

Soft integrals and soft anomalous dimensions at N³LO and beyond

Claude Duhr,^a Bernhard Mistlberger^b and Gherardo Vita,^b

^a*Bethe Center for Theoretical Physics, Universität Bonn,
Am Hof 1, Bonn D-53115, Germany*

^b*SLAC National Accelerator Laboratory, Stanford University,
2575 Sand Hill Road, Menlo Park 94025, U.S.A.*

E-mail: cduhr@uni-bonn.de, bernhard.mistlberger@gmail.com,
gherardo@slac.stanford.edu

ABSTRACT: We calculate soft phase-space and loop master integrals for the computation of color-singlet cross sections through N³LO in perturbative QCD. Our results are functions of homogeneous transcendental weight and include the first nine terms in the expansion in the dimensional regulator ϵ . We discuss the application of our results to the computation of deeply-inelastic scattering and e^+e^- annihilation processes. We use these results to compute the perturbative coefficient functions for the Drell-Yan and gluon-fusion Higgs boson production cross sections to higher orders in ϵ through N³LO in QCD in the limit where only soft partons are produced on top of the colorless final state. Furthermore, we extract the anomalous dimension of the inclusive threshold soft function and of the N -Jettiness beam and jet functions to N⁴LO in perturbative QCD.

KEYWORDS: Effective Field Theories of QCD, Factorization, Renormalization Group, Higher-Order Perturbative Calculations, Resummation

ARXIV EPRINT: [2205.04493](https://arxiv.org/abs/2205.04493)

Contents

1	Introduction	1
2	Setup	3
2.1	Final state phase space	4
2.2	General structure of soft integrals	6
3	Computing soft master integrals	7
3.1	Method	8
3.2	RR soft integrals with two final state partons	8
3.2.1	Differential RR soft master integrals	8
3.2.2	Inclusive RR soft master integrals	9
3.2.3	Relating inclusive and differential RR soft master integrals	10
3.3	RRR and RRV soft master integrals	10
4	Threshold limit of production cross sections	11
5	Threshold factorization and soft anomalous dimension	13
5.1	Threshold factorization	13
5.2	Results for the threshold anomalous dimension through N ⁴ LO	16
5.3	Thrust and N -jettiness anomalous dimensions to N ⁴ LO	18
6	Conclusions	19
A	Ultraviolet renormalization	21
B	Mass factorization	21
C	Color	22

1 Introduction

Analytic computations for scattering cross sections play a crucial role in the field of high energy particle physics phenomenology. In such computations, we use perturbative quantum field theory (QFT) in order to achieve precise predictions for observables that allow us to study the interactions of fundamental particles. The field of analytic computations is advancing rapidly and has produced many cutting-edge results, like predictions for production cross sections to third order in QCD perturbation theory for the Large Hadron Collider(LHC) [1–7], fourth order QCD results for the production of hadrons in electron-positron collisions [8, 9] or analytic formulae for event shape observables like the energy-energy correlation function [10–12]. On top of predictions for explicit cross sections, analytic

results play a crucial role to determine many universal quantities appearing as ingredients to the calculation of scattering cross sections. Shining examples are the splitting functions at third order in perturbative QCD [13–17], the so-called cusp anomalous dimension at fourth loop order [18, 19], or extraction of the universal infrared behavior of scattering amplitudes at third loop order [20, 21].

In this article, we discuss and extend a set of analytic ingredients for perturbative computations that have already found widespread application. The quantities in question are so-called *soft integrals*. These integrals are Feynman integrals for phase-space and loop integrals expanded around a certain kinematic limit — the so-called *soft* or *threshold* limit. Throughout this article, we work within the framework of dimensional regularization. Our soft integrals first made their appearance in the computation of hadronic production cross section for a Higgs boson at N³LO [22–25], where they played a two-fold role: first, using the framework of reverse unitarity [26–30], it is possible to express the threshold approximation of the production cross section [25, 31] as a linear combination of these soft master integrals. Second, the soft master integrals served as boundary conditions [6, 32] for differential equations [33–37] used to calculate the exact Higgs boson cross section at N³LO in QCD perturbation theory.

The same soft master integrals were subsequently used in the computation of several analytic results. First, the computation of the inclusive gluon fusion Higgs boson production cross section at the LHC was extended to the charged current and neutral current Drell-Yan cross sections [3, 4, 7], as well as to the production cross section of a Higgs boson from bottom quark fusion [2, 38]. In ref. [39] it was realized that analytic computations for more differential quantities, like the rapidity or transverse momentum distributions, can be carried out efficiently thanks to the knowledge of the very same analytic information. As a result, it was possible to perform a threshold expansion of differential cross sections for the production of a Higgs boson at N³LO [40] and to compute the rapidity distribution of the Higgs boson [5] using analytic results. In ref. [41] it was pointed out that soft integrals may serve as key analytic ingredients to determine so-called collinear master integrals. In turn, these results were then used to determine the so-called transverse momentum-dependent beam functions at N³LO in QCD [42] as well as the N -jettiness beam functions at the same order [43]. In ref. [44] it was realized that it is easy to analytically continue the soft integrals computed for a production cross section to serve as ingredients for the computation of a Deep Inelastic Scattering (DIS) process or a cross section relevant for an electron-positron annihilation experiment. As a consequence, the energy-energy correlation function was calculated in the large-angle limit at N³LO in QCD perturbation theory [10]. The above results have widespread implications on particle physics phenomenology, which demonstrates the importance of analytic results for soft integrals.

The soft integrals discussed above were presented in the literature as a Laurent series in the dimensional regulator up to the power required to perform computations at N³LO in QCD perturbation theory. Here, we extend this computation to include two additional powers in this Laurent expansion, and consequently, we obtain information that will be an ingredient for the computation of scattering cross sections beyond N³LO. In particular, we compute two classes of soft integrals: the first one is differential in the four-momentum

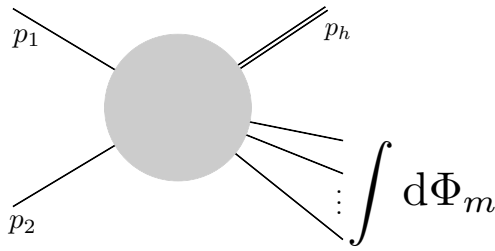


Figure 1. Schematic depiction of a partonic scattering cross involving $m+2$ partons and a color neutral particle h .

of the color-neutral particle, while the second one is integrated over the full final state phase space. Our basis of soft integrals is given in terms of pure functions of uniform transcendentality weight. Here we focus on integrals with two and three partons in the final state as results for single parton final state integrals can be found elsewhere [32, 45–47]. We then use these new results in order to determine the neutral current Drell-Yan and gluon fusion Higgs boson production cross section in the threshold limit to two orders beyond the finite term in the dimensional regulator at N³LO.

Threshold factorization [48–54] allows one to compute the threshold limit of any colorless production cross section once purely virtual corrections for this process and the so-called threshold soft function are known. We compute the threshold soft function through N⁴LO in perturbative QCD up to one undetermined constant. We extract explicitly the anomalous dimension of the soft function through N⁴LO as one of our results. We find agreement for the threshold anomalous dimension and soft function with the existing results [55]. Building on refs. [55–60], as a side product we are able to determine previously unknown coefficients of the Altarelli-Parisi splitting functions at third non-trivial order.

This article is organized as follows. In section 2 we introduce our notation and give our definitions of soft phase space and loop integrals. Next, we discuss our computation of soft loop and phase space integrals for integrals in section 3. We then apply these soft integrals to the computation of the Drell-Yan and Higgs boson production cross section in the threshold limit through N³LO in perturbative QCD in section 4. We generalize our results to generic production cross sections using threshold factorization and extract the threshold soft function and anomalous dimension in section 5. Finally, we draw our conclusions and summarize our results in section 6.

2 Setup

In this article we discuss Feynman integrals appearing in the computation of scattering cross sections in perturbative QFT. We are interested in scattering processes where two partons and one color-neutral particle (like a Higgs boson or an off-shell photon) scatter with m massless final state particles. Schematically, an amplitude for such a process is depicted in figure 1. We are interested in the case where all kinematic information of these m final state particles is integrated out. Defining all momenta to be in-going, momentum

conservation is given by

$$p_1 + p_2 + p_h + k = 0, \tag{2.1}$$

where we denote the collective momentum of the m massless final state partons by k . We define the following variables.

$$s = (p_1 + p_2)^2, \quad w_1 = -\frac{2p_1 k}{2p_1 p_2}, \quad w_2 = -\frac{2p_2 k}{2p_1 p_2}, \quad x = \frac{(2p_1 p_2) k^2}{(2p_1 k)(2p_2 k)}. \tag{2.2}$$

We are interested in the kinematic limit where the energies of the m final state particles are almost zero, i.e., they are soft, and consequently, we have $k \sim 0$. Furthermore, we are interested in the case where any loop momentum appearing in a virtual loop of our scattering process is low energetic as well. We refer to the resulting phase space and loop integrals as *soft integrals*. Constructing such a soft integral follows the method of regions [61], and details can be found in refs. [23, 24].

Above we only specified that the m soft particles are in the final state. Depending on whether the remaining external particles are in the final or initial state, the soft integrals contribute to different kinds of scattering processes. For example, the kinematic configuration where the momenta p_1 and p_2 in the initial state and p_h in the final state corresponds to a partonic production process of the colorless state h from hadron collisions at the LHC:

$$\textit{Production:} \quad p_1 + p_2 \rightarrow p_h + k. \tag{2.3}$$

If we consider p_h and p_1 in the initial state and p_2 in the final state, we obtain a kinematic configuration corresponding to a semi-inclusive Deeply Inelastic Scattering (DIS) process:

$$\textit{DIS:} \quad p_1 + p_h \rightarrow p_2 + k. \tag{2.4}$$

Finally, if only p_h is in the initial state, this may be recognized as a doubly-resolved scattering configuration in e^+e^- annihilation:

$$e^+e^- \textit{ Annihilation:} \quad p_h \rightarrow p_1 + p_2 + k. \tag{2.5}$$

In the remainder of this section, we will first discuss the final state phase space associated with the three different scattering configurations discussed above. Next, we will discuss the general structure of soft integrals, how to analytically continue it from one kinematic region to another, and finally, we will define inclusive soft integrals.

2.1 Final state phase space

For different scattering processes we define the following phase space measures:

1. *Production:*

$$d\Phi_{h+m} = (2\pi)^d \delta^d \left(p_1 + p_2 + p_h + \sum_{i=3}^{m+2} p_i \right) \frac{d^d p_h}{(2\pi)^d} (2\pi) \delta_+(p_h^2 - m_h^2) \prod_{i=3}^{m+2} \frac{d^d p_i}{(2\pi)^d} (2\pi) \delta_+(p_i^2). \tag{2.6}$$

2. *DIS*:

$$d\Phi_{1+m} = (2\pi)^d \delta^d \left(p_1 + p_h + \sum_{i=2}^{m+2} p_i \right) \prod_{i=2}^{m+2} \frac{d^d p_i}{(2\pi)^d} (2\pi) \delta_+ \left(p_i^2 \right). \quad (2.7)$$

3. e^+e^- *Annihilation*:

$$d\Phi_{2+m} = (2\pi)^d \delta^d \left(p_h + \sum_{i=1}^{m+2} p_i \right) \prod_{i=1}^{m+2} \frac{d^d p_i}{(2\pi)^d} (2\pi) \delta_+ \left(p_i^2 \right). \quad (2.8)$$

We work in dimensional regularization and denote the spacetime dimension by $d = 4 - 2\epsilon$, where ϵ is the dimensional regulator. Introducing the momentum k and using the variables we defined in eq. (2.2), we parametrize the phase space measures as follows:

1. *Production*:

$$\begin{aligned} d\Phi_{h+m} &= \frac{(s w_1 w_2)^{1-\epsilon}}{4(2\pi)^{3-2\epsilon}} (1-x)^{-\epsilon} d\Phi_m(k) dx dw_1 dw_2 d\Omega_{2-2\epsilon} \\ &\times \delta \left(\frac{m_h^2}{s} - (1-w_1-w_2+w_1 w_2 x) \right) \theta(x(1-x)) \theta(w_1) \theta(w_2). \end{aligned} \quad (2.9)$$

