

Partition functions of p -forms from Harish-Chandra characters

Justin R. David and Jyotirmoy Mukherjee

*Centre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India*

E-mail: justin@iisc.ac.in, jyotirmoym@iisc.ac.in

ABSTRACT: We show that the determinant of the co-exact p -form on spheres and anti-de Sitter spaces can be written as an integral transform of bulk and edge Harish-Chandra characters. The edge character of a co-exact p -form contains characters of anti-symmetric tensors of rank lower to p all the way to the zero-form. Using this result we evaluate the partition function of p -forms and demonstrate that they obey known properties under Hodge duality. We show that the partition function of conformal forms in even $d + 1$ dimensions, on hyperbolic cylinders can be written as integral transforms involving only the bulk characters. This supports earlier observations that entanglement entropy evaluated using partition functions on hyperbolic cylinders do not contain contributions from the edge modes. For conformal coupled scalars we demonstrate that the character integral representation of the free energy on hyperbolic cylinders and branched spheres coincide. Finally we propose a character integral representation for the partition function of p -forms on branched spheres.

KEYWORDS: Field Theories in Higher Dimensions, Conformal Field Theory, Gauge Symmetry

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1 Introduction

Partition functions of free fields on spheres, anti-de Sitter spaces, hyperbolic cylinders, branched spheres, de Sitter spaces are important ingredients for tests of AdS/CFT dualities, evaluation of entanglement entropies, quantum corrections to black hole entropy and evaluation of anomaly coefficients in even dimensions and the F -function in odd dimensions. There are numerous works in the literature which highlight the importance of these partition functions, a partial list of these references can be obtained from the recent work of [1]. It has always been useful to cast these one loop partition functions as integrals over characters. One of the early instances was in [2] where the coincident heat kernel for arbitrary spin fields on thermal AdS_3 was written as a transform of the Harish-Chandra character of $SL(2, C)$. In higher dimensional AdS character integral representations were found in [3–5] and [6, 7], in the former series of works, the integral also involved angular integrations over the additional Cartan directions.

Recently in [1] and [8] a very useful expression for the one loop partition function for scalars, fermions and integer higher spin fields on spheres/euclidean patch of de Sitter space as well as anti-de Sitter space was found. This integral representation has only one integral over the Harish-Chandra character of the element in $SO(1, 1)$ of these spaces just as the one found in [2]. The expression for the one loop path integral on the sphere for higher spin fields on S^{d+1} can be summarised as follows

$$\begin{aligned} \log \mathcal{Z} &= \log \mathcal{Z}_G + \log \mathcal{Z}_{\text{chr}}, \\ \log \mathcal{Z}_{\text{chr}} &= \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} (\chi_{\text{bulk}}(t) - \chi_{\text{edge}}(t)). \end{aligned} \tag{1.1}$$

Here \mathcal{Z}_G contains the dependence of the dimensionless coupling constant of the theory. The dimensionless coupling is obtained by considering an appropriate combination of the coupling and the radius of the sphere. \mathcal{Z}_G also contains dependence of the volumes of the gauge groups of the fields involved in the one loop partition function. The interesting contributions arise from $\log \mathcal{Z}_{\text{chr}}$ which can be split into 2 parts. The bulk contribution can be written as an integral transform of Harish-Chandra characters of the group $SO(1, d + 1)$, while the edge term can be written as integral transform of Harish-Chandra character but with $d \rightarrow d - 2$. The contributions from the characters are interesting. For example, in the case of even $d + 1$, the coefficient of the logarithmic divergence which is independent of regularization and can be extracted from the $1/t$ coefficient of the integrand in $\log \mathcal{Z}_{\text{chr}}$. In the case of odd $d + 1$ one can extract the IR finite contribution to the partition function by performing the integral using a regulator. In [1] expressions for $\log \mathcal{Z}_{\text{chr}}$ were obtained for scalars, fermions and higher spin fields in arbitrary dimensions. This was then extended in [8] to the one loop partition function for these fields in AdS_{d+1} where now the integrand involves Harish-Chandra characters for the group $SO(2, d)$ and $SO(2, d - 2)$ as bulk and edge contributions respectively.

In this paper we obtain the character integral representation of the one loop partition function for p -forms on spheres and anti-de Sitter spaces. It is known that after fixing gauge the path integral of p -forms on spheres can be written as [9–11]

$$\mathcal{Z}_p[S^{d+1}] = \left[\frac{1}{\det_T \Delta_p \det_T \Delta_{p-2}} \dots \left(\frac{\det_T \Delta_1}{\det' \Delta_0} \text{Vol } S^{d+1} \right)^{(-1)^p} \right]^{\frac{1}{2}}, \tag{1.2}$$

where Δ_p is the Hodge-de Rham operator acting on a form of degree p on the sphere. The subscript T in the determinant refers to the fact that the determinant is taken over co-exact or transverse p -forms. The prime in the determinant of the 0-form refers to the fact that it does not include the zero mode. Finally the $\text{Vol } S^{d+1}$ is the volume of the $d + 1$ dimensional sphere of radius R . From (1.2) we see that the key ingredient in the one loop partition function of the p -form is the determinant of the co-exact p -form. Starting from the eigen values of the Hodge-de Rham Laplacian of co-exact p -forms on S^{d+1} and their degeneracies, we show that the character part of the determinant of the co-exact p form

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\sum_{i=0}^p (-1)^i \chi_{(d-2i, p-i)}^{dS}(t) \right), \tag{1.3}$$

where $\chi_{(d,p)}^{dS}(t)$ is the $SO(1, d+1)$ Harish-Chandra character of the anti-symmetric tensor of rank p , $\Delta = \frac{d}{2} + i\nu$ representation with $i\nu = \frac{d}{2} - p$. This is given by

$$\chi_{(d,p)}^{dS}(t) = \binom{d}{p} \frac{e^{-(d-p)t} + e^{-pt}}{(1 - e^{-t})^d}. \quad (1.4)$$

From (1.3) we see the bulk and the edge characters on the sphere S^{d+1} for the co-exact p -form is given by

$$\chi_{\text{bulk},(d,p)}^{dS}(t) = \chi_{(d,p)}^{dS}(t), \quad \chi_{\text{edge}(d,p)}^{dS}(t) = - \sum_{i=1}^p (-1)^i \chi_{(d-2i,p-i)}^{dS}(t). \quad (1.5)$$

Note that the edge character involves characters in lower dimensions. The dimensions are lowered by units of 2, while the degree of the p -forms are lowered by units of 1 all the way to $p = 0$. It is clear that if $2p > d$, the character involves powers of e^{+t} , therefore these are the ‘naive’ characters in the sense of [1] and should be replaced by the corresponding ‘flipped’ character. Thus the character representation of the determinant of the co-exact p -form can be written as difference of the bulk and edge character. Using this representation of the determinant of the co-exact p -form, we evaluate the path integral of the p forms on the spheres S^{d+1} for $2 \leq d \leq 13$. It is clear from (1.2) and (1.3) that the partition function of the p -form is a difference of the bulk and the edge contribution. To emphasise this again, note that the path integral involves a product of determinants of co-exact forms from 0 to p (1.2). Therefore to evaluate the free energy of the p -form we would need to perform a sum of (1.3) from $p = 0$ to p with alternating signs.

The integral in (1.3) can be regularised using the methods of [1]. For even $d+1$ we show that the coefficient of the log divergence of the partition function which is renormalization group invariant can be extracted easily from the coefficient of the t^{-1} term in the small t expansion of the integrand. This coefficient precisely agrees with earlier results in literature. For odd $d+1$ we regulate the integral to obtain the infrared finite part of the partition function and again the results agree with previous results in literature. For both even and odd dimensions we observe that these results obey known properties under Hodge duality.

We then repeat the analysis for p -forms in anti-de Sitter space of $d+1$ dimensions. Here again the key ingredient to evaluate the partition function is the determinant of the co-exact p form on AdS_{d+1} . Starting from the Plancherel measure and the eigen values of the Hodge-deRham Laplacian, we show that the character part of determinant of co-exact p form on AdS_{d+1} for even $d+1$ is given by

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\chi_{\text{bulk}}^{\text{AdS}}(t) - \chi_{\text{edge}}^{\text{AdS}}(t) \right), \quad d+1 \text{ even} \quad (1.6)$$

where

$$\chi_{\text{bulk}}^{\text{AdS}}(t) = \chi_{(d,p)}^{\text{AdS}}(t), \quad \chi_{\text{edge}}^{\text{AdS}}(t) = - \sum_{i=1}^p (-1)^i \chi_{(d-2i,p-i)}^{\text{AdS}}(t). \quad (1.7)$$

$\chi_{d,p}^{AdS}(t)$ is the $SO(2, d)$ Harish-Chandra character of the anti-symmetric tensor of rank p , $\Delta = \frac{d}{2} + i\nu$ representation with $i\nu = \frac{d}{2} - p$ which is given by

$$\chi_{(d,p)}^{AdS}(t) = \binom{d}{p} \frac{e^{-(d-p)t}}{(1 - e^{-t})^d}. \tag{1.8}$$

Similarly for AdS_{d+1} with $d + 1$ odd is given by

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \frac{\log(\tilde{R})}{2\pi i} \oint_C \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\chi_{\text{bulk}}^{AdS}(t) - \chi_{\text{bulk}}^{AdS}(t) \right), \quad d + 1 \text{ odd}. \tag{1.9}$$

Here the contour C is a small circle around the origin, and \tilde{R} is the dimensionless IR cutoff. It can be taken as the ratio of the radial cutoff on AdS to the radius of AdS. The characters $\chi_{\text{bulk}}^{AdS}(t), \chi_{\text{bulk}}^{AdS}(t)$ are defined in (1.7). We perform the following consistency check on our results for the partition function of p -forms: one loop free energies on even dimensional spaces are determined by the trace anomaly of the theory leads to the prediction that ratio of free energies in AdS_{d+1} to S^{d+1} is given by

$$\frac{\log \mathcal{Z}_p[AdS_{d+1}]}{\log \mathcal{Z}_p[S^{d+1}]} = \frac{1}{2} \quad \text{for even } d + 1. \tag{1.10}$$

We verify that this relation is satisfied by the character integral representation of the partition function.

We then study conformal invariant fields on hyperbolic cylinders. The fact that hyperbolic cylinders can be conformally mapped to spheres suggests that partition function of these fields on these spaces should be identical. Indeed in [12] it was verified through direct calculations that partition functions of conformal scalars and fermions in $S^1 \times AdS_2$ precisely agrees with that on S^3 . More recently in [13], the free energies of conformal scalars on hyperbolic cylinders and spheres were shown to agree to $d = 100$ by explicit calculations.¹ Since we have found that partition functions are integral transforms of characters, we can ask that if the integrands that occur in the character representation of the partition functions on hyperbolic cylinders agree with the integrands of the partition functions on spheres. We find that indeed that for conformal scalars we can sum over all the Kaluza-Klein modes on the S^1 of hyperbolic cylinders and show that the character which to begin with was an AdS character becomes the character on the sphere. The character integral representations on spheres and hyperbolic cylinders coincide and therefore provides a proof that the free energies of conformal scalars on these spaces are same.

Conformal p -forms occur in even $d + 1$ dimensions. We show that the logarithmic divergence of the gauge invariant partition function of the conformal $\frac{d-1}{2}$ -form in $S^1 \times AdS_d$ can be written as the following transform of characters.

$$\log \mathcal{Z}_{\frac{d-1}{2}}[S^1 \times AdS_d] = \frac{\log(\tilde{R})}{2\pi i} \int_C \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \sum_{i=0}^{\frac{d-1}{2}} (-1)^i \chi_{(d, \frac{d-1}{2}-i)}^{dS}(t), \quad d + 1 \text{ even} \tag{1.11}$$

¹See statement around equation 4.52 of [13].

where $\chi_{(d,p)}^{[dS]}$ is given in (1.4). Note the integral contains dS characters associated with the sphere S^{d+1} , it does not contain edge characters. In fact the integrand is precisely the bulk character of the conformal form on the sphere S^{d+1} . This is consistent with earlier observations found by studying entanglement entropy [14–19]. Entanglement entropy of theories with gauge symmetries evaluated using partition functions on hyperbolic cylinders contain only the bulk contributions and misses out on the edge terms.

Finally we study the co-exact 1-form on the branched sphere S_q^{d+1} with branching q . Using the known spectrum we write its free energy in terms of characters. From this result we are led to propose that the free energy of the co-exact p -form on branched spheres is given by

$$-\frac{1}{2} \det_T(\Delta_p^{S_q^{d+1}}) = \int_0^\infty \frac{dt}{2t} \left\{ \left(\frac{1 + e^{-\frac{t}{q}}}{1 - e^{-\frac{t}{q}}} \right) \chi_{(d,p)}^{dS}(t) + \left(\frac{1 + e^{-t}}{1 - e^{-t}} \right) \sum_{i=1}^p (-1)^i \chi_{(d-2i,p)}^{dS}(t) \right\}. \tag{1.12}$$

Note that the branching affects only the kinematic factor in front of the bulk character. The kinematic factor in front the edge character remains invariant under branching. We verify the proposal in (1.12) by evaluating the partition function p -forms on branched spheres using (1.2) and comparing to existing results in the literature.

The organization of the paper is as follows: in section 2 we write the free energy of p -forms on spheres in terms of characters, evaluate trace anomaly coefficients for even dimensional spheres and F -terms for odd spheres. We also demonstrate that these partition functions satisfy Hodge duality properties known in literature. In section 3 we examine p -forms on AdS spaces. One of important steps in evaluating the p -form Free energy as a transform of Harish-Chandra characters is to construct the Fourier transform of the Plancherel measure of the co-exact p -forms in AdS_{d+1} . We do this in section 3.1. We also verify the prediction (1.10). Finally in section 4 we obtain character integral representations for free energies of conformal p -forms including the conformal scalar on hyperbolic cylinders. In the section 4.3 we determine the character integral representation for free energies of p -forms on branched sphere by extrapolating the result for the 1-form. The appendices contain a list of Harish-Chandra characters used in the paper and an alternate approach to evaluate the Fourier transform of the Plancherel measure.

2 p -forms on spheres

The partition function of gauge fixed p -form field on a sphere in total $d + 1$ dimension is given by [9–11]

$$\mathcal{Z}_p[S^{d+1}] = \left[\frac{1}{\det_T \Delta_p \det_T \Delta_{p-2}} \dots \left(\frac{\det_T \Delta_1}{\det' \Delta_0} \text{Vol}S^{d+1} \right)^{(-1)^p} \right]^{\frac{1}{2}}. \tag{2.1}$$

$\det_T \Delta_p$ denotes the determinant of Hodge de-Rham Laplacian of the co-exact p -forms. The prime in $\det' \Delta_0$ refers to the determinant of 0-form or scalar without the zero modes. $\text{Vol}S^{d+1}$ refers to the volume of the $d + 1$ dimensional sphere. This arises due to the

fact that the scalar has a zero mode, and integration over this zero mode results in the volume. Therefore the building block of the partition function of p -forms is the determinant of the co-exact p form. In section 2.1 we will show that upon choosing an appropriate regulator, we can write the determinant of the co-exact p -form in terms of Harish-Chandra characters. Then in section (2.2) use these determinants and evaluate the coefficient of the logarithmic divergence in partition function of p -forms on even dimensional spheres. We will demonstrate that the result obeys known properties under Hodge duality. In section (2.3) we evaluate the infrared finite term in the partition function for p -forms on odd dimensional spheres and observe that they satisfy Hodge duality.

