Integrable deformed $T^{1,1}$ sigma models from 4D Chern-Simons theory

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ABSTRACT: Recently, a variety of deformed $T^{1,1}$ manifolds, with which 2D non-linear sigma models (NLSMs) are classically integrable, have been presented by Arutyunov, Bassi and Lacroix (ABL) [46]. We refer to the NLSMs with the integrable deformed $T^{1,1}$ as the ABL model for brevity. Motivated by this progress, we consider deriving the ABL model from a 4D Chern-Simons (CS) theory with a meromorphic one-form with four double poles and six simple zeros. We specify boundary conditions in the CS theory that give rise to the ABL model and derive the sigma-model background with target-space metric and anti-symmetric two-form. Finally, we present two simple examples 1) an anisotropic $T^{1,1}$ model and 2) a G/H λ-model. The latter one can be seen as a one-parameter deformation of the Guadagnini-Martellini-Mintchev model.

Contents

1 Introduction

A significant subject in String Theory is the integrability in the AdS/CFT correspondence [1–3] (For a comprehensive review, see [4]). Although there are a lot of research directions, we are interested in the sigma-model classical integrability here. In the typical case of AdS/CFT, the string-theory side is basically described by a 2D non-linear sigma model (NLSM)[1] with target space AdS$_5 \times$S^5 together with the Virasoro constraints after fixing 2D diffeomorphism. Then the classical integrability is ensured by the fact that AdS$_5 \times$S^5 is described as a symmetric coset which exhibits the $\mathbb{Z}_2$-grading.

Many integrable backgrounds are known apart from AdS spaces and spheres. Some of the examples are γ-deformations of S^5 [5, 6], gravity duals of non-commutative gauge theory [7, 8] and Schrödinger spacetimes [9–11]. Such integrable backgrounds may be constructed by performing Yang-Baxter deformations [12, 13] of AdS$_5 \times$S^5 [14–16]. There are other integrable-deformation methods such as bi-Yang-Baxter deformations [13, 17] and λ-deformations [18–21]. It would be worth noting that 2D integrable NLSMs and integrable deformations of them can be described in a unified way based on a 4D Chern-Simons (CS) theory [22, 23] (For related progress, see [24–34]).

On the other hand, there are a lot of non-integrable backgrounds such as AdS black holes [35] and AdS solitons [36–38]. The $T^{1,1}$ background [39], whose metric is given by

$$ds^2 = \frac{1}{6} \sum_{r=1}^{2} \left(d\theta_r^2 + \sin^2 \theta_r \, d\phi_r^2 \right) + \frac{1}{9} (d\psi + \cos\theta_1 \, d\phi_1 + \cos\theta_2 \, d\phi_2) , \tag{1.1}$$

is also one of the non-integrable examples [40–42]. For the coset construction of $T^{1,1}$ and its Yang-Baxter deformation, see [43, 44]. The AdS$_5 \times T^{1,1}$ geometry is well studied because it is a gravity dual of $\mathcal{N} = 1$ superconformal field theory (SCFT) [45]. In relation to SCFT, possible generalizations or deformations of this geometry have intensively been studied.

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[1]We will concentrate on the bosonic part only here.

Recently, Arutyunov, Bassi and Lacroix (ABL) [46] have found a family of integrable deformed $T^{1,1}$ NLSMs.[2] We refer to the NLSMs with the integrable deformed $T^{1,1}$ as the ABL model for brevity. In this paper, we will derive the ABL model from a 4D CS theory. We start from a certain meromorphic one-form with four double poles and six simple zeros. Then by taking an appropriate boundary condition, we can reproduce the classical action of 2D NLSM with four parameters (up to the overall factor). This is nothing but the ABL model. Then we explicitly derive the sigma-model background with target-space metric and anti-symmetric two-form. Finally, we present two simple cases 1) an anisotropic $T^{1,1}$ model and 2) a G/H λ-model. The latter one can be seen as a one-parameter deformation of the Guadagnini-Martellini-Mintchev (GMM) model [52].

This paper is organized as follows. Section 2 is a short review of a derivation of 2D NLSMs from a 4D CS theory. This part is basically based on the seminal work [23]. Then in section 3, we derive the ABL model and the sigma-model background is explicitly computed. In addition, two simple examples are presented. Section 4 is devoted to conclusion and discussion. Appendix A explains a scaling limit of the ABL model which leads to the GMM model by following [46].

2 2D NLSM from 4D CS theory

In this section, we give a derivation of 2D NLSMs from a 4D CS theory by following [22, 23].

Let G be a Lie group with the Lie algebra $\mathfrak{g}$, and $\mathfrak{g}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$. We now consider a 4-dimensional space $\mathcal{M} \times \mathbb{C}P^1$, where $\mathcal{M}$ and $\mathbb{C}P^1$ are parametrized by coordinates (τ, σ) and $(z, \bar{z})$, respectively. A 4D CS action is defined as [22],[3]

$$S[A] = \frac{i}{4\pi} \int_{\mathcal{M} \times \mathbb{C}P^1} \omega \wedge \mathrm{CS}(A) \,, \tag{2.1}$$

where A is a $\mathfrak{g}^{\mathbb{C}}$-valued one-form and $\mathrm{CS}(A)$ is the CS three-form defined as

$$\mathrm{CS}(A) \equiv \left\langle A, dA + \frac{2}{3} A \wedge A \right\rangle \,. \tag{2.2}$$

$\langle \cdot, \cdot \rangle$ is a non-degenerate adjoint-invariant bilinear form $\mathfrak{g}^{\mathbb{C}} \times \mathfrak{g}^{\mathbb{C}} \to \mathbb{C}$. Then ω is a meromorphic one-form defined as

$$\omega \equiv \varphi(z) dz \tag{2.3}$$

and φ is a meromorphic function on $\mathbb{C}P^1$. This function is found to be a twist function characterizing the Poisson structure of the underlying integrable field theory [49]. The pole and zero structure of φ will be important in the following discussion. We denote the sets of poles and zeros by $\mathfrak{p}$ and $\mathfrak{z}$, respectively.

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[2]The discussion in [46] is based on an affine Gaudin model [47–49] and covers more general cases. This family of $T^{1,1}$ models is a special case of it. For the off-critical value of the B-field, classical chaos appears for some initial conditions [50, 51].

[3]For the notation and convention here, see [24].

Note that an extra gauge symmetry

$$A \mapsto A + \chi \, dz \tag{2.4}$$

can alway gauge away the z-component of A like

$$A = A_\sigma \, d\sigma + A_\tau \, d\tau + A_{\bar{z}} \, d\bar{z} \,. \tag{2.5}$$

By taking a variation of the action (2.1), we obtain the bulk equation of motion

$$\omega \wedge F(A) = 0 \,, \qquad F(A) \equiv dA + A \wedge A \tag{2.6}$$

and the boundary equation of motion

$$d\omega \wedge \langle A, \delta A \rangle = 0 \,. \tag{2.7}$$

The boundary conditions satisfying (2.7) play an important role to describe integrable deformations [22, 23]. Note that the boundary equation of motion (2.7) does not vanish only on $\mathcal{M} \times \mathfrak{p} \subset \mathcal{M} \times \mathbb{C}P^1$, because the relation

$$d\omega = \partial_{\bar{z}} \varphi(z) \, d\bar{z} \wedge dz \tag{2.8}$$

indicates that only the pole of φ can contribute as a distribution. This can be seen by rewriting the equation (2.7) to

$$\sum_{x \in \mathfrak{p}} \sum_{p \geq 0} (\mathrm{res}_x \, \xi_x^p \omega) \, \epsilon^{ij} \frac{1}{p!} \partial_{\xi_x}^p \langle A_i, \delta A_j \rangle \big|_{\mathcal{M} \times \{x\}} = 0 \,, \tag{2.9}$$

where ϵ^{ij} is the antisymmetric tensor. Here the local holomorphic coordinates ξ_x are defined as $\xi_x \equiv z - x$ for $x \in \mathfrak{p} \backslash \{\infty\}$ and $\xi_\infty \equiv 1/z$ if $\mathfrak{p}$ includes the point at infinity. The expression (2.9) manifestly shows that the boundary equation of motion has the support only on $\mathcal{M} \times \mathfrak{p}$.

In terms of the components, the bulk equation of motion (2.6) reads

$$\partial_\sigma A_\tau - \partial_\tau A_\sigma + [A_\sigma, A_\tau] = 0 \,, \tag{2.10}$$

$$\omega \left(\partial_{\bar{z}} A_\sigma - \partial_\sigma A_{\bar{z}} + [A_{\bar{z}}, A_\sigma] \right) = 0 \,, \tag{2.11}$$

$$\omega \left(\partial_{\bar{z}} A_\tau - \partial_\tau A_{\bar{z}} + [A_{\bar{z}}, A_\tau] \right) = 0 \,. \tag{2.12}$$

The factor ω is kept since $\partial_{\bar{z}} A_\sigma$ and $\partial_{\bar{z}} A_\tau$ are in general distributions on $\mathbb{C}P^1$ supported by $\mathfrak{z}$.

Lax form. By performing a formal gauge transformation

$$A = -d\hat{g}\hat{g}^{-1} + \hat{g}\mathcal{L}\hat{g}^{-1} \tag{2.13}$$

with a smooth function $\hat{g} : \mathcal{M} \times \mathbb{C}P^1 \to G^{\mathbb{C}}$, the $\bar{z}$-components of $\mathcal{L}$ can be taken to zero:

$$\mathcal{L}_{\bar{z}} = 0 \,. \tag{2.14}$$

Hence the one-form $\mathcal{L}$ takes the form

$$\mathcal{L} \equiv \mathcal{L}_\sigma d\sigma + \mathcal{L}_\tau d\tau \,. \tag{2.15}$$

The one-form will be specified as a Lax pair for 2D theory later, and so we refer to $\mathcal{L}$ as the Lax form.

