

Superconformal duality-invariant models and $\mathcal{N} = 4$ SYM effective action

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ABSTRACT: We present $\mathcal{N} = 2$ superconformal $U(1)$ duality-invariant models for an Abelian vector multiplet coupled to conformal supergravity. In a Minkowski background, such a nonlinear theory is expected to describe (the planar part of) the low-energy effective action for the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills (SYM) theory on its Coulomb branch where (i) the gauge group $SU(N)$ is spontaneously broken to $SU(N - 1) \times U(1)$; and (ii) the dynamics is captured by a single $\mathcal{N} = 2$ vector multiplet associated with the $U(1)$ factor of the unbroken group. Additionally, a local $U(1)$ duality-invariant action generating the $\mathcal{N} = 2$ super-Weyl anomaly is proposed. By providing a new derivation of the recently constructed $U(1)$ duality-invariant $\mathcal{N} = 1$ superconformal electrodynamics, we introduce its $SL(2, \mathbb{R})$ duality-invariant coupling to the dilaton-axion multiplet.

KEYWORDS: Extended Supersymmetry, Superspaces, Supersymmetry and Duality

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1 Introduction

It is believed that the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory is self-dual [1, 2] (see also [3]). This conjecture was originally put forward in the late 1970s as a duality between the conventional and soliton sectors of the theory. Twenty years later it was suggested¹ [7] that self-duality might be realised in terms of a low-energy effective action of the theory on its Coulomb branch. Here the gauge group $SU(N)$ is spontaneously broken to $SU(N-1) \times U(1)$ and the dynamics is described by a single $\mathcal{N} = 2$ vector multiplet associated with the $U(1)$ factor of the unbroken group. Two different realisations of self-duality for the $\mathcal{N} = 4$ SYM effective action in $\mathcal{N} = 2$ superspace were proposed: (i) self-duality under Legendre transformation [7]; and (ii) self-duality under $U(1)$ duality rotations [8].² So far, both

¹This was inspired in part by the Seiberg-Witten theory [4, 5] and also by the AdS/CFT correspondence [6].

²It was also conjectured by Schwarz [9] that the world-volume action of a probe D3-brane in an $AdS_5 \times S^5$ background of type IIB superstring theory, with one unit of flux, can be reinterpreted as the exact (or highly) effective action for $U(2)$ $\mathcal{N} = 4$ super Yang-Mills theory on the Coulomb branch.

proposals have not been derived from first principles, although each of them is consistent with the one-loop [10–14] and two-loop [15] calculations.³ It is worth pointing out that (ii) implies (i), see [8, 17] for the technical details.

Building on the influential 1981 work by Gaillard and Zumino [18], the general theory of U(1) duality-invariant models for nonlinear electrodynamics in four dimensions was developed in the mid 1990s [19–22] and the early 2000s [23–25] (see also [26]), including the case of duality-invariant theories with higher derivatives [17].⁴ The Gaillard-Zumino-Gibbons-Rasheed (GZGR) formalism [18–22] admits a natural extension to higher dimensions [31–33] (see also [17, 27, 34] for a review). In four dimensions, this setting has been properly generalised to formulate general U(1) duality-invariant $\mathcal{N} = 1$ and $\mathcal{N} = 2$ globally [8, 17] and locally [35–37] supersymmetric theories. In particular, extending the earlier proposal of [38, 39], the first consistent perturbative scheme to construct the $\mathcal{N} = 2$ supersymmetric Born-Infeld action was given in [17]. The formalism of nonlinear realisations for the partial $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ breaking of supersymmetry advocated in [40] reproduced [41] the results of [17]. Further progress toward the construction of the $\mathcal{N} = 2$ supersymmetric Born-Infeld action has been achieved in [42–44].

Eight years ago, the general formalism of supersymmetric duality rotations [8, 17, 35–37] was combined with the bosonic approach due to Ivanov and Zupnik⁵ [24–26] to develop new formulations for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric duality-invariant theories coupled to supergravity [45]. The method makes use of an auxiliary unconstrained chiral superfield (a spinor in the $\mathcal{N} = 1$ case and a scalar for $\mathcal{N} = 2$) and is characterised by the fundamental property that U(1) duality invariance is equivalent to the manifest U(1) invariance of the self-interaction. In the $\mathcal{N} = 1$ rigid supersymmetric case, analogous results were independently obtained in [46].⁶

A constructive perturbative scheme to compute $\mathcal{N} = 2$ superconformal U(1) duality-invariant actions for the $\mathcal{N} = 2$ vector multiplet was described in [8]. In the present paper we will give closed-form expressions for such actions using the formalism of ref. [45] in conjunction with the results of earlier publications [52, 53] devoted to the $\mathcal{N} = 2$ super-Gauss-Bonnet term and super-Weyl anomalies.

While ref. [8] described the $\mathcal{N} = 2$ superconformal U(1) duality-invariant actions, it did not discuss $\mathcal{N} = 0$ and $\mathcal{N} = 1$ duality-invariant models compatible with (super) conformal symmetry. The consideration in [8, 17] was restricted to those $\mathcal{N} = 0$ and $\mathcal{N} = 1$ self-dual systems which are well-defined in a weak-field limit.⁷ A year ago, it was shown that there exists a unique model for conformal duality-invariant electrodynamics [54] (see also [55]).

³For an alternative two-loop calculation see [16].

⁴Further aspects of duality-invariant theories with higher derivatives were studied, e.g., in [27–30].

⁵The Ivanov-Zupnik (IZ) approach was inspired by the structure of the $\mathcal{N} = 3$ supersymmetric Born-Infeld action proposed in [23].

⁶The IZ approach [24, 25] has also been generalised to higher dimensions in [47]. It has been used [48] to establish the relation between helicity conservation for the tree-level scattering amplitudes and the electromagnetic duality. In four dimensions a hybrid formulation has been developed [49] which combines the powerful features of the IZ approach with the Pasti-Sorokin-Tonin formalism [50, 51].

⁷This means that the interactions $\Lambda(z, \bar{z})$ in (2.9) and (A.2) were chosen in [8, 17] to be real analytic. As a result, (super)conformal nonlinear systems were automatically excluded.

Recently its $\mathcal{N} = 1$ superconformal extension has been introduced [56]. Below we will show how the $\mathcal{N} = 1$ superconformal duality-invariant model of [56] naturally occurs within the framework developed in [8, 17, 45].

This paper is organised as follows. As a warmup exercise, in section 2 we give a new derivation of the recently constructed $\mathcal{N} = 1$ superconformal duality-invariant model [56]. Section 3 describes the new $\mathcal{N} = 2$ superconformal duality-invariant models. The obtained results and some generalisations are discussed in section 4. The main body of the paper is accompanied by three technical appendices. Appendix A reviews the model for conformal duality-invariant electrodynamics [54] (see also [55]). Appendix B collects those results concerning $\mathcal{N} = 1$ supergravity and super-Weyl transformations, which are used in section 2. Appendix C contains similar material but for the $\mathcal{N} = 2$ case. Our two-component spinor notation and conventions correspond to [57, 58].

2 $\mathcal{N} = 1$ superconformal duality-invariant model

We consider a dynamical system describing an Abelian $\mathcal{N} = 1$ vector multiplet in curved superspace and denote by $S[W, \bar{W}]$ the corresponding action functional. The action is assumed to depend on the chiral spinor field strength W_α and its conjugate $\bar{W}_{\dot{\alpha}}$ which are constructed in terms of a real unconstrained gauge prepotential V [59, 60] as

$$W_\alpha = -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\mathcal{D}_\alpha V, \quad \bar{\mathcal{D}}_{\dot{\beta}} W_\alpha = 0. \tag{2.1}$$

The prepotential is defined modulo gauge transformations

$$\delta_\lambda V = \lambda + \bar{\lambda}, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \lambda = 0, \tag{2.2}$$

such that $\delta_\lambda W_\alpha = 0$. The gauge-invariant field strengths W_α and $\bar{W}_{\dot{\alpha}}$ obey the Bianchi identity

$$\mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \tag{2.3}$$

and thus W_α is a reduced chiral superfield. We assume that $S[W, \bar{W}]$ does not involve the combination $\mathcal{D}^\alpha W_\alpha$ as an independent variable, and therefore it can unambiguously be defined as a functional of a *general* chiral superfield W_α and its conjugate $\bar{W}_{\dot{\alpha}}$. Then, defining

$$i M_\alpha := 2 \frac{\delta}{\delta W_\alpha} S[W, \bar{W}], \tag{2.4}$$

the equation of motion for V is

$$\mathcal{D}^\alpha M_\alpha = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{M}^{\dot{\alpha}}. \tag{2.5}$$

Here the variational derivative $\delta S/\delta W^\alpha$ is defined by

$$\delta S = \int d^4x d^2\theta \mathcal{E} \delta W^\alpha \frac{\delta S}{\delta W^\alpha} + \text{c.c.}, \tag{2.6}$$

where \mathcal{E} denotes the chiral integration measure, and W^α is assumed to be an unrestricted covariantly chiral spinor.

Since the Bianchi identity (2.3) and the equation of motion (2.5) have the same functional form, one may consider U(1) duality rotations

$$\delta W_\alpha = \lambda M_\alpha, \quad \delta M_\alpha = -\lambda W_\alpha, \quad (2.7)$$

with $\lambda \in \mathbb{R}$ a constant parameter. The condition for duality invariance is the so-called self-duality equation

$$\text{Im} \int d^4x d^2\theta \mathcal{E} \{W^\alpha W_\alpha + M^\alpha M_\alpha\} = 0, \quad (2.8)$$

in which W_α is chosen to be a general chiral spinor.