2.1 Determinant of co-exact p -forms as character integrals

From the definition of the determinant in terms of its eigen values we have

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = - \sum_{n=1}^{\infty} \frac{1}{2} g_n^{(p)} \log(\lambda_n^{(p)}), \tag{2.2}$$

where the eigen values $\lambda_n^{(p)}$ and the degeneracies $g_n^{(p)}$ of the Hodge-deRham Laplacian of co-exact p -forms on S^{d+1} are given by [10]

$$\begin{aligned} \lambda_n^{(p)} &= (n+p)(n-p+d), \\ g_n^{(p)} &= \frac{(2n+d)\Gamma[n+d+1]}{\Gamma[p+1]\Gamma[d-p+1]\Gamma[n](n+p)(n+d-p)}. \end{aligned} \tag{2.3}$$

In (2.2) the summation is from $n = 1$ to infinity for all values of p . Note that this includes the case of $p = 0$ for which the zero eigen value is not part of the gauge fixed determinant. Here we have focussed on the part of the one loop determinant that is independent of the radius of the sphere as well as the coupling of the theory. We now replace the logarithm by the identity

$$-\log y = \int_0^{\infty} \frac{d\tau}{\tau} (e^{-y\tau} - e^{-\tau}). \tag{2.4}$$

Substituting this identity in (2.2), we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^{\infty} \frac{d\tau}{2\tau} \left(\sum_{n=1}^{\infty} g_n^{(p)} (e^{-\tau\lambda_n^{(p)}} - e^{-\tau}) \right). \tag{2.5}$$

Here we wish to point out that our treatment differs from that of [1] in which the identity (2.4) was not used and therefore the second term in (2.2) was not present for [1]. As such the integral in (2.5) is convergent at $\tau = 0$ provided the second term can be regularized. To regularize the second term we follow the approach introduced by [20]. First note that the large n behaviour of the degeneracies is given by

$$g_n^{(p)} = \left(\frac{1}{n}\right)^{-d} \left(\frac{2}{\Gamma(d-p+1)\Gamma(p+1)} + O\left(\frac{1}{n}\right) \right). \tag{2.6}$$

Therefore the sum over the degeneracies can be performed by first choosing d to be a sufficiently negative and continuing this result to positive values of d . We will refer to this

as ‘dimensional regularization’. The result is given by [20, 21]

$$\sum_{n=1}^{\infty} g_n^{(p)} = -\cos p\pi = (-1)^{p+1}. \quad (2.7)$$

Using this result in (2.5) we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{d\tau}{2\tau} \left(\sum_{n=1}^{\infty} g_n^{(p)} e^{-\tau \lambda_n^{(p)}} + (-1)^p e^{-\tau} \right). \quad (2.8)$$

Now that we have regulated the $\tau = 0$ limit using dimensional regularization of the sum there is no need of introducing a UV regulator as in [1]. However we find it convenient to introduce an ϵ regulator. This will help us keep track of the branch cuts in the τ -plane that arise in the integrand and will not serve as a UV regulator. Indeed finally we will take the $\epsilon \rightarrow 0$ limit. We will also indicate how the branch cuts are present if ϵ was not introduced. This results in

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \left(\sum_{n=1}^{\infty} g_n^{(p)} e^{-\tau \lambda_n^{(p)}} + (-1)^p \right). \quad (2.9)$$

For the second term ϵ is introduced by the change of variables $\tau \rightarrow \frac{\epsilon^2}{4\tau}$. We will soon see that this choice leads us to the character integral representation for the one loop partition function. What we will obtain is the ‘naive’ character in the sense of [1]. The integral is not regulated in the IR yet. To extract IR finite terms we need another regulator as in [1]. In section 2.3 we will discuss the procedure to extract the IR finite terms. Let us go back to the integral in (2.9), by analytically continuing in n , from (2.3) we notice that

$$\begin{aligned} g_n^{(p)} &= 0, & \text{for } -(p-1) < n < 0, \\ g_{-p}^{(p)} &= (-1)^p, & \text{and } \lambda_{-p}^{(p)} = 0. \end{aligned} \quad (2.10)$$

Using these properties of the degeneracies and eigen values, we can continue the sum in (2.5) to continue to $n = -p$ for the co-exact p -form. This extension resulted naturally due to the properties of the degeneracies given in (2.10), which as far as we are aware has not been noted earlier. We obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \left(\sum_{n=-p}^{\infty} g_n^{(p)} e^{-\tau \lambda_n^{(p)}} \right). \quad (2.11)$$

Such an analytical continuation of the sums were also seen for higher spin fields by [1] using a different approach. As we have mentioned our starting point (2.5) and the regulator used is different from that of [1]. To show our approach yields the same results we have repeated the analysis for partition function of massless symmetric rank s tensor using our approach in appendix C and obtained the same conclusions of ([1]).

We now follow the steps of [1] and carry out the sum over n , After writing $\lambda_n^{(p)}$ as difference of squares we get

$$\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \left(\sum_{n=-p}^{\infty} g_n^{(p)} e^{-\tau(n+\frac{d}{2})^2} e^{-\tau \nu_p^2} \right), \quad (2.12)$$

where $\nu_p^2 = -(p - \frac{d}{2})^2$. We can now linearise the sum over n by using the Hubbard-Stratonovich trick.

$$\sum_{n=-p}^{\infty} g_n^{(p)} e^{-\tau(n+\frac{d}{2})^2} = \int_C \frac{du}{\sqrt{4\pi\tau}} e^{-\frac{u^2}{4\tau}} f_p(u). \quad (2.13)$$

Here the contour C runs from $-\infty$ to ∞ slightly above the real axis as in figure 1. and

$$f_p(u) = \sum_{n=-p}^{\infty} g_n^{(p)} e^{iu(n+\frac{d}{2})}. \quad (2.14)$$

Substituting the degeneracies $g_n^{(p)}$ given in (2.3) this sum can be performed resulting in

$$\begin{aligned} f_p(u) &= \frac{e^{\frac{1}{2}i(d+2)u}\Gamma(d+3)}{(d-2p)\Gamma(d-p+1)} \\ &\times \left[\frac{(e^{iu})^{-d+p-1} (B(e^{iu}; d-p+1, -d-1) - 2B(e^{iu}; d-p+1, -d-2))}{\Gamma(p+1)} \right. \\ &\quad \left. + 2 [{}_2\tilde{F}_1(d+3, p+1; p+2; e^{iu})] - {}_2\tilde{F}_1(d+2, p+1; p+2; e^{iu}) \right] \\ &\quad + (-1)^p e^{iu(\frac{d}{2}-p)}, \end{aligned} \quad (2.15)$$

where $B(e^{iu}; d-p+1, -d-1)$ is the incomplete beta function defined as

$$B(z, a, b) = \int_0^z t^{a-1} (1-t)^{b-1} dt \quad (2.16)$$

and ${}_2\tilde{F}_1(a, b, c; z)$ is the regularised hypergeometric function.

$${}_2\tilde{F}_1(a, b, c; z) = \frac{{}_2F_1(a, b, c; z)}{\Gamma[c]}. \quad (2.17)$$

The last line of $f_p(u)$ in (2.15) comes from term $n = -p$ in the sum. We can consider this term as the contribution from the zero mode. Substituting this sum we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{d\tau}{2\tau} \int_C \frac{du}{\sqrt{4\pi\tau}} e^{-\frac{\epsilon^2+u^2}{4\tau}} e^{\tau\nu_p^2} f_p(u). \quad (2.18)$$

Let us now perform the τ integral which results in

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_C \frac{du}{2\sqrt{u^2+\epsilon^2}} \left(e^{-\nu_p\sqrt{u^2+\epsilon^2}} f_p(u) \right). \quad (2.19)$$

At this stage it is good to point out that all the previous steps could have also been performed with $\epsilon = 0$. However in the above step one would have obtained $e^{-\nu_p|u|}/|u|$ indicating the presence of branch cut. If one worked with $\epsilon = 0$ one needs to keep track of this, it is easier to do this with $\epsilon \neq 0$, therefore we continue as before. We deform the contour C from the real line to the contour C' which runs on the both sides of the branch cut on the imaginary axis originating at $u = i\epsilon$ on the u -plane as shown in figure 1. Substituting $u = it$ we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \left(e^{i\nu_p\sqrt{t^2-\epsilon^2}} + e^{-i\nu_p\sqrt{t^2-\epsilon^2}} \right) f_p(it). \quad (2.20)$$

We can now take $\epsilon \rightarrow 0$ and the resulting integral is on the positive real axis in the t - plane as shown in figure 2. Now it can be shown by explicit check that the following remarkable identity holds²

$$(e^{\frac{d}{2}-p)t} + e^{-\frac{d}{2}-p)t})f_p(it) = \frac{1 + e^{-t}}{1 - e^{-t}} \sum_{i=0}^p (-1)^i \chi_{(d-2i,p-i)}^{dS}(t), \quad (2.21)$$

$$\chi_{(d,p)}^{dS}(t) = \binom{d}{p} \frac{e^{-t(d-p)} + e^{-tp}}{(1 - e^{-t})^d}. \quad (2.22)$$

Therefore the partition function of the co-exact p -form reduces to

$$-\frac{1}{2} \log(\det_T \Delta_p^{S^{d+1}}) = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\sum_{i=0}^p (-1)^i \chi_{(d-2i,p-i)}^{dS}(t) \right) \quad (2.23)$$

which is an integral representation in terms of $\text{SO}(1, d+1-2i)$ Harish-Chandra characters. The term $i = 0$, is the ‘naive’ bulk character in the sense of [1] of the co-exact p -form, while all terms $i \geq 1$ constitute the ‘naive’ edge characters.

Just as a check, let us examine the co-exact 1-form, we obtain

$$-\frac{1}{2} \log(\det_T \Delta_1^{S^{d+1}}) = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(d \frac{e^{-(d-1)t} + e^{-t}}{(1 - e^{-t})^d} - \frac{1 + e^{-(d-2)t}}{(1 - e^{-t})^{d-2}} \right). \quad (2.24)$$

We can compare this expression for the co-exact 1-form to the partition function of the transverse traceless for $s = 1$. On comparing with (F.12) of [1] we see that we are missing the non-local term $\log Z_{\text{res}}$ which in the end cancels off in (F.16). We have obtained directly equation (F.16) of [1] which is local. This reason for this may be attributed to our different starting point in (2.5). Ref. [1] did not have the second term which resulted from the use of the representation of the logarithm in (2.4). As a consistency check of our approach we repeat the analysis for rank s symmetric traceless tensors in appendix C and obtain the same conclusions. It is interesting that our starting point directly gives the representation of the path integral for the co-exact p -forms as well as transverse traceless tensors directly without any non-local terms. It would also be interesting to repeat the analysis using the starting point of [1] for the co-exact p -forms and reproduce our results, it would require the identification of the number of killing tensors of these forms.

Let us now examine the entire bulk contribution to the partition function of the 1-form using the expression for the path integral in (2.1) which includes the ghosts. We obtain

$$\log \mathcal{Z}_1[S^{d+1}]|_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(d \frac{e^{-(d-1)t} + e^{-t}}{(1 - e^{-t})^d} - \frac{e^{-dt} + 1}{(1 - e^{-t})^d} \right) \quad (2.25)$$

Thus we see that the character in the integrand can be identified with the $\hat{\chi}_{\text{bulk},s}$ in equation (G.18) for [1], which is the naive bulk character. Now to obtain the ‘flipped’ character³ from (2.25) we need to subtract the coefficient which contributes at the $t \rightarrow \infty$

²We have checked this for all values of $p, d \leq 14$.

³We will review the procedure of obtaining the ‘flipped’ character in the examples discussed subsequently.

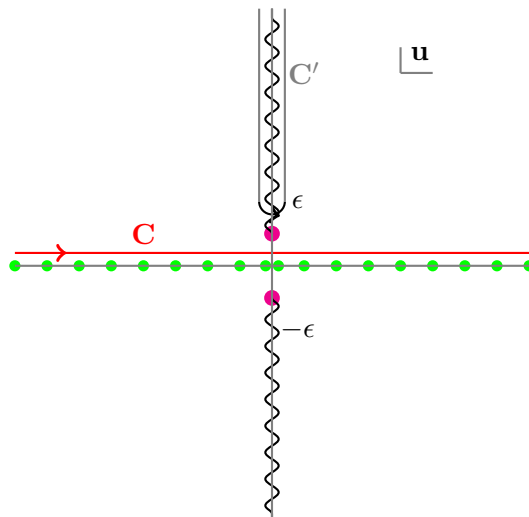


Figure 1. Integration contour in u -plane.

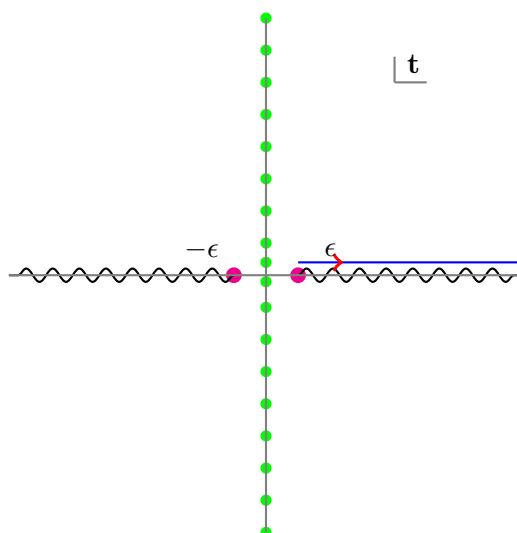


Figure 2. Integration contour in t -plane.

limit from the term in the curved bracket. This results in

$$\log \tilde{Z}_1[S^{d+1}]|_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(d \frac{e^{-(d-1)t} + e^{-t}}{(1 - e^{-t})^d} - \frac{e^{-dt} + 1}{(1 - e^{-t})^d} + 1 \right) \quad (2.26)$$

The procedure of ‘flipping’ does not change the coefficient of the $1/t$ term for even $d + 1$ or the IR finite term for odd $d + 1$ as discussed subsequently. However as noted in [1] the expression in the curved bracket of (2.26) coincides with the character of the unitary irreducible representation of $\text{SO}(1, d + 1)$ belonging to the exceptional series which is called the Harish-Chandra character of the massless 1-form or the rank one tensor. It is an interesting question whether the ‘flipped’ bulk character coincides with for all p -forms coincides with

characters of unitary irreducible representation of [1] belonging to the exceptional series. As a small step we perform this check for the 2-form in $d + 1 = 6$ below.

Finally note that the character sum in the integrand of (2.23) runs all the way over to the 0-form in $d - 2p$ dimensions. Therefore if $2p > d$, we would have terms which grow in t exponentially. We should then think of these characters as naive characters and convert them to flipped characters using the rules given in [1].

Let us now apply the expression (2.23) to obtain character integral representations of the one loop determinants of the p -form.

1-form on S^4 . From the expression given in (2.1) for the gauge fixed determinant of the 1-form on S^4 , the one loop partition function is given by

$$\log \mathcal{Z}_1[S^4] = -\frac{1}{2} \log(\det_T \Delta_1^{S^4}) + \frac{1}{2} \log(\det' \Delta_0^{S^4}). \quad (2.27)$$

Here we have ignored the dependence on the radius of the sphere. Substituting the expression in (2.23) we obtain

$$\begin{aligned} \log \mathcal{Z}_1[S^4] &= \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\sum_{i=0}^1 (-1)^i \chi_{(3-2i,1-i)}^{dS}(t) - \chi_{(3,0)}^{dS}(t) \right), \\ &= - \int_0^\infty \frac{dt}{2t} \left(\frac{2(e^{-t} + 1)^2 (-3e^{-t} + e^{-2t} + 1)}{(1 - e^{-t})^4} \right). \end{aligned} \quad (2.28)$$

The second line in the above equation is obtained by substituting the explicit values of the characters given in (2.22). The coefficient of the logarithmic divergence which is a renormalization group invariant can be obtained by examining the coefficient of $\frac{1}{t}$ of the integrand. This is given by

$$\log \mathcal{Z}_1[S^4] |_{\log \text{ divergence}} = -\frac{31}{45}. \quad (2.29)$$

As expected this agrees with the trace anomaly coefficient of the 1-form.⁴

2-form on S^6 . From (2.1), we see that partition function of the 2-form is given by the following combination of the determinant of co-exact forms

$$\log \mathcal{Z}_2[S^6] = -\frac{1}{2} \log(\det_T \Delta_2^{S^6}) + \frac{1}{2} \log(\det_T \Delta_1^{S^6}) - \frac{1}{2} \log(\det' \Delta_0^{S^6}). \quad (2.30)$$

Again we have ignored the dependence on the radius. Using the character integral representation of the partition function of the co-exact form we get

$$\log \mathcal{Z}_2[S^6] = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\sum_{i=0}^2 (-1)^i \chi_{(5-2i,2-i)}^{dS}(t) - \sum_{i=0}^1 (-1)^i \chi_{(5-2i,1-i)}^{dS}(t) + \chi_{(5,0)}^{dS}(t) \right). \quad (2.31)$$

⁴There are several references from which this coefficient can be read out from. One reference is [21]. This paper quotes values for $\tilde{F} = (-1)^{\frac{d+1}{2}} F$ for even $d + 1$. Note that we are looking at the $\log Z = -F$.

Finally substituting the Harish-Chandra characters from (2.23) we obtain

$$\log \mathcal{Z}_2[S^6] = \int_0^\infty \frac{dt}{2t} \frac{(e^{-t} + 1)^2 (3e^{-4t} - 16e^{-3t} + 32e^{-2t} - 16e^{-t} + 3)}{(1 - e^{-t})^6}. \quad (2.32)$$

The trace anomaly of the conformal 2-form can be read out easily from the coefficient of the $\frac{1}{t}$ term which is given by

$$\log \mathcal{Z}_2[S^6] \Big|_{\log \text{ divergence}} = \frac{221}{210}. \quad (2.33)$$

This again agrees with coefficient of the trace anomaly for the conformal 2-form in literature, see [21]

Let us verify that if indeed the bulk character for the 2-form coincides with that of the character of the corresponding unitary irreducible representation (UIR) of $SO(1,6)$ in the exceptional series. According to the notation of [22], the 2-form coincides with $p = 3, \Delta = 3, r = 2, \vec{x} = 0$ in the exceptional series.⁵ It corresponds to a Young tableaux of a single column with 2 boxes or $Y_3 = (1, 1)$. The contribution to the character comes from the second summation of equation (13) of [22]. We need to correct this by a factor of 2 as pointed out in [1]. With this correction, the character of the UIR is given by

$$\chi^{dS}(q = e^{-t}, \vec{x} = 0) = 2 \frac{q^5 - 5q^4 + 10q^3}{(1 - q)^5} \quad (2.34)$$

Here the coefficients in the numerator are the dimensions corresponding to the singlet, a vector and an anti-symmetric tensor of $SO(5)$. The overall factor of 2 is the correction noted by [1] which agrees with the older results of [23]. Now let us examine the bulk character of the 2-form in from our analysis, we use (2.30) and extract the bulk contribution from each of the determinants.

$$\log \mathcal{Z}_2[S^6] \Big|_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left[\binom{5}{2} \frac{e^{-2t} + e^{-3t}}{(1 - e^{-t})^5} - \binom{5}{1} \frac{e^{-t} + e^{-4t}}{(1 - e^{-t})^5} + \binom{5}{0} \frac{1 + e^{-5t}}{(1 - e^{-t})^5} \right] \quad (2.35)$$

We now subtract the coefficient which contributes at the $t \rightarrow \infty$ limit from the term in the square bracket to go over to the flipped character. We obtain

$$\log \tilde{\mathcal{Z}}_2[S^6] \Big|_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left[\binom{5}{2} \frac{e^{-2t} + e^{-3t}}{(1 - e^{-t})^5} - \binom{5}{1} \frac{e^{-t} + e^{-4t}}{(1 - e^{-t})^5} + \binom{5}{0} \frac{1 + e^{-5t}}{(1 - e^{-t})^5} - 1 \right] \quad (2.36)$$

Simplifying the terms in the square bracket we obtain

$$\log \tilde{\mathcal{Z}}_2[S^6] \Big|_{\text{bulk}} = \int_0^\infty \frac{dt}{2t} \frac{1 + e^{-t}}{1 - e^{-t}} \left(\frac{20e^{-3t} - 10e^{-4t} + 2e^{-5t}}{(1 - e^{-t})^5} \right) \quad (2.37)$$

We see that terms in the curved bracket of (2.37) precisely coincides with the character of the UIR (2.34) corresponding to the 2-form in $d + 1 = 6$. It will be interesting to verify whether such a statement holds for all p -forms in arbitrary dimensions, as it was demonstrated for the case of massless symmetric tensors in [1].⁶

⁵This ‘ p ’ refers to the p used in [22] not the p -form. In general for a given p -form, the ‘ p ’ of [22] is $p + 1$.