The bulk equations of motion (2.6) in terms of the Lax form $\mathcal{L}$ are expressed as

$$\partial_\tau \mathcal{L}_\sigma - \partial_\sigma \mathcal{L}_\tau + [\mathcal{L}_\tau, \mathcal{L}_\sigma] = 0 \,, \tag{2.16}$$

$$\omega \wedge \partial_{\bar{z}} \mathcal{L} = 0 \,. \tag{2.17}$$

These equations mean that $\mathcal{L}$ is a meromorphic one-form with poles at the zeros of ω, namely $\mathfrak{z}$ can be regarded as the set of poles of $\mathcal{L}$.

Reality condition. To ensure the reality of the 4D action (2.1) and the resulting action (2.22), we suppose some condition for the form of ω and the configuration of A [23].

For a complex coordinate z, an involution $\mu_{\rm t} : \mathbb{C}P^1 \to \mathbb{C}P^1$ is defined by complex conjugation $z \mapsto \bar{z}$. Let $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be an anti-linear involution which satisfies

$$\overline{\langle B, C \rangle} = \langle \tau B, \tau C \rangle \,, \qquad {}^\forall B, C \in \mathfrak{g}^{\mathbb{C}} \,. \tag{2.18}$$

Then a real Lie subalgebra $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ is given as the set of the fixed points under τ. The associated operation to the Lie group G is denoted by $\tilde{\tau} : G^{\mathbb{C}} \to G^{\mathbb{C}}$.

One can see that the action (2.1) is real if ω and A satisfy

$$\overline{\omega} = \mu_{\rm t}^* \omega \,, \tag{2.19}$$

$$\tau A = \mu_{\rm t}^* A \,. \tag{2.20}$$

Recalling the relation (2.13), the condition

$$\tilde{\tau} \hat{g} = \mu_{\rm t}^* \hat{g} \,, \qquad \tau \mathcal{L} = \mu_{\rm t}^* \mathcal{L} \,, \tag{2.21}$$

leads to (2.20).

From 4D to 2D via the archipelago conditions. The 4D action (2.1) can be reduced to a 2D action with the WZ term when $\hat{g}$ satisfies the archipelago condition [23]. By performing an integral over $\mathbb{C}P^1$, we obtain

$$S\left[\{g_x\}_{x \in \mathfrak{p}}\right] = \frac{1}{2} \sum_{x \in \mathfrak{p}} \int_{\mathcal{M}} \left\langle \mathrm{res}_x(\varphi \, \mathcal{L}), g_x^{-1} dg_x \right\rangle - \frac{1}{2} \sum_{x \in \mathfrak{p}} (\mathrm{res}_x \omega) \int_{\mathcal{M} \times [0, R_x]} I_{\rm WZ}[g_x] \,. \tag{2.22}$$

Here $I_{\rm WZ}[u]$ is the Wess-Zumino (WZ) three-form defined as

$$I_{\rm WZ}[u] \equiv \frac{1}{3} \langle u^{-1} du, u^{-1} du \wedge u^{-1} du \rangle \,, \tag{2.23}$$

where R_x is the radius of the open disk on $\mathbb{C}P^1$.

The action (2.22) is invariant under a gauge transformation

$$g_x \mapsto g_x h \,, \qquad \mathcal{L} \mapsto h^{-1} \mathcal{L} h + h^{-1} dh \,, \tag{2.24}$$

with a local function $h : \mathcal{M} \to G$. One can seen this as the residual gauge symmetry after taking the gauge (2.14).

3 The ABL model from 4D CS theory

In this section, we shall consider 2D NLSMs with a family of deformed $T^{1,1}$ manifolds, which have been presented by Arutyunov-Bassi-Lacroix [46]. We will refer to them as the ABL model for brevity as explained in Introduction.

Here, let us reproduce the ABL model from 4D CS theory. In the ABL model, the 2D surface $\mathcal{M}$ is embedded into the Lie group $G \times G$. By defining a subgroup $H \subset G$ as fixed points of an involutive automorphism, this model exhibits a gauge H_{diag}-symmetry, where $H_{\mathrm{diag}} = \{(h, h) \in G \times G \,|\, h \in H\}$ rather than $G_{\mathrm{diag}} = \{(g, g) \in G \times G \,|\, g \in G\}$. Then the target space is reduced to a coset $(G \times G)/H_{\mathrm{diag}}$.

Twist function. Let us start with the following meromorphic one-form,

$$\omega = \varphi_{\mathrm{ABL}}(z)\, dz = 2K \frac{z(z^2 - \zeta_+^2)(z^2 - \zeta_-^2)}{\prod_{i=1}^2 (z^2 - z_i^2)^2} dz\,, \tag{3.1}$$

where $\varphi_{\mathrm{ABL}}(z)$ is a twist function with $\zeta_\pm \in \mathbb{C}P^1$ and $z_1, z_2 \in \mathbb{R}$. This ω has the four double poles and the six simple zeros

$$\mathfrak{p} = \{\pm z_1, \pm z_2\}\,, \qquad \mathfrak{z} = \{0, \pm \zeta_+\,, \pm \zeta_-, \infty\}\,. \tag{3.2}$$

The twist function in (3.1) corresponds to the case with $N = 2$ and $T = 2$ in (3.14) in [46].

Boundary condition. In specifying a 2D integrable model associated with ω, we need to choose a solution to the boundary equations of motion,

$$\epsilon^{\alpha\beta} \langle\!\langle (A_\alpha, \partial_{\xi_p} A_\alpha), \delta(A_\beta, \partial_{\xi_p} A_\beta) \rangle\!\rangle_p = 0\,, \qquad p \in \mathfrak{p}\,. \tag{3.3}$$

Here we used the formula (2.9) with (3.1) and the double bracket is defined as

$$\begin{aligned}
\langle\!\langle (x, y), (x', y') \rangle\!\rangle_p &\equiv (\mathrm{res}_p\, \omega)\langle x, x' \rangle + (\mathrm{res}_p\, \xi_p \omega)\left(\langle x, y' \rangle + \langle x', y \rangle\right) \\
&= b_p \langle x, x' \rangle + c_p \left(\langle x, y' \rangle + \langle x', y \rangle\right)\,,
\end{aligned} \tag{3.4}$$

where the constants b_p and c_p $(p \in \mathfrak{p})$ are given by

$$\begin{aligned}
b_{\pm z_1} &= \frac{K((\zeta_+^2 + \zeta_-^2)(z_1^2 + z_2^2) - 2(z_1^2 z_2^2 + \zeta_+^2 \zeta_-^2))}{(z_1^2 - z_2^2)^3} = -b_{\pm z_2} \equiv -\Bbbk\,, \\
c_{\pm z_1} &= \pm \frac{K(z_1^2 - \zeta_+^2)(z_1^2 - \zeta_-^2)}{2z_1(z_1^2 - z_2^2)^2}\,, \qquad c_{\pm z_2} = \pm \frac{K(z_2^2 - \zeta_+^2)(z_2^2 - \zeta_-^2)}{2z_2(z_1^2 - z_2^2)^2}\,.
\end{aligned} \tag{3.5}$$

The boundary equations of motion (3.3) take the same form as in the PCM with the WZ term case [23].

To derive the ABL model, we take the following solution:

$$(A|_{z=\pm z_1}, \partial_z A|_{z=\pm z_1}) \in \{0\} \ltimes \mathfrak{g}_{\mathrm{ab}}\,, \qquad (A|_{z=\pm z_2}, \partial_z A|_{z=\pm z_2}) \in \{0\} \ltimes \mathfrak{g}_{\mathrm{ab}}\,, \tag{3.6}$$

where $\mathfrak{g}_{\mathrm{ab}}$ is an abelian copy of $\mathfrak{g}$. The boundary condition (3.6) is nothing but

$$A|_{z=\pm z_{1,2}} = 0\,. \tag{3.7}$$

3.1 Lax form

Before deriving the sigma model action, we shall derive the Lax pair by employing the boundary condition (3.6).

We take $\hat{g}$ at each pole of the twist function (3.1) as

$$\hat{g}(\tau, \sigma, z)|_{z=z_1} = g_1(\tau, \sigma), \qquad \hat{g}(\tau, \sigma, z)|_{z=-z_1} = \tilde{g}_1(\tau, \sigma),$$
$$\hat{g}(\tau, \sigma, z)|_{z=z_2} = g_2(\tau, \sigma), \qquad \hat{g}(\tau, \sigma, z)|_{z=-z_2} = \tilde{g}_2(\tau, \sigma), \tag{3.8}$$

where $g_k, \tilde{g}_k \in G$ ($k = 1, 2$). Note that g_k take values in G (not $G^{\mathbb{C}}$) due to the reality condition (2.21). The associated left-invariant currents are defined as

$$j_1 \equiv g_1^{-1} dg_1, \qquad \tilde{j}_1 \equiv \tilde{g}_1^{-1} d\tilde{g}_1, \qquad j_2 \equiv g_2^{-1} dg_2, \qquad \tilde{j}_2 \equiv \tilde{g}_2^{-1} d\tilde{g}_2, \tag{3.9}$$

and the relations between the gauge field A and the Lax form $\mathcal{L}$ at each pole are written as

$$A|_{z=z_1} = -dg_1 g_1^{-1} + \mathrm{Ad}_{g_1} \mathcal{L}|_{z=z_1}, \qquad A|_{z=-z_1} = -d\tilde{g}_1 \tilde{g}_1^{-1} + \mathrm{Ad}_{\tilde{g}_1} \mathcal{L}|_{z=-z_1},$$
$$A|_{z=z_2} = -dg_2 g_2^{-1} + \mathrm{Ad}_{g_2} \mathcal{L}|_{z=z_2}, \qquad A|_{z=-z_2} = -d\tilde{g}_2 \tilde{g}_2^{-1} + \mathrm{Ad}_{\tilde{g}_2} \mathcal{L}|_{z=-z_2}, \tag{3.10}$$

where the adjoint action $\mathrm{Ad}_g : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ is defined as $\mathrm{Ad}_g \mathcal{L} = g \mathcal{L} g^{-1}$.