2. *DIS*:

$$\begin{aligned} d\Phi_{1+m} &= \frac{(p_h^2 w_1 w_2)^{1-\epsilon}}{4(2\pi)^{3-2\epsilon}} (1-w_1)^{-3+2\epsilon} (1-x)^{-\epsilon} (1-w_1-w_2+w_1 w_2 x)^{-1+\epsilon} \\ &\times d\Phi_m(k) dx dw_1 dw_2 d\Omega_{2-2\epsilon} \theta(x(1-x)) \theta(-w_1) \theta(w_2) \theta(-p_h^2). \end{aligned} \quad (2.10)$$

3. e^+e^- *Annihilation*:

$$\begin{aligned} d\Phi_{2+m} &= \frac{\pi^{4-2\epsilon} (p_h^2)^{1-\epsilon} (p_h^2 w_1 w_2)^{1-\epsilon} (1-x)^{-\epsilon}}{(1-2\epsilon) \Gamma(1-2\epsilon)} (1-w_1-w_2+w_1 w_2 x)^{3\epsilon-4} \\ &\times \theta(x(1-x)) \theta(-w_1) \theta(-w_2) dw_1 dw_2 dx d\Phi_m(k), \end{aligned} \quad (2.11)$$

with

$$d\Phi_m(k) = (2\pi)^d \delta^d \left(k - \sum_{i=3}^{m+3} p_i \right) \prod_{i=3}^{m+3} \frac{d^d p_i}{(2\pi)^d} (2\pi) \delta_+ \left(p_i^2 \right). \quad (2.12)$$

In the soft limit, where all final state partons associated with the momenta p_3, \dots, p_{m+2} become low-energetic, we find

$$w_1 \rightarrow 0, \quad w_2 \rightarrow 0. \quad (2.13)$$

In this limit, the differential phase space measures become proportional to each other.

$$\begin{aligned}
 \lim_{w_{1,2} \rightarrow 0} \frac{d\Phi_{h+m}}{dw_1 dw_2 dx} &= \frac{d\Phi_{h+m}^{\text{soft}}}{dw_1 dw_2 dx} = \frac{(p_h^2 w_1 w_2)^{1-\epsilon}}{4(2\pi)^{3-2\epsilon}} (1-x)^{-\epsilon} d\Phi_m(k) d\Omega_{2-2\epsilon}, \\
 \lim_{w_{1,2} \rightarrow 0} \frac{d\Phi_{1+m}}{dw_1 dw_2 dx} &= \lim_{w_{1,2} \rightarrow 0} \frac{d\Phi_{h+m}}{dw_1 dw_2 dx}, \\
 \lim_{w_{1,2} \rightarrow 0} \frac{d\Phi_{2+m}}{dw_1 dw_2 dx} &= \frac{(4\pi)^{\epsilon-2} (p_h^2)^{1-\epsilon} \Gamma(1-\epsilon)}{(1-2\epsilon)\Gamma(1-2\epsilon)} \lim_{w_{1,2} \rightarrow 0} \frac{d\Phi_{h+m}}{dw_1 dw_2 dx}.
 \end{aligned} \tag{2.14}$$

Above, we implicitly set Kronecker delta constraints of external variables to unity.

2.2 General structure of soft integrals

We define a differential soft Feynman integral including L loops and m soft final state particles by

$$I^D(s, w_1, w_2, x) = c_{L+m} \int \frac{d\Phi_{h+m}^{\text{soft}}}{dw_1 dw_2 dx} \prod_{i=0}^L \frac{d^d p_{2+m+i}}{(2\pi)^d} \mathcal{I}^D(p_h, p_1, \dots, p_{2+L+m}, \epsilon). \tag{2.15}$$

We define the constant

$$c_L = (4\pi)^{-L\epsilon} e^{L\epsilon\gamma_E}. \tag{2.16}$$

The integrand \mathcal{I}^D is a ratio of polynomials in Lorentz invariant scalar products of the external momenta and loop momenta as well as the dimensional regulator. The superscript label ‘D’ indicates that we refer to these integrals as *differential* soft integrals. In contrast, we define *inclusive* soft integrals with a superscript ‘I’ as

$$\begin{aligned}
 I^I(s, \bar{z}) &= c_{L+m} \int d\Phi_{h+m}^{\text{soft}} \prod_{i=0}^L \frac{d^d p_{2+m+i}}{(2\pi)^d} \mathcal{I}^I(p_h, p_1, \dots, p_{2+L+m}, \epsilon) \\
 &= \int_0^1 dx \int_0^\infty dw_1 dw_2 \delta(\bar{z} - w_1 - w_2) I^D(s, w_1, w_2, x).
 \end{aligned} \tag{2.17}$$

For example, the inclusive soft phase space volume is given by

$$\begin{aligned}
 \int d\Phi_{h+m}^{\text{soft}} &= \int_0^1 dx \int_0^\infty dw_1 dw_2 \delta(\bar{z} - w_1 - w_2) \lim_{w_{1,2} \rightarrow 0} \frac{d\Phi_{h+m}}{dw_1 dw_2 dx} \\
 &= \frac{(4\pi)^{1-2\epsilon m} e^{\gamma_E \epsilon m} (p_h^2)^{-1+m-m\epsilon} \bar{z}^{-1+2m(1-\epsilon)} \Gamma(1-\epsilon)^m}{2\Gamma(2m(1-\epsilon))}.
 \end{aligned} \tag{2.18}$$

Properly chosen integrands of soft integrals are characterized by a rescaling symmetry.

$$\begin{aligned}
 \mathcal{I}^D(p_h, \lambda p_1, p_2, \dots, p_{2+L+m}, \epsilon) &= \lambda^{\alpha_1} \mathcal{I}^D(p_h, p_1, p_2, \dots, p_{2+L+m}, \epsilon), \\
 \mathcal{I}^D(p_h, p_1, \lambda p_2, \dots, p_{2+L+m}, \epsilon) &= \lambda^{\alpha_2} \mathcal{I}^D(p_h, p_1, p_2, \dots, p_{2+L+m}, \epsilon),
 \end{aligned} \tag{2.19}$$

for integer exponents α_1 and α_2 that can be determined easily from the specific integrands. This is most easily illustrated by looking at an example. For the following integrand of a

phase space integral with two additional partons in the final state, we find the exponents $\alpha_1 = \alpha_2 = -1$:

$$\mathcal{I}_{\text{example}}^{\text{D}}(p_h, p_1, p_2, p_3, p_4, \epsilon) = \frac{1}{(2p_1 p_3)(2p_2 p_4)}. \quad (2.20)$$

From the fact that s is the only Lorentz invariant variable in our set of variables of eq. (2.2), we conclude that any differential soft integral takes the form

$$I^{\text{D}}(s, w_1, w_2, x) = s^{\Lambda - (m+L)\epsilon} w_1^{\delta_1 - (m+L)\epsilon} w_2^{\delta_2 - (m+L)\epsilon} f(x, \epsilon). \quad (2.21)$$

Above, the integer mass dimension Λ and the integer exponents δ_1 and δ_2 can be determined from the integrand using the rescaling symmetry in conjunction with the dependence of the loop and phase space measure on the variable s , w_1 and w_2 . The integers m and L are the number of final state partons that were integrated out and the number of loops. The variable x is invariant under a rescaling of the momenta p_1 and p_2 , and consequently, our soft differential integrals have a non-trivial functional dependence on x in the form of a function $f(x, \epsilon)$. Inclusive soft master integrals then take the form

$$I^{\text{I}}(s, \bar{z}) = s^{\Lambda - (m+L)\epsilon} \bar{z}^{\delta_1 + \delta_2 + 1 - 2(m+L)\epsilon} \tilde{f}(\epsilon), \quad (2.22)$$

where \bar{z} is introduced via eq. (2.17) and $\tilde{f}(\epsilon)$ is a function of the dimensional regulator.

Having identified the structure of differential soft integrals, we can now discuss what happens when crossing from production to DIS or e^+e^- kinematics. First, we note that the variable x is by definition (eq. (2.2)) invariant under crossing the partons with momenta p_1 or p_2 from initial to final state or vice versa. We can collect the dependence of soft integrals on the remaining variables s , w_1 , and w_2 using eq. (2.21) into one prefactor.

$$I^{\text{D}} \sim (s w_1 w_2)^{-(m+L)\epsilon} = \left(\frac{(2p_1 k)(2p_2 k)}{(2p_1 p_2)} \right)^{-(m+L)\epsilon}. \quad (2.23)$$

From the above equation, we easily see that this factor is also invariant under crossing p_1 or p_2 from the initial to the final state or vice versa. Consequently, differential soft integrals are identical for production, DIS or e^+e^- scattering kinematics, up to an overall sign that can be determined from the integer powers Λ , δ_1 , and δ_2 .

3 Computing soft master integrals

We begin this section by outlining our method to compute soft integrals. After that, we show our explicit results for phase space integrals with two additional final state partons, ($m = 2$, $L = 0$ in eq. (2.15)). We refer to these integrals as double real (RR) phase space integrals. We then briefly discuss our results for soft integrals with three additional partons in the final state (triple real; RRR) and soft integrals with two additional partons in the final state and one loop integral (double-real virtual; RRV).

3.1 Method

Soft Feynman integrals can be related to each other via the framework of reverse unitarity [26–30] and IBP identities [62–64]. We construct a basis of master integrals for soft integrals involving a certain number of loop and phase space integrals. We then use methods developed in refs. [65, 66] to construct a basis for so-called canonical master integrals. We construct such a basis for both differential and inclusive soft integrals. We use differential equations [33–37] for differential soft master integrals to compute the functional dependence of these integrals on the variable x . Next, we use eq. (2.17) to relate the differential and inclusive soft master integrals to each other. Since in many cases the inclusive soft master integrals can be evaluated directly, without first computing their differential analogs, we can express the boundary conditions required for the solution of the differential equations for the differential soft master integrals in terms of the inclusive soft master integrals. This relation between inclusive and differential master integrals also constrains some of the inclusive master integrals. We elaborate on this below in section 3.2.3 using an example. Additional consistency conditions that can be determined from the system of differential equations and from relations of the differential soft master integrals to systems of differential equations appearing in the computation of refs. [42, 43], and this gives additional constraints on the inclusive soft master integrals. Ultimately, we determine the remaining inclusive soft master integrals using direct integration techniques developed in refs. [23, 24]. We express the inclusive soft master integrals as a Laurent series in the differential regulator with rational numbers and the multiple ζ values as coefficients (and they depend on p_h^2 and \bar{z} as shown in eq. (2.22)). The resulting differential master integrals depend on the variables p_h^2 , w_1 , and w_2 , as illustrated in eq. (2.21), and the function $f(x)$ is given by a Laurent series in the dimensional regulator and harmonic polylogarithms [67] with argument x and multiple ζ values as coefficients. As we choose canonical integrals as our basis integrals, the master integrals have uniform transcendental weight. In particular, we compute in this article all soft master integrals up to transcendental weight eight, or equivalently up to $\mathcal{O}(\epsilon^8)$ in the Laurent expansion.

3.2 RR soft integrals with two final state partons

As an example, we discuss explicitly in this section the soft master integrals for pure phase space integrals with two partons in the final state that are integrated out (RR). The corresponding inclusive soft master integrals were presented already to all orders in the dimensional regulator in ref. [22].