⁶We have also verified the fact that the UIR of the 2-form in $d + 1 = 7$ coincides with the corresponding flipped character.

2-form on S^4 . Let us now consider the case of 2-form on S^4 for which $2p > d$. From (2.1) we see that the partition function is given by

$$\log \mathcal{Z}_2[S^4] = -\frac{1}{2} \log(\det_T \Delta_2^{S^4}) + \frac{1}{2} \log(\det_T \Delta_1^{S^4}) - \frac{1}{2} \log(\det' \Delta_0^{S^4}). \quad (2.38)$$

Using the character representation in (2.23) for each of the co-exact p -forms that occur in the above expression we obtain

$$\log \mathcal{Z}_2[S^4] = \int_0^\infty \frac{dt}{2t} \frac{1+e^{-t}}{1-e^{-t}} \left(\sum_{i=0}^2 (-1)^i \chi_{(3-2i,2-i)}^{dS}(t) - \sum_{i=0}^1 (-1)^i \chi_{(3-2i,1-i)}^{dS}(t) + \chi_{(3,0)}^{dS}(t) \right). \quad (2.39)$$

Note that the characters that occur for the co-exact 2-form on S^4 are given by

$$\chi_{(3-2i,2-i)}^{dS} = \binom{3-2i}{2-i} \frac{(e^{-(2-i)t} + e^{-(1-i)t})}{(1-e^{-t})^{3-2i}}, \quad (2.40)$$

with i running from 0 to 2 at $i = 2$, this naive character which grow as e^t and therefore cannot be considered as character of a unitary representation of $\text{SO}(1, d+1)$. This feature also occurred in [1] for character representation of one loop determinants of massless higher spin fields with spins ≥ 2 . We can follow the same procedure developed in [1] to deal with such naive characters. We replace the naive character by the flipped character. Let $x = e^{-t}$, and consider the character $\chi = \sum_k c_k x^k$ with terms $k < 0$, then the flipped character is given by

$$[\chi]_+ = \chi - c_0 - c_k(x^k + x^{-k}). \quad (2.41)$$

As explained in [1], this procedure can be thought of as a contour deformation so that the integration is done over the negative t axis and removing zero modes from the path integral.

Let us demonstrate this for the character corresponding to the co-exact 2-form.

$$\begin{aligned} \chi &= \sum_{i=0}^2 (-1)^i \chi_{(3-2i,2-i)}^{dS}(t) \\ &= \frac{\binom{3}{2} (x^2 + x)}{(1-x)^3} - \frac{\binom{1}{1} (x+1)}{1-x} + \frac{\binom{-1}{0} (x^{-1} + 1)}{(1-x)^{-1}}, \\ &= \frac{1}{x} - 1 + 10x^2 + 25x^3 + 46x^4 + \dots \end{aligned} \quad (2.42)$$

The flipped character is then given by

$$\begin{aligned} [\chi]_+ &= \chi - (-1) - (x^{-1} + x), \\ &= \frac{x(-1 + 13x - 8x^2 + 2x^3)}{(1-x)^3}. \end{aligned} \quad (2.43)$$

Note the integral with the flipped character has the same UV divergence as the original naive character. The reason is that the additional contributions to convert the naive

$d+1 \backslash p$	2	4	6	8	10	12	14
0	$\frac{1}{3}$	$\frac{29}{90}$	$\frac{1139}{3780}$	$\frac{32377}{113400}$	$\frac{2046263}{7484400}$	$\frac{5389909963}{20432412000}$	$\frac{31374554287}{122594472000}$
1	-2	$-\frac{31}{45}$	$-\frac{1271}{1890}$	$-\frac{4021}{6300}$	$-\frac{456569}{748440}$	$-\frac{1199869961}{2043241200}$	$-\frac{893517041}{1571724000}$
2	+4	$+\frac{209}{90}$	$+\frac{221}{210}$	$+\frac{2603}{2520}$	$+\frac{13228}{13365}$	$+\frac{1296877349}{1362160800}$	$+\frac{2686807471}{2918916000}$
3		-4	$-\frac{5051}{1890}$	$-\frac{8051}{5670}$	$-\frac{5233531}{3742200}$	$-\frac{1417811}{1051050}$	$-\frac{456732097}{350269920}$
4		+6	$+\frac{16259}{3780}$	$+\frac{7643}{2520}$	$+\frac{1339661}{748440}$	$+\frac{15630799}{8845200}$	$+\frac{933250433}{544864320}$
5			-6	$-\frac{29221}{6300}$	$-\frac{12717931}{3742200}$	$-\frac{525793111}{243243000}$	$-\frac{7280049421}{3405402000}$
6			+8	$+\frac{712777}{113400}$	$+\frac{66688}{13365}$	$+\frac{33321199}{8845200}$	$+\frac{3698905481}{1459458000}$
7				-8	$-\frac{4947209}{748440}$	$-\frac{5622011}{1051050}$	$-\frac{14090853421}{3405402000}$
8				+10	$+\frac{61921463}{7484400}$	$+\frac{9469842149}{1362160800}$	$+\frac{3112707713}{544864320}$
9					-10	$-\frac{17545799561}{2043241200}$	$-\frac{2558351617}{350269920}$
10					+12	$+\frac{209714029963}{20432412000}$	$+\frac{26038135471}{2918916000}$
11						-12	$-\frac{16610757041}{1571724000}$
12						+14	$+\frac{1502508218287}{122594472000}$
13							-14

Table 1. Logarithmic divergence in the partition function of p -forms on even spheres.

character to the flipped character in (2.41) together with the factor $(1+x)/(1-x)$ in the integral transform is an odd function of t and therefore will not contribute to the coefficient of $\frac{1}{t}$.

Extracting the logarithmic divergence from (2.39) we obtain

$$\log \mathcal{Z}_2[S^4] \Big|_{\log \text{ divergence}} = \frac{209}{90}. \tag{2.44}$$

This coefficient agrees with the result quoted in [21].

2.2 Logarithmic divergence on even spheres

Proceeding as described in the previous section we can extract the coefficient of the logarithmic divergence for p forms on spheres in even $d+1$ with $2 \leq (d+1) \leq 14$. The result is summarised in table 1.

Note that the p -forms which are Hodge dual to each other do not have the same logarithmic divergence in the partition function. The difference in this coefficient between such Hodge dual pair of p forms have been seen to be integer multiples of 2. Consider the

(0, 2)-form pair on S^4 , from table (1) we see that the difference in the logarithmic divergence is -2 . Similarly the coefficients for (1, 3) and (2, 4)-form pairs on S^6 and S^8 differ by $+2$ and -2 respectively. Now consider the (0, 4), (1, 4)-pair on S^6 and S^8 , the coefficient of the logarithmic divergence differ by -4 and 4 respectively. These jumps in the trace anomaly coefficients agree with that noted earlier in [21].

2.3 IR finite term or the F-term on odd spheres

Consider the case $d + 1$ is odd, the character integral representation of the determinant of the co-exact p form is given by (2.23) with the characters given in (2.22). Using the definition of the characters it is easy to see that for d even, The characters are all even functions of t which makes the integrand in (2.23) is an even function since the remaining factor $(1 + e^{-t})/t(1 - e^{-t})$ is also an even function of t . Therefore the contour in t can be extended to the whole real line as shown in figure 3. Note that to ensure that the integrand is IR finite one always replaces the character by the flipped character. The additional terms one adds to flip the character is always even and the flipped character by construction converges as $t \rightarrow \infty$ in the complex plane. This enables us to close the contour using a large semi-circle in either the upper half or lower half plane. In figure 3 we have chosen to close the contour D in the upper half plane. The integral then can be evaluated by summing over the residues that occur on the imaginary axis. This results in the IR finite term. The IR finite term of the partition function is negative of what is known as the F-term in the literature. We illustrate this method in two examples.

2-form on S^7 . The character for the co-exact 2-form in $d = 6$ is given by

$$\begin{aligned} \chi(t) &= \sum_{i=0}^2 (-1)^i \chi_{(6-2i, 2-i)}^{dS}(t), \\ &= 1 - 2x + 3x^2 + 52x^3 + 242x^4 + \dots, \quad x = e^{-t} \end{aligned} \tag{2.45}$$

Therefore the flipped character is given by

$$[\chi(t)]_+ = \chi(t) - 1. \tag{2.46}$$

The IR finite term is obtained by evaluating the integral

$$-\frac{1}{2} \log(\det_T \Delta_2^{S^7}) \Big|_{\text{IR finite}} = \int_D \frac{dt}{4t} \frac{1 + e^{-t}}{1 - e^{-t}} [\chi(t)]_+ \tag{2.47}$$

Here D is the contour in the upper half plane as shown in figure 3. Then evaluating the residues we obtain

$$-\frac{1}{2} \log(\det_T \Delta_2^{S^7}) \Big|_{\text{IR finite}} = \frac{9}{32\pi^2} \zeta(3) + \frac{1}{16\pi^4} \zeta(5) - \frac{15}{64\pi^6} \zeta(7). \tag{2.48}$$

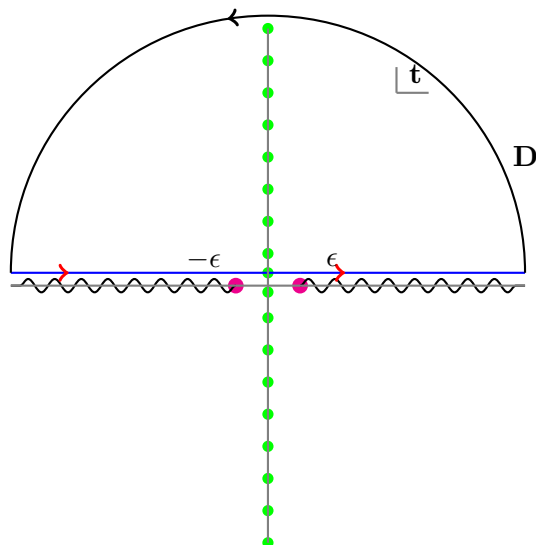


Figure 3. Contour for extracting the contribution of IR finite term in odd spheres.

2-form on S^3 . The naive character for the co-exact 2 form in $d = 2$ is given by

$$\begin{aligned} \chi(t) &= \sum_{i=0}^2 (-1)^i \chi_{(2-2i, 2-i)}^{dS}(t), \\ &= \frac{1}{x^2} - \frac{2}{x} + 3 + 5x^2 + 6x^3 + 8x^4 + \dots \end{aligned} \tag{2.49}$$

Therefore the flipped character is given by

$$[\chi(t)]_+ = \chi(t) - 3 + 2(e^t + e^{-t}) - (e^{-2t} + e^{-2t}). \tag{2.50}$$

Proceeding as before, the IR finite contribution to the one loop determinant is given by

$$\begin{aligned} -\frac{1}{2} \log(\det_T \Delta_2^{S^7}) \Big|_{\text{IR finite}} &= \int_D \frac{dt}{4t} \frac{1 + e^{-t}}{1 - e^{-t}} [\chi(t)]_+, \\ &= -\frac{1}{4\pi^2} \zeta(3). \end{aligned} \tag{2.51}$$

We use the method described to obtain the values of the IR finite terms for p forms for $0 \leq p \leq 7$ in $3 \leq d + 1 \leq 9$. We have compared the results of these finite terms wherever possible with the results of [21] and noted that they agree. For instance the results of the IR finite part evaluated in equations (2.29)–(2.34) of [21] agree with the corresponding values in table 2. There are differences in sign, due to the fact that what is quoted in [21] is a quantity called $\tilde{F} = (-1)^{d/2}$ times the sphere free energy in odd $d + 1$, while we have evaluated $\log \mathcal{Z}$ which is negative of the free energy. It is easy to observe the IR finite part respects Hodge duality as was seen earlier in [20, 21]. For example the values of the IR finite part for the Hodge-dual pair of forms $(0, 1)$ in $d = 2$ agree. The same property holds for the pairs $(0, 3), (1, 2)$ in $d = 4$, other Hodge-dual pairs can be seen in table 2.

$d + 1$	$p = 0$	$d + 1$	$p = 1$
3	$-\frac{\zeta(3)}{4\pi^2}$	3	$-\frac{\zeta(3)}{4\pi^2}$
5	$-\frac{23\zeta(3)}{48\pi^2} + \frac{\zeta(5)}{16\pi^4}$	5	$\frac{5\zeta(3)}{16\pi^2} + \frac{3\zeta(5)}{16\pi^4}$
7	$\frac{949\zeta(3)}{1440\pi^2} + \frac{13\zeta(5)}{48\pi^4} - \frac{\zeta(7)}{64\pi^6}$	7	$\frac{179\zeta(3)}{288\pi^2} + \frac{17\zeta(5)}{48\pi^4} - \frac{5\zeta(7)}{64\pi^6}$
9	$-\frac{16399\zeta(3)}{20160\pi^2} + \frac{2087\zeta(5)}{3840\pi^4} + \frac{\zeta(9)}{256\pi^8} - \frac{31\zeta(7)}{256\pi^6}$	9	$\frac{841\zeta(3)}{960\pi^2} + \frac{1249\zeta(5)}{3840\pi^4} + \frac{7\zeta(9)}{256\pi^8} - \frac{101\zeta(7)}{256\pi^6}$
$d + 1$	$p = 2$	$d + 1$	$p = 3$
5	$\frac{5\zeta(3)}{16\pi^2} + \frac{3\zeta(5)}{16\pi^4}$	5	$\frac{\zeta(5)}{16\pi^4} - \frac{23\zeta(3)}{48\pi^2}$
7	$\frac{49\zeta(3)}{144\pi^2} - \frac{7\zeta(5)}{24\pi^4} - \frac{5\zeta(7)}{32\pi^6}$	7	$-\frac{49\zeta(3)}{144\pi^2} - \frac{7\zeta(5)}{24\pi^4} - \frac{5\zeta(7)}{32\pi^6}$
9	$-\frac{1987\zeta(3)}{2880\pi^2} + \frac{21\zeta(9)}{256\pi^8} - \frac{2333\zeta(5)}{3840\pi^4} - \frac{71\zeta(7)}{256\pi^6}$	9	$\frac{205\zeta(3)}{576\pi^2} + \frac{91\zeta(5)}{256\pi^4} + \frac{75\zeta(7)}{256\pi^6} + \frac{35\zeta(9)}{256\pi^8}$
$d + 1$	$p = 4$	$d + 1$	$p = 5$
7	$\frac{179\zeta(3)}{288\pi^2} + \frac{17\zeta(5)}{48\pi^4} - \frac{5\zeta(7)}{64\pi^6}$	7	$-\frac{949\zeta(3)}{1440\pi^2} + \frac{13\zeta(5)}{48\pi^4} - \frac{\zeta(7)}{64\pi^6}$
9	$\frac{205\zeta(3)}{576\pi^2} + \frac{91\zeta(5)}{256\pi^4} + \frac{75\zeta(7)}{256\pi^6} + \frac{35\zeta(9)}{256\pi^8}$	9	$-\frac{1987\zeta(3)}{2880\pi^2} + \frac{21\zeta(9)}{256\pi^8} - \frac{2333\zeta(5)}{3840\pi^4} - \frac{71\zeta(7)}{256\pi^6}$
$d + 1$	$p = 6$	$d + 1$	$p = 7$
9	$\frac{841\zeta(3)}{960\pi^2} + \frac{1249\zeta(5)}{3840\pi^4} + \frac{7\zeta(9)}{256\pi^8} - \frac{101\zeta(7)}{256\pi^6}$	9	$-\frac{16399\zeta(3)}{20160\pi^2} + \frac{2087\zeta(5)}{3840\pi^4} + \frac{\zeta(9)}{256\pi^8} - \frac{31\zeta(7)}{256\pi^6}$

Table 2. IR finite part of the partition function on odd spheres.

3 p -forms on anti-de Sitter space

The construction of the partition function of gauge fixed p -form field on AdS_{d+1} proceeds along the same lines as that done for the sphere. The partition function is given by

$$\mathcal{Z}_p[AdS_{d+1}] = \left[\frac{1}{\det_T \Delta_p} \frac{\det_T \Delta_{p-1}}{\det_T \Delta_{p-2}} \dots \left(\frac{\det_T \Delta_1}{\det \Delta_0} \right)^{(-1)^p} \right]^{\frac{1}{2}}, \quad (3.1)$$

where $\det_T \Delta_p$ denotes the determinant of Hodge de-Rham Laplacian of co-exact p -form field and $\det \Delta_0$ is the determinant of 0-form on AdS. Note that the 0-form Laplacian does not have a discrete zero mode unlike the case for spheres. Therefore the one loop path integral does not contain the volume of AdS. The expression implies again that the key ingredient to evaluate the partition function is the determinant of the co-exact p -forms on AdS. In section 3.1 we will show that these determinants can be written in terms of Harish-Chandra characters of the anti-de Sitter group. In section 3.2 we evaluate the trace anomaly coefficient in AdS_{d+1} with $d + 1$ even and show that it is proportional to that of S^{d+1} as expected since these spaces are conformally flat.