Recall that $\mathcal{L}$ should have poles at the zeros of ω as mentioned just below (2.17). Hence, taking account of the configuration of the zeros (3.1), we suppose an ansatz for the Lax form as

$$\mathcal{L} = \left(U_+^{[1]} + z\, U_+^{[2]} + \frac{U_+^{[3]}}{z - \zeta_+} + \frac{U_+^{[4]}}{z + \zeta_+} \right) d\sigma^+ + \left(U_-^{[1]} + z^{-1} U_-^{[2]} + \frac{U_-^{[3]}}{z - \zeta_-} + \frac{U_-^{[4]}}{z + \zeta_-} \right) d\sigma^-, \tag{3.11}$$

where $U_\pm^{[k]}$ ($k = 1, \ldots, 4$) are undetermined smooth functions of τ and σ, and the light-cone coordinates are defined as

$$\sigma^\pm \equiv \frac{1}{2} (\tau \pm \sigma). \tag{3.12}$$

In order to obtain the explicit expression of the Lax pair under the boundary condition (3.6), we rewrite the relations in (3.10) as

$$j_{1,\pm} = U_\pm^{[1]} + z_1^{\pm 1} U_\pm^{[2]} + \frac{U_\pm^{[3]}}{z_1 - \zeta_\pm} + \frac{U_\pm^{[4]}}{z_1 + \zeta_\pm}, \tag{3.13}$$

$$\tilde{j}_{1,\pm} = U_\pm^{[1]} - z_1^{\pm 1} U_\pm^{[2]} - \frac{U_\pm^{[3]}}{z_1 + \zeta_\pm} - \frac{U_\pm^{[4]}}{z_1 - \zeta_\pm}, \tag{3.14}$$

$$j_{2,\pm} = U_\pm^{[1]} + z_2^{\pm 1} U_\pm^{[2]} + \frac{U_\pm^{[3]}}{z_2 - \zeta_\pm} + \frac{U_\pm^{[4]}}{z_2 + \zeta_\pm}, \tag{3.15}$$

$$\tilde{j}_{2,\pm} = U_\pm^{[1]} - z_2^{\pm 1} U_\pm^{[2]} - \frac{U_\pm^{[3]}}{z_2 + \zeta_\pm} - \frac{U_\pm^{[4]}}{z_2 - \zeta_\pm}. \tag{3.16}$$

By solving these equations with respect to $U_\pm^{[k]}$ ($k = 1, 2, 3, 4$), we obtain

$$U_\pm^{[1]} = \frac{(j_{1,\pm} + \tilde{j}_{1,\pm})(z_1^2 - \zeta_\pm^2) - (j_{2,\pm} + \tilde{j}_{2,\pm})(z_2^2 - \zeta_\pm^2)}{2(z_1^2 - z_2^2)}, \tag{3.17}$$

$$U_\pm^{[2]} = \frac{z_1^{\mp 1}(j_{1,\pm} - \tilde{j}_{1,\pm})\left(z_1^{\pm 2} - \zeta_\pm^{\pm 2}\right) - z_2^{\mp 1}(j_{2,\pm} - \tilde{j}_{2,\pm})\left(z_2^{\pm 2} - \zeta_\pm^{\pm 2}\right)}{2\left(z_1^{\pm 2} - z_2^{\pm 2}\right)}, \tag{3.18}$$

$$U_\pm^{[3]} = -\frac{(z_1 z_2)^{\mp 1}(z_1^2 - \zeta_\pm^2)(z_2^2 - \zeta_\pm^2)}{4\zeta_\pm(z_1^2 - z_2^2)}\left[z_2^{\pm 1}\left(j_{1,\pm}\left(z_1^{\pm 1} + \zeta_\pm^{\pm 1}\right) + \tilde{j}_{1,\pm}\left(z_1^{\pm 1} - \zeta_\pm^{\pm 1}\right)\right)\right.$$
$$\left. - z_1^{\pm 1}\left(j_{2,\pm}\left(z_2^{\pm 1} + \zeta_\pm^{\pm 1}\right) + \tilde{j}_{2,\pm}\left(z_2^{\pm 1} - \zeta_\pm^{\pm 1}\right)\right)\right], \tag{3.19}$$

$$U_\pm^{[4]} = \frac{(z_1^2 - \zeta_\pm^2)(z_2^2 - \zeta_\pm^2)}{4\zeta_\pm(z_1^2 - z_2^2)}\left[z_1^{\mp 1}\left(j_{1,\pm}\left(z_1^{\pm 1} - \zeta_\pm^{\pm 1}\right) + \tilde{j}_{1,\pm}\left(z_1^{\pm 1} + \zeta_\pm^{\pm 1}\right)\right)\right.$$
$$\left. - z_2^{\mp 1}\left(j_{2,\pm}\left(z_2^{\pm 1} - \zeta_\pm^{\pm 1}\right) + \tilde{j}_{2,\pm}\left(z_2^{\pm 1} + \zeta_\pm^{\pm 1}\right)\right)\right]. \tag{3.20}$$

Then the Lax pair can be rewritten as

$$\mathcal{L}_\pm(z) = \eta_{1,\pm}^{(0)}(z)J_{1,\pm}^{(0)} + \eta_{1,\pm}^{(1)}(z)J_{1,\pm}^{(1)} + \eta_{2,\pm}^{(0)}(z)J_{2,\pm}^{(0)} + \eta_{2,\pm}^{(1)}(z)J_{2,\pm}^{(1)}, \tag{3.21}$$

where $J_{s,\pm}^{(k)}$ ($k = 0, 1$, $s = 1, 2$) are defined as

$$J_{1,\pm}^{(0)} = \frac{j_{1,\pm} + \tilde{j}_{1,\pm}}{2}, \qquad J_{1,\pm}^{(1)} = \frac{j_{1,\pm} - \tilde{j}_{1,\pm}}{2},$$
$$J_{2,\pm}^{(0)} = \frac{j_{2,\pm} + \tilde{j}_{2,\pm}}{2}, \qquad J_{2,\pm}^{(1)} = \frac{j_{2,\pm} - \tilde{j}_{2,\pm}}{2}, \tag{3.22}$$

and the coefficients $\eta_{s,\pm}^{(k)}$ ($k = 0, 1$, $s = 1, 2$) are

$$\eta_{\pm,1}^{(0)}(z) = \frac{(z^2 - z_2^2)(z_1^2 - \zeta_\pm^2)}{(z^2 - \zeta_\pm^2)(z_1^2 - z_2^2)}, \qquad \eta_{\pm,1}^{(1)}(z) = \left(\frac{z}{z_1}\right)^{\pm 1}\eta_{\pm,1}^{(0)}(z),$$
$$\eta_{\pm,2}^{(0)}(z) = -\frac{(z^2 - z_1^2)(z_2^2 - \zeta_\pm^2)}{(z^2 - \zeta_\pm^2)(z_1^2 - z_2^2)}, \qquad \eta_{\pm,2}^{(1)}(z) = \left(\frac{z}{z_2}\right)^{\pm 1}\eta_{\pm,2}^{(0)}(z). \tag{3.23}$$

3.2 2D action

Next, let us derive the 2D action under the boundary condition (3.6).

For this purpose, we evaluate the residues of $\varphi_{\rm ABL}\mathcal{L}$ at $z = \pm z_1, \pm z_2$. By using the expression (3.21) of the Lax form, we obtain

$$\begin{aligned}
\mathrm{res}_{z=\pm z_1}(\varphi_{\rm ABL}\mathcal{L}) &= \left(-J_{1,+}^{(0)}\rho_{12}^{(0)} + J_{2,+}^{(0)}\rho_{21}^{(0)} \pm J_{1,+}^{(1)}c_{1,+}^{(1)} \pm J_{2,+}^{(1)}\rho_{21}^{(1)}\right)d\sigma^+ \\
&\quad + \left(J_{1,-}^{(0)}\rho_{21}^{(0)} - J_{2,-}^{(0)}\rho_{12}^{(0)} \mp J_{1,-}^{(1)}c_{1,-}^{(1)} \mp J_{2,-}^{(1)}\rho_{12}^{(1)}\right)d\sigma^-, \\
\mathrm{res}_{z=\pm z_2}(\varphi_{\rm ABL}\mathcal{L}) &= \left(J_{1,+}^{(0)}\rho_{12}^{(0)} - J_{2,+}^{(0)}\rho_{21}^{(0)} \pm J_{1,+}^{(1)}\rho_{12}^{(1)} \mp J_{2,+}^{(1)}c_{2,+}^{(1)}\right)d\sigma^+ \\
&\quad + \left(-J_{1,-}^{(0)}\rho_{21}^{(0)} + J_{2,-}^{(0)}\rho_{12}^{(0)} \mp J_{1,-}^{(1)}\rho_{21}^{(1)} \pm J_{2,-}^{(1)}c_{2,-}^{(1)}\right)d\sigma^-,
\end{aligned} \tag{3.24}$$

