Two-loop HTL-resummed thermodynamics for $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

Qianqian Du,[a,b] **Michael Strickland,**[b] **Ubaid Tantary**[b] **and Ben-Wei Zhang**[a]

[a]*Institute of Particle Physics and Key Laboratory of Quark and Lepton Physics (MOS),*
Central China Normal University,
Wuhan, 430079, China

[b]*Department of Physics, Kent State University,*
Kent, OH 44242, U.S.A.

E-mail: duqianqianstudent@mails.ccnu.edu.cn, mstrick6@kent.edu,
utantary@kent.edu, bwzhang@mail.ccnu.edu.cn

ABSTRACT: We compute the two-loop hard-thermal-loop (HTL) resummed thermodynamic potential for $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM). Our final result is manifestly gauge-invariant and was renormalized using only simple vacuum energy, gluon mass, scalar mass, and quark mass counter terms. The HTL mass parameters m_D, M_D, and m_q are then determined self-consistently using a variational prescription which results in a set of coupled gap equations. Based on this, we obtain the two-loop HTL-resummed thermodynamic functions of $\mathcal{N} = 4$ SYM. We compare our final result with known results obtained in the weak- and strong-coupling limits. We also compare to previously obtained approximately self-consistent HTL resummations and Padé approximants. We find that the two-loop HTL resummed results for the scaled entropy density is a quantitatively reliable approximation to the scaled entropy density for $0 \leq \lambda \lesssim 2$ and is in agreement with previous approximately self-consistent HTL resummation results for $\lambda \lesssim 6$.

Contents

1 Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) is the most famous example of a conformal field theory (CFT) in four dimensions, and is often taken as a model for hot QCD in the large number of colors N_c and strong 't Hooft coupling $\lambda = g^2 N_c$ limits. The strong coupling behavior of the free energy has been computed using the anti-de Sitter space/CFT (AdS/CFT) correspondence [1], with the result being

$$\frac{\mathcal{F}}{\mathcal{F}_{\mathrm{ideal}}} = \frac{\mathcal{S}}{\mathcal{S}_{\mathrm{ideal}}} = \frac{3}{4}\left[1 + \frac{15}{8}\zeta(3)\lambda^{-3/2} + \mathcal{O}(\lambda^{-2})\right], \tag{1.1}$$

where $\mathcal{F}_{\mathrm{ideal}} = -d_A \pi^2 T^4/6$ is the ideal or Stefan-Boltzmann limit of the free energy and $\mathcal{S}_{\mathrm{ideal}} = 2 d_A \pi^2 T^3/3$, with $d_A = N_c^2 - 1$ being the dimension of the adjoint representation.

In the weak-couping limit the $\mathcal{N} = 4$ SYM free energy has been calculated through order $\lambda^{3/2}$ giving [2–4]

$$\frac{\mathcal{F}}{\mathcal{F}_{\mathrm{ideal}}} = \frac{\mathcal{S}}{\mathcal{S}_{\mathrm{ideal}}} = 1 - \frac{3}{2\pi^2}\lambda + \frac{3 + \sqrt{2}}{\pi^3}\lambda^{3/2} + \mathcal{O}(\lambda^2)\,. \tag{1.2}$$

Note that, since the beta function of the $\mathcal{N} = 4$ SYM theory is zero, the coupling constant does not run and is independent of the temperature. As a result, we can vary the coupling between the two limits at each temperature.

One expects these two series to describe their respective asymptotic limits correctly, however, the radius of convergence of each of these series is unknown and, therefore, it is unclear to what degree each of these can be trusted away from their respective limits. In figure 1 we plot the scaled entropy density resulting from eqs. (1.1) and (1.2) as a function of λ along with a $R_{[4,4]}$ Padé approximant constructed from these results [3, 5].[1] From this figure, we can see that the two successive weak coupling approximations are only close to one another below $\lambda \sim 0.1$ and rapidly diverge beyond $\lambda \sim 1$. In the strong coupling limit, only the first two terms in the series are known. As can be seen from figure 1 the strong coupling result diverges quickly below $\lambda \sim 10$. The question then becomes, how can we systematically extend these two results into the *intermediate coupling* region $\lambda \sim 1 - 10$. In this paper, we present progress towards this goal in the weak-coupling limit

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[1]Although a Padé approximant might provide a convenient interpolation between the weak- and strong-coupling limits their construction is in no sense systematic. In particular, the resulting expressions are incomplete since we know that, at least at weak coupling, the series will contain logarithms of the coupling constant beyond $\mathcal{O}(\lambda^{3/2})$.
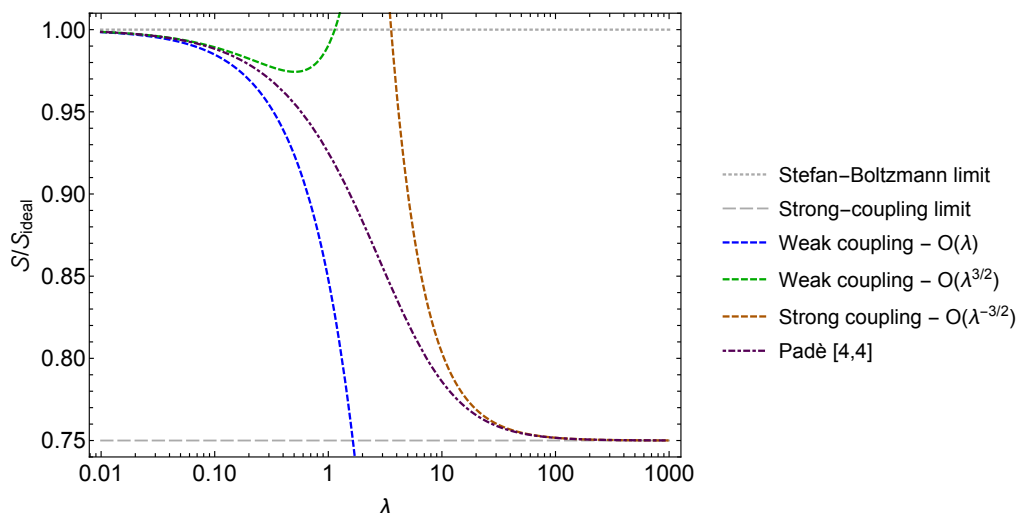
Figure 1. Weak and strong coupling results for the entropy density in $\mathcal{N} = 4$ SYM theory compared to a $R_{[4,4]}$ Padè approximation constructed from both limits.

using hard-thermal-loop (HTL) perturbation theory. Our study is complementary to the earlier work of Blaizot, Iancu, and Rebhan in which they applied HTL resummation using an approximately self-consistent scheme [5]. The key difference from this earlier work is that our result is based on the HTL perturbation theory (HTLpt) framework which can be used to extend such calculations to arbitrary loop order in a gauge-invariant manner [6–8].

The goal of our work is to improve the convergence of the successive weak-coupling approximations. One promising approach is to use a variational framework in which the free energy $\mathcal{F}$ is expressed as the variational minimum of the thermodynamic potential $\Omega(T, \lambda; m^2)$ that depends on one or more variational parameters that we denote collectively by $\mathbf{a}$

$$\mathcal{F}(T, \lambda) = \Omega(T, \lambda; \mathbf{a})\big|_{\partial\Omega/\partial\mathbf{a}=0} \, . \tag{1.3}$$

For example, the Φ-derivable approximation is a widely used variational method in which the propagator is used as an infinite set of variational parameters [9, 10]. The Φ-derivable thermodynamic potential Ω is given by the 2-particle-irreducible (2PI) effective action, which is the sum of all diagrams that are 2-particle-irreducible with respect to the complete propagator [11]. This method is difficult to apply to relativistic field theories except for the case where the self-energy is momentum-independent. Despite this, there still have some progress in applications to quantum chromodynamics (QCD) and electrodynamics (QED) [12–17]. Historically, the Φ-derivable approximation was first applied to QCD by Freedman and McLerran [18], who demonstrated that the thermodynamic potential Ω is gauge dependent beyond a given order in the coupling constant. The gauge parameter dependence appears at the same order in α_s as the series truncation when evaluated off the stationary point and at twice the order in α_s when evaluated at the stationary point [17, 19, 20]. Despite this issue, this method had been used as the starting point

for approximately self-consistent HTL resummation of the QCD entropy [21, 22] and the pressure [23].

The problems encountered when applying the Φ-derivable approximation to gauge theories motivated the use of alternative variational approximations. One such alternative, which in its simplest form involves a single variational parameter m, has been called optimized perturbation theory [24], variational perturbation theory [25, 26], or the linear δ expansion [27, 28]. This strategy has been successfully used for the thermodynamics of the massless ϕ^4 field theory up to four-loop order using "screened perturbation theory" [29–32], and spontaneously broken field theories at finite temperature [33–37]. One impediment to applying such ideas to gauge theories was that one cannot simply introduce a scalar mass for the gluon without breaking gauge invariance. The solution to this problem was introduced in refs. [7] and [8] in which it was demonstrated that one could generalize the linear delta expansion by adding and subtracting the full gauge-invariant HTL effective Lagrangian [38, 39].

The resulting scheme was called hard-thermal-loop perturbation theory (HTLpt) [7, 8]. The HTLpt method has been used to improve the convergence of weak coupling calculations of the free energy in QED [40] and QCD up to three-loop order at finite temperature and chemical potential [41–47]. When confronted with finite temperature and chemical potential lattice QCD data the HTLpt resummation scheme works remarkably well down to temperatures on the order of 200–300 MeV where the QCD coupling constant is on the order of $g_s \sim 2$ [46–48]. The method successfully describes all thermodynamic variables including second- and fourth-order quark susceptibilities.

Herein, we will take the first steps in applying this method to $\mathcal{N} = 4$ SYM in the hope that a similar improvement in convergence can be achieved in this theory at intermediate couplings. We will calculate the one- and two-loop HTLpt-resummed thermodynamic potential in $\mathcal{N} = 4$ SYM using the same method as was used to obtain the one- and two-loop QCD results in refs. [7, 8, 49, 50]. Importantly, in these papers it was demonstrated that it was possible to renormalize the resummed thermodynamic potential at two-loop order using simple vacuum and mass counterterms. Herein we will demonstrate the same occurs in $\mathcal{N} = 4$ SYM. In this theory the NLO contributions include scalar and scalar-gluon, scalar-quark interactions compared to the QCD calculation, however these are relatively straightforward to include. Additionally, in $\mathcal{N} = 4$ SYM instead of having only gluon and quark thermal masses, m_D and m_q, respectively, we will also have a thermal mass for the scalar particles, M_D. Our final results at one- and two-loop order are infinite series in λ which, when trucated at $\mathcal{O}(\lambda^{3/2})$, reproduce the weak-coupling results obtained previously in refs. [2–4]. In order to make the calculation tractable, we expand the HTLpt scalarized sum-integrals in a power series in the three mass parameters M_D, m_D, and m_q such that it includes terms that would naively contribute through $\mathcal{O}(\lambda^{5/2})$. Our final results indicate that, in $\mathcal{N} = 4$ SYM, NLO HTLpt provides a good approximation for the scaled entropy for couplings in the range $0 \leq \lambda \lesssim 2$.

We begin with a brief summary of HTLpt for $\mathcal{N} = 4$ SYM in section 2. In section 3, we give the expressions for the one- and two-loop diagrams contributing to the SYM thermodynamic potential. In section 4, we reduce these diagrams to scalar sum-integrals. As

mentioned in the prior paragraph, since it would be intractable to calculate the resulting sum-integrals analytically, in section 5 we expand these expressions by treating m_D, M_D and m_q as $\mathcal{O}(\lambda^{1/2})$ and expanding the integrals in powers of m_D/T, M_D/T and m_q/T, keeping all terms up to $\mathcal{O}(\lambda^{5/2})$. In section 6, we combine the results obtained in section 5 to obtain the complete expressions for the leading- (LO) and next-to-leading order (NLO) thermodynamic potentials. In section 7, we present our numerical results for the HTLpt-resummed LO and NLO scaled thermodynamic functions in $\mathcal{N} = 4$ SYM and compare to prior results in the literature. For details concerning the transformation to Euclidean space and the sum-integrals necessary we refer the reader to the appendices of refs. [49, 50].

Notation and conventions. We use lower-case letters for Minkowski space four-vectors, e.g. p, and upper-case letters for Euclidean space four-vectors, e.g. P. We use the mostly minus convention for the metric.

2 HTLpt for $\mathcal{N} = 4$ SYM

In $\mathcal{N} = 4$ SYM theory all fields belong to the adjoint representation of the $\mathrm{SU}(N_c)$ gauge group. For the fermionic fields, a massless two-component Weyl fermion ψ in four dimensions can be converted to four-component Majorana fermions [51–55]

$$\psi \equiv \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \qquad \text{and} \qquad \bar{\psi} \equiv \begin{pmatrix} \psi^\alpha & \bar{\psi}_{\dot{\alpha}} \end{pmatrix}, \tag{2.1}$$

where $\alpha = 1, 2$ and the Weyl spinors satisfy $\bar{\psi}^{\dot{\alpha}} \equiv [\psi^\alpha]^\dagger$. The conjugate spinor $\bar{\psi}$ is not independent, but is related to ψ via the Majorana condition $\psi = C\bar{\psi}$, where $C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$ is the charge conjugation operator with $\epsilon_{02} = -\epsilon_{11} \equiv -1$. In the following, we will use the indices $i, j = 1, 2, 3, 4$ to enumerate the Majorana fermions and use ψ_i to denote each bispinor.

The definition of gauge field is the same as QCD, and A_μ can be expanded as $A_\mu = A_\mu^a t^a$, with real coefficients A_μ^a, and Hermitian color generators t^a in the fundamental representation that satisfy

$$[t^a, t^b] = i f_{abc} t^c \quad \text{and} \quad \mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}, \tag{2.2}$$

where $a, b = 1, \cdots, N_c^2 - 1$, the structure constants f_{abc} are real and completely antisymmetric. The fermionic fields can similarly be expanded in the basis of color generators as $\psi_i = \psi_i^a t^a$. The coefficients ψ_i^a are four-component Grassmann-valued spinors.

There are six independent real scalar fields which are represented by a multiplet

$$\Phi \equiv (X_1, Y_1, X_2, Y_2, X_3, Y_3), \tag{2.3}$$

where $X_{\mathtt{p}}$ and $Y_{\mathtt{q}}$ hermitian, with $\mathtt{p}, \mathtt{q} = 1, 2, 3$. $X_{\mathtt{p}}$ and $Y_{\mathtt{q}}$ denote scalars and pseudoscalar fields, respectively. We will use a capital Latin index A to denote components of vector Φ. Therefore Φ_A, $X_{\mathtt{p}}$, and $Y_{\mathtt{q}}$ can be expanded as $\Phi_A = \Phi_A^a t^a$, with $A = 1, \cdots, 6$, and $X_{\mathtt{p}} = X_{\mathtt{p}}^a t^a$, $Y_{\mathtt{q}} = Y_{\mathtt{q}}^a t^a$.

The Lagrangian density that generates the perturbative expansion for $\mathcal{N} = 4$ SYM theory in Minkowski-space can be expressed as

$$\mathcal{L}_{\text{SYM}} = \text{Tr}\bigg[-\frac{1}{2}G_{\mu\nu}^2 + (D_\mu \Phi_A)^2 + i\bar{\psi}_i \slashed{D} \psi_i - \frac{1}{2}g^2(i[\Phi_A, \Phi_B])^2$$

$$-ig\bar{\psi}_i\big[\alpha_{ij}^{\text{p}} X_{\text{p}} + i\beta_{ij}^{\text{q}}\gamma_5 Y_{\text{q}}, \psi_j\big]\bigg] + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{gh}} + \Delta\mathcal{L}_{\text{SYM}}\,, \tag{2.4}$$

where the field strength tensor is $G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, and $D_\nu = \partial_\nu - ig[A_\nu, \cdot]$ is the covariant derivative in the adjoint representation. α^{p} and β^{q} are 4×4 matrices that satisfy

$$\{\alpha^{\text{p}}, \alpha^{\text{q}}\} = -2\delta^{\text{pq}}, \quad \{\beta^{\text{p}}, \beta^{\text{q}}\} = -2\delta^{\text{pq}}, \quad [\alpha^{\text{p}}, \beta^{\text{q}}] = 0\,, \tag{2.5}$$

and their explicit form can be given as

$$\alpha^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \qquad \alpha^2 = \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},$$

$$\beta^1 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \qquad \beta^2 = \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \qquad \beta^3 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \tag{2.6}$$

where σ_i with $i \in \{1, 2, 3\}$ are the 2×2 Pauli matrices. And α and β satisfies $\alpha_{ik}^{\text{p}}\alpha_{kj}^{\text{p}} = -3\delta_{ij}$ and $\beta_{ij}^{\text{q}}\beta_{ji}^{\text{p}} = -4\delta^{\text{pq}}$, with $\delta_{ii} = 4$ for four Majorana fermions and $\delta^{\text{pp}} = 3$ for three scalars.

