , \qquad N'_{\mathrm{S}_i} = -\sum_{i=1}^{k-1} (\mathsf{A}_{k-1}^{-1})_{ij} \, N_{\mathrm{S}_j} , \tag{D.17}$$

where the inverse A_{k-1} given by [56]

$$(\mathsf{A}_{k-1}^{-1})_i^j = \min\{i, j\} - \frac{ij}{k} . \tag{D.18}$$

We can thus interpret the flux quanta identifications in (3.13) as a change of basis between a natural 4-cycle and a natural 2-cycle basis, along with a reversal of indexing and sign flip in the south to preserve positivity of the fluxes, i.e.

$$N_{\widetilde{\beta}_i} = N'_{\mathrm{N}_i} , \qquad N_{\widetilde{\gamma}_i} = -N'_{\mathrm{S}_{k-i}} . \tag{D.19}$$

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