Orbifolds by 2-groups and decomposition

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Abstract: In this paper we study three-dimensional orbifolds by 2-groups with a trivially-acting one-form symmetry group BK. These orbifolds have a global two-form symmetry, and so one expects that they decompose into (are equivalent to) a disjoint union of other three-dimensional theories, which we demonstrate. These theories can be interpreted as sigma models on 2-gerbes, whose formal structures reflect properties of the orbifold construction.

Contents

1 Introduction

Decomposition, the observation that a local quantum field theory is sometimes a disjoint union of other local quantum field theories, has by now been extensively studied since its initial observation in [1] in two-dimensional gauge theories with trivially-acting subgroups, see for example [2–26]. A few reviews can be found in [27–31].

Although decomposition was originally observed in two dimensional theories, it has also been observed in four-dimensional theories, see for example [7, 8]. The purpose of this paper is to discuss examples in three dimensions, where it has not previously been studied.

Globally, decomposition is expected to take place in any theory in d spacetime dimensions with a global $(d-1)$-form symmetry (possibly realized noninvertibly) [7, 8]. One way to produce such a symmetry is via a suitable gauging. In broad brushstrokes, gauging a trivially-acting n-form symmetry results in a theory with a global $(n+1)$-form symmetry (distinct from the quantum symmetry), so one can hope to produce a d-dimensional theories with a decomposition by gauging a trivially-acting $(d-2)$-form symmetry.

For example, ordinary gauge theories with trivially-acting subgroups are a source of examples in two dimensions, as mentioned above, because such theories have a global one-form symmetry (distinct from the quantum symmetry, and tied specifically to the fact that the group acts trivially).

In this paper, we study three-dimensional gauge theories with gauged trivially-acting one-form symmetries. Gauging the trivially-acting one-form symmetry leads to a global two-form symmetry, hence, in three dimensions, a decomposition.

Specifically, in this paper we describe orbifolds of three-dimensional effective[1] field theories by 2-groups, which are extensions of ordinary (here, finite) groups by one-form symmetries. (See for example [32] for a mathematical introduction to 2-groups. These structures have a long history in both math and physics, see for example [33–58] for a few older instances, and [59–72] for some more recent physics descriptions and applications of 2-groups.) When those one-form symmetries act trivially, gauging them results in a global two-form symmetry, hence a decomposition as above, which we will check explicitly.

We begin in section 2 by reviewing two-dimensional orbifolds by central extensions of G by trivially-acting K, and how a decomposition arises in such orbifolds. In particular,

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[1]We emphasize that because we often discuss orbifolds of three-dimensional sigma models, we understand those sigma models as effective field theories, not necessarily renormalizable theories. Our methods also apply to more general three-dimensional theories, such as, for example, Chern-Simons theories.

decomposition implements a restriction on nonperturbative sectors. In an orbifold the nonperturbative sectors are the twisted sectors, and in these orbifolds those twisted sectors are restricted to those describing G bundles satisfying a condition. The restriction is implemented physically by a sum over G orbifolds, namely the decomposition, realizing a 'multiverse interference effect' between the constituent G orbifolds ('universes'). An important role in that decomposition is played by discrete torsion, so in section 3 we review three-dimensional analogues of discrete torsion, counted by $H^3(G, \mathrm{U}(1))$.

In section 4 we turn to the main content of this paper: we define and study orbifolds by 2-group extensions of ordinary (finite) groups G by trivially-acting one-form symmetry groups BK. Just as in two-dimensional cases, the nonperturbative sectors correspond to G bundles satisfying a condition. We argue that, also just as in two-dimensional cases, that restriction implies (and is implemented by) a decomposition of the three-dimensional theory, with universes indexed by irreducible representations of K, which we study explicitly in several examples.

In section 5 we interpret this structure formally in terms of a sigma model whose target is a 2-gerbe. In section 6 we outline higher-dimensional analogues and their interpretations.

In section 7 we briefly outline analogous decompositions in Chern-Simons theories with gauged one-form symmetry group actions, which will be further addressed in other work to appear.

In appendix A, we give mathematically rigorous derivations of statements about bundles of 2-groups. In appendix B we formally discuss decomposition as a duality transform, as a type of Fourier transform. Finally, in appendix C we collect some results on group cohomology that are used in computations in the main text.

Higher-dimensional orbifolds by ordinary groups have also been discussed in e.g. [73, 74]. However, so far as we can determine, those papers do not discuss orbifolds by higher groups, and do not discuss decomposition. We believe our observations in this paper (regarding orbifolds by 2-groups and decomposition) are novel.

2 Review: decomposition in ordinary orbifolds

In this section we will review decomposition of two-dimensional orbifolds in which a central subgroup of the orbifold group acts trivially. The fact that such orbifolds are equivalent to ('decompose into') disjoint unions of other theories was worked out in [1]; however, our presentation of the phenomenon here has not been previously published, and is the prototype for our discussion of decomposition in 2-group orbifolds later.

Let X be a space, and G a finite group acting on X. Let Γ be a central extension of G by a finite abelian group K:

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1. \tag{2.1}$$

Such extensions are classified by elements of $H^2(G, K)$. Briefly, the statement of decomposition here is that [1]

$$\mathrm{QFT}\left([X/\Gamma]\right) = \coprod_{\rho \in \hat{K}} \mathrm{QFT}\left([X/G]_{\rho(\omega)}\right), \tag{2.2}$$

where $\hat{K}$ denotes irreducible representations of K, and $\rho(\omega) \in H^2(G, \mathrm{U}(1))$ is the image $\rho \circ \omega$ of the extension class ω under $\rho \in \hat{K}$. (Decomposition is also defined for more general orbifolds [1–3], but for our purposes in this paper, the special case of central extensions above will suffice.)

Next, we establish this decomposition, by computing partition functions. First, recall that the extension Γ can be described set-wise as a product $G \times K$, with product deformed by an element $[\omega] \in H^2(G, K)$. Let $\gamma \in \Gamma$, and write Γ set-wise as the product $G \times K$, then the product in Γ is defined by

$$\gamma_1 \gamma_2 = (g_1, k_1)(g_2, k_2) = (g_1 g_2, k_1 k_2 \omega(g_1, g_2)). \tag{2.3}$$

In the partition function of a two-dimensional orbifold $[X/\Gamma]$ on T^2, we sum over commuting pairs of group elements in Γ, but clearly the condition for γ_1 and γ_2 to commute is equivalent to g_1 commuting with g_2 and

$$\frac{\omega(g_1, g_2)}{\omega(g_2, g_1)} = 1. \tag{2.4}$$

Define

$$\epsilon(g_1, g_2) = \frac{\omega(g_1, g_2)}{\omega(g_2, g_1)}, \tag{2.5}$$

then it is straightforward to demonstrate that

$$\epsilon(a, bc) = \epsilon(a, b)\epsilon(a, c), \tag{2.6}$$

(and symmetrically,) so as a consequence, ϵ is invariant under conjugation:[2]

$$
\begin{aligned}
\epsilon(hah^{-1}, hbh^{-1}) &= \epsilon(hah^{-1}, h)\, \epsilon(hah^{-1}, b)\, \epsilon(hah^{-1}, h^{-1}), \\
&= \epsilon(hah^{-1}, b), \\
&= \epsilon(h, b)\, \epsilon(a, b)\, \epsilon(h^{-1}, b), \\
&= \epsilon(a, b).
\end{aligned}
\tag{2.7}
$$

In particular, this descends to isomorphism classes of G bundles, which on T^2 are classified by $\mathrm{Hom}(\pi_1(T^2), G)/G$. We can view ϵ as assigning a phase to each such bundle.

Thus, the partition function of a two-dimensional $[X/\Gamma]$ orbifold looks like the partition function of a $[X/G]$ orbifold but with a restriction on the allowed sectors. We can implement that restriction on allowed sectors by inserting an operator

$$\delta(\epsilon - 1) = \frac{1}{|K|} \sum_{\rho \in \hat{K}} \epsilon_\rho(g_1, g_2), \tag{2.8}$$

where ϵ_ρ is the image of $\omega(g_1, g_2)/\omega(g_2, g_1)$ under $\rho : K \to \mathrm{U}(1)$. This is the origin of decomposition [1].

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[2]We restrict to the same h on each input because ϵ is only defined on commuting pairs.

Now, let us assemble these pieces. The partition function of a Γ orbifold on T^2 is, universally,

$$Z_{T^2}\left([X/\Gamma]\right) = \frac{1}{|\Gamma|} \sum_{\gamma\lambda=\lambda\gamma} Z(\gamma,\lambda), \tag{2.9}$$

where the sum is over commuting pairs $\gamma, \lambda \in \Gamma$, and $Z(\gamma,\lambda)$ is the contribution from a square with sides identified by γ, λ — known as the twisted sectors or partial traces. In the present circumstances, since $K \subset \Gamma$ acts trivially,

$$Z(\gamma,\lambda) = Z(g,h) \tag{2.10}$$

where $g = \pi(\gamma)$, $h = \pi(\lambda)$, for $\pi : \Gamma \to G$ the projection. Taking this into account, we then have

$$
\begin{aligned}
Z_{T^2}\left([X/\Gamma]\right) &= \frac{|K|^2}{|\Gamma|} \sum_{gh=hg,\epsilon=1} Z(g,h), \\
&= \frac{|K|^2}{|\Gamma|} \frac{|G|}{|K|} \sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho(\omega)}\right), \\
&= \sum_{\rho\in\hat{K}} Z_{T^2}\left([X/G]_{\rho(\omega)}\right),
\end{aligned} \tag{2.11}
$$

where

$$Z_{T^2}\left([X/G]_{\rho(\omega)}\right) = \frac{1}{|G|} \sum_{gh=hg} \epsilon_\rho(g,h)Z(g,h) \tag{2.12}$$

is the partition function of the G orbifold on T^2 with discrete torsion $\rho(\omega) \in H^2(G, \mathrm{U}(1))$. Thus, we see that partition functions are consistent with the prediction of decomposition (2.2).

In passing, note that in the case $G = \mathbb{Z}_2 = K$, $H^2(G,K) = \mathbb{Z}_2$ (and hence has nontrivial elements), but for all $[\omega] \in H^2(G,K)$, and all commuting pairs,

$$\frac{\omega(g_1,g_2)}{\omega(g_2,g_1)} = 1. \tag{2.13}$$

Thus, triviality of the ratio of cocycles can happen even if ω is a nontrivial cohomology class.

Our analysis above was specific to the case that the worldsheet is T^2, but it generalizes easily to other genus. Before considering general genus, let us next walk through the case of genus 2. Let $\gamma_i = (a_i,k_i) \in \Gamma$, $\lambda_i = (b_i,z_i) \in \Gamma$, $i \in \{1,2\}$, obeying the condition

$$[\gamma_1,\lambda_1]\,[\gamma_2,\lambda_2] = 1, \tag{2.14}$$

for

$$[g,h] = ghg^{-1}h^{-1}, \tag{2.15}$$

and define

$$\xi_1 = [a_1,b_1] = a_1 b_1 a_1^{-1} b_1^{-1}. \tag{2.16}$$

Then, using the fact that

$$\gamma_i^{-1} = \left(a_i^{-1}, k_i^{-1}\omega(a_i, a_i^{-1})^{-1}\right), \quad \lambda_i^{-1} = \left(b_i^{-1}, z_i^{-1}\omega(b_i, b_i^{-1})^{-1}\right), \quad (2.17)$$

it is straightforward to compute that

$$
\begin{aligned}
[\gamma_1, \lambda_1] = \Big(& [a_1, b_1], \omega(a_1, b_1)\, \omega(a_1 b_1, a_1^{-1})\, \omega(a_1 b_1 a_1^{-1}, b_1^{-1}) \\
& \cdot \omega(a_1, a_1^{-1})^{-1}\, \omega(b_1, b_1^{-1})^{-1} \Big),
\end{aligned}
\quad (2.18)
$$

$$
\begin{aligned}
[\gamma_1, \lambda_1]\, [\gamma_2, \lambda_2] = \Big(& [a_1, b_1][a_2, b_2], \omega(a_1, b_1)\, \omega(a_1 b_1, a_1^{-1})\, \omega(a_1 b_1 a_1^{-1}, b_1^{-1}) \\
& \cdot \omega(\xi_1, a_2)\, \omega(\xi_1 a_2, b_2)\, \omega(\xi_1 a_2 b_2, a_2^{-1}) \\
& \cdot \omega(\xi_1 a_2 b_2 a_2^{-1}, b_2^{-1}) \\
& \cdot \omega(a_1, a_1^{-1})^{-1}\, \omega(a_2, a_2^{-1})^{-1} \\
& \cdot \omega(b_1, b_1^{-1})^{-1}\, \omega(b_2, b_2^{-1})^{-1} \Big),
\end{aligned}
\quad (2.19)
$$

so we see that the closure condition (2.14) holds if and only if both

$$[a_1, b_1]\, [a_2, b_2] = 1 \quad (2.20)$$

and

$$
\begin{aligned}
1 = {} & \omega(a_1, b_1)\, \omega(a_1 b_1, a_1^{-1})\, \omega(a_1 b_1 a_1^{-1}, b_1^{-1}) \\
& \cdot \omega(\xi_1, a_2)\, \omega(\xi_1 a_2, b_2)\, \omega(\xi_1 a_2 b_2, a_2^{-1})\, \omega(\xi_1 a_2 b_2 a_2^{-1}, b_2^{-1}) \\
& \cdot \omega(a_1, a_1^{-1})^{-1}\, \omega(a_2, a_2^{-1})^{-1}\, \omega(b_1, b_1^{-1})^{-1}\, \omega(b_2, b_2^{-1})^{-1}.
\end{aligned}
\quad (2.21)
$$

Next, we generalize to arbitrary genus. Consider a Riemann surface of genus g, with boundary conditions determined by $\gamma_i = (a_i, k_i) \in \Gamma$, $\lambda_i = (b_i, z_i) \in \Gamma$, $i \in \{1, \cdots, g\}$. Define $\xi_i = [a_i, b_i]$, and

$$X = \left[\prod_i \omega(a_i, a_i^{-1}) \prod_i \omega(b_i, b_i^{-1})\right]^{-1}. \quad (2.22)$$

The condition that the group elements must obey to define boundary conditions on the Riemann surface is that

$$[\gamma_1, \lambda_1]\, [\gamma_2, \lambda_2] \cdots [\gamma_g, \lambda_g] = 1, \quad (2.23)$$

which implies that

$$[a_1, b_1]\, [a_2, b_2] \cdots [a_g, b_g] = 1 \quad (2.24)$$

(which are required for $a_i, b_i \in G$ to close on the Riemann surface) as well as

$$\epsilon(a_i, b_i) = 1 \quad (2.25)$$

for

$$
\begin{aligned}
\epsilon(a_i, b_i) \equiv {} & X\omega(a_1, b_1)\omega(a_1 b_1, a_1^{-1})\omega(a_1 b_1 a_1^{-1}, b_1^{-1})\omega(\xi_1, a_2)\omega(\xi_1 a_2, b_2)\omega(\xi_1 a_2 b_2, a_2^{-1}) \\
& \cdot \omega(\xi_1 a_2 b_2 a_2^{-1}, b_2^{-1})\omega(\xi_1 \xi_2, a_3) \cdots \omega(\xi_1 \cdots \xi_{g-1} a_g b_g a_g^{-1}, b_g^{-1}).
\end{aligned}
$$

(This can be obtained either by direct multiplication or by triangulating the Riemann surface into simplices and associating a factor of ω with each simplex, as in [75].) Thus, as before, the data required to define a Γ orbifold on a genus g Riemann surface is a restriction on the combinatorial data used to define a G orbifold on the same Riemann surface, a restriction of the form $\epsilon(a_i, b_i) = 1$. As for T^2, we can implement that restriction by inserting a projection operator Π, of the same form as before, with ϵ_ρ that are the image of the genus-g ϵ under an irreducible representation ρ. The resulting phases are the same as the phases defining discrete torsion on a genus g Riemann surface (see ([75], eq. (15)), [76]), again for discrete torsion given by the image of $H^2(G, K)$ under the irreducible representation $\rho : K \to \mathrm{U}(1)$. Thus, we see the story for T^2 generalizes immediately to other Riemann surfaces.