2.1 Formulation without auxiliary chiral variables

General duality-invariant supersymmetric theories with at most two derivatives at the component level were constructed in [8, 17, 35, 36]. They belong to the family of nonlinear vector multiplet theories of the general form

$$S[W, \bar{W}; \Upsilon] = \frac{1}{4} \int d^4x d^2\theta \mathcal{E} W^2 + \text{c.c.} \\ + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E \frac{W^2 \bar{W}^2}{\Upsilon^2} \Lambda \left(\frac{u}{\Upsilon^2}, \frac{\bar{u}}{\Upsilon^2} \right), \quad (2.9)$$

where $W^2 = W^\alpha W_\alpha$ and $\bar{W}^2 = \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}$, the complex variable u is defined by

$$u := \frac{1}{8} (\mathcal{D}^2 - 4\bar{R}) W^2, \quad (2.10)$$

and Υ is a nowhere vanishing real scalar with the super-Weyl transformation

$$\delta_\sigma \Upsilon = (\sigma + \bar{\sigma}) \Upsilon, \quad \bar{\mathcal{D}}_{\dot{\beta}} \sigma = 0, \quad (2.11)$$

with σ being the super-Weyl parameter, see appendix B for more details. The transformation law (2.11) implies that (2.9) is super-Weyl invariant. We remind the reader that V is inert under the super-Weyl transformations,

$$\delta_\sigma V = 0 \quad \implies \quad \delta_\sigma W_\alpha = \frac{3}{2} \sigma W_\alpha \quad \implies \quad \delta_\sigma (\mathcal{D}^\alpha W_\alpha) = (\sigma + \bar{\sigma}) \mathcal{D}^\alpha W_\alpha, \quad (2.12)$$

and therefore the following composite chiral scalar

$$(\mathcal{D}^2 - 4\bar{R}) \left(\frac{W^2}{\Upsilon^2} \right) \quad (2.13)$$

is super-Weyl invariant. In the functional (2.9), the expression in the first line is the free vector multiplet action, while the interaction effects are encoded by $\Lambda(z, \bar{z})$ which is a real function of a complex variable z .

Three different realisations of Υ are possible:

1. One option is that Υ is a composite superfield, which is constructed in terms of the chiral compensator S_0 of old minimal supergravity and matter chiral superfields φ^i ,

$$\Upsilon = S_0 \bar{S}_0 \exp \left(-\frac{1}{3} K(\varphi^i, \bar{\varphi}^{\bar{j}}) \right), \quad \bar{\mathcal{D}}_{\dot{\beta}} S_0 = 0, \quad \bar{\mathcal{D}}_{\dot{\beta}} \varphi^i = 0, \quad (2.14)$$

where $K(\varphi, \bar{\varphi})$ is the Kähler potential of a Kähler manifold. The super-Weyl transformations laws of the chiral compensator and matter chiral superfields are

$$\delta_\sigma S_0 = \sigma S_0, \quad \delta_\sigma \varphi^i = 0. \quad (2.15)$$

2. Another choice for Υ is the real linear compensator \mathbb{L} of new minimal supergravity,

$$(\bar{\mathcal{D}}^2 - 4R)\mathbb{L} = (\mathcal{D}^2 - 4\bar{R})\mathbb{L} = 0. \tag{2.16}$$

This constraint is only compatible with the super-Weyl transformation law

$$\delta_\sigma \mathbb{L} = (\sigma + \bar{\sigma})\mathbb{L}. \tag{2.17}$$

3. One more option is given by $\Upsilon = \mathcal{D}^\alpha W_\alpha = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}$, which corresponds to the family of superconformal vector multiplet models introduced in [61]:

$$S[W, \bar{W}] = \frac{1}{4} \int d^4x d^2\theta \mathcal{E} W^2 + \text{c.c.} + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E \frac{W^2 \bar{W}^2}{(\mathcal{D}W)^2} \mathfrak{H} \left(\frac{u}{(\mathcal{D}W)^2}, \frac{\bar{u}}{(\mathcal{D}\bar{W})^2} \right). \tag{2.18}$$

Here $\mathfrak{H}(z, \bar{z})$ is a real function of a complex variable. In general, this action explicitly depends on $\mathcal{D}^\alpha W_\alpha$, and thus there is no way to unambiguously define it as a functional of an unrestricted chiral spinor W_α and its conjugate $\bar{W}_{\dot{\alpha}}$; such models are not compatible with duality invariance. However, all dependence on $\mathcal{D}^\alpha W_\alpha$ disappears provided⁸

$$\mathfrak{H}(z, \bar{z}) = \frac{y}{\sqrt{z\bar{z}}} + \frac{1}{2}(1-x) \left(\frac{1}{z} + \frac{1}{\bar{z}} \right), \tag{2.19}$$

for real parameters x, y .

In what follows, it will be assumed that Υ is a compensating multiplet independent of the vector multiplet V , which excludes option 3 with the exception of (2.19).

The model (2.9) is U(1) duality-invariant if the interaction $\Lambda(u, \bar{u})$ satisfies the following differential equation

$$\text{Im} \left\{ \Gamma - \bar{u} \Gamma^2 \right\} = 0, \quad \Gamma := \frac{\partial(u\Lambda)}{\partial u}, \quad \Lambda = \Lambda(u, \bar{u}). \tag{2.20}$$

The function $\Lambda(u, \bar{u})$ was chosen in [8, 17] to be real analytic, in order for the model to be well-defined in the weak-field limit. However $\Lambda(u, \bar{u})$ is not required to be globally analytic. It should be pointed out that setting $\Upsilon = g^{-1} = \text{const}$ in (2.9) and choosing

$$\Lambda(u, \bar{u}) = \frac{1}{1 + \frac{1}{2}A + \sqrt{1 + A + \frac{1}{4}B^2}}, \quad A = u + \bar{u}, \quad B = u - \bar{u} \tag{2.21}$$

defines the $\mathcal{N} = 1$ supersymmetric Born-Infeld action [62]. This U(1) duality-invariant theory is a Goldstone multiplet action for partial $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking in Minkowski space [63, 64], as well as in the following maximally supersymmetric backgrounds [65]: (i) $\mathbb{R} \times S^3$; (ii) $\text{AdS}_3 \times \mathbb{R}$; and (iii) a supersymmetric plane wave.

⁸The most general function $\mathfrak{H}(z, \bar{z})$, for which $\mathcal{D}^\alpha W_\alpha$ drops from the action (2.18), is given by $\mathfrak{H}_{\text{SC}}(z, \bar{z}) = (z\bar{z})^{-1/2} f(u/\bar{u})$, with f a function on S^1 . However, it is the special choice (2.19) which is compatible with the self-duality equation (2.8).

2.2 Superconformal duality-invariant model

The model (2.9) is superconformal provided the functional form of $\Lambda(u, \bar{u})$ is such that (2.9) is independent of Υ .⁹ This condition implies that

$$\Lambda_{\text{SC}}(u, \bar{u}) = (u\bar{u})^{-\frac{1}{2}} f(u/\bar{u}), \quad f : S^1 \rightarrow \mathbb{R}. \quad (2.22)$$

However, only a special choice of f proves to be compatible with duality invariance, eq. (2.20), specifically

$$\Lambda_{\text{SC}}(u, \bar{u}) = \frac{y}{\sqrt{u\bar{u}}} + \frac{1}{2}(1-x) \left(\frac{1}{u} + \frac{1}{\bar{u}} \right), \quad (2.23a)$$

with x and y real parameters. Now the self-duality equation (2.20) is satisfied iff

$$x^2 - y^2 = 1 \quad \iff \quad x = \cosh \gamma, \quad y = \sinh \gamma. \quad (2.23b)$$

Due to the identity

$$\int d^4x d^2\theta d^2\bar{\theta} E \frac{W^2 \bar{W}^2}{\bar{u}} = -2 \int d^4x d^2\theta \mathcal{E} W^2, \quad (2.24)$$

the superconformal U(1) duality-invariant action takes the form

$$\begin{aligned} S[W, \bar{W}] &= \frac{1}{4} \cosh \gamma \int d^4x d^2\theta \mathcal{E} W^2 + \text{c.c.} \\ &\quad + \frac{1}{4} \sinh \gamma \int d^4x d^2\theta d^2\bar{\theta} E \frac{W^2 \bar{W}^2}{\sqrt{u\bar{u}}}. \end{aligned} \quad (2.25)$$

This is the model proposed in [56].

2.3 Formulation with auxiliary chiral variables

There exists a different formulation for the models discussed above. Following [45], we consider an action functional of the form

$$\begin{aligned} S[W, \bar{W}, \eta, \bar{\eta}; \Upsilon] &= \int d^4x d^2\theta \mathcal{E} \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \text{c.c.} \\ &\quad + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E \frac{\eta^2 \bar{\eta}^2}{\Upsilon^2} \mathfrak{F} \left(\frac{v}{\Upsilon^2}, \frac{\bar{v}}{\Upsilon^2} \right), \end{aligned} \quad (2.26a)$$

in which

$$v := \frac{1}{8} (\mathcal{D}^2 - 4\bar{R}) \eta^2, \quad (2.26b)$$

and the auxiliary spinor η_α is only constrained to be covariantly chiral, $\bar{\mathcal{D}}_{\dot{\beta}} \eta_\alpha = 0$. The action is super-Weyl invariant if η_α transforms as¹⁰

$$\delta_\sigma \eta_\alpha = \frac{3}{2} \sigma \eta_\alpha, \quad (2.27)$$

in conjunction with the transformation of Υ , eq. (2.11).

⁹The vector multiplet action (2.18) is superconformal. However it is not compatible with duality invariance since the integrand depends on $\mathcal{D}^\alpha W_\alpha$.