The ghost term $\mathcal{L}_{\text{gh}}$ depends on the choice of the gauge-fixing term $\mathcal{L}_{\text{gf}}$ and is the same as in QCD. Here we work in general covariant gauge, giving

$$\mathcal{L}_{\text{gf}} = -\frac{1}{\xi}\text{Tr}\big[(\partial^\mu A_\mu)^2\big],$$

$$\mathcal{L}_{\text{gh}} = -2\text{Tr}\big[\bar{\eta}\,\partial^\mu D_\mu \eta\big], \tag{2.7}$$

with ξ being the gauge parameter.

In general, perturbative expansion in powers of g in quantum field theory generates ultraviolet divergences. The renormalizability of perturbation theory guarantees that all divergences in physical quantities can be removed by the renormalization of masses and coupling constants. The coupling constant in $\mathcal{N} = 4$ SYM theory is denoted as $\lambda = g^2 N_c$. Unlike QCD, $\mathcal{N} = 4$ SYM theory does not run.

Similar to the case of QCD presented in refs. [49, 50], HTLpt is also a reorganziation of the perturbation series for the SYM theory, and can be defined by introducing an expansion parameter δ. The HTL "shifted" Lagrangian density can be written as

$$\mathcal{L} = (\mathcal{L}_{\text{SYM}} + \mathcal{L}_{\text{HTL}})|_{g\to\sqrt{\delta}g} + \Delta\mathcal{L}_{\text{HTL}}\,. \tag{2.8}$$

The HTL improvement term is

$$\mathcal{L}_{\text{HTL}} = -\frac{1}{2}(1-\delta)m_D^2 \text{Tr}\bigg(G_{\mu\alpha}\bigg\langle \frac{y^\alpha y^\beta}{(y\cdot D)^2}\bigg\rangle_{\hat{\mathbf{y}}} G_\beta^\mu \bigg)$$

$$+(1-\delta)im_q^2\text{Tr}\bigg(\bar{\psi}_j\gamma^\mu\bigg\langle \frac{y^\mu}{y\cdot D}\bigg\rangle_{\hat{\mathbf{y}}} \psi_j \bigg)$$

$$-(1-\delta)M_D^2\text{Tr}(\Phi_A^2)\,, \tag{2.9}$$

where $y^\mu = (1, \hat{\mathbf{y}})$ is a light-like four vector defined in appendix A, $j \in \{1 \ldots 4\}$ indexes the four Majorana fermions, $A \in \{1 \ldots 6\}$ indexes the scalar degrees of freedom, and $\langle \cdots \rangle_{\hat{\mathbf{y}}}$ represents the average over the direction of $\hat{\mathbf{y}}$. The parameters m_D and M_D are the electric screening masses for the gauge field and the adjoint scalar field, respectively. The parameter m_q can be seen as the induced finite temperature quark mass. We note that, if we set $\delta = 1$, the Lagrangian above (2.8) reduces to the vacuum $\mathcal{N} = 4$ SYM Lagrangian (2.4). HTLpt is defined by treating δ as a formal expansion parameter, expanding around $\delta = 0$ to a fixed order, and then setting $\delta = 1$. In the limit that this expansion is taken to all orders, one reproduces the QCD result by construction, however, the loop expansion is now shifted to be around the high-temperature minimum of the effective action, resulting in a reorganization of the perturbation series which has better convergence than the naive expansion loop expansion around the $T = 0$ vacuum. In addition, this reorganization eliminates all infrared divergences associated with the electric sector of the theory.

The HTLpt reorganization generates new ultraviolet (UV) divergences and, due to the renormalizability of perturbation theory, the ultraviolet divergences are constrained to have a form that can be canceled by the counterterm Lagrangian $\Delta\mathcal{L}_{\mathrm{HTL}}$. References [49, 50] demonstrated that at two-loop order the thermodynamic potential can be renormalized using a simple counterterm Lagrangian $\Delta\mathcal{L}_{\mathrm{HTL}}$ containing vacuum and mass counterterms. Although the general structure of the ultraviolet divergences is unknown, it has been demonstrated that one can renormalize the next-to-leading order HTLpt thermodynamic potential through three-loop order using only vacuum, gluon thermal mass, quark thermal mass, and coupling constant counterterms [44, 47]. In this paper, we demonstrate that the same method can be used for $\mathcal{N} = 4$ SYM and we compute the vacuum and screening mass counterterms necessary.

We find that the vacuum counterterm $\Delta_0\mathcal{E}_0$, which is the leading order counterterm in the δ expansion of the vacuum energy $\mathcal{E}_0$, can be obtained by calculating the free energy to leading order in δ. In section 6.1, we show that $\Delta_1\mathcal{E}_0$ can be obtained by expanding $\Delta\mathcal{E}_0$ to linear order in δ. As a result, the counterterm $\Delta\mathcal{E}_0$ has the form

$$\Delta\mathcal{E}_0 = \left(\frac{d_A}{128\pi^2\epsilon} + O(\lambda\delta)\right)(1-\delta)^2 m_D^4 + \left(\frac{3d_A}{32\pi^2\epsilon} + O(\lambda\delta)\right)(1-\delta)^2 M_D^4. \quad (2.10)$$

To calculate the NLO free energy we need to expand to order δ and we will need the counterterms $\Delta\mathcal{E}_0$, Δm_D^2, Δm_q^2, and ΔM_D^2 to order δ in order to cancel the UV divergences. We find that in order to remove the divergences to two-loop order, the mass counterterms should have the form

$$\Delta m_D^2 = \left(\frac{1}{16\pi^2\epsilon}\lambda\delta + O(\lambda^2\delta^2)\right)(1-\delta)m_D^2,$$

$$\Delta M_D^2 = \left(\frac{3}{8\pi^2\epsilon}\lambda\delta + O(\lambda^2\delta^2)\right)(1-\delta)M_D^2,$$

$$\Delta m_q^2 = \left(-\frac{1}{\pi^2\epsilon}\lambda\delta + O(\lambda^2\delta^2)\right)(1-\delta)m_q^2. \quad (2.11)$$

In the $\mathcal{N} = 4$ SYM theory, we will use the same method as in QCD to calculate physical observables in HTLpt, namely expanding the path-integral in powers of δ, truncating at

some specified order, and then setting $\delta = 1$. The results of the physical observables will depend on m_D, M_D, and m_q for any truncation of the expansion in δ, and some prescription is required to determine m_D, M_D, and m_q as a function of λ. In this work, we will follow the two-loop HTLpt QCD prescription and determine them by minimizing the free energy. If we use $\Omega_N(T, \lambda, m_D, M_D, m_q, \delta)$ to represent the thermodynamic potential expanded to N-th order in δ, then our full variational prescription is

$$\frac{\partial}{\partial m_D}\Omega_N(T, \lambda, m_D, M_D, m_q, \delta = 1) = 0,$$
$$\frac{\partial}{\partial M_D}\Omega_N(T, \lambda, m_D, M_D, m_q, \delta = 1) = 0,$$
$$\frac{\partial}{\partial m_q}\Omega_N(T, \lambda, m_D, M_D, m_q, \delta = 1) = 0. \qquad (2.12)$$

We will call eqs. (2.12) the *gap equations*. The free energy is obtained by evaluating the thermodynamic potential at the solution to the gap equations. Other thermodynamic functions can then be obtained by taking appropriate derivatives of free energy with respect to T.

3 Next-to-leading order thermodynamic potential

In the imaginary-time formalism, Minkoswski energies have discrete imaginary values $p_0 = i(2\pi nT)$, and the integrals over Minkowski space should be replaced by Euclidean sum integrals. There are two ways to do this which have been discussed in refs. [49, 50]. One is transforming the Feynman rules in Minkowski space given in appendix A into the form in Euclidean space firstly, then calculating the free energy. The other way is using the Feynman rules in Minkowski space to get the forms of free energy, after reducing these forms, transforming it into the form in Euclidean space. Results from the two methods must be the same.

The HTL perturbative thermodynamic potential at next-to-leading order in $\mathcal{N} = 4$ SYM can be expressed as

$$\Omega_{\text{NLO}} = \Omega_{\text{LO}} + \Omega_{\text{2-loop}} + \Omega_{\text{HTL}} + \Delta\Omega_{\text{NLO}}, \qquad (3.1)$$

where Ω_{LO} is the leading order thermodynamic potential, $\mathcal{O}(\delta^0)$, which includes the one-loop graphs shown in figure 2 and the LO vacuum renormalization counterterm. We first discuss the contributions at this order.

3.1 LO thermodynamic potential

In $\text{SU}(N_c)$ gauge theory with massless particles, Ω_{LO} can be expressed as

$$\Omega_{\text{LO}} = d_A \mathcal{F}_g + d_F \mathcal{F}_q + d_S \mathcal{F}_s + \Delta_0 \mathcal{E}_0, \qquad (3.2)$$

where $d_A = N_c^2 - 1$ is the dimension of the adjoint representation. There are four independent Majorana fermions in the adjoint representation, $d_F = 4d_A$, and $d_S = 6d_A$ for the six scalars.
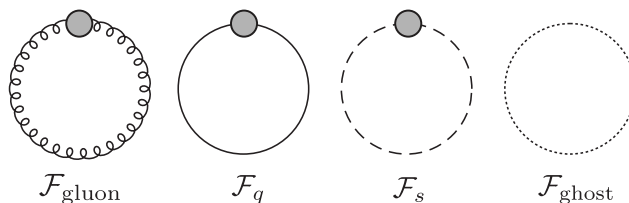
Figure 2. One loop Feynman diagrams for $\mathcal{N} = 4$ SYM theory in HTLpt. Dashed lines indicate a scalar field and dotted lines indicate a ghost field. Shaded circles indicate HTL-dressed propagators.

There are $D = d + 1$ polarization state for gluons, where d is the number of spatial dimensions. After canceling the two unphysical states using the ghost contribution, we obtain the HTL one-loop free energy of each of the color states of the gluon

$$\mathcal{F}_g = \mathcal{F}_{\text{gluon}} + \mathcal{F}_{\text{ghost}} = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \{(d-1)\log[-\Delta_T(P)] + \log \Delta_L(P)\}. \tag{3.3}$$

The transverse and longitudinal HTL propagators $\Delta_T(P)$ and $\Delta_L(P)$ are the HTL gluon propagator (A.13) in Euclidean space

$$\Delta_T(P) = \frac{-1}{P^2 + \Pi_T(P)},$$
$$\Delta_L(P) = \frac{1}{p^2 + \Pi_L(P)}. \tag{3.4}$$

The result above is the same as in QCD. The only difference is the definition of m_D^2 in the gluon propagator which now contains contributions from gluon, fermion, and scalar loops as detailed in appendix A.1.

Since the Majorana fermion is its own antiparticle, the fermionic contribution is reduced by a factor of two when comparing QCD and $\mathcal{N} = 4$ SYM. Our definition of m_q^2 is presented in appendix A.3. The one-loop fermionic free energy is

$$\mathcal{F}_q = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_{\{P\}} \log \det[\not{P} - \Sigma(P)], \tag{3.5}$$

where $\Sigma(P)$ is the HTL fermion self-energy (A.27) in Euclidean space. The scalar one-loop free energy is simply

$$\mathcal{F}_s = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \log[-\Delta_s^{-1}(P)], \tag{3.6}$$

where $\Delta_s^{-1}(P)$ is the inverse scalar propagator which is given in eq. (4.2). Finally, we note that the leading order counterterm $\Delta_0 \mathcal{E}_0$ cancels the divergent terms of the one-loop thermodynamic potential in $\mathcal{N} = 4$ SYM theory.

3.2 NLO thermodynamic potential

In eq. (3.1) $\Omega_{\text{2-loop}}$ corresponds to the two-loop contributions shown in figure 3. It can be expressed as

$$\Omega_{\text{2-loop}} = d_A \lambda \big[\mathcal{F}_{3g} + \mathcal{F}_{4g} + \mathcal{F}_{gh} + \mathcal{F}_{4s} + \mathcal{F}_{3gs} + \mathcal{F}_{4gs} + \mathcal{F}_{3qg} + \mathcal{F}_{4qg} + \mathcal{F}_{3qs}\big], \tag{3.7}$$

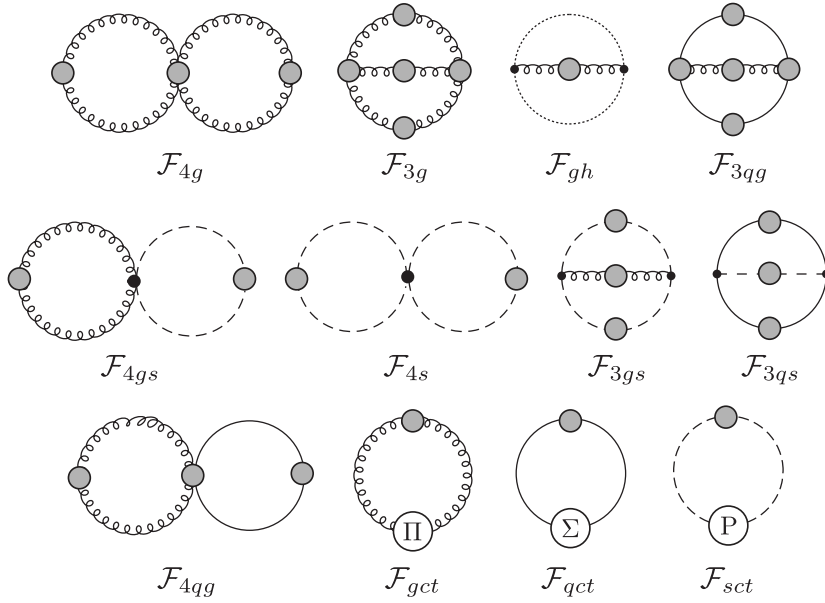where $\lambda = g^2 N_c$ is the 't Hooft coupling constant.

Figure 3. Two loop Feynman diagrams for $\mathcal{N} = 4$ SYM theory in HTLpt. Dashed lines indicate a scalar field and dotted lines indicate a ghost field. Shaded circles indicate HTL-dressed propagators and vertices.

The gluon propagator, the three-gluon vertex, the four-gluon vertex, and the gluon-ghost vertex are the same as in QCD up to the expression for the Debye mass m_D.[2] As a result, the purely gluonic and glue-ghost graphs given by $\mathcal{F}_{3g}$, $\mathcal{F}_{4g}$, and $\mathcal{F}_{gh}$ are, respectively,

$$
\mathcal{F}_{3g} = \frac{1}{12} \sum\!\!\!\!\!\!\!\int_{PQ} \Gamma^{\mu\lambda\rho}(P,Q,R)\Gamma^{\nu\sigma\tau}(P,Q,R)\Delta^{\mu\nu}(P)\Delta^{\lambda\sigma}(Q)\Delta^{\rho\tau}(R),
$$
$$
\mathcal{F}_{4g} = \frac{1}{8} \sum\!\!\!\!\!\!\!\int_{PQ} \Gamma^{\mu\nu,\lambda\sigma}(P,-P,Q,-Q)\Delta^{\mu\nu}(P)\Delta^{\lambda\sigma}(P),
$$
$$
\mathcal{F}_{gh} = \frac{1}{2} \sum\!\!\!\!\!\!\!\int_{PQ} \frac{1}{Q^2}\frac{1}{R^2}Q^\mu R^\nu \Delta^{\mu\nu}(P)\,, \tag{3.8}
$$

where $R + P + Q = 0$. Contributions come from the two-loop diagrams with four scalar vertex, scalar-gluon vertex and scalar-gluon four vertex are respectively

$$
\mathcal{F}_{4s} = \frac{15}{2} \sum\!\!\!\!\!\!\!\int_{PQ} \Delta_s(P)\Delta_s(Q),
$$
$$
\mathcal{F}_{3gs} = \frac{3}{2} \sum\!\!\!\!\!\!\!\int_{PQ} \Delta_s(R)\Delta_s(Q)\Delta_{\mu\nu}(P)(R+Q)_\mu(R+Q)_\nu,
$$
$$
\mathcal{F}_{4gs} = 3 \sum\!\!\!\!\!\!\!\int_{PQ} \Delta_s(Q)\Delta_{\mu\nu}(P)\delta_{\mu\nu}\,, \tag{3.9}
$$

where $R + P - Q = 0$. The contributions $\mathcal{F}_{3qg}$ and $\mathcal{F}_{4qg}$ involve only quarks and gluons and, since the Majorana fermion is its own antiparticle, their symmetry factor is $1/4$ instead of

---

[2]See A.1 for proof of this statement.

