On Itzykson-Zuber Ansatz

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Abstract: We apply the renormalized coupling constants and Virasoro constraints to derive the Itzykson-Zuber Ansatz on the form of the free energy in 2D topological gravity. We also treat the 1D topological gravity and the Hermitian one-matrix models in the same fashion. Some uniform behaviors are discovered in this approach.

Contents

How can I convert this from a problem of infinite number of degrees of freedom, which you can't deal with anyhow, to a problem which is finite even if it was so large that you would have to have an astronomical size computer. I just wanted to convert it from an infinite number of degrees of freedom to a finite number.

K. Wilson

1 Introduction

In recent years there have appeared various Gromov-Witten type theories. In all these theories we study their partition functions or free energy functions, expressed as formal power series in infinitely many formal variables. It is not convenient to study functions in infinitely many variables. The well-established methods in mathematics usually deal with only finitely many variables, especially when analyticity or smoothness of the functions are concerned. It is then very desirable to develop some methods to convert the problems that involves infinitely many variables to problems with only finitely many variables. Amazingly, such situations were faced by Wilson when he studied renormalization theory. He discovered that in doing renormalzations step by step, it is necessary to work with Hamiltonians with all possible coupling constants hence it is necessary to work with a problem of an infinite degrees of freedom. By considering the fixed points of the renormalization flow, a miracle happens: in the limit the theory becomes soluble with finitely many degrees of freedom. We will report in this work similar miracles happen in the case of some Gromov-Witten type theories. In this paper we will focus on three examples: 1D topological gravity, Hermitian one-matrix models, and 2D topological gravity. In subsequent work, we will make generalizations to other models.

To explain our results, let us begin with the case of two-dimensional topological gravity. Witten [26] interpreted the 2D topological gravity as the intersection theory of ψ-classes on the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$. The free energy of this theory is the generating function defined by:

$$F^{2D}(t) = \sum_{g=0}^{\infty} F_g^{2D} := \sum_{g=0}^{\infty} \sum_{n_0,n_1,n_2\cdots} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g^{2D} \frac{t_0^{n_0}}{n_0!} \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!} \cdots, \qquad (1.1)$$

where $\{t_i\}_{i \geqslant 0}$ are formal variables understood as the coupling constants, and they will be understood as coordinates on the big phase space of the 2D topological gravity. The partition function is defined by

$$Z^{2D} := e^{F^{2D}}. \qquad (1.2)$$

Witten conjectured that Z^{2D} is a tau-function of the KdV hierarchy, i.e., $u = \frac{\partial^2 F^{2D}}{\partial t_0^2}$ satisfies a system of nonlinear differential equations of the form:

$$\frac{\partial u}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1}(u, u_{t_0}, \dots), \qquad (1.3)$$

and furthermore, the free energy satisfies the string equation

$$\frac{\partial F^{2D}}{\partial t_0} = \frac{t_0^2}{2} + \sum_{n=0}^{\infty} t_{n+1} \frac{\partial F^{2D}}{\partial t_n}. \tag{1.4}$$

Conversely, together with the string equation, the KdV hierarchy completely determines the free energy. This conjecture was proved by Kontsevich [20, 21] by computing the partition function in a different way-the Kontsevich matrix model, so the partition function Z^{2D} is also known as the Witten-Kontsevich tau-function and denoted by τ_{WK}. For earlier matrix model approach to two-dimensional gravity, see e.g. Brezin-Kazakov [5], Douglas-Shenker [11] and Gross-Migdal [15].

There is another way to study the Witten-Kontsevich tau-function τ_{WK}. Douglas [10] proposed the connection to generalized KdV hierarchy (Gelfand-Dickey hierarchy) with the additional condition called the string equation. Dijkgraaf-Verlinde-Verlinde [8] derived from the KdV equation and the string equation that τ_{KW} satisfies a system of linear partial differential equations known as the Virasoro constraints, and furthermore is uniquely determined by these constraints. See also the independent work by Fukuma-Kawai-Nakayama [13]. For the derivation of the Virasoro constraints for the Hermitean matrix model, see e.g. [4, 6, 16, 24]. For the derivation in the Kontsevich model, see e.g. [14, 22, 27]. For further development for generalized Kontsevich matrix models, see e.g. [18, 19].

One can use either the characterization by KdV hierarchy plus string equation or the Virasoro constraints to compute the free energy functions F_g^{2D}. The results are of course some formal power series in the coupling constants $t_0, t_1, \ldots$ which are too complicated to expect any closed formulas directly. In [17], Itzykson and Zuber introduced another group of variables $I_0, I_1, \cdots$ defined by:

$$u_0(t) := \frac{\partial^2 F_0^{2D}}{\partial t_0^2}, \tag{1.5}$$

$$I_n(t) := \sum_{k \geqslant 0} t_{n+k} \frac{u_0^k}{k!}, \quad n \geqslant 0 \tag{1.6}$$

and they proved that $u_0 = I_0$, i.e.

$$I_n(t) = \sum_{k \geqslant 0} t_{n+k} \frac{I_0^k}{k!}, \quad n \geqslant 0. \tag{1.7}$$

With these definitions, they made an ansatz on the shape of F_g^{2D}:

$$F_g^{2D}(t) = \sum_{\sum_{2 \leqslant k \leqslant 3g-2}(k-1)l_k=3g-3} \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \right\rangle_g^{2D} \prod_{j=2}^{3g-2} \frac{1}{l_j!} \left(\frac{I_j}{(1-I_1)^{\frac{2j+1}{3}}} \right)^{l_j} \tag{1.8}$$

for $g \geqslant 2$. This is a finite sum of monomials in $\frac{I_j}{(1-I_1)^{(2j+1)/3}}$. By inserting this ansatz into KdV hierarchy, they got some concrete results of free energies of low genus and they remarked that "There is no difficulty to pursue these computations as far as one wishes".

Such idea of introducing new variables so that the free energy depending on an infinite set of coupling constants can be rearranged into polynomials of finitely many new variables is also developed and applied to the Hermitian matrix models in [2, 3]. Two sequences of moment variables $\{M_k\}_{k\geq 1}$ and $\{J_k\}_{k\geq 1}$ are introduced in the case of one-cut solutions. They reduce to only one sequence in the double scaling limit. Note the Itzykson-Zuber ansatz was used in [2] to derive the equivalence between the double scaling limit of Hermitian one-matrix model and Kontsevich model.

Inspired by the Itzykson-Zuber ansatz especially the introduction of I_k, the second named author carried out a comprehensive study of one-dimensional topological gravity in [28] in the hope that this would lead to a better understanding of the Itzykson-Zuber ansatz. He considered the action function:

$$S(x) = -\frac{1}{2}x^2 + \sum_{n=0}^{\infty} t_n \frac{x^{n+1}}{(n+1)!}, \tag{1.9}$$

and its renormalization by the iterated procedure of completing the square. In the limit of this procedure, one reaches the critical point x_∞ of the action function which satisfies the following equation ([28], proposition 2.1):

$$x_\infty = \sum_{n\geqslant 0} t_n \frac{x_\infty^n}{n!}, \tag{1.10}$$

and the Taylor expansion of $S(x)$ at $x = x_\infty$ is

$$S(x) = S(x_\infty) + \sum_{n=2}^{\infty} (I'_{n-1} - \delta_{n,2}) \frac{(x - x_\infty)^n}{n!}, \tag{1.11}$$

where

$$I'_n(t) = \sum_{k\geqslant 0} t_{n+k} \frac{x_\infty^k}{k!}, \quad n \geqslant 1. \tag{1.12}$$

One can see that x_∞ and I'_k are exactly I_0 and I_k by definition! In [28], by Lagrange inversion formula, the second named author derived an explicit formula for I_0 in terms of $\{t_n\}_{n\geq 0}$:

$$I_0 = t_0 + \sum_{k=2}^{\infty} \frac{1}{k} \sum_{\substack{p_1+\cdots+p_k=k-1 \\ p_i\geqslant 0, i=1,\cdots,k}} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_k}}{p_k!}. \tag{1.13}$$

By plugging this into (1.7), one can get similar expressions for I_k. He also considered the inverse transformation, and got the following result:

$$t_n = \sum_{k\geqslant 0} I_{n+k} \frac{(-1)^k I_0^k}{k!}, \quad n \geqslant 0. \tag{1.14}$$

As we have mentioned, $\{t_n\}_{n\geqslant 0}$ are understood as coordinates on the big phase space. Now $\{I_n\}_{n\geqslant 0}$ can be understood as new coordinates on the big phase space. They are called the renormalized coupling constants in [28]. The reason for this terminology is

because the striking analogy with Wilson's renormalization theory in the sense that one starts with an arbitrary set of infinitely many coupling constants and the fixed point of the renormalization flow gives one another set of infinitely coupling constants. Another analogy with Wilson's theory is that with the introduction of these renormalized constants makes the theory become soluble in finitely many degrees of freedom. This is our interpretation of the Itzykson-Zuber ansatz (1.8). In fact, inspired by Itzykson-Zuber's ansatz, the second author proved that for the 1D topological gravity theory,

$$F_g^{1D}(t) = \sum_{\sum_{2 \leqslant k \leqslant 2g-1}(k-1)l_k = 2g-2} \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{2g-1}^{l_{2g-1}} \right\rangle_g^{1D} \prod_{j=2}^{2g-1} \frac{1}{l_j!} \left(\frac{I_j}{(1-I_1)^{\frac{j+1}{2}}} \right)^{l_j} \quad (1.15)$$

for $g \geqslant 2$, where F_g^{1D} is the free energy of genus g of the 1D topological gravity.

Another example we treat in this work is the Hermitian one-matrix models of order N (i.e. the matrix is of $N \times N$). The second named author understood the 1D topological gravity as the Hermitian one-matrix models of order 1 and extended some results for 1D topological gravity to Hermitian one-matrix models in [29–31]. Using the same method, he proved that for Hermitian one-matrix models,

$$F_g^N(t) = \sum_{\sum_{2 \leqslant k \leqslant 2g-1}(k-1)l_k = 2g-2} \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{2g-1}^{l_{2g-1}} \right\rangle_g^N \prod_{j=2}^{2g-1} \frac{1}{l_j!} \left(\frac{I_j}{(1-I_1)^{\frac{j+1}{2}}} \right)^{l_j} \quad (1.16)$$

for $g \geqslant 2$, where F_g^N is the free energy of genus g.

The proofs of (1.15) and (1.16) in previous works of the second named author are based on rewriting the corresponding puncture equation and the dilaton equation in 1D topological gravity and Hermitian one-matrix models in terms of the I-coordinates. It was announced in [28] that the same method can be applied to establish the Itzykson-Zuber ansatz. In this work we will achieve more than that. More precisely, we will show that all the Virasoro operators $\{L_m\}_{m \geqslant -1}$ in all the above three examples can be rewritten in terms of the I-coordinates, and using this fact, one can also calculate the free energies of in arbitrary genus g recursively.

In [33] the second named author introduced some coupling constants t_{-n} for $n \geq 1$ and call them the ghost variables. They were used to defined the following extension of free energy F_0^{2D} in genus zero:

$$\tilde{F}_0^{2D} = F_0^{2D} + \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) t_{-n-1}. \quad (1.17)$$

This was used to make sense of the following special deformation of Airy curve introduced in [32]:

$$w^{2D} := z^{\frac{1}{2}} - \sum_{n=0}^{\infty} \frac{t_n}{(2n-1)!!} z^{n-\frac{1}{2}} - \sum_{n=0}^{\infty} (2n+1)!! \frac{\partial F_0^{2D}}{\partial t_n} z^{-n-\frac{3}{2}}. \quad (1.18)$$

See also section 5. In this paper we also consider the renormalizations of the ghost variables. They will be denoted by I_{-k} for $k \geq 1$:

$$I_{-k} = \sum_{n=0}^{\infty} t_{n-k} \frac{I_0^n}{n!}. \quad (1.19)$$

When we impose the condition $t_{-m} = 0$ for $m > 0$, the ghost variables I_{-n} can be expressed as formal series in $\{I_k\}_{k \geq 0}$:

$$I_{-n}|_{t_{-m}=0,m\geq 1} = \sum_{k=0}^{\infty} I_k \frac{(-1)^k I_0^{k+n}}{k!(n-1)!(k+n)}. \qquad (1.20)$$

It turns out that certain shift, denoted by $\tilde{I}_{-n}$ is more natural. An amazing result is that

$$F_0^{1D} = \tilde{I}_{-1}|_{t_{-m}=0,\ m\geq 1}, \qquad (1.21)$$

$$F_0^N = N \cdot \tilde{I}_{-1}|_{t_{-m}=0,\ m\geq 1}, \qquad (1.22)$$

$$F_0^{2D} = \frac{1}{2} \sum_{n \geqslant 1} (-1)^n \tilde{I}_n \tilde{I}_{-n-1}|_{t_{-m}=0,\ m\geq 1}. \qquad (1.23)$$

We also used the renormalized ghost variables to investigate the applications of the I-coordinates to the study of the special deformations of the emergent spectral curves in the three cases we consider in this paper. It turns out that they manifest some uniform behaviors related to (1.9) in this new perspective. We summarize them in the end of the paper where we present some concluding remarks.

In summary, we have studied the Itzykson-Zuber Ansatz and its analogues from the point of view of Wilson's renormalization theory. One can interpret the results stated in such Ansatz as generalizations of the constitutive relations in the mean field theory approach studied by Dijkgraaf and Witten [9]. It is interesting to see that renormalization leads to the derivations of results in mean field theory in these theory. We believe this should hold in general and hope to return to this in future investigations.

We arrange the rest of the paper in the following fashion. We treat the cases of 1D topological gravity and Hermitian one-matrix models in section 2 and section 3 respectively. We verify the Itzykson-Zuber Ansatz in section 4. We generalize the renormalized coupling constants to include the ghost coupling constants in section 5, and use the renormalized ghost variables to study the special deformation of the Airy curve induced by the Witten-Kontsevich tau-function. In section 6 we rederive the constitutive relations in genus zero due to Dijkgraaf and Witten [9] and derive their analogues for F_0^{1D} and F_0^N. In the final section 7 we comment on the uniform behavior of the special deformations of the spectral curves in the perspective of I-coordinates.

2 Computations in 1D topological gravity by Virasoro constraints in renormalized coupling constants

In this section we recall the 1D topological gravity [28]. The partition function of this theory satisfies Virasoro constraints derived in [25] and further studied in [28]. We rewrite the Virasoro constraints in the I-coordinates. Using these constraints, we derive a recursively way to solve free energy in I-coordinates. We also study the special deformation of the 1D gravity in I-coordinates.

2.1 Renormalized coupling constants in the 1D topological gravity

In order to understand how the I-coordinates in the Itzykson-Zuber Ansatz naturally arise, the second named author proposed in [28] to start with the 1D topological gravity and understand its action function from the point of view of Wilson's renormalization theory. The partition function of 1D gravity is the following formal Gaussian integral:

$$Z^{1D} = \frac{1}{\sqrt{2\pi}\lambda} \int e^{\frac{1}{\lambda^2}S(x)}dx, \tag{2.1}$$

where the action function of the 1D topological gravity is given by:

$$S(x) = -\frac{1}{2}x^2 + \sum_{n=0}^{\infty} t_n \frac{x^{n+1}}{(n+1)!}. \tag{2.2}$$

The coefficients $\{t_n\}_{n\geqslant 0}$ are the bare coupling constants of this theory. One can modify the action by completion of square:

$$S = \left(t_{-1} + \frac{1}{2}\frac{t_0^2}{1-t_1} + \sum_{n\geq 3}\frac{t_{n-1}}{n!}\left(\frac{t_0}{1-t_1}\right)^n\right) + \tilde{x}\sum_{n\geq 2}\frac{t_n}{n!}\left(\frac{t_0}{1-t_1}\right)^n$$
$$-\frac{1}{2}\left(1 - \sum_{n\geq 0}t_{n+1}\frac{1}{n!}\left(\frac{t_0}{1-t_1}\right)^n\right)\tilde{x}^2 + \sum_{m=3}^{\infty}\frac{\tilde{x}^m}{m!}\sum_{n\geq 0}t_{n+m-1}\frac{1}{n!}\left(\frac{t_0}{1-t_1}\right)^n,$$

where $\tilde{x} = x - x_1$. A new set of coupling constants is then obtained in this fashion. As explained in [28], by repeating this procedure for infinitely many times, one gets a limiting set of coupling constants which are fixed by the procedure of completing the square. More precisely, the action function has the following form:

$$S(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(I_k + \delta_{k,1})I_0^{k+1} + \sum_{n=2}^{\infty}(I_{n-1} - \delta_{n,2})\frac{(x-I_0)^n}{n!} \tag{2.3}$$

in the limit. The limiting set of coupling constants $\{I_n\}_{n\geq 0}$ will be referred to as the *renormalized coupling constants*. Here I_n are defined by:

$$I_k = \sum_{n\geq 0}t_{n+k}\frac{x_\infty^n}{n!}, \tag{2.4}$$

where $I_0 = x_\infty$ satisfies the equation

$$\frac{\partial S}{\partial x}(x_\infty) = 0,$$

or, more explicitly,

$$x_\infty = \sum_{n\geq 0}t_n\frac{x_\infty^n}{n!}. \tag{2.5}$$

This situation is analogous to Wilson's renormalization theory. First of all, Wilson considered the space of Hamiltonians with all possible coupling constants. In our case, we allow our action function S to have all possible coupling constants t_n, at the expense of

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considering only formal Gaussian integrals instead of addressing the issue of convergence of the Gaussian integrals. Secondly, Wilson started with a theory with arbitrary coupling constants and modified it to get a new theory of the same form with different coupling constants. In our case, we use the completion of square to modify our coupling constants. Thirdly, Wilson introduced the notion of a fixed point to describe the limiting theory of the renormalization flow. In our case the situation is similar. We reach the critical point of the action function and use the expansion there to obtain the limiting coupling constants. Furthermore, in [28] it was proved that the transformation from $\{t_n\}_{n \geq 0}$ to $\{I_n\}_{n \geq 0}$ can be regarded as a nonlinear change of coordinates, and the space of the theory of 1D topological gravity can be regarded as an infinite-dimensional manifold with at least two coordinate patches given by local coordinates $\{t_n\}_{n \geq 0}$ and $\{I_n\}_{n \geq 0}$ respectively.