3.2.1 Differential RR soft master integrals

We define three differential RR soft master integrals by

$$I_i^{\text{D-RR}} = c_2 \int \frac{d\Phi_{h+2}^{\text{soft}}}{dw_1 dw_2 dx} \mathcal{I}_i^{\text{D-RR}}. \tag{3.1}$$

The corresponding integrands are given by

$$\begin{aligned}\mathcal{I}_1^{\text{D-RR}} &= -\frac{s_{12}^2 \epsilon (s_{13} + s_{14})}{s_{14}}, \\ \mathcal{I}_2^{\text{D-RR}} &= \frac{s_{12}^2 \epsilon (s_{12}^2 s_{34} - s_{14} s_{23} - s_{14} s_{24})}{s_{14} s_{24}}, \\ \mathcal{I}_3^{\text{D-RR}} &= \frac{s_{12}^2 \epsilon (s_{12}^2 s_{34} - s_{12} s_{13} s_{23} - s_{12} s_{13} s_{24} - s_{12} s_{14} s_{23} - s_{12} s_{14} s_{24} + s_{13} s_{23} + s_{13} s_{24})}{s_{13} s_{24}}.\end{aligned}\tag{3.2}$$

Here, we used the notation

$$s_{ij} = (p_i + p_j)^2, \quad s_{ii} = p_i^2.\tag{3.3}$$

The integrated results in terms of our chosen variables of eq. (2.2) are given by

$$\begin{aligned}I_1^{\text{D-RR}} &= \frac{e^{2\gamma\epsilon} s_{12}^{-2\epsilon} w_1^{-2\epsilon} w_2^{-2\epsilon} (1-x)^{-\epsilon} x^{-\epsilon}}{128\pi^3 \Gamma(1-2\epsilon)}, \\ I_2^{\text{D-RR}} &= I_1^{\text{D-RR}} (1 - 2x {}_2F_1(1, 1; 1 - \epsilon; 1 - x)), \\ I_3^{\text{D-RR}} &= I_1^{\text{D-RR}} (-2x {}_2F_1(1, 1; \epsilon + 1; 1 - x) + 2x^\epsilon (1-x)^{-\epsilon} \Gamma(1-\epsilon) \Gamma(\epsilon+1) + 1).\end{aligned}\tag{3.4}$$

These results are valid to all orders in the dimensional regulator, and ${}_2F_1$ is the Gauss hypergeometric function,

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.\tag{3.5}$$

This function can be easily expanded in terms of a Laurent series in the dimensional regulator using the results of ref. [68].

3.2.2 Inclusive RR soft master integrals

To express a basis of double real inclusive soft master integrals we require two different master integrals.

$$I_i^{\text{I-RR}} = c_2 \int d\Phi_{h+2}^{\text{soft}} \mathcal{I}_i^{\text{I-RR}}.\tag{3.6}$$

The integrands are given by

$$\begin{aligned}\mathcal{I}_1^{\text{I-RR}} &= \epsilon^3 \frac{s_{12}}{s_{14} s_{34}}, \\ \mathcal{I}_1^{\text{I-RR}} &= \epsilon^3 \bar{z} \frac{s_{12}^2}{s_{13} s_{24} s_{34}}.\end{aligned}\tag{3.7}$$

The integrals are given by

$$\begin{aligned}I_1^{\text{I-RR}} &= (1-2\epsilon)(3-4\epsilon)(1-4\epsilon) \Phi_{h+2}^{\text{soft}} = \frac{e^{-2\gamma\epsilon} \bar{z}^{-4\epsilon} s_{12}^{-2\epsilon} \Gamma(1-\epsilon)^2}{256\pi^3 \Gamma(1-4\epsilon)}, \\ I_2^{\text{I-RR}} &= -\frac{3e^{-2\gamma\epsilon} \Gamma(1-2\epsilon)^2 \Gamma(1-\epsilon)}{128\pi^3 \Gamma(1-4\epsilon) \Gamma(1-3\epsilon)} \bar{z}^{-4\epsilon} s_{12}^{-2\epsilon} {}_3F_2(-\epsilon, -\epsilon, -\epsilon; 1-\epsilon, -3\epsilon; 1).\end{aligned}\tag{3.8}$$

3.2.3 Relating inclusive and differential RR soft master integrals

The fact that we compute a basis of master integrals simultaneously for the differential and inclusive cases can be very helpful. If we integrate the differential integrals over the inclusive soft phase space measure using IBP identities, we find that the result is related to the phase space volume.

$$\begin{aligned}
 c_2 \int d\Phi_{h+2}^{\text{soft}} \mathcal{I}_1^{\text{D-RR}} &= (1 - 2\epsilon)\Phi_{h+2}^{\text{soft}}, \\
 c_2 \int d\Phi_{h+2}^{\text{soft}} \mathcal{I}_2^{\text{D-RR}} &= -\Phi_{h+2}^{\text{soft}}, \\
 c_2 \int d\Phi_{h+2}^{\text{soft}} \mathcal{I}_3^{\text{D-RR}} &= \Phi_{h+2}^{\text{soft}}.
 \end{aligned}
 \tag{3.9}$$

Since we are computing the differential master integrals using the method of differential equations, we can fix the boundary conditions of the differential equations by performing the inclusive integration over all differential variables and demanding that the above equations are true. Conversely, we may integrate the integrand of an inclusive soft master integral over the differential soft measure.

$$c_2 \int \frac{d\Phi_{h+2}^{\text{soft}}}{dw_1 dw_2 dx} \mathcal{I}_2^{\text{I-RR}} = -\frac{s_{12}^2 \epsilon^2}{x(1-x)w_1 w_2} \left[I_1^{\text{D-RR}} + I_3^{\text{D-RR}} \right]
 \tag{3.10}$$

Subsequently integrating over the remaining variables w_1 , w_2 , and x we can determine the value of the inclusive master integral $I_2^{\text{I-RR}}$ and indeed find the solution of eq. (3.8).

$$I_2^{\text{I-RR}} = c_2 \int dx dw_1 dw_2 \delta(1 - w_1 - w_2) \int \frac{d\Phi_{h+2}^{\text{soft}}}{dw_1 dw_2 dx} \mathcal{I}_2^{\text{D-RR}}.
 \tag{3.11}$$

In this fashion, we determined all boundary conditions and inclusive RR soft master integrals by only computing the inclusive soft phase space volume (eq. (2.18)) explicitly. We observe in more complicated cases than the RR soft master integrals that additional boundary conditions need to be computed by other means.

3.3 RRR and RRV soft master integrals

One of the main results of this article is the computation of differential and inclusive soft master integrals through transcendental weight eight for RRV and RRR scattering configurations. To determine these integrals, we follow the steps outlined above, and we present our results in terms of computer-readable files together with the arXiv and journal submission of this article. In particular, we compute 24 differential and 14 inclusive RRV soft master integrals.

$$I_i^{\text{D-RRV}} = c_3 \int \frac{d\Phi_{h+2}^{\text{soft}}}{dw_1 dw_2 dx} \int \frac{d^d p_6}{(2\pi)^d} \mathcal{I}_i^{\text{D-RRV}},
 \tag{3.12}$$

$$I_i^{\text{I-RRV}} = c_3 \int d\Phi_{h+2}^{\text{soft}} \int \frac{d^d p_6}{(2\pi)^d} \mathcal{I}_i^{\text{I-RRV}}.
 \tag{3.13}$$

Furthermore, we compute 61 differential and 13 inclusive soft RRR master integrals.

$$I_i^{\text{D-RRR}} = c_3 \int \frac{d\Phi_{h+3}^{\text{soft}}}{dw_1 dw_2 dx} \mathcal{I}_i^{\text{D-RRR}}, \quad (3.14)$$

$$I_i^{\text{I-RRR}} = c_3 \int d\Phi_{h+3}^{\text{soft}} \mathcal{I}_i^{\text{I-RRR}}. \quad (3.15)$$

Many of our integrals were computed already through transcendental weight six for the purpose of refs. [23, 24, 40, 42, 43], and we find agreement with these past results. In order to ensure the correctness of our soft integrals, we furthermore explicitly derive numerical results for them through all calculated orders in ϵ using Mellin-Barnes (MB) techniques. Indeed, the properties of the integrands under rescaling in eq. (2.19) implies that for RRR we can easily integrate out the energies of the soft particles in terms of Γ functions. The remaining angular integrals can be performed in closed form as a MB integral [69]. Following this strategy, we can easily obtain a MB representation for all RRR soft integrals [23], which can be evaluated numerically as a Laurent series in the dimensional regulator using standard techniques [70, 71]. For the RRV integrals, we cannot immediately perform the integration over the energies in terms of Γ functions, because they are entangled with the loop integration. We can, however, easily introduce additional MB integrations for the (soft regions of the) one-loop integrals involved [24], and then evaluate numerically the MB representations for the combined phase space and loop integrations.

4 Threshold limit of production cross sections

In this section, we discuss the LHC cross sections for the production of a virtual photon or a Higgs boson in gluon fusion in the infinite top quark mass limit:

$$\sigma_B = \tau \hat{\sigma}_0^B C_B^2 \sum_{ij} f_i(\tau) \circ_\tau \eta_{ij}^B(\tau) \circ_\tau f_j(\tau), \quad B \in \{H, \gamma^*\}. \quad (4.1)$$

In the above equation the f_i are parton distribution function, $\hat{\sigma}_0^B$ represents the partonic Born cross section and we define the ratio $\tau = Q^2/S$, where Q is the virtuality of the produced boson¹ and S is the hadronic center-of-mass energy. The PDFs are convoluted with the partonic coefficient functions, see appendix B for details. The partonic coefficient functions are given by

$$\eta_{ij}^B(z) = \frac{\mathcal{N}_{ij}}{2Q^2 \hat{\sigma}_0^B} \sum_{m=0}^{\infty} \int d\Phi_{h+m} \mathcal{M}_{ij \rightarrow B+m}. \quad (4.2)$$

The initial state dependent normalization factor \mathcal{N}_{ij} is given by

$$\mathcal{N}_{gg} = \frac{1}{4(n_c^2 - 1)^2 (1 - \epsilon)^2}, \quad \mathcal{N}_{gq} = \frac{1}{4(n_c^2 - 1)n_c(1 - \epsilon)}, \quad \mathcal{N}_{q\bar{q}} = \frac{1}{4n_c^2}, \quad (4.3)$$

where g , q and \bar{q} represent a gluon, quark and anti-quark respectively, and n_c denotes the number of fundamental $\text{SU}(n_c)$ colors. The factor C_B is equal to one for the production

¹In case of the Higgs boson production cross section, Q is identical to the Higgs boson mass.

cross section of a virtual photon and equal to the Wilson coefficient [72–75] for the infinite top quark mass effective field theory [76–79]. $\mathcal{M}_{ij \rightarrow B+m}$ represents the color and spinor summed interference of scattering amplitudes describing the production of the desired boson B and m final state partons in the collision of initial state partons i and j . Expanding $\mathcal{M}_{ij \rightarrow B+m}$ in the strong coupling constant α_S gives the perturbative coefficient functions in QCD perturbation theory.

$$\eta_{ij}^B(z) = \sum_{i=0}^{\infty} a_S^i \eta_{ij}^{B,(i)}(z), \quad a_S = \frac{\alpha_S}{\pi} (4\pi)^\epsilon e^{-\epsilon\gamma_E}. \quad (4.4)$$

In the definition of a_S we include for convenience already a factor $(4\pi)^\epsilon e^{-\epsilon\gamma_E}$, anticipating later $\overline{\text{MS}}$ renormalization. At Born level we find

$$\eta_{gg}^{H,(0)}(z) = \frac{1}{1-\epsilon} \delta(\bar{z}), \quad \eta_{q\bar{q}}^{\gamma^*,(0)}(z) = (1-\epsilon) \delta(\bar{z}), \quad \bar{z} = 1-z. \quad (4.5)$$

For more details see refs. [4, 6].

In this article, we are interested in the threshold limit of the partonic coefficient function. This limit is characterized by the kinematic condition that all the radiation produced alongside the produced boson is very low energetic, and we consider the limit $\bar{z} \rightarrow 0$. In this limit, the partonic coefficient function factorizes as follows.

$$\lim_{\bar{z} \rightarrow 0} \eta_{ij}^B = H_{ij}^B \times S_{\text{thr.}}^{R_{ij}}(z). \quad (4.6)$$

Above, H_{ij}^B is the process-dependent hard function and $S_{\text{thr.}}^{R_{ij}}$ is the so-called threshold soft function that only depends on the color representation of the initial state partons R_{ij} . The hard function was computed for Higgs boson and photon production through three loops in refs. [80–83] and at fourth loop order in refs. [18, 19, 84–86]. Similarly, $S_{\text{thr.}}^{R_{ij}}$ was computed through N³LO in QCD perturbation theory in refs. [31, 87, 88] and we discuss partial fourth loop-order results below.