1/2 in QCD. Additionally, there are four Majorana fermions in $\mathcal{N} = 4$ SYM, so that $\mathcal{F}_{3qg}$ and $\mathcal{F}_{4qg}$ are 2 times the result obtained in QCD. As a result, we can substitute s_F and d_F in ref. [50] to $2N_c$ and $2d_A$, respectively, to obtain the $\mathcal{N} = 4$ SYM result. After this adjustment, the only other change required is to use the $\mathcal{N} = 4$ SYM definitions of m_D^2 and m_q^2. Based on the results contained in ref. [50] one obtains

$$
\begin{aligned}
\mathcal{F}_{3qg} &= -\sumint_{P\{Q\}} \Delta_{\mu\nu}(P)\mathrm{Tr}\big[\Gamma^\mu(P,Q,R)S(R)\Gamma^\nu(P,Q,R)S(Q)\big], \\
\mathcal{F}_{4qg} &= -\sumint_{P\{Q\}} \Delta_{\mu\nu}(P)\mathrm{Tr}\big[\Gamma^{\mu\nu}(P,-P,Q,Q)S(Q)\big].
\end{aligned}
\tag{3.10}
$$

The momentum conservation is $R + P - Q = 0$. One can also obtain these expressions using the Feynman rules contained in appendix A.

The final new graph, the quark-scalar diagram $\mathcal{F}_{3qs}$, can be split into two parts, one coming from the quark-scalar vertex, and the other coming from the quark-pseudoscalar vertex. Using the Feynman rules in appendix A, one finds that their contributions are the same. As a consequence, $\mathcal{F}_{3qs}$ can be written as

$$
\mathcal{F}_{3qs} = -6\sumint_{P\{Q\}} \mathrm{Tr}\big[S(R)S(Q)\big]\Delta_s(P),
\tag{3.11}
$$

where $R + P - Q = 0$.

The contribution Ω_{HTL} in eq. (3.1) is the sum of the gluon, quark and scalar HTL counterterms shown in figure 3. These enter in order to subtract contributions at lower loop orders and guarantee that naive perturbative results are recovered order by order if the expressions are truncated in λ. They can be expressed as

$$
\Omega_{\mathrm{HTL}} = d_A \mathcal{F}_{gct} + d_F \mathcal{F}_{qct} + d_S \mathcal{F}_{sct}.
\tag{3.12}
$$

There are two ways to get these three contributions, one is using the Feynman rules in appendix A, the other one is substituting $m_D^2 \to (1-\delta)m_D^2$, $M_D^2 \to (1-\delta)M_D^2$ and $m_q^2 \to (1-\delta)m_q^2$ in the one-loop expressions for $\mathcal{F}_g + \mathcal{F}_{\mathrm{ghost}}$, $\mathcal{F}_s$, and $\mathcal{F}_q$ and expanding them to linear order in δ. In terms of the first method, the contribution from the HTL gluon counterterm diagram is

$$
\begin{aligned}
\mathcal{F}_{gct} &= \frac{1}{2}\sumint_P \Pi^{\mu\nu}(P)\Delta^{\mu\nu}(P) \\
&= \frac{1}{2}\sumint_P \big[(d-1)\Pi_T(P)\Delta_T(P) - \Pi_L(P)\Delta_L(P)\big].
\end{aligned}
\tag{3.13}
$$

It is the same as in QCD up to the definition of m_D^2. The contribution from the HTL scalar counterterm diagram is

$$
\mathcal{F}_{sct} = \frac{1}{2}\sumint_P \Delta_s(P)\mathcal{P}_{aa}^{AA}(P),
\tag{3.14}
$$

where $\mathcal{P}_{aa}^{AA}(P) = M_D^2$ is referred in appendix A.6. The contribution from the HTL quark counterterm diagram is

$$
\mathcal{F}_{qct} = -\frac{1}{2}\sumint_{\{P\}} \mathrm{Tr}\big[\Sigma(P)S(P)\big].
\tag{3.15}
$$

Compared to QCD this is different by one half due to the Majorana nature of the SYM fermions. As usual, the quark mass should be adjusted to the SYM case.

Since HTL perturbation theory is renormalizable, the ultraviolet divergences of free energy at any order in δ can be cancelled by $\mathcal{E}$, m_D^2, m_q^2, and M_D^2 and the coupling constant λ. $\Delta\Omega_{\mathrm{NLO}}$ in eq. (3.1) is the renormalization contribution at first order in δ is used to cancel the next-to-leading order divergences. It can be expressed as

$$\Delta\Omega_{\mathrm{NLO}} = \Delta_1\mathcal{E}_0 + \left(\Delta_1 m_D^2 + \Delta_1 m_q^2 + \Delta_1 M_D^2\right)\frac{\partial}{\partial M_D^2}\Omega_{\mathrm{LO}}\,, \tag{3.16}$$

where $\Delta_1\mathcal{E}_0$, $\Delta_1 m_D^2$, $\Delta_1 m_q^2$, and $\Delta_1 M_D^2$ are the terms of order δ in the vacuum energy (2.10) and mass counterterms (2.11). The first term $\Delta_1\mathcal{E}_0$ can be obtained simply by expanding $\Delta_0\mathcal{E}_0$ to first order in δ. The second term in (3.16) is slightly different from the QCD result in refs. [49, 50]. This is because this term must be used to cancel the divergences of two-loop self energy. As we can see in (6.5), there are two mixed term $M_D m_D^2$ and $M_D m_q^2$ which comes from the (hs) contribution of $\mathcal{F}_{4gs}$ and $\mathcal{F}_{3qs}$ respectively. There are many ways to construct the mass renormalization form which is corresponding to the second part of eq. (3.16), but the only way to use one set of three counterterms $\Delta_1 m_D^2$, $\Delta_1 m_q^2$, and $\Delta_1 M_D^2$ is the form we have shown above.

In this work, we calculate the thermodynamic potential as an expansion in powers of m_D/T, m_q/T, and M_D/T to order g^5. We will show that, at order δ, all divergences in the two-loop thermodynamic potential plus HTL counterterms can be removed by these vacuum and mass counterterms. This means the method used in QCD can also be used in $\mathcal{N}=4$ SYM theory. This provides nontrivial evidence for the renormalizability of HTLpt at order δ in $\mathcal{N}=4$ SYM.

4 Reduction to scalar sum-integrals

Since we can make use of prior QCD results, we only need to calculate $\mathcal{F}_s$, $\mathcal{F}_{sct}$, $\mathcal{F}_{4s}$, $\mathcal{F}_{3gs}$, $\mathcal{F}_{4gs}$, $\mathcal{F}_{3qs}$, and the HTL counterterms contributing to eq. (3.12). The first step to calculate the new SYM contributions in figures 2 and 3 is to reduce the sum of these diagrams to scalar sum-integrals. In Euclidean space, by substituting p_0 to iP_0 the scalar propagator can be written as

$$\Delta_s(P) = \frac{-1}{P^2 + M_D^2}\,, \tag{4.1}$$

so its inverse is

$$\Delta_s^{-1}(P) = -(P^2 + M_D^2)\,. \tag{4.2}$$

The leading-order scalar contribution can be written as

$$\mathcal{F}_s = \frac{1}{2}\,\sumint_P \log[P^2 + M_D^2]\,. \tag{4.3}$$

The HTL scalar counterterm can be written as

$$\mathcal{F}_{sct} = -\frac{1}{2}\,\sumint_P \frac{M_D^2}{P^2 + M_D^2}\,. \tag{4.4}$$

We proceed to simplify the sum of formulas in eq. (3.9) in general covariant gauge parameterized by ξ. Using eqs. (4.1) and (A.24), we obtain

$$
\begin{aligned}
\mathcal{F}_{4s+3gs+4gs} = {} & \frac{15}{2} \oint_{PQ} \Delta_s(P)\Delta_s(Q) \\
& + \frac{3}{2} \oint_{PQ} \Delta_s(R)\Delta_s(Q)\bigg\{ \Delta_T(P)\bigg[2R^2 + 2Q^2 - P^2 - \frac{(Q^2-R^2)^2}{P^2} \bigg] \\
& + \Delta_X(P)\bigg[\frac{2}{P^2}(Q^2-R^2)(Q^2-R^2+r^2-q^2) - (2R^2+2Q^2-P^2) \\
& + (2r^2+2q^2-p^2) - \frac{P_0^2}{P^4}(Q^2-R^2)^2 \bigg] - \frac{\xi}{P^4}(Q^2-R^2)^2 \bigg\} \\
& + 3 \oint_{PQ} \Delta_s(Q)\bigg[d\Delta_T(P) - \frac{p^2}{P^2}\Delta_X(P) - \frac{\xi}{P^2} \bigg].
\end{aligned}
\tag{4.5}
$$

There are two terms which depend on ξ in (4.5), however, using $P + R - Q = 0$, one finds that they cancel each other, so that the sum of these contributions is gauge independent. Similarly, the results for $\mathcal{F}_{3g+4g+gh}$ and $\mathcal{F}_{3qg+4qg}$ are independent of gauge parameter ξ as shown in prior QCD calculations. Therefore, we have verified explicitly that the NLO HTLpt resummed thermodynamic potential in $\mathcal{N} = 4$ SYM is gauge independent.

Finally, we simplify eq. (3.11) to

$$
\mathcal{F}_{3qs} = -24 \oint_{P\{Q\}} \frac{\mathcal{A}_0(R)\mathcal{A}_0(Q) - \mathcal{A}_s(R)\mathcal{A}_s(Q)\hat{\mathbf{r}}\cdot\hat{\mathbf{q}}}{[\mathcal{A}_0^2(R) - \mathcal{A}_s^2(R)][\mathcal{A}_0^2(Q) - \mathcal{A}_s^2(Q)]}\Delta_s(P),
\tag{4.6}
$$

where in Euclidean space

$$
\begin{aligned}
\mathcal{A}_0(P) &= iP_0 - \frac{m_q^2}{iP_0}\mathcal{T}_P, \\
\mathcal{A}_s(P) &= p + \frac{m_q^2}{p}\big[1 - \mathcal{T}_P\big],
\end{aligned}
\tag{4.7}
$$

with $\mathcal{T}_P$ is the angular average $\mathcal{T}^{00}(p,-p)$ in Euclidean space, can be expressed as

$$
\mathcal{T}_P = \left\langle \frac{P_0^2}{P_0^2 + p^2 c^2} \right\rangle_c,
\tag{4.8}
$$

where the angular brackets denote an average over c defined by

$$
\big\langle f(c) \big\rangle_c \equiv \omega(\epsilon) \int_0^1 dc(1-c^2)^{-\epsilon} f(c),
\tag{4.9}
$$

where $\omega(\epsilon)$ is given in (A.15).

5 High temperature expansion

Having reduced $\mathcal{F}_s$, $\mathcal{F}_{4s+3gs+4gs}$, $\mathcal{F}_{3qs}$, and the HTL counterterm $\mathcal{F}_{sct}$ to scalar sum-integrals, we will now evaluate these sum-integrals approximately by expanding them in powers of m_D/T, m_q/T, and M_D/T. We will keep terms that contribute through $\mathcal{O}(g^5)$ if m_D, m_q and M_D are taken to be of order g at leading-order. Additionally, each of these terms can be divided into contributions from hard and soft momentum, so we will proceed to calculate their hard and soft contributions, respectively. In some cases, the results presented here were obtained in previous one- and two-loop QCD HTLpt papers [8, 49, 50]. When converting the prior QCD graphs involving quarks, as mentioned previously, one has to take into account that the four SYM quarks are Majorana fermions. Here we list results for all contributions to the $\mathcal{N} = 4$ SYM Feynman graphs and counterterms for completeness and ease of reference. In all cases, we use the integral and sum-integral formulas from refs. [49, 50] to obtain explicit expressions.

5.1 One-loop sum-integrals

The one-loop sum-integrals include the leading gluon, quark, and scalar contributions (3.3), (3.5), and (3.6) along with their corresponding counterterms (3.13), (3.15), and (3.14). In order to include all terms through $\mathcal{O}(g^5)$, we need to expand the one-loop contribution to order m_D^4, m_q^4, and M_D^4.

5.1.1 Hard contributions

The hard contribution from the gluon free energy (3.3) is [49]

$$
\begin{aligned}
\mathcal{F}_g^{(h)} = &-\frac{\pi^2}{45}T^4 + \frac{1}{24}\left[1 + \left(2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right]\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_D^2 T^2 \\
&- \frac{1}{128\pi^2}\left(\frac{1}{\epsilon} - 7 + 2\gamma + \frac{2\pi^2}{3}\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_D^4 .
\end{aligned} \tag{5.1}
$$

The hard contribution from the gluon HTL counterterm (3.13) is [49]

$$
\mathcal{F}_{gct}^{(h)} = -\frac{1}{24}m_D^2 T^2 + \frac{1}{64\pi^2}\left(\frac{1}{\epsilon} - 7 + 2\gamma + \frac{2\pi^2}{3}\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_D^4 . \tag{5.2}
$$

The hard contribution from the quark free energy (3.5) is

$$
\begin{aligned}
\mathcal{F}_q^{(h)} = &-\frac{7\pi^2}{360}T^4 + \frac{1}{12}\left[1 + \left(2 - 2\log 2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right]\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}m_q^2 T^2 \\
&+ \frac{1}{24\pi^2}\left(\pi^2 - 6\right)m_q^4 .
\end{aligned} \tag{5.3}
$$

The hard contribution from the quark HTL counterterm (3.15) is

$$
\mathcal{F}_{qct}^{(h)} = -\frac{1}{12}m_q^2 T^2 - \frac{1}{12\pi^2}\left(\pi^2 - 6\right)m_q^4 . \tag{5.4}
$$

Since scalars are bosons, the sum-integrals in (3.6) are the same those used for gluons. After expansion, we obtain the hard contribution to the LO scalar free energy

$$\mathcal{F}_s^{(h)} = \frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \log P^2 + \frac{1}{2}M_D^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} - \frac{1}{4}M_D^4 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^4}\,. \tag{5.5}$$

Using the results for sum-integrals contained in the appendices B and C of refs. [49, 50], eq. (5.5) reduces to

$$\begin{aligned}
\mathcal{F}_s^{(h)} = & -\frac{1}{90}\pi^2 T^4 + \frac{1}{24}\left[1 + \left(2 + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\epsilon\right]\left(\frac{\mu}{4\pi T}\right)^{2\epsilon} M_D^2 T^2 \\
& -\frac{1}{64\pi^2}\left[\frac{1}{\epsilon} + 2\gamma\right]\left(\frac{\mu}{4\pi T}\right)^{2\epsilon} M_D^4\,.
\end{aligned} \tag{5.6}$$

The scalar HTL counterterm is given in (3.14), after expansion, we get

$$\mathcal{F}_{sct}^{(h)} = -\frac{1}{2}M_D^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} + \frac{1}{2}M_D^4 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^4}\,, \tag{5.7}$$

which can be reduced to

$$\mathcal{F}_{sct}^{(h)} = -\frac{1}{24}M_D^2 T^2 + \frac{1}{32\pi^2}\left(\frac{1}{\epsilon} + 2\gamma\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon} M_D^4\,. \tag{5.8}$$

Note that the first terms in (5.1), (5.3) and (5.6) cancel the order-ϵ^0 term in the coefficient of mass squared in (5.2), (5.4) and (5.8), respectively.