3.1 Determinant of co-exact p -forms as character integrals

To evaluate determinants, we need both the eigen values and their degeneracies of the Hodge-deRham Laplacian. Since AdS is non-compact, the eigen values are part of continuous spectrum distributed through a measure known as the Plancherel measure. These

eigen values of the Laplacian are given by [24]

$$\Delta_p \psi_\lambda^{(u_i)} = -\left(\lambda^2 + \left(\frac{d}{2} - p\right)^2\right) \psi_\lambda^{(u_i)}, \quad (3.2)$$

where λ runs from 0 to ∞ and $\psi_\lambda^{(u_i)}$ is the basis of eigen functions for co-exact p -forms. The Plancherel measure is given by [24]

$$\mu_p(\lambda) = N(d) \hat{g}(p) \times \begin{cases} \frac{\lambda \tanh(\pi\lambda) \prod_{j=\frac{1}{2}}^{\frac{d}{2}} (j^2 + \lambda^2)}{\left(\frac{d}{2} - p\right)^2 + \lambda^2}, & \text{for } d \text{ odd} \\ \frac{\prod_{j=0}^{\frac{d}{2}} (j^2 + \lambda^2)}{\left(\frac{d}{2} - p\right)^2 + \lambda^2}, & \text{for } d \text{ even} \end{cases} \quad (3.3)$$

where the product runs over half integers for d odd and integers for d even. The normalizations are given by

$$N(d) = \frac{\text{Vol}(AdS_{d+1})}{2^d \Gamma\left(\frac{d+1}{2}\right) \pi^{\frac{d+1}{2}}}, \quad \hat{g}(p) = \frac{d!}{p!(d-p)!}. \quad (3.4)$$

By $\text{Vol}(AdS_{d+1})$ we refer to the regularised volume given by

$$\text{Vol}(AdS_{d+1}) = \begin{cases} \pi^{\frac{d}{2}} \Gamma\left(-\frac{d}{2}\right), & \text{for } d \text{ odd} \\ \frac{2(-\pi)^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)} \log \tilde{R} & \text{for } d \text{ even.} \end{cases} \quad (3.5)$$

Here \tilde{R} is the dimensionless IR cutoff, the ratio fo the radial cutoff to the radius of AdS.

Using these inputs let us proceed to evaluate the determinant of the co-exact p -form

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = -\frac{1}{2} \int_0^\infty d\lambda \mu_p(\lambda) \log \left[\lambda^2 + \left(\frac{d}{2} - p\right)^2 \right]. \quad (3.6)$$

We again replace the logarithm by the identity in (2.4) to obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \int_0^\infty \frac{d\tau}{2\tau} \int_0^\infty d\lambda \mu_p(\lambda) (e^{-\tau(\lambda^2 + (\frac{d}{2} - p)^2)} - e^{-\tau}). \quad (3.7)$$

The integral over the Plancherel measure vanishes

$$\int_0^\infty d\lambda \mu_p(\lambda) = 0. \quad (3.8)$$

Just as in the case of equation (2.7) for spheres, we obtain this result in (3.8) by choosing d to be sufficiently negative and then analytically continuing the result to positive d .⁷ Thus we can proceed by regulating only the first term which results in

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \int_0^\infty \frac{d\tau}{4\tau} e^{-\frac{\epsilon^2}{4\tau}} \int_{-\infty}^\infty d\lambda \mu_p(\lambda) e^{-\tau(\lambda^2 + (\frac{d}{2} - p)^2)}. \quad (3.9)$$

⁷Since $\mu_p(\lambda)$ is an even function of λ , it is equivalent to show $\int_{-\infty}^\infty d\lambda \mu_p(\lambda) = 0$. By replacing $\mu_p(\lambda)$ with its Fourier transform $W_p(u)$ and performing the integration over λ we see that we get $\int_{-\infty}^\infty d\lambda \mu_p(\lambda) = \lim_{u \rightarrow 0} W_p(u)$. From (3.24) we see that in this limit $W_p(u)$ vanishes for sufficiently large negative d .

Here we have used the fact that the Plancherel measure is symmetric in λ . We now write the Plancherel measure in terms of its Fourier transform.

$$\mu_p(\lambda) = \int_C \frac{du}{2\pi} e^{-i\lambda u} W_p(u). \tag{3.10}$$

Here C is a contour chosen to ensure the Fourier transform is well defined, which will be detailed subsequently. The contour differs when $d + 1$ is even or odd. We will denote this as C_e, C_o respectively. Substituting (3.10) in (3.9) and performing the integration over λ , we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \int_C du \int_0^\infty \frac{d\tau}{8(\pi\tau^3)^{\frac{1}{2}}} e^{-\frac{\epsilon^2+u^2}{4\tau} - \tau(\frac{d}{2}-p)^2} W_p(u). \tag{3.11}$$

After integration over τ we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \int_C \frac{du}{4\sqrt{u^2 + \epsilon^2}} e^{-(\frac{d}{2}-p)\sqrt{u^2+\epsilon^2}} W_p(u). \tag{3.12}$$

We can take the $\epsilon \rightarrow 0$ limit at the end.⁸ Thus the task of determining the one loop determinant is reduced to finding the Fourier transform $W_p(u)$.

Fourier transform of the Plancherel measure. To construct the Fourier transform $W_p(u)$, consider the ratio of the Plancherel measure of the p -form to the 0-form. From (3.3) we see that this is given by

$$\frac{\mu_p(\lambda)}{\mu_0(\lambda)} = \hat{g}(p) \frac{\lambda^2 + \left(\frac{d}{2}\right)^2}{\lambda^2 + \left(\frac{d}{2} - p\right)^2}. \tag{3.13}$$

From the definition of the Fourier transform in (3.10), the inverse is given by

$$W_p(u) = \int_{-\infty}^\infty d\lambda e^{i\lambda u} \mu_p(\lambda), \tag{3.14}$$

where the integral is on the real line. Using the relation in (3.13), we obtain the following differential equation satisfied by $W_p(u)$.

$$\frac{d^2}{du^2} W_p(u) - \left(p - \frac{d}{2}\right)^2 W_p(u) = \hat{g}(p) \left(\frac{d^2 W_0(u)}{du^2} - \frac{d^2}{4} W_0(u) \right). \tag{3.15}$$

Here we have assumed that the derivative with respect to u can be taken inside the integral involved in the defining the Fourier transform. $W_0(u)$ has been constructed in [8], On substituting $W_0(u)$ the differential equation in (3.15) becomes an inhomogenous second order ordinary differential equation for $W_p(u)$.

Before we solve the differential equation, let us recall the construction of $W_0(p)$ which was done in [8]. For both even and odd d it is given by

$$W_0(u) = \frac{1 + e^{-u}}{1 - e^{-u}} \frac{e^{-\frac{d}{2}u}}{(1 - e^{-u})^d}. \tag{3.16}$$

⁸After performing the integration over τ one gets the factor $|\frac{d}{2} - p|$ in the exponent. We have chosen one branch since we expect the answer to be analytic in d, p . This will be seen in the subsequent analysis.

However the contour relating $W_0(u)$ to the Plancherel measure in (3.10) is different in even and odd dimensions.

For $d + 1$ even: the contour is given by

$$\mu_0(\lambda) = \frac{1}{2} \left(\int_{R+i\delta} + \int_{R-i\delta} \right) \frac{du}{2\pi} e^{-i\lambda u} W_0(u), \quad d + 1 \text{ even.} \quad (3.17)$$

The contour is the sum of the line integrals above and below the real line. Since $W_0(u)$ has no branch cuts on the real line, the two line integrals can be replaced by a single line integral above the real line as shown in figure 4. We call this contour C_e . The normalization of W_0 is such that results in the factor $N(d)$ in (3.4) for $d + 1$ even in $\mu_0(\lambda)$.⁹

For $d + 1$ odd, the contour for obtaining the Plancherel measure from its Fourier transform is given by

$$\mu_0(\lambda) = -\frac{i}{\pi} \log(\tilde{R}) \left(\int_{R-i\delta} - \int_{R+i\delta} \right) \frac{du}{2\pi} e^{-i\lambda u} W_0(u). \quad (3.18)$$

Since $W_0(u)$ has no branch cuts on the real line, the contour can be deformed to a small circle around the origin as shown in figure 5. We call this contour C_o . Again, the normalization of W_0 and the factors in the equation (3.18) are such that results in the factor $N(d)$ in (3.4) for $d + 1$ odd.

We can now substitute W_0 in the r.h.s. of (3.15) we obtain

$$\frac{d^2}{du^2} W_p(u) - \left(p - \frac{d}{2} \right)^2 W_p(u) = \frac{(d+2)!}{p!(d-p)!} \frac{(1+e^{-u})e^{-\frac{(d+2)}{2}u}}{(1-e^{-u})^{d+3}}. \quad (3.19)$$

For $p \neq \frac{d}{2}$, the general solution of the differential equation is given by the following function depending on constants c_1, c_2 .

$$F(x) = c_1 x^{\frac{d}{2}-p} + c_2 x^{p-\frac{d}{2}} + \frac{x^{-\frac{d}{2}-p} \Gamma(d+3)}{(d-2p)\Gamma(p+1)\Gamma(d-p+1)} \left\{ x^{2p} [B(x; d-p+1, -d-1) - 2B(x; d-p+1, -d-2)] + x^d [2B(x; p+1, -d-2) - B(x; p+1, -d-1)] \right\}. \quad (3.20)$$

To fix the integration constants, let us first expand in small x , we get

$$F(x) = c_1 x^{p-\frac{d}{2}} + \frac{c_2 x^{\frac{1}{2}(d-2p)}}{d-2p} + \dots \quad (3.21)$$

To fix the boundary conditions, we demand that $W_p(u)$ is analytic in p . Therefore we demand that at $p = 0$, the function obeys the expansion obeyed by W_0 given in (3.17). This implies that we set $c_1 = 0$. Further more note that W_0 has a zero at $x = -1$, this must be true for W_p as well, since the factor $\frac{1+x}{1-x}$ arises due to the kinematic factor that the

⁹Our normalization of $W_0(u)$ differs from that in [8].

we are examining partition functions for bosons.¹⁰ Thus, setting $c_1 = 0$ and demanding that $W_p(x)$ admits a zero at $x = -1$ we obtain

$$c_2 = \frac{\Gamma(d+3)}{\Gamma(p+1)\Gamma(d-p+1)} \times \left\{ 2(-1)^{2p-d}B(-1; d-p+1, -d-2) + (-1)^{2p-d+1}B(-1; d-p+1, -d-1) - 2B(-1; p+1, -d-2) + B(-1; p+1, -d-1) \right\}. \quad (3.22)$$

Imposing these boundary conditions lead to the following expression for $W_p(x)$

$$W_p(x) = \frac{x^{-\frac{d}{2}-p}\Gamma(d+3)}{(d-2p)\Gamma(p+1)\Gamma(d-p+1)} \left\{ x^{2p}[B(x; d-p+1, -d-1) - 2B(x; d-p+1, -d-2)] + 2(-1)^{2p-d}x^d B(-1; d-p+1, -d-2) + (-1)^{-d+2p+1}x^d B(-1; d-p+1, -d-1) + x^d[2B(x; p+1, -d-2) - B(x; p+1, -d-1) - 2B(-1; p+1, -d-2) + B(-1; p+1, -d-1)] \right\}. \quad (3.23)$$

Though this expression for $W_p(x)$ seems non-illuminating, it can be written in terms of AdS Harish-Chandra characters as follows

$$\begin{aligned} W_p(u)e^{(-\frac{d}{2}+p)u} &= \frac{1+e^{-u}}{1-e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-2i}{p-i} \frac{e^{-(d-p-i)u}}{(1-e^{-u})^{d-2i}}, \\ &= \frac{1+e^{-u}}{1-e^{-u}} \sum_{i=0}^p \chi_{(d-2i, p-i)}^{\text{AdS}}, \\ \chi_{(d-2i, p-i)}^{\text{AdS}} &= \binom{d-2i}{p-i} \frac{e^{-(d-p-i)u}}{(1-e^{-u})^{d-2i}}. \end{aligned} \quad (3.24)$$

We have verified the above identity for all p -forms in dimensions $2 \leq d \leq 20$. We would like to emphasise that as a cross check we have also directly verified that $W_p(u)$ given in (3.24) solves the differential equation for $W_p(u)$ in (3.15). In the appendix B for $d+1$ even, we have performed the Fourier transform in (3.14) using the method of residues and have demonstrated that the result for $W_p(u)$ is as given in (3.24). Therefore the method of residues provides another cross check especially for the choice of the boundary conditions we used in the differential equation (3.15) to obtain $W_p(u)$.

Let us now examine the case $p = \frac{d}{2}$, this situation occurs only when $d+1$ is odd. This case needs a separate discussion one of the independent solution becomes logarithmic. For this case the differential equation determining $W_p(u)$ reduces to

$$\frac{d^2}{du^2} W_p(u) = \frac{\Gamma(2p+3)(e^{-u}+1)(e^{-(p+1)u})}{\Gamma(p+1)^2(1-e^{-u})^{2p+3}}, \quad p = \frac{d}{2}. \quad (3.25)$$

¹⁰The partition function of the simple harmonic oscillator also admits such a factor [1].

The most general solution of this differential equation is given by

$$F(x) = c_1 \log(x) + c_2 + \frac{x^{p+1} \Gamma(2p+3) {}_2\tilde{F}_1(p+1, 2p+2; p+2; x)}{\Gamma(p+2)}, \quad (3.26)$$

where $x = e^{-u}$ and

${}_2\tilde{F}_1(p+1, 2p+2; p+2; x)$ is the regularised hypergeometric function defined in (2.17). It is clear that we need to set $c_1 = 0$ if we wish to obtain an expression in terms of Harish-Chandra characters. Again demanding that $F(x)$ has a zero at $x = -1$, we obtain

$$\begin{aligned} c_2 &= \frac{(-1)^p \Gamma(2p+3)}{\Gamma(p+2)} {}_2\tilde{F}_1(p+1, 2p+2; p+2; -1), \\ &= (-1)^p. \end{aligned} \quad (3.27)$$

Substituting these boundary conditions, the solution of the differential equation for $p = \frac{d}{2}$ is given by

$$W_{p=\frac{d}{2}}(x) = \frac{\Gamma(2p+3)}{\Gamma(p+2)} x^{p+1} {}_2\tilde{F}_1(p+1, 2p+2; p+2; x) + (-1)^p, \quad p = \frac{d}{2} \quad (3.28)$$

Again it can be verified that the above expression can be written in terms of Harish-Chandra characters as follows

$$W_{p=\frac{d}{2}}(u) = \frac{1 + e^{-u}}{1 - e^{-u}} \sum_{i=0}^p (-1)^i \chi_{(d-2i, p-i)}^{\text{AdS}}, \quad p = \frac{d}{2} \quad (3.29)$$

where we have re-introduced $u = -\log x$. Note that in this case $p = \frac{d}{2}$, $\nu = \frac{d}{2} - p = 0$. As a cross check we have also verified that the Fourier transform in (3.29) satisfies the differential equation (3.25).

We conclude that for all value of p, d , the expression for the transform $W_p(u)$ is given in (3.24).

The one loop determinant. Let us now finally use $W_p(u)$ in equation (3.12) to write down the one loop determinant of the co-exact p -form on AdS_{d+1} . For this we need to specify the contour C . We have already used the fact that our results should be continuous in p . Therefore we choose the contour to be the same as the one used for the case $p = 0$ in (3.17) and (3.18) for $d+1$ even and $d+1$ odd respectively.