where the constants $\rho_{rs}^{(k)}(r, s = 1, 2, k = 0, 1)$ are defined as

$$\rho_{11}^{(0)} = \rho_{22}^{(0)} = \frac{K}{2}\frac{\zeta_-^2 - \zeta_+^2}{(z_1^2 - z_2^2)^2}, \qquad \rho_{12}^{(0)} = K\frac{(z_1^2 - \zeta_+^2)(z_2^2 - \zeta_-^2)}{(z_1^2 - z_2^2)^3},$$

$$\rho_{21}^{(0)} = -K\frac{(z_1^2 - \zeta_-^2)(z_2^2 - \zeta_+^2)}{(z_1^2 - z_2^2)^3}, \tag{3.25}$$

and

$$\rho_{11}^{(1)} = \frac{K}{2}\frac{(z_1^4 - 2\zeta_+^2 z_1^2 + \zeta_-^2\zeta_+^2)}{z_1^2(z_1^2 - z_2^2)^2}, \qquad \rho_{12}^{(1)} = K\frac{z_2(z_1^2 - \zeta_+^2)(z_2^2 - \zeta_-^2)}{z_1(z_1^2 - z_2^2)^3},$$

$$\rho_{21}^{(1)} = -K\frac{z_1(z_1^2 - \zeta_-^2)(z_2^2 - \zeta_+^2)}{z_2(z_1^2 - z_2^2)^3}, \qquad \rho_{22}^{(1)} = \frac{K}{2}\frac{(z_2^4 - 2\zeta_+^2 z_2^2 + \zeta_-^2\zeta_+^2)}{z_2^2(z_1^2 - z_2^2)^2}. \tag{3.26}$$

Furthermore, the constants $c_{s,\pm}^{(1)}$ $(s = 1, 2)$ are

$$c_{1,-}^{(1)} = \frac{K\left(z_1^2 - \zeta_-^2\right)}{2z_1^2\left(z_1^2 - z_2^2\right)^3}\left(\zeta_+^2\left(z_2^2 - 3z_1^2\right) + z_1^2\left(z_1^2 + z_2^2\right)\right),$$

$$c_{1,+}^{(1)} = \frac{K\left(z_1^2 - \zeta_+^2\right)}{2z_1^2\left(z_1^2 - z_2^2\right)^3}\left(z_1^2\left(z_1^2 - 3z_2^2\right) + \zeta_-^2\left(z_1^2 + z_2^2\right)\right),$$

$$c_{2,-}^{(1)} = \frac{K\left(z_2^2 - \zeta_-^2\right)}{2z_2^2\left(z_1^2 - z_2^2\right)^3}\left(\zeta_+^2\left(z_1^2 - 3z_2^2\right) + z_2^2\left(z_1^2 + z_2^2\right)\right),$$

$$c_{2,+}^{(1)} = \frac{K\left(z_2^2 - \zeta_+^2\right)}{2z_2^2\left(z_1^2 - z_2^2\right)^3}\left(z_2^2\left(z_2^2 - 3z_1^2\right) + \zeta_-^2\left(z_1^2 + z_2^2\right)\right). \tag{3.27}$$

Note that the above constants satisfy the relations

$$\frac{\rho_{12}^{(0)} + \rho_{21}^{(0)}}{2} = -\rho_{11}^{(0)} = -\rho_{22}^{(0)}, \qquad \frac{c_{1,+}^{(1)} + c_{1,-}^{(1)}}{2} = \rho_{11}^{(1)}, \qquad \frac{c_{2,+}^{(1)} + c_{2,-}^{(1)}}{2} = -\rho_{22}^{(1)}. \tag{3.28}$$

By using (3.28), we obtain

$$\left\langle \mathrm{res}_{z=z_1}(\varphi_{\mathrm{ABL}}\,\mathcal{L}), j_1\right\rangle + \left\langle \mathrm{res}_{z=-z_1}(\varphi_{\mathrm{ABL}}\,\mathcal{L}), \tilde{j}_1\right\rangle$$

$$= 2\sum_{k=0}^{1}\left(2\rho_{11}^{(k)}\left\langle J_{1,+}^{(k)}, J_{1,-}^{(k)}\right\rangle + \rho_{12}^{(k)}\left\langle J_{1,+}^{(k)}, J_{2,-}^{(k)}\right\rangle + \rho_{21}^{(k)}\left\langle J_{2,+}^{(k)}, J_{1,-}^{(k)}\right\rangle\right)d\sigma^+ \wedge d\sigma^-, \tag{3.29}$$

$$\left\langle \mathrm{res}_{z=z_2}(\varphi_{\mathrm{ABL}}\,\mathcal{L}), j_2\right\rangle + \left\langle \mathrm{res}_{z=-z_2}(\varphi_{\mathrm{ABL}}\,\mathcal{L}), \tilde{j}_2\right\rangle$$

$$= 2\sum_{k=0}^{1}\left(2\rho_{22}^{(k)}\left\langle J_{2,+}^{(k)}, J_{2,-}^{(k)}\right\rangle + \rho_{12}^{(k)}\left\langle J_{1,+}^{(k)}, J_{2,-}^{(k)}\right\rangle + \rho_{21}^{(k)}\left\langle J_{2,+}^{(k)}, J_{1,-}^{(k)}\right\rangle\right)d\sigma^+ \wedge d\sigma^-. \tag{3.30}$$

The residues of ω at each pole are

$$-\mathrm{res}_{z=\pm z_1}\omega = \mathrm{res}_{z=\pm z_2}\omega = \Bbbk = K\frac{2z_1^2 z_2^2 + 2\zeta_-^2\zeta_+^2 - (z_1^2 + z_2^2)(\zeta_-^2 + \zeta_+^2)}{(z_1^2 - z_2^2)^3}. \tag{3.31}$$

Then, the 2D action (2.22) is rewritten as

$$S[g_k] = \int_{\mathcal{M}} \sum_{r,s=1}^{2} \left(\rho_{rs}^{(0)} \left\langle J_{r,+}^{(0)}, J_{s,-}^{(0)} \right\rangle + \rho_{rs}^{(1)} \left\langle J_{r,+}^{(1)}, J_{s,-}^{(1)} \right\rangle \right) 2\, d\sigma^+ \wedge d\sigma^-$$
$$+ \frac{\hbar}{2} \int_{\mathcal{M}\times[0,R_r]} \left(I_{\mathrm{WZ}}[g_1] + I_{\mathrm{WZ}}[\tilde{g}_1] \right) - \frac{\hbar}{2} \int_{\mathcal{M}\times[0,R_s]} \left(I_{\mathrm{WZ}}[g_2] + I_{\mathrm{WZ}}[\tilde{g}_2] \right). \quad (3.32)$$

Here we would like to impose a relation between j_s and $\tilde{j}_s$ ($s = 1, 2$). Note that the resulting action (3.32) is invariant under the exchange of j_1 and $\tilde{j}_1$ (j_2 and $\tilde{j}_2$). This invariance is respected if $j_s, \tilde{j}_s$ are related as $\tilde{j}_1 = f_1(j_1)$, $\tilde{j}_2 = f_2(j_2)$ with involutive automorphisms $f_s : \mathfrak{g} \to \mathfrak{g}$ ($s = 1, 2$). The maps f_s thus satisfy

$$f_s([\mathrm{x}, \mathrm{y}]) = [f_s(\mathrm{x}), f_s(\mathrm{y})], \qquad f_s \circ f_s(\mathrm{x}) = \mathrm{x}, \qquad \mathrm{x} \in \mathfrak{g}. \quad (3.33)$$

It is significant to argue a consistent condition for f_s ($s = 1, 2$). The introduction of them was apparently independent but it seems likely that we should impose that $f_1 = f_2$ as a possible consistent condition. This condition is compatible with the preceding work [46] based on the dihedral affine Gaudin model [47–49]. There might be a possibility to remove this condition but we will not try to exhaust here. In the following, we will work under this condition.

By utilizing the involutions f_s, the vector space $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, i.e., the generators

$$\mathfrak{h} = \langle \mathbf{J}_{\hat{a}} \rangle, \qquad \mathfrak{m} = \langle \mathbf{P}_{\check{a}} \rangle, \qquad \hat{a} = 1, \ldots, \dim \mathfrak{h}, \quad \check{a} = 1, \ldots, \dim \mathfrak{m}, \quad (3.34)$$

are introduced so that

$$f_s(\mathbf{P}_{\check{a}}) = -\mathbf{P}_{\check{a}}, \qquad f_s(\mathbf{J}_{\hat{a}}) = \mathbf{J}_{\hat{a}}. \quad (3.35)$$

The vector subspace $\mathfrak{h}$ is also a subalgebra of $\mathfrak{g}$, and thus there exists the associated Lie subgroup H. Then the projection operators into $\mathfrak{h}, \mathfrak{m}$ are defined as

$$P_{(0)} : \mathfrak{g} \to \mathfrak{h}, \qquad P_{(1)} : \mathfrak{g} \to \mathfrak{m}, \quad (3.36)$$

and then $\tilde{j}_s$ and $J_s^{(k)}$ are expressed as

$$\tilde{j}_s = f_s(j_s) = f_s \left(P_{(0)}(j_s) + P_{(1)}(j_s) \right) = P_{(0)}(j_s) - P_{(1)}(j_s), \quad (3.37)$$
$$J_s^{(0)} = P_s^{(0)}(j_s), \qquad J_s^{(1)} = P_s^{(1)}(j_s). \quad (3.38)$$

By using the commutation relation of the Lie algebra for the symmetric coset and the orthogonality of $\mathfrak{m}$ and $\mathfrak{h}$ with respect to the bilinear form $\langle \cdot, \cdot \rangle$, we can see

$$\langle P_{(0)}(g_s^{-1} dg_s), P_{(0)}(g_s^{-1} dg_s) \wedge P_{(1)}(g_s^{-1} dg_s) \rangle = 0, \quad (3.39)$$
$$\langle P_{(1)}(g_s^{-1} dg_s), P_{(1)}(g_s^{-1} dg_s) \wedge P_{(1)}(g_s^{-1} dg_s) \rangle = 0. \quad (3.40)$$