For later use, we note that the discrete torsion here can equivalently be understood as a coupling to a discrete theta angle, defined by a characteristic class $x^*\omega$, for $x : \Sigma \to BG$ a map defining the twisted sector, in the notation of appendix A. One can rewrite such a discrete theta angle coupling

$$\int_\Sigma \langle \rho, x^*\omega \rangle \tag{2.26}$$

as a discrete torsion phase by triangulating the Riemann surface Σ and associating phases to each simplex as reviewed above and in [75, 77].

So far we have considered central extensions. Decomposition also exists for orbifolds by non-central extensions, see e.g. [1, 2]; however, its form is more complex. In this paper we focus on (analogues of) central extensions.

3 Three-dimensional analogues of discrete torsion

We have seen that two-dimensional orbifolds with trivially-acting subgroups decompose into disjoint unions of orbifolds with discrete torsion, a modular-invariant phase factor [1, 2]. Similarly, the three-dimensional version of decomposition will also generate theories twisted by a three-dimensional version of discrete torsion. Such analogues of discrete torsion were studied in [77] in the special case of orbifolds of points (forming Dijkgraaf-Witten theory), and more generally in [78]. In this section, we briefly review those constructions here, in both ordinary orbifolds and in orientifolds, to set up their appearance in three-dimensional versions of decomposition.

3.1 Ordinary orbifolds

First, recall that in two dimensions, discrete torsion in a G orbifold is classified by group cohomology, specifically $H^2(G, \mathrm{U}(1))$ with a trivial action on the coefficients. Similarly, in three dimensions [77, 78], the analogue of discrete torsion in a G orbifold is classified by $H^3(G, \mathrm{U}(1))$, again with a trivial action on the coefficients.

Furthermore, given $[\omega] \in H^2(G, \mathrm{U}(1))$, one can derive coboundary-invariant phases that weight Riemann surfaces. For example, on T^2, a twisted sector is defined by two commuting elements $g, h \in G$, and the corresponding coboundary-invariant phase is

$$\frac{\omega(g, h)}{\omega(h, g)}. \tag{3.1}$$

Analogous expressions on higher-genus Riemann surfaces can be found in [75].

Analogous constructions exist in three dimensions, which use $[\omega] \in H^3(G, \mathrm{U}(1))$ to assign a coboundary-invariant phase to three-manifolds. One construction [77] proceeds as follows. given a three-manifold Y, we pick a triangulation by simplices, and associate to each simplex a cocycle. We then take an alternating product of those associated cocycles (with exponent determined by orientation) to form a coboundary-invariant phase. For example, the triangulation of a cube into six simplices can be visualized by viewing the cube along a line through two corners, as
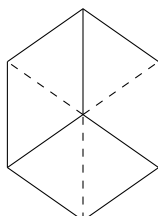
and then taking the tetrahedra cut out by the six interior lines projected through the cube in the figure above. See also [78] for an alternative construction in terms of C field holonomlies.

For example, on T^3, a twisted sector is defined by three commuting group elements g_1, g_2, g_3,
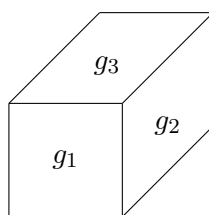
and here one multiplies $Z(g_1, g_2, g_3)$ by the phase ([77], eq. (6.35)), [78]

$$\epsilon_3(g_1, g_2, g_3) = \frac{\omega(g_1, g_2, g_3)}{\omega(g_2, g_1, g_3)} \frac{\omega(g_3, g_1, g_2)}{\omega(g_3, g_2, g_1)} \frac{\omega(g_2, g_3, g_1)}{\omega(g_1, g_3, g_2)}. \tag{3.2}$$

corresponding to $[\omega] \in H^3(G, \mathrm{U}(1))$. As noted in ([77], footnote 5), perhaps the simplest example in which this phase is nontrivial is the group $G = (\mathbb{Z}_2)^3$.

As discussed in [77, 78], this phase factor is invariant under both coboundaries as well as $\mathrm{SL}(3, \mathbb{Z})$ transformations of T^3, just as the discrete torsion phase factor is invariant under both coboundaries as well as $\mathrm{SL}(2, \mathbb{Z})$ transformations of T^2.

For another example, consider $S^1 \times \Sigma$ for Σ a genus-two surface. Here, the associated phase is

$$\xi_2 = \frac{\omega(a_1, b_1, g)}{\omega(\gamma b_1, a_1, g)\, \omega(\gamma, b_1, g)} \frac{\omega(\gamma, a_2, g)\, \omega(\gamma a_2, b_2, g)}{\omega(b_2, a_2, g)}$$
$$\cdot \frac{\omega(\gamma b_1, g, a_1)\, \omega(\gamma, g, b_1)}{\omega(a_1, g, b_1)} \frac{\omega(b_2, g, a_2)}{\omega(\gamma, g, a_2)\, \omega(\gamma a_2, g, b_2)}$$
$$\cdot \frac{\omega(g, a_1, b_1)}{\omega(g, \gamma b_1, a_1)\, \omega(g, \gamma, b_1)} \frac{\omega(g, \gamma, a_2)\, \omega(g, \gamma a_2, b_2)}{\omega(g, b_2, a_2)} \tag{3.3}$$

where

$$\gamma = a_1 b_1 a_1^{-1} b_1^{-1}, \quad \gamma a_2 b_2 a_2^{-1} b_2^{-1} = 1. \tag{3.4}$$

and g commutes with all a_i, b_i. It can be shown that this expression is invariant under coboundaries.

This expression is motivated by the two-dimensional genus-two phase ([75], eq. (15))

$$\frac{\omega(a_1, b_1)}{\omega(\gamma b_1, a_1)\,\omega(\gamma, b_1)}\frac{\omega(\gamma, a_2)\,\omega(\gamma a_2, b_2)}{\omega(b_2, a_2)}. \tag{3.5}$$

Also, in the special case that $\gamma = 1$, it correctly factorizes into the product of two T^3 phases:

$$\xi_2 = \frac{\omega(a_1, b_1, g)\,\omega(b_1, g, a_1)\,\omega(g, a_1, b_1)}{\omega(b_1, a_1, g)\,\omega(a_1, g, b_1)\,\omega(g, b_1, a_1)} \cdot \frac{\omega(a_2, b_2, g)\,\omega(b_2, g, a_2)\,\omega(g, a_2, b_2)}{\omega(b_2, a_2, g)\,\omega(a_2, g, b_2)\,\omega(g, b_2, a_2)}, \tag{3.6}$$

where without loss of generality we assume that the cocycle ω is normalized (so that $\omega = 1$ if any of its arguments is the identity).

In fact, it is also straightforward to conjecture the corresponding phase factor for $S^1 \times \Sigma_h$ for Σ_h a genus-h Riemann surface. Following [75], define

$$\gamma_i = a_i b_i a_i^{-1} b_i^{-1}, \quad \zeta_i = \gamma_1 \gamma_2 \cdots \gamma_{i-1}, \tag{3.7}$$

then the two-dimensional discrete torsion phase is ([75], eq. (15))

$$\xi_h = \frac{\omega(a_1, b_1)}{\omega(\gamma_1 b_1, a_1)\,\omega(\gamma_1, b_1)}\left(\prod_{i=2}^{h-1}\frac{\omega(\zeta_i, a_i)\,\omega(\zeta_i a_i, b_i)}{\omega(\zeta_i \gamma_i b_i, a_i)\,\omega(\zeta_i \gamma_i, b_i)}\right)\frac{\omega(\zeta_h, a_h)\,\omega(\zeta_h a_h, b_h)}{\omega(b_h, a_h)} \tag{3.8}$$

and we conjecture that the analogous three-dimensional phase on $S^1 \times \Sigma_h$ is

$$\begin{aligned}
\xi_h = {} & \frac{\omega(a_1, b_1, g)}{\omega(\gamma_1 b_1, a_1, g)\,\omega(\gamma_1, b_1, g)}\frac{\omega(\gamma_1 b_1, g, a_1)\,\omega(\gamma_1, g, b_1)}{\omega(a_1, g, b_1)}\frac{\omega(g, a_1, b_1)}{\omega(g, \gamma_1 b_1, a_1)\,\omega(g, \gamma_1, b_1)} \\
& \cdot\left(\prod_{i=2}^{h-1}\frac{\omega(\zeta_i, a_i, g)\,\omega(\zeta_i a_i, b_i, g)}{\omega(\zeta_i \gamma_i b_i, a_i, g)\,\omega(\zeta_i \gamma_i, b_i, g)}\frac{\omega(\zeta_i \gamma_i b_i, g, a_i)\,\omega(\zeta_i \gamma_i, g, b_i)}{\omega(\zeta_i, g, a_i)\,\omega(\zeta_i a_i, g, b_i)}\right. \\
& \left.\cdot\frac{\omega(g, \zeta_i, a_i)\,\omega(g, \zeta_i a_i, b_i)}{\omega(g, \zeta_i \gamma_i b_i, a_i)\,\omega(g, \zeta_i \gamma_i, b_i)}\right) \\
& \cdot\frac{\omega(\zeta_h, a_h, g)\,\omega(\zeta_h a_h, b_h, g)}{\omega(b_h, a_h, g)}\frac{\omega(b_h, g, a_h)}{\omega(\zeta_h, g, a_h)\,\omega(\zeta_h a_h, g, b_h)} \\
& \cdot\frac{\omega(g, \zeta_h, a_h)\,\omega(g, \zeta_h a_h, b_h)}{\omega(g, b_h, a_h)}
\end{aligned} \tag{3.9}$$

where $g \in G$ commutes with all a_i, b_i.

In passing, a general expression for S^1 reduction of such phases is discussed in ([79], supplementary section), ([80], section V).

In two dimensions, discrete torsion phases obey multiloop factorization (target space unitarity), which is the following constraint. If Σ is any Riemann surface, corresponding to a twisted sector of some orbifold, and Σ can degenerate into a product of Σ_1 and Σ_2

connected at one point (compatibly with the orbifold structure, in the sense that there are no twist fields at the connection), then the phase associated to Σ must equal the product of the phases associated to Σ_1 and Σ_2.

In two dimensions, for the genus-one phase

$$\epsilon_2(g, h) = \frac{\omega(g, h)}{\omega(h, g)}, \tag{3.10}$$

this is the property ([81], eq. (42))

$$\epsilon_2(x, ab) = \epsilon_2(x, a)\, \epsilon_2(x, b), \tag{3.11}$$

which can be demonstrated simply using

$$\frac{(d\omega)(x, a, b)\, (d\omega)(a, b, x)}{(d\omega)(a, x, b)} = \frac{\epsilon_2(x, ab)}{\epsilon_2(x, a)\, \epsilon_2(x, b)} \tag{3.12}$$

for x, a, b all mutually commuting. When combined with the fact that $\epsilon_2(1, -) = \epsilon_2(-, 1) = 1$, we see this means that ϵ_2 is a bihomomorphism from commuting pairs in G to U(1).

In three dimensions, there is a simple analogue of multiloop factorization: if a three-manifold $S^1 \times \Sigma$ can degenerate into $S^1 \times (\Sigma_1 \coprod \Sigma_2)$, the phase assigned to $S^1 \times \Sigma$ must match the product of the phases assigned to $S^1 \times \Sigma_1$, $S^1 \times \Sigma_2$. On such grounds, one then expects

$$\epsilon_3(x, y, ab) = \epsilon_3(x, y, a)\epsilon_3(x, y, b). \tag{3.13}$$

In fact, it is straightforward to check that this is a consequence of the identity

$$\frac{(d\omega)(y, x, a, b)}{(d\omega)(x, y, a, b)} \frac{(d\omega)(a, b, y, x)}{(d\omega)(a, b, x, y)} \frac{(d\omega)(y, a, b, x)}{(d\omega)(x, a, b, y)} \frac{(d\omega)(x, a, y, b)}{(d\omega)(y, a, x, b)} \frac{(d\omega)(a, x, b, y)}{(d\omega)(a, y, b, x)} \frac{(d\omega)(a, y, x, b)}{(d\omega)(a, x, y, b)}$$
$$= 1. \tag{3.14}$$

(See also ([77], section 6), where a different argument is given for the same result.)

One can use multiloop factorization to argue that discrete torsion(-like) phases descend to conjugacy classes. For example, in the case of the genus-one phase ϵ_2, from (3.11), it is easy to show that[3]

$$\epsilon_2(aga^{-1}, aha^{-1}) = \epsilon_2(aga^{-1}, a)\, \epsilon_2(aga^{-1}, h)\, \epsilon_2(aga^{-1}, a^{-1}) = \epsilon_2(aga^{-1}, h),$$
$$= \epsilon_2(a, h)\, \epsilon_2(g, h)\, \epsilon_2(a^{-1}, h),$$
$$= \epsilon_2(g, h). \tag{3.16}$$

Computing in exactly the same fashion, one can use (3.13) to show that

$$\epsilon_3(aga^{-1}, aha^{-1}, aka^{-1}) = \epsilon_3(g, h, k). \tag{3.17}$$

---

[3]In fact, formally both this expression and its three-dimensional analogue appear to generalize to independent conjugation on the parameters, as

$$\epsilon_2(aga^{-1}, bhb^{-1}) = \epsilon_2(g, h). \tag{3.15}$$

However, $\epsilon_2(g, h)$ is only defined for commuting g, h, so we restrict to the case $a = b$. Identical remarks apply to ϵ_3.