5.1.2 Soft contributions

The *soft* contributions to the thermodynamical potential come from the $n = 0$ Matsubara mode ($P_0 = 0$) in the resulting bosonic sum-integrals. For fermions, since $P_0 = (2n + 1)\pi T \neq 0$ for integer n, the quark momentum is always *hard*; therefore, quarks do not have a soft contribution. For gluons, in the soft limit $P \to (0, \mathbf{p})$, the HTL gluon self-energy functions reduce to $\Pi_T(P) = 0$ and $\Pi_L(P) = m_D^2$. For scalars, in this limit the propagator reduces to $\Delta_s(P) = -1/(p^2 + M_D^2)$ where, here $p^2 = \mathbf{p}^2$.

The soft contribution to the gluon free energy (3.3) is

$$\mathcal{F}_g^{(s)} = -\frac{1}{12\pi}\left(1 + \frac{8}{3}\epsilon\right)\left(\frac{\mu}{2m_D}\right)^{2\epsilon} m_D^3 T\,. \tag{5.9}$$

The soft contribution from the gluon HTL counterterm (3.13) is

$$\mathcal{F}_{gct}^{(s)} = \frac{1}{8\pi}m_D^3 T\,. \tag{5.10}$$

The soft contribution from scalar free energy (3.6) is

$$\mathcal{F}_s^{(s)} = \frac{1}{2}T \int_{\mathbf{p}} \log\left(p^2 + M_D^2\right)\,, \tag{5.11}$$

which can be reduced to

$$\mathcal{F}_s^{(s)} = -\frac{1}{12\pi}\left(1 + \frac{8}{3}\epsilon\right)\left(\frac{\mu}{2M_D}\right)^{2\epsilon} M_D^3 T \, . \tag{5.12}$$

The soft contribution from the scalar HTL counterterm (3.14) is

$$\mathcal{F}_{sct}^{(s)} = -\frac{1}{2}M_D^2 T \int_{\mathbf{p}} \frac{1}{p^2 + M_D^2} \, , \tag{5.13}$$

then it can be reduced as

$$\mathcal{F}_{sct}^{(s)} = \frac{1}{8\pi}M_D^3 T \, . \tag{5.14}$$

5.2 Two-loop sum-integrals

Since the two-loop sum-integrals have an explicit factor of λ, we only need to expand these sum-integrals to order m_D^3/T^3, M_D^3/T^3, $m_D m_q^2/T^3$, $M_D m_q^2/T^3$, $m_D^2 M_D/T^3$, and $M_D^2 m_D/T^3$ to include all terms through $\lambda^{5/2}$. Since these integrals involve two momentum integrations we will expand contributions from hard loop momentum and soft loop momentum for each momentum integral. For bosons, this gives three contributions which we will denote as (hh), (hs) and (ss). For fermions, since their momentum is always hard, there will be only two regions (hh) and (hs). In the (hh) region, all three momentum are hard $p \sim T$, while in the (ss) region, all the three momentum are soft, $p \sim gT$. In the (hs) region, two of the three momenta are hard and the other is soft.

5.2.1 Contributions from the (hh) region

In the (hh) region, the self energies are suppressed by m_D^2/T^2, M_D^2/T^2 and m_q^2/T^2, so we can expand in powers of Π_T, Π_L, Σ, and M_D^2.

The (hh) contribution from gluon self energy (3.8) is [49]

$$\mathcal{F}_{3g+4g+gh}^{(hh)} = \frac{1}{144}T^4 - \frac{7}{1152\pi^2}\left(\frac{1}{\epsilon} + 4.6216\right)\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} m_D^2 T^2 \, . \tag{5.15}$$

The (hh) contribution from quark self energy (3.10) is [49]

$$\mathcal{F}_{3qg+4qg}^{(hh)} = \frac{5}{144}T^4 - \frac{1}{144\pi^2}\left[\frac{1}{\epsilon} + 1.2963\right]\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} m_D^2 T^2$$
$$+ \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + 8.96751\right]\left(\frac{\mu}{4\pi T}\right)^{4\epsilon} m_q^2 T^2 \, . \tag{5.16}$$

The (hh) contribution from (3.9) can be expanded as

$$\mathcal{F}_{4s+3gs+4gs}^{(hh)} = \sum\!\!\!\!\!\!\!\int_{PQ} \frac{3(d+2)}{P^2Q^2} + M_D^2 \sum\!\!\!\!\!\!\!\int_{PQ}\left[-\frac{3(5+d)}{P^2Q^4} + \frac{6}{P^2Q^2R^2}\right]$$
$$+ \frac{m_D^2}{d-1}\sum\!\!\!\!\!\!\!\int_{PQ}\left\{\frac{3(1-d)}{P^4Q^2} + \frac{3(d-2)}{p^2P^2Q^2} + \frac{3(2+d)}{2p^2Q^2R^2} - \frac{3}{P^2Q^2R^2}\right.$$
$$- \frac{6dq^2}{p^4Q^2R^2} + \frac{6q^2}{p^2P^2Q^2R^2} + \frac{3d(Q\cdot R)}{p^4Q^2R^2}$$
$$\left. + \left[\frac{3(1-d)}{p^2P^2Q^2} - \frac{3(1+d)}{2p^2Q^2R^2} + \frac{6dq^2}{p^4Q^2R^2} - \frac{3d(Q\cdot R)}{p^4Q^2R^2}\right]\mathcal{T}_P\right\} \, , \tag{5.17}$$

where $P + R - Q = 0$. Using the sum-integral formulas in appendix C of ref. [50] this reduces to

$$\mathcal{F}^{(hh)}_{4s+3gs+4gs} = \frac{5}{48}T^4 - \frac{1}{8\pi^2}\left(\frac{1}{\epsilon} + 2\gamma + 5.72011\right)\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}M_D^2 T^2$$
$$-\frac{7}{384\pi^2}\left(\frac{1}{\epsilon} + 5.61263\right)\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}m_D^2 T^2\,. \tag{5.18}$$

The (hh) contribution to (3.11) can be expanded as

$$\mathcal{F}^{(hh)}_{3qs} = 24\left[\sum\!\!\!\!\!\!\int_{P\{Q\}}\frac{-1}{P^2Q^2} + \sum\!\!\!\!\!\!\int_{\{PQ\}}\frac{1}{2P^2Q^2}\right] + 24M_D^2\sum\!\!\!\!\!\!\int_{P\{Q\}}\left[\frac{1}{P^4Q^2} - \frac{1}{2P^2Q^2R^2}\right]$$
$$+24m_q^2\sum\!\!\!\!\!\!\int_{P\{Q\}}\left\{\frac{2}{P^2Q^4} + \frac{1}{P^2Q^2R^2} + \frac{p^2-r^2}{P^2Q^2R^2q^2} + \left[\frac{-1}{P^2Q^2Q_0^2}\right.\right.$$
$$\left.\left.+\frac{r^2-p^2}{P^2R^2Q_0^2q^2}\right]\mathcal{T}_Q\right\} + 24m_q^2\sum\!\!\!\!\!\!\int_{\{PQ\}}\left[\frac{-2}{P^2Q^4} + \frac{1}{P^2Q^2Q_0^2}\mathcal{T}_Q\right]\,, \tag{5.19}$$

where $P + R - Q = 0$. Using the sum-integral formulas in appendix C of ref. [50] this reduces to

$$\mathcal{F}^{(hh)}_{3qs} = \frac{5}{48}T^4 - \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + 5.73824\right)\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}M_D^2 T^2$$
$$+\frac{3}{16\pi^2}\left(\frac{1}{\epsilon} + 9.96751\right)\left(\frac{\mu}{4\pi T}\right)^{4\epsilon}m_q^2 T^2\,. \tag{5.20}$$

5.2.2 Contributions from the (hs) region

In the (hs) region, the soft momentum can be any bosonic momentum. The functions that multiply the soft propagators $\Delta_T(0,\mathbf{p})$, $\Delta_X(0,\mathbf{p})$, or $\Delta_s(0,\mathbf{p})$ can be expanded in powers of the soft momentum $\mathbf{p}$. In terms involving $\Delta_T(0,\mathbf{p})$, the resulting integrals over $\mathbf{p}$ have no scale and vanish in dimensional regularization. The integration measure $\int_{\mathbf{p}}$ scales like m_D^3 for gluon momentum and M_D^3 for scalar momentum, respectively. The soft propagators $\Delta_X(0,\mathbf{p})$ and $\Delta_s(0,\mathbf{p})$ scale like $1/m_D^2$ and $1/M_D^2$, respectively, and every power of p in the numerator scales like m_D for gluon momentum and M_D for scalar momentum.

The (hs) contribution from the gluonic free energy graphs (3.8) is [49]

$$\mathcal{F}^{(hs)}_{3g+4g+gh} = -\frac{1}{24\pi}m_D T^3 - \frac{11}{384\pi^3}\left(\frac{1}{\epsilon} + 2\gamma + \frac{27}{11}\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left(\frac{\mu}{2m_D}\right)^{2\epsilon}m_D^3 T\,. \tag{5.21}$$

The (hs) contribution from quark self energy (3.10) is

$$\mathcal{F}^{(hs)}_{3qg+4qg} = -\frac{1}{12\pi}m_D T^3 + \frac{1}{4\pi^3}m_D m_q^2 T$$
$$+\frac{1}{48\pi^3}\left(\frac{1}{\epsilon} + 2\gamma + 1 + 4\log 2\right)\left(\frac{\mu}{4\pi T}\right)^{2\epsilon}\left(\frac{\mu}{2m_D}\right)^{2\epsilon}m_D^3 T\,. \tag{5.22}$$

Note that the sign on the second term differs from ref. [50]. This is due to an incorrect sign in the HTL-corrected quark-gluon three vertex in ref. [50], which we discuss in the appendix A.9.

For the (hs) contributions to (3.9) and (3.11), like QCD, after expansion there will be terms of contributing at $\mathcal{O}(g)$ and higher. For terms that are already of order g^3, we can set $R = Q$ for soft momentum P. For terms that are $\mathcal{O}(g)$, we must expand the sum-integral to second order in $\mathbf{p}$, and then perform the angular integration for $\mathbf{p}$, where the linear terms in $\mathbf{p}$ vanish and quadratic terms of the form $p^i p^j$ can be replaced by $p^2 \delta^{ij}/d$. Therefore, the (hs) contribution from (3.9) can be written as

$$
\mathcal{F}^{(hs)}_{4s+3gs+4gs} = T \int_{\mathbf{p}} \frac{1}{p^2 + M_D^2} \sum_{Q}\!\!\!\!\!\!\int \left[\frac{3(d+5)}{Q^2} - M_D^2 \left(\frac{3}{Q^4} + \frac{12q^2}{dQ^6} \right) - m_D^2 \frac{3}{Q^4} \right]
$$
$$
+ T \int_{\mathbf{p}} \frac{1}{p^2 + m_D^2} \sum_{Q}\!\!\!\!\!\!\int \left\{ \frac{6q^2}{Q^4} - \frac{3}{Q^2} + M_D^2 \left(\frac{9}{Q^4} - \frac{12q^2}{Q^6} \right) \right\}
$$
$$
+ m_D^2 \left[-\frac{6}{Q^4} + 6(d+4)\frac{q^2}{dQ^6} - \frac{24q^4}{dQ^8} \right], \tag{5.23}
$$

where $P + R - Q = 0$. Using the sum-integral formulas from appendix C of ref. [50] this reduces to

$$
\mathcal{F}^{(hs)}_{4s+3gs+4gs} = -T^3 \left(\frac{m_D}{8\pi} + \frac{M_D}{2\pi} \right) - \frac{3}{32\pi^3} M_D^2 m_D T
$$
$$
+ \left[\frac{3}{64\pi^3} M_D m_D^2 T + \frac{3}{32\pi^3} M_D^3 T \right] \left(\frac{1}{\epsilon} + 2 + 2\gamma \right) \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} \left(\frac{\mu}{2M_D} \right)^{2\epsilon}
$$
$$
+ \frac{1}{128\pi^3} \left(\frac{1}{\epsilon} + 4 + 2\gamma \right) \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} \left(\frac{\mu}{2m_D} \right)^{2\epsilon} m_D^3 T. \tag{5.24}
$$

The (hs) contribution from (3.11) can be written as

$$
\mathcal{F}^{(hs)}_{3qs} = 24T \int_{\mathbf{p}} \frac{1}{p^2 + M_D^2} \sum_{\{Q\}}\!\!\!\!\!\!\int \left[-\frac{1}{Q^2} + M_D^2 \left(-\frac{1}{Q^4} + \frac{2q^2}{dQ^6} \right) + m_q^2 \frac{2}{Q^4} \right], \tag{5.25}
$$

where $P + R - Q = 0$. Again using the sum-integral formulas from appendix C of ref. [50] this reduces to

$$
\mathcal{F}^{(hs)}_{3qs} = -\frac{1}{4\pi} M_D T^3 + \left[\frac{3}{16\pi^3} M_D^3 T - \frac{3}{4\pi^3} M_D m_q^2 T \right]
$$
$$
\times \left(\frac{1}{\epsilon} + 2 + 2\gamma + 4\log 2 \right) \left(\frac{\mu}{4\pi T} \right)^{2\epsilon} \left(\frac{\mu}{2M_D} \right)^{2\epsilon}. \tag{5.26}
$$

5.2.3 Contributions from the (ss) region

In the (ss) region, all bosonic momentum are soft, and the gluonic HTL correction functions $\mathcal{T}_P$, $\mathcal{T}^{000}$, and $\mathcal{T}^{0000}$ vanish. The gluonic self-energy functions at zero-frequency are $\Pi_T(0, \mathbf{p}) = 0$ and $\Pi_L(0, \mathbf{p}) = m_D^2$. The scales in the integrals come from the gluonic longitudinal propagator $\Delta_L(0, \mathbf{p}) = 1/(p^2 + m_D^2)$ and scalar propagator $\Delta_s(0, \mathbf{p}) = -1/(p^2 + M_D^2)$. Therefore for bosons, in dimensional regularization, at least one such propagator is required in order for the integral to be nonzero, and there is no (ss) contributions coming from fermionic diagrams.

The (hs) contribution to the gluonic free energy graphs (3.8) is

$$\mathcal{F}^{(ss)}_{3g+4g+gh} = \frac{1}{64\pi^2}\left(\frac{1}{\epsilon}+3\right)\left(\frac{\mu}{2m_D}\right)^{4\epsilon} m_D^2 T^2 \,. \tag{5.27}$$

The (ss) contribution to (3.9) can be expanded as

$$\mathcal{F}^{(ss)}_{4s+3gs+4gs} = T^2 \int_{\mathbf{pq}} \left[\frac{3}{(p^2+m_D^2)(q^2+M_D^2)} + \frac{6M_D^2+9p^2}{p^2(q^2+M_D^2)(r^2+M_D^2)}\right], \quad (5.28)$$

where $P + R - Q = 0$. Again using the sum-integral formulas from appendix C of ref. [50] this reduces to

$$\mathcal{F}^{(ss)}_{4s+3gs+4gs} = \frac{3}{16\pi^2}M_D m_D T^2 + \frac{3}{32\pi^2}\left(\frac{1}{\epsilon}+8\right)\left(\frac{\mu}{2M_D}\right)^{4\epsilon} M_D^2 T^2 \,. \tag{5.29}$$

6   HTL thermodynamic potential

In this section, we calculate the thermodynamic potential $\Omega(T, \lambda, m_D, M_D, m_q, \delta = 1)$ explicity, first to LO in the δ expansion and then to NLO.