¹⁰This super-Weyl transformation law coincides with that of W_α and the chiral spinor prepotential of the tensor multiplet given in section 6.7 of [58].

Making use of the equations of motion for η_α and $\bar{\eta}_{\dot{\alpha}}$ allows one to integrate out these variable to end up with an action of the form (2.9). For the interaction $\Lambda(u, \bar{u})$ one obtains

$$\Lambda(u, \bar{u}) := \frac{\mathfrak{F} + \bar{v}[\partial_v(v\mathfrak{F})]^2 + v[\partial_{\bar{v}}(\bar{v}\mathfrak{F})]^2}{[1 + \partial_v(v\bar{v}\mathfrak{F})]^2 [1 + \partial_{\bar{v}}(v\bar{v}\mathfrak{F})]^2}, \quad (2.28)$$

see [45] for the technical details. Here the variables u and v are related to each other as follows

$$u \approx v[1 + \partial_v(v\bar{v}\mathfrak{F})]^2, \quad (2.29)$$

where the symbol \approx is used to indicate that the result holds modulo terms proportional to η_α and $\bar{\eta}_{\dot{\alpha}}$ or, equivalently, to W_α and $\bar{W}_{\dot{\alpha}}$ (such terms do not contribute to the action).

Our model (2.26) possesses U(1) duality invariance under the condition

$$\mathfrak{F}(v, \bar{v}) = \mathfrak{F}(v\bar{v}), \quad (2.30)$$

see [45] for the technical details.

2.4 Superconformal duality-invariant model

The duality-invariant model defined by eqs. (2.26) and (2.30) is superconformal if the action is independent of Υ , which means that the functional form of \mathfrak{F} is fixed modulo a single real parameter,

$$\mathfrak{F}_{\text{SC}}(v\bar{v}) = \frac{\kappa}{\sqrt{v\bar{v}}}. \quad (2.31)$$

The auxiliary variables η_α and $\bar{\eta}_{\dot{\alpha}}$ can be integrated out using the equation of motion for η_α ,

$$W_\alpha = \eta_\alpha \left\{ 1 + \frac{1}{8}(\bar{\mathcal{D}}^2 - 4R) \left[\bar{\eta}^2 \left(\mathfrak{F}_{\text{SC}} + \frac{1}{8}(\mathcal{D}^2 - 4\bar{R})(\eta^2 \partial_v \mathfrak{F}_{\text{SC}}) \right) \right] \right\}, \quad (2.32)$$

and its conjugate. Rather lengthy calculations lead to

$$S[W, \bar{W}] = \frac{1}{4} \int d^4x d^2\theta \mathcal{E} W^2 + \text{c.c.} + \frac{1}{4} \int d^4x d^2\theta d^2\bar{\theta} E W^2 \bar{W}^2 \Lambda_{\text{SC}}(u, \bar{u}), \quad (2.33)$$

where $\Lambda_{\text{SC}}(u, \bar{u})$ has the form (2.23), with the parameters x and y being given by

$$x = \frac{1 + (\kappa/2)^2}{1 - (\kappa/2)^2}, \quad y = \frac{\kappa}{1 - (\kappa/2)^2}. \quad (2.34)$$

It is interesting to compare the results obtained in subsections 2.2 and 2.4. Within the approach without auxiliary chiral variables, the most general superconformal coupling is given by eq. (2.22) and involves an arbitrary real function f on S^1 . Requiring the superconformal coupling to obey the self-duality equation (2.20) fixes the functional form of $\Lambda_{\text{SC}}(u, \bar{u})$ to be given by eq. (2.23). The remaining freedom in the choice of $\Lambda_{\text{SC}}(u, \bar{u})$ is the single real parameter κ . On the other hand, when the formulation with auxiliary chiral variables is used, duality invariance is guaranteed by the condition (2.30). Upon imposing this condition, there exists a unique functional form for superconformal coupling, eq. (2.31), and the only remaining freedom is again the coupling constant κ .

2.5 Superconformal invariance

To conclude this section, we comment on rigid symmetries of the superconformal models (2.18), including the duality-invariant theory (2.25), in a background curved superspace. We remind the reader that, within the superconformal approach to supergravity-matter systems [66], every theory of Einstein supergravity interacting with supersymmetric matter is realised as a coupling of the same matter multiplets to conformal supergravity and a superconformal compensator, see, e.g., [67]. Truly superconformal theories are independent of any compensator.

The gauge group of conformal supergravity is spanned by general coordinate (ξ^B), local Lorentz ($K^{\beta\gamma}$ and $\bar{K}^{\dot{\beta}\dot{\gamma}}$) and super-Weyl (σ and $\bar{\sigma}$) transformations. In the infinitesimal case, they act on the covariant derivatives by the rule

$$\delta\mathcal{D}_A = \delta_{\mathcal{K}}\mathcal{D}_A + \delta_{\sigma}\mathcal{D}_A, \quad (2.35)$$

where the general coordinate and local Lorentz transformation is given by

$$\delta_{\mathcal{K}}\mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A], \quad \mathcal{K} = \xi^B\mathcal{D}_B + K^{\beta\gamma}M_{\beta\gamma} + \bar{K}^{\dot{\beta}\dot{\gamma}}\bar{M}_{\dot{\beta}\dot{\gamma}} = \bar{\mathcal{K}} \quad (2.36)$$

and the super-Weyl variation $\delta_{\sigma}\mathcal{D}_A$ is defined in (B.4). The local transformation (2.35) acts on a primary tensor superfield \mathfrak{T} (with its indices suppressed) as follows

$$\delta\mathfrak{T} = \delta_{\mathcal{K}}\mathfrak{T} + \delta_{\sigma}\mathfrak{T} = \mathcal{K}\mathfrak{T} + (p\sigma + q\bar{\sigma})\mathfrak{T}. \quad (2.37)$$

Here p and q are constant parameters which are known as the super-Weyl weights of \mathfrak{T} , with $(p + q)$ being the dimension of \mathfrak{T} . The action of any supergravity-matter system is invariant under the transformations (2.35) and (2.37), where \mathfrak{T} is the collective notation for the matter multiplets and, perhaps, the compensator (the latter is present if the theory is not superconformal).

Let us consider a background curved superspace $(\mathcal{M}^{4|4}, \mathcal{D})$. In accordance with section 6.4 of [58], a supervector field $\xi = \xi^B E_B$ on $(\mathcal{M}^{4|4}, \mathcal{D})$ is called conformal Killing if there exists a symmetric spinor $K^{\gamma\delta}$ and a covariantly chiral scalar σ such that

$$(\delta_{\mathcal{K}} + \delta_{\sigma})\mathcal{D}_A = 0. \quad (2.38)$$

In other words, the coordinate transformation generated by ξ can be accompanied by certain Lorentz and super-Weyl transformations such that the superspace geometry does not change. It turns out that the parameters ξ^α , $K^{\alpha\beta}$ and σ are completely determined in terms of ξ^a and its covariant derivatives,

$$\xi^\alpha = -\frac{i}{8}\bar{\mathcal{D}}_{\dot{\alpha}}\xi^{\alpha\dot{\alpha}}, \quad K^{\alpha\beta}[\xi] = \mathcal{D}^{(\alpha}\xi^{\beta)} + \frac{i}{2}\xi^{(\alpha}\dot{\mathcal{D}}^{\beta)}G^{\dot{\beta}\dot{\alpha}}, \quad (2.39a)$$

$$\sigma[\xi] = -\frac{1}{3}(\mathcal{D}^\alpha\xi_\alpha + 2\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} - i\xi^a G_a), \quad (2.39b)$$

The real vector ξ^a proves to obey the superconformal Killing equation

$$\mathcal{D}_{(\alpha}\xi_{\beta)\dot{\beta}} = 0 \iff \bar{\mathcal{D}}_{(\dot{\alpha}}\xi_{\beta\dot{\beta})} = 0. \quad (2.40)$$

We denote by $\delta_{\mathcal{K}[\xi]}$ and $\delta_{\sigma[\xi]}$ the transformations associated with the conformal Killing supervector field $\xi^A = (\xi^a, -\frac{i}{8}\bar{\mathcal{D}}_{\dot{\beta}}\xi^{\alpha\dot{\beta}}, -\frac{i}{8}\mathcal{D}^{\dot{\beta}}\xi_{\beta\dot{\alpha}})$. The set of all conformal Killing supervector fields forms the superconformal algebra of $(\mathcal{M}^{4|4}, \mathcal{D})$. If the superspace is conformally flat (for instance, Minkowski $\mathbb{M}^{4|4}$ or anti-de Sitter $\text{AdS}^{4|4}$ superspace), its superconformal algebra is isomorphic to $\mathfrak{su}(2, 2|1)$.

Given a superconformal field theory defined on the background superspace $(\mathcal{M}^{4|4}, \mathcal{D})$, its action functional is invariant under superconformal transformations

$$\delta_{\xi}\mathfrak{T} = (\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\mathfrak{T} . \tag{2.41}$$

However, if the theory under consideration is not superconformal, then the set \mathfrak{T} also includes a conformal compensator Ξ which is non-dynamical and, instead, it is a part of the background supergravity multiplet $(\mathcal{M}^{4|4}, \mathcal{D}, \Xi)$. In this case the equations (2.39) and (2.40) must be accompanied by the additional condition

$$(\delta_{\mathcal{K}[\xi]} + \delta_{\sigma[\xi]})\Xi = 0 , \tag{2.42}$$

which singles out the Killing supervector fields of $(\mathcal{M}^{4|4}, \mathcal{D}, \Xi)$.