For even $d+1$ substituting (3.24) in (3.12) and using the contour in (3.17) we obtain

$$\begin{aligned} -\frac{1}{2} \log(\det_T \Delta_p^{\text{AdS}_{d+1}}) &= \frac{1}{2} \left(\int_{R+i\delta} + \int_{R-i\delta} \right) \frac{du}{4\sqrt{u^2 + \epsilon^2}} e^{(-\frac{d}{2}+p)\sqrt{u^2 + \epsilon^2}} W_p(u), \\ W_p(u) e^{(-\frac{d}{2}+p)u} &= \frac{1 + e^{-u}}{1 - e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-2i}{p-i} \frac{e^{-(d-p-i)u}}{(1 - e^{-u})^{d-2i}}, \quad d+1 \text{ even.} \end{aligned} \quad (3.30)$$

Since there is no branch cut on the real line, the contour below the real line and above the real line are equivalent. The contour is shown in figure 4. Furthermore note that $W_p(u) = W_p(-u)$ when $d+1$ is even. The remaining terms in the integrand are symmetric

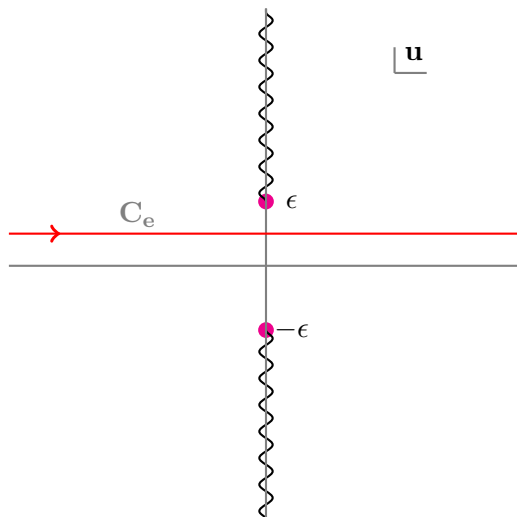


Figure 4. Contour C_e in the u -plane for even AdS_{d+1} .

in u . Thus we can restrict the integral over the positive real axis. Using these inputs and finally taking the $\epsilon \rightarrow 0$ limit, we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \int_0^\infty \frac{du}{2u} \frac{1 + e^{-u}}{1 - e^{-u}} \sum_{i=0}^p (-1)^i \chi_{(d-2i, p-i)}^{AdS}(u), \quad d + 1 \text{ even.} \quad (3.31)$$

For $d + 1$ odd, we substitute (3.24) in (3.12) and use the contour (3.17). This results in the following

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \frac{1}{2\pi i} \log(\tilde{R}) \left(\int_{R-i\delta} - \int_{R+i\delta} \right) \frac{du}{2\sqrt{u^2 + \epsilon^2}} e^{(-\frac{d}{2}+p)\sqrt{u^2+\epsilon^2}} W_p(u), \quad d + 1 \text{ odd.} \quad (3.32)$$

The branch cut in the integrand occurs only on the imaginary axis, choosing $\delta \ll \epsilon$, we can deform the contour to a small circle around $u = 0$ and then take $\epsilon \rightarrow 0$. The contour is shown in figure 5. Therefore we obtain

$$-\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}}) = \frac{1}{2\pi i} \log(\tilde{R}) \int_{C_0} \frac{du}{2u} \frac{1 + e^{-u}}{1 - e^{-u}} \sum_{i=0}^p (-1)^i \chi_{(d-2i, p-i)}^{AdS}(u), \quad d + 1 \text{ odd.} \quad (3.33)$$

In the next section we will verify that (3.31) is indeed the expression for the one loop partition of all p -forms in even dimensional AdS space by comparing it to corresponding partition function on even dimensional spheres. For the odd dimensions we will provide a cross check that (3.33) is the partition function for $p = \frac{d}{2}$ in section 4.2.

3.2 Trace anomaly cross check for even AdS

In conformally flat backgrounds and in even dimensions, free energies are proportional to the trace anomaly coefficient a_{d+1} , that appears with the Euler density as defined by the

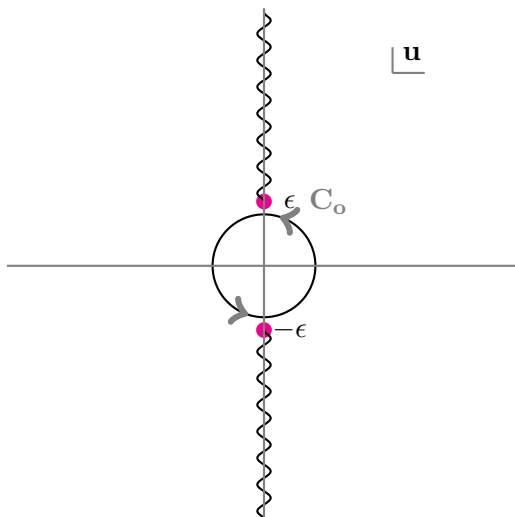


Figure 5. Contour C_o in the u -plane for even AdS_{d+1} .

expectation value of the trace of the stress tensor

$$\langle T_{\mu}^{\mu} \rangle = \frac{1}{(4\pi)^{\frac{d+1}{2}}} \left(\sum_j c_{(d+1)j} I_{j+1}^{(d+1)} - (-1)^{\frac{(d+1)}{2}} a_{d+1} E_{d+1} \right). \quad (3.34)$$

Here E_{d+1} is the Euler density which is given by

$$E_{d+1} = \frac{1}{2^{\frac{d+1}{2}}} \delta_{\mu_1 \dots \mu_d}^{\nu_1 \dots \nu_{d+1}} R_{\nu_1 \nu_2}^{\mu_1 \mu_2} \dots R_{\nu_d \nu_{d+1}}^{\mu_d \mu_{d+1}} \quad (3.35)$$

and $I_j^{(d+1)}$ are independent Weyl invariants of weight $-(d+1)$. It is known that for instance in $d+1 = 4$ dimensions, the variation of the partition function in conformally flat background with respect to the metric is given by [25, 26]

$$\begin{aligned} \frac{2}{\sqrt{g}} \frac{\delta \log \mathcal{Z}}{\delta g_{\mu\nu}} &= \langle T^{\mu\nu} \rangle, \\ &= -\frac{a_4}{(4\pi)^2} \left(g^{\mu\nu} \left(\frac{R^2}{2} - R_{\lambda\rho}^2 \right) + 2R^{\mu\lambda} R_{\lambda}^{\nu} - \frac{4}{3} R R^{\mu\nu} \right). \end{aligned} \quad (3.36)$$

This equation is true for both S^4 and AdS_4 , both these spaces are conformally flat. Integrating the above equation, we obtain

$$\log \mathcal{Z} = a_4 \int d^4 x \sqrt{g} S(R^{(2)}). \quad (3.37)$$

Here $S(R^{(2)})$, refers to a function made of the various quadratic invariants of the curvature tensor. Now the curvature tensor on spheres and anti-de Sitter spaces of unit radius differ just by a sign.

$$R_{\mu\nu\rho\sigma}|_{S^{d+1}} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}, \quad R_{\mu\nu\rho\sigma}|_{AdS_{d+1}} = g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}. \quad (3.38)$$

This implies that $S(R^{(2)})$ evaluated on AdS_4 and S^4 would be identical. Furthermore since both the spaces are Einstein, we see that $S(R^{(2)})$ would be constant in the space. This leads us to conclude

$$\frac{\log \mathcal{Z}[AdS_4]}{\log \mathcal{Z}[S^4]} = \frac{\text{Vol}(AdS_4)}{\text{Vol}(S^4)} = \frac{1}{2}. \quad (3.39)$$

To obtain the last line we have substituted the regularised volume of AdS^4 given in (3.5) and the volume of sphere given by

$$\text{Vol}(S^{d+1}) = 2 \frac{\pi^{\frac{d}{2}+1}}{\Gamma\left(\frac{d}{2} + 1\right)}, \quad (3.40)$$

which results in

$$\frac{\text{Vol}(AdS_{d+1})}{\text{Vol}(S^{d+1})} = \frac{(-1)^{\frac{d+1}{2}}}{2}. \quad (3.41)$$

We can repeat this analysis in $d + 1 = 6$. In this case, the equation corresponding to (3.38) is known and is given by [26]

$$\begin{aligned} \frac{2}{\sqrt{g}} \frac{\delta \log \mathcal{Z}}{\delta g_{\mu\nu}} &= \langle T^{\mu\nu} \rangle, \\ &= -\frac{a_6}{(4\pi)^3} \left[\frac{3}{2} R^\mu_\lambda R^\nu_\sigma R^{\lambda\sigma} - \frac{3}{4} R^{\mu\nu} R^\lambda_\sigma R^\sigma_\lambda - \frac{1}{2} g^{\mu\nu} R^\sigma_\lambda R^\lambda_\rho R^\rho_\sigma, \right. \\ &\quad \left. - \frac{21}{20} R^{\mu\lambda} R^\nu_\lambda R + \frac{21}{40} g^{\mu\nu} R^\sigma_\lambda R^\lambda_\sigma R + \frac{39}{100} R^{\mu\nu} R^2 - \frac{1}{10} g^{\mu\nu} R^3 \right]. \end{aligned} \quad (3.42)$$

Integrating this equation we get

$$\log \mathcal{Z} = a_6 \int d^6 x \sqrt{g} S(R^{(3)}), \quad (3.43)$$

where the $S(R^{(3)})$ is a function of cubic invariants constructed out of the curvature tensor. Using the fact that the curvature tensor on AdS and the sphere differ by a sign as in (3.38), we see that

$$S(R^{(3)})|_{AdS_6} = -S(R^{(3)})|_{S^6}. \quad (3.44)$$

Using (3.43), (3.41) and (3.44) we conclude that

$$\frac{\log \mathcal{Z}[AdS_6]}{\log \mathcal{Z}[S^6]} = -\frac{\text{Vol}(AdS_6)}{\text{Vol}(S^6)} = \frac{1}{2}. \quad (3.45)$$

Indeed noting the fact that the Euler density in (3.35) is a homogenous polynomial composed of curvature tensors of order $\frac{d+1}{2}$ allows us to conclude that in $d + 1$ dimensions that the free energies are given by

$$\log \mathcal{Z} = a_{d+1} \int d^{d+1} x \sqrt{g} S(R^{(\frac{d+1}{2})}). \quad (3.46)$$

Here $S\left(R^{\left(\frac{d+1}{2}\right)}\right)$ is a function invariants made of $\frac{d+1}{2}$ powers of the curvature tensor. From the properties of the curvature in AdS and spheres we have

$$S\left(R^{\left(\frac{d+1}{2}\right)}\right)|_{AdS_{d+1}} = (-1)^{\frac{d+1}{2}} S\left(R^{\left(\frac{d+1}{2}\right)}\right)|_{S^{d+1}}. \tag{3.47}$$

Then from (3.46), (3.41) and (3.47) we conclude

$$\frac{\log \mathcal{Z}[AdS_{d+1}]}{\log \mathcal{Z}[S^{d+1}]} = (-1)^{d+1} \frac{\text{Vol}(AdS_{d+1})}{\text{Vol}(S^{d+1})} = \frac{1}{2}. \tag{3.48}$$

Thus from the trace anomaly we can conclude that free energies in two conformally flat backgrounds is the same functional of the curvatures. This leads to the prediction that the ratio of free energies in even dimensional AdS and spheres is half. Since we have written down the partition function in both these spaces in terms of Harish-Chandra characters we can verify that this prediction is indeed true.

Let us examine the coefficient of the logarithmic divergence of the partition function. This is the only term which does not depend any prescription to regulate the integral in (3.31). The logarithmic divergence is determined by the coefficient of the $\frac{1}{u}$ term of the integrand. This can be extracted by considering a small circle C_r of radius r around the origin. Therefore the coefficient of the logarithmic divergence of the partition function of the co-exact p -form in even AdS_{d+1} given in (3.31)

$$\begin{aligned} -\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}})|_{\log \text{ divergence}} &= \frac{1}{2\pi i} \int_{C_r} \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-2i}{p-i} \frac{e^{-u(d-p-i)}}{(1-e^{-u})^{d-2i}} \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{du}{2u} \frac{1+e^u}{1-e^u} \sum_{i=0}^p (-1)^i \binom{d-2i}{p-i} \frac{e^{u(p-i)}}{(1-e^u)^{d-2i}}. \end{aligned} \tag{3.49}$$

In the last line we have also used the fact that $d+1$ is even. Note that in this integration $u = r^{i\theta}$, and θ runs from 0 to 2π . The result is invariant if we start the contour at θ from π and take it all the way to 3π in the counter-clock wise direction. The contour still remains the same. This effectively is a change of variable $\theta \rightarrow \hat{\theta} + \pi$, where $\hat{\theta}$ runs from 0 to 2π . But performing this change of variables sends $u \rightarrow -u$. Therefore we see that

$$\begin{aligned} -\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}})|_{\log \text{ divergence}} & \tag{3.50} \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-2i}{p-i} \frac{e^{-u(p-i)}}{(1-e^{-u})^{d-2i}}. \end{aligned}$$

As a cross check we have explicitly verified that indeed that coefficient of the $\frac{1}{u}$ term in (3.50) agrees with that of (3.49). Now adding (3.49) and (3.50) we see that

$$\begin{aligned} -\frac{1}{2} \log(\det_T \Delta_p^{AdS_{d+1}})|_{\log \text{ divergence}} &= \frac{1}{2} \frac{1}{2\pi i} \int_{C_r} \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \left(\sum_{i=0}^p (-1)^i \chi_{d-2i,p-i}^{dS}(u) \right), \\ &= -\frac{1}{4} \log(\det_T \Delta_p^{S^{d+1}})|_{\log \text{ divergence}}. \end{aligned} \tag{3.51}$$

In the first line of the above equation we have used the definition of the $SO(1, d+1)$ Harish-Chandra character given in (2.22). We have shown that the coefficient of the logarithmic divergence in the partition function of the co-exact p -form on even dimensional AdS_{d+1} is half that on the even sphere S^{d+1} . Therefore it is clear that the same fact holds for the logarithmic divergence of the partition function of the p -forms in these spaces since the co-exact forms form the key ingredient of the partitions functions as given by (2.1) and (3.1). This concludes our check of prediction given in (3.48).

4 Hyperbolic cylinders and branched spheres

In this section we study one loop partition functions of conformal scalars as well as conformal p -forms on hyperbolic cylinders. We show these partition functions can be written in terms of Harish-Chandra characters. Partition functions on hyperbolic cylinders naturally arise on evaluating entanglement entropy of conformal field theories across a spherical entangling surface [27]. When the ratio of the radius of S^1 to that of AdS is q the cylinder is referred to as $S_q^1 \times AdS_d$. Then the Rényi entropy across a spherical entangling surface can be evaluated given the partition function of a conformal field theory on the hyperbolic cylinder. The work of [1, 8] and the previous sections in this paper have demonstrated that one loop partition functions on spheres and anti-de Sitters spaces have nice character integral representations. It is natural to ask the question whether the same can be said about the partition functions on hyperbolic cylinders.

Hyperbolic cylinders $S_q^1 \times AdS_d$ are known to be conformally equivalent to branched spheres S_q^{d+1} [27]. Therefore we expect that the character representation of the partition function of conformal scalars on hyperbolic cylinders to agree with that of the branched sphere. In section 4.1 we indeed show that this expectation is true. In section 4.2 we show that the character integral representation of conformal p -form partition functions on the hyperbolic cylinder agrees with only the bulk character of the corresponding partition function on the sphere. This is consistent with the earlier observations of [14–16, 19, 28, 29] that partition function on hyperbolic cylinders miss out the edge modes or the non-extractable classical contribution to entanglement entropy. The character integral representation of the conformal p -form we derive generalises the observation seen first for 1-forms, to conformal p -forms in arbitrary dimensions. Finally in section 4.3, we obtain a character integral representation for 1-forms in arbitrary dimensions on branched spheres. Using this input, together with the observations for character representation of conformal p -forms on hyperbolic cylinders $S_q^1 \times AdS_d$ we propose the character integral representation of the partition functions of co-exact p -form on branched spheres in all dimensions. We verify that this proposal agrees with previous evaluations of these partition functions by [18].

4.1 Conformal scalars

We will first evaluate the partition function of conformal scalars on the hyperbolic cylinder and show that it can be written in terms of Harish-Chandra characters. We will see these characters turn out to be characters for the sphere and the partition function is identical to that of conformal scalars on the branched sphere.

Conformal scalars on $S_q^1 \times AdS_d$. The Weyl invariant action of the real scalar in $d+1$ dimensions is given by

$$S = -\frac{1}{2} \int d^{d+1}x \sqrt{g} \left(\partial_\mu \phi \partial^\mu \phi + \frac{d-1}{4d} R \phi^2 \right). \quad (4.1)$$

Here R is the curvature. The metric on $S_q^1 \times AdS_d$ is given by

$$ds_{S_q^1 \times AdS_d}^2 = d\tau^2 + du^2 + \sinh^2 u d\Omega_{d-1}^2, \quad (4.2)$$

where τ is the coordinate on the circle with the identification $\tau \sim \tau + 2\pi q$. The radial coordinate on AdS_d is u and Ω_{d-1} refers to the $d-1$ sphere. Using this metric, we can evaluate the curvature scalar on the hyperbolic cylinder which is given by

$$R|_{S_q^1 \times AdS_d} = -d(d-1). \quad (4.3)$$

The partition function of Weyl invariant scalar on this background is given by the determinant

$$\mathcal{Z}[S_q^1 \times AdS_d] = \left(\frac{1}{\det(-\partial_\tau^2 - \Delta_0 + m_{S_q^1 \times AdS_d}^2)} \right)^{\frac{1}{2}}, \quad (4.4)$$

where the mass arises from the curvature coupling and is given by

$$m_{S_q^1 \times AdS_d}^2 = -\left(\frac{d-1}{2} \right)^2. \quad (4.5)$$

Here $\Delta_{(0)}$ is the spin-0 Laplacian on AdS_d . We decompose the scalar using the eigen modes of the spin-0 Laplacian on AdS_d and the Kaluza-Klein modes on the circle S^1 . The eigen values of the spin-0 Laplacian on AdS_d are given by

$$\Delta_{(0)} \psi^{\{\lambda, u\}} = -\left[\lambda^2 + \left(\frac{d-1}{2} \right)^2 \right] \psi_\lambda^{\{\lambda, u\}}, \quad (4.6)$$

$\psi^{\{\lambda, u\}}$ are the corresponding eigen functions, $\{u\}$ labels other quantum numbers on AdS_d . Using these eigen values and the Kaluza-Klein decomposition of the partition function. we obtain

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = -\frac{1}{4} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda \mu_0^{(d)}(\lambda) \log \left(\frac{n^2}{q^2} + \lambda^2 \right). \quad (4.7)$$

Note that the shift in the eigen value of the Laplacian in (4.6) precisely cancels the mass due to the curvature coupling. The mass in (4.5) saturates the Brietenholer-Freedman bound. We have labelled the Plancherel measure for scalars with the superscript (d) to indicate that we are in AdS_d . This measure is given by the expressions in (3.3), with $p=0$ and $d \rightarrow d-1$ since we are in AdS_d . We replace the logarithm by the identity in (2.4) to obtain

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \frac{1}{4} \int_0^\infty \frac{d\tau}{\tau} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda \mu_0^{(d)}(\lambda) \left(e^{-\tau \left(\lambda^2 + \frac{n^2}{q^2} \right)} - e^{-\tau} \right). \quad (4.8)$$

Just as in the case of (3.8), by analytically continuing from large negative d we have

$$\int_0^\infty d\lambda \mu_0^{(d)} = 0. \quad (4.9)$$

Therefore we can proceed by regulating the first term as

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \int_0^\infty \frac{d\tau}{4\tau} e^{-\frac{\epsilon^2}{4\tau}} \int_{-\infty}^\infty d\lambda \mu_0^{(d)}(\lambda) \left(e^{-\tau\lambda^2} + 2 \sum_{n=1}^\infty e^{-\tau\left(\lambda^2 + \frac{n^2}{q^2}\right)} \right). \quad (4.10)$$

Now replace the Plancherel measure by its Fourier transform given by

$$W_0^{(d)}(u) = \frac{1 + e^{-u}}{1 - e^{-u}} \frac{e^{-\frac{d-1}{2}u}}{(1 - e^{-u})^{d-1}}. \quad (4.11)$$

Note that here d in equation (3.16) has been replaced by $d-1$ as we are in AdS_d . Then following the same steps as in equations (3.9), (3.11) and (3.12) we are led to

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \int_C \frac{du}{4\sqrt{\epsilon^2 + u^2}} \left(1 + 2 \sum_{n=1}^\infty e^{-\frac{n}{q}\sqrt{\epsilon^2 + u^2}} \right) W_0^{(d)}(u). \quad (4.12)$$

Here the contour is C_o or C_e as defined in figure 4 and figure 5 depending on whether d is odd or even respectively.