6.1   Leading order

As we mentioned in section 3, the leading order thermodynamic potential is the sum of the contributions from one-loop diagrams and the leading order vacuum energy counterterm. The contributions coming from the one-loop diagrams is the sum of (5.1), (5.3), (5.6), (5.9) and (5.12). After multiplying by the appropriate coefficients in (3.2), one obtains

$$\Omega_{1\text{-loop}} = \mathcal{F}_{\text{ideal}}\bigg\{ 1 - \hat{m}_D^2 + 4\hat{m}_D^3 - 6\hat{M}_D^2 + 24\hat{M}_D^3 - 8\hat{m_q}^2 + 16\hat{m_q}^4(6-\pi^2)$$

$$+\frac{3}{4}\hat{m}_D^4\left[\frac{1}{\epsilon} - 7 + 2\gamma + \frac{2\pi^2}{3} + 2\log\frac{\hat{\mu}}{2}\right] + 9\hat{M}_D^4\left[\frac{1}{\epsilon} + 2\gamma + 2\log\frac{\hat{\mu}}{2}\right] \bigg\}, \tag{6.1}$$

where $\mathcal{F}_{\text{ideal}}$ is the free energy density of $\mathcal{N} = 4$ SYM in the ideal gas limit and $\hat{m}_D$, $\hat{M}_D$, $\hat{m}_q$ and $\hat{\mu}$ are dimensionless variables, defined as

$$\hat{m}_D = \frac{m_D}{2\pi T}\,,$$
$$\hat{M}_D = \frac{M_D}{2\pi T}\,,$$
$$\hat{m}_q = \frac{m_q}{2\pi T}\,,$$
$$\hat{\mu} = \frac{\mu}{2\pi T}\,. \tag{6.2}$$

Since the leading order vacuum energy counterterm $\Delta_0\mathcal{E}_0$ should cancel the divergences in the one-loop free energy, we obtain

$$\Delta_0\mathcal{E}_0 = d_A\left(\frac{1}{128\pi^2\epsilon}m_D^4 + \frac{3}{32\pi^2\epsilon}M_D^4\right) = \mathcal{F}_{\text{ideal}}\left(-\frac{3}{4\epsilon}\hat{m}_D^4 - \frac{9}{\epsilon}\hat{M}_D^4\right). \tag{6.3}$$

After adding the leading order vacuum renormalization counterterm, our final result for the renormalized LO HTLpt thermodynamic potential is

$$\Omega_{\text{LO}} = \mathcal{F}_{\text{ideal}}\bigg\{1 - \hat{m}_D^2 + 4\hat{m}_D^3 - 6\hat{M}_D^2 + 24\hat{M}_D^3 - 8\hat{m}_q^2 + 16\hat{m}_q^4(6 - \pi^2)$$

$$+ \frac{3}{4}\hat{m}_D^4\bigg[-7 + 2\gamma + \frac{2\pi^2}{3} + 2\log\frac{\hat{\mu}}{2}\bigg] + 18\hat{M}_D^4\bigg[\gamma + \log\frac{\hat{\mu}}{2}\bigg]\bigg\}. \tag{6.4}$$

6.2 Next-to-leading order

The next-to-leading order corrections to the thermodynamic potential include all of the two-loop free energy diagrams, the gluon, quark, scalar counterterms in figure 3, and the renormalization counterterms. The contributions from the two-loop diagrams include all terms through order g^5 is the sum of (5.15), (5.16), (5.18), (5.20), (5.21), (5.22), (5.24), (5.26), (5.27) and (5.29) multiplied by λd_A. Adding these gives

$$\Omega_{\text{2-loop}} = \mathcal{F}_{\text{ideal}}\frac{\lambda}{\pi^2}\bigg\{-\frac{3}{2} + 3\hat{m}_D + 9\hat{M}_D - \frac{9}{2}\hat{M}_D\hat{m}_D + \frac{9}{2}\hat{m}_D\hat{M}_D^2$$

$$-12\hat{m}_D\hat{m}_q^2 + \frac{3}{8}\hat{m}_D^2\bigg[\frac{1}{\epsilon} + 4\log\hat{m}_D + 4\log\frac{\hat{\mu}}{2} + 5.93198468\bigg]$$

$$+\frac{9}{4}\hat{M}_D^2\bigg[\frac{1}{\epsilon} + 4\log\hat{M}_D + 4\log\frac{\hat{\mu}}{2} + 4.99154798\bigg] + \frac{1}{8}\hat{m}_D^3\bigg[7 - 32\log 2\bigg] \tag{6.5}$$

$$-6\hat{m}_q^2\bigg[\frac{1}{\epsilon} + 4\log\frac{\hat{\mu}}{2} + 9.71751112\bigg] - 36\log 2\hat{M}_D^3 + 144\log 2\hat{M}_D\hat{m}_q^2$$

$$+\bigg[-\frac{27}{2}\hat{M}_D^3 - \frac{9}{4}\hat{M}_D\hat{m}_D^2 + 36\hat{M}_D\hat{m}_q^2\bigg]\bigg[2 + \frac{1}{\epsilon} + 2\gamma - 2\log\hat{M}_D + 4\log\frac{\hat{\mu}}{2}\bigg]\bigg\}.$$

The total NLO HTL counterterm contribution is the sum of (5.2), (5.4), (5.8), (5.10) and (5.14) multiplied by the Casimirs in (3.12)

$$\Omega_{\text{HTL}} = \mathcal{F}_{\text{ideal}}\bigg\{\hat{m}_D^2 - 6\hat{m}_D^3 + 6\hat{M}_D^2 - 36\hat{M}_D^3 + 8\hat{m}_q^2 + 32\hat{m}_q^4(\pi^2 - 6)$$

$$-\frac{3}{2}\hat{m}_D^4\bigg[\frac{1}{\epsilon} - 7 + 2\gamma + 2\log\frac{\hat{\mu}}{2} + \frac{2\pi^2}{3}\bigg] - 18\hat{M}_D^4\bigg[\frac{1}{\epsilon} + 2\gamma + 2\log\frac{\hat{\mu}}{2}\bigg]\bigg\}. \tag{6.6}$$

The ultraviolet divergences in (6.5) and (6.6) will be removed by the renormalization of the vacuum energy density $\mathcal{E}_0$ and the HTL mass parameters m_D, M_D, and m_q. The renormalization counterterm contribution at linear order in δ is denoted by $\Delta\Omega_{\text{NLO}}$ in eqs. (3.1) and (3.16). We cannot obtain its form directly from ref. [50] due to the fact that there are contributions coming from the scalar fields in $\mathcal{N} = 4$ SYM theory, but we can use the same method as in QCD. The form of $\Delta_1\mathcal{E}_0$ can be obtained by expanding (2.10) to first order in δ, which is

$$\Delta_1\mathcal{E}_0 = -d_A\bigg(\frac{m_D^4}{64\pi^2\epsilon} + \frac{3M_D^4}{16\pi^2\epsilon}\bigg) = \mathcal{F}_{\text{ideal}}\bigg(\frac{3}{2\epsilon}\hat{m}_D^4 + \frac{18}{\epsilon}\hat{M}_D^4\bigg). \tag{6.7}$$

This renormalization counterterm cancels the divergences in Ω_{HTL} (6.6). For the two-loop self energy, as we can see, the divergent terms are

$$\mathcal{F}_{\text{ideal}}\frac{\lambda}{\pi^2}\bigg[\frac{3}{8\epsilon}\hat{m}_D^2 + \frac{9}{4\epsilon}\hat{M}_D^2 - \frac{6}{\epsilon}\hat{m}_q^2 - \frac{27}{2\epsilon}\hat{M}_D^3 - \frac{9}{4\epsilon}\hat{M}_D\hat{m}_D^2 + \frac{36}{\epsilon}\hat{M}_D\hat{m}_q^2\bigg]. \tag{6.8}$$

Since there are two mixed terms $\hat{M}_D \hat{m}_D^2$ and $\hat{M}_D \hat{m}_q^2$ which is a big difference from QCD, we cannot use the following formula

$$\Delta_1 m_D^2 \frac{\partial}{\partial m_D^2} \Omega_{\text{LO}} + \Delta_1 m_q^2 \frac{\partial}{\partial m_q^2} \Omega_{\text{LO}} + \Delta_1 M_D^2 \frac{\partial}{\partial M_D^2} \Omega_{\text{LO}}. \tag{6.9}$$

This is because we cannot get the two mixed terms using this form. In order to cancel the ultraviolet divergence for two-loop self energy, the simplest form is given in eq. (3.16). Using (2.11), (6.7), and (3.16), one finds

$$\Delta\Omega_{\text{NLO}} = \mathcal{F}_{\text{ideal}} \bigg\{ \frac{3}{2\epsilon} \hat{m}_D^4 + \frac{18}{\epsilon} \hat{M}_D^4 + \frac{\lambda}{\pi^2} \bigg[\bigg(-\frac{3}{8} \hat{m}_D^2 - \frac{9}{4} \hat{M}_D^2 + 6\hat{m}_q^2 \bigg) $$
$$\times \bigg(\frac{1}{\epsilon} + 2 + 2\frac{\zeta'(-1)}{\zeta(-1)} + 2\log\frac{\hat{\mu}}{2} \bigg) + \bigg(\frac{27}{2} \hat{M}_D^3 + \frac{9}{4} \hat{M}_D \hat{m}_D^2 - 36\hat{M}_D \hat{m}_q^2 \bigg) $$
$$\times \bigg(\frac{1}{\epsilon} + 2 - 2\log\hat{M}_D + 2\log\frac{\hat{\mu}}{2} \bigg) \bigg] \bigg\}. \tag{6.10}$$

Adding the leading order thermodynamic potential in (6.4), the two-loop free energy in (6.5), the HTL gluon and quark counterterms in (6.6), and the HTL vacuum and mass renormalizations in (6.10), our final expression for the NLO HTLpt thermodynamic potential in $\mathcal{N} = 4$ SYM is

$$\Omega_{\text{NLO}} = \mathcal{F}_{\text{ideal}} \bigg\{ 1 - 2\hat{m}_D^3 - 12\hat{M}_D^3 + 16\hat{m}_q^4(\pi^2 - 6) - 18\hat{M}_D^4 \bigg(\gamma + \log\frac{\hat{\mu}}{2} \bigg) $$
$$- \frac{3}{2} \hat{m}_D^4 \bigg(-\frac{7}{2} + \gamma + \frac{\pi^2}{3} + \log\frac{\hat{\mu}}{2} \bigg) + \frac{\lambda}{\pi^2} \bigg[-\frac{3}{2} + 3\hat{m}_D + 9\hat{M}_D $$
$$- \frac{9}{2} \hat{m}_D \hat{M}_D + \frac{9}{2} \hat{m}_D \hat{M}_D^2 - 12\hat{m}_D \hat{m}_q^2 - 12\hat{m}_q^2 \bigg(1.87370184 + \log\frac{\hat{\mu}}{2} \bigg) $$
$$+ \frac{3}{4} \hat{m}_D^2 \bigg(-0.01906138 + 2\log\hat{m}_D + \log\frac{\hat{\mu}}{2} \bigg) + \frac{1}{8} \hat{m}_D^3 \big(7 - 32\log 2 \big) $$
$$+ \frac{9}{2} \hat{M}_D^2 \bigg(-0.489279733 + 2\log\hat{M}_D + \log\frac{\hat{\mu}}{2} \bigg) - \frac{9}{2} \hat{M}_D \hat{m}_D^2 \bigg(\gamma + \log\frac{\hat{\mu}}{2} \bigg) $$
$$+ 72\hat{M}_D \hat{m}_q^2 \bigg(\gamma + 2\log 2 + \log\frac{\hat{\mu}}{2} \bigg) - 9\hat{M}_D^3 \bigg(3\gamma + 4\log 2 + 3\log\frac{\hat{\mu}}{2} \bigg) \bigg] \bigg\}. \tag{6.11}$$

Note that this result reproduces the perturbative expansion given in (1.2) through $\mathcal{O}(\lambda^{3/2})$ in the weak-coupling limit. This can be verified by taking $\hat{m}_D$, $\hat{M}_D$, and $\hat{m}_q$ to be given by their leading-order expressions (6.16) and truncating the resulting expansion in the 't Hooft coupling at $\mathcal{O}(\lambda^{3/2})$.

6.3 Gap equations

The gluon, scalar, and quark mass parameters m_D, M_D, and m_q are determined by using the variational method, requiring that the derivative of Ω_{NLO} with respect to each

parameter is zero

$$\frac{\partial}{\partial m_q}\Omega_{\text{NLO}}(T,\lambda,m_D,M_D,m_q,\delta=1) = 0\,,$$

$$\frac{\partial}{\partial m_D}\Omega_{\text{NLO}}(T,\lambda,m_D,M_D,m_q,\delta=1) = 0\,,$$

$$\frac{\partial}{\partial M_D}\Omega_{\text{NLO}}(T,\lambda,m_D,M_D,m_q,\delta=1) = 0\,. \qquad (6.12)$$

The first equation gives

$$\hat{m}_q^2\big(\pi^2-6\big) = \frac{\lambda}{4\pi^2}\left[\frac{3}{2}\hat{m}_D + \frac{3}{2}\left(1.87370184 + \log\frac{\hat{\mu}}{2}\right) - 9\hat{M}_D\left(\gamma + 2\log 2 + \log\frac{\hat{\mu}}{2}\right)\right]. \quad (6.13)$$

The second equation gives

$$\hat{m}_D^2 + \hat{m}_D^3\left(-\frac{7}{2} + \gamma + \frac{\pi^2}{3} + \log\frac{\hat{\mu}}{2}\right) =$$
$$\frac{\lambda}{4\pi^2}\left[2 - 3\hat{M}_D + 3\hat{M}_D^2 - 8\hat{m}_q^2 + \frac{7}{4}\hat{m}_D^2\left(1 - \frac{32}{7}\log 2\right)\right.$$
$$\left. -6\hat{m}_D\hat{M}_D\left(\gamma + \log\frac{\hat{\mu}}{2}\right) + \hat{m}_D\left(0.980939 + 2\log\hat{m}_D + \log\frac{\hat{\mu}}{2}\right)\right]. \qquad (6.14)$$

The third equation gives

$$\hat{M}_D^2 + 2\hat{M}_D^3\left(\gamma + \log\frac{\hat{\mu}}{2}\right) = \frac{\lambda}{4\pi^2}\left[1 - \frac{1}{2}\hat{m}_D + \hat{m}_D\hat{M}_D - \frac{1}{2}\hat{m}_D^2\left(\gamma + \log\frac{\hat{\mu}}{2}\right)\right.$$
$$-3\hat{M}_D^2\left(4\log 2 + 3\gamma + 3\log\frac{\hat{\mu}}{2}\right) + 8\hat{m}_q^2\left(\gamma + 2\log 2 + \log\frac{\hat{\mu}}{2}\right)$$
$$\left. +\hat{M}_D\left(0.51072 + 2\log\hat{M}_D + \log\frac{\hat{\mu}}{2}\right)\right]. \qquad (6.15)$$

Note that the terms proportional to $\hat{m}_q^2$ in eqs. (6.14) and (6.15) can be written in terms of $\hat{M}_D$ and $\hat{m}_D$ by using (6.13).