The above discussion can be naturally modified to be applicable to the $\mathcal{N} = 2$ superconformal theories studied in the next section. Similar symmetry considerations also hold for supergravity-matter theories in $d \leq 6$ with up to eight supercharges, where off-shell conformal supergravity always exists, see [68] for the technical details.

3 $\mathcal{N} = 2$ superconformal duality-invariant models

Let $S[W, \bar{W}]$ be the action for an Abelian $\mathcal{N} = 2$ vector multiplet coupled to supergravity. As a curved-superspace extension of the Grimm-Sohnius-Wess formulation [69], the vector multiplet is described by a covariantly chiral scalar superfield W ,

$$\bar{\mathcal{D}}_{\dot{\alpha}i}W = 0 , \tag{3.1}$$

subject to the Bianchi identity

$$(\mathcal{D}^{ij} + 4S^{ij})W = (\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})\bar{W} , \tag{3.2}$$

where $\mathcal{D}^{ij} := \mathcal{D}^{\alpha(i}\mathcal{D}_{\alpha}^{j)}$ and $\bar{\mathcal{D}}^{ij} := \bar{\mathcal{D}}_{\dot{\alpha}(i}\bar{\mathcal{D}}^{j)\dot{\alpha}}$, while S^{ij} and its conjugate \bar{S}^{ij} are special components of the superspace torsion. Constraint (3.2) defines a reduced $\mathcal{N} = 2$ chiral scalar superfield. The superspace formulation for $\mathcal{N} = 2$ conformal supergravity developed in [70] is used in this section, see appendix C for the technical details.

The reduced chiral scalar W is a gauge-invariant field strength constructed in terms of Mezincescu's prepotential [71, 72], $V_{ij} = V_{ji}$, which is an unconstrained real $\text{SU}(2)$ triplet, $\bar{V}_{ij} = V^{ij} = \varepsilon^{ik}\varepsilon^{jl}V_{kl}$. The expression for W in terms of V_{ij} was found in [73] to be

$$W = \bar{\Delta}(\mathcal{D}^{ij} + 4S^{ij})V_{ij} . \tag{3.3}$$

Here $\bar{\Delta}$ is the covariantly chiral projecting operator (C.5). The prepotential V_{ij} is defined modulo gauge transformations [74]

$$\delta_{\Lambda}V^{ij} = \mathcal{D}_k^{\alpha}\Lambda_{\alpha}{}^{kij} + \bar{\mathcal{D}}_{\dot{\alpha}k}\bar{\Lambda}^{\dot{\alpha}kij} , \quad \Lambda_{\alpha}{}^{kij} = \Lambda_{\alpha}{}^{(kij)} , \quad \bar{\Lambda}^{\dot{\alpha}kij} := \overline{\Lambda_{\alpha}{}^{kij}} , \tag{3.4}$$

with the gauge parameter Λ_α^{kij} being arbitrary modulo the algebraic condition given.¹¹ It is an instructive exercise to show that $\delta_\Lambda W = 0$.

The super-Weyl transformation law of V_{ij} [37] is

$$\delta_\sigma V_{ij} = -(\sigma + \bar{\sigma})V_{ij} . \tag{3.5}$$

Making use of the relations collected in appendix C, we then deduce the super-Weyl transformation law of the field strength W [70]

$$\delta_\sigma W = \sigma W . \tag{3.6}$$

3.1 Formulation without auxiliary superfields

To describe the equations of motion, we introduce a covariantly chiral scalar superfield M defined as¹²

$$iM := 4 \frac{\delta}{\delta W} S[W, \bar{W}], \quad \bar{\mathcal{D}}_{\dot{\alpha}i} M = 0 . \tag{3.7}$$

In terms of M and its conjugate \bar{M} , the equation of motion for V_{ij} is

$$\left(\mathcal{D}^{ij} + 4S^{ij}\right)M = \left(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\right)\bar{M} . \tag{3.8}$$

Making use of (3.6) and the super-Weyl transformation of the chiral density [70], eq. (C.8), we obtain the super-Weyl transformation of M :

$$\delta_\sigma M = \sigma M . \tag{3.9}$$

We see that W and M have the same super-Weyl transformation.

Since the Bianchi identity (3.2) and the equation of motion (3.8) have the same functional form, one can consider infinitesimal $U(1)$ duality rotations

$$\delta W = \lambda M , \quad \delta M = -\lambda W , \tag{3.10}$$

with $\lambda \in \mathbb{R}$ a constant parameter. The theory under consideration is duality invariant under the condition [37]

$$\text{Im} \int d^4x d^4\theta \mathcal{E} \left(W^2 + M^2 \right) = 0 . \tag{3.11}$$

In the rigid superspace limit, this reduces to the $\mathcal{N} = 2$ self-duality equation derived in [8].

3.2 Formulation with auxiliary chiral variables

The duality-invariant models discussed in the previous subsection possess an important reformulation [45]. We consider a locally supersymmetric theory with action

$$S[W, \bar{W}, \eta, \bar{\eta}] = \frac{1}{2} \int d^4x d^4\theta \mathcal{E} \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \text{c.c.} + \mathfrak{S}_{\text{int}}[\eta, \bar{\eta}] , \tag{3.12}$$

¹¹In the flat-superspace limit, the gauge transformation law (3.4) reduces to that given in [71].

¹²It is assumed here that $S[W, \bar{W}]$ is consistently defined as a functional of a general $\mathcal{N} = 2$ chiral scalar W and its conjugate.

in which the scalar superfield η is only constrained to be chiral, $\bar{\mathcal{D}}_{\dot{\beta}}\eta = 0$. We require $S[W, \bar{W}, \eta, \bar{\eta}]$ to be super-Weyl invariant, and therefore the corresponding transformation of η is¹³

$$\delta_{\sigma}\eta = \sigma\eta . \tag{3.13}$$

For $\mathfrak{S}_{\text{int}}[\eta, \bar{\eta}]$ to be super-Weyl invariant,

$$\delta_{\sigma}\mathfrak{S}_{\text{int}}[\eta, \bar{\eta}] = 0 , \tag{3.14}$$

it may depend, in general, on the supergravity compensators. However, in this paper we are mostly interested in superconformal dynamical systems which possess no dependence on the compensators.

We assume that η and its conjugate $\bar{\eta}$ are auxiliary superfields in the sense that the equation of motion for η ,

$$W = \eta - 2\frac{\delta}{\delta\eta}\mathfrak{S}_{\text{int}}[\eta, \bar{\eta}] , \tag{3.15}$$

and its conjugate may be solved, at least in perturbation theory, to express η as a functional of the field strength W and its conjugate, $\eta = \eta[W, \bar{W}]$. As a result, we end up with the action

$$S[W, \bar{W}] = S[W, \bar{W}, \eta, \bar{\eta}] \Big|_{\eta=\eta[W, \bar{W}]} , \tag{3.16}$$

which describes the dynamics of the vector multiplet.

The above action, eq. (3.12), defines a U(1) duality-invariant system provided $\mathfrak{S}_{\text{int}}[\eta, \bar{\eta}]$ is invariant under U(1) rigid transformations,

$$\mathfrak{S}_{\text{int}}[e^{i\varphi}\eta, e^{-i\varphi}\bar{\eta}] = \mathfrak{S}_{\text{int}}[\eta, \bar{\eta}] , \quad \varphi \in \mathbb{R} . \tag{3.17}$$

3.3 Superconformal duality-invariant models

We introduce the super-Weyl invariant functional

$$\begin{aligned} 2\mathfrak{S}_{\text{int}}[\eta, \bar{\eta}] &= \int d^4x d^4\theta \mathcal{E} \left\{ (c - a)W^{\alpha\beta}W_{\alpha\beta} + a\Xi \right\} \ln \eta + \text{c.c.} \\ &+ 2a \int d^4x d^4\theta d^4\bar{\theta} E \ln \eta \ln \bar{\eta} - \Gamma_{\text{eff}} , \end{aligned} \tag{3.18}$$

where $W_{\alpha\beta}$ is the super-Weyl tensor, the chiral scalar Ξ is given by eq. (C.9), and a and c are real anomaly coefficients. In accordance with [53], Γ_{eff} is a nonlocal functional of the background supergravity multiplet which generates the super-Weyl anomaly, with its super-Weyl variation being

$$\delta_{\sigma}\Gamma_{\text{eff}} = \int d^4x d^4\theta \mathcal{E} \sigma \left\{ (c - a)W^{\alpha\beta}W_{\alpha\beta} + a\Xi \right\} + \text{c.c.} \tag{3.19}$$

By construction, Γ_{eff} is independent of η and $\bar{\eta}$.

¹³This super-Weyl transformation law coincides with that of W and the chiral scalar prepotential of the $\mathcal{N} = 2$ tensor multiplet [75].

The functional (3.18) satisfies the condition (3.17) due to the following two properties. Firstly, for any covariantly chiral scalar ϕ , it holds that

$$\bar{D}_i^{\dot{\alpha}} \phi = 0 \implies \int d^4x d^4\theta d^4\bar{\theta} E \phi = 0. \quad (3.20)$$

Secondly, the functionals

$$\mathfrak{T}_1 = \int d^4x d^4\theta \mathcal{E} \{W^{\alpha\beta} W_{\alpha\beta} - \Xi\}, \quad (3.21a)$$

$$\mathfrak{T}_2 = \text{Im} \int d^4x d^4\theta \mathcal{E} W^{\alpha\beta} W_{\alpha\beta} \quad (3.21b)$$

are topological invariants. In the case of $\mathcal{N} = 4$ SYM theory, the anomaly coefficients a and c coincide, $a = c$. In what follows, we will assume this relation and make use of c .