It turns out that the analogue of the Itzykson-Zuber Ansatz also holds in 1D topological gravity. The free energy of 1D topological gravity is defined by:

$$F^{1D} = \log Z^{1D}. \tag{2.6}$$

There is a genus expansion for F^{1D}:

$$F^{1D} = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g^{1D}, \tag{2.7}$$

where $\{F_g^{1D}\}_{g \geq 0}$ are formal power series of $t_0, t_1, \cdots$. By [28], if we define

$$\deg t_n = n - 1, \quad n = 0, 1, 2, \cdots \tag{2.8}$$

then F_g^{1D} is weighted homogeneous in $t_0, t_1, \cdots$ with

$$\deg F_g^{1D} = 2g - 2, \quad g = 0, 1, 2, \cdots \tag{2.9}$$

By (2.4), it is natural to define

$$\deg I_n = n - 1, \quad n = 0, 1, 2, \cdots \tag{2.10}$$

and then F_g^{1D} can be viewed as weighted homogeneous formal series of degree $2g - 2$ in I-coordinates. In [28], the second named author used two different methods, the Feynman diagram technique and the Virasoro constraints, to get the following results:

Theorem 2.1.1. *([28, theorem 6.4 and 6.6])*

$$F_0^{1D} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}, \tag{2.11}$$

$$F_1^{1D} = \frac{1}{2} \log \frac{1}{1 - I_1}, \tag{2.12}$$

$$F_g^{1D} = \sum_{\sum_{2 \leq k \leq 2g-1} m_k(k-1)=2g-2} \left\langle \prod_{j=2}^{2g-1} \tau_j^{m_j} \right\rangle_g^{1D} \cdot \prod_{j=2}^{2g-1} \frac{1}{m_j!} \left(\frac{I_j}{(1-I_1)^{\frac{j+1}{2}}} \right)^{m_j}, \; g \geq 2, \tag{2.13}$$

where the correlators are defined by:

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^{1D} = \left. \frac{\partial^n F_g^{1D}}{\partial t_{a_1} \cdots \partial t_{a_n}} \right|_{t=0}. \tag{2.14}$$

2.2 Virasoro constraints for 1D topological gravity in I-coordinates

Let us now recall how to prove the above theorem by Virasoro constraints. First one has the following theorem:

Theorem 2.2.1. *(Virasoro constraints [28]) The partition function Z^{1D} of 1D topological gravity satisfies the following equations for $m \geqslant -1$:*

$$L_m^{1D} Z^{1D} = 0, \tag{2.15}$$

where

$$L_{-1}^{1D} = \frac{t_0}{\lambda^2} + \sum_{n \geqslant 1}(t_n - \delta_{n,1})\frac{\partial}{\partial t_{n-1}}, \tag{2.16}$$

$$L_0^{1D} = 1 + \sum_{n \geqslant 0}(n+1)(t_n - \delta_{n,1})\frac{\partial}{\partial t_n}, \tag{2.17}$$

$$L_m^{1D} = \lambda^2(m+1)!\frac{\partial}{\partial t_{m-1}} + \sum_{n \geqslant 0}\frac{(m+n+1)!}{n!}(t_n - \delta_{n,1})\frac{\partial}{\partial t_{m+n}}, \tag{2.18}$$

for $m \geqslant 1$. Furthermore, $\{L_m^{1D}\}_{m \geqslant -1}$ satisfies the following commutation relations:

$$\left[L_m^{1D}, L_n^{1D}\right] = (m-n)L_{m+n}^{1D}, \tag{2.19}$$

for $m, n \geqslant -1$.

Next the change of coordinates between $\{t_n\}_{n \geq 0}$ and $\{I_n\}_{n \geq 0}$ induces the linear transformations between $\{\frac{\partial}{\partial t_n}\}_{n \geq 0}$ and $\{\frac{\partial}{\partial I_n}\}_{n \geq 0}$. The concrete expressions are given by:

Lemma 2.2.1. *([28], corollary 2.7 and proposition 2.8) The vector fields $\{\frac{\partial}{\partial I_k}\}_{k \geqslant 0}$ can be expressed in terms of the vector fields $\{\frac{\partial}{\partial t_k}\}_{k \geqslant 0}$ as follows:*

$$\frac{\partial}{\partial I_0} = \frac{\partial}{\partial t_0} - \sum_{k \geqslant 0} t_{k+1}\frac{\partial}{\partial t_k}, \tag{2.20}$$

$$\frac{\partial}{\partial I_l} = \sum_{k=0}^{l}\frac{(-1)^{l-k}I_0^{l-k}}{(l-k)!}\frac{\partial}{\partial t_k}, \quad l \geqslant 1 \tag{2.21}$$

and conversely,

$$\frac{\partial}{\partial t_k} = \frac{1}{1-I_1}\frac{I_0^k}{k!}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1}\frac{\partial}{\partial I_l}\right) + \sum_{1 \leqslant l \leqslant k}\frac{I_0^{k-l}}{(k-l)!}\frac{\partial}{\partial I_l}, \quad k \geqslant 0 \tag{2.22}$$

Using these expressions, the following formula for L_{-1}^{1D} and L_0^{1D} has been proved [28, (192), (196)]:

$$L_{-1}^{1D} = -\frac{\partial}{\partial I_0} + \frac{1}{\lambda^2}\sum_{n=0}^{\infty}\frac{(-1)^n I_0^n}{n!}I_n, \tag{2.23}$$

$$L_0^{1D} = -I_0\frac{\partial}{\partial I_0} - 2\frac{\partial}{\partial I_1} + \sum_{l \geqslant 1}(l+1)I_l\frac{\partial}{\partial I_l} + 1. \tag{2.24}$$

From these one can derive (2.11)–(2.13). The main result of this section is that by express-ing L_m^{1D} for higher m in I-coordinates, one can find an algorithm to explicitly compute F_g^{1D}. In this subsection we first express L_m^{1D} in I-coordinates. The applications to the computations of F_g^{1D} will be presented in the next subsection.

Theorem 2.2.2. *In the I-coordinates, the Virasoro operators L_m^{1D} ($m \geq 1$) for 1D topo-logical gravity can be rewritten as follows:*

$$L_m^{1D} = \lambda^2(m+1)! \left(\frac{1}{1-I_1} \frac{I_0^{m-1}}{(m-1)!} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) + \sum_{1 \leqslant l \leqslant m-1} \frac{I_0^{m-1-l}}{(m-1-l)!} \frac{\partial}{\partial I_l} \right)$$
$$- I_0^{m+1} \frac{\partial}{\partial I_0} + \sum_{i=1}^{m+1} \binom{m+1}{i} I_0^{m+1-i} \sum_{p=1}^{\infty} \frac{(p+i)!}{p!} (I_p - \delta_{p,1}) \frac{\partial}{\partial I_{p+i-1}}. \tag{2.25}$$

Proof. The first term in the definition of L_m^{1D} in (2.18) is:

$$\frac{\partial}{\partial t_{m-1}} = \frac{1}{1-I_1} \frac{I_0^{m-1}}{(m-1)!} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) + \sum_{1 \leqslant l \leqslant m-1} \frac{I_0^{m-1-l}}{(m-1-l)!} \frac{\partial}{\partial I_l}. \tag{2.26}$$

This is the first term of right hand of (2.25). By (1.14) and (2.22),

$$\sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} t_n \frac{\partial}{\partial t_{m+n}} \tag{2.27}$$

$$= \sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} \sum_{k \geqslant 0} \frac{(-1)^k I_0^k}{k!} I_{n+k} \frac{I_0^{m+n}}{(m+n)!} \frac{1}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right)$$

$$+ \sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} \sum_{k=0}^{\infty} \frac{(-1)^k I_0^k}{k!} I_{n+k} \left(\sum_{1 \leqslant l \leqslant m+n} \frac{I_0^{m+n-l}}{(m+n-l)!} \frac{\partial}{\partial I_l} \right)$$

$$= \sum_{p \geqslant 0} \sum_{n+k=p} \frac{m+n+1}{n!k!} (-1)^k I_0^{m+n+k} I_{n+k} \frac{1}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right)$$

$$+ \sum_{l \geqslant 1} \sum_{p \geqslant 0} \sum_{n+k=p} \frac{(m+n+1)!}{n!k!(m+n-l)!} (-1)^k I_0^{m+n+k-l} I_{n+k} \frac{\partial}{\partial I_l}$$

$$= \frac{(m+1+I_1)I_0^{m+1}}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) + \sum_{l \geqslant 1, p \geqslant 0} \sum_{i=0}^{l+1} \binom{m+1}{i} \binom{l+1}{i} i! \delta_{p,l+1-i} I_0^{m+p-l} I_p \frac{\partial}{\partial I_l}$$

$$= \frac{(m+1+I_1)I_0^{m+1}}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) + \sum_{l \geqslant 1} \sum_{i=0}^{l+1} \binom{m+1}{i} \binom{l+1}{i} i! I_0^{m+1-i} I_{l+1-i} \frac{\partial}{\partial I_l}.$$

We have used the following identity:

$$\sum_{n+k=p} \frac{(m+n+1)!}{n!k!(m+n-l)!} (-1)^k = \sum_{i=0}^{l+1} \binom{m+1}{i} \binom{l+1}{i} i! \delta_{p,l+1-i}. \tag{2.28}$$

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This can be proved as follows:

$$\sum_{n+k=p} \frac{(m+n+1)!}{n!k!(m+n-l)!}(-1)^k = \sum_{n+k=p} \binom{m+n+1}{l+1} \frac{(l+1)!}{n!k!}(-1)^k$$

$$= \sum_{n+k=p}\sum_{i+j=l+1} \binom{m+1}{i}\binom{n}{j}\frac{(l+1)!}{n!k!}(-1)^k$$

$$= \sum_{i+j=l+1} \binom{m+1}{i}\frac{(l+1)!}{j!} \sum_{n+k=p} \frac{(-1)^k}{(n-j)!k!}$$

$$= \sum_{i=0}^{l+1} \binom{m+1}{i}\binom{l+1}{i}i!\delta_{p,l+1-i}.$$

By (2.22):

$$\frac{\partial}{\partial t_{m+1}} = \frac{1}{1-I_1}\frac{I_0^{m+1}}{(m+1)!}\frac{\partial}{\partial I_0} + \frac{I_0^{m+1}}{(m+1)!}\sum_{l\geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l} + \sum_{1\leqslant l\leqslant m+1}\frac{I_0^{m+1-l}}{(m+1-l)!}\frac{\partial}{\partial I_l}. \qquad (2.29)$$

Therefore,

$$\sum_{n\geqslant 0}\frac{(m+n+1)!}{n!}(t_n-\delta_{n,1})\frac{\partial}{\partial t_{m+n}}$$

$$= \frac{(m+1+I_1)I_0^{m+1}}{1-I_1}\left(\frac{\partial}{\partial I_0}+\sum_{l\geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right)+\sum_{l\geqslant 1}\sum_{i=0}^{l+1}\binom{m+1}{i}\frac{(l+1)!}{(l+1-i)!}I_0^{m+1-i}I_{l+1-i}\frac{\partial}{\partial I_l}$$

$$\quad -\frac{(m+2)I_0^{m+1}}{1-I_1}\left(\frac{\partial}{\partial I_0}+\sum_{l\geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right)-\sum_{1\leqslant l\leqslant m+1}\frac{(m+2)!}{(m+1-l)!}I_0^{m+1-l}\frac{\partial}{\partial I_l}$$

$$= -I_0^{m+1}\left(\frac{\partial}{\partial I_0}+\sum_{l\geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right)+\sum_{l\geqslant 1}\sum_{i=1}^{l}\binom{m+1}{i}\frac{(l+1)!}{(l+1-i)!}I_0^{m+1-i}I_{l+1-i}\frac{\partial}{\partial I_l}$$

$$\quad +\sum_{l\geqslant 1}I_0^{m+1}I_{l+1}\frac{\partial}{\partial I_l}+\sum_{l\geqslant 1}\binom{m+1}{l+1}\frac{(l+1)!}{0!}I_0^{m+1-l}\frac{\partial}{\partial I_l}-\sum_{1\leqslant l\leqslant m+1}\frac{(m+2)!}{(m+1-l)!}I_0^{m+1-l}\frac{\partial}{\partial I_l}$$

$$= -I_0^{m+1}\frac{\partial}{\partial I_0}+\sum_{i=1}^{m+1}\binom{m+1}{i}I_0^{m+1-i}\sum_{l\geqslant i}\frac{(l+1)!}{(l+1-i)!}I_{l+1-i}\frac{\partial}{\partial I_l}-\sum_{l=1}^{m+1}\frac{(m+1)!}{(m+1-l)!}(l+1)I_0^{m+1-l}\frac{\partial}{\partial I_l}$$

$$= -I_0^{m+1}\frac{\partial}{\partial I_0}+\sum_{i=1}^{m+1}\binom{m+1}{i}I_0^{m+1-i}\sum_{p=1}^{\infty}\frac{(p+i)!}{p!}I_p\frac{\partial}{\partial I_{p+i-1}}-\sum_{l=1}^{m+1}\binom{m+1}{l}(l+1)!I_0^{m+1-l}\frac{\partial}{\partial I_l}$$

$$= -I_0^{m+1}\frac{\partial}{\partial I_0}+\sum_{i=1}^{m+1}\binom{m+1}{i}I_0^{m+1-i}\sum_{p=1}^{\infty}\frac{(p+i)!}{p!}(I_p-\delta_{p,1})\frac{\partial}{\partial I_{p+i-1}}.$$

Together with (2.22), the proof is completed. $\qquad\square$

2.3 Computations of F_g^{1D} by Virasoro constraints in I-coordinates

As applications of the expressions of the Virasoro operators expressed in I-coordinates derived in last subsection, we use them in this subsection to compute F_g^{1D}. The formulas for F_0^{1D} and F_1^{1D} in the I-coordinate are already known. We will focus on $F_g^{1D}(t)$ for $g \geqslant 2$.

Theorem 2.3.1. *For free energy of 1D gravity of genus $g \geqslant 2$, the following equations hold:*

$$\frac{\partial F_g^{1D}}{\partial I_1} = \frac{1}{2(1-I_1)} \sum_{l=2}^{2g-1} (l+1)I_l \frac{\partial F_g^{1D}}{\partial I_l}, \tag{2.30}$$

$$\frac{\partial F_g^{1D}}{\partial I_2} = \frac{1}{3(1-I_1)} \left(\frac{1}{1-I_1} \sum_{l=1}^{2g-3} I_{l+1} \frac{\partial F_{g-1}^{1D}}{\partial I_l} + \sum_{p=2}^{2g-2} \binom{p+2}{p} I_p \frac{\partial F_g^{1D}}{\partial I_{p+1}} \right), \tag{2.31}$$

$$\frac{\partial F_g^{1D}}{\partial I_{k+2}} = \frac{1}{(k+3)(1-I_1)} \left(\frac{\partial F_{g-1}^{1D}}{\partial I_k} + \sum_{p=2}^{2g-2-k} \binom{p+k+2}{p} I_p \frac{\partial F_g^{1D}}{\partial I_{k+p+1}} \right), \tag{2.32}$$

where $k = 1, 2, \cdots, 2g-3$.