Hence, we obtain

$$I_{\mathrm{WZ}}[g_1] = I_{\mathrm{WZ}}[\tilde{g}_1], \qquad I_{\mathrm{WZ}}[g_2] = I_{\mathrm{WZ}}[\tilde{g}_2]. \quad (3.41)$$

Then, by using the expressions of $\tilde{j}_s$ in (3.37), the 2D action can be further rewritten as

$$S[g_1, g_2] = \sum_{r,s=1}^{2} \int d^2\sigma \left(\rho_{rs}^{(0)} \left\langle J_{r,+}^{(0)}, J_{s,-}^{(0)} \right\rangle + \rho_{rs}^{(1)} \left\langle J_{r,+}^{(1)}, J_{s,-}^{(1)} \right\rangle \right) + \Bbbk \, I_{\mathrm{WZ}}[g_1] - \Bbbk \, I_{\mathrm{WZ}}[g_2] \,.$$
(3.42)

The Lax form (3.21) becomes

$$\mathcal{L}_\pm(z) = \sum_{r=1}^{2} \sum_{k=0}^{1} \eta_{\pm,r}^{(k)}(z) \, J_{r,\pm}^{(k)} \,.$$
(3.43)

The expressions (3.42) and (3.43) are the same as the classical action and the associated Lax pair given in [46].

Gauge invariance. The action (3.42) exhibits a local H_{diag}-symmetry, which is regarded as a gauge symmetry. The diagonal subgroup $H_{\mathrm{diag}} = \{(h,h) \in G \times G \,|\, h \in H\}$ acts on $G \times G$ as

$$g_1 \mapsto g_1 h \,, \qquad g_2 \mapsto g_2 h \,,$$
(3.44)

where h is a smooth map $\mathcal{M} \to H$. Noting that the Wess-Zumino terms vary according to the Polyakov-Wigmann formula [53], we can see that the action (3.42) is invariant if the following conditions hold:

$$\begin{cases} \sum_{r,s}^{1,2} \rho_{rs}^{(0)} = 0 \,, \\ \rho_{11}^{(0)} + \rho_{12}^{(0)} - \frac{\Bbbk}{2} = \rho_{21}^{(0)} + \rho_{22}^{(0)} + \frac{\Bbbk}{2} = 0 \,, \\ \rho_{11}^{(0)} + \rho_{21}^{(0)} + \frac{\Bbbk}{2} = \rho_{12}^{(0)} + \rho_{22}^{(0)} - \frac{\Bbbk}{2} = 0 \,, \end{cases}$$
(3.45)

$$\Leftrightarrow \quad \rho_{11}^{(0)} = \rho_{22}^{(0)} \,, \quad \rho_{12}^{(0)} - \rho_{21}^{(0)} = \Bbbk \,, \quad \frac{\rho_{12}^{(0)} + \rho_{21}^{(0)}}{2} + \rho_{11}^{(0)} = 0 \,.$$
(3.46)

These relations are indeed satisfied by the parametrization (3.26) and (3.31). The gauge invariance under (3.44) is nothing but the unbroken part of the 2D gauge invariance under (2.24). Although the original gauge group is $G_{\mathrm{diag}} = \{(g,g) \in G \times G \,|\, g \in G\}$, the grading condition (3.37) explicitly break $G_{\mathrm{diag}}/H_{\mathrm{diag}}$ and only H_{diag} survives.

3.3 Examples

The resulting action (3.42) and Lax form (3.43) are a bit abstract and complicated. Hence, it is instructive to see a simple case with $G = \mathrm{SU}(2)$ and $H = \mathrm{U}(1)$. Then it is easy to read off the background metric and B-field.

The generators of $\mathfrak{su}(2)$ are represented by $\{i\sigma_a/2 \,, i = 1, 2, 3\}$, where σ_a are the Pauli matrices. The bilinear form $\langle \cdot, \cdot \rangle$ becomes the trace operation. In this case, the involutive automorphisms f_s $(s = 1, 2)$ are defined as

$$f_s \left(\frac{i\sigma_1}{2} \right) = -\frac{i\sigma_1}{2} \,, \qquad f_s \left(\frac{i\sigma_2}{2} \right) = -\frac{i\sigma_2}{2} \,, \qquad f_s \left(\frac{i\sigma_3}{2} \right) = \frac{i\sigma_3}{2} \,.$$
(3.47)

We choose the parameters $\{\phi_1, \theta_1, \psi_1, \phi_2, \theta_2, \psi_2\} = \{x^\mu\}\, (\mu = 1, \ldots, 6)$ to express $(g_1, g_2) \in$ SU(2) $\times$ SU(2) as

$$g_1 = \exp\!\left(\frac{i\sigma_3}{2}\phi_1\right) \exp\!\left(\frac{i\sigma_2}{2}\theta_1\right) \exp\!\left(\frac{i\sigma_3}{2}\psi_1\right),$$
$$g_2 = \exp\!\left(-\frac{i\sigma_3}{2}\phi_2\right) \exp\!\left(-\frac{i\sigma_2}{2}\theta_2\right) \exp\!\left(-\frac{i\sigma_3}{2}\psi_2\right). \tag{3.48}$$

Then the gauge transformation (3.44) corresponds to the shift

$$(\psi_1, \psi_2) \mapsto (\psi_1 + \alpha, \psi_2 - \alpha). \tag{3.49}$$

The ABL background. By substituting the parametrization (3.48), the resulting action is given by

$$S[x^\mu] = -\frac{1}{4} \int d^2\sigma \left(G_{\mu\nu} + B_{\mu\nu}\right) \partial_- x^\mu \partial_+ x^\nu, \tag{3.50}$$

where $G_{\mu\nu}$ and $B_{\mu\nu}$ are the background metric and B-field, respectively. By using the relations (3.46), $G_{\mu\nu}$ and $B_{\mu\nu}$ are expressed as, respectively,

$$\begin{aligned}
&\frac{1}{2}G_{\mu\nu}dx^\mu dx^\nu\\
&= \sum_{r=1}^{2} \Big[\rho_{rr}^{(1)}\left(d\theta_r^2 + \sin^2\theta_r d\phi_r^2\right) + \rho_{rr}^{(0)}\left(\cos\theta_r d\phi_r + d\psi_r\right)^2\Big]\\
&\quad - \left(\rho_{12}^{(0)} + \rho_{21}^{(0)}\right)\left(\cos\theta_1 d\phi_1 + d\psi_1\right)\left(\cos\theta_2 d\phi_2 + d\psi_2\right)\\
&\quad - \left(\rho_{12}^{(1)} + \rho_{21}^{(1)}\right)\Big[-\sin\theta_1\sin\theta_2\cos(\psi_1+\psi_2)d\phi_1 d\phi_2 + \cos(\psi_1+\psi_2)d\theta_1 d\theta_2\\
&\qquad\qquad\qquad\qquad + \sin\theta_1\sin(\psi_1+\psi_2)d\phi_1 d\theta_2 + \sin\theta_2\sin(\psi_1+\psi_2)d\theta_1 d\phi_2\Big]\\
&= \sum_{r=1}^{2} \rho_{rr}^{(1)}\left(d\theta_r^2 + \sin^2\theta_r d\phi_r^2\right) + \rho_{11}^{(0)}\left(d\psi_1 + d\psi_2 + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2\\
&\quad - \left(\rho_{12}^{(1)} + \rho_{21}^{(1)}\right)\Big[-\sin\theta_1\sin\theta_2\cos(\psi_1+\psi_2)d\phi_1 d\phi_2 + \cos(\psi_1+\psi_2)d\theta_1 d\theta_2\\
&\qquad\qquad\qquad\qquad + \sin\theta_1\sin(\psi_1+\psi_2)d\phi_1 d\theta_2 + \sin\theta_2\sin(\psi_1+\psi_2)d\theta_1 d\phi_2\Big], \tag{3.51}
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu\\
&= \Bbbk\cos\theta_1\, d\phi_1 \wedge d\psi_1 + \Bbbk\cos\theta_2\, d\psi_2 \wedge d\phi_2\\
&\quad + \left(\rho_{12}^{(0)} - \rho_{21}^{(0)}\right)\left(\cos\theta_1 d\phi_1 + d\psi_1\right) \wedge \left(\cos\theta_2 d\phi_2 + d\psi_2\right)\\
&\quad + \left(\rho_{12}^{(1)} - \rho_{21}^{(1)}\right)\Big[-\sin\theta_1\sin\theta_2\cos(\psi_1+\psi_2)\, d\phi_1 \wedge d\phi_2 + \cos(\psi_1+\psi_2)\, d\theta_1 \wedge d\theta_2\\
&\qquad\qquad\qquad\qquad + \sin\theta_1\sin(\psi_1+\psi_2)\, d\phi_1 \wedge d\theta_2 + \sin\theta_2\sin(\psi_1+\psi_2)\, d\theta_1 \wedge d\phi_2\Big]\\
&= \Bbbk\left(d\psi_1 + d\psi_2 + \cos\theta_1 d\phi_1\right) \wedge \left(d\psi_1 + d\psi_2 + \cos\theta_2 d\phi_2\right)\\
&\quad + \left(\rho_{12}^{(1)} - \rho_{21}^{(1)}\right)\Big[-\sin\theta_1\sin\theta_2\cos(\psi_1+\psi_2)\, d\phi_1 \wedge d\phi_2 + \cos(\psi_1+\psi_2)\, d\theta_1 \wedge d\theta_2\\
&\qquad\qquad\qquad\qquad + \sin\theta_1\sin(\psi_1+\psi_2)\, d\phi_1 \wedge d\theta_2 + \sin\theta_2\sin(\psi_1+\psi_2)\, d\theta_1 \wedge d\phi_2\Big]. \tag{3.52}
\end{aligned}$$