Another family of U(1)-invariant couplings is given by

$$\tilde{\mathfrak{S}}_{\text{int}}[\eta, \bar{\eta}] = \int d^4x d^4\theta d^4\bar{\theta} E \mathfrak{F}(\Omega\bar{\Omega}), \quad (3.22)$$

where we have introduced the super-Weyl inert chiral superfield

$$\Omega := \frac{1}{\eta^2} \left(\bar{\Delta} \ln \bar{\eta} + \frac{1}{2} \Xi \right), \quad \bar{D}_i^{\dot{\alpha}} \Omega = 0, \quad \delta_{\sigma} \Omega = 0. \quad (3.23)$$

This is a curved superspace generalisation of the conformal primary weight-zero chiral superfield introduced in [76]. The superfield $\eta^2 \Omega$ naturally originates within the approach developed in [52].

With the contribution $(-\Gamma_{\text{eff}})$ included, the action defined by eqs. (3.12) and (3.18) is super-Weyl invariant but nonlocal. Instead of working with the nonlocal functional, we can consider the following U(1) duality-invariant action

$$S = \text{Re} \int d^4x d^4\theta \mathcal{E} \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 + \left((c - a) W^{\alpha\beta} W_{\alpha\beta} + a \Xi \right) \ln \eta \right\} \\ + a \int d^4x d^4\theta d^4\bar{\theta} E \ln \eta \ln \bar{\eta}. \quad (3.24)$$

This action generates the $\mathcal{N} = 2$ super-Weyl anomaly, and therefore it can be viewed as the Goldstone multiplet action for spontaneously broken $\mathcal{N} = 2$ superconformal symmetry.¹⁴ It was proposed in [53] to describe the $\mathcal{N} = 2$ dilaton effective action in terms of the reduced chiral scalar W . An alternative realisation was put forward in ref. [79] where the $\mathcal{N} = 2$ Goldstone multiplet was identified with an unrestricted chiral scalar, such as η . Both superfields are present in (3.24).

3.4 Superconformal duality-invariant models in flat space

In the flat-superspace limit, the supergravity covariant derivatives \mathcal{D}_A , eq. (C.1), turn into the flat ones $D_A = (\partial_a, D_{\alpha}^i, \bar{D}_{\dot{\alpha}i}^{\dot{\alpha}})$, with $i = \underline{1}, \underline{2}$, which satisfy the graded commutation relations

$$\{D_{\alpha}^i, D_{\beta}^j\} = \{\bar{D}_{\dot{\alpha}i}^{\dot{\alpha}}, \bar{D}_{\dot{\beta}j}^{\dot{\beta}}\} = 0, \quad \{D_{\alpha}^i, \bar{D}_{\dot{\alpha}j}^{\dot{\alpha}}\} = -2i\delta_j^i (\sigma^a)_{\alpha\dot{\alpha}} \partial_a. \quad (3.25)$$

¹⁴The $\mathcal{N} = 1$ dilaton effective action was proposed in [77], see also [78].

In Minkowski superspace, the duality-invariant model (3.24) for the $a = c$ choice, which corresponds to the $\mathcal{N} = 4$ SYM, takes the form

$$S[W, \bar{W}, \eta, \bar{\eta}] = \text{Re} \int d^4x d^4\theta \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + c \int d^4x d^4\theta d^4\bar{\theta} \ln \eta \ln \bar{\eta}. \quad (3.26)$$

This action is invariant under standard $\mathcal{N} = 2$ superconformal transformations, see [76] for the technical details. The equation of motion for η is

$$\eta = W + \frac{2c}{\eta} \bar{\Delta} \ln \bar{\eta}. \quad (3.27)$$

Assuming the validity of (3.27), the action (3.26) turns into

$$S[W, \bar{W}] = \frac{1}{4} \text{Re} \int d^4x d^4\theta W^2 + c \int d^4x d^4\theta d^4\bar{\theta} \ln \eta \ln \bar{\eta} - c^2 \int d^4x d^4\theta d^4\bar{\theta} \left\{ \frac{1}{\eta^2} \ln \bar{\eta} \bar{\Delta} \ln \bar{\eta} + \frac{1}{\bar{\eta}^2} \ln \eta \Delta \ln \eta \right\}, \quad (3.28)$$

where η is assumed to be a functional of W and its conjugate, $\eta = \eta[W, \bar{W}]$, obtained by solving the equation (3.27). The latter can be solved by iterations as a series in powers of c ,

$$\eta = W + \frac{2c}{W} \bar{\Delta} \ln \bar{W} + \mathcal{O}(c^2). \quad (3.29)$$

Making use of this solution in (3.28) gives

$$S[W, \bar{W}] = \frac{1}{4} \text{Re} \int d^4x d^4\theta W^2 + c \int d^4x d^4\theta d^4\bar{\theta} \ln W \ln \bar{W} + c^2 \int d^4x d^4\theta d^4\bar{\theta} \left\{ \frac{1}{W^2} \ln \bar{W} \bar{\Delta} \ln \bar{W} + \frac{1}{\bar{W}^2} \ln W \Delta \ln W \right\} + \mathcal{O}(c^3). \quad (3.30)$$

This agrees with the perturbative solution derived in [8].

It is interesting to point out that the equation (3.27) is analogous to the $\mathcal{N} = 1$ chiral constraint (in the notation of [17])

$$X + \frac{1}{4} X \bar{D}^2 \bar{X} = W^\alpha W_\alpha, \quad (3.31)$$

which is at the heart of the Bagger-Galperin construction [63] of the vector Goldstone multiplet action for partial $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ supersymmetry breaking in Minkowski space.

The kinetic terms in (3.30) is purely classical. As is known, it receives no perturbative [80–82] and non-perturbative [83, 84] quantum corrections. The c -term in (3.30), originally introduced in [80], contains four-derivative quantum corrections at the component level; these include an F^4 term, with F the U(1) field strength. This term is believed to be generated only at one loop in the $\mathcal{N} = 4$ SYM theory [85]. In particular it is known that non-perturbative F^4 quantum corrections do not occur in this theory [83, 84]. The one-loop calculation of c was carried out by several groups [7, 10–14], and the absence of two-loop corrections was shown in [82].

The c^2 -term in (3.30), which was originally introduced in [7] and later proved to be superconformal [76], contains six-derivative quantum corrections at the component level, including an F^6 term. Such a quantum correction is absent at one loop in the $\mathcal{N} = 4$ SYM theory [76]. It is generated at the two-loop order and the numerical coefficient computed in [15] agrees with that predicted by duality invariance.¹⁵

Our superconformal duality-invariant model (3.26) admits a natural generalisation described by the following action

$$S[W, \bar{W}, \eta, \bar{\eta}] = \text{Re} \int d^4x d^4\theta \left\{ \eta W - \frac{1}{2} \eta^2 - \frac{1}{4} W^2 \right\} + \int d^4x d^4\theta d^4\bar{\theta} \left\{ c \ln \eta \ln \bar{\eta} + \mathfrak{F}(\Omega \bar{\Omega}) \right\}, \quad (3.32)$$

where $\Omega := \eta^{-2} \bar{\Delta} \ln \bar{\eta}$ is the flat-superspace counterpart of (3.23). Here the functional in the second line is invariant under rigid $U(1)$ transformations

$$\eta \rightarrow e^{i\varphi} \eta, \quad \bar{\eta} \rightarrow e^{-i\varphi} \bar{\eta}, \quad \varphi \in \mathbb{R}. \quad (3.33)$$

The action (3.32) includes higher-derivative structures as compared with (3.26).

Modulo higher derivative terms, we identify (3.26) with (the planar part of) the low-energy effective action of $\mathcal{N} = 4$ SYM on the Coulomb branch. There are several reasons for this identification. Firstly, upon elimination of the auxiliary variables η and $\bar{\eta}$, the resulting action (3.30) agrees with known structure of the $\mathcal{N} = 4$ SYM effective action to the sixth order in derivatives. Secondly, the theory (3.26) possesses $U(1)$ duality invariance. Thirdly, the action (3.26) involves a single coupling constant, the anomaly coefficient c , which is similar to the D3-brane action in $AdS_5 \times S^5$, eq. (4.2). The latter is expected to be related to the low-energy effective action of $\mathcal{N} = 4$ SYM (see the discussion in the next section), however only upon a nontrivial field redefinition at the component level [7, 15]. It remains possible that one has to include a c -dependent higher-derivative term $\mathfrak{F}(\Omega \bar{\Omega})$ in (3.32) in order to get a complete agreement between the model. This is still an open problem.

4 Discussion

Similar to its bosonic sector (A.7), the $\mathcal{N} = 1$ superconformal duality-invariant model is uniquely defined by eq. (2.25) or, equivalently, eqs. (2.26) and (2.31). The only free parameter of the theory is the coupling constant κ in (2.31). In the $\mathcal{N} = 2$ case, however, the superconformal duality-invariant model is given by the action (3.32), which involves an arbitrary function of a real variable, $\mathfrak{F}(x)$. The reason why the functional freedom is larger in the $\mathcal{N} = 2$ case can be explained by looking at the $\mathcal{N} = 1$ components of the $\mathcal{N} = 2$ vector multiplet field strength W . Given an $\mathcal{N} = 2$ superfield $U(x, \theta_i, \bar{\theta}^i)$, its $\mathcal{N} = 1$ sub-multiplets can be introduced with the aid of the $\mathcal{N} = 1$ bar-projection defined by $U| = U(\theta_i, \bar{\theta}^i)|_{\theta_2 = \bar{\theta}^2 = 0}$. The $\mathcal{N} = 2$ chiral field strength W contains two independent chiral $\mathcal{N} = 1$ components

$$\sqrt{2}\Phi := W|, \quad 2iW_\alpha := D_\alpha^2 W| \quad \implies \quad (D^2)^2 W| = \sqrt{2} \bar{D}^2 \bar{\Phi}. \quad (4.1)$$

¹⁵A different value for the F^6 -term coefficient was derived in [16]. Unfortunately, an error was made in [16] in the process of evaluating the harmonic supergraphs, as explained in [15].