The threshold soft function can be computed by considering the strict soft limit of the partonic coefficient function of one of Higgs boson or photon production.

$$S_{\text{thr.}}^{\text{adjoint}}(z) = (1-\epsilon) \lim_{\text{strict soft}} \eta_{ij}^H(z),$$

$$S_{\text{thr.}}^{\text{fundamental}}(z) = \frac{1}{(1-\epsilon)} \lim_{\text{strict soft}} \eta_{ij}^{\gamma^*}(z). \quad (4.7)$$

The strict soft limit is defined by taking all final state parton momenta to be very low energetic and all loop momenta to be uniformly low energetic as well. It is now easy to see that the partonic cross section in this limit can be expressed in terms of soft master integrals as introduced in previous sections. In practice, this is achieved by following the method of regions [61] and using techniques introduced in refs. [23, 24]. To compute the threshold soft function contribution at n^{th} perturbative order, matrix elements with up to n additional soft partons in the final state must be included. Contributions with one additional parton can be extracted from the computation of the one emission current at one and two loop order [46, 89–91]. The integrand for two and three additional partons

required for computations up to N³LO in QCD perturbation theory was determined for the purposes of refs. [23, 24, 31, 40, 42, 43], and we build on these results here. In particular, we use our newly computed soft master integrals to compute $S_{\text{thr.}}^{R_{ij}}(z)$ at N³LO in QCD perturbation theory. This result was previously obtained in refs. [31, 87, 88, 92] through finite order in the dimensional regulator ϵ at N³LO, and we find agreement. We extend these results here to include two additional orders in the Laurent expansion in the dimensional regulator, which will serve as a key ingredient for a future computation of the Higgs boson and Drell-Yan production cross section at N⁴LO. In particular, we compute results for the bare partonic cross section $\eta_{ij}^B(z)$ in the threshold limit to $\mathcal{O}(\epsilon^{8-2n})$ at NⁿLO in QCD perturbation theory for $n \in \{0, 1, 2, 3\}$. We attach our results in electronically readable form alongside the arXiv and journal submission of this article.

5 Threshold factorization and soft anomalous dimension

In this section we discuss the universal nature of the partonic scattering cross section at threshold. Next, we use results from existing literature to extract anomalous dimensions which play a crucial role in the resummation of large logarithm in a diverse range of scattering processes.

5.1 Threshold factorization

The inclusive cross section for the production of a colorless final state factorizes in the limit where the hadronic center-of-mass energy becomes similar to the invariant mass of the colorless system, i.e., $\tau \rightarrow 1$. This was realized in refs. [48–51] for QCD and derived in the language of soft-collinear effective theory (SCET) [93–97] in refs. [52–54]. Mathematically, we may write eq. (4.1) in this limit as²

$$\sigma_B = \hat{\sigma}_0^B \sum_{ij} H_{ij}^B f_i^{\text{th}}(\tau) \otimes_{\tau} S_{\text{thr.}}^r(\tau) \otimes_{\tau} f_j^{\text{th}}(\tau) + \mathcal{O}(1 - \tau). \tag{5.1}$$

Above, the product \otimes_{τ} is defined in the appendix in eq. (B.5) and $r = R_{ij}$ denotes the color representation of the initial state partons. The hard function $H_{ij}^B(\mu^2)$ is the squared Wilson coefficient of the leading power hard scattering operator that couples the color singlet to the partons i and j . It is related to the form factor of such an operator.

We can define the soft function as a squared matrix element of soft Wilson lines [98, 99] involving a measurement over a complete set of soft states $|X_s\rangle$,

$$S_{\text{thr.}}^r(z) = \frac{1}{C_r} \sum_{X_s} \text{tr} \langle 0 | Y_{\bar{n}}^{\dagger r}(0) Y_n^r(0) \delta(\hat{E}^2 - Q^2(1 - z)) | X_s \rangle \langle X_s | Y_n^{\dagger r}(0) Y_{\bar{n}}^r(0) | 0 \rangle, \tag{5.2}$$

where the operator \hat{E} picks up the total energy of the real emissions, the Wilson lines are taken in the representation r of the scattering partons and along their light-like direction n, \bar{n} and C_r is the quadratic Casimir of the gauge group for the representation r . Note that for the rest of this section, we will drop the label r for the representation, but the

²For simplicity, we set here $C_B = 1$ as it can easily be absorbed in the hard function.

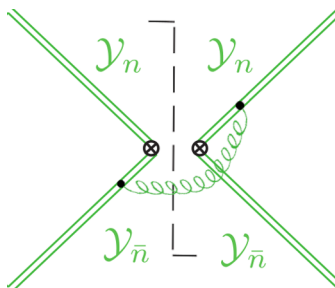


Figure 2. Cut diagram for the one real emission correction to the threshold soft function S_{thr}^r for color singlet production. Doubled lines represent Wilson lines that trace the path of the initial state partons involved in the hard scattering process. The wiggled line represents a radiated gluon which crosses the final state phase space cut indicated by the dashed line.

following discussion straightforwardly applies to both the case of adjoint and fundamental representation. Explicitly the Wilson lines take the form

$$Y_n(x) = \mathbf{P} \exp \left[ig \int_{-\infty}^0 dt n \cdot A_s(x + tn) \right]. \tag{5.3}$$

The contributions to $S_{\text{thr}}^r(z)$ can be represented, order by order in perturbation theory, in terms of cut diagrams and can be calculated in terms of eikonal Feynman rules, see for example figure 2.

The quantities in eq. (5.1) are bare quantities that are individually ultraviolet and infrared divergent, and the divergences manifest themselves as poles in the dimensional regulator ϵ . We implement the renormalization of the strong coupling constant via the operator \mathbf{Z}_{α_S} (see appendix A), which expresses the bare strong coupling constant in terms of its renormalized counterpart. We absorb infrared singularities of the hard function into $Z_H(\mu^2)$. The threshold PDFs $f_i^{\text{th}}(\tau)$ absorb collinear initial state singularities via a standard mass factorization counter term (see appendix B for details).

$$\begin{aligned} H_{ij}^B(\mu^2) &= Z_H(\mu^2) \mathbf{Z}_{\alpha_S} H_{ij}^B, \\ f_i^{\text{th}}(\tau, \mu^2) &= \Gamma^r(\tau) \otimes_{\tau} f_i^{\text{th}}(\tau), \\ S_{\text{thr}}^r(\tau, \mu^2) &= Z_H(\mu^2)^{-1} \Gamma^r(\tau)^{-1} \otimes_{\tau} \mathbf{Z}_{\alpha_S} S_{\text{thr}}^r(\tau) \otimes_{\tau} \Gamma^r(\tau)^{-1}. \end{aligned} \tag{5.4}$$

Note, that we indicate renormalized and finite objects by explicitly indicating their dependence on the scale μ^2 .

In the context of SCET, the divergences appearing in the bare soft function can be interpreted as UV divergences in the effective theory. Therefore, we can absorb these divergences in an $\overline{\text{MS}}$ counterterm $Z_S(z, \mu^2)$ for this operator and obtain a renormalized soft function. Extracting from eq. (5.4) we find

$$\begin{aligned} S_{\text{thr}}^r(z, \mu^2) &= Z_S(z, \mu^2) \otimes_z \mathbf{Z}_{\alpha_S} S_{\text{thr}}^r(z), \\ Z_S(z, \mu^2) &= Z_H(\mu^2)^{-1} \Gamma^r(z)^{-1} \otimes_z \Gamma^r(z)^{-1}. \end{aligned} \tag{5.5}$$

The hard function obeys the renormalization group equation (RGE)

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} H_{ij}^B(\mu^2) &= \gamma_H^r(\alpha_S(\mu^2), \mu^2) H_{ij}^B(\mu^2), \\ \gamma_H^r(\alpha_S(\mu^2), \mu^2) &= \Gamma_{\text{cusp}}^r(\alpha_S(\mu)) \ln \frac{Q^2}{\mu^2} + \frac{1}{2} \gamma_H^r(\alpha_S(\mu^2)). \end{aligned} \quad (5.6)$$

Above $\Gamma_{\text{cusp}}^r(\alpha_S(\mu^2))$ is the cusp anomalous dimension [98]. Furthermore, we distinguish $\gamma_H^r(\alpha_S(\mu^2))$, the non-cusp part of the anomalous dimension, from the entire anomalous dimension $\gamma_H^r(\alpha_S(\mu^2), \mu^2)$ by the number of arguments. The renormalization group equation for the threshold PDFs is given by the DGLAP evolution equation [100–102] in the limit of $z \rightarrow 1$. Explicitly, the anomalous dimension of the threshold PDFs is the limit of $z \rightarrow 1$ of the Altarelli-Parisi splitting functions:

$$\begin{aligned} \gamma_f^{\text{fundamental}}(z, \alpha_S(\mu^2)) &= \lim_{z \rightarrow 1} P_{qq}(z, \alpha_S(\mu^2)), \\ \gamma_f^{\text{adjoint}}(z, \alpha_S(\mu^2)) &= \lim_{z \rightarrow 1} P_{gg}(z, \alpha_S(\mu^2)). \end{aligned} \quad (5.7)$$

With this we find

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} f_i^{\text{th}}(z, \mu^2) &= \gamma_f^r(z, \alpha_S(\mu^2)) \otimes_z f_i^{\text{th}}(z, \mu^2), \\ \gamma_f^r(z, \alpha_S(\mu^2)) &= \Gamma_{\text{cusp}}^r(\alpha_S(\mu^2)) \left[\frac{1}{1-z} \right]_+ + \frac{1}{2} \gamma_f^r(\alpha_S(\mu^2)) \delta(1-z). \end{aligned} \quad (5.8)$$

Above, the plus distribution is defined via its action on a test function as

$$\int_0^1 dz \left[\frac{1}{1-z} \right]_+ \phi(z) = \int_0^1 dz \left(\frac{1}{1-z} \right) (\phi(z) - \phi(1)). \quad (5.9)$$

Since the hadronic cross section is independent of the scale μ^2 , the threshold soft function also satisfies an RGE that can be derived by consistency:

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} S_{\text{thr}}^r(z, \mu^2) &= \gamma_{\text{thr}}^r(z, \alpha_S(\mu^2)) \otimes_z S_{\text{thr}}^r(z, \mu^2), \\ \gamma_{\text{thr}}^r(z, \alpha_S(\mu^2)) &= -\Gamma_{\text{cusp}}^r(\alpha_S(\mu^2)) \left(2 \left[\frac{1}{1-z} \right]_+ + \delta(1-z) \log \frac{Q^2}{\mu^2} \right) \\ &\quad + \frac{1}{2} \gamma_{\text{thr}}^r(\alpha_S(\mu^2)) \delta(1-z). \end{aligned} \quad (5.10)$$

In order for the hadronic cross section in eq. (5.1) to be independent of the scale, the following equation has to be satisfied.

$$\gamma_{\text{thr}}^r(\alpha_S(\mu^2)) = -2\gamma_f^r(\alpha_S(\mu^2)) - \gamma_H^r(\alpha_S(\mu^2)). \quad (5.11)$$

5.2 Results for the threshold anomalous dimension through N⁴LO

All but one ingredient to compute the threshold limit of the partonic coefficient function for DY and Higgs boson production at N⁴LO are currently available. The one missing ingredient is the coefficient of $\delta(\bar{z})$ of the threshold soft function at N⁴LO. The hard function can be extracted from the computation of the purely virtual matrix elements computed through four loops in QCD in refs. [19, 80–86]. The ultraviolet renormalization counterterms are given in terms of the QCD beta function [103–108]. The cusp anomalous dimension is known through fourth loop order [18, 19]. The mass factorization counterterm at threshold can be determined from the Altarelli-Parisi splitting functions, which have been determined through second [13–17, 109, 110] and third [55–60] non-trivial order. The last ingredient is not yet available in fully analytic form, and some constants have been determined only numerically, based on an extraction from computation of several moments of the full splitting functions and a leading color computation. The consequence is that the splitting functions are afflicted by an, albeit small, numerical uncertainty. Our computation of the soft function to higher orders in the dimensional regulator through third order enables us to determine all contributions to the soft function at fourth order originating from ultraviolet and infrared counter terms.