3.2 Manifolds with boundaries

For completeness, let us also quickly outline the case of manifolds with boundary, that will be of use in our subsequent works.

Let's begin with an overview of how this works in two-dimensional theories and ordinary discrete torsion, before describing an example in three dimensions.

Consider a genus-one correlation function in an orbifold $[X/G]$ with a single insertion of an operator associated to $g \in G$. In effect, we have a T^2 with a puncture corresponding to g. If we let $a, b \in G$ denote group elements corresponding to the usual T^2 boundary conditions, then we can sketch the construction of the punctured torus as in the diagram below:
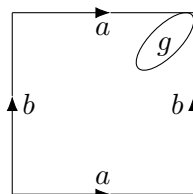
with a hole cut out in the upper right corner, or equivalently,
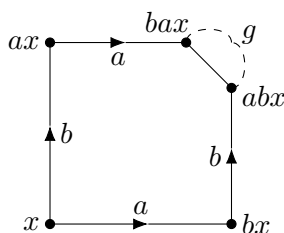
In the presence of the puncture, a and b no longer commute, but instead obey

$$abg = ba. \tag{3.18}$$

Alternatively, we can say that if Σ is a punctured T^2, then to specify an element of $\text{Hom}(\pi_1(\Sigma), G)$ we can first assign a group element g to the loop circling the puncture, and then the group elements a and b assigned to the non-contractible cycles of the torus will need to satisfy (3.18), since the cycle associated to $a^{-1}b^{-1}ab$ is homotopic to the cycle circling the puncture. Then bundles on Σ are classified by $\text{Hom}(\pi_1(\Sigma), G)/G$, as usual.

One more perspective comes from consideration of topological defect lines. The a and b twists on the T^2 are implemented by wrapping a and b lines around the cycles. Saying that our inserted operator is associated to g is equivalent to saying that it sits at the end of a defect line labeled by g. The other end of that line must terminate somewhere on the first two defect lines. The simplest possibility is to connect everything at a single junction of degree five. In order for that junction to remain topological (i.e. to avoid an extra non-topological insertion), we need the cyclic product of lines coming in to give the identity, which again leads to (3.18).

In any event, the discrete torsion phase assigned to a punctured T^2 is not the same as that assigned to T^2 itself — the contribution to the boundary conditions from the

puncture modifies the phase. Applying methods of [75], we see that the phase associated to this diagram is

$$\xi_{1,1} \equiv \frac{\omega(a,b)}{\omega(b,a)} \omega(ab,g). \tag{3.19}$$

Now, if we add a coboundary α, this phase changes:

$$\xi_{1,1} \mapsto \xi_{1,1} \alpha(g). \tag{3.20}$$

This is not quite invariant under coboundaries; however, the coboundary $\alpha(g)$ can be absorbed into the operator at the puncture, so taking that into account, the phase is well-defined. Proceeding in this fashion, one is led to correlation functions, see e.g. [82] for examples in the case of orbifolds of a point (Dijkgraaf-Witten theory).

Now, let us turn to three-dimensional analogues. We consider a T^3 with a hole, of boundary T^2. If we let the three primary sides be related by g_1, g_2, g_3, and the new edge defining the hole by k, then graphically,
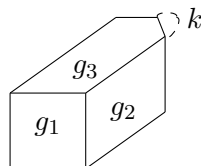
The cut-out corner, seen edge-on, is the square
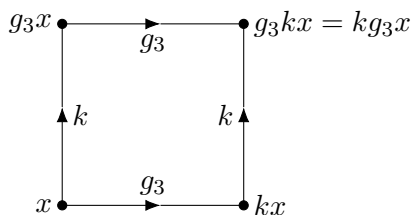
In order for the diagram to close, the four group elements $g_1, g_2, g_3, k \in G$ obey

$$g_1 g_3 = g_3 g_1, \quad g_2 g_3 = g_3 g_2, \quad g_1 g_2 k = g_2 g_1, \quad g_3 k = k g_3. \tag{3.21}$$

Applying the same methods as [78], we find that the phase factor associated with this diagram is

$$\frac{\omega(g_1,g_2,g_3)}{\omega(g_2,g_1,g_3)} \frac{\omega(g_3,g_1,g_2)}{\omega(g_1,g_3,g_2)} \frac{\omega(g_2,g_3,g_1)}{\omega(g_3,g_2,g_1)} \frac{\omega(g_1 g_2, k, g_3) \omega(g_3, g_1 g_2, k)}{\omega(g_1 g_2, g_3, k)}. \tag{3.22}$$

As for the T^2 with boundary, this is not quite coboundary-invariant, but rather picks up a phase

$$\frac{\alpha(k,g_3)}{\alpha(g_3,k)}, \tag{3.23}$$

which has the same appearance as the phase one would assign to a T^2 with the same boundary conditions. We interpret this as before, as a contribution that would be absorbed by a defect inserted at the puncture, precisely in the spirit of anomaly inflow (see e.g. [83]). It is also extremely reminiscent of the relationship between three-dimensional Chern-Simons theories and WZW models on boundaries, see e.g. [84].

3.3 Orientifolds

Now, consider the case that a subgroup of the orbifold group G acts, in part, by reversing orientations, to form an orientifold. C fields on orientifolds were analyzed in [85], section 6), in the same pattern as in [78] for C fields on ordinary orbifolds and [85] for B fields on orientifolds. Briefly, the conclusion was that the analogue of C field discrete torsion on orientifolds is counted by $H^3(G, U(1))$ with a nontrivial action on the coefficients, encoded in a homomorphism $\epsilon : G \to \mathbb{Z}_2$ expressing whether a given element acts trivially.

One example discussed in ([85], section 6.2) is a cube, with sides identified by three group elements $g_1, g_2, g_3 \in G$, in which one of the group elements reverses the orientation. The three group elements must be related by

$$g_2 g_3 = g_3 g_2, \quad g_1 g_3 = g_3 g_1, \quad g_1 = g_2 g_1 g_2, \tag{3.24}$$

where of the three, g_1 reverses orientation, but the other two do not. It was argued there that the corresponding partition function phase factor is

$$\frac{\omega(g_1, g_2^{-1}, g_3)\,\omega(g_2, g_3, g_1)\,\omega(g_3, g_1, g_2^{-1})}{\omega(g_2, g_1, g_3)\,\omega(g_1, g_3, g_2^{-1})\,\omega(g_3, g_2, g_1)} \frac{\omega(g_3, g_2, g_2^{-1})\,\omega(g_2, g_2^{-1})}{\omega(g_2, g_3, g_2^{-1})}, \tag{3.25}$$

which is invariant under coboundaries.

As another example, consider $S^1 \times \mathbb{RP}^2$. Let $g \in G$ be orientation-reversing, with $g^2 = 1$, and $h \in G$ any other element that commutes with g. It was argued in ([85], section 5.2) that on the real projective plane $\mathbb{RP}^2$, with sides identified by g, the discrete torsion phase is $\omega(g, g)$, which for $g^2 = 1$ is easily checked to be coboundary-invariant. For $S^1 \times \mathbb{RP}^2$, where the real projective plane is again constructed with g, the C field discrete torsion phase can be shown to be

$$\frac{\omega(g, g, h)\,\omega(h, g, g)}{\omega(g, h, g)}, \tag{3.26}$$

which is easily checked to be coboundary-invariant.

4 Orbifolds by 2-groups

In this section we will discuss three-dimensional[4] orbifolds by 2-group extensions. We saw in section 2 that an ordinary orbifold by a central group extension of G by trivially-acting K involves a restriction on permitted G bundles, which is implemented by the sum over universes. We shall see an analogous structure here: the 2-group orbifold will involve a restriction on permitted G bundles, which is implemented by a sum over universes. In this fashion we will derive a decomposition, which we will check in examples.

In passing, we should mention that just as Dijkgraaf-Witten topological field theory [77] can be interpreted as an orbifold of a point, at least naively the Yetter model [33–38] appears to be interpretable as a 2-group (or higher) group orbifold of a point. We will not pursue that in this paper, however.

---

[4]As also noted in the introduction, throughout we have in mind effective field theories as prototypes, though our methods also apply more generally.

4.1 General aspects

4.1.1 Notions of 2-groups and their gauging

A 2-group is, roughly, a group in which associativity holds only up to isomorphisms. In this section we will outline orbifolds by 2-groups, and their decomposition.

Briefly, from ([32], section 8.3), given a group G and an abelian group K, to specify a (coherent) 2-group one specifies an action $\alpha : G \to \text{Aut}(K)$ plus an element of $H^3(G, K)$, where the group cohomology is defined with the action of G on K given by α. In this section we will restrict to the analogue of a central extension, for which the map α is trivial, and for which $H^3(G, K)$ is defined with trivial action on the coefficients.

We will describe 2-groups as extensions of the form

$$1 \longrightarrow BK \longrightarrow \tilde{\Gamma} \longrightarrow G \longrightarrow 1, \tag{4.1}$$

for finite abelian K. These are classified by $[\omega] \in H^3(G, K)$.

In broad brushstrokes, to gauge a 2-group $\tilde{\Gamma}$ means that the path integral

- sums over K gerbes, and within that, for each K gerbe,

- sums over G bundles twisted by the action of the K gerbe, in the sense of e.g. [86].

(In general there may also be other mutual twistings, as in e.g. ([64], eqs. (1.10), (1.14)), implementing a Green-Schwarz mechanism, in which case one would not have for example precisely a path integral over ordinary K gerbes, but rather over slightly different objects forming a torsor under K gerbes.) Examples in which the K gerbe acts nontrivially (via the action of BK on line operators, for example) include the gauging that arose in [13], and also in discussions of gauging BK in Chern-Simons theories for K the center of the gauge group.

In this paper, we will be focused on the case in which the one-form symmetry group being gauged acts completely trivially on the three-dimensional theory, meaning that line operators are invariant[5] under BK, meaning for example that associated line operators have no braiding with one another or with any of the line operators in the theory being gauged. In this case, relevant for us in this paper, we will see that gauging a 2-group $\tilde{\Gamma}$ means that the path integral (modulo mutual twistings subtleties as above),

- sums over K gerbes, and for each K gerbe,

- sums over ordinary G bundles — no longer twisted by K, as BK now acts trivially, but with a more subtle restriction on allowed G bundles, a shadow of the fact that we are gauging a nontrivial extension of G by BK.

---

[5]For example, consider SU(2) Chern-Simons theory in three dimensions. This has a $B\mathbb{Z}_2$ one-form symmetry, inherited from the center of SU(2). However, that one-form symmetry multiplies Wilson lines by phases, and so we would not characterize SU(2) Chern-Simons as invariant under this $B\mathbb{Z}_2$. One could in principle consider a different $B\mathbb{Z}_2$, unrelated to the central $\mathbb{Z}_2$, which leaves all Wilson lines invariant. In that case, that $B\mathbb{Z}_2$ could be said to act trivially.

These two cases can be subsumed into a more general picture which is most conveniently related by describing the 2-groups differently, in terms of what are called crossed modules. In any event, in this paper we will gauge finite 2-group extensions involving trivially-acting BK, for which the second notion of gauging is a more apt description. We will study more general cases in upcoming work.

4.1.2 Decomposition conjecture

Consider, as above, gauging a 2-group $\tilde{\Gamma}$ described formally as an extension of a finite group G by BK for K finite and abelian,

$$1 \longrightarrow BK \longrightarrow \tilde{\Gamma} \longrightarrow G \longrightarrow 1. \tag{4.2}$$

This extension determines an element $[\omega] \in H^3(G, K)$.

Because we are gauging a trivially-acting BK, one expects that the theory should possess a global two-form symmetry (distinct from the quantum symmetry), and so should decompose.

We conjecture that such three-dimensional theories decompose in the form

$$\text{QFT}\left([X/\tilde{\Gamma}]\right) = \text{QFT}\left(\coprod_{\rho \in \hat{K}} [X/G]_{\rho(\omega)}\right), \tag{4.3}$$

where $\rho(\omega) \in H^3(G, \text{U}(1))$ represents a discrete theta angle, formally involving a term in the action of the form

$$\int_M \langle \rho, x^*\omega \rangle, \tag{4.4}$$

for $x^*\omega$ as defined in appendix A.2. As we will discuss later, at least on Seifert fibered three-manifolds, this can be rewritten as a discrete-torsion-like phase (of the form discussed in section 3) given by the image of ω under the map

$$H^3(G, K) \xrightarrow{\rho} H^3(G, \text{U}(1)). \tag{4.5}$$

This is a three-dimensional version of decomposition [1], whose existence reflects the fact that $[X/\tilde{\Gamma}]$ has a 2-form symmetry, due to the trivially-acting BK.

Next, we will justify this decomposition conjecture by computing partition functions for gauged finite 2-groups, and also studying operator spectra. In subsequent sections we will check the details in examples.

4.1.3 Partition functions

In this section we will compute partition functions for $\tilde{\Gamma}$ orbifolds in three dimensions (for $\tilde{\Gamma}$ a 2-group extension of a finite group G by a trivially-acting BK). These are (weighted) sums over G bundles restricted so that an invariant vanishes (see appendix A.2). We will see that the resulting partition functions are equivalent to sums of partition functions of ordinary G orbifolds, weighted by C field analogues of discrete torsion,

$$Z\left([X/\tilde{\Gamma}]\right) = \sum_{\rho \in \hat{K}} Z\left([X/G]_{\rho(\omega)}\right), \tag{4.6}$$

in accordance with decomposition (4.3).