Modulo certain technical complications related to the need to perform a nonlinear superfield redefinition [7, 15] (when switching from the $\mathcal{N} = 2$ to $\mathcal{N} = 1$ superfield descriptions), the $\mathcal{N} = 1$ counterpart of the system (3.32) is given by the relations (2.26) and (2.30), with $\Upsilon \propto \Phi\bar{\Phi}$, modulo terms involving derivatives of Φ and $\bar{\Phi}$. The model for $\mathcal{N} = 1$ superconformal duality-invariant electrodynamics (2.25) does not appear to have a $\mathcal{N} = 2$ counterpart.

The AdS/CFT correspondence provides the main evidence to believe in self-duality of the low-energy effective action for the $\mathcal{N} = 4$ $SU(N)$ SYM theory on its Coulomb branch where the gauge group $SU(N)$ is spontaneously broken to $SU(N - 1) \times U(1)$. The AdS/CFT correspondence predicts [6, 16, 86, 87] (a more comprehensive list of references is given in [16]) that the $\mathcal{N} = 4$ SYM effective action (in the large- N limit) is related to the D3-brane action in $AdS_5 \times S^5$

$$S = T_3 \int d^4x \left(h^{-1} - \sqrt{-\det(g_{mn} + F_{mn})} \right),$$

$$g_{mn} = h^{-1/2} \eta_{mn} + h^{1/2} \partial_m X^I \partial_n X^I, \quad h = \frac{Q}{(X^I X^I)^2}, \quad (4.2)$$

where X^I , $I = 1, \dots, 6$, are transverse coordinates, $T_3 = (2\pi g_s)^{-1}$ and $Q = g_s(N - 1)/\pi$. The action S/T_3 is self-dual in the sense that it enjoys invariance under electromagnetic $U(1)$ duality rotations [19–22]. Self-duality of the D3-brane action is a fundamental property related to the S-duality of type IIB string theory [88, 89]. If the $\mathcal{N} = 4$ SYM effective action and the D3-brane action in $AdS_5 \times S^5$ are indeed related, the former should possess some form of self-duality. The $U(1)$ duality-invariant action (4.2) involves a single coupling constant, Q , the same as the $\mathcal{N} = 2$ superconformal $U(1)$ duality-invariant action (3.26). The D3-brane parameter Q is proportional to the anomaly coefficient c .

Considered as a field theory in Minkowski space \mathbb{M}^4 , the D3-brane action (4.2) is manifestly scale-invariant, but it is not invariant under the standard linear special conformal transformations. It is instead invariant under deformed conformal transformations being nonlinear in the fields [6]. At the component level, the complete effective action of the $\mathcal{N} = 4$ SYM also proves to be invariant under quantum-corrected superconformal transformations [90–92] which differ from the ordinary linear superconformal transformations that leave the classical action invariant.

In conclusion, we describe simple generalisations of the models for (super)conformal duality-invariant electrodynamics discussed in this paper. Given a $U(1)$ duality-invariant model for nonlinear electrodynamics, $L(F_{ab}) = L(\omega, \bar{\omega})$, described in appendix A, its compact duality group $U(1)$ can be enhanced to the non-compact $SL(2, \mathbb{R})$ group by coupling the electromagnetic field to the dilaton φ and axion \mathbf{a} [20–22]. Specifically, this is achieved by replacing $L(\omega, \bar{\omega})$ with

$$L(\omega, \bar{\omega}, \mathcal{S}, \bar{\mathcal{S}}) = L(\mathcal{S}_2 \omega, \mathcal{S}_2 \bar{\omega}) + \frac{i}{2} \mathcal{S}_1 (\bar{\omega} - \omega), \quad \mathcal{S} = \mathcal{S}_1 + i \mathcal{S}_2 = \mathbf{a} + i e^{-\varphi}. \quad (4.3)$$

The duality group acts by the rule

$$\begin{pmatrix} G' \\ F' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}, \quad \mathcal{S}' = \frac{a\mathcal{S} + b}{c\mathcal{S} + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad (4.4)$$

where $G_{ab}(F)$ is defined by

$$\tilde{G}_{ab}(F) \equiv \frac{1}{2} \varepsilon_{abcd} G^{cd}(F) = 2 \frac{\partial L(F)}{\partial F^{ab}}, \quad (4.5)$$

see [17] for more details. In the case of the conformal model described by eq. (A.7), this leads to

$$L_{\text{conf}}(\omega, \bar{\omega}, \mathcal{S}, \bar{\mathcal{S}}) = -\frac{1}{2} \cosh \gamma (\omega + \bar{\omega}) \mathcal{S}_2 + \sinh \gamma \sqrt{\omega \bar{\omega}} \mathcal{S}_2 + \frac{i}{2} (\bar{\omega} \bar{\mathcal{S}} - \omega \mathcal{S}). \quad (4.6)$$

In curved space, the model defined by the action $S = \int d^4x e L_{\text{conf}}(\omega, \bar{\omega}, \mathcal{S}, \bar{\mathcal{S}})$ is Weyl-invariant provided \mathcal{S} is inert under the Weyl transformations. We can add to (4.6) a higher-derivative Lagrangian for the dilaton and axion (see, e.g., [93])

$$L = \frac{\kappa}{2(\text{Im } \mathcal{S})^2} \left[\mathfrak{D}^2 \mathcal{S} \mathfrak{D}^2 \bar{\mathcal{S}} - 2(R^{ab} - \frac{1}{3} \eta^{ab} R) \nabla_a \mathcal{S} \nabla_b \bar{\mathcal{S}} \right] + \frac{1}{12(\text{Im } \mathcal{S})^4} \left[\alpha \nabla^a \mathcal{S} \nabla_a \mathcal{S} \nabla^b \bar{\mathcal{S}} \nabla_b \bar{\mathcal{S}} + \beta \nabla^a \mathcal{S} \nabla_a \bar{\mathcal{S}} \nabla^b \mathcal{S} \nabla_b \bar{\mathcal{S}} \right] \quad (4.7)$$

where

$$\mathfrak{D}^2 \mathcal{S} := \nabla^a \nabla_a \mathcal{S} + \frac{i}{\text{Im } \mathcal{S}} \nabla^a \mathcal{S} \nabla_a \mathcal{S}, \quad (4.8)$$

∇_a is the torsion-free Lorentz covariant derivative, and κ, α and β are numerical parameters. The Lagrangian (4.7) is manifestly invariant under $\text{SL}(2, \mathbb{R})$ transformations (4.4). The Weyl invariance follows from the fact that the Fradkin-Tseytlin operator [94, 95]

$$\Delta_0 = (\nabla^a \nabla_a)^2 + 2 \nabla^a (R_{ab} \nabla^b - \frac{1}{3} R \nabla_a) \quad (4.9)$$

is conformal.¹⁶

Within the $\mathcal{N} = 1$ Poincaré supersymmetry, $\text{SL}(2, \mathbb{R})$ duality-invariant couplings of the dilaton-axion multiplet to general models for self-dual supersymmetric nonlinear electrodynamics were described in [17], while the case of the supersymmetric Born-Infeld action [62, 63] was first considered in [98]. The results of [17] were generalised to supergravity in [35]. These results can be used to couple the $\mathcal{N} = 1$ superconformal $\text{U}(1)$ duality-invariant model to the dilaton-axion multiplet described by a chiral scalar Φ and its conjugate $\bar{\Phi}$. One obtains

$$S_{\text{SC}}[W, \bar{W}, \Phi, \bar{\Phi}] = \frac{1}{4} \int d^4x d^2\theta \mathcal{E} \Phi W^2 + \text{c.c.} \quad (4.10) - \frac{1}{16} \int d^4x d^2\theta d^2\bar{\theta} E (\Phi - \bar{\Phi})^2 W^2 \bar{W}^2 \Lambda_{\text{SC}} \left(\frac{i}{2} (\Phi - \bar{\Phi}) u, \frac{i}{2} (\Phi - \bar{\Phi}) \bar{u} \right),$$

where $\Lambda_{\text{SC}}(u, \bar{u})$ is given by eq. (2.23). It is assumed that Φ parametrises the lower half-plane, and therefore $(\Phi - \bar{\Phi})^{-1}$ exists.¹⁷ One can add to (4.11) a higher-derivative super-Weyl and $\text{SL}(2, \mathbb{R})$ invariant action for the dilaton-axion multiplet [99]

$$S_{\text{DA}}[\Phi, \bar{\Phi}] = -\frac{\kappa}{16} \int d^4x d^2\theta d^2\bar{\theta} E \frac{1}{(\Phi - \bar{\Phi})^2} \left\{ \nabla^2 \Phi \bar{\nabla}^2 \bar{\Phi} - 8 \mathcal{D}^\alpha \Phi G_{\alpha\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Phi} \right\} + \frac{\alpha}{8} \int d^4x d^2\theta d^2\bar{\theta} E \frac{1}{(\Phi - \bar{\Phi})^4} \mathcal{D}^\alpha \Phi \mathcal{D}_\alpha \Phi \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Phi} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Phi}, \quad (4.11)$$

¹⁶This operator was re-discovered by Paneitz in 1983 [96] and Riegert in 1984 [97].