Proof. The Virasoro constraints for partition function

$$L_m^{1D} Z^{1D} = 0, \quad m \geqslant 0$$

can be rewritten as the following equations for free energy in I-coordinate:

$$0 = -I_0 \frac{\partial F^{1D}}{\partial I_0} - 2 \frac{\partial F^{1D}}{\partial I_1} + \sum_{l \geqslant 1} (l+1)I_l \frac{\partial F^{1D}}{\partial I_l} + 1, \tag{2.33}$$

$$0 = \lambda^2 (m+1)m \frac{I_0^{m-1}}{1-I_1} \left(\frac{\partial F^{1D}}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial F^{1D}}{\partial I_l} + (1-I_1) \sum_{1 \leqslant l \leqslant m-1} \binom{m-1}{l} l! I_0^{-l} \frac{\partial F^{1D}}{\partial I_l} \right)$$
$$- I_0^{m+1} \frac{\partial F^{1D}}{\partial I_0} + \sum_{i=1}^{m+1} \binom{m+1}{i} \sum_{p=1}^{\infty} \frac{(i+p)!}{p!} (I_p - \delta_{p,1}) I_0^{m-i+1} \frac{\partial F^{1D}}{\partial I_{i+p-1}}. \tag{2.34}$$

Recall the genus expansion of the free energy:

$$F^{1D} = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g^{1D},$$

we have by (2.33):

$$I_0 \frac{\partial F_0^{1D}}{\partial I_0} + 2 \frac{\partial F_0^{1D}}{\partial I_1} = \sum_{l \geqslant 1} (l+1)I_l \frac{\partial F_0^{1D}}{\partial I_l} \tag{2.35}$$

$$2 \frac{\partial F_1^{1D}}{\partial I_1} = \sum_{l \geqslant 1} (l+1)I_l \frac{\partial F_1^{1D}}{\partial I_l} + 1 \tag{2.36}$$

$$2 \frac{\partial F_g^{1D}}{\partial I_1} = \sum_{l \geqslant 1} (l+1)I_l \frac{\partial F_g^{1D}}{\partial I_l}, \quad g > 1 \tag{2.37}$$

and by (2.34):

$$0 = (m+1)m\frac{I_0^{m-1}}{1-I_1}\left(\sum_{l\geqslant 1}I_{l+1}\frac{\partial F_{g-1}^{1D}}{\partial I_l} + (1-I_1)\sum_{1\leqslant l\leqslant m-1}\binom{m-1}{l}l!I_0^{-l}\frac{\partial F_{g-1}^{1D}}{\partial I_l}\right) \quad (2.38)$$

$$+ \sum_{i=1}^{m+1}\binom{m+1}{i}\sum_{p=1}^{\infty}\frac{(i+p)!}{p!}(I_p-\delta_{p,1})I_0^{m-i+1}\frac{\partial F_g^{1D}}{\partial I_{i+p-1}},$$

for $m\geqslant 1$, $g\geqslant 2$. Comparing the coefficients of I_0^k in (2.38) for $k=m+1,m-1,m-2,\cdots,0$, we have:

$$0 = \sum_{p\geqslant 1}(p+1)(I_p-\delta_{p,1})\frac{\partial F_g^{1D}}{\partial I_p} \quad (2.39)$$

$$0 = \sum_{l\geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g-1}^{1D}}{\partial I_l} + \sum_{p\geqslant 1}\binom{p+2}{2}(I_p-\delta_{p,1})\frac{\partial F_g^{1D}}{\partial I_{p+1}} \quad (2.40)$$

$$0 = \frac{\partial F_{g-1}^{1D}}{\partial I_l} + \sum_{p\geqslant 1}\binom{p+l+2}{l+2}(I_p-\delta_{p,1})\frac{\partial F_g^{1D}}{\partial I_{p+l+1}}. \quad (2.41)$$

By (2.13), F_g^{1D} depends only on $I_1, I_2, \cdots, I_{2g-1}$, this completes the proof. □

Remark 2.3.1. Theorem 2.3.1 can be derived in another way: one can firstly rewrite L_{-1}^{1D} in I-coordinates, this will give F_0^{1D} and show that F_g^{1D} is independent of I_0 for $g > 0$. To solve F_g^{1D} for $g > 0$, one can let $I_0 = 0$, in this case, $t_0 = 0$,

$$t_k = I_k, \quad k > 0 \quad (2.42)$$

and

$$\frac{\partial}{\partial t_0} = \frac{1}{1-I_1}\frac{\partial}{\partial I_0} + \sum_{l\geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l},$$

$$\frac{\partial}{\partial t_k} = \frac{\partial}{\partial I_k}, \quad k > 0 \quad (2.43)$$

hence one can rewrite $\{L_m^{1D}\}_{m\geqslant 0}$ in I-coordinates in a simpler way.

Now we explain how to use theorem 2.3.1 to calculate free energies of higher genus. By (2.13), F_g^{1D} depends only on $I_1,\ldots,I_{2g-1}$ for $g\geqslant 2$. By (2.32), we have:

$$\frac{\partial F_g^{1D}}{\partial I_{2g-1}} = \frac{1}{2g(1-I_1)}\left(\frac{\partial F_{g-1}^{1D}}{\partial I_{2g-3}}\right), \quad (2.44)$$

$$\frac{\partial F_g^{1D}}{\partial I_{2g-2}} = \frac{1}{(2g-1)(1-I_1)}\left(\frac{\partial F_{g-1}^{1D}}{\partial I_{2g-4}} + \binom{2g}{2}I_2\frac{\partial F_g^{1D}}{\partial I_{2g-1}}\right), \quad (2.45)$$

$$\vdots$$

$$\frac{\partial F_g^{1D}}{\partial I_3} = \frac{1}{4(1-I_1)}\left(\frac{\partial F_{g-1}^{1D}}{\partial I_1} + \sum_{p=1}^{2g-3}\binom{p+3}{p}I_p\frac{\partial F_g^{1D}}{\partial I_{p+2}}\right). \quad (2.46)$$

For $\frac{\partial F_g^{1D}}{\partial I_2}$ and $\frac{\partial F_g^{1D}}{\partial I_1}$ we have by (2.30) and (2.31):

$$\frac{\partial F_g^{1D}}{\partial I_2} = \frac{1}{3(1 - I_1)} \left(\frac{1}{1 - I_1} \sum_{l=1}^{2g-3} I_{l+1} \frac{\partial F_{g-1}^{1D}}{\partial I_l} + \sum_{p=2}^{2g-2} \binom{p+2}{p} I_p \frac{\partial F_g^{1D}}{\partial I_{p+1}} \right), \qquad (2.47)$$

$$\frac{\partial F_g^{1D}}{\partial I_1} = \frac{1}{2(1 - I_1)} \sum_{l=2}^{2g-1} (l+1) I_l \frac{\partial F_g^{1D}}{\partial I_l}. \qquad (2.48)$$

By equations (2.44)–(2.48), we can solve $\left\{ \frac{\partial F_g^{1D}}{\partial I_k} \right\}_{g \geqslant 2, k=2g-1, 2g-2, \cdots, 1}$ recursively, given the computation for F_{g-1}^{1D}. Since F_g^{1D} is weighted homogeneous of degree $2g - 2$ in I_k with $\deg I_k = k - 1$, we have

$$F_g^{1D} = \frac{1}{2g - 2} \sum_{k=1}^{2g-1} (k-1) I_k \frac{\partial F_g^{1D}}{\partial I_k}. \qquad (2.49)$$

Example 2.3.1. *Let us compute F_2^{1D} by the above procedure. We have*

$$\frac{\partial F_2^{1D}}{\partial I_3} = \frac{1}{4(1 - I_1)} \left(\frac{\partial F_1^{1D}}{\partial I_1} \right) = \frac{1}{8(1 - I_1)^2}, \qquad (2.50)$$

$$\frac{\partial F_2^{1D}}{\partial I_2} = \frac{1}{3(1 - I_1)} \left(\frac{1}{1 - I_1} I_2 \frac{\partial F_1^{1D}}{\partial I_1} + \binom{4}{2} I_2 \frac{\partial F_2^{1D}}{\partial I_3} \right) = \frac{5}{12} \frac{I_2}{(1 - I_1)^3}, \qquad (2.51)$$

$$\frac{\partial F_2^{1D}}{\partial I_1} = \frac{1}{2(1 - I_1)} \left(3 I_2 \frac{\partial F_2^{1D}}{\partial I_2} + 4 I_3 \frac{\partial F_2^{1D}}{\partial I_3} \right) = \frac{5}{8} \frac{I_2^2}{(1 - I_1)^4} + \frac{1}{4} \frac{I_3}{(1 - I_1)^3}, \qquad (2.52)$$

therefore

$$F_2^{1D} = \frac{1}{2} \left(I_2 \frac{\partial F_2^{1D}}{\partial I_2} + 2 I_3 \frac{\partial F_2^{1D}}{\partial I_3} \right) = \frac{5}{24} \frac{I_2^2}{(1 - I_1)^3} + \frac{1}{8} \frac{I_3}{(1 - I_1)^2}. \qquad (2.53)$$

Similarly, we have with the help of a computer program:

$$F_3^{1D} = \frac{15}{48} \frac{I_2^4}{(1-I_1)^6} + \frac{25}{48} \frac{I_2^2 I_3}{(1-I_1)^6} + \frac{1}{12} \frac{I_3^2}{(1-I_1)^4} + \frac{7}{48} \frac{I_2 I_4}{(1-I_1)^4} + \frac{1}{48} \frac{I_5}{(1-I_1)^3}, \qquad (2.54)$$

$$F_4^{1D} = \frac{1105}{1152} \frac{I_2^6}{(1-I_1)^9} + \frac{985}{384} \frac{I_2^4 I_3}{(1-I_1)^8} + \frac{445}{288} \frac{I_2^2 I_3^2}{(1-I_1)^7} + \frac{11}{96} \frac{I_3^3}{(1-I_1)^6} + \frac{161}{192} \frac{I_2^3 I_4}{(1-I_1)^7} + \frac{7}{12} \frac{I_2 I_3 I_4}{(1-I_1)^6} \qquad (2.55)$$

$$+ \frac{21}{640} \frac{I_4^2}{(1-I_1)^5} + \frac{113}{576} \frac{I_2^2 I_5}{(1-I_1)^6} + \frac{5}{96} \frac{I_3 I_5}{(1-I_1)^5} + \frac{1}{32} \frac{I_2 I_6}{(1-I_1)^5} + \frac{1}{384} \frac{I_7}{(1-I_1)^4}.$$

2.4 Special deformation of the spectral curve of 1D topological gravity in I-coordinates

In [28], the second named author defined the special deformation of 1D topological gravity by:

$$y^{1D} = \frac{1}{\sqrt{2}} \sum_{n \geqslant 0} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n \geqslant 1} \frac{n!}{z^{n+1}} \frac{\partial F_0^{1D}}{\partial t_{n-1}}. \qquad (2.56)$$

This is a deformation of the Catalan curve [31]. Now we rewrite it in I-coordinates. We get the following result:

Theorem 2.4.1. *In the I-coordinates, the special deformation of the spectral curve of 1D topological gravity can be written as:*

$$y^{1D} = \frac{\sqrt{2}}{z - I_0} + \frac{1}{\sqrt{2}} \sum_{n \geqslant 1} \frac{I_n - \delta_{n,1}}{n!} (z - I_0)^n. \tag{2.57}$$

Proof. This is just the $N = 1$ case of [31, theorem 2.1]. $\qquad\square$

3 Computations in Hermitian one-matrix models by the renormalized coupling constants

In this section we recall the results on Hermitian one-matrix models in [29–31]. The partition function of this theory satisfies Virasoro constraints. Similar to the case of 1D topological gravity, we rewrite the Virasoro constraints for Hermitian one-matrix models in I-coordinates and use them to derive the explicit formulas for the free energy in I-coordinates.

3.1 Free energy functions of the Hermitian one-matrix models

For standard references on matrix models, see e.g. [7, 23]. Here we follow the notations in [29–31]. The partition function of the Hermitian $N \times N$-matrix model is defined by the formal Gaussian integral:

$$Z^N = \frac{\int_{\mathbb{H}_N} dM \exp\left(\frac{1}{g_s} tr\left(-\frac{1}{2}M^2 + \sum_{n=0}^{\infty} t_n \frac{M^{n+1}}{(n+1)!}\right)\right)}{\int_{\mathbb{H}_N} dM \exp\left(-\frac{1}{2g_s} tr M^2\right)}, \tag{3.1}$$

where $\mathbb{H}_N$ is the space of Hermitian $N \times N$-matrices. One can see that for $N = 1$,

$$Z^{N=1} = Z^{1D}. \tag{3.2}$$

The following result is well known, for the proof, one can see [7, 23].

Proposition 3.1.1. *For the Hermitian one-matrix integrals, one has:*

$$\int_{\mathbb{H}_N} dM \exp\left(\frac{1}{g_s} tr V(M)\right) = \int_{\mathbb{R}^N} \prod_{1 \leqslant i < j \leqslant N} (\lambda_i - \lambda_j)^2 \exp\left(\frac{1}{g_s} \sum_{i=1}^{N} V(\lambda_i)\right) \prod_{i=1}^{N} d\lambda_i, \tag{3.3}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_N$ are eigenvalues of M.

By taking $V(M) = -\frac{1}{2}M^2 + \sum_{n=0}^{\infty} t_n \frac{M^{n+1}}{(n+1)!}$ in this proposition, we get the following analogue of the renormalization of 1D topological gravity:

Proposition 3.1.2. *For the Hermitian one-matrix integrals, one has:*

$$\int_{\mathbb{H}_N} dM \exp\left(\frac{1}{g_s} tr\left(-\frac{1}{2}M^2 + \sum_{n=0}^{\infty} t_n \frac{M^{n+1}}{(n+1)!}\right)\right) \tag{3.4}$$

$$= \exp\left(\frac{N}{g_s} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}\right) \cdot \int_{\mathbb{H}_N} dM \exp\left(\frac{1}{g_s} tr\left(-\frac{1}{2}M^2 + \sum_{n=1}^{\infty} I_n \frac{M^{n+1}}{(n+1)!}\right)\right).$$

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Proof. By 3.1.1 and renormalization of the action function of 1D gravity, one has:

$$\int_{\mathbb{H}_N} dM \exp\left(\frac{1}{g_s} tr\left(-\frac{1}{2}M^2 + \sum_{n=0}^{\infty} t_n \frac{M^{n+1}}{(n+1)!}\right)\right)$$

$$= \int_{\mathbb{R}^N} \prod_{1 \leqslant i < j \leqslant N} (\lambda_i - \lambda_j)^2 \exp\left(\frac{1}{g_s} \sum_{i=1}^{N}\left(-\frac{1}{2}\lambda_i^2 + \sum_{n=0}^{\infty} t_n \frac{\lambda_i^{n+1}}{(n+1)!}\right)\right) \prod_{i=1}^{N} d\lambda_i$$

$$= \int_{\mathbb{R}^N} \prod_{1 \leqslant i < j \leqslant N} (\lambda_i - I_0 - (\lambda_j - I_0))^2 \exp\left(\frac{1}{g_s} \sum_{i=1}^{N} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}\right)$$

$$\cdot \exp\left(\frac{1}{g_s} \sum_{i=1}^{N}\left(-\frac{1}{2}(\lambda_i - I_0)^2 + \sum_{n=1}^{\infty} I_n \frac{(\lambda_i - I_0)^{n+1}}{(n+1)!}\right)\right) \prod_{i=1}^{N} d\lambda_i$$

$$= \exp\left(\frac{N}{g_s} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}\right)$$

$$\cdot \int_{\mathbb{R}^N} \prod_{1 \leqslant i < j \leqslant N} (\lambda_i - \lambda_j)^2 \exp\left(\frac{1}{g_s} \sum_{i=1}^{N}\left(-\frac{1}{2}\lambda_i^2 + \sum_{n=1}^{\infty} I_n \frac{\lambda_i^{n+1}}{(n+1)!}\right)\right) \prod_{i=1}^{N} d\lambda_i$$

$$= \exp\left(\frac{N}{g_s} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}\right) \cdot \int_{\mathbb{H}_N} dM \exp\left(\frac{1}{g_s} tr\left(-\frac{1}{2}M^2 + \sum_{n=1}^{\infty} I_n \frac{M^{n+1}}{(n+1)!}\right)\right).$$

$\square$

The free energy F^N of Hermitian one-matrix models is defined by:

$$F^N := \log Z^N. \tag{3.5}$$

There is a genus expansions for F^N:

$$F^N = \sum_{g=0}^{\infty} g_s^{g-1} F_g^N, \tag{3.6}$$

where $\{F_g^N\}_{g \geqslant 0}$ are formal power series of $t_0, t_1, \cdots$. This is called the thin genus expansion in [30]. By a result in [30], if we define

$$\deg t_n = n - 1, \quad n = 0, 1, 2, \cdots \tag{3.7}$$

then F_g^N is weighted homogeneous in $t_0, t_1, \cdots$ with

$$\deg F_g^N = 2g - 2, \quad g = 0, 1, 2, \cdots \tag{3.8}$$

The following analogue of the Itzykson-Zuber Ansatz and theorem 2.1.1 is proved in [30]:

Theorem 3.1.1. *([30, theorem 5.1 and 5.2])*

$$F_0^N = N \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1}, \tag{3.9}$$

$$F_1^N = \frac{N^2}{2} \log \frac{1}{1-I_1}, \tag{3.10}$$

$$F_g^N = \sum_{\sum_{2 \leqslant k \leqslant 2g-1} m_k(k-1)=2g-2} \left\langle \prod_{j=2}^{2g-1} \tau_j^{m_j} \right\rangle_g^N \cdot \prod_{j=2}^{2g-1} \frac{1}{m_j!} \left(\frac{I_j}{(1-I_1)^{\frac{j+1}{2}}} \right)^{m_j}, \ g \geqslant 2 \tag{3.11}$$

where the correlators are defined by:

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g^N = \left. \frac{\partial^n F_g^N}{\partial t_{a_1} \cdots \partial t_{a_n}} \right|_{t=0}. \tag{3.12}$$

In the literature another type of genus expansion is used. By introducing the 't Hooft coupling constant

$$t = Ng_s, \tag{3.13}$$

F^N can be rewritten as:

$$F^N = \sum_{g=0}^{\infty} g_s^{2g-2} F_g^t, \tag{3.14}$$

where $\{F_g^t\}_{g \geqslant 0}$ are formal power series of $t_0, t_1, \cdots$ and t. This is called the fat genus expansion in [30]. We will study both thin and fat genus expansions for the free energy function of Hermitian one-matrix model by the correspondence Virasoro constraints.