In general terms, this is a consequence of the fact, explained in appendix A.2, that $\tilde{\Gamma}$ bundles on three-manifolds M map to G bundles obeying the constraint $x^*\omega = 1 \in H^3(M, K)$, where $\omega \in H^3(G, K)$ determines the extension $\tilde{\Gamma}$, and $x : M \to BG$ determines the G bundle. Such a constraint is implemented by a projector, proportional to

$$\sum_{\rho \in \hat{K}} \exp \left(\int_M \langle \rho, x^*\omega \rangle \right). \tag{4.7}$$

Summing over $\rho \in \hat{K}$ effectively cancels out contributions from any G bundle for which $x^*\omega \neq 1$. As we saw for ordinary central extensions in section 2, inserting such a projection operator in a path integral is equivalent to working with a sum of theories, one for each $\rho \in \hat{K}$, each of which is modified by a discrete theta angle defined by $\rho \in \hat{K}$ and coupling to $x^*\omega \in H^3(M, K)$. This gives rise to the present version of decomposition (4.3).

At least for Seifert fibered three-manifolds, it is straightforward to give this construction a much more concrete description, by describing $x^*\omega$ explicitly in terms of phases derived from the group cocycle ω. To do so, we follow the same[6] procedure used in ([77], section 6.5). Briefly, given a triangulation of the three-manifold M, associate a phase $\omega(g_1, g_2, g_3)$ to each simplex, and use an ordering to determine whether to multiply or divide the phase. (We specialize to Seifert fibered manifolds solely because of potential practical difficulties in explicitly construction a triangulation. Given a triangulation, the method of [77] is otherwise general.) The result is that $\langle \rho, x^*\omega \rangle$ can be identified with a discrete-torsion-like phase [78], as described in section 3, for a class in $H^3(G, \mathrm{U}(1))$ given by the image of $\omega \in H^3(G, K)$ under ρ, or schematically,

$$\begin{aligned} H^3(G, K) &\xrightarrow{\rho} H^3(G, \mathrm{U}(1)), \\ \omega &\mapsto \rho \circ \omega = \rho(\omega). \end{aligned} \tag{4.8}$$

We have that on a connected three-manifold M,

$$Z_M\left([X/\tilde{\Gamma}]\right) = \sum_{\rho \in \hat{K}} Z_M\left([X/G]_{\rho(\omega)}\right), \tag{4.9}$$

matching the prediction of decomposition 4.3, with the sum over universes implementing the restriction to G bundles such that $x^*\omega = 1$.

Next, we specialize to the case of $M = T^3$. As everything can be computed explicitly in this case, we will walk through all the details in order to better explain the idea.

Ordinarily, in a G orbifold on T^3, one would sum over commuting triples $g_1, g_2, g_3 \in G$. Here, however, because of the 2-group extension, only some triples are consistent, much as we saw in the case of ordinary central extensions in section 2. As mentioned above, and as described in detail in appendix A.2, the constraint on G bundles is that $x^*\omega = 1 \in H^3(T^3, K)$.

---

[6]Our notations differ, but the procedure is identical. Specifically, the $\gamma : M \to BG$ used in [77] is the same as $x : M \to BG$ here, and the $\alpha \in H^3(G, \mathrm{U}(1))$ used there coincides with $\omega \in H^3(G, K)$ here. Their analysis is done for $\mathrm{U}(1)$ coefficients, but essentially because K is abelian and in both cases, the group action on the coefficients is trivial, the argument is otherwise the same.

To understand the result, we outline here a slightly sloppy computation for the special case of T^3, which will reproduce the T^3 result derived rigorously in appendix A.3. To make the 2-group $\tilde{\Gamma}$ more concrete, we imagine associating K-valued wavefunctions ψ_g to $g \in G$, which can then be multiplied by K-valued cocycles, where associativity holds up to the cocycle ω as

$$\psi_{g_1 g_2} \psi_{g_3} = \omega(g_1, g_2, g_3) \, \psi_{g_1} \psi_{g_2 g_3}. \tag{4.10}$$

(Note that adding coboundaries to ω merely multiplies the products by phases.) Then, we can derive a consistency condition on commuting triples, as follows.

$$
\begin{aligned}
\psi_{g_1 g_2} \psi_{g_3} &= \psi_{g_2 g_1} \psi_{g_3}, \\
&= \omega(g_2, g_1, g_3) \psi_{g_2} \psi_{g_1 g_3}, \\
&= \omega(g_2, g_1, g_3) \psi_{g_2} \psi_{g_3 g_1}, \\
&= \frac{\omega(g_2, g_1, g_3)}{\omega(g_2, g_3, g_1)} \psi_{g_2 g_3} \psi_{g_1}, \\
&= \frac{\omega(g_2, g_1, g_3)}{\omega(g_2, g_3, g_1)} \psi_{g_3 g_2} \psi_{g_1}.
\end{aligned}
\tag{4.11}
$$

It also equals

$$
\begin{aligned}
\psi_{g_1 g_2} \psi_{g_3} &= \omega(g_1, g_2, g_3) \psi_{g_1} \psi_{g_2 g_3}, \\
&= \omega(g_1, g_2, g_3) \psi_{g_1} \psi_{g_3 g_2}, \\
&= \frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \psi_{g_1 g_3} \psi_{g_2}, \\
&= \frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \psi_{g_3 g_1} \psi_{g_2}, \\
&= \frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \omega(g_3, g_1, g_2) \psi_{g_3} \psi_{g_1 g_2}, \\
&= \frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \omega(g_3, g_1, g_2) \psi_{g_3} \psi_{g_2 g_1}, \\
&= \frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \frac{\omega(g_3, g_1, g_2)}{\omega(g_3, g_2, g_1)} \psi_{g_3 g_2} \psi_{g_1}.
\end{aligned}
\tag{4.12}
$$

In order for these two expressions to match, we must require

$$\frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \frac{\omega(g_3, g_1, g_2)}{\omega(g_3, g_2, g_1)} \frac{\omega(g_2, g_3, g_1)}{\omega(g_2, g_1, g_3)} = 1 \tag{4.13}$$

as an element of K, which is the same condition derived mathematically in appendix A.3. (We suspect it may also be possible to use topological defect lines to give a simple argument, but we leave that for future work.)

We can therefore understand a $\tilde{\Gamma}$ bundle as a collection of K gerbes and G bundles on T^3 defined by commuting triples (g_1, g_2, g_3) subject to the constraint

$$\epsilon(g_1, g_2, g_3) = 1 \tag{4.14}$$

for

$$\epsilon(g_1, g_2, g_3) = \frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \frac{\omega(g_3, g_1, g_2)}{\omega(g_3, g_2, g_1)} \frac{\omega(g_2, g_3, g_1)}{\omega(g_2, g_1, g_3)}. \tag{4.15}$$

For the same reasons as discussed for $H^3(G, \mathrm{U}(1))$ in section 3, it is straightforward to demonstrate that

$$\epsilon(g_1, g_2, g_3 g_4) = \epsilon(g_1, g_2, g_3)\epsilon(g_1, g_2, g_4) \tag{4.16}$$

(and symmetrically), hence using the same argument as in the two-dimensional case, ϵ is invariant under simultaneous conjugation,[7]

$$\epsilon(h g_1 h^{-1}, h g_2 h^{-1}, h g_3 h^{-1}) = \epsilon(g_1, g_2, g_3). \tag{4.17}$$

The partition function of the $\tilde{\Gamma}$ orbifold on T^3 then takes the form[8] [87]

$$\begin{aligned}
Z_{T^3}\left([X/\tilde{\Gamma}]\right) &= \frac{|H^0(T^3, K)|}{|H^1(T^3, K)|} \frac{1}{|H^0(T^3, G)|} \sum_{z_1, z_2, z_3 \in K} \sum_{g_1, g_2, g_3 \in G}{}' Z(g_1, g_2, g_3), \\
&= \frac{1}{|K|^2 |G|} \sum_{z_1, z_2, z_3 \in K} \sum_{g_1, g_2, g_3 \in G}{}' Z(g_1, g_2, g_3),
\end{aligned} \tag{4.18}$$

where the prime indicates that the sum over triples in G is constrained to commuting triples such that $\epsilon(g_1, g_2, g_3) = 1$.

Now, we can enforce the condition that $\epsilon = 1$ by inserting a projector

$$\frac{1}{|K|} \sum_{\rho \in \hat{K}} \epsilon_\rho(g_1, g_2, g_3) \tag{4.19}$$

where ϵ_ρ is the image of ϵ under $\rho : K \to \mathrm{U}(1)$. The partition function then has the form

$$\begin{aligned}
Z_{T^3}\left([X/\tilde{\Gamma}]\right) &= \frac{1}{|K|^2 |G|}|K|^3 \sum_{g_1, g_2, g_3 \in G} \frac{1}{|K|} \sum_{\rho \in \hat{K}} \epsilon_\rho(g_1, g_2, g_3) Z(g_1, g_2, g_3), \\
&= \sum_{\rho \in \hat{K}} Z_{T^3}\left([X/G]_{\epsilon_\rho}\right),
\end{aligned} \tag{4.20}$$

where

$$Z_{T^3}\left([X/G]_{\epsilon_\rho}\right) = \frac{1}{|G|} \sum_{g_1, g_2, g_3 \in G} \epsilon_\rho(g_1, g_2, g_3) Z(g_1, g_2, g_3), \tag{4.21}$$

using a standard normalization (compare e.g. ([73], eq. (5.14))). Each factor ϵ_ρ is precisely a C field analogue of discrete torsion, as reviewed in section 3, and coincides with the quantity we earlier labelled $\rho(\omega)$.

Thus, we see that for the special case of T^3, partition functions are consistent with the decomposition conjecture 4.3. As outlined at the beginning, the same argument applies for

---

[7]We restrict to the same h on each factor because ϵ is only defined on commuting triples, meaning each pair obeys $g_i g_j = g_j g_i$.

[8]The overall factor of $1/|G|$ is standard in orbifolds and ultimately reflects the fact that the sum is counting bundles with automorphisms, see e.g. ([73], eq. (5.14)). The factors involving K can be found in e.g. ([38], eq. (5.20)), ([88], eqs. (2.31), (2.32)) ([89], eq. (9.1)).

any three-manifold. The only real difference on other three-manifolds is that there may be dilaton-type Euler counterterm shifts, as discussed in e.g. [1], which vanish on T^3 as $\chi(T^3) = 0$. Modulo such trivial counterterms, on any connected three-manifold,

$$Z\left([X/\tilde{\Gamma}]\right) = \sum_{\rho \in \hat{K}} Z\left([X/G]_{\epsilon_\rho}\right). \tag{4.22}$$

This is precisely the statement of decomposition (4.3), at the level of partition functions.

To summarize, we see that inserting a projection operator to enforce the constraint on G-twisted sectors makes manifest the statement that the partition function of the 2-group orbifolds equals the partition function for a sum of three-dimensional orbifolds, each twisted by an ϵ_ρ which is [78] a three-dimensional analogue of discrete torsion. In this fashion, we recover decomposition (4.3), at the level of partition functions, in close analogy with the description in section 2 of decomposition in two-dimensional orbifolds.

As an aside, previously in two-dimensional theories with a one-form symmetry given by a trivially-acting K, we saw universes enumerated by irreducible representations of K, see e.g. [1]. Here, since we have a 2-form symmetry and trivially-acting BK, one might have naively guessed that universes would be enumerated by representations of BK, at variance with the conjecture above. However, we examine decomposition for both 1-form and 2-form symmetries formally in appendix B, and observe there that in both cases, universes appear to be enumerated by representations of K, so the form of the conjecture above is consistent.

4.1.4 Local operators

So far we have given a general justification of the decomposition conjecture for gauged 2-groups using partition functions. Let us briefly outline an analogous argument using local operators. In two dimensional orbifolds with trivially-acting subgroups, the twist fields associated to trivially-acting group elements form dimension-zero operators, and the projectors (onto universes) are constructed from linear combinations of those projectors. In three dimensions, when gauging a one-form symmetry, from the general theory of topological defect lines, the theory contains monopole operators, which play an analogous role. Briefly, the monopole operators are endpoints of real codimension two lines corresponding to the gauged one-form symmetry, just as gauging an ordinary (zero-form) symmetry results in real codimension one walls. Two-spheres surrounding the monopole operators have K gerbes, just as circles surrounding two-dimensional twist fields carry bundles.

In any event, given a trivially-acting gauged BK symmetry, the resulting three-dimensional theory will contain monopole operators, which are closely analogous to two-dimensional twist fields, and can be used to build projectors.

For example, in a gauged $B\mathbb{Z}_k$, the monopole operators will generate $\mathbb{Z}_k$ gerbes on S^2, which are classified by $H^2(S^2, \mathbb{Z}_k) = \mathbb{Z}_k$. As those gerbes on S^2 are all generated by powers of one gerbe, there will be one monopole operator which generates the others, call it $\hat{z}$, and which obeys $\hat{z}^k = 1$. Given such operators, one can build projectors, as linear

combinations of the form

$$\Pi_m = \frac{1}{k} \sum_{j=0}^{k-1} \xi^{jm} \hat{z}^j, \tag{4.23}$$

for $\xi = \exp(2\pi i/k)$, which from $\hat{z}^k = 1$ are easily checked to obey

$$\Pi_m \Pi_n = \Pi_m \delta_{m,n}, \quad \sum_{m=0}^{k-1} \Pi_m = 1. \tag{4.24}$$

4.2 Example: $G = 1$, $K = \mathbb{Z}_2$

Let us consider the orbifold $[X/B\mathbb{Z}_2]$ for a moment, where the $B\mathbb{Z}_2$ acts trivially, in the sense that all line operators in the theory are invariant under the $B\mathbb{Z}_2$.

Then, at a path integral level, the orbifold $[X/B\mathbb{Z}_2]$ involves a sum over $\mathbb{Z}_2$ gerbes, but each of the gerbe sectors is identical, much as in a two-dimensional orbifold by a group that acts completely trivially.

At the level of operators, gauging the $B\mathbb{Z}_2$ results in monopole operators, which generate $\mathbb{Z}_2$ gerbes on spheres surrounding the operators, much as twist fields generate branch cuts and hence bundles on surrounding circles in two-dimensional theories.

Since the $B\mathbb{Z}_2$ acts trivially, the monopole operators commute with all local operators present in the original theory, we see that the full set of operators in the gauged theory is just two copies of the operators of the original theory. Furthermore, since the monopole operators generate $\mathbb{Z}_2$ gerbes on surrounding S^2's, and the product of a nontrivial $\mathbb{Z}_2$ gerbe with itself is trivial, we see that if $\hat{z}$ denotes a monopole operator, then $\hat{z}^2 = 1$, and so we can build projection operators

$$\Pi_\pm = \frac{1}{2} \left(1 \pm \hat{z} \right), \tag{4.25}$$

which implement a decomposition.