For the case when d is even substituting W_0^d and using the contour C_e as shown in figure 4 we obtain

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \frac{1}{2} \int_{R+i\delta} \frac{du}{2u} \frac{1 + e^{-u}}{1 - e^{-u}} \frac{e^{-\frac{d-1}{2}u}}{(1 - e^{-u})^{d-1}} \frac{1 + e^{-\frac{1}{q}\sqrt{\epsilon^2 + u^2}}}{1 - e^{-\frac{1}{q}\sqrt{\epsilon^2 + u^2}}}. \quad (4.13)$$

We have summed the geometric series in (4.12). We can use the fact that the integrand is even to integrate only over the positive real axis, and then take the $\epsilon \rightarrow 0$ limit and then take $\delta \rightarrow 0$. Thus we arrive at

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \int_0^\infty \frac{du}{2u} \frac{1 + e^{-\frac{u}{q}}}{1 - e^{-\frac{u}{q}}} \chi_{(d,0)}^{dS} \text{conf}(u), \quad (4.14)$$

where the $SO(1, d+1)$ Harish-Chandra character is given by

$$\chi_{(d,0)}^{dS} \text{conf}(u) = \frac{e^{-\frac{(d-1)}{2}t} + e^{-\frac{d+1}{2}t}}{(1 - e^{-t})^d}. \quad (4.15)$$

We observe that this character corresponds to that of the conformal scalar on S^{d+1} .

The $SO(1, d+1)$ Harish-Chandra character of the scalar of mass m is given by

$$\chi_{(d,0)}^{dS} \nu(u) = \frac{e^{-(\frac{d}{2} + i\nu)t} + e^{-(\frac{d}{2} - i\nu)t}}{(1 - e^{-t})^d} \quad (4.16)$$

and ν is related to the mass by the following

$$i\nu = \sqrt{\frac{d^2}{4} - m^2}. \quad (4.17)$$

This mass induced by the curvature coupling on the sphere S^{d+1} can be read out from (4.1) with $R = d(d + 1)$. We see that this mass is

$$m_{S^{d+1}}^2 = \frac{d^2 - 1}{4}. \tag{4.18}$$

Substituting this mass in (4.17) we see that the Harish-Chandra character reduces to that seen in (4.14).

We can proceed with the same analysis for the case of d is odd. The analysis is identical except that the contour C_e is replaced by C_o of figure (5). The result is given by

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \frac{\log(\tilde{R})}{2\pi i} \int_{C_o} \frac{du}{2u} \frac{1 + e^{-\frac{u}{q}}}{1 - e^{-\frac{u}{q}}} \chi_{(d,0)}^{dS} \text{conf}(u). \tag{4.19}$$

The interesting thing to note from (4.14) and (4.19) is that we started with the $SO(2, d - 1)$ Harish-Chandra characters on AdS_d . The sum over all the Kaluza-Klein modes resulted in $SO(1, d + 1)$ Harish-Chandra characters. Note that the we have included the zero Kaluza-Klein mode as well. In literature usually this is omitted since it just results it is q independent [30]. However we see here that it is including this term, that the integrand organises as a character. Furthermore observe that at $q = 1$, these results are identical to partition function of the conformal scalar on sphere S^{d+1} . Indeed, we will show that the partition function of conformal scalars on branched spheres S_q^{d+1} precisely agrees with that obtained in (4.14) and (4.19) for $S_q^1 \times AdS_d$.

Conformal scalars on branched spheres S_q^{d+1} . The metric on the branched sphere is given by

$$ds^2|_{S_q^{d+1}} = \cos^2 \phi d\tau^2 + d\phi^2 + \sin^2 \phi d\Omega_{d-1}^2, \tag{4.20}$$

where $\tau \sim \tau + 2\pi q$ and $0 \leq \phi \leq \frac{\pi}{2}$. Given the Weyl invariant action (4.1) and the Ricci curvature $R = d(d + 1)$, the partition function of conformal scalar on the branched sphere can be written as

$$\mathcal{Z}[S_q^{d+1}] = \frac{1}{\det(-\Delta_0 + m_{S_q^{d+1}}^2)^{\frac{1}{2}}}. \tag{4.21}$$

The curvature induced mass can be read off from (4.18).

The eigenvalue and their corresponding degeneracies for the scalar Laplacian on the branched sphere are known [31]. They are labelled by 2 integers

$$\lambda_{n,m}^{(0)} = (n + \frac{m}{q})(n + \frac{m}{q} + d), \quad n, m \in \{0, \dots, \infty\} \tag{4.22}$$

with degeneracies

$$g_{n,m=0}^{(0)} = \binom{d+n-1}{d-1}, \quad n \in \{0, \dots, \infty\} \tag{4.23}$$

$$g_{n,m>0}^{(0)} = 2 \binom{d+n-1}{d-1}, \quad n \in \{0, \dots, \infty\}, \quad m \in \{1, \dots, \infty\}$$

Therefore the free energy of the conformal scalar

$$\begin{aligned} \log \mathcal{Z}[S_q^{d+1}] &= -\frac{1}{2} \sum_{n,m=0}^{\infty} g_{n,m}^{(0)} \log \left(\lambda_{n,m}^{(0)} + \frac{d^2 - 1}{4} \right), \\ &= \int_0^{\infty} \frac{d\tau}{2\tau} \sum_{n,m=0}^{\infty} g_{n,m}^{(0)} (e^{-\tau(\lambda_{n,m}^{(0)})} - e^{-\tau}). \end{aligned} \quad (4.24)$$

In the second line of the above equation we have again used the identity (2.4). It can be shown by looking at sufficiently large negative d and using zeta function regularization for the summation over m we obtain

$$\sum_{n,m=0}^{\infty} g_{n,m}^{(0)} = 0. \quad (4.25)$$

Therefore we can proceed by regulating the first term

$$\log \mathcal{Z}[S_q^{d+1}] = \int_0^{\infty} \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \sum_{n,m=0}^{\infty} g_{n,m}^{(0)} e^{-\tau(\lambda_{n,m}^{(0)})}. \quad (4.26)$$

We can now perform the Hubbard-Stratonovich trick given in (2.13) and following the same steps as in equation (2.11) to (2.20) we obtain

$$\log \mathcal{Z}[S_q^{d+1}] = \int_{\epsilon}^{\infty} \frac{dt}{2\sqrt{t^2 - \epsilon^2}} \left(e^{i\nu\sqrt{t^2 - \epsilon^2}} + e^{-i\nu\sqrt{t^2 - \epsilon^2}} \right) f_q^{(0)}(it). \quad (4.27)$$

with $\nu = \frac{i}{2}$ and

$$\begin{aligned} f_q^{(0)}(u) &= \sum_{n,m=0}^{\infty} g_{n,m}^{(0)} e^{i(n + \frac{m}{q} + \frac{d}{2})u} \\ &= \frac{e^{\frac{iud}{2}}}{(1 - e^{iu})^d} \frac{1 + e^{\frac{i}{q}u}}{1 - e^{\frac{i}{q}u}}. \end{aligned} \quad (4.28)$$

Substituting these expressions in (4.27) and taking the $\epsilon \rightarrow 0$ limit the partition function becomes

$$\log \mathcal{Z}[S_q^{d+1}] = \int_0^{\infty} \frac{dt}{2t} \frac{1 + e^{-\frac{t}{q}}}{1 - e^{-\frac{t}{q}}} \chi_{(d,0)}^{dS} \text{conf}(u), \quad (4.29)$$

with the character as given in (4.15).

For $d+1$ odd, comparison with equation (4.14), with (4.29) we see that the integrands are identical. For the case when $d+1$ is even comparison of (4.19) with (4.29) shows that the regularization independent quantity, the logarithmic divergence of the expression in (4.29) and that given in (4.19) are identical provided we relate the cut off to regulate the integral in (4.29) to \tilde{R} in (4.19). Indeed as recently mentioned in [13], the agreement of free energies of the conformal scalar on the hyperbolic cylinder and the branched sphere coincides was verified till $d = 100$.¹¹ It is interesting to note here that since their character

¹¹See [32] for a discussion for fermions.

D	$-\mathcal{F}_q$	S_{EE}
2	$\frac{q^2+1}{6q}$	$\frac{1}{3}$
4	$-\frac{3q^4+1}{360q^3}$	$-\frac{1}{90}$
6	$\frac{31q^6+7q^2+2}{30240q^5}$	$\frac{1}{756}$
8	$-\frac{289q^8+56q^4+20q^2+3}{1814400q^7}$	$-\frac{23}{113400}$
10	$\frac{6657q^{10}+1188q^6+462q^4+99q^2+10}{239500800q^9}$	$\frac{263}{7484400}$
12	$-\frac{6803477q^{12}+1153152q^8+469040q^6+117117q^4+18200q^2+1382}{130767436800q^{11}}$	$-\frac{133787}{20432412000}$
14	$\frac{16018495q^{14}+2620800q^{10}+1095952q^8+298155q^6+56420q^4+6910q^2+420}{15692092416000q^{13}}$	$\frac{157009}{122594472000}$

Table 3. Logarithmic divergence of the partition function of conformal scalars on branched spheres and universal terms in entanglement entropies.

$d+1$	$-\mathcal{F}_{q=1}$
3	$\frac{3\zeta(3)-2\pi^2\log(2)}{16\pi^2}$
5	$\frac{2\pi^2\zeta(3)-15\zeta(5)+2\pi^4\log(2)}{256\pi^4}$
7	$-\frac{82\pi^4\zeta(3)-150\pi^2\zeta(5)-945\zeta(7)+60\pi^6\log(2)}{61440\pi^6}$
9	$\frac{1588\pi^6\zeta(3)-210\pi^4\zeta(5)-13230\pi^2\zeta(7)-26775\zeta(9)+1050\pi^8\log(2)}{6881280\pi^8}$
11	$-\frac{70146\pi^8\zeta(3)+48500\pi^6\zeta(5)-383670\pi^4\zeta(7)-1338750\pi^2\zeta(9)-1611225\zeta(11)+44100\pi^{10}\log(2)}{1651507200\pi^{10}}$
13	$\frac{7157604\pi^{10}\zeta(3)+8436890\pi^8\zeta(5)-24019380\pi^6\zeta(7)-124289550\pi^4\zeta(9)-248128650\pi^2\zeta(11)-212837625\zeta(13)+4365900\pi^{12}\log(2)}{871995801600\pi^{12}}$

Table 4. IR finite term or the ‘F’-term on odd spheres for conformal scalars.

integral representations coincide and therefore the agreement of the partitions functions is manifest.

Given the partition function on the branched cylinder, one can evaluate the universal contribution to Rényi entropy across spherical entangling surfaces in even dimensions from the logarithmic divergence of the free energy \mathcal{F}_q on hyperbolic cylinders. The Rényi entropy S_q and the entanglement entropy S_{EE} are given by

$$S_q = \frac{-\mathcal{F}_q + q\mathcal{F}_{q=1}}{1-q}, \quad S_{\text{EE}} = \lim_{q \rightarrow 1} S_q. \quad (4.30)$$

In table 3 we have listed both these entropies for the conformal scalar are listed for even $4 \leq d+1 \leq 14$. These have been evaluated using the character integral representation. We have seen that they precisely agree with earlier evaluations in [33]. The IR finite term when $d+1$ is odd is also a regularization independent term. We have evaluated this finite part from the character integral representation for the corresponding partition function in table 4. These precisely agree with that evaluated in table 1 of [34].

4.2 Conformal p -forms

In [19] a procedure to fix gauge on hyperbolic cylinders was introduced. Using this method the gauge invariant partition functions of the conformal 1-form on $S_q^1 \times AdS_3$ and the 2-form on $S_q^1 \times AdS_5$ was obtained. The conclusion in both the cases was the gauge invariant partition function can be thought of as the partition function of the tower of Kaluza-Klein tower of co-exact p -forms with the Hodge-de Rham Laplacian along the AdS directions. Though this observation was explicitly demonstrated only for the 1-form in $d+1 = 4$ and 2-form in $d+1 = 6$, the method developed in [19] is such that the result can be extrapolated to conformal $p = \frac{d-1}{2}$ on the hyperbolic cylinder $S_q^1 \times AdS_d$. To summarise the gauge invariant partition function of conformal p forms is given by the following determinant

$$\mathcal{Z}[S_q^1 \times AdS_d] = \left[\frac{1}{\det_T(-\partial_\tau^2 - \Delta_p)} \right]^{\frac{1}{2}}, \quad p = \frac{d-1}{2} \quad (4.31)$$

Here Δ_p is the Hodge-deRham Laplacian acting on co-exact forms. The operator ∂_τ^2 picks out the Kaluza-Klein mass along the S^1 direction.

Let us follow the same analysis we carried for conformal scalars on hyperbolic cylinders. The eigen values of the Hodge-deRham operator acting on co-exact p -forms are given by

$$\Delta_p \psi_{i_1 i_2 \dots i_p}^{\{\lambda, u\}} = -\lambda^2 \psi_{i_1 i_2 \dots i_p}^{\{\lambda, u\}}. \quad (4.32)$$

Here $\{u\}$ refer to other quantum numbers on AdS_d and ψ refers to the eigen functions. Using these eigen values and the Fourier expansion on S^1 , the partition function can be written as

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \int d\lambda \mu_p^{(d)}(\lambda) \log \left(\frac{n^2}{q^2} + \lambda^2 \right). \quad (4.33)$$

Following the same steps as in equations (4.7) to (4.12) we get

$$\log \mathcal{Z}[S_q^1 \times AdS_d] = \frac{1}{2\pi i} \log \tilde{R} \int_{C_o} \frac{du}{2u} \frac{1 + e^{-\frac{u}{q}}}{1 - e^{-\frac{u}{q}}} W_p^{(d)}(u), \quad (4.34)$$

where $W_p^{(d)}(u)$ is the Fourier transform of the Plancherel measure of the $p = \frac{d-1}{2}$ -form in AdS_d . From (3.24) we see that this is given by

$$W_p^{(d)}(u) = \frac{1 + e^{-u}}{1 - e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-1-2i}{p-i} \frac{e^{-u(d-1-p-i)}}{(1 - e^{-u})^{d-1-2i}}, \quad p = \frac{d-1}{2} \quad (4.35)$$

Here we have replaced $d \rightarrow d-1$ in (3.24) and the exponential factor on the l.h.s. becomes trivial since $p = \frac{d-1}{2}$. Now remarkably the equation in (4.35) can be re-written as

$$\begin{aligned} W_p^{(d)}(u) &= \frac{1 + e^{-u}}{1 - e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-1-2i}{p-i} \frac{e^{-u(d-1-p-i)}}{(1 - e^{-u})^{d-1-2i}}, \\ &= \sum_{i=0}^p (-1)^i \binom{d}{p-i} \frac{e^{-u(p-i)} + e^{-u(d-p+i)}}{(1 - e^{-u})^d}, \quad p = \frac{d-1}{2} \end{aligned} \quad (4.36)$$

What is important to note is that in the second line consists of sum of $SO(1, d + 1)$ Harish-Chandra characters of the same dimension d . Substituting this expression into (4.34) we obtain

$$\begin{aligned} \log \mathcal{Z}[S_q^1 \times AdS_d] &= \frac{1}{2\pi i} \log \tilde{R} \int_{C_o} \frac{du}{2u} \frac{1 + e^{-\frac{u}{q}}}{1 - e^{-\frac{u}{q}}} \sum_{i=0}^p (-1)^i \binom{d}{p-i} \frac{e^{-u(p-i)} + e^{-u(d-p+i)}}{(1 - e^{-u})^d}, \\ &= \int_{C_o} \frac{du}{2u} \frac{1 + e^{-\frac{u}{q}}}{1 - e^{-\frac{u}{q}}} \sum_{i=0}^p (-1)^p \chi_{(d,p-i)}^{dS}(u), \quad p = \frac{d-1}{2} \end{aligned} \quad (4.37)$$

From the last line we see that at $q = 1$ the gauge invariant partition function of the conformal p -form on $S_q^1 \times AdS_d$ can be written as a character integral of only the bulk characters of the partition function of the p forms on the sphere S^{d+1} . That is consider the determinant of all the co-exact forms that appear in the one loop partition function in (2.1). Then only the bulk character of each of the p -form appears in (4.37), the edge modes are missing. This behaviour was observed in the study of the entanglement entropies of the Maxwell in $d + 1 = 4$ [14–16, 28, 29] field as well as the 2 form in $d + 1 = 6$ [17–19] and seen by Partition functions on the corresponding hyperbolic cylinders miss the edge modes. Here we see that this is true for conformal forms in all even dimensions and it can be seen at the level of the integrand in the character integral representation.