By taking a gauge choice $\psi_2 = 0$, these are further simplified as

$$
\begin{aligned}
\frac{1}{2}G_{\mu\nu}dx^\mu dx^\nu &= \sum_{r=1}^{2} \rho_{rr}^{(1)} \left(d\theta_r^2 + \sin^2\theta_r d\phi_r^2 \right) + \rho_{11}^{(0)} \left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 \right)^2 \\
&\quad - \left(\rho_{12}^{(1)} + \rho_{21}^{(1)} \right) \Big[-\sin\theta_1 \sin\theta_2 \cos\psi \, d\phi_1 d\phi_2 + \cos\psi \, d\theta_1 d\theta_2 \\
&\qquad\qquad\qquad\qquad + \sin\theta_1 \sin\psi \, d\phi_1 d\theta_2 + \sin\theta_2 \sin\psi \, d\theta_1 d\phi_2 \Big],
\end{aligned}
\tag{3.53}
$$

$$
\begin{aligned}
B &= \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu \\
&= \Bbbk \left(d\psi + \cos\theta_1 d\phi_1 \right) \wedge \left(d\psi + \cos\theta_2 d\phi_2 \right) \\
&\quad + \left(\rho_{12}^{(1)} - \rho_{21}^{(1)} \right) \Big[-\sin\theta_1 \sin\theta_2 \cos\psi \, d\phi_1 \wedge d\phi_2 + \cos\psi \, d\theta_1 \wedge d\theta_2 \\
&\qquad\qquad\qquad\qquad + \sin\theta_1 \sin\psi \, d\phi_1 \wedge d\theta_2 + \sin\theta_2 \sin\psi \, d\theta_1 \wedge d\phi_2 \Big],
\end{aligned}
\tag{3.54}
$$

where we have renamed ψ_1 as ψ.

In the following, we will consider two specific cases by taking some parametrization.

Example 1) Anisotropic $T^{1,1}$ model. For simplicity, let us first impose the following condition:

$$
\rho_{12}^{(1)} + \rho_{21}^{(1)} = 0 \,.
\tag{3.55}
$$

This condition (3.55) is solved in terms of z_1, z_2 as

$$
\zeta_+^2 = \frac{z_1^2 z_2^2}{z_1^2 + z_2^2 - \zeta_-^2} \quad \Leftrightarrow \quad \zeta_-^2 = \frac{-z_1^2 z_2^2 + z_1^2 \zeta_+^2 + z_2^2 \zeta_+^2}{\zeta_+^2} \,.
\tag{3.56}
$$

By introducing a new quantity r defined as

$$
r \equiv \rho_{22}^{(1)} / \rho_{11}^{(1)} = \frac{-z_2^2 + \zeta_-^2}{z_1^2 - \zeta_-^2} \,,
\tag{3.57}
$$

the coefficients are expressed as

$$
\begin{aligned}
\rho_{11}^{(1)} &= \frac{K}{2} \frac{1}{z_1^2 + r z_2^2} \,, \\
\rho_{22}^{(1)} &= r \rho_{11}^{(1)} \,, \\
\rho_{11}^{(0)} &= \frac{r}{1+r} \rho_{11}^{(1)} \,, \\
\rho_{12}^{(1)} - \rho_{21}^{(1)} &= -\frac{4r z_1 z_2}{(1+r)(z_1^2 - z_2^2)} \rho_{11}^{(1)} \,, \\
\Bbbk &= -\frac{2r(z_1^2 + z_2^2)}{(1+r)(z_1^2 - z_2^2)} \rho_{11}^{(1)} \,.
\end{aligned}
\tag{3.58}
$$

Then the metric, B-field and the twist function are given by, respectively,

$$\frac{1}{2}G_{\mu\nu}dx^\mu dx^\nu = \rho_{11}^{(1)}\Big[\big(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\big) + r\big(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\big)$$
$$+ \frac{r}{1+r}\big(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\big)^2\Big], \tag{3.59}$$

$$\frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu = \rho_{11}^{(1)}\Big[-\frac{2r(z_1^2 + z_2^2)}{(1+r)(z_1^2 - z_2^2)}\big(d\psi + \cos\theta_1 d\phi_1\big) \wedge \big(d\psi + \cos\theta_2 d\phi_2\big)$$
$$\pm -\frac{4rz_1z_2}{(1+r)(z_1^2 - z_2^2)}$$
$$\Big(-\sin\theta_1 \sin\theta_2 \cos\psi\, d\phi_1 \wedge d\phi_2 + \cos\psi\, d\phi_1 \wedge d\theta_2 \tag{3.60}$$
$$+ \sin\theta_1 \sin\psi\, d\phi_1 \wedge d\theta_2 + \sin\theta_2 \sin\psi\, d\theta_1 \wedge d\phi_2\Big)\Big],$$

$$\varphi_{\mathrm{ABL}} = 2K\frac{z\big(rz_1^2 + z_2^2 - (1+r)z^2\big)\big((1+r)z_1^2 z_2^2 - (z_1^2 + rz_2^2)z^2\big)}{(1+r)(z_1^2 + rz_2^2)(z^2 - z_1^2)^2(z^2 - z_2^2)^2}. \tag{3.61}$$

There remain three independent parameters $\rho_{11}^{(1)}$, r and z_1/z_2 now. Note here that the original $T^{1,1}$ case is not included in this example because the vanishing B-field means that $r = 0$. The GMM model is also not included. The parameter r may be rather seen as an anisotropic parameter. In the isotropic case with $r = 1$, the coefficient of the U(1)-fiber is fixed as $1/2$ and the B-field remains complicated.

Finally, the Lax pair is given by

$$\mathcal{L}_+ = \frac{1}{(1+r)z_1^2 z_2^2 - (z_1^2 + rz_2^2)z^2} \times$$
$$\times \Big[z_1^2(z_2^2 - z^2)\Big(J_{1,+}^{(0)} + \frac{z}{z_1}J_{1,+}^{(1)}\Big) + rz_2^2(z_1^2 - z^2)\Big(J_{2,+}^{(0)} + \frac{z}{z_2}J_{2,+}^{(1)}\Big)\Big], \tag{3.62}$$
$$\mathcal{L}_- = \frac{1}{rz_1^2 + z_2^2 - (1+r)z^2}\Big[(z_2^2 - z^2)\Big(J_{1,-}^{(0)} + \frac{z_1}{z}J_{1,-}^{(1)}\Big) + r(z_1^2 - z^2)\Big(J_{2,-}^{(0)} + \frac{z_2}{z}J_{2,-}^{(1)}\Big)\Big].$$

Example 2) G/H λ-model. As the second example, let us suppose the following conditions:

$$z_1^2 = \zeta_+^2, \quad z_2^2 = \zeta_-^2. \tag{3.63}$$

Then the parameters in (3.25) and (3.26) are expressed as

$$\mathcal{k} = K\frac{\zeta_-^2 - \zeta_+^2}{(z_1^2 - z_2^2)^2} = -K\frac{1}{z_1^2 - z_2^2},$$
$$\rho_{11}^{(0)} = \rho_{22}^{(0)} = \rho_{11}^{(1)} = \rho_{22}^{(1)} = \frac{\mathcal{k}}{2}, \tag{3.64}$$
$$\rho_{21}^{(0)} = -\mathcal{k}, \qquad \rho_{21}^{(1)} = \frac{z_1}{z_2}\rho_{21}^{(0)},$$
$$\rho_{12}^{(0)} = \rho_{12}^{(1)} = 0.$$

This parametrization corresponds to the limit where the poles and zeros of the twist function (3.1) coincide. In this case, the resulting action and Lax pair are given by, respectively,

$$S[g_1, g_2] = \left\{ \sum_{r=1}^{2} \int d^2\sigma \frac{\hbar}{2} \operatorname{tr}(J_{r,+}, J_{r,-}) + \hbar I_{\mathrm{WZ}}[g_1] - \hbar I_{\mathrm{WZ}}[g_2] \right.$$

$$\left. - \hbar \int d^2\sigma \operatorname{tr}\left(J_{2,+}^{(0)} J_{1,-}^{(0)}\right) - \frac{z_1}{z_2} \hbar \int d^2\sigma \operatorname{tr}\left(J_{2,+}^{(1)} J_{1,-}^{(1)}\right) \right\}, \qquad (3.65)$$

$$\mathcal{L} = \left(J_{2,+}^{(0)} + z J_{2,+}^{(1)}\right) d\sigma^+ + \left(J_{1,-}^{(0)} + \frac{z_1}{z_2 z} J_{1,-}^{(1)}\right) d\sigma^- . \qquad (3.66)$$

This is nothing but the Lagrangian and Lax pair of a $G \times G/H$ sigma model related to the tripled G/H λ-model formulation[4] [54, 55]. Notably, in the limit $z_1/z_2 \to 0$, the action (3.65) reduces to that of the GMM model [52]. Another family including the GMM model can be obtained by considering a scaling limit described in appendix A. See (A.6).