In particular, in these circumstances,

$$[X/B\mathbb{Z}_2] = X \coprod X, \tag{4.26}$$

as expected from decomposition (4.3). (Here, we use the fact that $\rho(\omega) = 1$ for all $\rho \in \hat{K}$, as ω itself is trivial.)

4.3 Example: $G = \mathbb{Z}_2 = K$

Let us begin with a very simple example. Consider the case of a two-group extension of the form

$$1 \longrightarrow B\mathbb{Z}_2 \longrightarrow \tilde{\Gamma} \longrightarrow \mathbb{Z}_2 \longrightarrow 1. \tag{4.27}$$

As discussed in appendix C.1, $H^3(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, so there is a nontrivial 2-group extension $\tilde{\Gamma}$ of this form.

In this case, it is straightforward to check that $\epsilon(g_1, g_2, g_3)$ is the identity in $\mathbb{Z}_2$ for all triples $g_{1-3} \in \mathbb{Z}_2$, so there is no additional constraint on G bundles on T^3 (beyond pairwise commutivity) to lift to a $\tilde{\Gamma}$ bundle.

It is then straightforward to compute the T^3 partition function from (4.18), yielding

$$
\begin{aligned}
Z_{T^3}\left([X/\tilde{\Gamma}]\right) &= \frac{1}{|K|^2|G|} \sum_{z_1,z_2,z_3 \in K} \sum_{g_1,g_2,g_3 \in G} Z(g_1,g_2,g_3), \\
&= \frac{|K|}{|G|} \sum_{g_1,g_2,g_3 \in G} Z(g_1,g_2,g_3), \\
&= Z_{T^3}\left(\coprod_{\hat{K}}[X/G]\right),
\end{aligned}
\tag{4.28}
$$

as expected from decomposition (4.3).

In this case, $\rho(\omega) = 1$ for all $\rho \in \hat{K}$. Although $H^3(G, \mathrm{U}(1)) = \mathbb{Z}_2$ for $G = \mathbb{Z}_2$, the group $G = \mathbb{Z}_2$ is in some sense too small to have any nontrivial phases resulting from analogues of discrete torsion.

The reader should also note that we get this decomposition for both 2-group extensions $\tilde{\Gamma}$ indexed by $H^2(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$, implying that they are physically equivalent to one another. (Analogous relations were seen in decomposition of two-dimensional theories with one-form symmetries in [1], in which different gerbes are described by the same physical theory.)

4.4 Example: $G = (\mathbb{Z}_2)^3$, $K = \mathbb{Z}_2$

Write $G = (\mathbb{Z}_2)^3 = \langle a, b, c \rangle$. Let us pick an extension of G by BK corresponding to the element of $H^3(G, K)$ given by $(-)^{a_1 b_2 c_3}$ in appendix C.3.

Then, the commuting triples g_{1-3} for which $\epsilon(g_1, g_2, g_3) \neq 1 \in K$ include, for example, (ax, by, cz) and their permutations, where

$$
x \in \{1, b, c, bc\}, \quad y \in \{1, a, c, ac\}, \quad z \in \{1, a, b, ab\}.
\tag{4.29}
$$

The partition function of $[X/\tilde{\Gamma}]$ then has the form

$$
Z_{T^3}\left([X/\tilde{\Gamma}]\right) = \frac{1}{|K|^2|G|} \sum_{z_{1-3} \in K} \sum_{g_{1-3} \in G}{}' Z(g_1,g_2,g_3) = \frac{|K|}{|G|} \sum_{g_{1-3} \in G}{}' Z(g_1,g_2,g_3),
\tag{4.30}
$$

where the prime indicates that some of the G-twisted sectors are omitted.

For the trivial representation $1 \in \hat{K}$, $\epsilon_1(g_1, g_2, g_3) = 1$, but for the nontrivial representation $1 \in \hat{K}$, $\epsilon_\rho(g_1, g_2, g_3)$ corresponds to the discrete-torsion-like phase (3.2) corresponding to the cocycle $\omega_4 \in H^3(G, \mathrm{U}(1))$ listed in appendix C.3, essentially because the ω_4 cocycle has the same form as the chosen element of $H^3(G, K)$ above: $\omega_4(g_1, g_2, g_3) = (-)^{a_1 b_2 c_3}$ also. That discrete-torsion-like phase equals -1 on precisely the triples that are omitted from the $[X/\tilde{\Gamma}]$ orbifold, namely sectors of the form (ax, by, cz) and their permutations, for x, y, z as in (4.29). Sectors that are not omitted include (g, g, g) for g any element of $(\mathbb{Z}_2)^3$.

Putting this together, we see

$$
Z_{T^3}\left([X/\tilde{\Gamma}]\right) = Z_{T^3}\left([X/G]\coprod [X/G]_{\omega_4}\right),
\tag{4.31}
$$

matching the prediction of decomposition (4.3) for this case. The sectors that are omitted in the $\tilde{\Gamma}$ orbifold cancel out between the two G orbifolds, realizing a 'multiverse interference effect' as usual.

4.5 Example: $G = (\mathbb{Z}_2)^2 = K$

In this case, it is straightforward to check that the discrete-torsion-like phase factors $\omega(\rho)$ are all trivial for any extension class in $H^3(G, K)$ and any $\rho \in \hat{K}$, hence in this case our conjecture (4.3) predicts

$$\mathrm{QFT}\left([X/\tilde{\Gamma}]\right) = \mathrm{QFT}\left(\coprod_{\rho \in \hat{K}} [X/G]\right). \tag{4.32}$$

We can check this by computing the T^3 partition function. In this case, for $G = K = (\mathbb{Z}_2)^2$, it is straightforward to check that $\epsilon = 1$ holds automatically for every $[\omega] \in H^3(G, K)$, so there is no constraint on commuting triples (g_1, g_2, g_3). Then, from the general formula (4.18),

$$\begin{aligned} Z_{T^3}\left([X/\tilde{\Gamma}]\right) &= \frac{1}{|K|^2|G|} \sum_{z_1, z_2, z_3 \in K} \sum_{g_1, g_2, g_3 \in G} Z(g_1, g_2, g_3), \\ &= \frac{|K|}{|G|} \sum_{g_1, g_2, g_3 \in G} Z(g_1, g_2, g_3), \\ &= |K| Z_{T^3}\left([X/G]\right), \end{aligned} \tag{4.33}$$

which is consistent with the prediction of decomposition.

5 Interpretation: sigma models on 2-gerbes

These orbifolds by 2-groups have a more formal description as realizations of sigma models on 2-gerbes, closely analogous to sigma models on gerbes as described in [90–92].

Briefly, gerbes are closely analogous to principal bundles. A n-(G-)gerbe is essentially a fiber bundle whose fibers are 'groups' $B^n G$ of higher-form symmetries. As a result, a sensibly-defined sigma model with target such a gerbe should admit a global $B^n G$ symmetry, corresponding to translations along the fibers of the gerbe.

Because the 'group' $BG = [\mathrm{point}/G]$, a G-gerbe — a fiber bundle with fiber BG — can be locally presented as a quotient in which a subgroup acts trivially. This was utilized in the previous work [90–92] to construct sigma models on gerbes, presented as orbifolds and gauge theories with trivially-acting subgroups.

Now, this glosses over a number of subtleties, including questions about non-uniqueness of presentations (dealt with by identifying a sigma model on a stack or gerbe with a universality class of RG flow), potential modular invariance and unitarity issues in orbifolds, seeming moduli mismatches, and most important for decomposition, violations of the cluster decomposition axiom, which were discussed in [1, 90–92].

In any event, from the same reasoning, orbifolds by 2-groups with trivially-acting one-form symmetries appear to be presentations of sigma models on 2-gerbes, just as sigma models on ordinary gerbes are realized in terms of gauge theories with trivially-acting (ordinary) subgroups [90–92].

As discussed in ([22], section 2), a map $f : Y \to \mathcal{G}$, for $\mathcal{G}$ a (banded) G-gerbe over M (G assumed finite), defines[9] a map $\tilde{f} : Y \to M$ with a trivialization of $\tilde{f}^*\mathcal{G}$. If $\dim Y = 2$, this gives a restriction on the degree of $\tilde{f}$. Explicitly, let $\pi : \mathcal{G} \to M$ be projection, then $\tilde{f} = \pi \circ f$, and $\tilde{f}^*\mathcal{G}$ has a canonical trivialization. This trivialization may be clearer to the reader in the closely related case of bundles. Given a map $g : Y \to E$ for some bundle $\pi : E \to M$, we can define $\tilde{g} = \pi^* g$, and then as

$$\tilde{g}^* E = \{(y, e) \in Y \times E \,|\, \tilde{g}(y) = \pi(e)\}, \tag{5.1}$$

there is a trivialization $Y \to \tilde{g}^* E$ given by $y \mapsto (y, \tilde{g}(y))$. The same analysis applies to gerbes.

So, we have that a map $f : Y \to \mathcal{G}$ defines a map $\tilde{f} : Y \to M$ such that $\tilde{f}^*\mathcal{G}$ is trivializable. As discussed in ([22], section 2), if $\dim Y = 2$, this implies a restriction on degrees. If the characteristic class of $\mathcal{G}$ is $\omega \in H^2(M, G)$ (G finite), then $\tilde{f}^*\omega = 0 \in H^2(Y, G)$. For example, if $Y = \mathbb{P}^1$ and $M = \mathbb{P}^N$, with $\tilde{f} : \mathbb{P}^1 \to \mathbb{P}^N$ of degree d, and $G = \mathbb{Z}_k$, then $\tilde{f}^*\omega = d\omega$, and $d\omega = 0 \in H^2(\mathbb{P}^N, \mathbb{Z}_k)$ means $d\omega \equiv 0 \mod k$, that the product of d and the characteristic class is divisible by k.

If the dimension of Y is not two, then one still has a constraint that $\tilde{f}^*\mathcal{G}$ is trivializable, which does restrict the possible maps $\tilde{f}$; however, that restriction will not be describable as simply as a restriction on map degrees.

Briefly, the same formal arguments apply to (banded analogues of) 2-gerbes. Just as for ordinary gerbes, a map $f : Y \to \mathcal{G}$, for $\mathcal{G}$ a 2-(G-)gerbe over M, from essentially the same argument as before, one gets a map $\tilde{f} : Y \to M$ with a restriction on degrees, following from the statement that $\tilde{f}^*\mathcal{G}$ is trivializable (and so has vanishing characteristic class in $H^3(Y, G)$).

6 Analogues in other dimensions and other degrees

6.1 Decomposition in higher-dimensional orbifolds

In this section, we make some conjectures for how this program could be continued into higher dimensions, by observing that the arguments we have applied to ordinary central extensions and 2-group extensions also apply, with only minor modifications, to higher-group extensions.

Consider orbifolds in d dimensions. Specifically, consider gauging a higher-group extension

$$1 \longrightarrow B^{d-2}K \longrightarrow \tilde{\Gamma} \longrightarrow G \longrightarrow 1, \tag{6.1}$$

for K a finite abelian group, classified by an element $[\omega] \in H^d(G, K)$. The orbifold $[X/\tilde{\Gamma}]$ has the structure of a $[X/G]$ orbifold but with a restriction on the G sectors, namely that they trivialize a coboundary-invariant constructed from ω, or explicitly $x^*\omega = 1$ in the notation of appendix A. For example, on T^d, we require that commuting d-tuples $g_1, \cdots, g_d$ also obey

$$\epsilon(g_1, \cdots, g_d) = 1 \in K, \tag{6.2}$$

---

[9]In fact, the map f is equivalent to the map $\tilde{f}$ plus a specific choice of trivialization of $\tilde{f}^*\mathcal{G}$.

for

$$\epsilon(g_1, \cdots, g_d) = \prod_{\text{perm's } \sigma} \omega(g_{\sigma(1)}, \cdots, g_{\sigma(d)})^{\text{sgn } \sigma}, \tag{6.3}$$

as outlined in appendix A.3.

The reader should note in passing that the phase ϵ above, for coefficients in any abelian group, obeys standard properties of discrete-torsion-like phases, specifically,

- the phase ϵ is invariant under coboundaries, and so is well-defined on cohomology $H^d(G, \mathrm{U}(1))$,

- the phase ϵ is a homomorphism in the sense that

$$\epsilon(ab, g_3, \cdots, g_{d+1}) = \epsilon(a, g_3, \cdots, g_{d+1}) \, \epsilon(b, g_3, \cdots, g_{d+1}), \tag{6.4}$$

  (and similarly for products in other positions, from the antisymmetry of ϵ), as can be verified from the identity

$$\prod_{\text{perm's } \sigma}{}' (d\omega) \left(g_{\sigma(1)}, \cdots, g_{\sigma(d+1)} \right)^{\text{sgn } \sigma} = 1, \tag{6.5}$$

  for permutations of the $(d+1)$-tuple $(a, b, g_3, \cdots, g_{d+1})$, where the prime indicates that we restrict to permutations preserving the order of a, b,

- the phase $\epsilon(g_1, \cdots, g_d)$ is invariant under $\mathrm{SL}(n, \mathbb{Z})$ actions on the group elements, as is straightforward to verify from the homomorphism property.

Returning to partition functions, the restriction above on G bundles can be implemented by inserting a projector, which (as discussed previously) is equivalent to a decomposition into universes $[X/G]$ weighted by a discrete theta angle coupling to $x^*\omega$, in the notation of appendix A.

In the special case of T^d, the restriction above to d-tuples obeying (6.2) is equivalent to inserting a projection operator in an ordinary $[X/G]$ orbifold, with projector which on T^d takes the form

$$\frac{1}{|K|} \sum_{\rho \in \hat{K}} \epsilon_\rho(g_1, \cdots, g_d), \tag{6.6}$$

where $\epsilon_\rho \in \mathrm{U}(1)$ is the image of ϵ under $\rho : K \to \mathrm{U}(1)$. The resulting T^d partition function is the same as that of a sum of partition functions of $[X/G]$ orbifolds, each with a discrete-torsion-like phase factor defined by ϵ_ρ.

Thus, in higher dimensions, based on the partition function analysis above, we expect that the $[X/\tilde{\Gamma}]$ orbifold decomposes:

$$\mathrm{QFT}\left([X/\tilde{\Gamma}]\right) = \mathrm{QFT}\left(\coprod_{\rho \in \hat{K}} [X/G]_{\rho(\omega)} \right), \tag{6.7}$$

(for $\rho(\omega)$ indicating a discrete theta angle ρ coupled to $x^*\omega$,) which at least in special cases can be expressed in the form

$$\mathrm{QFT}\left([X/\tilde{\Gamma}]\right) = \mathrm{QFT}\left(\coprod_{\rho \in \hat{K}} [X/G]_{\rho(C)} \right), \tag{6.8}$$

for $\rho(C)$ expressing elements of higher-dimensional analogues of discrete torsion. (Interpreted literally as a sigma model, this theory should only be understood as a low-energy effective action, of course, though this should also be a prototype for theories in d dimensions.)