3.2 Virasoro constraints of Hermitian one-matrix model

See the Introduction for more references on Virasoro constraints of Hermitian one-matrix models.

Theorem 3.2.1. *(Virasoro constraints for thin genus expansion [30]) The partition function Z^N of Hermitian one-matrix model satisfies the following equations for $m \geqslant -1$:*

$$L_m^N Z^N = 0, \tag{3.15}$$

where

$$L_{-1}^N = \frac{Nt_0}{g_s} + \sum_{n \geqslant 1} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{n-1}}, \tag{3.16}$$

$$L_0^N = N^2 + \sum_{n \geqslant 0} (n+1)(t_n - \delta_{n,1}) \frac{\partial}{\partial t_n}, \tag{3.17}$$

$$L_m^N = 2Ng_s m! \frac{\partial}{\partial t_{m-1}} + \sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{m+n}} + g_s^2 \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial}{\partial t_{k-1}} \frac{\partial}{\partial t_{m-k-1}}. \tag{3.18}$$

for $m \geqslant 1$. Furthermore, $\{L_m^N\}_{m \geqslant -1}$ satisfies the following commutation relations:

$$\left[L_m^N, L_n^N \right] = (m-n) L_{m+n}^N, \tag{3.19}$$

for $m, n \geqslant -1$.

Theorem 3.2.2. *(Virasoro constraints for fat genus expansion [30]) The partition function* Z^N *of Hermitian one-matrix model with* $t = Ng_s$ *satisfies the following equations for* $m \geqslant -1$:

$$L_m^t Z^N = 0, \tag{3.20}$$

where

$$L_{-1}^t = \frac{tt_0}{g_s^2} + \sum_{n \geqslant 1}(t_n - \delta_{n,1})\frac{\partial}{\partial t_{n-1}}, \tag{3.21}$$

$$L_0^t = \frac{t^2}{g_s^2} + \sum_{n \geqslant 0}(n+1)(t_n - \delta_{n,1})\frac{\partial}{\partial t_n}, \tag{3.22}$$

$$L_m^t = 2tm!\frac{\partial}{\partial t_{m-1}} + \sum_{n \geqslant 0}\frac{(m+n+1)!}{n!}(t_n - \delta_{n,1})\frac{\partial}{\partial t_{m+n}} + g_s^2\sum_{k=1}^{m-1}k!(m-k)!\frac{\partial}{\partial t_{k-1}}\frac{\partial}{\partial t_{m-k-1}}. \tag{3.23}$$

for $m \geqslant 1$. *Furthermore,* $\{L_m^t\}_{m \geqslant -1}$ *satisfies the following commutation relations:*

$$\left[L_m^t, L_n^t\right] = (m-n)L_{m+n}^t, \tag{3.24}$$

for $m, n \geqslant -1$.

Similarly as in the 1D topological gravity theory, one can rewrite the Virasoro operators in I-coordinates:

Theorem 3.2.3. *The Virasoro operators for thin genus expansion can be written in I-coordinates as follows:*

$$L_{-1}^N = -\frac{\partial}{\partial I_0} + \frac{N}{g_s}\sum_{n=0}^{\infty}\frac{(-1)^n I_0^n}{n!}I_n, \tag{3.25}$$

$$L_0^N = -I_0\frac{\partial}{\partial I_0} - 2\frac{\partial}{\partial I_1} + \sum_{l \geqslant 1}(l+1)I_l\frac{\partial}{\partial I_l} + N^2, \tag{3.26}$$

$$L_m^N = 2Ng_s m!\left(\frac{1}{1-I_1}\frac{I_0^{m-1}}{(m-1)!}d_X + \sum_{1 \leqslant l \leqslant m-1}\frac{I_0^{m-1-l}}{(m-1-l)!}\frac{\partial}{\partial I_l}\right) \tag{3.27}$$

$$-I_0^{m+1}\frac{\partial}{\partial I_0} + \sum_{i=1}^{m+1}\binom{m+1}{i}I_0^{m+1-i}\sum_{p=1}^{\infty}\frac{(p+i)!}{p!}(I_p - \delta_{p,1})\frac{\partial}{\partial I_{p+i-1}}$$

$$+g_s^2\binom{m+1}{3}\left(\frac{(m-2)I_0^{m-3}}{1-I_1}d_X\frac{\partial}{\partial I_1} + \frac{I_0^{m-2}}{(1-I_1)^2}d_X^2 + \left(\frac{(m-2)I_0^{m-3}}{(1-I_1)^2} + \frac{I_0^{m-2}I_2}{(1-I_1)^3}\right)d_X\right)$$

$$+2g_s^2\sum_{j=2}^{m-2}\binom{m+1}{j+3}(j+1)!\frac{I_0^{m-2-j}}{1-I_1}\left(\frac{\partial}{\partial I_{j-1}} + d_X\frac{\partial}{\partial I_j}\right)$$

$$+g_s^2\sum_{i=1}^{m-2}\sum_{j=1}^{m-2-i}\binom{m+1}{i+j+3}(i+1)!(j+1)!I_0^{m-2-i-j}\frac{\partial}{\partial I_i}\frac{\partial}{\partial I_j},$$

where

$$d_X = \frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}. \tag{3.28}$$

Moreover, by taking $N = \frac{t}{g_s}$, one get the Virasoro operators for fat genus expansion in I-coordinates:

$$L_{-1}^t = -\frac{\partial}{\partial I_0} + \frac{t}{g_s^2} \sum_{n=0}^\infty \frac{(-1)^n I_0^n}{n!} I_n, \tag{3.29}$$

$$L_0^t = -I_0 \frac{\partial}{\partial I_0} - 2\frac{\partial}{\partial I_1} + \sum_{l \geqslant 1}(l+1)I_l\frac{\partial}{\partial I_l} + \frac{t^2}{g_s^2}, \tag{3.30}$$

$$L_m^t = 2tm! \left(\frac{1}{1-I_1}\frac{I_0^{m-1}}{(m-1)!}d_X + \sum_{1 \leqslant l \leqslant m-1} \frac{I_0^{m-1-l}}{(m-1-l)!}\frac{\partial}{\partial I_l} \right) \tag{3.31}$$

$$-I_0^{m+1}\frac{\partial}{\partial I_0} + \sum_{i=1}^{m+1}\binom{m+1}{i}I_0^{m+1-i}\sum_{p=1}^\infty \frac{(p+i)!}{p!}(I_p - \delta_{p,1})\frac{\partial}{\partial I_{p+i-1}}$$

$$+g_s^2\binom{m+1}{3}\left(\frac{(m-2)I_0^{m-3}}{1-I_1}d_X\frac{\partial}{\partial I_1} + \frac{I_0^{m-2}}{(1-I_1)^2}d_X^2 + \left(\frac{(m-2)I_0^{m-3}}{(1-I_1)^2} + \frac{I_0^{m-2}I_2}{(1-I_1)^3} \right)d_X \right)$$

$$+2g_s^2\sum_{j=2}^{m-2}\binom{m+1}{j+3}(j+1)!\frac{I_0^{m-2-j}}{1-I_1}\left(\frac{\partial}{\partial I_{j-1}} + d_X\frac{\partial}{\partial I_j} \right)$$

$$+g_s^2\sum_{i=1}^{m-2}\sum_{j=1}^{m-2-i}\binom{m+1}{i+j+3}(i+1)!(j+1)!I_0^{m-2-i-j}\frac{\partial}{\partial I_i}\frac{\partial}{\partial I_j},$$

Proof. We have proved the following identity in the proof of theorem 2.2.2:

$$\sum_{n \geqslant 0}\frac{(m+n+1)!}{n!}(t_n - \delta_{n,1})\frac{\partial}{\partial t_{m+n}} = -I_0^{m+1}\frac{\partial}{\partial I_0} + \sum_{i=1}^{m+1}\binom{m+1}{i}I_0^{m+1-i}\sum_{p=1}^\infty \frac{(p+i)!}{p!}(I_p - \delta_{p,1})\frac{\partial}{\partial I_{p+i-1}}. \tag{3.32}$$

Now we rewrite $\sum_{k=1}^{m-1}k!(m-k)!\frac{\partial}{\partial t_{k-1}}\frac{\partial}{\partial t_{m-k-1}}$ in I-coordinates:

$$\sum_{k=1}^{m-1}k!(m-k)!\frac{\partial}{\partial t_{k-1}}\frac{\partial}{\partial t_{m-k-1}}$$

$$= \sum_{k=1}^{m-1}k!(m-k)!\left(\frac{1}{1-I_1}\frac{I_0^{k-1}}{(k-1)!}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l} \right) + \sum_{1 \leqslant i \leqslant k-1}\frac{I_0^{k-1-i}}{(k-1-i)!}\frac{\partial}{\partial I_i} \right)$$

$$\cdot \left(\frac{1}{1-I_1}\frac{I_0^{m-k-1}}{(m-k-1)!}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l} \right) + \sum_{1 \leqslant j \leqslant m-k-1}\frac{I_0^{m-k-1-j}}{(m-k-1-j)!}\frac{\partial}{\partial I_j} \right)$$

$$= \sum_{k=1}^{m-1}\left(\frac{kI_0^{k-1}}{1-I_1}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l} \right) + \sum_{1 \leqslant i \leqslant k-1}\frac{k!I_0^{k-1-i}}{(k-1-i)!}\frac{\partial}{\partial I_i} \right)$$

$$\cdot \left(\frac{(m-k)I_0^{m-k-1}}{1-I_1}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l} \right) + \sum_{1 \leqslant j \leqslant m-k-1}\frac{(m-k)!I_0^{m-k-1-j}}{(m-k-1-j)!}\frac{\partial}{\partial I_j} \right)$$

$$= \sum_{k=1}^{m-1} k(m-k) \frac{I_0^{m-2}}{(1-I_1)^2} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right)^2$$

$$+ \sum_{k=1}^{m-1} \sum_{1 \leqslant j \leqslant m-k-1} \frac{k(m-k)!}{(m-k-1-j)!} \frac{I_0^{m-2-j}}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) \frac{\partial}{\partial I_j}$$

$$+ \sum_{k=1}^{m-1} \sum_{1 \leqslant i \leqslant k-1} \frac{(m-k)k!}{(k-1-i)!} \frac{I_0^{m-2-i}}{1-I_1} \frac{\partial}{\partial I_i} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right)$$

$$+ \sum_{k=1}^{m-1} \sum_{i=1}^{k-1} \sum_{j=1}^{m-k-1} \frac{(m-k)!k!}{(k-1-i)!(m-k-1-j)!} I_0^{m-2-i-j} \frac{\partial}{\partial I_i} \frac{\partial}{\partial I_j}$$

$$+ \sum_{k=1}^{m-1} k(m-k) \frac{I_0^{m-3}}{(1-I_1)^3} \left((m-k-1)(1-I_1) + I_0 I_2 \right) \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right)$$

$$+ \sum_{k=1}^{m-1} \sum_{1 \leqslant j \leqslant m-k-1} \frac{k(m-k)!}{(m-k-2-j)!} \frac{I_0^{m-3-j}}{1-I_1} \frac{\partial}{\partial I_j}$$

$$+ \sum_{k=2}^{m-1} \frac{(m-k)k!}{(k-2)!} \frac{I_0^{m-3}}{(1-I_1)^2} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right).$$

Denote these summations by (a)-(g) respectively, then using the following identity:

$$\sum_{k=a}^{m-b} \frac{k!(m-k)!}{(k-a)!(m-k-b)!} = \binom{m+1}{a+b+1} a! b!, \qquad (3.33)$$

we have:

$$(a) = \binom{m+1}{3} \frac{I_0^{m-2}}{(1-I_1)^2} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right)^2, \qquad (3.34)$$

$$(b) = \sum_{j=1}^{m-2} \binom{m+1}{j+3} I_0^{m-2-j} \frac{1}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) (j+1)! \frac{\partial}{\partial I_j}, \qquad (3.35)$$

$$(c) = \sum_{i=1}^{m-2} \binom{m+1}{i+3} I_0^{m-2-i} \frac{1}{1-I_1} (i+1)! \frac{\partial}{\partial I_i} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) \qquad (3.36)$$

$$= \sum_{i=1}^{m-2} \binom{m+1}{i+3} I_0^{m-2-i} \frac{1}{1-I_1} \left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} I_{l+1} \frac{\partial}{\partial I_l} \right) (i+1)! \frac{\partial}{\partial I_i}$$

$$+ \sum_{i=2}^{m-2} \binom{m+1}{i+3} I_0^{m-2-i} \frac{1}{1-I_1} (i+1)! \frac{\partial}{\partial I_{i-1}},$$

$$(d) = \sum_{i=1}^{m-2} \sum_{j=1}^{m-2-i} \sum_{k=i+1}^{m-j-1} \frac{(m-k)!k!}{(k-1-i)!(m-k-1-j)!} I_0^{m-2-i-j} \frac{\partial}{\partial I_i} \frac{\partial}{\partial I_j} \qquad (3.37)$$

$$= \sum_{i=1}^{m-2} \sum_{j=1}^{m-2-i} \binom{m+1}{i+j+3} I_0^{m-2-i-j} (i+1)! (j+1)! \frac{\partial}{\partial I_i} \frac{\partial}{\partial I_j},$$

$$(e) = \left(2\binom{m+1}{4}\frac{I_0^{m-3}}{(1-I_1)^2} + \binom{m+1}{3}\frac{I_0^{m-2}I_2}{(1-I_1)^3}\right)\left(\frac{\partial}{\partial I_0} + \sum_{l\geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right), \tag{3.38}$$

$$(f) = \sum_{j=1}^{m-3}\binom{m+1}{j+4}\frac{I_0^{m-3-j}}{1-I_1}(j+2)!\frac{\partial}{\partial I_j}, \tag{3.39}$$

$$(g) = 2\binom{m+1}{4}\frac{I_0^{m-3}}{(1-I_1)^2}\left(\frac{\partial}{\partial I_0} + \sum_{l\geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right). \tag{3.40}$$

Plus all these equations together, we complete the proof. $\qquad\square$

3.3 Computations of F_g^N by Virasoro constraints for thin genus expansion in I-coordinates

As applications of the expressions of the Virasoro operators expressed in I-coordinates derived in last subsection, we use them in this subsection to compute F_g^N.

Theorem 3.3.1. *For the free energies $\{F_g^N\}_{g\geqslant 2}$ of the Hermitian one-matrix model, the following equations hold:*

$$\frac{\partial F_g^N}{\partial I_1} = \frac{1}{2(1-I_1)}\sum_{n=2}^{2g-1}(n+1)I_n\frac{\partial F_g^N}{\partial I_n}, \tag{3.41}$$

$$\frac{\partial F_g^N}{\partial I_2} = \frac{N}{3(1-I_1)}\frac{1}{1-I_1}d_X\left(F_{g-1}^N\right) + \sum_{n=2}^{2g-2}\frac{(n+2)!}{3!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{n+1}}, \tag{3.42}$$

$$\frac{\partial F_g^N}{\partial I_3} = \frac{N}{6(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_1} + \sum_{n=2}^{2g-3}\frac{(n+3)!}{4!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{2+n}} + \frac{1}{4!(1-I_1)^3}\sum_{g_1=1}^{g-2}d_X\left(F_{g_1}^N\right)d_X\left(F_{g-1-g_1}^N\right)$$

$$+ \frac{1}{4!(1-I_1)}\left(\frac{1}{1-I_1}d_X\right)^2\left(F_{g-2}^N\right) + \delta_{g,2}\frac{N}{4!(1-I_1)^2}, \tag{3.43}$$

$$\frac{\partial F_g^N}{\partial I_4} = \frac{N}{10(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_2} + \sum_{n=2}^{2g-4}\frac{(n+4)!}{5!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{n+3}} + \frac{1}{30(1-I_1)^2}\sum_{g_1=1}^{g-2}\frac{\partial F_{g_1}^N}{\partial I_1}d_X\left(F_{g-1-g_1}^N\right)$$

$$+ \frac{1}{60(1-I_1)^3}\left(d_X\left(F_{g-2}^N\right) + 2(1-I_1)d_X\left(\frac{\partial F_{g-2}^N}{\partial I_1}\right)\right), \tag{3.44}$$

$$\frac{\partial F_g^N}{\partial I_{p+4}} = \frac{2N(p+3)!}{(p+5)!(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_{p+2}} + \sum_{n=2}^{2g-4-p}\frac{(p+n+4)!}{n!(p+5)!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{p+n+3}} \tag{3.45}$$

$$+ \sum_{g_1=1}^{g-2}\sum_{k=2}^{p+1}\frac{k!(p+3-k)!}{(p+5)!(1-I_1)}\frac{\partial F_{g_1}^N}{\partial I_{k-1}}\frac{\partial F_{g-1-g_1}^N}{\partial I_{p+2-k}} + \frac{2}{(1-I_1)^2}\frac{(p+2)!}{(p+5)!}\sum_{g_1=1}^{g-2}\frac{\partial F_{g_1}^N}{\partial I_{p+1}}d_X\left(F_{g-1-g_1}^N\right)$$

$$+ \sum_{k=2}^{p+1}\frac{k!(p+3-k)!}{(p+5)!(1-I_1)}\frac{\partial^2 F_{g-2}^N}{\partial I_{k-1}\partial I_{p+2-k}} + \frac{1}{(1-I_1)^2}\frac{(p+2)!}{(p+5)!}\left(\frac{\partial F_{g-2}^N}{\partial I_p} + 2d_X\left(\frac{\partial F_{g-2}^N}{\partial I_{p+1}}\right)\right),$$

where $p = 1, 2, \cdots, 2g - 5$.