The flatness condition for this Lax pair is obtained as

$$0 = \partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-]$$

$$= \partial_+ J_{1,-}^{(0)} - \partial_- J_{2,+}^{(0)} + \left[J_{2,+}^{(0)}, J_{1,-}^{(0)}\right] + \frac{z_1}{z_2} \left[J_{2,+}^{(1)}, J_{1,-}^{(1)}\right]$$

$$+ \frac{z_1}{z_2 z} \left(\partial_+ J_{1,-}^{(1)} + \left[J_{2,+}^{(0)}, J_{1,-}^{(1)}\right]\right) + z \left(-\partial_- J_{2,+}^{(1)} + \left[J_{2,+}^{(1)}, J_{1,-}^{(0)}\right]\right) , \qquad (3.67)$$

and this is equivalent to the equations of motion of the model. Note that as we take the limit $z_1/z_2 \to 0$, the term with $1/z$ is lost, and thus we cannot reproduce all of the equations of motions. A possible way to care this point would be to prepare another Lax pair by scaling the spectral parameter as $z' = z z_2/z_1$ [46].

The background metric and B-field for the model (3.65) are given by, respectively,

$$G_{\mu\nu} dx^\mu dx^\nu = \hbar \sum_{r=1}^{2} \left(d\theta_r^2 + \sin^2\theta_r d\phi_r^2\right) + \hbar \left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2$$

$$+ \frac{2z_1}{z_2} \hbar \Big[- \sin\theta_1 \sin\theta_2 \cos\psi \, d\phi_1 d\phi_2 + \cos\psi \, d\theta_1 d\theta_2 \qquad (3.68)$$

$$+ \sin\theta_1 \sin\psi \, d\phi_1 d\theta_2 + \sin\theta_2 \sin\psi \, d\theta_1 d\phi_2 \Big] ,$$

$$B = \hbar \left(d\psi + \cos\theta_1 d\phi_1\right) \wedge \left(d\psi + \cos\theta_2 d\phi_2\right)$$

$$+ \frac{z_1}{z_2} \hbar \Big[- \sin\theta_1 \sin\theta_2 \cos\psi \, d\phi_1 \wedge d\phi_2 + \cos\psi \, d\theta_1 \wedge d\theta_2 \qquad (3.69)$$

$$+ \sin\theta_1 \sin\psi \, d\phi_1 \wedge d\theta_2 + \sin\theta_2 \sin\psi \, d\theta_1 \wedge d\phi_2 \Big] .$$

Notably, this result coincides with a parafermionic deformation of the GMM background presented in [56]. When $z_1/z_2 = 0$, the target space of the original GMM model is reproduced.

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[4]The deformation parameter z_1/z_2 corresponds to $\tilde{\lambda}$ in [54].

4 Conclusion and discussion

In this paper, we have derived the ABL model from a 4D CS theory with a meromorphic one-form (3.1) with four double poles and six simple zeros by specifying a boundary condition. Then we have explicitly derived the sigma-model background with metric and anti-symmetric two-form (i.e., the ABL background). As its special cases, we have presented an anisotropic $T^{1,1}$ model and a G/H λ-model. The latter can be regarded as a one-parameter integrable deformation of the GMM model.

It would be very interesting to consider the ABL background in the context of AdS/CFT. The first task is to find out a possible string-theory embedding of the ABL background. It is nice to try to identify the remaining components of type IIB supergravity for the ABL background. A possibility is to consider a variant of the ABL model for $G = \mathrm{SL}(2)$ and $H = \mathrm{U}(1)$. This is a natural extension of the work [57], which considered the GMM model for $G = \mathrm{SU}(2) \times \mathrm{SL}(2)$ and $H = \mathrm{U}(1) \times \mathrm{U}(1)$ so as to be a supergravity background. Once the ABL background has been embedded into string theory, it would be nice to explore the dual gauge theory.

Another future direction is to consider a relation between the present result and 6D holomorphic CS theory [58, 59]. Along this line, it may be possible to derive a new family of 4D integrable systems. We would like to report some results in the near future in another place.

We hope that the ABL background we have derived would open up a new arena in the study of the integrability in AdS/CFT.

A   A scaling limit of the ABL model

It is helpful to give a brief explanation about a scaling limit of the ABL model considered in [46].

Let us first redefine the parameters of the model as

$$z_1 = 1\,, \qquad z_2 = \frac{1}{\alpha}\,, \qquad K = \frac{\lambda_2^2}{\alpha^2}\,, \qquad \zeta_1 = \frac{\lambda}{\lambda_2 \alpha}\,, \qquad \zeta_2 = \frac{\lambda_1}{\lambda}\,, \qquad (A.1)$$

and then take the $\alpha \to 0$ limit while keeping λ_1, λ_2 and λ fixed. In this limit, the twist function (3.1) becomes

$$\varphi_1(z) = \frac{2z(\lambda_1^2 - \lambda^2 z^2)}{(z^2 - 1)^2} . \tag{A.2}$$

The poles and zeros of $\varphi_1(z)$ are listed as

$$\mathfrak{p} = \{\pm 1, \infty\}, \qquad \mathfrak{z} = \left\{0, \pm\frac{\lambda_1}{\lambda}\right\} . \tag{A.3}$$

Here, $z = \pm 1$ are double poles and $z = \infty$ is a simple pole, $z = 0, \pm\frac{\lambda_1}{\lambda}$ are simple zeros. The Lax pair (3.43) reduces to

$$\mathcal{L}_+(z) = \frac{1}{\lambda^2 z^2 - \lambda_1^2} \left((\lambda^2 - \lambda_1^2) \left(J_{1,+}^{(0)} + z\, J_{1,+}^{(1)} \right) + \lambda^2(z^2 - 1)\, J_{2,+}^{(0)} \right) , \tag{A.4}$$

$$\mathcal{L}_-(z) = J_{1,-}^{(0)} + z^{-1}\, J_{1,-}^{(1)} , \tag{A.5}$$

and the resulting action is given by

$$S[g_1, g_2] = \sum_{r=1}^{2} \int d^2\sigma \left(\frac{\lambda^2}{2} \left\langle J_{r,+}^{(0)}, J_{r,-}^{(0)} \right\rangle + \frac{\lambda_r^2}{2} \left\langle J_{r,+}^{(1)}, J_{r,-}^{(1)} \right\rangle \right) - \lambda^2 \left\langle J_{2,+}^{(0)}, J_{1,-}^{(0)} \right\rangle$$
$$+ \lambda^2\, I_{\mathrm{WZ}}[g_1] - \lambda^2\, I_{\mathrm{WZ}}[g_2] . \tag{A.6}$$

Note here that the flatness condition of the Lax pair (A.5) does not describe all the equations of motion of the action (A.6). The specific choice $\lambda^2 = \lambda_1^2 = \lambda_2^2$ leads to the GMM model [52, 57]. This choice corresponds to the limit $z_1/z_2 \to 0$ in (3.65).

The other equations of motion come from the flatness condition of another Lax pair which can be obtained by taking a different scaling limit. To see this, let us redefine z as z/α, and then take the same limit (A.1). Then, the twist function (3.1) becomes

$$\varphi_2(z) = -\frac{2(\lambda^2 - \lambda_2^2 z^2)}{z(z^2 - 1)^2} . \tag{A.7}$$

The poles and zeros of $\varphi_2(z)$ are listed as

$$\mathfrak{p} = \{0, \pm 1\}, \qquad \mathfrak{z} = \left\{\infty, \pm\frac{\lambda}{\lambda_2}\right\} , \tag{A.8}$$

where $z = \pm 1$ are double poles and $z = 0$ is a simple pole, $z = \infty, \pm\frac{\lambda_1}{\lambda}$ are simple zeros. The Lax pair (3.43) becomes

$$\tilde{\mathcal{L}}_+(z) = J_{2,+}^{(0)} + z\, J_{2,+}^{(1)} , \tag{A.9}$$

$$\tilde{\mathcal{L}}_-(z) = \frac{1}{\lambda_2^2 z^2 - \lambda^2} \left(\lambda^2(z^2 - 1)\, J_{-,1}^{(0)} + (\lambda_2^2 - \lambda^2) \left(z^2 J_{-,2}^{(0)} + z\, J_{-,2}^{(1)} \right) \right) . \tag{A.10}$$

We can check that the flatness condition of (A.10) leads to the remaining equations of motion of the action (A.6).

References

[1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [`hep-th/9711200`] [ɪɴSPIRE].

[2] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`] [ɪɴSPIRE].

[3] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [`hep-th/9802109`] [ɪɴSPIRE].

[4] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [`arXiv:1012.3982`] [ɪɴSPIRE].

[5] O. Lunin and J.M. Maldacena, *Deforming field theories with* $U(1) \times U(1)$ *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [`hep-th/0502086`] [ɪɴSPIRE].

[6] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [`hep-th/0503201`] [ɪɴSPIRE].

[7] A. Hashimoto and N. Itzhaki, *Noncommutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett. B* **465** (1999) 142 [`hep-th/9907166`] [ɪɴSPIRE].

[8] J.M. Maldacena and J.G. Russo, *Large N limit of noncommutative gauge theories*, *JHEP* **09** (1999) 025 [`hep-th/9908134`] [ɪɴSPIRE].

[9] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, *JHEP* **11** (2008) 080 [`arXiv:0807.1099`] [ɪɴSPIRE].

[10] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non-relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [`arXiv:0807.1100`] [ɪɴSPIRE].

[11] A. Adams, K. Balasubramanian and J. McGreevy, *Hot Spacetimes for Cold Atoms*, *JHEP* **11** (2008) 059 [`arXiv:0807.1111`] [ɪɴSPIRE].

[12] C. Klimčík, *Yang-Baxter sigma models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [`hep-th/0210095`] [ɪɴSPIRE].

[13] C. Klimčík, *On integrability of the Yang-Baxter sigma-model*, *J. Math. Phys.* **50** (2009) 043508 [`arXiv:0802.3518`] [ɪɴSPIRE].

[14] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the* $AdS_5 \times S^5$ *superstring action*, *Phys. Rev. Lett.* **112** (2014) 051601 [`arXiv:1309.5850`] [ɪɴSPIRE].