¹⁷The axion $\mathfrak{a}(x)$ and dilaton $\varphi(x)$ are the component fields of $\Phi(x, \theta)$ defined by $\Phi|_{\theta=0} = \mathfrak{a} - i e^{-\varphi}$.

where κ and α are real parameters, and

$$\nabla^2\Phi = \mathcal{D}^2\Phi - 2\frac{\mathcal{D}^\alpha\Phi\mathcal{D}_\alpha\Phi}{\Phi - \bar{\Phi}}, \quad \bar{\nabla}^2\bar{\Phi} = \bar{\mathcal{D}}^2\bar{\Phi} + 2\frac{\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Phi}\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\Phi}}{\Phi - \bar{\Phi}} \quad (4.12)$$

are $\text{SL}(2, \mathbb{R})$ covariant derivatives. The complete action $S_{\text{SC}}[W, \bar{W}, \Phi, \bar{\Phi}] + S_{\text{DA}}[\Phi, \bar{\Phi}]$ has the following fundamental properties: (i) it is super-Weyl invariant; and (ii) it is $\text{SL}(2, \mathbb{R})$ duality invariant.

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A Conformal duality-invariant electrodynamics

In this appendix we re-derive the conformal $\text{U}(1)$ duality-invariant electrodynamics constructed in [54, 55] following the non-supersymmetric approaches described in [17] and [24, 25].

Given a model for nonlinear electrodynamics, its Lagrangian $L(F_{ab})$ can be expressed in terms of the two independent invariants of the electromagnetic field,

$$\alpha = \frac{1}{4} F^{ab} F_{ab}, \quad \beta = \frac{1}{4} F^{ab} \tilde{F}_{ab}. \quad (A.1)$$

Introducing the complex combination $\omega = \alpha + i\beta$, the Lagrangian can be rewritten in the form

$$L(\omega, \bar{\omega}) = -\frac{1}{2} (\omega + \bar{\omega}) + \omega \bar{\omega} \Lambda(\omega, \bar{\omega}). \quad (A.2)$$

Then, the GZGR condition for $\text{U}(1)$ duality invariance is given by

$$\text{Im} \left\{ \frac{\partial(\omega \Lambda)}{\partial\omega} - \bar{\omega} \left(\frac{\partial(\omega \Lambda)}{\partial\omega} \right)^2 \right\} = 0, \quad (A.3)$$

see [17] for the technical details.

Requiring the action functional to be conformally invariant constrains $\Lambda(\omega, \bar{\omega})$ in (A.2) to have the following functional form

$$\Lambda_{\text{conf}}(\omega, \bar{\omega}) = (\omega \bar{\omega})^{-\frac{1}{2}} f(\omega/\bar{\omega}), \quad f : S^1 \rightarrow \mathbb{R}. \quad (A.4)$$

Only a restricted class of such models prove to be compatible with $\text{U}(1)$ duality invariance, specifically

$$\Lambda_{\text{conf}}(\omega, \bar{\omega}) = \frac{y}{\sqrt{\omega \bar{\omega}}} - x \left(\frac{1}{\omega} + \frac{1}{\bar{\omega}} \right), \quad (A.5)$$

with x and y being real coefficients. These coefficients turn out to be constrained by requiring the self-duality equation (A.3) to hold, specifically

$$\Lambda_{\text{conf}}(\omega, \bar{\omega}) = \frac{\sinh \gamma}{\sqrt{\omega \bar{\omega}}} - \frac{1}{2}(\cosh \gamma - 1) \left(\frac{1}{\omega} + \frac{1}{\bar{\omega}} \right), \quad (\text{A.6})$$

with γ a real parameter. The resulting Lagrangian is

$$L_{\text{conf}}(\omega, \bar{\omega}) = -\frac{1}{2} \cosh \gamma (\omega + \bar{\omega}) + \sinh \gamma \sqrt{\omega \bar{\omega}}, \quad (\text{A.7})$$

which is exactly the model proposed in [54] (see also [55]). It was called “ModMax electrodynamics” in [54].

It is instructive to re-derive the conformal duality-invariant model using the IZ approach [24, 25]. Their reformulation of nonlinear electrodynamics is obtained by replacing $L(F_{ab}) \rightarrow \tilde{L}(F_{ab}, V_{ab})$, where $V_{ab} = -V_{ba}$ is an auxiliary unconstrained bivector, which is equivalent to a pair of symmetric rank-2 spinors, $V_{\alpha\beta} = V_{\beta\alpha}$ and its conjugate $\bar{V}_{\dot{\alpha}\dot{\beta}}$. The new Lagrangian \tilde{L} is at most quadratic with respect to the electromagnetic field strength F_{ab} , while the self-interaction is described by a nonlinear function of the auxiliary variables, $L_{\text{int}}(V_{ab})$,

$$\tilde{L}(F_{ab}, V_{ab}) = \frac{1}{4} F^{ab} F_{ab} + \frac{1}{2} V^{ab} V_{ab} - V^{ab} F_{ab} + L_{\text{int}}(V_{ab}). \quad (\text{A.8})$$

The original theory $L(F_{ab})$ is obtained from $\tilde{L}(F_{ab}, V_{ab})$ by integrating out the auxiliary variables. In terms of $\tilde{L}(F_{ab}, V_{ab})$, the condition of $U(1)$ duality invariance was shown [24, 25] to be equivalent to the requirement that the self-interaction

$$L_{\text{int}}(V_{ab}) = L_{\text{int}}(\nu, \bar{\nu}), \quad \nu := V^{\alpha\beta} V_{\alpha\beta} \quad (\text{A.9})$$

is invariant under linear $U(1)$ transformations $\nu \rightarrow e^{i\varphi} \nu$, with $\varphi \in \mathbb{R}$, therefore

$$L_{\text{int}}(\nu, \bar{\nu}) = f(\nu \bar{\nu}). \quad (\text{A.10})$$

A unique conformal duality-invariant model corresponds to the choice

$$L_{\text{int,conf}} = \kappa \sqrt{\nu \bar{\nu}}, \quad (\text{A.11})$$

with κ a coupling constant. Integrating out the auxiliary variables lead to the model (A.7), in which

$$\sinh \gamma = \frac{\kappa}{1 - (\kappa/2)^2}. \quad (\text{A.12})$$

B Elements of $\mathcal{N} = 1$ supergravity

General $\mathcal{N} = 1$ supergravity-matter systems can be naturally described in superspace using the Grimm-Wess-Zumino geometry [100, 101] in conjunction with the super-Weyl

transformations discovered in [102]. The structure group in this setting is $SL(2, \mathbb{C})$, and the geometry of curved superspace is described by the covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A^M \partial_M + \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}, \quad (\text{B.1})$$

where $M_{\beta\gamma} = M_{\gamma\beta}$ and $\bar{M}_{\dot{\beta}\dot{\gamma}} = \bar{M}_{\dot{\gamma}\dot{\beta}}$ are the Lorentz generators. The covariant derivatives obey the graded commutation relations (see [58] for the derivation)

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = -2i\mathcal{D}_{\alpha\dot{\alpha}}, \quad (\text{B.2a})$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R}M_{\alpha\beta}, \quad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 4R\bar{M}_{\dot{\alpha}\dot{\beta}}, \quad (\text{B.2b})$$

$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}] = i\varepsilon_{\alpha\beta} \left(\bar{R}\bar{\mathcal{D}}_{\dot{\beta}} + G_{\beta\dot{\beta}}^\gamma \mathcal{D}_\gamma - \mathcal{D}^\gamma G_{\beta\dot{\beta}}^\delta M_{\gamma\delta} + 2\bar{W}_{\beta\dot{\beta}}^{\dot{\gamma}\delta} \bar{M}_{\dot{\gamma}\delta} \right) + i\bar{\mathcal{D}}_{\dot{\beta}} \bar{R} M_{\alpha\beta}, \quad (\text{B.2c})$$

$$[\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = -i\varepsilon_{\dot{\alpha}\dot{\beta}} \left(R\mathcal{D}_\beta + G_{\beta\dot{\beta}}^{\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}} - \bar{\mathcal{D}}^{\dot{\gamma}} G_{\beta\dot{\beta}}^{\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}} + 2W_{\beta\dot{\beta}}^{\gamma\delta} M_{\gamma\delta} \right) - i\mathcal{D}_\beta R \bar{M}_{\dot{\alpha}\dot{\beta}}. \quad (\text{B.2d})$$

Here the torsion tensors R , $G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy the Bianchi identities:

$$\bar{\mathcal{D}}_{\dot{\alpha}} R = 0, \quad \bar{\mathcal{D}}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0, \quad (\text{B.3a})$$

$$\bar{\mathcal{D}}^{\dot{\gamma}} G_{\alpha\dot{\gamma}} = \mathcal{D}_\alpha R, \quad \mathcal{D}^\gamma W_{\alpha\beta\gamma} = i\mathcal{D}_{(\alpha} \bar{\mathcal{D}}^{\dot{\gamma}} G_{\beta)\dot{\gamma}}. \quad (\text{B.3b})$$

An infinitesimal super-Weyl transformation of the covariant derivatives [102]

$$\delta_\sigma \mathcal{D}_\alpha = \left(\bar{\sigma} - \frac{1}{2}\sigma \right) \mathcal{D}_\alpha + \mathcal{D}^\beta \sigma M_{\alpha\beta}, \quad (\text{B.4a})$$

$$\delta_\sigma \bar{\mathcal{D}}_{\dot{\alpha}} = \left(\sigma - \frac{1}{2}\bar{\sigma} \right) \bar{\mathcal{D}}_{\dot{\alpha}} + (\bar{\mathcal{D}}^{\dot{\beta}} \bar{\sigma}) \bar{M}_{\dot{\alpha}\dot{\beta}} \quad (\text{B.4b})$$

induces the following variations of the torsion superfields

$$\delta_\sigma R = 2\sigma R + \frac{1}{4}(\bar{\mathcal{D}}^2 - 4R)\bar{\sigma}, \quad (\text{B.5a})$$

$$\delta_\sigma G_{\alpha\dot{\alpha}} = \frac{1}{2}(\sigma + \bar{\sigma})G_{\alpha\dot{\alpha}} + i\mathcal{D}_{\alpha\dot{\alpha}}(\sigma - \bar{\sigma}), \quad (\text{B.5b})$$

$$\delta_\sigma W_{\alpha\beta\gamma} = \frac{3}{2}\sigma W_{\alpha\beta\gamma}. \quad (\text{B.5c})$$

Here the super-Weyl parameter σ is a covariantly chiral scalar superfield, $\bar{\mathcal{D}}_{\dot{\alpha}}\sigma = 0$. The super-Weyl transformations belong to the gauge group of conformal supergravity.