In practice, one must solve these three equations simultaneously in order to obtain the gap equation solutions for $\hat{m}_q^2(\lambda)$, $\hat{m}_D^2(\lambda)$, and $\hat{M}_D^2(\lambda)$. In figure 4 we present our numerical solutions to these three gap equations scaled by the corresponding leading-order weak-coupling limits

$$\hat{m}_{q,\text{LO}}^2 = \frac{\lambda}{8\pi^2}\,,$$
$$\hat{m}_{D,\text{LO}}^2 = \frac{\lambda}{2\pi^2}\,,$$
$$\hat{M}_{D,\text{LO}}^2 = \frac{\lambda}{4\pi^2}\,. \qquad (6.16)$$

In all three panels, the black line is the solution when taking the renormalization scale $\hat{\mu} = 1$, the red dashed line is $\hat{\mu} = 1/2$, and the blue long-dashed line is $\hat{\mu} = 2$. As can be
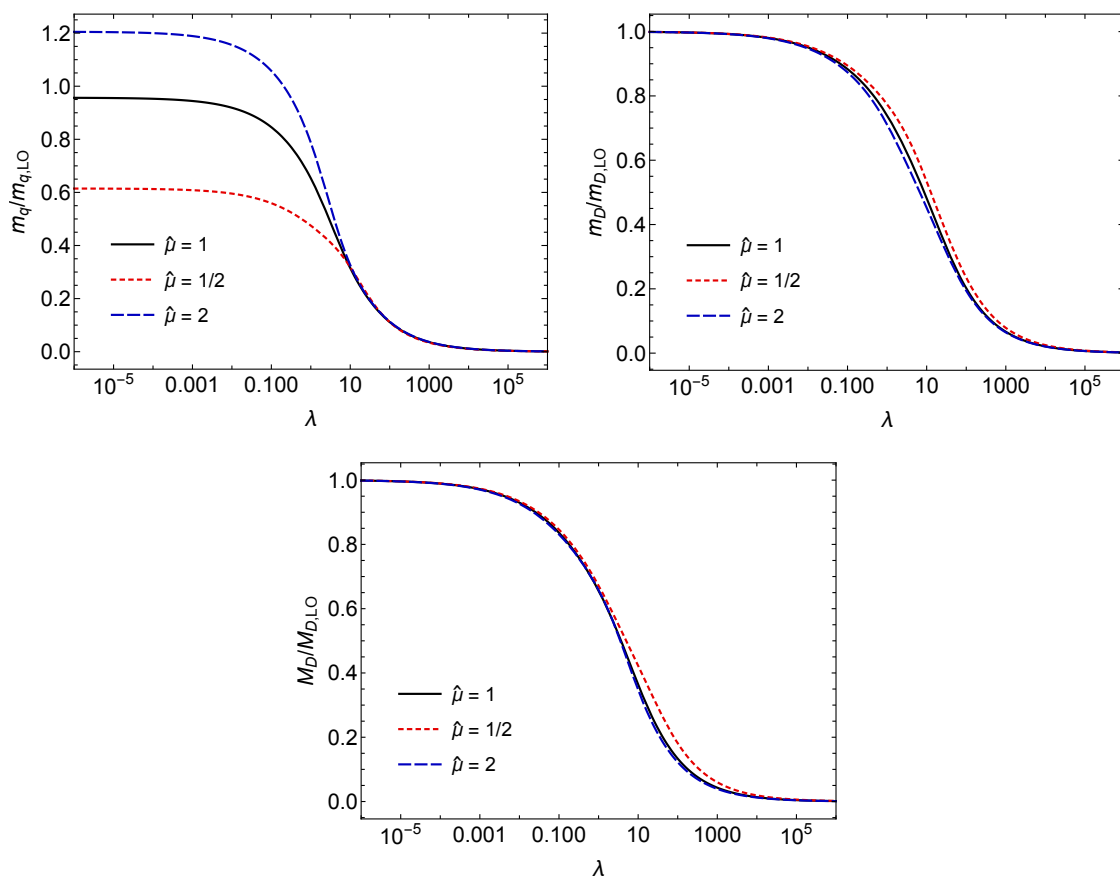
Figure 4. Numerical solution of gap equations for m_q, m_D, and M_D as a function of λ. In each panel the results are scaled by their corresponding leading-order weak-coupling limits.

seen from figure 4, the gap equation solution for $\hat{m}_q$ does not approach its perturbative limit when λ is approaches zero.

Similar to what was found in NLO HTLpt applied to QCD [50], for the SUSY theory $\hat{m}_q$ does not go to the perturbative value in the limit $\lambda \to 0$. This is due to the fact that the perturbative limit of the quark gap equation comes from terms which are proportional to the coupling constant λ, these terms are not included completely at NLO in HTLpt. Unfortunately, this problem cannot be completely solved by calculating to higher order in HTLpt, because the fermionic sector is infrared safe and therefore only even powers of $\hat{m}_q$ will appear at each order since fermions will not have soft contributions. At NNLO all terms contributing at $\mathcal{O}(\lambda^2)$ will be modifed and the $\hat{m}_q$ dependence will be pushed up to $O(\lambda^3)$. All these behaviors above will exist at all orders in HTLpt, so the weak-coupling limit of the gap equation quark mass will be scale dependent at any order, and will never approach to its perturbative value. Fundamentally, this is due to the fact that the fermionic sector is infrared safe.

In order to avoid this problem, one can simply remove the quark contribution from the HTL effective Lagrangian by setting $m_q = 0$. Since the fermionic sector is infrared
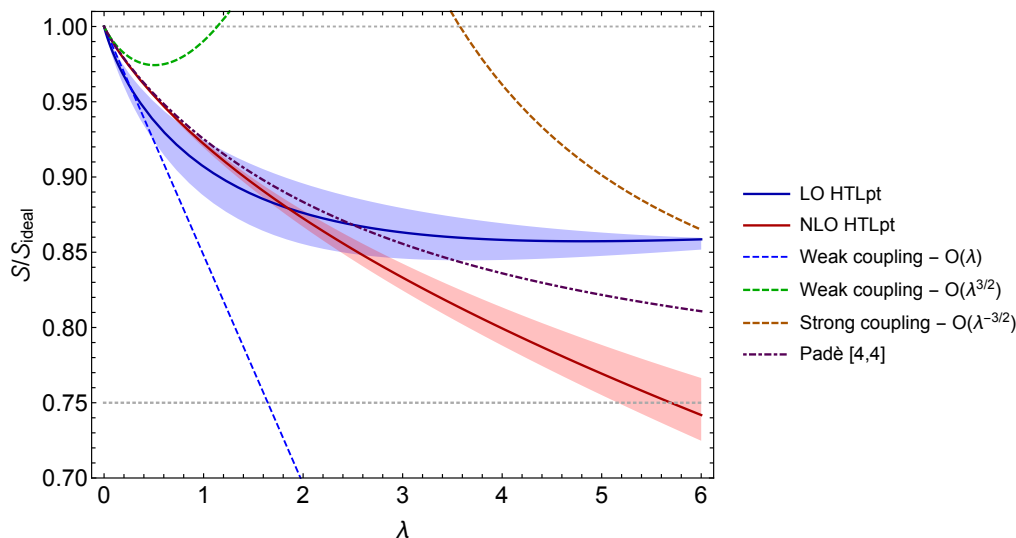
Figure 5. Comparison of the LO and NLO HTLpt results for the scaled entropy density with prior results from the literature. A detailed description of the various lines can be found in the text.

safe, the calculation is still well-defined and amounts to an alternative reorganization of the perturbative series. In the main body of the text we will use the full gap equation solutions with $m_q \neq 0$, but in appendix B we present the effect of using the alternative $m_q = 0$ prescription.

7 Thermodynamic functions

The NLO HTLpt approximation to the free energy is obtained by evaluating the NLO HTLpt thermodynamic potential (6.11) at the solution of the gap equations (6.12)

$$\mathcal{F}_{\text{NLO}} = \Omega_{\text{NLO}}(T, \lambda, m_D^{\text{gap}}, M_D^{\text{gap}}, m_q^{\text{gap}}, \delta = 1). \tag{7.1}$$

The pressure, entropy density, and energy density can then be obtained using

$$\begin{aligned} \mathcal{P} &= -\mathcal{F}, \\ \mathcal{S} &= -\frac{d\mathcal{F}}{dT}, \\ \mathcal{E} &= \mathcal{F} - T\frac{d\mathcal{F}}{dT}. \end{aligned} \tag{7.2}$$

Note that due the conformality of the SYM theory, in all three of these functions, the only dependence on T is contained in the overall factor of $\mathcal{F}_{\text{ideal}}$. As a result, when scaled by their ideal limits, the ratios of all of these quantities are the same, i.e. $\mathcal{P}/\mathcal{P}_{\text{ideal}} = \mathcal{S}/\mathcal{S}_{\text{ideal}} = \mathcal{E}/\mathcal{E}_{\text{ideal}}$.

7.1 Numerical results

In figure 5 we present our final results for the scaled entropy density in $\mathcal{N} = 4$ SYM. The red solid line with a red shaded band is the NLO HTLpt result and the blue solid line

with a blue shaded band is the LO HTLpt result determined by evaluating eq. (6.4) at the solution to the NLO mass gap equations (6.12). The HTLpt shaded bands result from variation of the renormalization scale $\hat{\mu}$. Herein, we take $\hat{\mu} \in \{1/2, 1, 2\}$ with the central value of the renormalization scale plotted as solid blue and red lines for the LO and NLO results, respectively. The blue dotted line is the weak-coupling result (1.2) truncated at order λ, the green dotted line is the weak-coupling result (1.2) truncated at order $\lambda^{3/2}$, the dark-orange dotted line is the strong-coupling result (1.1). The purple dot-dashed line is the result of constructing a $R_{[4,4]}$ Padé approximant which interpolates between the weak and strong coupling limits [5]. Finally, the grey dotted lines indicate the strong and weak coupling limits of 3/4 and 1, respectively.

As can be seen from figure 5, the LO and NLO HTLpt predictions are close to one another out to $\lambda \lesssim 2$, however, one observes that the bands obtained by scale variation for the two results (blue and red shaded bands) do not necessarily overlap in this range. Computing the ratio of the NLO and LO results, we find that they are within $\sim 5\%$ of one another in this range. This is a much smaller change from LO to NLO than is found using the naive weak-coupling expansion. We also observe that the size of the scale variation (shown as shaded red and blue bands) decreases as one goes from LO to NLO. Comparing the bands at $\lambda = 1$ we find that the LO HTLpt variation around $\hat{\mu} = 1$ is on the order of 2%, whereas the NLO order HTLpt variation is 0.3%. For $\lambda \gtrsim 6$ the NLO HTLpt result is below the value expected in the strong coupling limit. At smaller couplings, $\lambda \lesssim 1$, we observe that the NLO HTLpt result is very close to the $R_{[4,4]}$ Padé approximant. This could be coincidental, however, it is suggestive that somehow the $R_{[4,4]}$ Padé approximant may provide a reasonable approximation to $\mathcal{N} = 4$ SYM thermodynamics despite its ad hoc construction.

A similar conclusion was obtained in a prior study of HTL resummation in $\mathcal{N} = 4$ SYM thermodynamics [5]. In their work, Blaizot, Iancu, Kraemmer, and Rebhan (BIKR) used the Φ-derivable framework to obtain an approximately self-consistent approximation to the scaled entropy density. In their approach, the one-loop Φ-derivable result for the entropy density has approximate next-to-leading-order accuracy (NLA), so it should be comparable to the our NLO HTLpt result. In figure 6 we present a comparison of our NLO HTLpt result with the BIKR NLA result [5]. In this figure, the blue line with a blue shaded band is our NLO HTLpt result and the solid red line with a red shaded band is the NLA result from ref. [5]. The dashed and dotted lines are the same as the previous figure. We find that, for $\hat{\mu} = 1$, the two calculations are within $\leq 2\%$ of one another for $\lambda \lesssim 6$. We observe that the NLO HTLpt result has a smaller scale variation than the NLA result at all couplings shown.

Finally, in figure 7 we present a comparison of all results for $\lambda \leq 2$. The various weak-coupling lines in this figure are the same as in figure 5. As can be seen from this figure, there is excellent agreement between the NLA calculation of BIKR and NLO HTLpt in this coupling range. We also see that the $R_{[4,4]}$ Padé approximant overlaps with both calculations at smaller λ. Given the agreement between our NLO results and the BIKR NLA results in this range of 't Hooft coupling, one can try to estimate the range of temperatures this might map to in a real-world QGP. This is a fraught endeavor, however, since one can
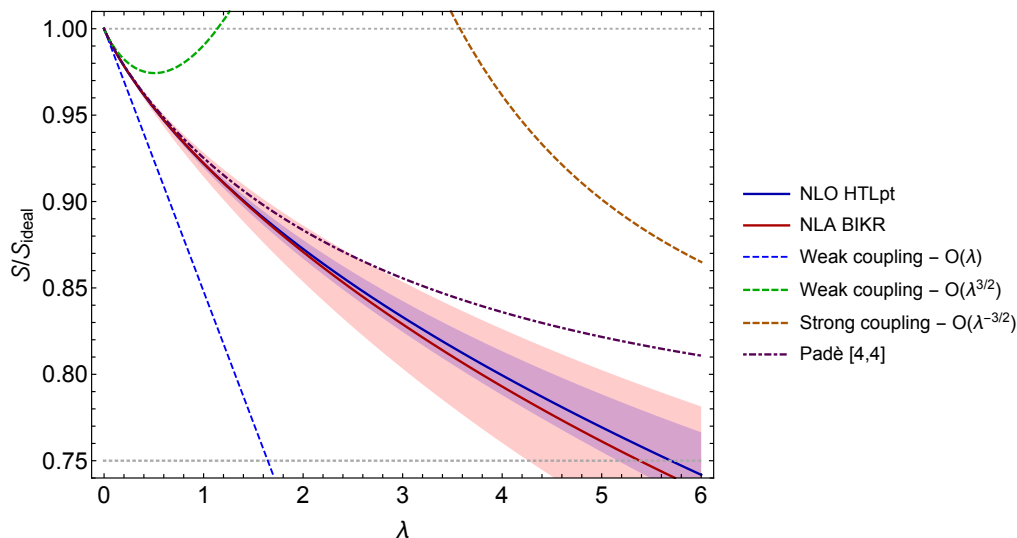
Figure 6. Comparison of our NLO HTLpt result for the scaled entropy density with the prior NLA work of Blaizot, Iancu, Kraemmer, and Rebhan (BIKR) [5]. A detailed description of the various lines can be found in the text.
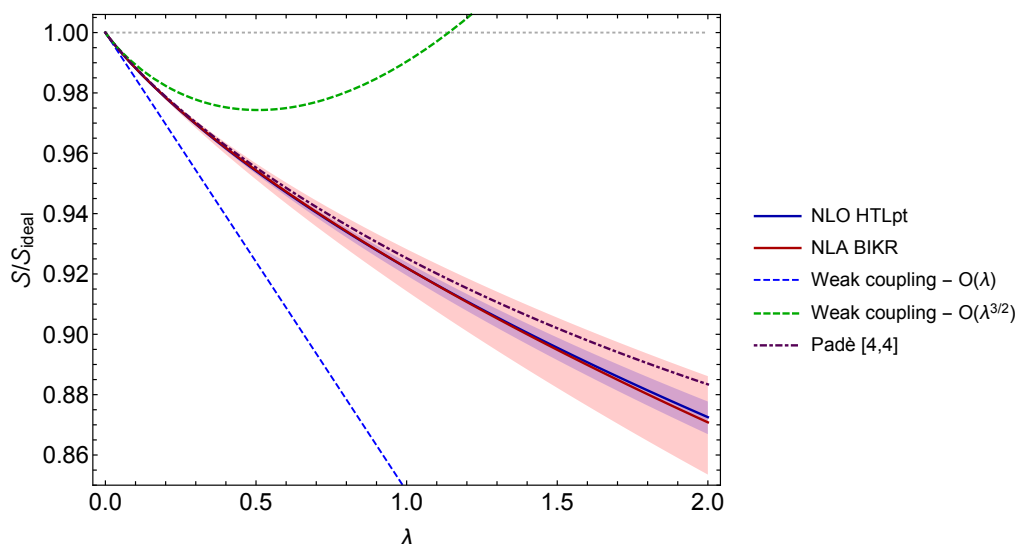

Figure 7. Comparison of our NLO HTLpt result for the scaled entropy density with prior results at small λ. Lines are the same as in figure 6.

choose to match a variety of quantities with, to our knowledge, no unique prescription. In ref. [5] the authors advocated matching the scaled entropy density. For the purposes of a ball-park estimate, we will follow their suggestion. State-of-the-art lattice data for the scaled entropy density indicates that corrections to the ideal limit saturate above approximately $T \sim 3T_c \sim 450\,\mathrm{MeV}$ at value of $\mathcal{S}_{\mathrm{QCD}}/\mathcal{S}_{\mathrm{QCD},0} \sim 0.8 - 0.85$ [56]. Matching this to the same ratio in $\mathcal{N} = 4$ SYM, one finds from figure 6 that this requires $\lambda \sim 3 - 4$ using

$\hat{\mu} = 1$.[3] Our results suggest that the NLO HTLpt result for $\mathcal{N} = 4$ SYM can be trusted to with high accuracy for $\lambda \lesssim 2$. This provides motivation for extending our calculation to NNLO.