[15] F. Delduc, M. Magro and B. Vicedo, *Derivation of the action and symmetries of the q-deformed* $AdS_5 \times S^5$ *superstring*, *JHEP* **10** (2014) 132 [`arXiv:1406.6286`] [ɪɴSPIRE].

[16] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the* $AdS_5 \times S^5$ *superstring*, *JHEP* **04** (2014) 153 [`arXiv:1401.4855`] [ɪɴSPIRE].

[17] C. Klimčík, *Integrability of the bi-Yang-Baxter sigma-model*, *Lett. Math. Phys.* **104** (2014) 1095 [`arXiv:1402.2105`] [ɪɴSPIRE].

[18] K. Sfetsos, *Integrable interpolations: From exact CFTs to non-Abelian T-duals*, *Nucl. Phys. B* **880** (2014) 225 [`arXiv:1312.4560`] [ɪɴSPIRE].

[19] S. Demulder, K. Sfetsos and D.C. Thompson, *Integrable λ-deformations: Squashing Coset CFTs and $AdS_5 \times S^5$*, *JHEP* **07** (2015) 019 [`arXiv:1504.02781`] [ɪɴSPIRE].

[20] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *An Integrable Deformation of the $AdS_5 \times S^5$ Superstring*, *J. Phys. A* **47** (2014) 495402 [`arXiv:1409.1538`] [ɪɴSPIRE].

[21] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *Integrable Deformations of Strings on Symmetric Spaces*, *JHEP* **11** (2014) 009 [`arXiv:1407.2840`] [ɪɴSPIRE].

[22] K. Costello and M. Yamazaki, *Gauge Theory And Integrability, III*, `arXiv:1908.02289` [ɪɴSPIRE].

[23] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *A unifying 2d action for integrable σ-models from 4d Chern-Simons theory*, *Lett. Math. Phys.* **110** (2020) 1645 [`arXiv:1909.13824`] [ɪɴSPIRE].

[24] O. Fukushima, J.-i. Sakamoto and K. Yoshida, *Comments on η-deformed principal chiral model from 4D Chern-Simons theory*, *Nucl. Phys. B* **957** (2020) 115080 [`arXiv:2003.07309`] [ɪɴSPIRE].

[25] O. Fukushima, J.-i. Sakamoto and K. Yoshida, *Yang-Baxter deformations of the $AdS_5 \times S^5$ supercoset sigma model from 4D Chern-Simons theory*, *JHEP* **09** (2020) 100 [`arXiv:2005.04950`] [ɪɴSPIRE].

[26] O. Fukushima, J.-i. Sakamoto and K. Yoshida, *Faddeev-Reshetikhin model from a 4D Chern-Simons theory*, *JHEP* **02** (2021) 115 [`arXiv:2012.07370`] [ɪɴSPIRE].

[27] K. Costello and B. Stefański, *Chern-Simons Origin of Superstring Integrability*, *Phys. Rev. Lett.* **125** (2020) 121602 [`arXiv:2005.03064`] [ɪɴSPIRE].

[28] D.M. Schmidtt, *Holomorphic Chern-Simons theory and lambda models: PCM case*, *JHEP* **04** (2020) 060 [`arXiv:1912.07569`] [ɪɴSPIRE].

[29] J. Tian, *Comments on λ-deformed models from 4D Chern-Simons theory*, `arXiv:2005.14554` [ɪɴSPIRE].

[30] J. Tian, Y.-J. He and B. Chen, *λ-deformed $AdS_5 \times S^5$ superstring from 4D Chern-Simons theory*, `arXiv:2007.00422` [ɪɴSPIRE].

[31] B. Chen, Y.-J. He and J. Tian, *Deformed Integrable Models from Holomorphic Chern-Simons Theory*, `arXiv:2105.06826` [ɪɴSPIRE].

[32] S. Lacroix and B. Vicedo, *Integrable $\mathcal{E}$-Models, 4d Chern-Simons Theory and Affine Gaudin Models. I. Lagrangian Aspects*, *SIGMA* **17** (2021) 058 [`arXiv:2011.13809`] [ɪɴSPIRE].

[33] M. Benini, A. Schenkel and B. Vicedo, *Homotopical analysis of 4d Chern-Simons theory and integrable field theories*, `arXiv:2008.01829` [ɪɴSPIRE].

[34] V. Caudrelier, M. Stoppato and B. Vicedo, *On the Zakharov-Mikhailov action: 4d Chern-Simons origin and covariant Poisson algebra of the Lax connection*, *Lett. Math. Phys.* **111** (2021) 82 [`arXiv:2012.04431`] [ɪɴSPIRE].

[35] L.A. Pando Zayas and C.A. Terrero-Escalante, *Chaos in the Gauge/Gravity Correspondence*, *JHEP* **09** (2010) 094 [`arXiv:1007.0277`] [ɪɴSPIRE].

[36] P. Basu, D. Das and A. Ghosh, *Integrability Lost*, *Phys. Lett. B* **699** (2011) 388 [`arXiv:1103.4101`] [ɪɴSPIRE].

[37] P. Basu, D. Das, A. Ghosh and L.A. Pando Zayas, *Chaos around Holographic Regge Trajectories*, *JHEP* **05** (2012) 077 [`arXiv:1201.5634`] [ɪɴSPIRE].

[38] T. Ishii, K. Murata and K. Yoshida, *Fate of chaotic strings in a confining geometry*, *Phys. Rev. D* **95** (2017) 066019 [arXiv:1610.05833] [INSPIRE].

[39] P. Candelas and X.C. de la Ossa, *Comments on Conifolds*, *Nucl. Phys. B* **342** (1990) 246 [INSPIRE].

[40] P. Basu and L.A. Pando Zayas, *Chaos rules out integrability of strings on $AdS_5 \times T^{1,1}$*, *Phys. Lett. B* **700** (2011) 243 [arXiv:1103.4107] [INSPIRE].

[41] Y. Asano, D. Kawai, H. Kyono and K. Yoshida, *Chaotic strings in a near Penrose limit of $AdS_5 \times T^{1,1}$*, *JHEP* **08** (2015) 060 [arXiv:1505.07583] [INSPIRE].

[42] P. Basu and L.A. Pando Zayas, *Analytic Non-integrability in String Theory*, *Phys. Rev. D* **84** (2011) 046006 [arXiv:1105.2540] [INSPIRE].

[43] P.M. Crichigno, T. Matsumoto and K. Yoshida, *Deformations of $T^{1,1}$ as Yang-Baxter sigma models*, *JHEP* **12** (2014) 085 [arXiv:1406.2249] [INSPIRE].

[44] L. Rado, V.O. Rivelles and R. Sánchez, *Yang-Baxter deformations of the $AdS_5 \times T^{1,1}$ superstring and their backgrounds*, *JHEP* **02** (2021) 126 [arXiv:2010.14081] [INSPIRE].

[45] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, *Nucl. Phys. B* **536** (1998) 199 [hep-th/9807080] [INSPIRE].

[46] G. Arutyunov, C. Bassi and S. Lacroix, *New integrable coset sigma models*, *JHEP* **03** (2021) 062 [arXiv:2010.05573] [INSPIRE].

[47] B. Vicedo, *On integrable field theories as dihedral affine Gaudin models*, *Int. Math. Res. Not.* **2020** (2020) 4513 [arXiv:1701.04856] [INSPIRE].

[48] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *Assembling integrable σ-models as affine Gaudin models*, *JHEP* **06** (2019) 017 [arXiv:1903.00368] [INSPIRE].

[49] B. Vicedo, *Holomorphic Chern-Simons theory and affine Gaudin models*, arXiv:1908.07511 [INSPIRE].

[50] K.S. Rigatos, *Nonintegrability of $L^{a,b,c}$ quiver gauge theories*, *Phys. Rev. D* **102** (2020) 106022 [arXiv:2009.11878] [INSPIRE].

[51] T. Ishii, S. Kushiro and K. Yoshida, *Chaotic string dynamics in deformed $T^{1,1}$*, arXiv:2103.12416 [INSPIRE].

[52] E. Guadagnini, M. Martellini and M. Mintchev, *Scale invariance sigma models on homogeneous spaces*, *Phys. Lett. B* **194** (1987) 69 [INSPIRE].

[53] A.M. Polyakov and P.B. Wiegmann, *Theory of Nonabelian Goldstone Bosons*, *Phys. Lett. B* **131** (1983) 121 [INSPIRE].

[54] B. Hoare, N. Levine and A.A. Tseytlin, *Integrable sigma models and 2-loop RG flow*, *JHEP* **12** (2019) 146 [arXiv:1910.00397] [INSPIRE].

[55] N. Levine and A.A. Tseytlin, *Integrability vs. RG flow in $G \times G$ and $G \times G/H$ sigma models*, *JHEP* **05** (2021) 076 [arXiv:2103.10513] [INSPIRE].

[56] G. Georgiou, E. Sagkrioti, K. Sfetsos and K. Siampos, *An exact symmetry in λ-deformed CFTs*, *JHEP* **01** (2020) 083 [arXiv:1911.02027] [INSPIRE].

[57] L.A. Pando Zayas and A.A. Tseytlin, *Conformal sigma models for a class of $T^{p,q}$ spaces*, *Class. Quant. Grav.* **17** (2000) 5125 [hep-th/0007086] [INSPIRE].

[58] K. Costello, *Topological strings, twistors and Skyrmions*, (2020), http://web.math.ucsb.edu/~drm/WHCGP/.

[59] R. Bittleston and D. Skinner, *Twistors, the ASD Yang-Mills equations, and 4d Chern-Simons theory*, arXiv:2011.04638 [INSPIRE].