We first construct the finite, renormalized partonic coefficient function through N⁴LO in QCD using the definitions of the previous sections. We define

$$\eta_{ij, \text{thr}}^B(z, \mu^2) = \lim_{z \rightarrow 1} \eta_{ij}^B(z, \mu^2) = H_{ij}^B(\mu^2) S_{\text{thr}}^{R_{ij}}(z, \mu^2). \quad (5.12)$$

We can then compare this with the existing results of ref. [55] and find agreement. Next, we want to exploit that the soft function is described by Wilson lines, as described above. This implies that the difference between the threshold soft function determined from the DY and Higgs boson cross section is only given by the color representation of the Wilson lines associated with the ingoing partons. To make this statement manifest we take the logarithm of the soft function and replace color factors as indicated below.

$$\begin{aligned} & \log\left(\eta_{ij, \text{thr}}^{DY}(z, \mu^2) / H_{ij}^{DY}(\mu^2)\right) \\ & \stackrel{!}{=} \frac{C_F}{C_A} \log\left(\eta_{ij, \text{thr}}^{\text{Higgs}}(z, \mu^2) / H_{ij}^{\text{Higgs}}(\mu^2)\right) \Bigg|_{C_{AA}^4 \rightarrow C_A / C_F C_{AF}^4, C_{AF}^4 \rightarrow C_A / C_F C_{FF}^4} + \mathcal{O}(\alpha_S^5), \end{aligned} \quad (5.13)$$

where $C_{R_1 R_2}^4$ denote quartic Casimir operators (see appendix C). The principle governing the above identity is often referred to as generalized Casimir scaling [59], which is a direct consequence of the fact that the soft function can be written as a matrix element of soft Wilson lines. Note, that the simple change of Casimir invariants as indicated above is particular to fourth order results.

We use eq. (5.13) to constrain some of the currently unknown coefficients of the third-order splitting function or to derive relations among them. In refs. [55, 60] the unknown coefficients of the $\gamma_f^{r, (4)}$ (half the coefficient of $\delta(\bar{z})$ in the four loop quark and gluon splitting function) are organized in terms of the contributing color factors. For example,

$$\frac{1}{2} \gamma_f^{\text{fundamental}, (4)} = C_F^4 b_{q, C_F}^4 + C_F^3 C_A b_{q, C_F^3 C_A}^4 + n_f C_F^3 b_{q, C_F^3 n_f}^4 + \dots \quad (5.14)$$

In the end, we are able to determine all but four unknown coefficients for both the gluon and quark splitting function. Explicitly we determine the following previously unknown constants:

$$\begin{aligned}
 b_{q, C_F^4}^4 &= -384\zeta_2\zeta_5 - 120\zeta_2\zeta_3 + 64\zeta_4\zeta_3 - 450\zeta_2 - 342\zeta_4 - 2111\zeta_6 \\
 &\quad + 5880\zeta_7 - 2520\zeta_5 - 1152\zeta_3^2 + 2004\zeta_3 + \frac{4873}{24}. \tag{5.15} \\
 b_{q, C_F^3 C_A}^4 &= 2064\zeta_2\zeta_5 - \frac{1988}{3}\zeta_2\zeta_3 + 128\zeta_4\zeta_3 + 1167\zeta_2 \\
 &\quad + 2167\zeta_4 + \frac{79297\zeta_6}{18} - 10920\zeta_7 - 976\zeta_5 + 3220\zeta_3^2 - 3260\zeta_3 - \frac{2085}{4}. \\
 b_{q, C_F^2 C_A^2}^4 &= -2104\zeta_2\zeta_5 + \frac{2096}{9}\zeta_2\zeta_3 - 32\zeta_4\zeta_3 - \frac{46771\zeta_2}{27} \\
 &\quad - \frac{60850\zeta_4}{27} - \frac{5497\zeta_6}{2} + 8610\zeta_7 + \frac{5354\zeta_5}{9} - \frac{7102\zeta_3^2}{3} + \frac{129662\zeta_3}{27} + \frac{29639}{36}.
 \end{aligned}$$

Furthermore, we find the following relations.

$$\begin{aligned}
 b_{g, n_f C_F^2 C_A}^4 &= b_{q, n_f C_F C_A^2}^4. \tag{5.16} \\
 b_{g, C_{AA}^4}^4 &= b_{q, C_{AF}^4}^4 - 272\zeta_2\zeta_3 + \frac{1184\zeta_2}{3} - \frac{508\zeta_4}{3} + \frac{748\zeta_6}{9} \\
 &\quad + \frac{760\zeta_5}{3} - \frac{784\zeta_3}{3} - \frac{800}{9}. \\
 b_{g, C_A^4}^4 &= -\frac{1}{24}b_{q, C_{AF}^4}^4 + 80\zeta_2\zeta_5 - \frac{3902}{9}\zeta_2\zeta_3 + 168\zeta_4\zeta_3 + \frac{2098\zeta_2}{27} + \frac{8965\zeta_4}{54} \\
 &\quad - \frac{19129\zeta_6}{54} + 700\zeta_7 - \frac{14617\zeta_5}{9} + \frac{682\zeta_3^2}{3} + \frac{48088\zeta_3}{27} + \frac{50387}{486}.
 \end{aligned}$$

The identified coefficients and relations are consistent with the numerical values found in refs. [55, 60]. The four remaining coefficients are known only numerically as determined by table 1 of ref. [60] and we show their values here.

$$\begin{aligned}
 b_{q, n_f C_F^2 C_A}^4 &= -455.247 \pm 0.005. \tag{5.17} \\
 b_{q, C_{AF}^4}^4 &= -998.0 \pm 0.2. \\
 b_{q, C_{FF}^4}^4 &= -143.6 \pm 0.2. \\
 b_{q, n_f C_F^3}^4 &= 80.780 \pm 0.005.
 \end{aligned}$$

Finally, we are able to determine the threshold anomalous dimension analytically up to the four unknown coefficients. We define the perturbative expansion of the threshold anomalous dimension as

$$\gamma_{\text{thr.}}^r = \sum_{i=0}^{\infty} \left(\frac{\alpha_S(\mu^2)}{\pi} \right)^i \gamma_{\text{thr.}}^{r,(i)}. \tag{5.18}$$

Numerically, we find that

$$\gamma_{\text{thr.}}^{\text{fundamental},(4)} = -41.8 \pm 0.01\%, \quad \gamma_{\text{thr.}}^{\text{adjoint},(4)} = -114.964 \pm 0.04\%. \tag{5.19}$$

We include the threshold anomalous dimension and the threshold soft function in analytic form as electronically readable files together with the arXiv submission of this article.

5.3 Thrust and N -jettiness anomalous dimensions to $N^4\text{LO}$

Consistency relations among SCET factorization theorems allow us to relate the threshold anomalous dimension to the anomalous dimensions driving the logarithmic behavior of SCET_I observables, which is a large class of observables including DIS at large- x , the jet-mass observable, as well as the thrust, C -parameter, and N -jettiness [111] event shapes.

Given the universality of SCET_I anomalous dimensions [112], it suffices to find a relation for one of these observables. For this, we take the factorization theorem of the non-singlet structure function at threshold in DIS [48, 50, 98, 113]

$$F_2^{\text{ns}}(x, Q^2) = \sum_q e_q^2 Q^2 H(Q^2, \mu^2) \int_x^1 d\xi J_q\left(Q^2 \frac{\xi - x}{x}, \mu^2\right) f_q^{\text{th}}(\xi, \mu^2), \quad (5.20)$$

where J_q is the SCET_I jet function that also appears in the factorization theorem for thrust [114–116] and N -Jettiness [111]. Its RGE reads

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} J_i(s, \mu^2) &= \int ds' \gamma_J^i(s - s', \mu^2) J_i(s', \mu^2), \\ \gamma_J^i(s, \mu^2) &= -\Gamma_{\text{cusp}}^i(\alpha_S(\mu^2)) \frac{1}{\mu^2} \left[\frac{\mu^2}{s} \right]_+ + \frac{1}{2} \gamma_J^i(\alpha_S(\mu^2)) \delta(s). \end{aligned} \quad (5.21)$$

The RGE invariance of the factorized cross section immediately implies that the non-cusp part of $\gamma_J^i(s, \mu^2)$ is

$$\gamma_J^i(\alpha_S(\mu^2)) = -\gamma_f^i(\alpha_S(\mu^2)) - \gamma_H^i(\alpha_S(\mu^2)), \quad (5.22)$$

and by using eq. (5.11) we can rewrite it in terms of the threshold and collinear anomalous dimension³

$$2\gamma_J^i(\alpha_S(\mu^2)) = \gamma_{\text{thr.}}^i(\alpha_S(\mu^2)) - \gamma_H^i(\alpha_S(\mu^2)). \quad (5.23)$$

We would like to emphasize that we extracted $\gamma_H^i(\alpha_S(\mu^2))$ from ref. [86] and $\gamma_f^i(\alpha_S(\mu^2))$ from refs. [55, 60] in combination with the relations found in section 5.2. For the SCET_I beam function [112] $B_i(t, z, \mu^2)$, one can either use the equivalence between the SCET_I jet

³Note also that, since from the thrust factorization it is trivial to show that

$$\gamma_H^i(\alpha_S(\mu^2)) + 2\gamma_J^i(\alpha_S(\mu^2)) + \gamma_S^i(\alpha_S(\mu^2)) = 0,$$

with $\gamma_S^i(\alpha_S(\mu^2))$ being the anomalous dimension of the thrust soft function, eq. (5.23) implies that the threshold soft function anomalous dimension is the opposite of the thrust soft function anomalous dimension

$$\gamma_S^i(\alpha_S(\mu^2)) = -\gamma_{\text{thr.}}^i(\alpha_S(\mu^2)).$$

and beam function anomalous dimensions and eq. (5.23), or repeat this exercise using the generalized threshold factorization theorem [117], to show that the following relation holds

$$\mu^2 \frac{d}{d\mu^2} B_i(t, z, \mu^2) = \int dt' \gamma_B^i(t-t', \mu^2) B_i(t', \mu^2), \quad (5.24)$$

$$\gamma_B^i(t, \mu^2) = -\Gamma_{\text{cusp}}^i(\alpha_S(\mu^2)) \frac{1}{\mu^2} \left[\frac{\mu^2}{t} \right]_+ + \frac{1}{4} \underbrace{\left(\gamma_{\text{thr.}}^i(\alpha_S(\mu^2)) - \gamma_H^i(\alpha_S(\mu^2)) \right)}_{2\gamma_B^i(\alpha_S(\mu^2))} \delta(t).$$

For completeness, we include the jet/beam function anomalous dimension in analytic form as electronically readable files together with the arXiv and journal submission of this article.

6 Conclusions

Throughout this paper, we computed analytic results for so-called soft master integrals. These integrals play a crucial part in the analytic computation of scattering cross sections involving two identified hadrons, and we discussed how these integrals relate to LHC production cross sections, semi-inclusive deeply inelastic scattering, and e^+e^- annihilation. Our integrals are essential ingredients to compute perturbative cross sections through N³LO and beyond in QCD and QED perturbation theory.

We have presented explicit analytic results for differential and inclusive soft master integrals as a Laurent series in the dimensional regulator for partonic scattering processes involving two initial state partons and two or three soft final state partons on top of a colorless final state. Our calculation extends available results in the literature, as we include the first nine terms in the expansion in the dimensional regulator.

Our differential master integrals are integrated analytically over the final state parton momenta and retain all differential dependence on the four-momentum of the colorless final state. These integrals depend in a non-trivial fashion on one dimensionless variable in terms of functions expressed as harmonic polylogarithms. Our inclusive soft master integrals are in addition integrated over the degrees of freedom of the colorless final state particle and are given by linear combinations of multiple zeta values. Our differential and inclusive master integrals are so-called pure functions of uniform transcendental weight. We discuss explicitly how the computation of such soft master integrals is greatly facilitated by the simultaneous computation of differential and inclusive soft master integrals in conjunction with the use of the method of differential equations.