It is also straightforward to outline the origin of projectors in this language. In two dimensional orbifolds, the projectors onto the universes are constructed as linear combinations of the twist fields associated to trivially-acting group elements. Now, in a d dimensional theory, if we gauge a p-form symmetry, then in the language of topological defect lines (see e.g. [93]), one gets a real codimension $(p+1)$ object that generalizes the branch cuts of an orbifold, and which terminates on a real codimension $(p+2)$ object, which is the analogue of a twist field.

So, work in d dimensions, and gauge a (trivially-acting) $(d-2)$-form symmetry. In principle, this should result in a theory with a global $(d-1)$-form symmetry, and hence a decomposition. Because we have gauged a $(d-2)$-form symmetry, we get a real codimension $(d-1)$ object, an analogue of the two-dimensional branch cut, which terminates at a real codimension d object (an analogue of a twist field), which in d dimensions is pointlike. Those pointlike objects, those analogues of twist fields, could then be used to construct projectors.

6.2 Interpretation: higher-dimensional sigma models

In this paper we have discussed how maps from 2-manifolds into ordinary gerbes and maps from 3-manifolds into 2-gerbes define maps into spaces with restrictions on degrees (following from the constraint that the pullback of the gerbe be trivial).

There is a very closely analogous story for higher gerbes, which we outline in this section (slightly generalizing ([22], section 2)). Maps into $(m\text{-})G$-gerbes are closely related to maps into underlying spaces with restrictions on degrees. Consider a map f from a space Y into a $(m\text{-})$gerbe $\mathcal{G} \to M$. Composing with the projection gives a map $\tilde{f} : Y \to M$. The map f defines a section of $\tilde{f}^*\mathcal{G}$, almost by definition, hence it trivializes $\tilde{f}^*\mathcal{G}$.

As a consequence, the map $\tilde{f}$ induces

$$\tilde{f}^* : \ H^{m+1}(M, G) \ \longrightarrow \ H^{m+1}(Y, G). \tag{6.9}$$

The characteristic class of the m-gerbe $\mathcal{G}$ must be in the kernel of that map, hence there is a restriction on possible maps $\tilde{f}$.

In particular, a map $f : Y \to \mathcal{G}$ is equivalent to a map $\tilde{f} : Y \to M$, trivializing the characteristic class of the gerbe, together with a specific choice of trivialization of the m-gerbe $\tilde{f}^*\mathcal{G}$, which is an $(m-1)$-gerbe over B.

Depending upon the circumstances, this may imply a restriction on the map $\tilde{f}$. For example, if $\mathcal{G}$ is an m-gerbe and $\dim Y \leq m$, the map $\tilde{f}$ is unconstrained, since the pullback of the characteristic class is an element of $H^{m+1}(Y, G) = 0$, so all maps are in the kernel.

On the other hand, suppose we have an m-gerbe and $\dim Y > m$. (For example, a four-dimensional low-energy effective sigma model mapping into a 1-gerbe, 2-gerbe, or 3-gerbe.) In this case, the map $\tilde{f}$ is constrained, but depending upon the relative values of m and $\dim Y$, the restriction may be on e.g. lower homotopy.

7 Analogues in Chern-Simons theories in three dimensions

It is well-known that gauging the $B\mathbb{Z}_2$ central symmetry of SU(2) Chern-Simons theory in three dimensions results in an SO(3) Chern-Simons theory. Briefly, the path integral sums over $\mathbb{Z}_2$ gerbes and gerbe-twisted SU(2) bundles with connection, for which bundle transition functions only close up to gerbe transition functions on triple overlaps; the resulting path integral is precisely a path integral over SO(3) bundles with connection, for which the second Stiefel-Whitney class w_2 coincides with the gerbe characteristic class, and the third Stiefel-Whitney class is determined by a Steenrod square as $w_3 = \mathrm{Sq}^1(w_2)$.

In that case, the $B\mathbb{Z}_2$ acted nontrivially on line operators, specifically as phases determined by the n-ality of the representation (partially) defining the Wilson line.

We could consider more general situations, in which the one-form symmetry group maps to an action on the center, but with a nonzero kernel. In general, consider a 2-group Γ defined by a crossed module $\{d : A \to H\}$, where A is abelian and the image of d is contained within the center of the group H. If we let K denote the kernel of d, and $G = H/\mathrm{im}\,A$, then

$$1 \longrightarrow K \longrightarrow A \longrightarrow H \longrightarrow G \longrightarrow 1, \tag{7.1}$$

which defines an element $\omega \in H^3(G, K)$. In principle, if G is, for example, a Lie group, but we are only concerned with flat bundles, then the same homotopy computations of appendix A.2 imply that (flat) Γ bundles map to (flat) G bundles obeying the constraint that $\phi^* \omega = 0$.

Such a constraint can be implemented via a decomposition, and flat bundles arise in Chern-Simons theories, so we have a prediction:

$$\text{Chern-Simons}(H)/BA = \coprod_{\theta \in \hat{K}} \text{Chern-Simons}(G)_\theta, \tag{7.2}$$

where the θ are discrete theta angles coupling to $\phi^* \omega$, and for levels such that the Chern-Simons theories are defined.

For example, consider an SU(2) Chern-Simons with an action of $B\mathbb{Z}_4$, which maps to the central one-form symmetry of SU(2), with a $B\mathbb{Z}_2$ kernel which leaves all line operators invariant. In this case, we predict

$$\text{Chern-Simons}(SU(2))/B\mathbb{Z}_4 = \text{Chern-Simons}(SO(3))_+ \coprod \text{Chern-Simons}(SO(3))_-. \tag{7.3}$$

This form of decomposition will be discussed in detail in upcoming work.

8 Conclusions

In this paper we have discussed 2-group orbifolds and their decomposition. Because these theories involve the gauging of a trivially-acting one-form symmetry, they possess a global two-form symmetry, implying a decomposition. The pattern followed is very similar to two dimensions: the twisted sectors of the 2-group orbifolds look like twisted sectors of ordinary orbifolds obeying a constraint, and that constraint is implemented by the decomposition.

In our analysis, we specialized to 2-groups that were analogues of central extensions, defined in part by trivial group actions of G on K. It would be interesting to consider more general cases; such analyses are left for future work.

One direction that would be interesting to pursue would be to deform 2-group orbifolds by turning on C field flux, in the same way that one can turn on discrete torsion to deform ordinary two-dimensional orbifolds. Decomposition in orbifolds with discrete torsion was discussed in [2]. A related direction that would be interesting to pursue would be analogues of quantum symmetries in 2-group orbifolds, generalizing the results of [3].

Acknowledgments

A Homotopy theory

In this section we will give more rigorous justifications of statements appearing in the main text that various Γ bundles project to G bundles obeying a restriction, utilizing homotopy theory. We first describe such restrictions in the case of ordinary central extensions, as a warm-up exercise for the reader, then turn to 2-group extensions.

A.1 Classification of bundles for central extensions

Let $\omega \in H^2(G, K)$, for G and K finite groups and K abelian, and associate a central extension Γ:

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1. \tag{A.1}$$

We want to understand Γ bundles, which is to say, $\mathrm{Map}(M, B\Gamma)$.

Now, from the surjective map in the central extension, there is a map $B\Gamma \to BG$.

Furthermore, since Γ is a central extension of G by K, $\Gamma \to G$ is a principal K bundle, hence classified by a map $G \to BK$, meaning that Γ is the fiber product

$$\begin{array}{ccc} \Gamma & \longrightarrow & \text{point} \\ \downarrow & & \downarrow \\ G & \longrightarrow & BK \end{array} \tag{A.2}$$

Taking B, we get the fiber product

$$\begin{array}{ccc} B\Gamma & \longrightarrow & \text{point} \\ \downarrow & & \downarrow \\ BG & \overset{\omega}{\longrightarrow} & K(K, 2) \end{array} \tag{A.3}$$

Put another way, if we apply the functor $B(-)$ to the short exact sequence (A.1) we get that $B\Gamma \to BG$ is a BK principal bundle, and so classified by a map $BG \to B^2K = K(K,2)$, which gives the diagram above.

In passing, if K is abelian but not central, then the G action on K gives a twisted form of K, a sheaf of groups $\mathcal{K} \to BG$ which is locally isomorphic to $K \times BG$. The extension class ω is then an element of $H^2(BG, \mathcal{K})$ (instead of $H^2(BG, K)$), equivalently a section of $\Gamma(BG, K(\mathcal{K}, 2))$. In this case, $B\Gamma$ is the homotopy intersection of this section and the zero section of $K(\mathcal{K}, 2)$.

Returning to central extensions, one has the diagram

$$
\begin{array}{ccc}
\mathrm{Map}(M, B\Gamma) & \longrightarrow & \mathrm{Map}(M, \mathrm{point}) = \mathrm{point} \\
\downarrow & & \downarrow \\
\mathrm{Map}(M, BG) & \longrightarrow & \mathrm{Map}(M, K(K,2)) = H^2(M, K),
\end{array}
\tag{A.4}
$$

where the bottom map sends $x \in \mathrm{Map}(M, BG) \mapsto x^*\omega$. An element $x \in \mathrm{Map}(M, BG)$ will be in the image of an element of $\mathrm{Map}(M, B\Gamma)$ precisely when $x^*\omega = 1 \in H^2(M, K)$. We will examine the implications of this on tori in section A.3.

It may be helpful to observe that $x^*\omega$ can also be understood as the image of the isomorphism class of the G bundle in $H^1(M, G)$ under the Bockstein homomorphism

$$
H^1(M, G) \longrightarrow H^2(M, K).
\tag{A.5}
$$

If we momentarily drop the assumption that G be finite, then for $G = \mathrm{SO}(n)$, $\Gamma = \mathrm{Spin}(n)$, the quantity we label $x^*\omega$ would coincide with the second Stiefel-Whitney class of the G bundle.

A.2 Classification of 2-group bundles

Now, let us repeat that analysis for bundles of 2-groups constructed analogously as extensions.

Let $\omega \in H^3(G, K) = H^3(BG, K)$. Associated to this is a 2-group, which we describe as a crossed module $\Gamma_\bullet$,

$$
\Gamma_\bullet = \{\Gamma_1 \xrightarrow{d} \Gamma_0\},
\tag{A.6}
$$

which sits in the sequence

$$
1 \longrightarrow K \longrightarrow \Gamma_1 \xrightarrow{d} \Gamma_0 \longrightarrow G \longrightarrow 1.
\tag{A.7}
$$

The classifying 2-stack $B\Gamma_\bullet$ is connected, with homotopy groups

$$
\pi_0(B\Gamma_\bullet) = 0, \quad \pi_1(B\Gamma_\bullet) = G, \quad \pi_2(B\Gamma_\bullet) = K.
\tag{A.8}
$$

Let M be a compact oriented manifold. We want to understand $\mathrm{Map}(M, B\Gamma_\bullet)$.

To that end, note that there is a map

$$
B\Gamma_\bullet \longrightarrow BG,
\tag{A.9}
$$

arising as the first stage of the Postnikov tower of $B\Gamma_\bullet$, and in fact $B\Gamma_\bullet$ is a fiber square

$$\begin{array}{ccc} B\Gamma_\bullet & \longrightarrow & \text{point} \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\ \omega\ } & K(K,3), \end{array} \qquad (A.10)$$

(from the definition of $B\Gamma_\bullet$ as a homotopy type). In the bottom map, we interpret $\omega \in H^3(G,K)$ by writing

$$H^3(G,K) = H^3(BG,K) = \text{Map}\left(BG, K(K,3)\right). \qquad (A.11)$$

From the fiber square above, we derive the square

$$\begin{array}{ccc} \text{Map}(M, B\Gamma_\bullet) & \longrightarrow & \text{Map}(M, \text{point}) = \text{point} \\ \downarrow & & \downarrow \\ \text{Map}(M, BG) & \longrightarrow & \text{Map}(M, K(K,3)) = H^3(M,K). \end{array} \qquad (A.12)$$

which constrains possible maps (hence possible G bundles on M).

Note that since

$$\omega \ \in \ H^3(G,K) = H^3(BG,K) = \text{Map}\left(BG, K(K,3)\right), \qquad (A.13)$$

we see $x^*\omega \in \text{Map}(M, K(K,3)) = H^3(M,K)$, so the restriction above is that $x^*\omega$ is trivial as an element of $H^3(M,K)$:

$$x^*\omega = 1. \qquad (A.14)$$

We will examine the implications of this on tori in section A.3.

Furthermore, the fiber in $\text{Map}(M, B\Gamma_\bullet)$ over such as x is just the fiber of the Postnikov tower, namely

$$\text{Map}(M, K(K,2)) = H^2(M,K). \qquad (A.15)$$

Thus, fibered over every G bundle are the K gerbes, much as one would expect physically when gauging a 2-group.

A.3 Computations on tori

So far we have argued that for both ordinary central extensions and 2-group central extensions of a finite group G, for a G bundle to be in the image of a bundle whose structure group is the extension, the G bundle must have the property that $x^*\omega = 1$, where ω is an element of group cohomology characterizing the extension, and $x \in \text{Map}(M, BG)$ encodes the G bundle.

In this section we will unpack that conclusion for the case that M is a torus.

Recall that for a torus T (of any dimension) we have

$$H^k(T, \mathbb{Z}) = \text{Hom}(\wedge^k H, \mathbb{Z}), \qquad (A.16)$$

where $H = H_1(T, \mathbb{Z})$, and the right hand side above can be viewed as skew-symmetric abelian group maps from the direct product of k copies of $H \to \mathbb{Z}$.

Another way to say this is as follows. The cochains $C^k(T, \mathbb{Z})$ are the group of poly linear maps $H^{\times k} \to \mathbb{Z}$, and the cocycles $Z^k(T, \mathbb{Z})$ are the subgroup of maps killed by the Hocschild differential. Each cocycle is cohomologous to a unique skew-symmetric cocycle and that skew-symmetric cocycle gives a preferred representative in the corresponding cohomology class.