Proof. The first equation has been proved in [30]. For the rest equations, we first rewrite Virasoro constraints for partition function into Virasoro constraints for free energy:

$$0 = (Z^N)^{-1} L_m^N Z^N = 2N g_s m! \frac{\partial F^N}{\partial t_{m-1}} + \sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} (t_n - \delta_{n,1}) \frac{\partial F^N}{\partial t_{m+n}} \tag{3.46}$$

$$+ g_s^2 \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial F^N}{\partial t_{k-1}} \frac{\partial F^N}{\partial t_{m-k-1}} + g_s^2 \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial^2 F^N}{\partial t_{k-1} \partial t_{m-k-1}},$$

Let

$$\tilde{F}^N = \sum_{g \geqslant 1} g_s^{g-1} F_g^N, \tag{3.47}$$

then $\tilde{F}^N$ does not depend on I_0, and

$$0 = 2Nm! \frac{\partial F_0^N}{\partial t_{m-1}} + \frac{1}{g_s} \sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} (t_n - \delta_{n,1}) \frac{\partial F_0^N}{\partial t_{m+n}} + \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial F_0^N}{\partial t_{k-1}} \frac{\partial F_0^N}{\partial t_{m-k-1}}$$

$$+ 2g_s \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial F_0^N}{\partial t_{k-1}} \frac{\partial \tilde{F}^N}{\partial t_{m-k-1}} + g_s \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial^2 F_0^N}{\partial t_{k-1} \partial t_{m-k-1}},$$

$$+ 2N g_s m! \frac{\partial \tilde{F}^N}{\partial t_{m-1}} + \sum_{n \geqslant 0} \frac{(m+n+1)!}{n!} (t_n - \delta_{n,1}) \frac{\partial \tilde{F}^N}{\partial t_{m+n}}$$

$$+ g_s^2 \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial \tilde{F}^N}{\partial t_{k-1}} \frac{\partial \tilde{F}^N}{\partial t_{m-k-1}} + g_s^2 \sum_{k=1}^{m-1} k!(m-k)! \frac{\partial^2 \tilde{F}^N}{\partial t_{k-1} \partial t_{m-k-1}}. \tag{3.48}$$

Now we let $I_0 = 0$, under this condition, one has:

$$t_0 = 0,$$
$$t_k = I_k, \quad k > 0$$

and

$$\frac{\partial}{\partial t_0} = \frac{1}{1 - I_1} \frac{\partial}{\partial I_0} + \sum_{l \geqslant 1} \frac{I_{l+1}}{1 - I_1} \frac{\partial}{\partial I_l}, \tag{3.49}$$

$$\frac{\partial}{\partial t_k} = \frac{\partial}{\partial I_k}, \quad k > 0 \tag{3.50}$$

By (3.9)

$$\frac{\partial F_0^N}{\partial I_0} = N t_0, \tag{3.51}$$

$$\frac{\partial F_0^N}{\partial I_k} = N \frac{(-1)^k I_0^{k+1}}{(k+1)!}, \quad k \geqslant 1. \tag{3.52}$$

So one has:

$$\left. \frac{\partial F_0^N}{\partial I_0} \right|_{I_0=0} = \left. \frac{\partial F_0^N}{\partial I_k} \right|_{I_0=0} = 0, \tag{3.53}$$

and therefore,

$$\frac{\partial F_0^N}{\partial t_0}\bigg|_{I_0=0} = \frac{\partial F_0^N}{\partial t_k}\bigg|_{I_0=0} = 0. \tag{3.54}$$

Moreover,

$$\frac{\partial^2 F_0^N}{\partial I_0^2} = N(1-t_1), \tag{3.55}$$

$$\frac{\partial^2 F_0^N}{\partial I_0 \partial I_k} = N\frac{(-1)^k I_0^k}{k!}, \quad k \geqslant 1 \tag{3.56}$$

$$\frac{\partial^2 F_0^N}{\partial I_k \partial I_l} = 0, \quad k,l \geqslant 1. \tag{3.57}$$

When restrict $I_0 = 0$, these equations give

$$\frac{\partial^2 F_0^N}{\partial I_0^2}\bigg|_{I_0=0} = N(1-I_1), \tag{3.58}$$

$$\frac{\partial^2 F_0^N}{\partial I_0 \partial I_k}\bigg|_{I_0=0} = \frac{\partial^2 F_0^N}{\partial I_k \partial I_l}\bigg|_{I_0=0} = 0, \quad k,l \geqslant 1. \tag{3.59}$$

Therefore

$$\frac{\partial^2 F_0^N}{\partial t_0^2}\bigg|_{I_0=0} = \left(\frac{1}{1-I_1}\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l}\right)^2 (F_0^N)\bigg|_{I_0=0} \tag{3.60}$$

$$= \frac{1}{(1-I_1)^2}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right)^2 (F_0^N)\bigg|_{I_0=0} + \frac{I_2}{(1-I_1)^3}\left(\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}I_{l+1}\frac{\partial}{\partial I_l}\right)(F_0^N)\bigg|_{I_0=0}$$

$$= \frac{1}{(1-I_1)^2}\frac{\partial^2 F_0^N}{\partial I_0^2}\bigg|_{I_0=0}$$

$$= \frac{N}{1-I_1},$$

$$\frac{\partial^2 F_0^N}{\partial t_0 \partial t_k}\bigg|_{I_0=0} = \left(\frac{1}{1-I_1}\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l}\right)\left(\frac{\partial F_0^N}{\partial I_k}\right)\bigg|_{I_0=0} \tag{3.61}$$

$$= \left(\frac{1}{1-I_1}\frac{\partial}{\partial I_0} + \sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l}\right)\left(N\frac{(-1)^k I_0^{k+1}}{(k+1)!}\right)\bigg|_{I_0=0}$$

$$= 0 \quad k \geqslant 1,$$

$$\frac{\partial^2 F_0^N}{\partial t_k \partial t_l}\bigg|_{I_0=0} = \frac{\partial^2 F_0^N}{\partial I_k \partial I_l}\bigg|_{I_0=0} = 0, \quad k,l \geqslant 1. \tag{3.62}$$

Hence (3.48) becomes

$$0 = \delta_{m,2}\frac{Ng_s}{1-I_1} + 2Ng_s m!\frac{\partial \tilde{F}^N}{\partial t_{m-1}} + \sum_{n \geqslant 1}\frac{(m+n+1)!}{n!}(t_n - \delta_{n,1})\frac{\partial \tilde{F}^N}{\partial t_{m+n}} \tag{3.63}$$

$$+ g_s^2 \sum_{k=1}^{m-1} k!(m-k)!\frac{\partial \tilde{F}^N}{\partial t_{k-1}}\frac{\partial \tilde{F}^N}{\partial t_{m-k-1}} + g_s^2 \sum_{k=1}^{m-1} k!(m-k)!\frac{\partial^2 \tilde{F}^N}{\partial t_{k-1}\partial t_{m-k-1}},$$

or

$$0 = \delta_{m,2}\delta_{g,1}\frac{N}{1-I_1} + 2Nm!\frac{\partial F_g^N}{\partial t_{m-1}} + \sum_{n \geqslant 0}\frac{(m+n+1)!}{n!}(t_n - \delta_{n,1})\frac{\partial F_{g+1}^N}{\partial t_{m+n}} \tag{3.64}$$

$$+ \sum_{g_1+g_2=g}\sum_{k=1}^{m-1}k!(m-k)!\frac{\partial F_{g_1}^N}{\partial t_{k-1}}\frac{\partial F_{g_2}^N}{\partial t_{m-k-1}} + \sum_{k=1}^{m-1}k!(m-k)!\frac{\partial^2 F_{g-1}^N}{\partial t_{k-1}\partial t_{m-k-1}},$$

with

$$\frac{\partial}{\partial t_0} = \frac{1}{1-I_1}d_X, \tag{3.65}$$

$$\frac{\partial}{\partial t_k} = \frac{\partial}{\partial I_k}, \quad k > 0 \tag{3.66}$$

for $m = 1$:

$$0 = 2N\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_g^N}{\partial I_l} + \sum_{n \geqslant 1}\frac{(n+2)!}{n!}(I_n - \delta_{n,1})\frac{\partial F_{g+1}^N}{\partial I_{n+1}}, \tag{3.67}$$

for $m = 2$:

$$0 = 4N\frac{\partial F_g^N}{\partial I_1} + \sum_{n \geqslant 1}\frac{(n+3)!}{n!}(I_n - \delta_{n,1})\frac{\partial F_{g+1}^N}{\partial I_{2+n}} + \sum_{g_1=1}^{g-1}\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g_1}^N}{\partial I_l}\sum_{k \geqslant 1}\frac{I_{k+1}}{1-I_1}\frac{\partial F_{g-g_1}^N}{\partial I_k}$$

$$+ \left(\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l}\right)^2(F_{g-1}^N) + \delta_{g,1}\frac{N}{1-I_1}, \tag{3.68}$$

for $m = 3$:

$$0 = 12N\frac{\partial F_g^N}{\partial I_2} + \sum_{n \geqslant 0}\frac{(n+4)!}{n!}(I_n - \delta_{n,1})\frac{\partial F_{g+1}^N}{\partial I_{3+n}} + 4\sum_{g_1=1}^{g-1}\frac{\partial F_{g_1}^N}{\partial I_1}\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g-g_1}^N}{\partial I_l}$$

$$+ 4\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial^2 F_{g-1}^N}{\partial I_l\partial I_1} + 2\sum_{l \geqslant 1}\frac{I_{l+1}}{(1-I_1)^2}\frac{\partial F_{g-1}^N}{\partial I_l}, \tag{3.69}$$

for $m \geqslant 4$:

$$0 = 2Nm!\frac{\partial F_g^N}{\partial I_{m-1}} + \sum_{n \geqslant 0}\frac{(m+n+1)!}{n!}(I_n-\delta_{n,1})\frac{\partial F_{g+1}^N}{\partial I_{m+n}} + \sum_{g_1=1}^{g-1}\sum_{k=2}^{m-2}k!\frac{\partial F_{g_1}^N}{\partial I_{k-1}}(m-k)!\frac{\partial F_{g-g_1}^N}{\partial I_{m-k-1}}$$

$$+ 2\sum_{g_1=1}^{g-1}(m-1)!\frac{\partial F_{g_1}^N}{\partial I_{m-2}}\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g-g_1}^N}{\partial I_l} + \sum_{k=2}^{m-2}k!(m-k)!\frac{\partial^2 F_{g-1}^N}{\partial I_{k-1}\partial I_{m-k-1}}$$

$$+ 2(m-1)!\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial^2 F_{g-1}^N}{\partial I_l\partial I_{m-2}} + (m-1)!\frac{1}{1-I_1}\frac{\partial F_{g-1}^N}{\partial I_{m-3}}. \tag{3.70}$$

The proof is completed. $\qquad\square$

Now we explain how to use theorem 3.3.1 to calculate free energies of higher genus. By (3.11), F_g^N depends only on $I_1, \ldots, I_{2g-1}$ for $g \geqslant 2$. By (3.45), we have:

$$\frac{\partial F_g^N}{\partial I_{2g-1}} = \frac{2N(2g-2)!}{(2g)!(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_{2g-3}} + \frac{1}{(1-I_1)^2}\frac{(2g-3)!}{(2g)!}\frac{\partial F_{g-2}^N}{\partial I_{2g-5}} + \sum_{g_1=1}^{g-2}\sum_{k=2}^{2g-4}\frac{k!(2g-2-k)!}{(2g)!(1-I_1)}\frac{\partial F_{g_1}^N}{\partial I_{k-1}}\frac{\partial F_{g-1-g_1}^N}{\partial I_{2g-3-k}}$$

(3.71)

$$+ \frac{2(2g-3)!}{(2g)!(1-I_1)^2}\sum_{g_1=1}^{g-2}\frac{\partial F_{g_1}^N}{\partial I_{2g-4}}d_X\left(F_{g-1-g_1}^N\right) + \sum_{k=2}^{2g-4}\frac{k!(2g-2-k)!}{(2g)!(1-I_1)}\frac{\partial^2 F_{g-2}^N}{\partial I_{k-1}\partial I_{2g-3-k}},$$

$$\frac{\partial F_g^N}{\partial I_{2g-2}} = \frac{2N(2g-3)!}{(2g-1)!(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_{2g-4}} + \frac{(2g)!}{2!(2g-1)!}\frac{I_2}{1-I_1}\frac{\partial F_g^N}{\partial I_{2g-1}}$$

(3.72)

$$+ \sum_{g_1=1}^{g-2}\sum_{k=2}^{2g-5}\frac{k!(2g-3-k)!}{(2g-1)!(1-I_1)}\frac{\partial F_{g_1}^N}{\partial I_{k-1}}\frac{\partial F_{g-1-g_1}^N}{\partial I_{2g-4-k}} + \frac{2}{(1-I_1)^2}\frac{(2g-4)!}{(2g-1)!}\frac{\partial F_{g-2}^N}{\partial I_{2g-5}}d_X\left(F_1^N\right)$$

$$+ \sum_{k=2}^{2g-5}\frac{k!(2g-3-k)!}{(2g-1)!(1-I_1)}\frac{\partial^2 F_{g-2}^N}{\partial I_{k-1}\partial I_{2g-4-k}} + \frac{1}{(1-I_1)^2}\frac{(2g-4)!}{(2g-1)!}\left(\frac{\partial F_{g-2}^N}{\partial I_{2g-6}} + 2d_X\left(\frac{\partial F_{g-2}^N}{\partial I_{2g-5}}\right)\right),$$

$$\vdots$$

$$\frac{\partial F_g^N}{\partial I_5} = \frac{N}{15(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_3} + \sum_{n=2}^{2g-5}\frac{(n+5)!}{6!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{n+4}}$$

(3.73)

$$+ \sum_{g_1=1}^{g-2}\frac{1}{180(1-I_1)}\frac{\partial F_{g_1}^N}{\partial I_1}\frac{\partial F_{g-1-g_1}^N}{\partial I_1} + \frac{1}{60(1-I_1)^2}\sum_{g_1=1}^{g-2}\frac{\partial F_{g_1}^N}{\partial I_2}d_X\left(F_{g-1-g_1}^N\right)$$

$$+ \frac{1}{180(1-I_1)}\frac{\partial^2 F_{g-2}^N}{\partial I_1\partial I_1} + \frac{1}{120(1-I_1)^2}\left(\frac{\partial F_{g-2}^N}{\partial I_1} + 2d_X\left(\frac{\partial F_{g-2}^N}{\partial I_2}\right)\right).$$

For $\frac{\partial F_g^N}{\partial I_i}$, $i = 4, 3, 2, 1$, we have by (3.41)–(3.44):

$$\frac{\partial F_g^N}{\partial I_4} = \frac{N}{10(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_2} + \sum_{n=2}^{2g-4}\frac{(n+4)!}{5!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{n+3}} + \frac{1}{30(1-I_1)^2}\sum_{g_1=1}^{g-2}\frac{\partial F_{g_1}^N}{\partial I_1}d_X\left(F_{g-1-g_1}^N\right)$$

$$+ \frac{1}{60(1-I_1)^3}\left(d_X\left(F_{g-2}^N\right) + 2(1-I_1)d_X\left(\frac{\partial F_{g-2}^N}{\partial I_1}\right)\right),$$

(3.74)

$$\frac{\partial F_g^N}{\partial I_3} = \frac{N}{6(1-I_1)}\frac{\partial F_{g-1}^N}{\partial I_1} + \sum_{n=2}^{2g-3}\frac{(n+3)!}{4!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{2+n}} + \frac{1}{4!(1-I_1)^3}\sum_{g_1=1}^{g-2}d_X\left(F_{g_1}^N\right)d_X\left(F_{g-1-g_1}^N\right)$$

$$+ \frac{1}{4!(1-I_1)}\left(\frac{1}{1-I_1}d_X\right)^2\left(F_{g-2}^N\right) + \delta_{g,2}\frac{N}{4!(1-I_1)^2},$$

(3.75)

$$\frac{\partial F_g^N}{\partial I_2} = \frac{N}{3(1-I_1)}\frac{1}{1-I_1}d_X\left(F_{g-1}^N\right) + \sum_{n=2}^{2g-2}\frac{(n+2)!}{3!n!}\frac{I_n}{1-I_1}\frac{\partial F_g^N}{\partial I_{n+1}},$$

(3.76)

$$\frac{\partial F_g^N}{\partial I_1} = \frac{1}{2(1-I_1)}\sum_{n=2}^{2g-1}(n+1)I_n\frac{\partial F_g^N}{\partial I_n}.$$