Thus entanglement entropy across spherical entangling surfaces evaluated using partition functions on hyperbolic cylinders miss out the edge modes. In [35, 36] it was shown that the edge modes of the 1-form in $d + 1 = 4$ dimensions contribute only to the classical part of the entanglement entropy is non-extractable. It will be interesting to repeat this analysis for p -forms and demonstrate that the edge contributions are classical and non-extractable.

4.3 p -forms on branched spheres

In section 4.1 we derived the character integral representation of the one loop partition function of the conformal scalar on branched spheres. In this section we wish to generalise that computation to arbitrary p -forms. For this purpose we would need the eigen values and the corresponding degeneracies of the Hodge-deRham Laplacian on branched spheres. As far as we are aware it is only for the 1-form, these properties are known explicitly in arbitrary dimensions. Though there are generating functions for degeneracies, from which one can possibly obtain the degeneracies for other p -forms [18]. Therefore we will first focus on the co-exact 1-form in arbitrary dimensions and obtain the character integral representation of the one loop partition function. From this result we propose the character integral for arbitrary co-exact p -forms on branched spheres. We will demonstrate that the proposal agrees with earlier evaluations of the partition functions of p -forms on branched spheres.

1-form. From [31] we see that the eigen values of the Hodge-deRham Laplacian of the 1-form on the branched sphere S_q^{d+1} are labelled by 2 integers and are given by

$$\lambda_{n,m}^{(1)} = \left(n + \frac{m}{q}\right) \left(n + \frac{m}{q} + d\right) - 1 + d, \quad \text{with } n + m \geq 1, \quad n, m \in \{0, 1, 2, \dots\} \quad (4.38)$$

The result for the eigen values in [31] was written for the vector Laplacian, but we have shifted it by d which arises due to the curvature terms relating the Hodge-deRham Laplacian and the usual Laplacian. These eigen values have degeneracies

$$g_{n,m=0}^{(1)} = \frac{1}{n+1} \binom{d+n-2}{n-1} \{(d+1)(n-4) + (d+1)^2 - n + 5\}, \quad n = 1, 2, \dots, \quad (4.39)$$

$$g_{n,m}^{(1)} = 2d \binom{d+n-1}{d-1}, \quad n = 0, 1, \dots \quad m = 1, 2, \dots$$

The partition function is therefore given by

$$-\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} = -\frac{1}{2} \sum_{n,m=0}^{\infty} g_{n,m}^{(1)} \log(\lambda_{n,m}^{(1)}). \quad (4.40)$$

Again using the identity (2.4) we rewrite the partition function as

$$-\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} = \int_0^{\infty} \frac{d\tau}{2\tau} \sum_{n,m=0}^{\infty} g_{n,m}^{(1)} (e^{-\tau \lambda_{n,m}^{(1)}} - e^{-\tau}). \quad (4.41)$$

Now the second term involves the sum over degeneracies. To regulate that term, we look at sufficiently negative d and perform the sum over m using zeta function regularization. We obtain

$$\sum_{n=1}^{\infty} \sum g_{n,m=0}^{(1)} = 1, \quad \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} g_{n,m}^{(1)} = 0. \quad (4.42)$$

Therefore the equation (4.41) reduces to

$$-\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} = \int_0^{\infty} \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \left(\sum_{n,m=0}^{\infty} g_{n,m}^{(1)} - 1 \right), \quad (4.43)$$

Note that the above equation is similar to (2.9) with $p = 1$. In fact the second term in (4.43) can be absorbed in the first term by noting from (4.39) that

$$g_{0,m=0}^{(1)} = 0, \quad g_{-1,m=0}^{(1)} = 1, \quad \lambda_{-1,m=0}^{(1)} = 0. \quad (4.44)$$

Therefore we rewrite (4.43) as

$$-\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} = \int_0^{\infty} \frac{d\tau}{2\tau} e^{-\frac{\epsilon^2}{4\tau}} \left(\sum_{n=-1}^{\infty} g_{n,m=0}^{(1)} e^{-\lambda_{n,m=0}^{(1)} \tau} + \sum_{n=0,m>0}^{\infty} g_{n,m}^{(1)} e^{-\lambda_{n,m}^{(1)} \tau} \right). \quad (4.45)$$

We can now follow the same steps as in (2.11) to (2.20) to obtain

$$-\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} = \int_{\epsilon}^{\infty} \frac{dt}{2\sqrt{t^2 - \epsilon^2}} \left(e^{(\frac{d}{2}-1)\sqrt{t^2 - \epsilon^2}} + e^{-(\frac{d}{2}-1)\sqrt{t^2 - \epsilon^2}} \right) (f_1(it) + f_2(it)), \quad (4.46)$$

where

$$f_1(u) = \sum_{n=-1}^{\infty} g_{n,m=0}^{(1)} e^{iu((n+\frac{d}{2}))} = \frac{e^{\frac{1}{2}i(d-2)u} (de^{iu} + e^{2iu} - 1)}{(1 - e^{iu})^d}, \quad (4.47)$$

$$f_2(u) = \sum_{m=1,n=0}^{\infty} g_{n,m}^{(1)} e^{iu((n+\frac{m}{q}+\frac{d}{2}))} = \frac{2de^{\frac{iu(dq+2)}{2q}}}{(1 - e^{iu})^d (1 - e^{\frac{iu}{q}})}. \quad (4.48)$$

$d+1$	$p=0$	$d+1$	$p=1$
2	$\frac{q^2+1}{6q}$	2	-2
4	$\frac{57q^4+60q^2-1}{360q^3}$	4	$-\frac{33q^4+30q^2+1}{180q^3} - \frac{1}{3}$
6	$\frac{4315q^6+5040q^4-245q^2+2}{30240q^5}$	6	$\frac{2-5q^2(271q^4+252q^2+7)}{7560q^5} - \frac{29}{90}$
8	$\frac{7q^2(33953q^6+43200q^4-3248q^2+100)-3}{1814400q^7}$	8	$-\frac{7(7297q^6+7200q^4+98q^2-40)q^2+3}{302400q^7} - \frac{1139}{3780}$
$d+1$	$p=2$	$d+1$	$p=3$
4	$\frac{57q^4+60q^2-1}{360q^3} + 2$	4	-4
6	$\frac{955q^6+840q^4+35q^2+2}{5040q^5} + \frac{31}{45}$	6	$\frac{2-5q^2(271q^4+252q^2+7)}{7560q^5} - \frac{209}{90}$
8	$\frac{7q^2(3233q^6+2880q^4+112q^2+4)-3}{120960q^7} + \frac{1271}{1890}$	8	$-\frac{7(2497q^6+2160q^4+98q^2+8)q^2+3}{90720q^7} - \frac{221}{210}$
$d+1$	$p=4$	$d+1$	$p=5$
6	$\frac{4315q^6+5040q^4-245q^2+2}{30240q^5} + 4$	6	-6
8	$\frac{7q^2(3233q^6+2880q^4+112q^2+4)-3}{120960q^7} + \frac{5051}{1890}$	8	$-\frac{7(7297q^6+7200q^4+98q^2-40)q^2+3}{302400q^7} - \frac{16259}{3780}$

Table 5. Coefficient of the logarithmic divergence of the partition function of p -forms on branched spheres in even dimension using (4.50).

Substituting these expressions in (4.46), rearranging the terms and taking the $\epsilon \rightarrow 0$ limit, one loop determinant of the co-exact 1-form on branched spheres S_q^{d+1} can be written as

$$\begin{aligned}
 -\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} &= \int_0^\infty \frac{dt}{2t} \left\{ \frac{1 + e^{-\frac{t}{q}}}{1 - e^{-\frac{t}{q}}} \left[\binom{d}{1} \frac{e^{-t} + e^{-(d-1)t}}{(1 - e^{-t})^d} \right] \right. \\
 &\quad \left. - \frac{1 + e^{-t}}{1 - e^{-t}} \left[\binom{d-2}{0} \frac{1 + e^{-(d-2)t}}{(1 - e^{-t})^{d-2}} \right] \right\} \\
 &= \int_0^\infty \frac{dt}{2t} \left\{ \frac{1 + e^{-\frac{t}{q}}}{1 - e^{-\frac{t}{q}}} \chi_{(d,1)}^{dS}(t) - \frac{1 + e^{-t}}{1 - e^{-t}} \chi_{(d-2,0)}^{dS}(t) \right\}. \tag{4.49}
 \end{aligned}$$

Proposal for the determinant of co-exact p -forms on branched spheres. From the explicit calculation of the determinant of the 1-form in (4.49), we see that it is only the kinematic factor of the bulk character which acquires dependence of the branching parameter q for spheres S^{d+1} . The kinematic factor of the edge character is blind to the branching. Using this input, we propose that the determinant of co-exact p -forms on branched spheres is given by

$$-\frac{1}{2} \log \det_T \Delta_1^{S_q^{d+1}} = \int_0^\infty \frac{dt}{2t} \left\{ \frac{1 + e^{-\frac{t}{q}}}{1 - e^{-\frac{t}{q}}} \chi_{(d,2p)}^{[dS]}(t) + \frac{1 + e^{-t}}{1 - e^{-t}} \sum_{i=1}^p (-1)^i \chi_{(d-2i,p-i)}^{[dS]} \right\}. \tag{4.50}$$

Using the determinant of the co-exact p -form on the branched sphere, one can evaluate the partition function of the p -form by using this input in the ghosts for ghosts expression of (2.1). In table 5 we have listed the coefficient of the logarithmic divergence for p -form partition functions on branched spheres using the proposal (4.50) in (2.1).

As far as we are aware there are few explicit calculations of one loop determinants of p -forms on branched spheres with $p > 1$. In even $d + 1$, [18] has put forward a method to evaluate these partition functions. It does not rely on the explicit knowledge of the degeneracies of the Hodge-deRham Laplacian. As a check of the proposal in (4.50) we have compared the coefficient of the logarithmic divergence of the partition functions in table 5 for $p = 2$ in $d + 1 = 4, 6$, $p = 3$ in $d + 1 = 6, 8$ to that given in equation (13) of [18]. Our values coincides with that of [18] upon identification of $q_{\text{ours}} \rightarrow \frac{1}{q_{\text{Dowker}}}$. It is interesting to note that q deformation does not change the Hodge-duality properties of the p -forms. For instance the $(0, 2)$ pair on S_q^4 differ by -2 . Similarly the $(1, 3)$ and $(2, 4)$ pair on S_q^6 differs by 2 and -2 respectively.

5 Conclusions

In this paper we have generalized the construction given in [1, 8] of the one loop partition function on spheres and anti-de Sitter space in terms of Harish-Chandra characters to p -forms. We have also seen how character integral representations make relations between partition functions manifest. For instance the equivalence of the conformal scalar partition function on the hyperbolic cylinder and the branched sphere was manifest from the character integral representations. It also provided insights in entanglement entropy for conformal p -forms.

In [8] the character integral representation was useful to evaluate the partition function of higher-spin Vasiliev theories. In this context it would be important to obtain character integral representations of fermionic higher spin fields. This would enable the revisiting various one loop calculations in the literature. Some examples of these are one loop calculations in $AdS_4 \times S^7$ done by [37] or that done around black holes with near horizon geometry $AdS_2 \times S^2$ in [38, 39]. Partition functions of supersymmetric higher spin theories can also be evaluated. The character integral representation enables writing down the partition function once the field content and the mass spectrum of the theory is known. Therefore such partition functions can be obtained with considerable ease.

The representation of the partition function of conformal p -forms in terms of Harish-Chandra characters on the hyperbolic cylinder has allowed us to show that partition function on hyperbolic cylinders capture the bulk contribution to the entanglement entropy. In [35, 36] it was shown for the 1-form in 4 dimensions the contribution of the edge modes to the entanglement entropy are classical and non-extractable. It would be interesting to repeat this analysis for the arbitrary p -form. A similar question can be addressed for gravitons. In [19] it was shown that the entanglement entropy of linearised gravitons or higher spin fields in $d + 1 = 4$ evaluated using the hyperbolic cylinder method and that from the branched sphere differ by the partition function on the sphere S^2 which constitute the edge modes. Here too it would be interesting to show that these modes are non-extractable or classical.

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A Harish-Chandra characters

Here we briefly list the various Harish-Chandra characters used in the paper. For more details see [1, 8, 22].

AdS characters. The easiest to understand are the $SO(2, d)$ characters associated with the minimally scalars on AdS_{d+1} . Let the mass of the scalar in AdS_{d+1} be m , The scaling dimension or the value of the Cartan H in a $SO(2, 1)$ of the lowest weight of this representation is given by

$$\Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}. \tag{A.1}$$

Then the Harish-Chandra character of this massive representation is given by

$$\chi_{(d,0)m}^{\text{AdS}}(t) = \text{Tr}_{\mathcal{H}_{\Delta_+}}(e^{-Ht}) = \frac{e^{-t\Delta_+}}{(1 - e^{-t})^d}. \tag{A.2}$$

Here the trace is taken over \mathcal{H}_{Δ_+} all in the representation. For simplicity let us take t real and $t > 0$. Setting $m = 0$ in (A.2) we obtain

$$\chi_{(d,0)}^{\text{AdS}}(t) = \frac{e^{-td}}{(1 - e^{-t})^d}. \tag{A.3}$$

We call this the character of the 0 form.

Consider the co-exact p -form which satisfy the equations of motion

$$(\Delta_T + m^2)A_{i_1, \dots, i_p} = 0, \tag{A.4}$$

where Δ_T is the Hodge-deRham Laplacian on AdS_{d+1} . The scaling dimension of the lowest weight of this representation is given by [40]

$$\Delta_+ = \frac{d}{2} + \sqrt{\left(\frac{d}{2} - p\right)^2 + m^2}. \tag{A.5}$$

The Harish-Chandra character of this representation is given by

$$\chi_{(d,p)m}^{\text{AdS}}(t) = \binom{d}{p} \frac{e^{-t\Delta_+}}{(1 - e^{-t})^d}. \tag{A.6}$$

Setting $m^2 = 0$ in this expression we obtain

$$\chi_{(d,p)}^{\text{AdS}}(t) = \binom{d}{p} \frac{e^{-t(d-p)}}{(1 - e^{-t})^d}. \tag{A.7}$$

Here we would like to mention that though we have obtain the above expression by setting $m = 0$ in the Harish-Chandra character corresponding to the massive representation it does not correspond to the characters of the unitary irreducible representations belonging to the exceptional series. In this paper we will loosely refer to the above limit of the massive character of the p form as a Harish-Chandra character. This character forms the basic building blocks of all the character integral representations for p -forms on AdS-spaces.

dS characters. Let us discuss more about the de Sitter group $SO(1, d + 1)$. Consider a minimally coupled scalar of mass

$$m^2 = \frac{d^2}{4} + \nu^2. \tag{A.8}$$

Then the corresponding Harish-Chandra character of this representation is given by

$$\chi_{(d,0)\nu}^{dS}(t) = \frac{e^{-t\Delta_+} + e^{-t\Delta_-}}{(1 - e^{-t})^d}, \quad \Delta_{\pm} = \frac{d}{2} \pm i\nu. \tag{A.9}$$

Setting $m = 0$ we obtain

$$\chi_{(d,0)}^{dS}(t) = \frac{e^{-td} + 1}{(1 - e^{-t})^d}, \tag{A.10}$$

As mentioned for the AdS case we will still refer to the above character as Harish-Chandra character though it does not correspond to a character of any UIR. While the conformally coupled scalar on S^{d+1} has mass

$$m_{\text{conf}}^2 = \frac{d^2 - 1}{4}, \quad i\nu_{\text{conf}} = \frac{1}{2}. \tag{A.11}$$

Therefore its character becomes

$$\chi_{(d,0)\text{conf}}^{dS}(t) = \frac{e^{-\frac{d-1}{2}t} + e^{-\frac{d+1}{2}t}}{(1 - e^{-t})^d}. \tag{A.12}$$

Finally consider the p -form with satisfies the equation (A.5) on S^{d+1} . Here we define ν such that

$$m^2 = \left(\frac{d}{2} - p\right)^2 + \nu^2. \tag{A.13}$$

The character of corresponding to the p -form is given by

$$\chi_{(d,p)\nu}^{dS}(t) = \binom{d}{p} \frac{e^{-t\Delta_+} + e^{-t\Delta_-}}{(1 - e^{-t})^d}, \tag{A.14}$$

where ν is obtained by solving (A.13). Setting $m = 0$ in (A.14) we obtain

$$\chi_{(d,p)}^{dS}(t) = \binom{d}{p} \frac{e^{-t(d-p)} + e^{-tp}}{(1 - e^{-t})^d}. \tag{A.15}$$

Again we wish to emphasise that in this paper we will still refer to the above expression as a Harish-Chandra character for convenience though it does not correspond to any character of UIR of massless p -forms. The building blocks of the integral representations for the partition functions of p -forms on spheres are the characters in (A.15). The reason the characters of the de Sitter group appear though we are on the sphere is due to the fact that the Wick rotated de Sitter space in the static patch is a sphere.