We build on a prior computation of the inclusive cross section for the production of a Higgs boson or a lepton pair at the LHC at the production threshold and express the corresponding partonic coefficient function in terms of our soft master integrals. With this, we compute threshold corrections to these partonic cross sections to two higher powers in the dimensional regulator. These results form a crucial ingredient for a future determination of the Drell-Yan and Higgs boson production cross section at N⁴LO in perturbative QCD.

Finally, we recap the factorization of cross sections describing the production of a colorless final state in the threshold limit. We extract all required anomalous dimensions and find agreement with previous results. We then explicitly determine the so-called threshold

soft function at N⁴LO in perturbative QCD up to one constant that is yet to be determined from a genuine computation at this order. Furthermore, we extract the threshold anomalous dimension through fourth loop order. Finally, using the generalized Casimir scaling property of the threshold soft function, we obtain new analytic results for several coefficients in the four-loop Altarelli-Parisi splitting functions in the term proportional to $\delta(1-z)$.

We present many of our results as ancillary files in electronically readable form appended to the arXiv and as supplementary material attached to this article, and we enumerate them here:

1. Definitions and solutions for canonical inclusive soft master integrals for RR, RRV, and RRR production cross sections. In our solutions we set $\bar{z} = Q^2 = 1$ as the functional dependence on these variables is easily restored by multiplying the solutions with $(\bar{z}Q)^{-2(m+L)\epsilon}$, where m and L are the number of soft partons and loops respectively.
2. Definitions and solutions for canonical differential soft master integrals for RR, RRV, and RRR production cross sections. In our solutions we set $w_1 = w_2 = Q^2 = 1$ as the functional dependence on these variables is easily restored by multiplying the solutions with $(w_1 w_2 Q^2)^{-(m+L)\epsilon}$, where m and L are the number of soft partons and loops respectively. Furthermore, we include the matrices A for the canonical differential equations of our differential soft master integrals I , which take the form

$$d\vec{I} = dA \cdot \vec{I}. \tag{6.1}$$

3. The bare inclusive soft-virtual cross section for the production of a Higgs boson or a Drell-Yan lepton pair through N³LO in perturbative QCD and including two additional powers in the dimensional regulator, i.e., in total the first nine terms in the expansion in ϵ at every perturbative order.
4. The renormalized, finite soft function (see eq. (5.5)) for the production of a colorless final state by scattering of quarks or gluons through N⁴LO in perturbative QCD. The soft function is determined up to one remaining constant at N⁴LO multiplying a Dirac delta distribution of \bar{z} .
5. The threshold anomalous dimension through N⁴LO in perturbative QCD. In particular, we include $\gamma_{\text{thr.}}^r(\alpha_S(\mu^2))$ of eq. (5.11).
6. The N-Jettiness beam function anomalous dimension $\gamma_J^r(\alpha_S(\mu^2))$ through N⁴LO in perturbative QCD of eq. (5.23).

Acknowledgments

GV and BM are supported by the United States Department of Energy, Contract DE-AC02-76SF00515.

A Ultraviolet renormalization

The strong coupling constant renormalizes as

$$\alpha_S = \alpha_S(\mu^2) (4\pi)^{-\epsilon} e^{\epsilon\gamma_E} Z_\alpha(\mu^2), \quad (\text{A.1})$$

with

$$\begin{aligned} Z_{\alpha_S}(\mu^2) = & 1 + a_S(\mu^2) \left[-\frac{\beta_0}{4\epsilon} \right] \\ & + a_S^2(\mu^2) \left[\frac{\beta_0^2}{16\epsilon^2} - \frac{\beta_1}{16\epsilon} \right] \\ & + a_S^3(\mu^2) \left[\frac{-\beta_0^3}{64\epsilon^3} + \frac{7\beta_1\beta_0}{384\epsilon^2} - \frac{\beta_2}{192\epsilon} \right] \\ & + a_S^4(\mu^2) \left[\frac{\beta_0^4}{256\epsilon^4} - \frac{23\beta_1\beta_0^2}{3072\epsilon^3} + \frac{3\beta_1^2}{2048\epsilon^2} - \frac{5\beta_0\beta_2}{1536\epsilon^2} - \frac{\beta_3}{1024\epsilon} \right] + \mathcal{O}(a_S^4(\mu^2)), \end{aligned} \quad (\text{A.2})$$

with $a_S(\mu^2) = \alpha_S(\mu^2)/\pi$ and β_i the coefficients of the QCD beta function [103–108]. The Wilson coefficient [72–75] for the heavy top effective theory renormalizes as

$$C_t = Z_t(\mu^2) C_t(\alpha_S(\mu^2), m_t^2, \mu^2), \quad Z_t(\mu^2) = \frac{1}{1 - \frac{\beta(\alpha_S(\mu^2))}{\epsilon}}. \quad (\text{A.3})$$

B Mass factorization

The mass factorization counter term absorbs collinear singularities into a suitable redefinition of the parton distribution functions. It is defined in terms of the following differential equation

$$\partial_{\mu^2} \Gamma_{ij}(x, \mu^2) = -a_S(\mu^2) \Gamma_{ik}(x, \mu^2) \circ_x P_{kj}(x, \mu^2), \quad \partial_{\mu^2} = \frac{d}{d \log(\mu^2)}. \quad (\text{B.1})$$

Here, P_{ij} are the Altarelli-Parisi splitting functions [13–17] and the above differential equation is derived from DGLAP evolution of PDFs [100–102]. We note that

$$\partial_{\mu^2} a_S(\mu^2) = -\epsilon a_S(\mu^2) - \beta(a_S(\mu^2)). \quad (\text{B.2})$$

We expand $\Gamma_{ij}(x, \mu^2)$ in the strong coupling constant and define

$$\Gamma_{ij}(x, \mu^2) = \sum_{o=0}^{\infty} a_S(\mu^2)^o \Gamma_{ij}^{(o)}(x, \mu^2), \quad \Gamma_{ij}^{(0)}(x, \mu^2) = \delta_{ij} \delta(1-x). \quad (\text{B.3})$$

In eq. (B.1) we make use of the Mellin convolution

$$f(z) \circ_z g(z) = \int_z^1 \frac{dx}{x} f(x) g\left(\frac{z}{x}\right). \quad (\text{B.4})$$

In the main part of this article we are interested in the soft limit of functions which enter such Mellin convolutions. A convolution of two functions $f(x)$ and $g(x)$ which have both

been computed in the limit $x \rightarrow 1$ will introduce power suppressed terms in $(1-x)$. It is thus useful to introduce another convolution which maintains the correct leading term of the convolution but does not introduce additional power suppressed terms.

$$f(z) \otimes_z g(z) = \int_z^1 dx f(x)g(1-x+z). \tag{B.5}$$

The above definition is easily found by expanding the original Mellin transform around the limit of $z \rightarrow 1$. We also define the mass factorization counter term in this limit to be

$$\Gamma^{\text{adjoint}}(x, \mu^2) = \lim_{x \rightarrow 1} \Gamma_{gg}(x, \mu^2), \quad \Gamma^{\text{fundamental}}(x, \mu^2) = \lim_{x \rightarrow 1} \Gamma_{qq}(x, \mu^2). \tag{B.6}$$

C Color

We define

$$C_A = n_c, \quad C_F = \frac{n_c^2 - 1}{2n_c}, \tag{C.1}$$

where n_c is the number of colors. We define the Casimir values by

$$C_{R_1 R_2}^n = \frac{1}{(n!)^2} \left(T_{R_1}^{\{a_1 \dots a_n\}} \right) \left(T_{R_2}^{\{a_1 \dots a_n\}} \right), \tag{C.2}$$

where the curly brackets indicate the fully symmetric trace over the terms and T_R^a is a generator of $SU(n_c)$ in representation R . We find for

- $n = 3$:

$$C_{FF}^3 = \frac{1}{16} \frac{n_c^2 - 4}{n_c} (n_c^2 - 1), \quad C_{FA}^3 = 0, \quad C_{AA}^3 = 0. \tag{C.3}$$

- $n = 4$:

$$C_{FF}^4 = \frac{1}{96} \frac{18 - 6n_c^2 + n_c^4}{n_c^2} (n_c^2 - 1), \tag{C.4}$$

$$C_{FA}^4 = \frac{1}{48} n_c (6 + n_c^2) (n_c^2 - 1),$$

$$C_{AA}^4 = \frac{1}{24} n_c^2 (36 + n_c^2) (n_c^2 - 1).$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

- [1] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, *Higgs Boson Gluon-Fusion Production in QCD at Three Loops*, *Phys. Rev. Lett.* **114** (2015) 212001 [[arXiv:1503.06056](https://arxiv.org/abs/1503.06056)] [[INSPIRE](#)].
- [2] C. Duhr, F. Dulat and B. Mistlberger, *Higgs Boson Production in Bottom-Quark Fusion to Third Order in the Strong Coupling*, *Phys. Rev. Lett.* **125** (2020) 051804 [[arXiv:1904.09990](https://arxiv.org/abs/1904.09990)] [[INSPIRE](#)].

- [3] C. Duhr, F. Dulat and B. Mistlberger, *Charged current Drell-Yan production at N^3LO* , *JHEP* **11** (2020) 143 [[arXiv:2007.13313](#)] [[INSPIRE](#)].
- [4] C. Duhr, F. Dulat and B. Mistlberger, *Drell-Yan Cross Section to Third Order in the Strong Coupling Constant*, *Phys. Rev. Lett.* **125** (2020) 172001 [[arXiv:2001.07717](#)] [[INSPIRE](#)].
- [5] F. Dulat, B. Mistlberger and A. Pelloni, *Precision predictions at N^3LO for the Higgs boson rapidity distribution at the LHC*, *Phys. Rev. D* **99** (2019) 034004 [[arXiv:1810.09462](#)] [[INSPIRE](#)].
- [6] B. Mistlberger, *Higgs boson production at hadron colliders at N^3LO in QCD*, *JHEP* **05** (2018) 028 [[arXiv:1802.00833](#)] [[INSPIRE](#)].
- [7] C. Duhr and B. Mistlberger, *Lepton-pair production at hadron colliders at N^3LO in QCD*, *JHEP* **03** (2022) 116 [[arXiv:2111.10379](#)] [[INSPIRE](#)].
- [8] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On Higgs decays to hadrons and the R -ratio at N^4LO* , *JHEP* **08** (2017) 113 [[arXiv:1707.01044](#)] [[INSPIRE](#)].
- [9] P.A. Baikov, K.G. Chetyrkin, J.H. Kuhn and J. Rittinger, *Complete $\mathcal{O}(\alpha_s^4)$ QCD Corrections to Hadronic Z -Decays*, *Phys. Rev. Lett.* **108** (2012) 222003 [[arXiv:1201.5804](#)] [[INSPIRE](#)].
- [10] M.A. Ebert, B. Mistlberger and G. Vita, *The Energy-Energy Correlation in the back-to-back limit at N^3LO and N^3LL'* , *JHEP* **08** (2021) 022 [[arXiv:2012.07859](#)] [[INSPIRE](#)].
- [11] L.J. Dixon, M.-X. Luo, V. Shtabovenko, T.-Z. Yang and H.X. Zhu, *Analytical Computation of Energy-Energy Correlation at Next-to-Leading Order in QCD*, *Phys. Rev. Lett.* **120** (2018) 102001 [[arXiv:1801.03219](#)] [[INSPIRE](#)].
- [12] L.J. Dixon, I. Moulton and H.X. Zhu, *Collinear limit of the energy-energy correlator*, *Phys. Rev. D* **100** (2019) 014009 [[arXiv:1905.01310](#)] [[INSPIRE](#)].
- [13] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The 3-loop pure singlet heavy flavor contributions to the structure function $F_2(x, Q^2)$ and the anomalous dimension*, *Nucl. Phys. B* **890** (2014) 48 [[arXiv:1409.1135](#)] [[INSPIRE](#)].
- [14] J. Ablinger, A. Behring, J. Blümlein, A. De Freitas, A. von Manteuffel and C. Schneider, *The three-loop splitting functions $P_{qg}^{(2)}$ and $P_{gg}^{(2, N_F)}$* , *Nucl. Phys. B* **922** (2017) 1 [[arXiv:1705.01508](#)] [[INSPIRE](#)].
- [15] J. Blümlein, P. Marquard, C. Schneider and K. Schönwald, *The three-loop unpolarized and polarized non-singlet anomalous dimensions from off shell operator matrix elements*, *Nucl. Phys. B* **971** (2021) 115542 [[arXiv:2107.06267](#)] [[INSPIRE](#)].
- [16] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case*, *Nucl. Phys. B* **688** (2004) 101 [[hep-ph/0403192](#)] [[INSPIRE](#)].
- [17] A. Vogt, S. Moch and J.A.M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case*, *Nucl. Phys. B* **691** (2004) 129 [[hep-ph/0404111](#)] [[INSPIRE](#)].
- [18] J.M. Henn, G.P. Korchemsky and B. Mistlberger, *The full four-loop cusp anomalous dimension in $\mathcal{N} = 4$ super Yang-Mills and QCD*, *JHEP* **04** (2020) 018 [[arXiv:1911.10174](#)] [[INSPIRE](#)].
- [19] A. von Manteuffel, E. Panzer and R.M. Schabinger, *Cusp and collinear anomalous dimensions in four-loop QCD from form factors*, *Phys. Rev. Lett.* **124** (2020) 162001 [[arXiv:2002.04617](#)] [[INSPIRE](#)].