This works up to torsion with arbitrary coefficients. In particular if K is a finite abelian group we have the universal coefficient theorem short exact sequence

$$0 \longrightarrow \text{Ext}^1(H_{k-1}(T, \mathbb{Z}), K) \longrightarrow H^k(T, K) \longrightarrow \text{Hom}(\wedge^k H, K) \longrightarrow 0. \tag{A.17}$$

However, note that $H_{k-1}(T, \mathbb{Z})$ is a free finitely generated abelian group and so

$$\text{Ext}^1(H_{k-1}(T, \mathbb{Z}), K) = 0, \tag{A.18}$$

which implies

$$H^k(T, K) = \text{Hom}(\wedge^k H, K). \tag{A.19}$$

Thus, for a K-valued cocycle, on the torus T its cohomology class is uniquely determined by its projection to its skew-symmetric part.

Now, let us consider particular examples. Earlier in section A.1 we argued that a G bundle arose from a Γ bundle for Γ an (ordinary) central extension determined by $\omega \in H^2(G, K)$ if and only if $x^*\omega = 1 \in H^2(M, K)$. From the analysis above, we see that if $M = T^2$, $x^*\omega$ is trivial if and only if

$$\frac{\omega(g_1, g_2)}{\omega(g_2, g_1)} = 1 \tag{A.20}$$

for commuting pairs $g_1, g_2 \in G$ defining a G bundle (up to isomorphism). In this fashion we recover the constraint described earlier in section 2.

For 2-groups and $\omega \in H^3(G, K)$, we can proceed similarly for $M = T^3$, and see that $x^*\omega = 1$ implies that

$$\frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)} \frac{\omega(g_3, g_1, g_2)}{\omega(g_3, g_2, g_1)} \frac{\omega(g_2, g_3, g_1)}{\omega(g_2, g_1, g_3)} = 1 \tag{A.21}$$

as previously outlined in section 4.1.

Concretely, given k and Γ defined by a central extension

$$1 \longrightarrow B^{k-2}K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1, \tag{A.22}$$

on a torus T^k with $H = H_1(T^k, \mathbb{Z})$, with central extension corresponding to a class $[\omega] \in H^k(G, K)$, we have

$$\text{Hom}(H, \Gamma) \longrightarrow \text{Hom}(H, G) \longrightarrow \text{Hom}(\wedge^k H, K), \tag{A.23}$$

where $\wedge^k H \cong \mathbb{Z}$, and so the homomorphism $\wedge^k H \to K$ is determined by the element of K which is the image of $1 \in \wedge^k H = \mathbb{Z}$. That element of K is the total skew-symmetrization

of the cocycle ω when evaluated on the commuting k-tuple of elements of G describing an element of $\mathrm{Hom}(H, G)$. In other words, for $k = 2$, the image of the generator of $\wedge^2 H$ is

$$\frac{\omega(g_1, g_2)}{\omega(g_2, g_1)},\tag{A.24}$$

and for $k = 3$, the image of the generator of $\wedge^3 H$ is

$$\frac{\omega(g_1, g_2, g_3)}{\omega(g_1, g_3, g_2)}\frac{\omega(g_3, g_1, g_2)}{\omega(g_3, g_2, g_1)}\frac{\omega(g_2, g_3, g_1)}{\omega(g_2, g_1, g_3)}.\tag{A.25}$$

We have focused on ordinary groups and 2-groups, but formally analogous results arise for $k > 3$. In particular, for a k-torus T and a degree k cohomology class $[\omega] \in H^k(G, K)$, we will have a similar statement. From the same analysis as above, for any k, the image of the generator of $\wedge^k H$

$$\prod_{\text{permutations } \sigma} \omega\left(g_{\sigma(1)}, g_{\sigma(2)}, \cdots, g_{\sigma(k)}\right)^{\mathrm{sgn}\,\sigma},\tag{A.26}$$

for $[\omega] \in H^k(G, K)$.

In mathematics, discussions of bundles for higher groups and related notions can be found in e.g. [96] and references therein.

B Decomposition as duality

In this appendix we describe decomposition, formally, as a kind of Fourier transform, and then apply that idea to three-dimensional examples to argue that in orbifolds $[X/\tilde{\Gamma}]$, where $\tilde{\Gamma}$ is an extension of G by BK, the universes are indexed by representations of K rather than BK, which is what we observe physically.

B.1 Ordinary decomposition

In this section we will describe decomposition as a form of duality. This is a special case of the duality discussed in [97]; see also [98].

Let Γ be a central extension

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow G \longrightarrow 1,\tag{B.1}$$

where both G and K are finite. Note $[X/\Gamma] \to [X/G]$ is a K gerbe, a principal BK bundle. Now, because K is abelian, $BK = K[1]$ is a stacky abelian group, and so a principal BK bundle is a torsor over BK, classified by a class in $H^1([X/G], BK) = \mathrm{Ext}^1_{[X/G]}(\mathbb{Z}, BK)$. Hence the extension (B.1) can be interpreted as a complex of sheaves of abelian groups on $[X/G]$ which is an extension of $\mathbb{Z}$ by BK (over $[X/G]$).

Let $\mathcal{X}$ be that extension. It is a family of (complexes of) abelian groups over $[X/G]$ which sits in an exact sequence

$$1 \longrightarrow BK \longrightarrow \mathcal{X} \longrightarrow \mathbb{Z} \longrightarrow 1.\tag{B.2}$$

Note that this means $\mathcal{X}$, as a space (rather than an abelian group) is a disjoint union of stacks $\mathcal{X}_n$ over $[X/G]$, where $\mathcal{X}_n$ is the preimage of $n \in \mathbb{Z}$. Each $\mathcal{X}_n$ is a K gerbe. For example,

$$\mathcal{X}_0 = [X/G] \times BK, \quad \mathcal{X}_1 = [X/\Gamma], \tag{B.3}$$

and for $n > 1$, $\mathcal{X}_n$ is the nth power of $\mathcal{X}_1 = [X/\Gamma]$ as a K gerbe over $[X/G]$.

Now, we can dualize, by taking homomorphisms into $\mathbb{Z}$. (Note that the usual dual of K is $\mathrm{Hom}(K, \mathbb{Z})$, whereas the Pontryagin dual of K is $\mathrm{Hom}(K, S^1)$. Also note $S^1 = B\mathbb{Z} = \mathbb{Z}[1]$.)

So, take the short exact sequence (B.2), and dualize to BS^1. This becomes

$$1 \longrightarrow \mathrm{Hom}(\mathbb{Z}, BS^1) \longrightarrow \mathrm{Hom}(\mathcal{X}, BS^1) \longrightarrow \mathrm{Hom}(BK, BS^1) \longrightarrow 1. \tag{B.4}$$

(More generally, there are higher Ext's on the right, which can be shown to vanish here.)

Define the dual group $\hat{\mathcal{X}} = \mathrm{Hom}(\mathcal{X}, BS^1)$, and use the fact that

$$\mathrm{Hom}(\mathbb{Z}, BS^1) = BS^1, \tag{B.5}$$

$$\mathrm{Hom}(BK, BS^1) = \mathrm{Hom}(K, S^1), \tag{B.6}$$

to rewrite the sequence above as

$$1 \longrightarrow BS^1 \longrightarrow \hat{\mathcal{X}} \longrightarrow \mathrm{Hom}(K, S^1) \longrightarrow 1. \tag{B.7}$$

Since $\mathrm{Hom}(K, S^1)$ is just the characters of K, we see that $\hat{\mathcal{X}}$ is a family of abelian groups, extending the characters by BS^1, hence is decomposed by characters:

$$\hat{\mathcal{X}} = \coprod_\lambda \hat{\mathcal{X}}_\lambda, \tag{B.8}$$

where $\hat{\mathcal{X}}_\lambda$ is an S^1-gerbe on $[X/G] \times \lambda$, for any character λ. The part corresponding to $\mathcal{X}_1$ is $\hat{\mathcal{X}}_\lambda$ for λ the tautological character.

So far we have discussed ordinary decomposition at a very formal level as a mathematical duality. This description has two ingredients:

- The data labelling components of the dual, namely characters of K, and

- the classes of S^1 gerbes on $[X/G] \times \lambda$, which are

    - images under λ of the original extension class of Γ,

    - images under λ of the characteristic class of the principal BK bundle $[X/\Gamma] \to [X/G]$, an element of $H^2(G, K)$,

    - images under λ of the extension class of $\mathbb{Z}$ by BK, namely $\mathrm{Ext}^1(\mathbb{Z}, BK)$.

B.2 Decomposition for two-group extensions

In this section we will outline a formal understanding of the decomposition appearing elsewhere in this paper. In particular, we will argue that, for at least one version of

decomposition for two-group extensions, the universes should be classified by irreducible representations of K, and not[10] BK.

Consider the 2-group extension

$$1 \longrightarrow BK \longrightarrow \tilde{\Gamma} \longrightarrow G \longrightarrow 1. \tag{B.9}$$

Here, $[X/\tilde{\Gamma}] \to [X/G]$ is a principal B^2K bundle.

Note $BK = K[1], B^2K = K[2]$.

As in section B.1, the extension given by formula (B.9) is equivalent to specifying a complex of abelian groups $\tilde{\mathcal{X}}$ on $[X/G]$ given as an extension

$$1 \longrightarrow K[2] \longrightarrow \tilde{\mathcal{X}} \longrightarrow \mathbb{Z} \longrightarrow 1. \tag{B.10}$$

Now, we can dualize, but there are several possible targets, such as $\mathbb{Z}$, $\mathbb{Z}[1] = B\mathbb{Z} = S^1$, $\mathbb{Z}[2] = BS^1$, $\mathbb{Z}[3] = B^2S^1$.

We will 'dualize' by taking Hom's into $\mathbb{Z}[3]$. Applying this to sequence (B.10), we get

$$1 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}[3]) \longrightarrow \mathrm{Hom}(\tilde{\mathcal{X}}, \mathbb{Z}[3]) \longrightarrow \mathrm{Hom}(K[2], \mathbb{Z}[3]) \longrightarrow 1. \tag{B.11}$$

Define $\widehat{\tilde{\mathcal{X}}} = \mathrm{Hom}(\tilde{\mathcal{X}}, \mathbb{Z}[3])$, and note

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}[3]) = \mathbb{Z}[3], \tag{B.12}$$

$$\mathrm{Hom}(K[2], \mathbb{Z}[3]) = \mathrm{Hom}(K, \mathbb{Z}[1]) = \mathrm{Hom}(K, S^1), \tag{B.13}$$

to simplify that sequence to

$$1 \longrightarrow \mathbb{Z}[3] \longrightarrow \widehat{\tilde{\mathcal{X}}} \longrightarrow \mathrm{Hom}(K, S^1) \longrightarrow 1, \tag{B.14}$$

so we see that $\widehat{\tilde{\mathcal{X}}}$ is fibered over characters of K, just as in the previous case. Note furthermore that $\widehat{\tilde{\mathcal{X}}}$ is an S^1 2-gerbe over each component, exactly as expected.

In this section we have made one choice of dualization, dualizing by taking Hom's to $\mathbb{Z}[3]$, to understand decomposition. In principle, there exist other dualizations, to $\mathbb{Z}[k]$ for other k. We leave an examination of the physical interpretation of such duals, if any, for future work.

C Some results in group cohomology

In this appendix we collect some results on group cohomology of various groups, which are used in the main text.

For reference, recall in group cohomology that coboundaries are determined in degree two by

$$(\delta\alpha)(g_1, g_2, g_3) = \frac{g_1 \cdot \alpha(g_2, g_3)}{\alpha(g_1 g_2, g_3)} \frac{\alpha(g_1, g_2 g_3)}{\alpha(g_1, g_2)}, \tag{C.1}$$

---

[10]See e.g. ([52], appendix A), [94, 95] for perspectives on representations of BK.

and in degree three by

$$(\delta\omega)(g_1, g_2, g_3, g_4) = \frac{g_1 \cdot \omega(g_2, g_3, g_4)}{\omega(g_1 g_2, g_3, g_4)} \frac{\omega(g_1, g_2 g_3, g_4)}{\omega(g_1, g_2, g_3 g_4)} \omega(g_1, g_2, g_3). \qquad (C.2)$$

As most of the computations in this paper involve group cohomology with trivial action on the coefficients, we will assume so unless otherwise noted. That said, orientifolds do involve group cohomology with nontrivial action on the coefficients, so on occasion we will use that group cohomology instead.

C.1 $\mathbb{Z}_2$

In this section we will collect some useful results on the group cohomology of $\mathbb{Z}_2$, which will be useful in setting a pattern for results later in this appendix for more general products of $\mathbb{Z}_2$'s.

First,

$$H^n(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2 \qquad (C.3)$$

for all (positive) n, where the group cohomology has trivial action on the coefficients, which is assumed throughout this appendix. Writing the elements of $\mathbb{Z}_2$ as $\{0, 1\}$, the only possibly nonzero normalized cochains are $x = \omega(1, 1, \cdots, 1)$. In this case,

$$d\omega(1, 1, \cdots, 1) = \begin{cases} 2x & n \text{ odd,} \\ 0 & n \text{ even.} \end{cases} \qquad (C.4)$$

However, for $\mathbb{Z}_2$ coefficients, $2x = 0$, hence $d\omega = 0$ in all cases.

Note that for U(1) coefficients, for example, $H^{\text{even}}(\mathbb{Z}_2, \text{U}(1)) = 0$, so the existence of these cocycles is tied to $\mathbb{Z}_2$ coefficients specifically.

C.2 $\mathbb{Z}_2 \times \mathbb{Z}_2$

In this section we will collect some useful results on the group cohomology of $\mathbb{Z}_2 \times \mathbb{Z}_2$.

First, consider the group

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2) = (\mathbb{Z}_2)^3. \qquad (C.5)$$

We represent the elements as $\mathbb{Z}_2$-valued normalized[11] cocycles $C(g, h)$, $g, h \in \mathbb{Z}_2 \times \mathbb{Z}_2$. Write

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}, \qquad (C.6)$$

and let x, y, z denote the generators of each of the three $\mathbb{Z}_2$'s in the cohomology group, then normalized cocycles are listed in table 1.

In particular, for $\mathbb{Z}_2$ coefficients and normalized cocycles, $C(g, g)$ is coboundary-invariant, and from table 1, we see that

$$C(a, a) = x, \quad C(b, b) = y, \quad C(ab, ab) = z \qquad (C.7)$$

naturally encode the generators of each of the three $\mathbb{Z}_2$'s in $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$.

---

[11]Throughout this paper, a normalized cocycle is one which is the identity if any group element among its arguments is the identity.

	1	a	b	ab
1	1	1	1	1
a	1	x	1	x
b	1	xyz	y	xz
ab	1	yz	y	z

Table 1. Representative normalized cocycles for $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$. For example, $C(a,b) = 1$, $C(b,a) = xyz$.