Integrals over the full superspace and its chiral subspace are related as follows:

$$\int d^4x d^2\theta d^2\bar{\theta} E U = -\frac{1}{4} \int d^4x d^2\theta \mathcal{E} (\bar{\mathcal{D}}^2 - 4R)U, \quad E^{-1} = \text{Ber}(E_A^M), \quad (\text{B.6})$$

where \mathcal{E} is the chiral density (see, e.g., [58] for the derivation). The super-Weyl transformation laws of the integration measures are

$$\delta_\sigma E = -(\sigma + \bar{\sigma})E, \quad \delta_\sigma \mathcal{E} = -3\sigma\mathcal{E}. \quad (\text{B.7})$$

C Elements of $\mathcal{N} = 2$ supergravity

In this appendix we give a summary of the so-called SU(2) superspace, the off-shell formulation for $\mathcal{N} = 2$ conformal supergravity developed in [70]. The structure group in this setting is $\text{SL}(2, \mathbb{C}) \times \text{SU}(2)$, and the covariant derivatives are

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_i^{\dot{\alpha}}) = E_A^M \partial_M + \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} + \Phi_A^{kl} J_{kl} . \quad (\text{C.1})$$

Here J_{kl} are the generators of the group SU(2). The spinor covariant derivatives obey the anti-commutation relations

$$\begin{aligned} \{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} &= 4S^{ij} M_{\alpha\beta} + 2\varepsilon^{ij} \varepsilon_{\alpha\beta} Y^{\gamma\delta} M_{\gamma\delta} + 2\varepsilon^{ij} \varepsilon_{\alpha\beta} \bar{W}^{\dot{\gamma}\dot{\delta}} \bar{M}_{\dot{\gamma}\dot{\delta}} \\ &\quad + 2\varepsilon_{\alpha\beta} \varepsilon^{ij} S^{kl} J_{kl} + 4Y_{\alpha\beta} J^{ij} , \end{aligned} \quad (\text{C.2a})$$

$$\begin{aligned} \{\bar{\mathcal{D}}_i^{\dot{\alpha}}, \bar{\mathcal{D}}_j^{\dot{\beta}}\} &= -4\bar{S}_{ij} \bar{M}^{\dot{\alpha}\dot{\beta}} - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{Y}^{\gamma\delta} \bar{M}_{\gamma\delta} - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} W^{\gamma\delta} M_{\gamma\delta} \\ &\quad - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{S}^{kl} J_{kl} - 4\bar{Y}^{\dot{\alpha}\dot{\beta}} J_{ij} , \end{aligned} \quad (\text{C.2b})$$

$$\{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_j^{\dot{\beta}}\} = -2i\delta_j^i (\sigma^c)_\alpha{}^{\dot{\beta}} \mathcal{D}_c + 4\delta_j^i G^{\delta\dot{\beta}} M_{\alpha\delta} + 4\delta_j^i G_{\alpha\dot{\gamma}} \bar{M}^{\dot{\gamma}\dot{\beta}} + 8G_\alpha{}^{\dot{\beta}} J^i{}_j . \quad (\text{C.2c})$$

Here the real four-vector $G_{\alpha\dot{\alpha}}$, the complex symmetric tensors $S^{ij} = S^{ji}$, $W_{\alpha\beta} = W_{\beta\alpha}$, $Y_{\alpha\beta} = Y_{\beta\alpha}$ and their complex conjugates $\bar{S}_{ij} := \overline{S^{ij}}$, $\bar{W}_{\dot{\alpha}\dot{\beta}} := \overline{W_{\alpha\beta}}$, $\bar{Y}_{\dot{\alpha}\dot{\beta}} := \overline{Y_{\alpha\beta}}$ obey additional differential constraints implied by the Bianchi identities and given in [70].

In SU(2) superspace, the gauge group of conformal supergravity includes super-Weyl transformations. An infinitesimal super-Weyl transformation of \mathcal{D}_A [70] is

$$\delta_\sigma \mathcal{D}_\alpha^i = \frac{1}{2} \bar{\sigma} \mathcal{D}_\alpha^i + (\mathcal{D}^{\gamma i} \sigma) M_{\gamma\alpha} - (\mathcal{D}_{\alpha k} \sigma) J^{ki} , \quad (\text{C.3a})$$

$$\delta_\sigma \bar{\mathcal{D}}_{\dot{\alpha}i} = \frac{1}{2} \sigma \bar{\mathcal{D}}_{\dot{\alpha}i} + (\bar{\mathcal{D}}_i^{\dot{\gamma}} \bar{\sigma}) \bar{M}_{\dot{\gamma}\dot{\alpha}} + (\bar{\mathcal{D}}_{\dot{\alpha}}^k \bar{\sigma}) J_{ki} , \quad (\text{C.3b})$$

where the parameter σ is an arbitrary covariantly chiral superfield, $\bar{\mathcal{D}}_i^{\dot{\alpha}} \sigma = 0$. The dimension-1 torsion superfields transform as follows:

$$\delta_\sigma S^{ij} = \bar{\sigma} S^{ij} - \frac{1}{4} \mathcal{D}^{\gamma(i} \mathcal{D}^{j)} \sigma , \quad (\text{C.4a})$$

$$\delta_\sigma Y_{\alpha\beta} = \bar{\sigma} Y_{\alpha\beta} - \frac{1}{4} \mathcal{D}_{(\alpha}^k \mathcal{D}_{\beta)k} \sigma , \quad (\text{C.4b})$$

$$\delta_\sigma G_{\alpha\dot{\beta}} = \frac{1}{2} (\sigma + \bar{\sigma}) G_{\alpha\dot{\beta}} - \frac{i}{4} \mathcal{D}_{\alpha\dot{\beta}} (\sigma - \bar{\sigma}) , \quad (\text{C.4c})$$

$$\delta_\sigma W_{\alpha\beta} = \sigma W_{\alpha\beta} . \quad (\text{C.4d})$$

Eq. (C.4d) shows that $W_{\alpha\beta}$ is the super-Weyl tensor.

The covariantly chiral projecting operator [103, 104] is

$$\begin{aligned} \bar{\Delta} &= \frac{1}{96} \left((\bar{\mathcal{D}}^{ij} + 16\bar{S}^{ij}) \bar{\mathcal{D}}_{ij} - (\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} - 16\bar{Y}^{\dot{\alpha}\dot{\beta}}) \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} \right) \\ &= \frac{1}{96} \left(\bar{\mathcal{D}}_{ij} (\bar{\mathcal{D}}^{ij} + 16\bar{S}^{ij}) - \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} (\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} - 16\bar{Y}^{\dot{\alpha}\dot{\beta}}) \right) , \end{aligned} \quad (\text{C.5})$$

where $\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} := \bar{\mathcal{D}}_k^{(\dot{\alpha}\dot{\beta})k}$. The fundamental property of $\bar{\Delta}$ is that $\bar{\Delta}U$ is covariantly chiral, for any scalar and isoscalar superfield U , that is $\bar{\mathcal{D}}_i^{\dot{\alpha}}\bar{\Delta}U = 0$. For any super-Weyl inert scalar U it holds that

$$\delta_\sigma U = 0 \implies \delta_\sigma \bar{\Delta}U = 2\sigma \bar{\Delta}U, \tag{C.6}$$

The operator $\bar{\Delta}$ relates an integral over the full superspace to that over its chiral subspace:

$$\int d^4x d^4\theta d^4\bar{\theta} E U = \int d^4x d^4\theta \mathcal{E} \bar{\Delta}U, \quad E^{-1} = \text{Ber}(E_A^M), \tag{C.7}$$

with \mathcal{E} the chiral density.¹⁸ The super-Weyl transformation laws of the integration measures are

$$\delta_\sigma E = 0, \quad \delta_\sigma \mathcal{E} = -2\sigma \mathcal{E}. \tag{C.8}$$

The superfield Ξ in (3.18) is the following composite scalar [52]:

$$\Xi := \frac{1}{6} \bar{\mathcal{D}}^{ij} \bar{S}_{ij} + \bar{S}^{ij} \bar{S}_{ij} + \bar{Y}_{\dot{\alpha}\dot{\beta}} \bar{Y}^{\dot{\alpha}\dot{\beta}}, \tag{C.9}$$

The fundamental properties of Ξ are as follows [52]:

- (i) Ξ is covariantly chiral,

$$\bar{\mathcal{D}}_i^{\dot{\alpha}} \Xi = 0; \tag{C.10a}$$

- (ii) the super-Weyl variation of Ξ is

$$\delta_\sigma \Xi = 2\sigma \Xi - 2\bar{\Delta}\bar{\sigma}; \tag{C.10b}$$

- (iii) the functional (3.21a) is a topological invariant related to the difference of the Gauss-Bonnet and Pontryagin invariants.

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¹⁸A derivation of (C.7) is given [104].

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