B $W_p(u)$ by direct Fourier transform

In this appendix we evaluate the Fourier transform of the Plancherel measure $\mu_p(\lambda)$ directly by re-writing the transform as a sum of residues. This can be done for $d+1$ even. We will see that the result agrees with (3.24) obtain by the differential equation method developed in the main text. We start with the Fourier transform.

$$W_p(u) = \int_{-\infty}^{\infty} d\lambda e^{i\lambda u} \mu_p(\lambda). \quad (\text{B.1})$$

We can choose $u > 0$ and close the contour to be the large semi-circle in the upper half plane. Therefore the integral reduces to the sum over residues at $\lambda = i(n + \frac{1}{2})$ and for $u < 0$ we sum over residues at $\lambda = -i(n + \frac{1}{2})$. The result is analytic in u . Let us choose the upper half plane to perform the integration. We obtain

$$\begin{aligned} W_p &= \sum_{n=0}^{\infty} \text{Res} \left(e^{i\lambda u} \mu_p(\lambda) \right) \Big|_{\lambda=i(n+\frac{1}{2})} \\ &= \sum_{n=0}^{\infty} - \frac{\left((2n+1)\Gamma(d+1)e^{-(n+\frac{1}{2})u} \prod_{j=\frac{1}{2}}^d \left(j^2 + \left(i \left(n + \frac{1}{2} \right) \right)^2 \right) \right)}{\left(\left(\frac{d}{2} - p \right)^2 + \left(i \left(n + \frac{1}{2} \right) \right)^2 \right) (\Gamma(p+1)\Gamma(d+1-p))} \\ &= e^{(\frac{d}{2}-p)u} \frac{1+e^{-u}}{1-e^{-u}} \sum_{i=0}^p (-1)^i \binom{d-2i}{p-i} \frac{e^{-u(d-p-i)}}{(1-e^{-u})^{d-2i}}. \end{aligned} \quad (\text{B.2})$$

Observe that this result coincides with (3.24) obtained using the differential equation method. It is analytic in p, d and therefore can be extended to all values of p, d .

C Massless symmetric traceless tensors of rank spin s

The partition function of massless symmetric spin s field on S^{d+1} is given by [41]

$$\mathcal{Z}_s = \left(\frac{-\Delta_{(s-1)\perp} - (s-1)(s+d-2)}{-\Delta_{(s)\perp} + s - (s-2)(s+d-2)} \right)^{\frac{1}{2}} \quad (\text{C.1})$$

Here $\Delta_{(s)\perp}$ refers to the Laplacian on S^{d+1} acting on the transverse traceless spin s field. The mass term arises due to the curvature coupling. The eigen-value and the degeneracy is given by

$$\begin{aligned} \lambda_n^{(s)} &= n(n+d) - s \\ g_n^{(s)} &= \frac{(d+2n)(d+2s-2)(n-s+1)(d+n-2)!(d+s-3)!(d+n+s-1)}{(d-2)!d!(n+1)!s!} \end{aligned} \quad (\text{C.2})$$

Using the same procedure followed in section 2, the free energy can be written as

$$\begin{aligned} -\log \mathcal{Z}_s &= \int_0^\infty \frac{d\tau}{2\tau} \left[\left(\sum_{n=s}^{\infty} g_n^{(s)} (e^{-\tau(\lambda_n^{(s)} + s - (s-2)(s+d-2))} - e^{-\tau}) \right) \right. \\ &\quad \left. - \left(\sum_{n=s-1}^{\infty} g_n^{(s-1)} (e^{-\tau(\lambda_n^{(s-1)} - (s-1)(s+d-2))} - e^{-\tau}) \right) \right]. \end{aligned} \quad (\text{C.3})$$

s	$g_n^{(s)}$	$\sum_{n=s}^{\infty} g_n^{(s)}$
0	$\frac{(d+2n)\Gamma(d+n)}{d!n!}$	0
1	$\frac{n(d+n)(d+2n)\Gamma(d+n-1)}{\Gamma(d)\Gamma(n+2)}$	1
2	$\frac{(d-1)(d+2)(n-1)(d+n+1)(d+2n)\Gamma(d+n-1)}{2d!\Gamma(n+2)}$	$\frac{1}{2}(d+2)(d+3)$
3	$\frac{(d+4)(n-2)(d+n+2)(d+2n)\Gamma(d+n-1)}{6\Gamma(d-1)\Gamma(n+2)}$	$\frac{1}{12}d(d+3)(d+4)(d+5)$
4	$\frac{(d+1)(d+6)(n-3)(d+n+3)(d+2n)\Gamma(d+n-1)}{24\Gamma(d-1)\Gamma(n+2)}$	$\frac{1}{144}d(d+1)(d+2)(d+5)(d+6)(d+7)$

Table 6. Examples illustrating $\sum_{n=s}^{\infty} g_n^{(s)} = -\sum_{n=-1}^{s-1} g_n^{(s)}$.

The sum over the degeneracy $g_n^{(s)}$ can again be performed by working at sufficiently negative d and analytically continuing the result to positive d . Note that the large n behaviour of the degeneracy is

$$g_n^{(s)} = \left(\frac{1}{n}\right)^{-d} \left(\frac{2(d+2s-4)(d+s-5)!}{(d-4)!(d-2)!s!n^2} + O\left(\left(\frac{1}{n}\right)^{5/2}\right) \right) \quad (C.4)$$

Therefore the sum $\sum_{n=s}^{\infty} g_n^{(s)}$ converges in the large negative value of $d \leq -2$. We evaluate the sum and observe that

$$\begin{aligned} \sum_{n=s}^{\infty} g_n^{(s)} &= \frac{(d+2s-3)(d+2s-2)(d+2s-1)\Gamma(d+s-2)\Gamma(d+s-1)}{s!\Gamma(d)\Gamma(d+2)\Gamma(s)} \\ &= -\sum_{n=-1}^{s-1} g_n^{(s)} \end{aligned} \quad (C.5)$$

This remarkable relation allows us to extend the sum from $n = -1$ to $n = \infty$. Just to be explicit, we present the sum of degeneracy for few cases in a table 6.

Due to the relation (C.5) we can re-write (C.3) as

$$\begin{aligned} -\log \mathcal{Z}_s &= \int_0^\infty \frac{d\tau}{2\tau} \left[\left(\sum_{n=-1}^{\infty} g_n^{(s)} (e^{-\tau(\lambda_n^{(s)} + s - (s-2)(s+d-2))}) \right) \right. \\ &\quad \left. - \left(\sum_{n=-1}^{\infty} g_n^{(s-1)} (e^{-\tau(\lambda_n^{(s-1)} - (s-1)(s+d-2))}) \right) \right]. \end{aligned} \quad (C.6)$$

We now perform the Hubbard-Stratonovich trick given in (2.13) and follow the same steps as in equation (2.11) to (2.20) and obtain

$$\begin{aligned} \log \mathcal{Z}_s &= \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \left[\left(e^{i\nu_s\sqrt{t^2-\epsilon^2}} + e^{-i\nu_s\sqrt{t^2-\epsilon^2}} \right) f^{(s)}(it) \right. \\ &\quad \left. + \left(e^{i\nu'_{s-1}\sqrt{t^2-\epsilon^2}} + e^{-i\nu'_{s-1}\sqrt{t^2-\epsilon^2}} \right) f^{(s-1)}(it) \right] \end{aligned} \quad (C.7)$$

with $\nu_s = i(s + \frac{d-4}{2})$, $\nu'_{s-1} = i(s + \frac{d-2}{2})$ and

$$f^{(s)}(u) = \sum_{n=-1}^{\infty} g_n^{(s)} e^{iu(n+\frac{d}{2})} \quad (C.8)$$

Substituting (C.8) in (C.7) and taking the limit $\epsilon \rightarrow 0$ we obtain

$$\log \mathcal{Z}_s = \int_0^\infty \frac{dt}{2t} \left[\sum_{n=-1}^\infty g_n^{(s)} (e^{-t(n-s+2)} + e^{-t(n+s+d-2)}) - \sum_{n=-1}^\infty g_n^{(s-1)} (e^{-t(n+s+d-1)} + e^{-t(n-s+1)}) \right] \quad (\text{C.9})$$

This agrees with the (G.9) of [1]. Therefore we obtain directly the ‘naive’ character of massless symmetric rank s tensor on S^{d+1} . Note that first two terms correspond in (C.9) correspond to the one loop determinant of the transverse traceless spin s field. It does not have any non-local terms just as in the case of the co-exact p -form discussed in section 2.

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References

- [1] D. Anninos, F. Denef, Y.T.A. Law and Z. Sun, *Quantum de Sitter horizon entropy from quasicanonical bulk, edge, sphere and topological string partition functions*, [arXiv:2009.12464](https://arxiv.org/abs/2009.12464) [[INSPIRE](#)].
- [2] J.R. David, M.R. Gaberdiel and R. Gopakumar, *The Heat Kernel on AdS_3 and its Applications*, *JHEP* **04** (2010) 125 [[arXiv:0911.5085](https://arxiv.org/abs/0911.5085)] [[INSPIRE](#)].
- [3] J.-B. Bae, E. Joung and S. Lal, *One-loop test of free $SU(N)$ adjoint model holography*, *JHEP* **04** (2016) 061 [[arXiv:1603.05387](https://arxiv.org/abs/1603.05387)] [[INSPIRE](#)].
- [4] T. Basile, E. Joung, S. Lal and W. Li, *Character Integral Representation of Zeta function in AdS_{d+1} : I. Derivation of the general formula*, *JHEP* **10** (2018) 091 [[arXiv:1805.05646](https://arxiv.org/abs/1805.05646)] [[INSPIRE](#)].
- [5] T. Basile, E. Joung, S. Lal and W. Li, *Character integral representation of zeta function in AdS_{d+1} . Part II. Application to partially-massless higher-spin gravities*, *JHEP* **07** (2018) 132 [[arXiv:1805.10092](https://arxiv.org/abs/1805.10092)] [[INSPIRE](#)].
- [6] M. Beccaria, X. Bekaert and A.A. Tseytlin, *Partition function of free conformal higher spin theory*, *JHEP* **08** (2014) 113 [[arXiv:1406.3542](https://arxiv.org/abs/1406.3542)] [[INSPIRE](#)].
- [7] E.D. Skvortsov and T. Tran, *AdS/CFT in Fractional Dimension and Higher Spin Gravity at One Loop*, *Universe* **3** (2017) 61 [[arXiv:1707.00758](https://arxiv.org/abs/1707.00758)] [[INSPIRE](#)].
- [8] Z. Sun, *AdS one-loop partition functions from bulk and edge characters*, [arXiv:2010.15826](https://arxiv.org/abs/2010.15826) [[INSPIRE](#)].
- [9] Y.N. Obukhov, *The geometrical approach to antisymmetric tensor field theory*, *Phys. Lett. B* **109** (1982) 195 [[INSPIRE](#)].
- [10] E.J. Copeland and D.J. Toms, *Quantized Antisymmetric Tensor Fields and Selfconsistent Dimensional Reduction in Higher Dimensional Space-times*, *Nucl. Phys. B* **255** (1985) 201 [[INSPIRE](#)].
- [11] A. Cappelli and G. D’Appollonio, *On the trace anomaly as a measure of degrees of freedom*, *Phys. Lett. B* **487** (2000) 87 [[hep-th/0005115](https://arxiv.org/abs/hep-th/0005115)] [[INSPIRE](#)].

- [12] I.R. Klebanov, S.S. Pufu, S. Sachdev and B.R. Safdi, *Rényi Entropies for Free Field Theories*, *JHEP* **04** (2012) 074 [[arXiv:1111.6290](#)] [[INSPIRE](#)].
- [13] T. Nishioka and Y. Sato, *Free energy and defect C-theorem in free scalar theory*, *JHEP* **05** (2021) 074 [[arXiv:2101.02399](#)] [[INSPIRE](#)].
- [14] K.-W. Huang, *Central Charge and Entangled Gauge Fields*, *Phys. Rev. D* **92** (2015) 025010 [[arXiv:1412.2730](#)] [[INSPIRE](#)].
- [15] W. Donnelly and A.C. Wall, *Entanglement entropy of electromagnetic edge modes*, *Phys. Rev. Lett.* **114** (2015) 111603 [[arXiv:1412.1895](#)] [[INSPIRE](#)].
- [16] W. Donnelly and A.C. Wall, *Geometric entropy and edge modes of the electromagnetic field*, *Phys. Rev. D* **94** (2016) 104053 [[arXiv:1506.05792](#)] [[INSPIRE](#)].
- [17] J. Nian and Y. Zhou, *Rényi entropy of a free (2,0) tensor multiplet and its supersymmetric counterpart*, *Phys. Rev. D* **93** (2016) 125010 [[arXiv:1511.00313](#)] [[INSPIRE](#)].
- [18] J.S. Dowker, *Rényi entropy and C_T for p -forms on even spheres*, [arXiv:1706.04574](#) [[INSPIRE](#)].
- [19] J.R. David and J. Mukherjee, *Hyperbolic cylinders and entanglement entropy: gravitons, higher spins, p -forms*, *JHEP* **01** (2021) 202 [[arXiv:2005.08402](#)] [[INSPIRE](#)].
- [20] S. Giombi, I.R. Klebanov and G. Tarnopolsky, *Conformal QED_d , F -Theorem and the ϵ Expansion*, *J. Phys. A* **49** (2016) 135403 [[arXiv:1508.06354](#)] [[INSPIRE](#)].
- [21] H. Raj, *A note on the sphere free energy of p -form gauge theory and Hodge duality*, *Class. Quant. Grav.* **34** (2017) 247001 [[arXiv:1611.02507](#)] [[INSPIRE](#)].
- [22] T. Basile, X. Bekaert and N. Boulanger, *Mixed-symmetry fields in de Sitter space: a group theoretical glance*, *JHEP* **05** (2017) 081 [[arXiv:1612.08166](#)] [[INSPIRE](#)].
- [23] T. Hirai, *The characters of irreducible representations of the Lorentz group of n -th order*, *Proc. Japan. Acad.* **41** (1965) 526.
- [24] R. Camporesi and A. Higuchi, *The Plancherel measure for p -forms in real hyperbolic spaces*, *J. Geom. Phys.* **15** (1994) 57.
- [25] L.S. Brown and J.P. Cassidy, *Stress Tensors and their Trace Anomalies in Conformally Flat Space-Times*, *Phys. Rev. D* **16** (1977) 1712 [[INSPIRE](#)].
- [26] C.P. Herzog and K.-W. Huang, *Stress Tensors from Trace Anomalies in Conformal Field Theories*, *Phys. Rev. D* **87** (2013) 081901 [[arXiv:1301.5002](#)] [[INSPIRE](#)].
- [27] H. Casini, M. Huerta and R.C. Myers, *Towards a derivation of holographic entanglement entropy*, *JHEP* **05** (2011) 036 [[arXiv:1102.0440](#)] [[INSPIRE](#)].
- [28] H. Casini, M. Huerta and J.A. Rosabal, *Remarks on entanglement entropy for gauge fields*, *Phys. Rev. D* **89** (2014) 085012 [[arXiv:1312.1183](#)] [[INSPIRE](#)].
- [29] H. Casini and M. Huerta, *Entanglement entropy of a Maxwell field on the sphere*, *Phys. Rev. D* **93** (2016) 105031 [[arXiv:1512.06182](#)] [[INSPIRE](#)].
- [30] L.-Y. Hung, R.C. Myers and M. Smolkin, *Twist operators in higher dimensions*, *JHEP* **10** (2014) 178 [[arXiv:1407.6429](#)] [[INSPIRE](#)].
- [31] L. De Nardo, D.V. Fursaev and G. Miele, *Heat kernel coefficients and spectra of the vector Laplacians on spherical domains with conical singularities*, *Class. Quant. Grav.* **14** (1997) 1059 [[hep-th/9610011](#)] [[INSPIRE](#)].

- [32] Y. Sato, *Free energy and defect C-theorem in free fermion*, *JHEP* **05** (2021) 202 [[arXiv:2102.11468](#)] [[INSPIRE](#)].
- [33] H. Casini and M. Huerta, *Entanglement entropy for the n-sphere*, *Phys. Lett. B* **694** (2011) 167 [[arXiv:1007.1813](#)] [[INSPIRE](#)].
- [34] I.R. Klebanov, S.S. Pufu and B.R. Safdi, *F-Theorem without Supersymmetry*, *JHEP* **10** (2011) 038 [[arXiv:1105.4598](#)] [[INSPIRE](#)].
- [35] R.M. Soni and S.P. Trivedi, *Entanglement entropy in (3 + 1)-d free U(1) gauge theory*, *JHEP* **02** (2017) 101 [[arXiv:1608.00353](#)] [[INSPIRE](#)].
- [36] U. Moitra, R.M. Soni and S.P. Trivedi, *Entanglement Entropy, Relative Entropy and Duality*, *JHEP* **08** (2019) 059 [[arXiv:1811.06986](#)] [[INSPIRE](#)].
- [37] S. Bhattacharyya, A. Grassi, M. Mariño and A. Sen, *A One-Loop Test of Quantum Supergravity*, *Class. Quant. Grav.* **31** (2014) 015012 [[arXiv:1210.6057](#)] [[INSPIRE](#)].
- [38] S. Banerjee, R.K. Gupta, I. Mandal and A. Sen, *Logarithmic Corrections to N = 4 and N = 8 Black Hole Entropy: A One Loop Test of Quantum Gravity*, *JHEP* **11** (2011) 143 [[arXiv:1106.0080](#)] [[INSPIRE](#)].
- [39] A. Sen, *Logarithmic Corrections to N = 2 Black Hole Entropy: An Infrared Window into the Microstates*, *Gen. Rel. Grav.* **44** (2012) 1207 [[arXiv:1108.3842](#)] [[INSPIRE](#)].
- [40] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [41] S. Giombi, I.R. Klebanov and A.A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin AdS_{d+1}/CFT_d*, *Phys. Rev. D* **90** (2014) 024048 [[arXiv:1402.5396](#)] [[INSPIRE](#)].