The cohomology groups of $\mathbb{Z}_2 \times \mathbb{Z}_2$ also include

$$H^2((\mathbb{Z}_2)^2, \mathrm{U}(1)) = \mathbb{Z}_2, \quad H^3((\mathbb{Z}_2)^2, \mathrm{U}(1)) = (\mathbb{Z}_2)^3, \quad H^4((\mathbb{Z}_2)^2, \mathrm{U}(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2. \quad \text{(C.8)}$$

Degree three cohomology was recently discussed in detail in [4] and ([5], appendix A), giving both representatives as well as invariants that distinguish different cohomology classes.

For use elsewhere, let us characterize the elements of $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$ more precisely. Let α denote a normalized 4-cocycle, meaning $\alpha(g_1, g_2, g_3, g_4) = 1$ if any $g_i = 1$. (In effect, this is a gauge choice, which requires in evaluating coboundaries that 3-cochains equal 1 if any of their arguments is 1.)

If we write each $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ as $g = (x, y)$ for $x, y \in \{0, 1\}$, then normalized cocycles $\alpha_{0, \cdots, 3}$ representing different elements of $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$ are as follows:

$$\alpha_0((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = +1, \quad \text{(C.9)}$$

$$\alpha_1((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = (-1)^{x_1 y_2 y_3 y_4}, \quad \text{(C.10)}$$

$$\alpha_2((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = (-1)^{x_1 x_2 x_3 y_4}, \quad \text{(C.11)}$$

$$\alpha_3((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = (-1)^{x_1 y_4 (x_2 x_3 + y_2 y_3)}. \quad \text{(C.12)}$$

As elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, α_0 is the identity and $\alpha_3 = \alpha_1 \alpha_2$.

For any pair $(g, h) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, we can define an invariant $A(g, h)$ of normalized 4-cocycles, invariant under coboundaries, as

$$A(g, h) = \frac{\alpha(g, g, g, h)}{\alpha(g, g, h, g)} \frac{\alpha(g, h, g, g)}{\alpha(h, g, g, g)}. \quad \text{(C.13)}$$

Applying these invariants to the normalized cocycles above, and writing $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$, with $a = (1, 0)$, $b = (0, 1)$, we compute invariants corresponding to elements of $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$ as in table 2. This can be useful in distinguishing elements of $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$, as cocycles are only defined up to coboundaries.

C.3 $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Write $\Gamma = (\mathbb{Z}_2)^3 = \langle a, b, c \rangle$, $G = \mathbb{Z}_2 = \langle a \rangle$, $K = (\mathbb{Z}_2)^2 = \langle b, c \rangle$.

Now,

$$H^3(\Gamma, \mathrm{U}(1)) = (\mathbb{Z}_2)^7, \quad \text{(C.14)}$$

α	$A(a,b)$	$A(a,ab)$	$A(b,a)$	$A(b,ab)$	$A(ab,a)$	$A(ab,b)$
α_0	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
α_1	$+1$	$+1$	-1	-1	-1	-1
α_2	-1	-1	$+1$	$+1$	-1	-1
α_3	-1	-1	-1	-1	$+1$	$+1$

Table 2. Invariants computed from elements of $H^4(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathrm{U}(1))$.

which we can understand as arising from the Lyndon-Hochschild-Serre spectral sequence as[12]

$$H^3(K, \mathrm{U}(1)) = (\mathbb{Z}_2)^3, \tag{C.15}$$

$$H^1(G, H^2(K, \mathrm{U}(1))) = H^1(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2, \tag{C.16}$$

$$H^2(G, H^1(K, \mathrm{U}(1))) = H^2(\mathbb{Z}_2, (\mathbb{Z}_2)^2) = (\mathbb{Z}_2)^2, \tag{C.17}$$

$$H^3(G, \mathrm{U}(1)) = \mathbb{Z}_2. \tag{C.18}$$

Thus three of the generators of $H^3(\Gamma, \mathrm{U}(1))$, call them ω_1, ω_2, ω_3, are pullbacks from $H^3(K, \mathrm{U}(1))$ under the projection $\Gamma = G \times K \to K$.

Another generator, call it ω_7, is similarly a pullback from $H^3(G, \mathrm{U}(1))$, and is given by

$$\omega_7((a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3)) = (-1)^{a_1 a_2 a_3} \tag{C.19}$$

where here we identify $a, b, c \in \{0, 1\}$.

One more generator, call it ω_4, comes from $H^1(G, H^2(K, \mathrm{U}(1)))$ and can be represented as

$$\omega_4 = (-1)^{a_1 b_2 c_3}. \tag{C.20}$$

The final two generators of $H^3(\Gamma, \mathrm{U}(1))$, call them $\omega_{5,6}$, come from $H^2(G, H^1(K, \mathrm{U}(1)))$, and can be represented as

$$\omega_5 = (-1)^{a_1 a_2 b_3}, \tag{C.21}$$

$$\omega_6 = (-1)^{a_1 a_2 c_3}. \tag{C.22}$$

One can check that all of the ω_i's are co-closed and that they are not cohomologous (they differ on coboundary invariants $\omega(g, g, g)$ for some $g \in \Gamma$). Furthermore, it is also easy to check that the C field discrete torsion phase (3.2) on T^3 are nontrivial for ω_4 evaluated on triples of the form $(a_1 x, b_2 y, c_3 z)\}$ and their permutations for

$$x \in \{1, b_1, c_1, b_1 c_1\}, \quad y \in \{1, a_2, c_2, a_2 c_2\}, \quad z \in \{1, a_2, b_2, a_2 b_2\}. \tag{C.23}$$

The group

$$H^3((\mathbb{Z}_2)^3, \mathbb{Z}_2) = (\mathbb{Z}_2)^{10}. \tag{C.24}$$

---

[12]Since Γ is just a direct sum, the extension class vanishes, and so all of the maps d_n in the spectral sequence are trivial and so the sequence stabilizes at $E_2^{p,1}$.

From Lyndon-Hochschild-Serre as before, we can write this as

$$H^3(K, \mathbb{Z}_2) = H^3((\mathbb{Z}_2)^2, \mathbb{Z}_2) = (\mathbb{Z}_2)^4, \tag{C.25}$$

$$H^1(G, H^2(K, \mathbb{Z}_2)) = \mathrm{Hom}(\mathbb{Z}_2, (\mathbb{Z}_2)^3) = (\mathbb{Z}_2)^3, \tag{C.26}$$

$$H^2(G, H^1(K, \mathbb{Z}_2)) = H^2(\mathbb{Z}_2, (\mathbb{Z}_2)^2) = (\mathbb{Z}_2)^2, \tag{C.27}$$

$$H^3(G, \mathbb{Z}_2) = \mathbb{Z}_2. \tag{C.28}$$

C.4 $(\mathbb{Z}_2)^k$

In this appendix we give a basis of cocycles for $H^n((\mathbb{Z}_2)^k, \mathbb{Z}_2)$ for any n, k.

Represent $g \in (\mathbb{Z}_2)^k$ as $g = (x^1, \cdots, x^k)$ with $x^i \in \{0, 1\}$. Pick k nonnegative integers $m_1, \cdots, m_k$ such that

$$m_1 + m_2 + \cdots + m_k = n. \tag{C.29}$$

There will be

$$N = \binom{n + k - 1}{k - 1} \tag{C.30}$$

possibilities, each of which corresponds to a cocycle. In particular, we will see that

$$H^n((\mathbb{Z}_2)^k, \mathbb{Z}_2) = (\mathbb{Z}_2)^N. \tag{C.31}$$

Define a function

$$f_m : \{1, \cdots, n\} \longrightarrow \{1, \cdots, k\} \tag{C.32}$$

$(m \in \{1, \cdots, N\})$ by

$$f_m(a) = j \tag{C.33}$$

for j such that

$$m_1 + m_2 + \cdots + m_{j-1} < a \leq m_1 + m_2 + \cdots + m_j \tag{C.34}$$

(in conventions in which $m_0 = 0$).

Then define

$$\omega_m(g_1, \cdots, g_n) = (-)^\alpha \tag{C.35}$$

for

$$\alpha = \prod_{a=1}^n x_a^{f_m(a)}. \tag{C.36}$$

For example, consider $H^n(\mathbb{Z}_2, \mathbb{Z}_2)$. In this case, $N = 1$ for all n, and $f_1(a) = 1$ for all $a \in \{1, \cdots, n\}$. In each case, if we write $\mathbb{Z}_2 = \langle a \rangle$, then a normalized cocycle for $H^n(\mathbb{Z}_2, \mathbb{Z}_2)$ is $(-)^a$, for any n.

For another example, consider the group cohomology of $(\mathbb{Z}_2)^2 = \langle a, b \rangle$, starting with $H^2((\mathbb{Z}_2)^2, \mathbb{Z}_2)$. Here, $N = 3$, corresponding to the three sums

$$1 + 1, \quad 2 + 0, \quad 0 + 2. \tag{C.37}$$

Corresponding respectively to those three sums we have the functions

$$\begin{aligned} f_{1+1}(1) &= 1, \quad f_{1+1}(2) = 2, \\ f_{2+0}(1) &= 1, \quad f_{2+0}(2) = 1, \\ f_{0+2}(1) &= 2, \quad f_{0+2}(2) = 2, \end{aligned} \tag{C.38}$$

which correspond to the three cocycles

$$(-)^{a_1 b_2}, \quad (-)^{a_1 a_2}, \quad (-)^{b_1 b_2}. \tag{C.39}$$

One can compute $H^3((\mathbb{Z}_2)^2, \mathbb{Z}_2)$ similarly. Here, $N = 4$, corresponding to the four sums

$$2 + 1, \quad 1 + 2, \quad 3 + 0, \quad 0 + 3, \tag{C.40}$$

and corresponding to those sums are the functions

$$\begin{array}{lll}
f_{2+1}(1) = 1, & f_{2+1}(2) = 1, & f_{2+1}(3) = 2, \\
f_{1+2}(1) = 1, & f_{1+2}(2) = 2, & f_{1+2}(3) = 2, \\
f_{3+0}(1) = 1, & f_{3+0}(2) = 1, & f_{3+0}(3) = 1, \\
f_{0+3}(1) = 2, & f_{0+3}(2) = 2, & f_{0+3}(3) = 2.
\end{array} \tag{C.41}$$

The corresponding cocycles are

$$\omega_{2+1} = (-)^{a_1 a_2 b_3}, \quad \omega_{1+2} = (-)^{a_1 b_2 b_3}, \quad \omega_{3+0} = (-)^{a_1 a_2 a_3}, \quad \omega_{0+3} = (-)^{b_1 b_2 b_3}. \tag{C.42}$$

For another example, for $(\mathbb{Z}_2)^3 = \langle a, b, c \rangle$, there is a basis of cocycles given by

$$H^1((\mathbb{Z}_2)^3, \mathbb{Z}_2) = \{(-)^a, (-)^b, (-)^c\} \text{ (3 elements)}, \tag{C.43}$$

$$H^2((\mathbb{Z}_2)^3, \mathbb{Z}_2) = \{(-1)^{a_1 a_2}, (-1)^{a_1 b_2}, (-1)^{a_1 c_2}, (-1)^{b_1 b_2}, (-1)^{b_1 c_2}, (-1)^{c_1 c_2}\}$$
$$\text{(6 elements)}, \tag{C.44}$$

$$H^3((\mathbb{Z}_2)^3, \mathbb{Z}_2) = \{(-1)^{a_1 a_2 a_3}, (-1)^{a_1 a_2 b_3}, (-1)^{a_1 a_2 c_3}, (-1)^{a_1 b_2 b_3}, (-1)^{a_1 b_2 c_3}, (-1)^{a_1 c_2 c_3},$$
$$(-1)^{b_1 b_2 b_3}, (-1)^{b_1 b_2 c_3}, (-1)^{b_1 c_2 c_3}, (-1)^{c_1 c_2 c_3}\}$$
$$\text{(10 elements)}. \tag{C.45}$$

Furthermore, $H^n((\mathbb{Z}_2)^3, \mathbb{Z}_2)$ has a basis of

$$\binom{n+2}{2} = \frac{(n+1)(n+2)}{2} \tag{C.46}$$

cocycles.

C.5   Dihedral groups

Let D_n denote the dihedral group of order $2n$. Then,

$$H^3(D_n, \mathrm{U}(1)) = \begin{cases} \mathbb{Z}_{2n} & \text{n odd}, \\ (\mathbb{Z}_2)^2 \times \mathbb{Z}_n & \text{n even}. \end{cases} \tag{C.47}$$

Below we give an explicit cocycle for $H^2(D_4, \mathbb{Z}_2) = (\mathbb{Z}_2)^3$. The group D_4 is presented in terms of generators a, b, with relations

$$a^2 = 1 = b^4, \quad aba = b^{-1} = b^3, \tag{C.48}$$

and the results are expressed in terms of x, y, z which generate each of three $\mathbb{Z}_2$'s in table 3. The corresponding invariant phases $\epsilon(g, h)$ only depend on one of the generators, x, as shown in table 4.

	1	b	b^2	b^3	a	ba	b^2a	b^3a
1	1	1	1	1	1	1	1	1
b	1	1	1	x	1	1	1	x
b^2	1	1	x	x	1	1	x	x
b^3	1	x	x	x	1	x	x	x
a	1	y	x	y	z	yz	xz	yz
ba	1	xy	x	y	z	xyz	xz	yz
b^2a	1	xy	1	y	z	xyz	z	yz
b^3a	1	xy	1	xy	z	xyz	z	xyz

Table 3. Table of cocycles representing elements of $H^2(D_4, \mathbb{Z}_2) = (\mathbb{Z}_2)^3$. The variables x, y, z generate the three $\mathbb{Z}_2$'s.

	1	b	b^2	b^3	a	ba	b^2a	b^3a
1	1	1	1	1	1	1	1	1
b	1	1	1	1	$-$	$-$	$-$	$-$
b^2	1	1	1	1	x	x	x	x
b^3	1	1	1	1	$-$	$-$	$-$	$-$
a	1	$-$	x	$-$	1	$-$	x	$-$
ba	1	$-$	x	$-$	$-$	1	$-$	x
b^2a	1	$-$	x	$-$	x	$-$	1	$-$
b^3a	1	$-$	x	$-$	$-$	x	$-$	1

Table 4. Table of invariant phases $\epsilon(g,h) = C(g,h)/C(h,g)$ using the cocycles in table 3. An entry '$-$' indicates a non-commuting pair.

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