(3.77)

By equations (3.71)–(3.77), we can solve $\left\{\frac{\partial F_g^N}{\partial I_k}\right\}_{g\geqslant 2,k=2g-1,2g-2,\cdots,1}$ recursively, given the computation for $F_{k<g}^N$. Since F_g^N is weighted homogeneous of degree $2g-2$ in I_k with $\deg I_k = k-1$, we have

$$F_g^N = \frac{1}{2g-2}\sum_{k=1}^{2g-1}(k-1)I_k\frac{\partial F_g^N}{\partial I_k}. \tag{3.78}$$

Example 3.3.1. *Let us compute F_2^N by the above procedure. We have*

$$\frac{\partial F_2^N}{\partial I_3} = \frac{N}{6(1-I_1)}\frac{\partial F_1^N}{\partial I_1} + \frac{N}{4!(1-I_1)^2} = \frac{N+2N^3}{24(1-I_1)^2}, \tag{3.79}$$

$$\frac{\partial F_2^N}{\partial I_2} = \frac{N}{3(1-I_1)}\frac{1}{1-I_1}d_X\left(F_1^N\right) + 2\frac{I_2}{1-I_1}\frac{\partial F_2^N}{\partial I_3} = \frac{(N+4N^3)I_2}{12(1-I_1)^3}, \tag{3.80}$$

$$\frac{\partial F_2^N}{\partial I_1} = \frac{1}{2(1-I_1)}\sum_{l=2}^{3}(l+1)I_l\frac{\partial F_2^N}{\partial I_l} = \frac{(N+4N^3)I_2^2}{8(1-I_1)^4} + \frac{(N+2N^3)I_3}{12(1-I_1)^3}, \tag{3.81}$$

therefore

$$F_2^N = \frac{1}{2}\left(I_2\frac{\partial F_2^N}{\partial I_2} + 2I_3\frac{\partial F_2^N}{\partial I_3}\right) = \frac{(N+4N^3)I_2^2}{24(1-I_1)^3} + \frac{(N+2N^3)I_3}{24(1-I_1)^2}. \tag{3.82}$$

Similarly, we have with the help of a computer program:

$$F_3^N = \left(216N^4 + 189N^2\right)\hat{I}_2^4 + \left(216N^4 + 234N^2\right)\hat{I}_2^2\hat{I}_3 + \left(18N^4 + 30N^2\right)\hat{I}_3^2 \tag{3.83}$$
$$+ \left(45N^4 + 60N^2\right)\hat{I}_2\hat{I}_4 + \left(5N^4 + 10N^2\right)\hat{I}_5,$$

$$F_4^N = \left(13608N^5 + 26892N^3 + \frac{8505}{2}N\right)\hat{I}_2^6 + \left(22032N^5 + 49248N^3 + 8505N\right)\hat{I}_2^4\hat{I}_3$$

$$+ \left(7776N^5 + 20304N^3 + 3960N\right)\hat{I}_2^2\hat{I}_3^2 + \left(288N^5 + 1056N^3 + 240N\right)\hat{I}_3^3$$

$$+ \left(5400N^5 + 13770N^3 + 2565N\right)\hat{I}_2^3\hat{I}_4 + \left(2160N^5 + 6480N^3 + 1440N\right)\hat{I}_2\hat{I}_3\hat{I}_4$$

$$+ \left(90N^5 + 300N^3 + \frac{165}{2}N\right)\hat{I}_4^2 + \left(1080N^5 + 3330N^3 + 675N\right)\hat{I}_2^2\hat{I}_5$$

$$+ \left(144N^5 + 600N^3 + 156N\right)\hat{I}_3\hat{I}_5 + \left(168N^5 + 630N^3 + 147N\right)\hat{I}_2\hat{I}_6$$

$$+ \left(14N^5 + 70N^3 + 21N\right)\hat{I}_7, \tag{3.84}$$

where

$$\hat{I}_k := \frac{1}{(k+1)!}\frac{I_k}{(1-I_1)^{(k+1)/2}} \tag{3.85}$$

3.4 Special deformation in thin genus expansion of the Hermitian one-matrix models in I-coordinates

In [31], the second named author studied the special deformation of the Hermitian one-matrix models which is defined by:

$$y^N = \frac{1}{\sqrt{2}}\sum_{n\geqslant 0}\frac{t_n - \delta_{n,1}}{n!}z^n + \frac{\sqrt{2}N}{z} + \sqrt{2}\sum_{n\geqslant 1}\frac{n!}{z^{n+1}}\frac{\partial F_0^N}{\partial t_{n-1}}. \tag{3.86}$$

Now we rewrite it in I-coordinates, we have:

Theorem 3.4.1. *In I-coordinates, the special deformation of the Hermitian one-matrix models is as follows:*

$$y^N = \frac{\sqrt{2}N}{z - I_0} + \frac{1}{\sqrt{2}} \sum_{n \geqslant 1} \frac{I_n - \delta_{n,1}}{n!} (z - I_0)^n. \tag{3.87}$$

Proof. This is just [31, theorem 2.1]. $\qquad\qquad\square$

3.5 Special deformation in fat genus expansion of the Hermitian one-matrix models in I-coordinates

In [31], the second named author also studied another special deformation of the Hermitian one-matrix models based on the fat genus expansion, which is defined by

$$y^t = \frac{1}{\sqrt{2}} \sum_{n \geqslant 0} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}t}{z} + \sqrt{2} \sum_{n \geqslant 1} \frac{n!}{z^{n+1}} \frac{\partial F_0^t}{\partial t_{n-1}}. \tag{3.88}$$

We can also rewrite this special deformation in I-coordinates, and we get:

Theorem 3.5.1. *In I-coordinates, the special deformation (3.88) can be written as:*

$$y^t = \frac{\sqrt{2}t}{z - I_0} + \frac{1}{\sqrt{2}} \sum_{n \geqslant 1} \frac{I_n - \delta_{n,1}}{n!} (z - I_0)^n + \frac{1}{(z - I_0)^2} \frac{1}{1 - I_1} d_X(\tilde{F}_0^t) + \sum_{l \geqslant 1} \frac{(l+1)!}{(z - I_0)^{l+2}} \frac{\partial \tilde{F}_0^t}{\partial I_l}. \tag{3.89}$$

Here $\tilde{F}_0^t$ is defined as follows:

$$\tilde{F}_0^t = F_0^t - F_{0,0}^t, \tag{3.90}$$

where F_0^t is free energy of genus 0 in fat genus expansion and $F_{0,0}^t$ is defined as follows:

$$F_{0,0}^t := t \int t_0 dI_0 = t F_0^{1D}. \tag{3.91}$$

Proof. Similar to the proof of theorem 2.4.1. $\qquad\qquad\square$

In fact, in [30, section 7], the second named author has pointed out that one can get $\tilde{F}_0^t$ from the thin genus expansion, and one has the following result:

Proposition 3.5.1. *$\tilde{F}_0^t$ has an expansion in 't Hooft coupling constant t:*

$$\tilde{F}_0^t = \sum_{k \geqslant 1} t^{k+1} F_{0,k}^t, \tag{3.92}$$

with $F_{0,k}^t$ weighted homogeneous of degree $2k-2$, and it satisfies the following equations:

$$\frac{\partial F_{0,k}^t}{\partial I_1} = \frac{1}{2(1-I_1)} \sum_{n=2}^{2g-1} (n+1) I_n \frac{\partial F_{0,k}^t}{\partial I_n} + \delta_{k,1} \frac{1}{2(1-I_1)}, \tag{3.93}$$

$$\frac{\partial F_{0,k}^t}{\partial I_2} = \frac{1}{3(1-I_1)} \frac{1}{1-I_1} d_X \left(F_{0,k-1}^t \right) + \sum_{n=2}^{2k-2} \frac{(n+2)!}{3!n!} \frac{I_n}{1-I_1} \frac{\partial F_{0,k}^t}{\partial I_{n+1}}, \tag{3.94}$$

$$\frac{\partial F_{0,k}^t}{\partial I_3} = \frac{1}{6(1-I_1)} \frac{\partial F_{0,k-1}^t}{\partial I_1} + \sum_{n=2}^{2k-3} \frac{(n+3)!}{4!n!} \frac{I_n}{1-I_1} \frac{\partial F_{0,k}^t}{\partial I_{2+n}} + \frac{1}{4!(1-I_1)^3} \sum_{k_1=1}^{k-2} d_X \left(F_{0,k_1}^t \right) d_X \left(F_{0,k-1-k_1}^t \right), \tag{3.95}$$

$$\frac{\partial F_{0,k}^t}{\partial I_4} = \frac{1}{10(1-I_1)} \frac{\partial F_{0,k-1}^t}{\partial I_2} + \sum_{n=2}^{2k-4} \frac{(n+4)!}{5!n!} \frac{I_n}{1-I_1} \frac{\partial F_{0,k}^t}{\partial I_{n+3}} + \frac{1}{30(1-I_1)^2} \sum_{k_1=1}^{g-2} \frac{\partial F_{0,k_1}^t}{\partial I_1} d_X \left(F_{0,k-1-k_1}^t \right), \tag{3.96}$$

$$\frac{\partial F_{0,k}^t}{\partial I_{p+4}} = \frac{2(p+3)!}{(p+5)!(1-I_1)} \frac{\partial F_{0,k-1}^t}{\partial I_{p+2}} + \sum_{n=2}^{2k-4-p} \frac{(p+n+4)!}{n!(p+5)!} \frac{I_n}{1-I_1} \frac{\partial F_{0,k}^t}{\partial I_{p+n+3}}$$
$$+ \sum_{k_1=1}^{k-2} \sum_{i=2}^{p+1} \frac{i!(p+3-i)!}{(p+5)!(1-I_1)} \frac{\partial F_{0,k_1}^t}{\partial I_{i-1}} \frac{\partial F_{0,k-1-k_1}^t}{\partial I_{p+2-i}} + \frac{2}{(1-I_1)^2} \frac{(p+2)!}{(p+5)!} \sum_{k_1=1}^{k-2} \frac{\partial F_{0,k_1}^t}{\partial I_{p+1}} d_X \left(F_{0,k-1-k_1}^t \right). \tag{3.97}$$

This proposition gives a recursively way to compute $F_{0,g}^t$, which is similar to the computation of F_g^N. We have:

$$F_{0,1}^t = \frac{1}{2} \log \frac{1}{1-I_1}, \tag{3.98}$$

$$F_{0,2}^t = \frac{1}{6} \frac{I_2^2}{(1-I_1)^3} + \frac{1}{12} \frac{I_3}{(1-I_1)^2}, \tag{3.99}$$

$$F_{0,3}^t = \frac{1}{6} \frac{I_2^4}{(1-I_1)^6} + \frac{1}{4} \frac{I_2^2 I_3}{(1-I_1)^6} + \frac{1}{32} \frac{I_3^2}{(1-I_1)^4} + \frac{1}{16} \frac{I_2 I_4}{(1-I_1)^4} + \frac{1}{144} \frac{I_5}{(1-I_1)^3}, \tag{3.100}$$

$$F_{0,4}^t = \frac{7}{24} \frac{I_2^6}{(1-I_1)^9} + \frac{17}{24} \frac{I_2^4 I_3}{(1-I_1)^8} + \frac{3}{8} \frac{I_2^2 I_3^2}{(1-I_1)^7} + \frac{1}{48} \frac{I_3^3}{(1-I_1)^6} + \frac{5}{24} \frac{I_2^3 I_4}{(1-I_1)^7} + \frac{1}{8} \frac{I_2 I_3 I_4}{(1-I_1)^6}$$
$$+ \frac{1}{160} \frac{I_4^2}{(1-I_1)^5} + \frac{1}{24} \frac{I_2^2 I_5}{(1-I_1)^6} + \frac{1}{120} \frac{I_3 I_5}{(1-I_1)^5} + \frac{1}{180} \frac{I_2 I_6}{(1-I_1)^5} + \frac{1}{2880} \frac{I_7}{(1-I_1)^4}. \tag{3.101}$$

In general,

$$F_{0,k}^t = a_1 \frac{I_{2k-1}}{(1-I_1)^k} + a_2 \frac{I_2 I_{2k-2}}{(1-I_1)^{k+1}} + a_3 \frac{I_3 I_{2k-3}}{(1-I_1)^{k+1}} + \cdots. \tag{3.102}$$

$$a_1 = \frac{1}{k!(k+1)!}, \tag{3.103}$$

$$a_2 = \frac{k^2}{k!(k+1)!}, \tag{3.104}$$

$$a_3 = \frac{g^2(k-1)}{2 \cdot k!(k+1)!}. \tag{3.105}$$

Remark 3.5.1. One can also get the proposition 3.5.1 by Virasoro constraints for fat genus expansion as follows. By proposition 3.1.2, one can rewrite the partition function Z^N as

$$Z^N = e^{\frac{1}{g_s}F_0^N} \tilde{Z}^N, \tag{3.106}$$

with $\tilde{Z}^N$ does not depend on I_0. Then $\tilde{Z}^N$ can be viewed as partition function of Hermitian one-matrix model with $g_1 = 0$ and $g_k = \frac{I_{k-1}}{(k-1)!}$. The free energy function $\tilde{F}^N = \log \tilde{Z}^N$ has a genus expansion (the fat genus expansion)

$$\tilde{F}^N = \sum_{g \geqslant 0} g_s^{2g-2} \tilde{F}_g^t, \tag{3.107}$$

then the two definitions of $\tilde{F}_0^t$ are coincident. By the proof of theorem 3.3.1, $\tilde{Z}^N$ satisfies the following constraints:

$$\tilde{L}_m^t \tilde{Z}^N = 0, \quad m \geqslant 0 \tag{3.108}$$

where

$$\tilde{L}_0^t = -2\frac{\partial}{\partial I_1} + \sum_{l \geqslant 1}(l+1)I_l\frac{\partial}{\partial I_l} + \frac{t^2}{g_s^2}, \tag{3.109}$$

$$\tilde{L}_1^t = \frac{2t}{1-I_1}d_X + \sum_{n \geqslant 1}\frac{(n+2)!}{n!}(I_n - \delta_{n,1})\frac{\partial}{\partial I_{n+1}}, \tag{3.110}$$

$$\tilde{L}_2^t = 4t\frac{\partial}{\partial I_1} + \sum_{n \geqslant 1}\frac{(n+3)!}{n!}(I_n - \delta_{n,1})\frac{\partial}{\partial I_{n+2}} + g_s^2\left(\frac{1}{1-I_1}d_X\right)^2 + \frac{t}{1-I_1}, \tag{3.111}$$

$$\tilde{L}_m^t = 2tm!\frac{\partial}{\partial I_{m-1}} + \sum_{n \geqslant 1}\frac{(m+n+1)!}{n!}(I_n - \delta_{n,1})\frac{\partial}{\partial I_{m+n}} \tag{3.112}$$

$$+ g_s^2(m-1)!\left(\frac{1}{1-I_1}d_X\frac{\partial}{\partial I_{m-2}} + \frac{\partial}{\partial I_{m-2}}\frac{1}{1-I_1}d_X\right)$$

$$+ g_s^2\sum_{k=2}^{m-2}k!(m-k)!\frac{\partial}{\partial I_{k-1}}\frac{\partial}{\partial I_{m-k-1}} + \delta_{m,2}\frac{t}{1-I_1},$$

for $m \geqslant 3$. This gives another proof of the proposition 3.5.1.

4 Itzykson-Zuber Ansatz in 2D topological gravity

In this section we will prove the validity of Itzykson-Zuber Ansatz in 2D topological gravity.

4.1 Preliminary results of the 2D topological gravity

According to Witten [26], the mathematical theory of the 2D topological gravity studies the following intersection numbers on the Deligne-Mumford moduli spaces:

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g^{2D} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \wedge \cdots \wedge \psi_n^{d_n}. \tag{4.1}$$

The free energy of the 2D topological gravity is the generating series of these intersection numbers:

$$F^{2D}(t) = \sum_{g=0}^{\infty} F_g^{2D} := \sum_{g=0}^{\infty} \sum_{n_0,n_1,n_2\cdots} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g^{2D} \frac{t_0^{n_0}}{n_0!} \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!} \cdots . \qquad (4.2)$$

It is well known that intersection number $\langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g^{2D}$ satisfies the following selection rule:

$$\sum_i (i-1)n_i = 3g - 3. \qquad (4.3)$$

In other words, if we define

$$\deg t_i = i - 1, \qquad (4.4)$$

then F_g^{2D} is weighted homogeneous of degree $3g - 3$. The partition function of the 2D topological gravity is defined by:

$$Z^{2D} := e^{F^{2D}}. \qquad (4.5)$$

4.2 Virasoro constraints for 2D topological gravity

The following theorem is a consequence of the famous Witten Conjecture/Kontsevich theorem [20, 26]. See also [8] or [17]. More references can be found in the Introduction.