8 Conclusions

In this paper we have extended the LO and NLO HTLpt calculation of the thermodynamic potential in QCD to $\mathcal{N} = 4$ SYM theory. We have presented results for the LO and NLO HTLpt predictions for the thermodynamics of $\mathcal{N} = 4$ SYM for arbitrary N_c. We found that it is possible to extend the range of applicability of perturbative calculations of thermodynamics in $\mathcal{N} = 4$ SYM theory to intermediate couplings, albeit using involved resummations. We compared our NLO HTLpt results to approximately self-consistent resummations obtained previously in ref. [5] and found them to be in excellent agreement with our NLO HTLpt results for the scaled entropy density for $\lambda \lesssim 6$. Compared to the method used in ref. [5], our HTLpt results are manifestly gauge-invariant at all loop orders and the HTLpt framework allows for systematic extension of the calculation to higher loop orders. It would be interesting to extend the HTLpt results obtained here to NNLO as has been done in QCD. For this purpose, it seems necessary to first establish the naive perturbative corrections to SUSY thermodynamics at orders g^4 and g^5. Work along these lines is in progress.

Acknowledgments

A HTL Feynman rules for $\mathcal{N} = 4$ SYM

In this appendix, we will present the Feynman rules for HTLpt applied to $\mathcal{N} = 4$ SYM. The Feynman rules are given in Minkowski space to facilitate future applications to real-time processes. A Minkowski momentum is denoted by $p = (p_0, \mathbf{p})$, and satisfies $p \cdot q = p_0 q_0 - \mathbf{p} \cdot \mathbf{q}$. The vector that specifies the thermal rest frame is $n = (1, \mathbf{0})$.

---

[3]Note that all such estimates should be taken with care since in QCD, unlike $\mathcal{N} = 4$ SYM, there is conformal symmetry breaking which causes, e.g., $\mathcal{S}_{\text{QCD}}/\mathcal{S}_{\text{QCD},0} \neq \mathcal{P}_{\text{QCD}}/\mathcal{P}_{\text{QCD},0}$. If one were to use the scaled pressure instead, one for find a different limit for λ. Additionally, our choice of $\hat{\mu} = 1$ is somewhat arbitrary and varying this scale will result in further variation of the constraint on the effective 't Hooft coupling.

A.1 Gluon polarization tensor

In $\mathcal{N} = 4$ SYM, there are six diagrams that contribute to the LO gluon self energy. In the HTL limit, for massless bosons and fermions, the gluon polarization tensor was derived in ref. [57]

$$\Pi^{\mu\nu}_{ab}(p) = -g^2 N_c \delta_{ab} \int \frac{d^3k}{(2\pi)^3} \frac{f(\mathbf{k})}{|\mathbf{k}|} \frac{p^2 k^\mu k^\nu - (p \cdot k)(p^\mu k^\nu + k^\mu p^\nu) + (p \cdot k)^2 g^{\mu\nu}}{(p \cdot k)^2} \,, \quad \text{(A.1)}$$

where $f(\mathbf{k}) \equiv 2n_g(\mathbf{k}) + 8n_q(\mathbf{k}) + 6n_s(\mathbf{k})$ is the effective one-particle distribution function for an $\mathcal{N} = 4$ SYM theory. The coefficients of n_g, n_q, n_s are equal to the number of degrees of freedom of the gauge field, fermions, and scalars. Since $\Pi^{\mu\nu}_{ab}(p)$ is symmetric and transverse in its Lorentz indices, it is gauge independent.

We can define the Debye mass for the gauge field using

$$m_D^2 = -g_{\mu\nu}\Pi^{\mu\nu}_{aa}(p) = 2g^2 N_c \int \frac{d^3k}{(2\pi)^3} \frac{f(\mathbf{k})}{|\mathbf{k}|} = 2\lambda T^2, \quad \lambda = g^2 N_c \,, \quad \text{(A.2)}$$

which has been given previously in refs. [3] and [57]. Then using integration by parts from ref. [39] applied to (A.1), we obtain

$$\Pi^{\mu\nu}(p) = -g^2 N_c \int \frac{d^3k}{(2\pi)^3} \frac{\partial f(\mathbf{k})}{\partial |\mathbf{k}|} \left[y^\mu y^\nu \frac{p \cdot n}{p \cdot y} - n^\mu n^\nu \right], \quad \text{(A.3)}$$

where $y^\mu \equiv k^\mu/|\mathbf{k}| = (1, \mathbf{k}/|\mathbf{k}|) \equiv (1, \hat{\mathbf{y}})$. After integration over the length of momentum $|\mathbf{k}|$, the HTL gluon polarization tensor can be written as

$$\Pi^{\mu\nu}(p) = m_D^2 \left[\mathcal{T}^{\mu\nu}(p, -p) - n^\mu n^\nu \right]. \quad \text{(A.4)}$$

Where we have introduced a rank-two tensor $\mathcal{T}^{\mu\nu}(p, q)$ which is defined only when $p + q = 0$ as

$$\mathcal{T}^{\mu\nu}(p, -p) = \left\langle y^\mu y^\nu \frac{p \cdot n}{p \cdot y} \right\rangle_{\hat{\mathbf{y}}}. \quad \text{(A.5)}$$

The angular brackets indicate averaging over the spatial direction of the light-like vector y. The tensor $\mathcal{T}^{\mu\nu}$ is symmetric in μ and ν, and satisfies the "Ward identity"

$$p_\mu \mathcal{T}^{\mu\nu}(p, -p) = (p \cdot n) n^\nu. \quad \text{(A.6)}$$

As a result, the polarization tensor $\Pi^{\mu\nu}$ is also symmetric in μ and ν and satisfies

$$\begin{aligned} p_\mu \Pi^{\mu\nu}(p) &= 0, \\ g_{\mu\nu} \Pi^{\mu\nu}(p) &= -m_D^2. \end{aligned} \quad \text{(A.7)}$$

The gluon polarization tensor can also be expressed in terms of two scalar functions, the transverse and longitudinal polarization functions Π_T and Π_L, defined by

$$\begin{aligned} \Pi_T(p) &= \frac{1}{d-1} \left(\delta^{ij} - \hat{p}^i \hat{p}^j \right) \Pi^{ij}(p), \\ \Pi_L(p) &= -\Pi^{00}(p), \end{aligned} \quad \text{(A.8)}$$

where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ is the unit vector in the direction of $\mathbf{p}$. The gluon polarization tensor can be written in terms of these two functions

$$\Pi^{\mu\nu}(p) = -\Pi_T(p)T_p^{\mu\nu} - \frac{1}{n_p^2}\Pi_L(p)L_p^{\mu\nu}\,, \tag{A.9}$$

where the tensor T_p and L_p are

$$T_p^{\mu\nu} = g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n_p^\mu n_p^\nu}{n_p^2},$$
$$L_p^{\mu\nu} = \frac{n_p^\mu n_p^\nu}{n_p^2}\,. \tag{A.10}$$

Above, the four-vector n_p^μ is

$$n_p^\mu = n^\mu - \frac{n \cdot p}{p^2}p^\mu\,, \tag{A.11}$$

which satisfies $p \cdot n_p = 0$ and $n_p^2 = 1 - (n \cdot p)^2/p^2$. Then (A.7) reduces to the identity

$$(d-1)\Pi_T(p) + \frac{1}{n_p^2}\Pi_L(p) = m_D^2\,, \tag{A.12}$$

at the same time, we can use $\mathcal{T}^{00}$ to represent

In the HTL limit, the polarization functions $\Pi_T(p)$ and $\Pi_L(p)$ can we written in terms of $\mathcal{T}^{00}$

$$\Pi_T(p) = \frac{m_D^2}{(d-1)n_p^2}\big[\mathcal{T}^{00}(p,-p) - 1 + n_p^2\big],$$
$$\Pi_L(p) = m_D^2\big[1 - \mathcal{T}^{00}(p,-p)\big]\,. \tag{A.13}$$

Note that it is essential to take the angular average in $d = 3 - 2\epsilon$ in (A.5), and then analytically continue to $d = 3$ only after all poles in ϵ have been elimimated. The expression for $\mathcal{T}^{00}$ is

$$\mathcal{T}^{00}(p,-p) = \frac{\omega(\epsilon)}{2}\int_{-1}^1 dc\,(1-c^2)^{-\epsilon}\frac{p_0}{p_0 - |\mathbf{p}|c}\,, \tag{A.14}$$

where the weight function $\omega(\epsilon)$

$$\omega(\epsilon) = \frac{\Gamma(2-2\epsilon)}{\Gamma^2(1-\epsilon)}2^{2\epsilon} = \frac{\Gamma\left(\frac{3}{2}-\epsilon\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(1-\epsilon)}\,. \tag{A.15}$$

The integral in (A.14) must be defined so that it is analytic at $|p_0| = \infty$. It then has a branch cut running from $p_0 = -|\mathbf{p}|$ to $p_0 = |\mathbf{p}|$. If we take the limit $\epsilon \to 0$, it reduces to its $d = 3$ form

$$\mathcal{T}^{00}(p,-p) = \frac{p_0}{2|\mathbf{p}|}\log\frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|}\,. \tag{A.16}$$

From the results above, we see that the definition of the gluon self energy in (A.1) and (A.4) is the same as in QCD in ref. [49] up to the definition of m_D. Furthermore, as shown in ref. [57], this means that the HTL three-gluon vertex, four-gluon vertex, and ghost-gluon vertex are also the same as obtained in QCD after adjustment of m_D.

A.2 Gluon propagator

The Feynman rule for the gluon propagator is

$$i\delta^{ab}\Delta_{\mu\nu}(p)\,,\tag{A.17}$$

where the gluon propagator tensor $\Delta_{\mu\nu}$ depends on the choice of gauge fixing. In the limit $\xi \to \infty$, the its inverse reduces to

$$
\begin{aligned}
\Delta_\infty^{-1}(p)^{\mu\nu} &= -p^2 g^{\mu\nu} + p^\mu p^\nu - \Pi^{\mu\nu}(p) \\
&= -\frac{1}{\Delta_T(p)}T_p^{\mu\nu} + \frac{1}{n_p^2 \Delta_L(p)}L_p^{\mu\nu}\,,
\end{aligned}
\tag{A.18}
$$

where Δ_T and Δ_L are the transverse and longitudinal propagators

$$
\begin{aligned}
\Delta_T(p) &= \frac{1}{p^2 - \Pi_T(p)}, \\
\Delta_L(p) &= \frac{1}{-n_p^2 p^2 + \Pi_L(p)}\,.
\end{aligned}
\tag{A.19}
$$

The inverse propagator for general ξ is

$$\Delta^{-1}(p)^{\mu\nu} = \Delta_\infty^{-1}(p)^{\mu\nu} - \frac{1}{\xi}p^\mu p^\nu\,,\tag{A.20}$$

then by inverting the tensor $\Delta^{-1}(p)^{\mu\nu}$, we can get

$$\Delta^{\mu\nu}(p) = -\Delta_T(p)T_p^{\mu\nu} + \Delta_L(p)n_p^\mu n_p^\nu - \xi\frac{p^\mu p^\nu}{(p^2)^2}\,.\tag{A.21}$$

In the course of the calculation it proved to be convenient to introduce the following propagators

$$\Delta_X(p) = \Delta_L(p) + \frac{1}{n_p^2}\Delta_T(p)\,.\tag{A.22}$$

Using (A.12) and (A.19), it can also be expressed as

$$\Delta_X(p) = \left[m_D^2 - d\Pi_T(p)\right]\Delta_L(p)\Delta_T(p)\,,\tag{A.23}$$

which vanishes in the limit $m_D \to 0$. Using this form, gluon propagator tensor can be written as

$$
\begin{aligned}
\Delta^{\mu\nu}(p) = {}& \left[-\Delta_T(p)g^{\mu\nu} + \Delta_X(p)n^\mu n^\nu\right] - \frac{n\cdot p}{p^2}\Delta_X(p)(p^\mu n^\nu + n^\mu p^\nu) \\
& + \left[\Delta_T(p) + \frac{(n\cdot p)^2}{p^2}\Delta_X(p) - \frac{\xi}{p^2}\right]\frac{p^\mu p^\nu}{p^2}\,.
\end{aligned}
\tag{A.24}
$$

A.3 Quark self-energy

In $\mathcal{N} = 4$ SYM theory, there are three diagrams that contribute to the quark self energy. In HTL limit, the quark self energy was computed in ref. [57] for massless bosons and fermions

$$\Sigma_{ab}^{ij}(p) = \frac{g^2}{2} N_c \delta_{ab} \delta^{ij} \int \frac{d^3k}{(2\pi)^3} \frac{f(\mathbf{k})}{|\mathbf{k}|} \frac{\not{k}}{p \cdot k} \,. \tag{A.25}$$

This form is not complicated and we can divide $|\mathbf{k}|$ directly for the last part in (A.25) giving

$$\Sigma_{ab}^{ij}(p) = \frac{g^2}{2} N_c \delta_{ab} \delta^{ij} \int \frac{k^2 dk}{2\pi^2} \frac{f(\mathbf{k})}{|\mathbf{k}|} \int \frac{d\Omega}{4\pi} \frac{\not{y}}{p \cdot y} \,, \tag{A.26}$$

after integration for momentum $|\mathbf{k}|$, the HTL quark self energy can be written as

$$\Sigma(p) = m_q^2 \mathcal{T}(p), \qquad m_q^2 = \frac{1}{2} \lambda T^2 \,, \tag{A.27}$$

where we have suppressed the trivial Kronecker deltas and

$$\mathcal{T}^\mu(p) \equiv \left\langle \frac{y^\mu}{p \cdot y} \right\rangle_{\hat{\mathbf{y}}} \,, \tag{A.28}$$

and m_q^2 is the quark mass in super symmetry, satisfies $m_q^2 = 1/4 m_D^2$.

Similar to the gluon polarization tensor, the angular average in $\mathcal{T}^\mu$ can be expressed as

$$\mathcal{T}^\mu(p) = \frac{\omega(\epsilon)}{2} \int_{-1}^{1} dc (1 - c^2)^{-\epsilon} \frac{y^\mu}{p_0 - |\mathbf{p}| c} \,. \tag{A.29}$$

The integral in (A.29) must be defined, so that it is analytic at $|p_0| = \infty$. It then has a branch cut running from $p_0 = -|\mathbf{p}|$ to $p_0 = |\mathbf{p}|$. In three dimensions, it can be written as

$$\Sigma(p) = \frac{m_q^2}{2|\mathbf{p}|} \gamma_0 \log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} + \frac{m_q^2}{|\mathbf{p}|} \gamma \cdot \hat{\mathbf{p}} \left(1 - \frac{p_0}{2|\mathbf{p}|} \log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} \right) . \tag{A.30}$$

We can see that the definition of quark self energy in (A.27) is the same as in QCD [50] up to the definition of m_q and taking into account that there are four Majorana fermions indexed by i. In practice, this means that the quark propagator, quark-gluon three vertex and quark-gluon four vertex in HTLpt are the same as in QCD after the appropriate adjustment of the group structure constants. We will take the results for these from ref. [50] with the understanding that the finite-temperature quark mass should be understood to that of the SYM theory.

A.4 Quark propagator

The Feynman rule for the quark propagator is

$$i\delta^{ab}\delta^{ij} S(p) \qquad \text{with} \qquad S(p) = \frac{1}{\not{p} - \Sigma(p)} \,, \tag{A.31}$$

where i, j index the Majorana fermion being considered. As a result, the inverse quark propagator can be written as

$$S^{-1}(p) = \not{p} - \Sigma(p) \equiv \mathcal{A}(p) \,, \tag{A.32}$$

where $\mathcal{A}_\mu(p) = (\mathcal{A}_0(p), \mathcal{A}_s(p)\hat{\mathbf{p}})$ with

$$\mathcal{A}_0(p) = p_0 - \frac{m_q^2}{p_0}\mathcal{T}_p\,,$$
$$\mathcal{A}_s(p) = |\mathbf{p}| + \frac{m_q^2}{|\mathbf{p}|}\big[1 - \mathcal{T}_p\big]. \tag{A.33}$$

A.5 HTL quark counterterm

The insertion of an HTL quark counterterm into a quark propagator is

$$i\delta^{ab}\delta^{ij}\Sigma(p)\,, \tag{A.34}$$

where $\Sigma(p)$ is the HTL quark self energy given in (A.27).