- [20] O. Almelid, C. Duhr, E. Gardi, A. McLeod and C.D. White, *Bootstrapping the QCD soft anomalous dimension*, *JHEP* **09** (2017) 073 [[arXiv:1706.10162](#)] [[INSPIRE](#)].
- [21] O. Almelid, C. Duhr and E. Gardi, *Three-loop corrections to the soft anomalous dimension in multileg scattering*, *Phys. Rev. Lett.* **117** (2016) 172002 [[arXiv:1507.00047](#)] [[INSPIRE](#)].
- [22] C. Anastasiou, S. Buehler, C. Duhr and F. Herzog, *NNLO phase space master integrals for two-to-one inclusive cross sections in dimensional regularization*, *JHEP* **11** (2012) 062 [[arXiv:1208.3130](#)] [[INSPIRE](#)].
- [23] C. Anastasiou, C. Duhr, F. Dulat and B. Mistlberger, *Soft triple-real radiation for Higgs production at N^3LO* , *JHEP* **07** (2013) 003 [[arXiv:1302.4379](#)] [[INSPIRE](#)].
- [24] C. Anastasiou, C. Duhr, F. Dulat, E. Furlan, F. Herzog and B. Mistlberger, *Soft expansion of double-real-virtual corrections to Higgs production at N^3LO* , *JHEP* **08** (2015) 051 [[arXiv:1505.04110](#)] [[INSPIRE](#)].
- [25] C. Anastasiou et al., *Higgs Boson Gluon Fusion Production Beyond Threshold in N^3LO QCD*, *JHEP* **03** (2015) 091 [[arXiv:1411.3584](#)] [[INSPIRE](#)].
- [26] C. Anastasiou and K. Melnikov, *Higgs boson production at hadron colliders in NNLO QCD*, *Nucl. Phys. B* **646** (2002) 220 [[hep-ph/0207004](#)] [[INSPIRE](#)].
- [27] C. Anastasiou and K. Melnikov, *Pseudoscalar Higgs boson production at hadron colliders in NNLO QCD*, *Phys. Rev. D* **67** (2003) 037501 [[hep-ph/0208115](#)] [[INSPIRE](#)].
- [28] C. Anastasiou, L.J. Dixon and K. Melnikov, *NLO Higgs boson rapidity distributions at hadron colliders*, *Nucl. Phys. B Proc. Suppl.* **116** (2003) 193 [[hep-ph/0211141](#)] [[INSPIRE](#)].
- [29] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, *Dilepton rapidity distribution in the Drell-Yan process at NNLO in QCD*, *Phys. Rev. Lett.* **91** (2003) 182002 [[hep-ph/0306192](#)] [[INSPIRE](#)].
- [30] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, *High precision QCD at hadron colliders: Electroweak gauge boson rapidity distributions at NNLO*, *Phys. Rev. D* **69** (2004) 094008 [[hep-ph/0312266](#)] [[INSPIRE](#)].
- [31] C. Anastasiou et al., *Higgs boson gluon-fusion production at threshold in N^3LO QCD*, *Phys. Lett. B* **737** (2014) 325 [[arXiv:1403.4616](#)] [[INSPIRE](#)].
- [32] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog and B. Mistlberger, *Real-virtual contributions to the inclusive Higgs cross-section at N^3LO* , *JHEP* **12** (2013) 088 [[arXiv:1311.1425](#)] [[INSPIRE](#)].
- [33] A.V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett. B* **254** (1991) 158 [[INSPIRE](#)].
- [34] A.V. Kotikov, *Differential equations method: The Calculation of vertex type Feynman diagrams*, *Phys. Lett. B* **259** (1991) 314 [[INSPIRE](#)].
- [35] A.V. Kotikov, *Differential equation method: The Calculation of N point Feynman diagrams*, *Phys. Lett. B* **267** (1991) 123 [*Erratum ibid.* **295** (1992) 409] [[INSPIRE](#)].
- [36] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys. B* **580** (2000) 485 [[hep-ph/9912329](#)] [[INSPIRE](#)].
- [37] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601 [[arXiv:1304.1806](#)] [[INSPIRE](#)].

- [38] C. Duhr, F. Dulat, V. Hirschi and B. Mistlberger, *Higgs production in bottom quark fusion: matching the 4- and 5-flavour schemes to third order in the strong coupling*, *JHEP* **08** (2020) 017 [[arXiv:2004.04752](#)] [[INSPIRE](#)].
- [39] F. Dulat, S. Lionetti, B. Mistlberger, A. Pelloni and C. Specchia, *Higgs-differential cross section at NNLO in dimensional regularisation*, *JHEP* **07** (2017) 017 [[arXiv:1704.08220](#)] [[INSPIRE](#)].
- [40] F. Dulat, B. Mistlberger and A. Pelloni, *Differential Higgs production at N^3LO beyond threshold*, *JHEP* **01** (2018) 145 [[arXiv:1710.03016](#)] [[INSPIRE](#)].
- [41] M.A. Ebert, B. Mistlberger and G. Vita, *Collinear expansion for color singlet cross sections*, *JHEP* **09** (2020) 181 [[arXiv:2006.03055](#)] [[INSPIRE](#)].
- [42] M.A. Ebert, B. Mistlberger and G. Vita, *Transverse momentum dependent PDFs at N^3LO* , *JHEP* **09** (2020) 146 [[arXiv:2006.05329](#)] [[INSPIRE](#)].
- [43] M.A. Ebert, B. Mistlberger and G. Vita, *N -jettiness beam functions at N^3LO* , *JHEP* **09** (2020) 143 [[arXiv:2006.03056](#)] [[INSPIRE](#)].
- [44] M.A. Ebert, B. Mistlberger and G. Vita, *TMD fragmentation functions at N^3LO* , *JHEP* **07** (2021) 121 [[arXiv:2012.07853](#)] [[INSPIRE](#)].
- [45] F. Dulat and B. Mistlberger, *Real-Virtual-Virtual contributions to the inclusive Higgs cross section at N^3LO* , [arXiv:1411.3586](#) [[INSPIRE](#)].
- [46] C. Duhr and T. Gehrmann, *The two-loop soft current in dimensional regularization*, *Phys. Lett. B* **727** (2013) 452 [[arXiv:1309.4393](#)] [[INSPIRE](#)].
- [47] C. Duhr, T. Gehrmann and M. Jaquier, *Two-loop splitting amplitudes and the single-real contribution to inclusive Higgs production at N^3LO* , *JHEP* **02** (2015) 077 [[arXiv:1411.3587](#)] [[INSPIRE](#)].
- [48] G.F. Sterman, *Summation of Large Corrections to Short Distance Hadronic Cross-Sections*, *Nucl. Phys. B* **281** (1987) 310 [[INSPIRE](#)].
- [49] S. Catani, D. de Florian, M. Grazzini and P. Nason, *Soft gluon resummation for Higgs boson production at hadron colliders*, *JHEP* **07** (2003) 028 [[hep-ph/0306211](#)] [[INSPIRE](#)].
- [50] S. Catani and L. Trentadue, *Resummation of the QCD Perturbative Series for Hard Processes*, *Nucl. Phys. B* **327** (1989) 323 [[INSPIRE](#)].
- [51] S. Catani and L. Trentadue, *Comment on QCD exponentiation at large x* , *Nucl. Phys. B* **353** (1991) 183 [[INSPIRE](#)].
- [52] V. Ahrens, T. Becher, M. Neubert and L.L. Yang, *Renormalization-Group Improved Prediction for Higgs Production at Hadron Colliders*, *Eur. Phys. J. C* **62** (2009) 333 [[arXiv:0809.4283](#)] [[INSPIRE](#)].
- [53] V. Ahrens, T. Becher, M. Neubert and L.L. Yang, *Origin of the Large Perturbative Corrections to Higgs Production at Hadron Colliders*, *Phys. Rev. D* **79** (2009) 033013 [[arXiv:0808.3008](#)] [[INSPIRE](#)].
- [54] V. Ahrens, T. Becher, M. Neubert and L.L. Yang, *Updated Predictions for Higgs Production at the Tevatron and the LHC*, *Phys. Lett. B* **698** (2011) 271 [[arXiv:1008.3162](#)] [[INSPIRE](#)].
- [55] G. Das, S. Moch and A. Vogt, *Approximate four-loop QCD corrections to the Higgs-boson production cross section*, *Phys. Lett. B* **807** (2020) 135546 [[arXiv:2004.00563](#)] [[INSPIRE](#)].

- [56] A. Vogt, F. Herzog, S. Moch, B. Ruijl, T. Ueda and J.A.M. Vermaseren, *Anomalous dimensions and splitting functions beyond the next-to-next-to-leading order*, *PoS LL2018* (2018) 050 [[arXiv:1808.08981](#)] [[INSPIRE](#)].
- [57] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Four-Loop Non-Singlet Splitting Functions in the Planar Limit and Beyond*, *JHEP* **10** (2017) 041 [[arXiv:1707.08315](#)] [[INSPIRE](#)].
- [58] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *Low moments of the four-loop splitting functions in QCD*, *Phys. Lett. B* **825** (2022) 136853 [[arXiv:2111.15561](#)] [[INSPIRE](#)].
- [59] S. Moch, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *On quartic colour factors in splitting functions and the gluon cusp anomalous dimension*, *Phys. Lett. B* **782** (2018) 627 [[arXiv:1805.09638](#)] [[INSPIRE](#)].
- [60] G. Das, S.-O. Moch and A. Vogt, *Soft corrections to inclusive deep-inelastic scattering at four loops and beyond*, *JHEP* **03** (2020) 116 [[arXiv:1912.12920](#)] [[INSPIRE](#)].
- [61] M. Beneke and V.A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, *Nucl. Phys. B* **522** (1998) 321 [[hep-ph/9711391](#)] [[INSPIRE](#)].
- [62] F.V. Tkachov, *A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions*, *Phys. Lett. B* **100** (1981) 65 [[INSPIRE](#)].
- [63] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β -functions in 4 Loops*, *Nucl. Phys. B* **192** (1981) 159 [[INSPIRE](#)].
- [64] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087 [[hep-ph/0102033](#)] [[INSPIRE](#)].
- [65] R.N. Lee, *Multiloop calculations with differential equations in ϵ -form*, *Eur. Phys. J. Web Conf.* **125** (2016) 05020.
- [66] J. Henn, B. Mistlberger, V.A. Smirnov and P. Wasser, *Constructing d -log integrands and computing master integrals for three-loop four-particle scattering*, *JHEP* **04** (2020) 167 [[arXiv:2002.09492](#)] [[INSPIRE](#)].
- [67] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys. A* **15** (2000) 725 [[hep-ph/9905237](#)] [[INSPIRE](#)].
- [68] T. Huber and D. Maître, *HypExp: A Mathematica package for expanding hypergeometric functions around integer-valued parameters*, *Comput. Phys. Commun.* **175** (2006) 122 [[hep-ph/0507094](#)] [[INSPIRE](#)].
- [69] G. Somogyi, *Angular integrals in d dimensions*, *J. Math. Phys.* **52** (2011) 083501 [[arXiv:1101.3557](#)] [[INSPIRE](#)].
- [70] M. Czakon, *Automatized analytic continuation of Mellin-Barnes integrals*, *Comput. Phys. Commun.* **175** (2006) 559 [[hep-ph/0511200](#)] [[INSPIRE](#)].
- [71] A.V. Smirnov and V.A. Smirnov, *On the Resolution of Singularities of Multiple Mellin-Barnes Integrals*, *Eur. Phys. J. C* **62** (2009) 445 [[arXiv:0901.0386](#)] [[INSPIRE](#)].
- [72] K.G. Chetyrkin, B.A. Kniehl and M. Steinhauser, *Decoupling relations to $O(\alpha_S^3)$ and their connection to low-energy theorems*, *Nucl. Phys. B* **510** (1998) 61 [[hep-ph/9708255](#)] [[INSPIRE](#)].
- [73] Y. Schröder and M. Steinhauser, *Four-loop decoupling relations for the strong coupling*, *JHEP* **01** (2006) 051 [[hep-ph/0512058](#)] [[INSPIRE](#)].