Theorem 4.2.1. *The partition function Z^{2D} of the 2D topological gravity satisfies the following equations:*

$$L_n^{2D} Z^{2D} = 0, \quad n \geqslant -1 \qquad (4.6)$$

where

$$L_{-1}^{2D} = \frac{t_0^2}{2} + \sum_{n \geqslant 1} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{n-1}}, \qquad (4.7)$$

$$L_0^{2D} = \frac{1}{8} + \sum_{n \geqslant 0} (2n+1)(t_n - \delta_{n,1}) \frac{\partial}{\partial t_n}, \qquad (4.8)$$

$$L_m^{2D} = \sum_{n \geqslant 0} \frac{(2n+2m+1)!!}{(2n-1)!!} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{m+n}} + \frac{1}{2} \sum_{k+l=m-1} (2k+1)!!(2l+1)!! \frac{\partial^2}{\partial t_k \partial t_l}, \quad (4.9)$$

for $m \geqslant 1$. Furthermore, $\{L_m^{2D}\}_{m \geqslant -1}$ satisfies the following commutation relations:

$$\left[L_m^{2D}, L_n^{2D}\right] = (m-n)L_{m+n}^{2D}, \qquad (4.10)$$

for $m, n \geqslant -1$.

4.3 The Itzykson-Zuber Ansatz

It was announced in [28] that the first two Virasoro constraints L_{-1}^{2D} and L_0^{2D} in I-coordinates can be used to solve free energies in genus $g = 0, 1$ and establish the Itzykson-Zuber ansatz. Here we present some details which are similar to the 1D topological gravity case in [28].

Theorem 4.3.1. *(Itzykson-Zuber Ansatz [17]) For free energy of 2D gravity, we have:*

$$F_0^{2D} = \frac{1}{6}I_0^3 - \sum_{n \geqslant 0} \frac{(-1)^n I_0^{n+2}}{(n+2)!} I_n + \frac{1}{2} \sum_{j,k \geqslant 0} \frac{(-1)^{j+k} I_0^{j+k+1}}{j!k!(j+k+1)} I_j I_k, \tag{4.11}$$

$$F_1^{2D} = \frac{1}{24} \log \frac{1}{1-I_1}, \tag{4.12}$$

and for $g \geq 2$,

$$F_g^{2D}(t) = \sum_{\sum_{2 \leqslant k \leqslant 3g-2}(k-1)l_k = 3g-3} \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \right\rangle_g^{2D} \prod_{j=2}^{3g-2} \frac{1}{l_j!} \left(\frac{I_j}{(1-I_1)^{\frac{2j+1}{3}}} \right)^{l_j}. \tag{4.13}$$

Proof. We first rewrite L_{-1}^{2D} in I-coordinates:

$$L_{-1}^{2D} = -\frac{\partial}{\partial I_0} + \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n \right)^2. \tag{4.14}$$

This gives

$$\frac{\partial F^{2D}}{\partial I_0} = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n \right)^2. \tag{4.15}$$

By analyzing degree of F_g^{2D}, one has

$$\frac{\partial F_0^{2D}}{\partial I_0} = \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_n \right)^2, \tag{4.16}$$

$$\frac{\partial F_g^{2D}}{\partial I_0} = 0, \quad g > 0. \tag{4.17}$$

This gives

$$F_0^{2D} = \frac{1}{6}I_0^3 - \sum_{n \geqslant 0} \frac{(-1)^n I_0^{n+2}}{(n+2)!} I_n + \frac{1}{2} \sum_{n,k \geqslant 0} \frac{(-1)^{n+k} I_0^{n+k+1}}{n!k!(n+k+1)} I_n I_k, \tag{4.18}$$

and F_g^{2D} is independent on I_0. By

$$L_0^{2D} Z^{2D} = 0,$$

one has

$$0 = \frac{1}{8} + \sum_{n \geqslant 0}(2n+1)(t_n - \delta_{n,1})\frac{\partial F^{2D}}{\partial t_n}. \tag{4.19}$$

Let $I_0 = 0$, we have

$$0 = \frac{1}{8} + \sum_{n \geqslant 1}(2n+1)(I_n - \delta_{n,1})\frac{\partial F^{2D}}{\partial I_n}. \tag{4.20}$$

This is equal to

$$0 = \sum_{n \geqslant 1}(2n+1)(I_n - \delta_{n,1})\frac{\partial F_g^{2D}}{\partial I_n} + \frac{1}{8}\delta_{g,1}. \tag{4.21}$$

Write F_g^{2D} as formal power series in I_1, with coefficients a priori formal series in $I_2, I_3, \cdots$:

$$F_g^{2D} = \sum_{n=0}^{\infty} a_{g,n}(I_2, I_3, \cdots) I_1^n. \tag{4.22}$$

Write

$$a_{g,0} = \sum \alpha_{l_2, l_3, \cdots, l_m} \prod_{i=2}^{m} \frac{I_i^{l_i}}{l_i!}. \tag{4.23}$$

Since $I_0 = 0$ is equivalent to $t_0 = 0$ and $t_k = I_k (k > 0)$,

$$\alpha_{l_2, l_3, \cdots, l_m} = \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_m^{l_m} \right\rangle_g^{2D}. \tag{4.24}$$

This vanishes unless the following selection rule is satisfied:

$$\sum_{i=2}^{m} (i-1) l_i = 3g - 3. \tag{4.25}$$

So $l_i = 0$ unless $i \leqslant 3g - 2$. Therefore

$$a_{g,0} = \sum \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \right\rangle_g^{2D} \prod_{i=2}^{3g-2} \frac{I_i^{l_i}}{l_i!}, \tag{4.26}$$

where the summation is taken over all nonnegative integers $l_2, l_3, \cdots, l_{3g-2}$ such that

$$\sum_{i=2}^{3g-2} (i-1) l_i = 3g - 3. \tag{4.27}$$

Let

$$E = \sum_{k \geqslant 2} \frac{2k+1}{3} I_k \frac{\partial}{\partial I_k}. \tag{4.28}$$

Then the equation (4.21) gives us the following recursion relations:

$$a_{g,1} = E(a_{g,0}) + \frac{1}{24} \delta_{g,1}, \tag{4.29}$$

$$m a_{g,m} = (m-1) a_{g,m-1} + E(a_{g,m-1}). \tag{4.30}$$

When $g = 1$, we have $a_{1,0} = 0$ by selection rule. One can see that $a_{1,n} = \frac{1}{24n}$ for $n \geqslant 1$. Therefore,

$$F_1^{2D} = \sum_{n=1}^{\infty} \frac{1}{24n} I_1^n = \frac{1}{24} \log \frac{1}{1 - I_1}. \tag{4.31}$$

When $g > 1$, one finds

$$a_{g,n} = \sum_{\sum_{j=2}^{3g-2} (j-1) l_j = 3g-3} \left\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \right\rangle_g^{2D} (-1)^n \binom{g - 1 - \sum_{j=2}^{3g-2} j l_j}{n} \prod_{i=2}^{3g-2} \frac{I_i^{l_i}}{l_i!}. \tag{4.32}$$

This proves

$$
F_g^{2D} = \sum_{\sum_{j=2}^{3g-2}(j-1)l_j=3g-3} \left\langle \tau_2^{l_2}\tau_3^{l_3}\cdots\tau_{3g-2}^{l_{3g-2}} \right\rangle_g^{2D} \frac{1}{(1-I_1)^{\sum_{j=2}^{3g-2} jl_j - g+1}} \prod_{i=2}^{3g-2} \frac{I_i^{l_i}}{l_i!} \qquad (4.33)
$$

$$
= \sum_{\sum_{j=2}^{3g-2}(j-1)l_j=3g-3} \left\langle \tau_2^{l_2}\tau_3^{l_3}\cdots\tau_{3g-2}^{l_{3g-2}} \right\rangle_g^{2D} \prod_{i=2}^{3g-2} \frac{1}{l_i!}\left(\frac{I_j}{(1-I_1)^{(2j+1)/2}} \right)^{l_j}.
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.3.1. The formula for F_0^{2D} in (4.11) is equivalent to the version announced in [28]. It is different from the original formula given by Itzyskon-Zuber [17]. Their version will be proved in corollary 5.3.1.

In [28], the second named author studied another kind of coordinates $\{\frac{\partial^n I_0}{\partial t_0^n}\}_{n\geqslant 0}$, and proved the following transformation equations between this coordinates and I-coordinates:

$$
\frac{\partial^n I_0}{\partial t_0^n} = \sum_{\sum_{j\geqslant 1} jm_j = n-1} \frac{(\sum_j (j+1)m_j)!}{\prod_j ((j+1)!)^{m_j} m_j!} \cdot \frac{\prod_j I_{j+1}^{m_j}}{(1-I_1)^{\sum_j (j+1)m_j+1}}, \qquad (4.34)
$$

$$
I_n = -\sum_{\sum_{j\geqslant 1} jm_j = n-1} \frac{(\sum_j (j+1)m_j)!}{\prod_j ((j+1)!)^{m_j} m_j!} \cdot \frac{\prod_j \left(-\frac{\partial^{j+1} I_0}{\partial t_0^{j+1}}\right)^{m_j}}{\left(\frac{\partial I_0}{\partial t_0}\right)^{\sum_j (j+1)m_j+1}}. \qquad (4.35)
$$

In an earlier work by Euguchi, Yamada and Yang [12], the Itzykson-Zuber Ansatz is written in two forms:

$$
F_g^{2D} = \sum_{\sum_{j=2}^{3g-2}(j-1)l_j=3g-3} a_{l_2\cdots l_{3g-2}} \frac{(u'')^{l_2}\cdots(u^{(3g-2)})^{l_{3g-2}}}{(u')^{2(1-g)+\sum_{j=2}^{3g-2} jl_j}} \qquad (4.36)
$$

$$
= \sum_{\sum_{j=2}^{3g-2}(j-1)l_j=3g-3} b_{l_2\cdots l_{3g-2}} \frac{I_2^{l_2}\cdots I_{3g-2}^{l_{3g-2}}}{(u')^{2(1-g)+\sum_{j=2}^{3g-2} l_j}}, \qquad (4.37)
$$

where $u = I_0$, and $u^{(n)} = \frac{\partial^n I_0}{\partial t_0^n}$. They proved (4.36) by KdV hierarchy plus string equation, and derived (4.37) by (4.36) following a proposal by Itzyson and Zuber [17]. In this subsection, we have proved (4.37) by Virasoro constraints, and by (4.35), one can derives (4.36) by (4.37). Our proof seems to be simpler because we use only two linear equations while the KdV hierarchy is a sequence of nonlinear equations. Furthermore, in our proof, we have only used L_{-1}^{2D} and L_0^{2D} in I-coordinates. In the next subsection, we will show that it is possible to use the higher Virasoro constraints in I-coordinates to derive a recursive method to solve free energies in higher genera.

4.4 Computations of F_g^{2D} by Virasoro constraints in I-coordinates

In the above we have shown that for $g \geq 2$, F_g^{2D} is independent of I_0. It follows that after we rewrite the Virasoro constraints in I-coordinates, we can take $I_0 = 0$ to compute F_g^{2D}.

Theorem 4.4.1. *For free energy of the 2D topological gravity of* $g \geqslant 2$, *the following equations hold:*

$$\frac{\partial F_g^{2D}}{\partial J_1} = \sum_{p=2}^{3g-2} (2p+1) \frac{J_p}{1-3J_1} \frac{\partial F_g^{2D}}{\partial J_p}, \tag{4.38}$$

$$\frac{\partial F_g^{2D}}{\partial J_2} = \sum_{p=2}^{3g-3} (2p+1) \frac{J_p}{1-3J_1} \frac{\partial F_g^{2D}}{\partial J_{1+p}} + \frac{1}{2} \frac{1}{1-3J_1} \left(\sum_{l=1}^{3g-4} (2l+3) \frac{J_{l+1}}{1-3J_1} \frac{\partial}{\partial J_l} \right)^2 \left(F_{g-1}^{2D} \right) \tag{4.39}$$

$$+ \frac{1}{2} \frac{1}{1-3J_1} \sum_{g_1=1}^{g-1} \sum_{l_1=1}^{3g_1-2} (2l_1+3) \frac{J_{l_1+1}}{1-3J_1} \frac{\partial F_{g_1}^{2D}}{\partial J_{l_1}} \sum_{l_2=1}^{3g-3g_1-2} (2l_2+3) \frac{J_{l_2+1}}{1-3J_1} \frac{\partial F_{g-g_1}^{2D}}{\partial J_{l_2}},$$

$$\frac{\partial F_g^{2D}}{\partial J_3} = \sum_{p=2}^{3g-4} (2p+1) \frac{J_p}{1-3J_1} \frac{\partial F_g^{2D}}{\partial J_{2+p}} + \sum_{g_1=1}^{g-1} \frac{\partial F_{g_1}^{2D}}{\partial J_1} \sum_{l=1}^{3g-3g_1-2} (2l+3) \frac{J_{l+1}}{(1-3J_1)^2} \frac{\partial F_{g-g_1}^{2D}}{\partial J_l} \tag{4.40}$$

$$+ \sum_{l=1}^{3g-5} (2l+3) \frac{J_{l+1}}{(1-3J_1)^2} \frac{\partial^2 F_{g-1}^{2D}}{\partial J_l \partial J_1} + 3 \sum_{l=1}^{3g-5} (2l+3) \frac{J_{l+1}}{(1-3J_1)^3} \frac{\partial F_{g-1}^{2D}}{\partial J_l},$$

$$\frac{\partial F_g^{2D}}{\partial J_{r+1}} = \sum_{p=2}^{3g-2-r} (2p+1) \frac{J_p}{1-3J_1} \frac{\partial F_g^{2D}}{\partial J_{r+p}} + \sum_{g_1=1}^{g-1} \frac{\partial F_{g_1}^{2D}}{\partial J_{r-1}} \sum_{l=1}^{3g-3g_1-2} (2l+3) \frac{J_{l+1}}{(1-3J_1)^2} \frac{\partial F_{g-g_1}^{2D}}{\partial J_l} \tag{4.41}$$

$$+ \sum_{l=1}^{3g-3-r} (2l+3) \frac{J_{l+1}}{(1-3J_1)^2} \frac{\partial^2 F_{g-1}^{2D}}{\partial J_l \partial J_{r-1}} + \frac{2r-1}{(1-3J_1)^2} \frac{\partial F_{g-1}^{2D}}{\partial J_{r-2}}$$

$$+ \frac{1}{2} \frac{1}{1-3J_1} \left(\sum_{g_1=1}^{g-1} \sum_{l_1=1}^{r-2} \frac{\partial F_{g_1}^{2D}}{\partial J_{l_1}} \frac{\partial F_{g-g_1}^{2D}}{\partial J_{r-1-l_1}} + \sum_{l_1=1}^{r-2} \frac{\partial^2 F_{g-1}^{2D}}{\partial J_{l_1} \partial J_{r-1-l_1}} \right),$$

for $r = 3, 4, \cdots 3g - 3$, *where*

$$J_k = \frac{I_k}{(2k+1)!!}.$$

Proof. By the constraint $L_m^{2D} Z^{2D} = 0$ for $m \geq 1$, one gets:

$$0 = \sum_{n \geqslant 0} \frac{(2n+2m+1)!!}{(2n-1)!!} (t_n - \delta_{n,1}) \frac{\partial F_g^{2D}}{\partial t_{m+n}} + \frac{1}{2} \sum_{k+l=m-1} (2k+1)!!(2l+1)!! \frac{\partial^2 F_{g-1}^{2D}}{\partial t_k \partial t_l} \tag{4.42}$$

$$+ \frac{1}{2} \sum_{g_1=0}^{g} \sum_{k+l=m-1} (2k+1)!!(2l+1)!! \frac{\partial^2 F_{g_1}^{2D}}{\partial t_k} \frac{\partial F_{g-g_1}^{2D}}{\partial t_l}.$$

We want to rewrite these equations in I-coordinates under the condition $I_0 = 0$. By (4.11),

$$\frac{\partial F_0^{2D}}{\partial I_0} = \frac{1}{2} \left(\sum_{k \geqslant 0} \frac{(-1)^k I_0^k}{k!} I_k \right)^2, \tag{4.43}$$

$$\frac{\partial F_0^{2D}}{\partial I_n} = \frac{(-1)^{n+1} I_0^{n+2}}{(n+2)!} + \sum_{k \geqslant 0} \frac{(-1)^{n+k} I_0^{n+k+1}}{n!k!(n+k+1)} I_k, \quad n > 0. \tag{4.44}$$

Hence

$$\left.\frac{\partial F_0^{2D}}{\partial I_0}\right|_{I_0=0} = \left.\frac{\partial F_0^{2D}}{\partial I_n}\right|_{I_0=0} = 0, \tag{4.45}$$

and this gives

$$\left.\frac{\partial F_0^{2D}}{\partial t_0}\right|_{I_0=0} = \left.\frac{\partial F_0^{2D}}{\partial t_n}\right|_{I_0=0} = 0. \tag{4.46}$$