A.6 Scalar self-energy

There are four diagrams that contribute to the scalar self energy in $\mathcal{N} = 4$ SYM theory. In the HTL limit, the scalar self energy $\mathcal{P}_{ab}^{AB}$ was computed in ref. [57] for massless bosons and fermions

$$\mathcal{P}_{ab}^{AB}(p) = g^2 N_c \delta_{ab}\delta^{AB} \int \frac{d^3k}{(2\pi)^3}\frac{f(\mathbf{k})}{|\mathbf{k}|}\,. \tag{A.35}$$

After integration over the length of the three-momentum $|\mathbf{k}|$, the HTL scalar self energy reduces to

$$\mathcal{P}_{aa}^{AA}(p) = g^2 N_c T^2 = \lambda T^2 = M_D^2\,, \tag{A.36}$$

where M_D^2 is the adjoint scalar mass, which has been given in refs. [3, 57]. We can see that it satisfies $M_D^2 = m_D^2/2 = 2m_q^2$.

Note that the scalar self energy is a constant, which means that it only affects the scalar propagator and not the scalar-gluon and scalar-quark vertices in $\mathcal{N} = 4$ SYM theory. This is due to the fact that the HTL Lagrangian density $\mathcal{L}_{\text{HTL}}$ is a combination of the fields and their corresponding covariantized self energies and the HTL vertices are obtained by expanding the covariant derivatives D_μ appearing in the HTL effective Lagrangian in powers of the gauge field A_μ. Since there are no covariant derivatives appearing in the scalar contribution to the HTL effective action (2.9), the scalar-gluon vertices will not receive corrections in HTLpt.[4,5]

A.7 Scalar propagator

The Feynman rule for the scalar propagator is

$$i\delta^{ab}\delta^{AB}\Delta_s(p)\,, \tag{A.37}$$

---

[4]Ref. [39] details the steps necessary to obtain the QCD HTLpt propagators and vertices from the QCD HTL effective action for both equilibrium and non-equilibrium systems.

[5]We are grateful for the authors of ref. [57] for bringing this to our attention.

where

$$\Delta_s(p) = \frac{1}{p^2 - M_D^2}\,,\tag{A.38}$$

and its inverse is

$$\Delta_s^{-1}(p) = p^2 - M_D^2\,.\tag{A.39}$$

A.8 HTL scalar counterterm

The insertion of an HTL scalar counterterm into a scalar propagator is

$$-i\delta^{ab}\delta^{AB}\mathcal{P}_{aa}^{AA}(p)\,,\tag{A.40}$$

where $\mathcal{P}_{aa}^{AA}(p)$ is the HTL scalar self energy given in (A.36).

A.9 Quark-gluon vertex

The quark-gluon vertex with incoming gluon momentum p, incoming quark momentum r, and outgoing quark momentum q, Lorentz index μ, and color indices a, b, c is

$$\begin{aligned}
\Gamma_{abc}^{\mu,ij}(p,q,r) &= -gf_{abc}\delta^{ij}\big[\gamma^\mu + m_q^2\tilde{\mathcal{T}}^\mu(p,q,r)\big] \\
&= -gf_{abc}\delta^{ij}\Gamma^\mu(p,q,r)\,.
\end{aligned}\tag{A.41}$$

Note that the sign on the second term differs from ref. [50]. This appears to be a typo in the original reference. The rank-one tensor $\tilde{\mathcal{T}}^\mu$ in the HTL correction term is only defined for $p + r - q = 0$

$$\tilde{\mathcal{T}}^\mu(p,q,r) = \left\langle y^\mu\bigg(\frac{\slashed{y}}{(y\cdot r)(y\cdot q)}\bigg)\right\rangle_{\hat{\mathbf{y}}}\,,\tag{A.42}$$

and is even under the permutation of q and r. It satisfies the "Ward identity"

$$p_\mu\tilde{\mathcal{T}}^\mu(p,q,r) = \mathcal{T}(r) - \mathcal{T}(q)\,.\tag{A.43}$$

Note that the overall sign here differs from ref. [50]. This appears to be a typo in the original reference. The quark-gluon vertex therefore satisfies the Ward identity

$$p_\mu\Gamma^\mu(p,q,r) = S^{-1}(q) - S^{-1}(r)\,.\tag{A.44}$$

A.10 Quark-gluon four vertex

The quark-gluon four vertex with outgoing gluon momentum p, q, incoming quark momentum r, and outgoing quark momentum s is

$$\Gamma_{abcd}^{\mu\nu,ij}(p,q,r,s) = -ig^2\delta^{ij}m_q^2\tilde{\mathcal{T}}_{abcd}^{\mu\nu}(p,q,r,s)\,,\tag{A.45}$$

where we note that compared to QCD, the SYM theory has only quark indices in the adjoint representation. There is no tree-level term. The rank-two tensor $\tilde{\mathcal{T}}^{\mu\nu}$ is only defined for $p + q + s - r = 0$

$$\begin{aligned}
\tilde{\mathcal{T}}_{abcd}^{\mu\nu}(p,q,r,s) &= f_{cde}f_{bae}\left\langle y^\mu y^\nu\frac{\slashed{y}}{(y\cdot r)(y\cdot s)[y\cdot(r-p)]}\right\rangle \\
&+ f_{bde}f_{cae}\left\langle y^\mu y^\nu\frac{\slashed{y}}{(y\cdot r)(y\cdot s)[y\cdot(s+p)]}\right\rangle,
\end{aligned}\tag{A.46}$$

and satisfies

$$\delta^{ij}\delta^{ad}\delta^{bc}\Gamma^{\mu\nu}_{abcd,ij}(p,q,r,s) = -4ig^2 N_c d_A \Gamma^{\mu\nu}(p,q,r,s)\,, \tag{A.47}$$

where

$$\Gamma^{\mu\nu}(p,q,r,s) = m_q^2 \left\langle y^\mu y^\nu \left(\frac{1}{y\cdot r} + \frac{1}{y\cdot s}\right) \frac{\slashed{y}}{[y\cdot(r-p)][y\cdot(s+p)]} \right\rangle. \tag{A.48}$$

This tensor is symmetric in μ and ν, and satisfies the Ward identity

$$p_\mu \Gamma^{\mu\nu}(p,q,r,s) = \Gamma^\nu(q,r-p,s) - \Gamma^\nu(q,r,s+p)\,. \tag{A.49}$$

A.11 Four-scalar vertex

The four-scalar vertex does not depend on the momentum and is

$$\begin{aligned}
\Gamma^{ABCD}_{abcd}(p,q,r,s) = -ig^2 \Big[& f_{abe}f_{cde}\left(\delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}\right) \\
& + f_{ace}f_{bde}\left(\delta^{AB}\delta^{CD} - \delta^{AD}\delta^{BC}\right) \\
& + f_{ade}f_{bce}\left(\delta^{AB}\delta^{CD} - \delta^{AC}\delta^{BD}\right) \Big]\,.
\end{aligned} \tag{A.50}$$

This vertex satisfies

$$\delta^{bd}\delta^{ac}\delta^{AC}\delta^{BD}\Gamma^{ABCD}_{abcd}(p,q,r,s) = (-ig^2)(60 N_c d_A)\,, \tag{A.51}$$

where $\delta^{AA} = 6$ for six scalars in this theory.

A.12 Scalar-gluon vertex

The scalar-gluon vertex with incoming gluon momentum p, incoming scalar momentum r, and outgoing scalar momentum q is

$$\Gamma^{\mu,AB}_{abc}(p,q,r) = g f_{abc}\delta^{AB}(r+q)^\mu\,. \tag{A.52}$$

A.13 Scalar-gluon four vertex

The scalar-gluon four vertex is independent on the direction of the momentum, and it can be expressed as

$$\Gamma^{\mu\nu,AB}_{abcde}(p,q,r,s) = -2ig^2 g^{\mu\nu}\delta^{AB} f_{ade}f_{bce}\,, \tag{A.53}$$

and satisfies

$$\delta^{ac}\delta^{bd}\delta^{AB}\Gamma^{\mu\nu,AB}_{abcde}(p,q,r,s) = (2ig^2)(6 N_c d_A)g^{\mu\nu}\,. \tag{A.54}$$
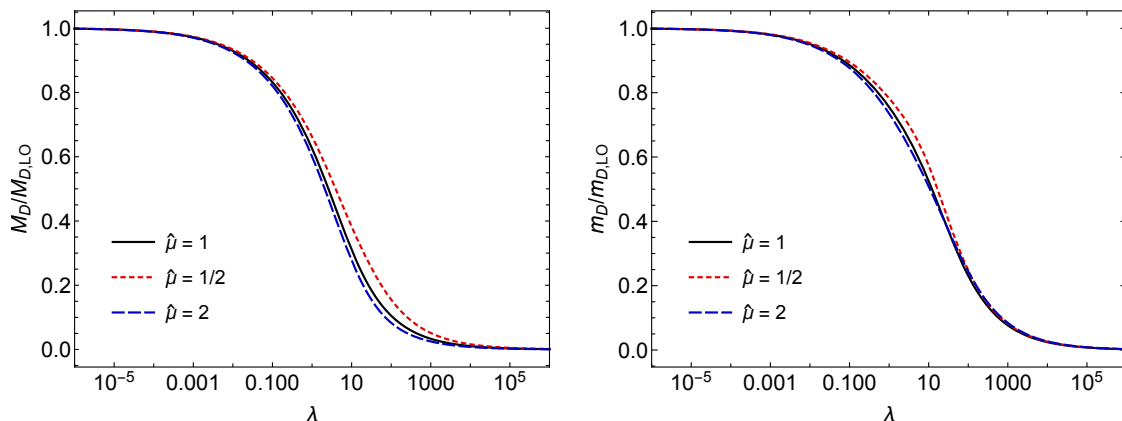
Figure 8. Numerical solution of the gap equations for M_D and m_D as a function of λ for the case $m_q = 0$. In both panels the results are scaled by their corresponding leading-order weak-coupling limits.

A.14 Quark-scalar vertex

Since fermions have different interactions with the scalar (X_{p}) and pseudoscalar (Y_{q}) degrees of freedom, there are two kinds of vertex needed. One is quark-scalar vertex, with incoming scalar momentum p, outgoing quark momentum q and incoming quark momentum r, and their corresponding colors a, b, c respectively. This vertex can be written as

$$\Gamma^{\mathsf{p}}_{abc,ij}(p, q, r) = -igf_{abc}\alpha^{\mathsf{p}}_{ij}. \tag{A.55}$$

The other one is quark-pseudoscalar vertex

$$\Gamma^{\mathsf{q}}_{abc,ij}(p, q, r) = gf_{abc}\beta^{\mathsf{q}}_{ij}\gamma_5. \tag{A.56}$$

B The comparison of the results under $m_q = 0$ and $m_q = m_q^{\mathrm{gap}}$

As discussed in the main body of the text, it is not formally necessary to include the quark contribution in the HTL effective Lagrangian since the fermionic sector is infrared safe. As a result, one can consider an alternative reorganization which starts from a purely-gluonic HTL effective Lagrangian. Formally, this can be accomplished by setting $m_q = 0$ from the beginning. This choice affects the solutions to the scalar and gluon masses and hence results in a different prediction for the thermodynamic functions. In figure 8 we present a comparison of the scalar and gluon masses resulting from the choice $m_q(\lambda) = 0$. We find that the solutions with $m_q = 0$ are qualitatively similar to those with $m_q = m_q^{\mathrm{gap}}$ shown in figure 4, however, small quantitative differences remain. In order to assess the differences between the two quark mass schemes, in figure 9 we present the ratio of the solutions obtained with $m_q = 0$ to those obtained with $m_q = m_q^{\mathrm{gap}}$ (6.13). As can be seen from this figure, the differences between the two schemes are maximally on the order of 40%, with the largest deviations occurring at very large couplings. Note that, despite these differences, both schemes have $M_D \to 0$ and $m_D \to 0$ in the large coupling limit as
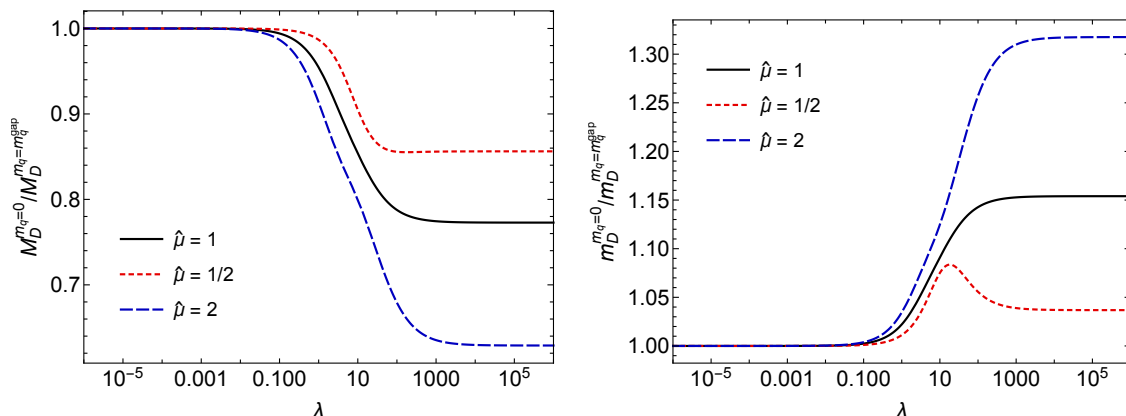
Figure 9. Ratio of the numerical solutions of the M_D and m_D gap equations for the two quark mass schemes: $m_q = 0$ and $m_q = m_q^{\text{gap}}$ (6.13).
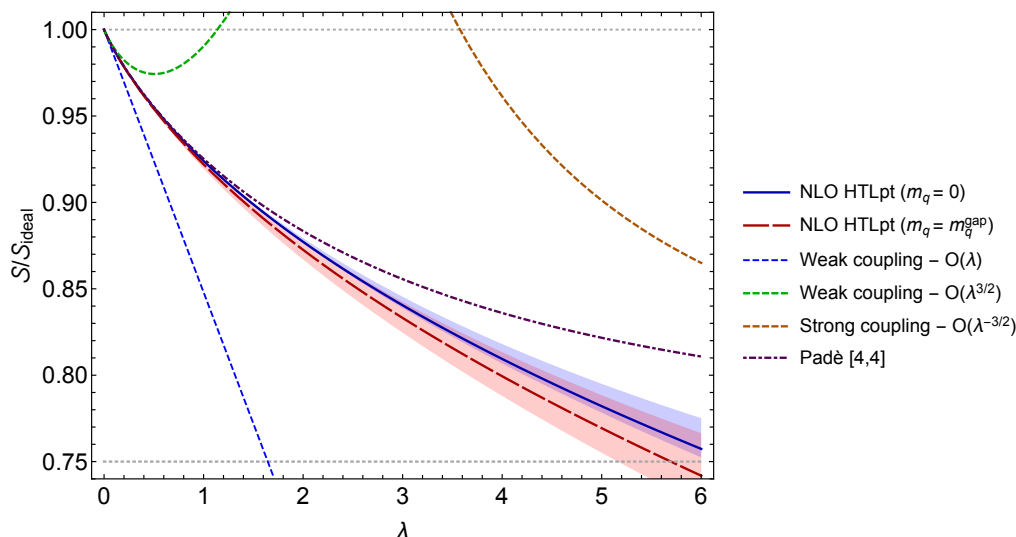

Figure 10. Comparison of two different quark mass prescriptions: $m_q = 0$ and $m_q = m_q^{\text{gap}}$. NLO HTLpt results with these prescriptions are shown as solid blue and long-dashed red lines, respectively. We also include plots of the results obtained in the strong and weak coupling limits along with the [4,4] Padé approximant.

can be seen in figures 4 and 8. Finally, in figure 10 we present a comparison of the final result for the scaled entropy using the two different quark mass prescriptions: $m_q = 0$ and $m_q = m_q^{\text{gap}}$. The NLO HTLpt results with these two prescriptions are shown as solid blue and long-dashed red lines, respectively. We also include plots of the results obtained in the strong and weak coupling limits along with the [4,4] Padé approximant. As can be seen from this figure, there is a dependence on the quark mass scheme, however, it is on the order of the size of the uncertainty bands of the full NLO HTLpt calculation. If one were to go to NNLO, presumably the difference between the two schemes would be diminished further.

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