- [74] K.G. Chetyrkin, J.H. Kuhn and C. Sturm, *QCD decoupling at four loops*, *Nucl. Phys. B* **744** (2006) 121 [[hep-ph/0512060](#)] [[INSPIRE](#)].
- [75] M. Krämer, E. Laenen and M. Spira, *Soft gluon radiation in Higgs boson production at the LHC*, *Nucl. Phys. B* **511** (1998) 523 [[hep-ph/9611272](#)] [[INSPIRE](#)].
- [76] T. Inami, T. Kubota and Y. Okada, *Effective Gauge Theory and the Effect of Heavy Quarks*, *Z. Phys. C* **18** (1983) 69.
- [77] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Remarks on Higgs Boson Interactions with Nucleons*, *Phys. Lett. B* **78** (1978) 443 [[INSPIRE](#)].
- [78] V.P. Spiridonov and K.G. Chetyrkin, *Nonleading mass corrections and renormalization of the operators $m \bar{\psi} \psi$ and $g_{(\mu\nu)}^2$* , *Sov. J. Nucl. Phys.* **47** (1988) 522 [[INSPIRE](#)].
- [79] F. Wilczek, *Decays of Heavy Vector Mesons Into Higgs Particles*, *Phys. Rev. Lett.* **39** (1977) 1304 [[INSPIRE](#)].
- [80] T. Gehrmann, T. Huber and D. Maître, *Two-loop quark and gluon form-factors in dimensional regularisation*, *Phys. Lett. B* **622** (2005) 295 [[hep-ph/0507061](#)] [[INSPIRE](#)].
- [81] P.A. Baikov, K.G. Chetyrkin, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Quark and gluon form factors to three loops*, *Phys. Rev. Lett.* **102** (2009) 212002 [[arXiv:0902.3519](#)] [[INSPIRE](#)].
- [82] T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli and C. Studerus, *Calculation of the quark and gluon form factors to three loops in QCD*, *JHEP* **06** (2010) 094 [[arXiv:1004.3653](#)] [[INSPIRE](#)].
- [83] T. Gehrmann, E.W.N. Glover, T. Huber, N. Ikizlerli and C. Studerus, *The quark and gluon form factors to three loops in QCD through to $O(\epsilon^2)$* , *JHEP* **11** (2010) 102 [[arXiv:1010.4478](#)] [[INSPIRE](#)].
- [84] B. Agarwal, A. von Manteuffel, E. Panzer and R.M. Schabinger, *Four-loop collinear anomalous dimensions in QCD and $N = 4$ super Yang-Mills*, *Phys. Lett. B* **820** (2021) 136503 [[arXiv:2102.09725](#)] [[INSPIRE](#)].
- [85] R.N. Lee, A. von Manteuffel, R.M. Schabinger, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Fermionic corrections to quark and gluon form factors in four-loop QCD*, *Phys. Rev. D* **104** (2021) 074008 [[arXiv:2105.11504](#)] [[INSPIRE](#)].
- [86] R.N. Lee, A. von Manteuffel, R.M. Schabinger, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Quark and Gluon Form Factors in Four-Loop QCD*, *Phys. Rev. Lett.* **128** (2022) 212002 [[arXiv:2202.04660](#)] [[INSPIRE](#)].
- [87] Y. Li, A. von Manteuffel, R.M. Schabinger and H.X. Zhu, *Soft-virtual corrections to Higgs production at N^3LO* , *Phys. Rev. D* **91** (2015) 036008 [[arXiv:1412.2771](#)] [[INSPIRE](#)].
- [88] T. Ahmed, M. Mahakhud, N. Rana and V. Ravindran, *Drell-Yan Production at Threshold to Third Order in QCD*, *Phys. Rev. Lett.* **113** (2014) 112002 [[arXiv:1404.0366](#)] [[INSPIRE](#)].
- [89] S. Catani and M. Grazzini, *The soft gluon current at one loop order*, *Nucl. Phys. B* **591** (2000) 435 [[hep-ph/0007142](#)] [[INSPIRE](#)].
- [90] Y. Li and H.X. Zhu, *Single soft gluon emission at two loops*, *JHEP* **11** (2013) 080 [[arXiv:1309.4391](#)] [[INSPIRE](#)].
- [91] L.J. Dixon, E. Herrmann, K. Yan and H.X. Zhu, *Soft gluon emission at two loops in full color*, *JHEP* **05** (2020) 135 [[arXiv:1912.09370](#)] [[INSPIRE](#)].

- [92] M. Bonvini and L. Rottoli, *Three loop soft function for N^3LL' gluon fusion Higgs production in soft-collinear effective theory*, *Phys. Rev. D* **91** (2015) 051301 [[arXiv:1412.3791](#)] [[INSPIRE](#)].
- [93] C.W. Bauer, S. Fleming and M.E. Luke, *Summing Sudakov logarithms in $B \rightarrow X_s \gamma$ in effective field theory*, *Phys. Rev. D* **63** (2000) 014006 [[hep-ph/0005275](#)] [[INSPIRE](#)].
- [94] C.W. Bauer, S. Fleming, D. Pirjol and I.W. Stewart, *An Effective field theory for collinear and soft gluons: Heavy to light decays*, *Phys. Rev. D* **63** (2001) 114020 [[hep-ph/0011336](#)] [[INSPIRE](#)].
- [95] C.W. Bauer and I.W. Stewart, *Invariant operators in collinear effective theory*, *Phys. Lett. B* **516** (2001) 134 [[hep-ph/0107001](#)] [[INSPIRE](#)].
- [96] C.W. Bauer, D. Pirjol and I.W. Stewart, *Soft collinear factorization in effective field theory*, *Phys. Rev. D* **65** (2002) 054022 [[hep-ph/0109045](#)] [[INSPIRE](#)].
- [97] C.W. Bauer, S. Fleming, D. Pirjol, I.Z. Rothstein and I.W. Stewart, *Hard scattering factorization from effective field theory*, *Phys. Rev. D* **66** (2002) 014017 [[hep-ph/0202088](#)] [[INSPIRE](#)].
- [98] G.P. Korchemsky and G. Marchesini, *Resummation of large infrared corrections using Wilson loops*, *Phys. Lett. B* **313** (1993) 433 [[INSPIRE](#)].
- [99] A.V. Belitsky, *Two loop renormalization of Wilson loop for Drell-Yan production*, *Phys. Lett. B* **442** (1998) 307 [[hep-ph/9808389](#)] [[INSPIRE](#)].
- [100] G. Altarelli and G. Parisi, *Asymptotic Freedom in Parton Language*, *Nucl. Phys. B* **126** (1977) 298 [[INSPIRE](#)].
- [101] Y.L. Dokshitzer, *Calculation of the Structure Functions for Deep Inelastic Scattering and e^+e^- Annihilation by Perturbation Theory in Quantum Chromodynamics*, *Sov. Phys. JETP* **46** (1977) 641 [[INSPIRE](#)].
- [102] V.N. Gribov and L.N. Lipatov, *Deep inelastic $e p$ scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 438 [[INSPIRE](#)].
- [103] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, *Five-Loop Running of the QCD coupling constant*, *Phys. Rev. Lett.* **118** (2017) 082002 [[arXiv:1606.08659](#)] [[INSPIRE](#)].
- [104] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren and A. Vogt, *The five-loop β -function of Yang-Mills theory with fermions*, *JHEP* **02** (2017) 090 [[arXiv:1701.01404](#)] [[INSPIRE](#)].
- [105] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, *The Four loop β -function in quantum chromodynamics*, *Phys. Lett. B* **400** (1997) 379 [[hep-ph/9701390](#)] [[INSPIRE](#)].
- [106] M. Czakon, *The Four-loop QCD β -function and anomalous dimensions*, *Nucl. Phys. B* **710** (2005) 485 [[hep-ph/0411261](#)] [[INSPIRE](#)].
- [107] O.V. Tarasov, A.A. Vladimirov and A.Y. Zharkov, *The Gell-Mann-Low Function of QCD in the Three Loop Approximation*, *Phys. Lett. B* **93** (1980) 429 [[INSPIRE](#)].
- [108] S.A. Larin and J.A.M. Vermaseren, *The Three loop QCD β -function and anomalous dimensions*, *Phys. Lett. B* **303** (1993) 334 [[hep-ph/9302208](#)] [[INSPIRE](#)].
- [109] S. Moch, J.A.M. Vermaseren and A. Vogt, *On γ_5 in higher-order QCD calculations and the NNLO evolution of the polarized valence distribution*, *Phys. Lett. B* **748** (2015) 432 [[arXiv:1506.04517](#)] [[INSPIRE](#)].

- [110] S. Moch, J.A.M. Vermaseren and A. Vogt, *The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case*, *Nucl. Phys. B* **889** (2014) 351 [[arXiv:1409.5131](#)] [[INSPIRE](#)].
- [111] I.W. Stewart, F.J. Tackmann and W.J. Waalewijn, *N-Jettiness: An Inclusive Event Shape to Veto Jets*, *Phys. Rev. Lett.* **105** (2010) 092002 [[arXiv:1004.2489](#)] [[INSPIRE](#)].
- [112] I.W. Stewart, F.J. Tackmann and W.J. Waalewijn, *The Quark Beam Function at NNLL*, *JHEP* **09** (2010) 005 [[arXiv:1002.2213](#)] [[INSPIRE](#)].
- [113] T. Becher, M. Neubert and B.D. Pecjak, *Factorization and Momentum-Space Resummation in Deep-Inelastic Scattering*, *JHEP* **01** (2007) 076 [[hep-ph/0607228](#)] [[INSPIRE](#)].
- [114] T. Becher and M.D. Schwartz, *A precise determination of α_s from LEP thrust data using effective field theory*, *JHEP* **07** (2008) 034 [[arXiv:0803.0342](#)] [[INSPIRE](#)].
- [115] R. Abbate, M. Fickinger, A.H. Hoang, V. Mateu and I.W. Stewart, *Thrust at N^3LL with Power Corrections and a Precision Global Fit for $\alpha_s(m_Z)$* , *Phys. Rev. D* **83** (2011) 074021 [[arXiv:1006.3080](#)] [[INSPIRE](#)].
- [116] R. Abbate, M. Fickinger, A.H. Hoang, V. Mateu and I.W. Stewart, *Precision Thrust Cumulant Moments at N^3LL* , *Phys. Rev. D* **86** (2012) 094002 [[arXiv:1204.5746](#)] [[INSPIRE](#)].
- [117] G. Lusterians, J.K.L. Michel and F.J. Tackmann, *Generalized Threshold Factorization with Full Collinear Dynamics*, [arXiv:1908.00985](#) [[INSPIRE](#)].