Now we restrict equation (4.42) in $I_0 = 0$, and $g \geqslant 2$, we have:

for $m = 1$:

$$0 = \sum_{n \geqslant 1} \frac{(2n+3)!!}{(2n-1)!!}(I_n - \delta_{n,1})\frac{\partial F_g^{2D}}{\partial I_{n+1}} + \frac{1}{2}\left(\sum_{l=1}^{2g-1} \frac{I_{l+1}}{1-I_1}\frac{\partial}{\partial I_l}\right)^2 (F_{g-1}^{2D}) \tag{4.47}$$
$$+ \frac{1}{2}\sum_{g_1=1}^{g-1}\left(\sum_{l=1}^{2g-1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g_1}^{2D}}{\partial I_l}\right)\left(\sum_{l=1}^{2g-1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g-g_1}^{2D}}{\partial I_l}\right),$$

for $m = 2$:

$$0 = \sum_{n \geqslant 1}\frac{(2n+5)!!}{(2n-1)!!}(I_n-\delta_{n,1})\frac{\partial F_g^{2D}}{\partial I_{n+2}} + 3!!\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial^2 F_{g-1}^{2D}}{\partial I_l \partial I_1} + \frac{3!!}{2}\sum_{l \geqslant 1}\frac{I_{l+1}}{(1-I_1)^2}\frac{\partial F_{g-1}^{2D}}{\partial I_l}$$
$$+ \sum_{g_1=1}^{g-1}3!!\frac{\partial^2 F_{g_1}^{2D}}{\partial I_1}\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g-g_1}^{2D}}{\partial I_l}, \tag{4.48}$$

for $m \geqslant 3$:

$$0 = \sum_{n \geqslant 1}\frac{(2n+2m+1)!!}{(2n-1)!!}(I_n-\delta_{n,1})\frac{\partial F_g^{2D}}{\partial I_{m+n}} + \frac{1}{2}\sum_{k=1}^{m-2}(2k+1)!!(2m-2k-1)!!\frac{\partial^2 F_{g-1}^{2D}}{\partial I_k \partial I_{m-k-1}} \tag{4.49}$$
$$+ (2m-1)!!\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial^2 F_{g-1}^{2D}}{\partial I_l \partial I_{m-1}} + \frac{(2m-1)!!}{2}\frac{\partial F_{g-1}^{2D}}{\partial I_{m-2}}$$
$$+ \frac{1}{2}\sum_{g_1=1}^{g-1}\sum_{k=1}^{m-2}(2k+1)!!(2m-2k-1)!!\frac{\partial^2 F_{g_1}^{2D}}{\partial I_k}\frac{\partial F_{g-g_1}^{2D}}{\partial I_{m-k-1}} + \sum_{g_1=1}^{g-1}(2m-1)!!\frac{\partial^2 F_{g_1}^{2D}}{\partial I_{m-1}}\sum_{l \geqslant 1}\frac{I_{l+1}}{1-I_1}\frac{\partial F_{g-g_1}^{2D}}{\partial I_l}.$$

Together with theorem 4.3.1, we complete the proof. $\qquad\square$

We explain how to use these equations to solve F_g^{2D} for $g \geqslant 2$. By above theorem,

$$\frac{\partial F_g^{2D}}{\partial J_{3g-2}} = \frac{6g-7}{(1-3J_1)^2}\frac{\partial F_{g-1}^{2D}}{\partial J_{3(g-1)-2}} + \frac{1}{2}\frac{1}{1-3J_1}\left(\sum_{l_1=1}^{3g-5}\frac{\partial^2 F_{g-1}^{2D}}{\partial J_{l_1}\partial J_{3g-4-l_1}} + \sum_{g_1=1}^{g-1}\sum_{l_1=1}^{3g-5}\frac{\partial F_{g_1}^{2D}}{\partial J_{l_1}}\frac{\partial F_{g-g_1}^{2D}}{\partial J_{3g-4-l_1}}\right), \quad (4.50)$$

$$\frac{\partial F_g^{2D}}{\partial J_{3g-3}} = \frac{5J_2}{1-3J_1}\frac{\partial F_g^{2D}}{\partial J_{3g-2}} + \frac{5J_2}{(1-3J_1)^2}\left(\frac{\partial F_1^{2D}}{\partial J_1}\frac{\partial F_{g-1}^{2D}}{\partial J_{3(g-1)-2}} + \frac{\partial^2 F_{g-1}^{2D}}{\partial J_1\partial J_{3(g-1)-2}}\right) \quad (4.51)$$

$$+ \frac{6g-9}{(1-3J_1)^2}\frac{\partial F_{g-1}^{2D}}{\partial J_{3g-6}} + \frac{1}{2}\frac{1}{1-3J_1}\left(\sum_{g_1=1}^{g-1}\sum_{l_1=1}^{3g-6}\frac{\partial F_{g_1}^{2D}}{\partial J_{l_1}}\frac{\partial F_{g-g_1}^{2D}}{\partial J_{3g-5-l_1}} + \sum_{l_1=1}^{3g-6}\frac{\partial^2 F_{g-1}^{2D}}{\partial J_{l_1}\partial J_{3g-5-l_1}}\right),$$

$$\vdots$$

$$\frac{\partial F_g^{2D}}{\partial J_4} = \frac{5}{(1-3J_1)^2}\frac{\partial F_{g-1}^{2D}}{\partial J_1} + \sum_{p=2}^{3g-5}\frac{(2p+1)J_p}{1-3J_1}\frac{\partial F_g^{2D}}{\partial J_{p+3}} + \sum_{g_1=1}^{g-1}\frac{\partial F_{g_1}^{2D}}{\partial J_2}\sum_{l=1}^{3g-3g_1-2}\frac{(2l+3)J_{l+1}}{(1-3J_1)^2}\frac{\partial F_{g-g_1}^{2D}}{\partial J_l} \quad (4.52)$$

$$+ \sum_{l=1}^{3g-6}\frac{(2l+3)J_{l+1}}{(1-3J_1)^2}\frac{\partial^2 F_{g-1}^{2D}}{\partial J_l\partial J_2} + \frac{1}{2}\frac{1}{1-3J_1}\left(\sum_{g_1=1}^{g-1}\frac{\partial F_{g_1}^{2D}}{\partial J_1}\frac{\partial F_{g-g_1}^{2D}}{\partial J_1} + \frac{\partial^2 F_{g-1}^{2D}}{\partial J_1^2}\right),$$

$$\frac{\partial F_g^{2D}}{\partial J_3} = \sum_{p=2}^{3g-4}(2p+1)\frac{J_p}{1-3J_1}\frac{\partial F_g^{2D}}{\partial J_{2+p}} + \sum_{g_1=1}^{g-1}\frac{\partial F_{g_1}^{2D}}{\partial J_1}\sum_{l=1}^{3g-3g_1-2}(2l+3)\frac{J_{l+1}}{(1-3J_1)^2}\frac{\partial F_{g-g_1}^{2D}}{\partial J_l} \quad (4.53)$$

$$+ \sum_{l=1}^{3g-5}(2l+3)\frac{J_{l+1}}{(1-3J_1)^2}\frac{\partial^2 F_{g-1}^{2D}}{\partial J_l\partial J_1} + 3\sum_{l=1}^{3g-5}(2l+3)\frac{J_{l+1}}{(1-3J_1)^3}\frac{\partial F_{g-1}^{2D}}{\partial J_l},$$

$$\frac{\partial F_g^{2D}}{\partial J_2} = \sum_{p=2}^{3g-3}(2p+1)\frac{J_p}{1-3J_1}\frac{\partial F_g^{2D}}{\partial J_{1+p}} + \frac{1}{2}\frac{1}{1-3J_1}\left(\sum_{l=1}^{3g-4}(2l+3)\frac{J_{l+1}}{1-3J_1}\frac{\partial}{\partial J_l}\right)^2\left(F_{g-1}^{2D}\right) \quad (4.54)$$

$$+ \frac{1}{2}\frac{1}{1-3J_1}\sum_{g_1=1}^{g-1}\sum_{l_1=1}^{3g_1-2}(2l_1+3)\frac{J_{l_1+1}}{1-3J_1}\frac{\partial F_{g_1}^{2D}}{\partial J_{l_1}}\sum_{l_2=1}^{3g-3g_1-2}(2l_2+3)\frac{J_{l_2+1}}{1-3J_1}\frac{\partial F_{g-g_1}^{2D}}{\partial J_{l_2}},$$

$$\frac{\partial F_g^{2D}}{\partial J_1} = \sum_{p=2}^{3g-2}(2p+1)\frac{J_p}{1-3J_1}\frac{\partial F_g^{2D}}{\partial J_p}. \quad (4.55)$$

Since F_g^{2D} is weighted homogeneous of degree $3g-3$ with $\deg J_k = k-1$, one has:

$$F_g^{2D} = \frac{1}{3g-3}\sum_{i=1}^{3g-2}(i-1)J_i\frac{\partial F_g^{2D}}{\partial J_i}. \quad (4.56)$$

Example 4.4.1. *Let us compute F_2^{2D} by the above procedure.*

$$\frac{\partial F_2^{2D}}{\partial J_4} = \frac{5}{(1-3J_1)^2}\frac{\partial F_1^{2D}}{\partial J_1} + \frac{1}{2}\frac{1}{1-3J_1}\left(\frac{\partial^2 F_1^{2D}}{\partial J_1^2} + \left(\frac{\partial F_1^{2D}}{\partial J_1}\right)^2\right) = \frac{105}{128}\frac{1}{(1-3J_1)^3}, \quad (4.57)$$

$$\frac{\partial F_2^{2D}}{\partial J_3} = \frac{5J_2}{1-3J_1}\frac{\partial F_2^{2D}}{\partial J_4} + \frac{5J_2}{(1-3J_1)^2}\left(\left(\frac{\partial F_1^{2D}}{\partial J_1}\right)^2 + \frac{\partial^2 F_1^{2D}}{\partial J_1^2}\right) + \frac{15J_2}{(1-3J_1)^3}\frac{\partial F_1^{2D}}{\partial J_1} \quad (4.58)$$

$$= \frac{1015}{128}\frac{J_2}{(1-3J_1)^4},$$

$$\frac{\partial F_2^{2D}}{\partial J_2} = \frac{5J_2}{1-3J_1}\frac{\partial F_2^{2D}}{\partial J_3} + \frac{7J_3}{1-3J_1}\frac{\partial F_2^{2D}}{\partial J_4} + \frac{1}{2}\frac{1}{1-3J_1}\left(\frac{5J_2}{1-3J_1}\frac{\partial}{\partial J_1} + \frac{7J_3}{1-3J_1}\frac{\partial}{\partial J_2}\right)^2\left(F_1^{2D}\right)$$

$$+ \frac{1}{2}\frac{25J_2^2}{(1-3J_1)^3}\left(\frac{\partial F_1^{2D}}{\partial J_1}\right)^2$$

$$= \frac{6300}{128}\frac{J_2^2}{(1-3J_1)^5} + \frac{1015}{128}\frac{J_3}{(1-3J_1)^4}. \tag{4.59}$$

so we have

$$F_2^{2D} = \frac{1}{3}\left(\frac{105}{128}\frac{3J_4}{(1-3J_1)^3} + \frac{1015}{128}\frac{2J_2J_3}{(1-3J_1)^4} + \frac{6300}{128}\frac{J_2^3}{(1-3J_1)^5} + \frac{1015}{128}\frac{J_2J_3}{(1-3J_1)^4}\right) \tag{4.60}$$

$$= \frac{2100}{128}\frac{J_2^3}{(1-3J_1)^5} + \frac{1015}{384}\frac{J_2J_3}{(1-3J_1)^4} + \frac{105}{128}\frac{J_4}{(1-3J_1)^3}.$$

This can be written in I-coordinates as:

$$F_2^{2D} = \frac{7}{1440}\breve{I}_2^3 + \frac{29}{5760}\breve{I}_2\breve{I}_3 + \frac{1}{1152}\breve{I}_4, \tag{4.61}$$

where

$$\breve{I}_j = \frac{I_j}{(1-I_1)^{(2j+1)/3}}. \tag{4.62}$$

Similarly, we have with the help of a computer program:

$$F_3^{2D} = \frac{245}{20736}\breve{I}_2^6 + \frac{193}{6912}\breve{I}_2^4\breve{I}_3 + \frac{205}{13824}\breve{I}_2^2\breve{I}_3^3 + \frac{583}{580608}\breve{I}_3^3 + \frac{53}{6912}\breve{I}_2^3\breve{I}_4 + \frac{1121}{241920}\breve{I}_2\breve{I}_3\breve{I}_4 \tag{4.63}$$

$$+ \frac{607}{2903040}\breve{I}_4^2 + \frac{17}{11520}\breve{I}_2^2\breve{I}_5 + \frac{503}{1451520}\breve{I}_3\breve{I}_5 + \frac{77}{414720}\breve{I}_2\breve{I}_6 + \frac{1}{82944}\breve{I}_7,$$

$$F_4^{2D} = \frac{259553}{2488320}\breve{I}_2^9 + \frac{475181}{1244160}\breve{I}_2^7\breve{I}_3 + \frac{145693}{331776}\breve{I}_2^5\breve{I}_3^2 + \frac{43201}{248832}\breve{I}_2^3\breve{I}_3^3 + \frac{134233}{7962624}\breve{I}_2\breve{I}_3^4 + \frac{14147}{124416}\breve{I}_2^6\breve{I}_4$$

$$+ \frac{83851}{414720}\breve{I}_2^4\breve{I}_3\breve{I}_4 + \frac{26017}{331776}\breve{I}_2^2\breve{I}_3^2\breve{I}_4 + \frac{185251}{49766400}\breve{I}_3^3\breve{I}_4 + \frac{5609}{276480}\breve{I}_2^3\breve{I}_4^2 + \frac{177}{20480}\breve{I}_3\breve{I}_4^2$$

$$+ \frac{175}{995328}\breve{I}_4^3 + \frac{21329}{829440}\breve{I}_2^5\breve{I}_5 + \frac{13783}{414720}\breve{I}_2^3\breve{I}_3\breve{I}_5 + \frac{1837}{259200}\breve{I}_2\breve{I}_3^2\breve{I}_5 + \frac{7597}{1382400}\breve{I}_2^2\breve{I}_4\breve{I}_5$$

$$+ \frac{719}{829440}\breve{I}_3\breve{I}_4\breve{I}_5 + \frac{533}{1935360}\breve{I}_2\breve{I}_5^2 + \frac{2471}{552960}\breve{I}_2^4\breve{I}_6 + \frac{7897}{2073600}\breve{I}_2^2\breve{I}_3\breve{I}_6 + \frac{1997}{6635520}\breve{I}_3^2\breve{I}_6$$

$$+ \frac{1081}{2322432}\breve{I}_2\breve{I}_4\breve{I}_6 + \frac{487}{18579456}\breve{I}_5\breve{I}_6 + \frac{4907}{8294400}\breve{I}_2^3\breve{I}_7 + \frac{16243}{58060800}\breve{I}_2\breve{I}_3\breve{I}_7 + \frac{1781}{92897280}\breve{I}_4\breve{I}_7$$

$$+ \frac{53}{921600}\breve{I}_2^2\breve{I}_8 + \frac{947}{92897280}\breve{I}_3\breve{I}_8 + \frac{149}{39813120}\breve{I}_2\breve{I}_9 + \frac{1}{7962624}\breve{I}_{10}. \tag{4.64}$$

5 Special deformation of the Airy curve in renormalized coupling constants

In this section we reformulate the special deformation of Airy curve introduced in the setting of 2D quantum gravity in [32] using renormalized coupling constants. It is remarkable that we need to consider ghost variables introduced in [33] and consider their renormalizations.

5.1 Special deformation of the Airy curve and ghost variables in 2D topological gravity

In [32], the second named author studied the following special deformation of Airy curve defined by:

$$w^{2D} := z^{\frac{1}{2}} - \sum_{n=0}^{\infty} \frac{t_n}{(2n-1)!!} z^{n-\frac{1}{2}} - \sum_{n=0}^{\infty} (2n+1)!! \frac{\partial F_0^{2D}}{\partial t_n} z^{-n-\frac{3}{2}}. \tag{5.1}$$

When all t_n are set to be equal to 0, one gets a plane algebraic curve

$$(w^{2D})^2 = z. \tag{5.2}$$

This is called the Airy curve because its quantization gives the Airy equation.

To make sense of the right-hand side of (5.1), the following extension of the free energy F_0^{2D} was introduced in [33]:

$$\tilde{F}_0^{2D} = F_0^{2D} + \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) t_{-n-1}, \tag{5.3}$$

where $t_{-1}, t_{-2}, \ldots$ are formal variables referred to as the ghost variables. The reason for adding these extra terms to F_0^{2D} is that the moduli spaces $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ do not make sense geometrically, however, from the following formula for $n \geq 3$:

$$\sum_{m_1,\ldots,m_n \geq 0} \langle \tau_{m_1} \cdots \tau_{m_n} \rangle_0 x_1^{m_1} \cdots x_n^{m_n} = (x_1 + \cdots + x_n)^{n-3}, \tag{5.4}$$

it is customary to use the following conventions:

$$\langle \tau_n \rangle_0 = \delta_{n,-2},$$
$$\langle \tau_k \tau_{-k-1} \rangle_0 = (-1)^k.$$

Consider the generating series

$$\sum_{n \in \mathbb{Z}} \frac{\partial \tilde{F}_0^{2D}}{\partial t_n} \cdot x^{n+1} = \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) x^{-n} + \sum_{n \geq 0} \left(\frac{\partial F_0^{2D}}{\partial t_n}(\mathbf{t}) + (-1)^n t_{-n-1} \right) \cdot x^{n+1}, \tag{5.5}$$

and its Laplace transform:

$$\sum_{n \in \mathbb{Z}} \frac{\partial \tilde{F}_0^{2D}}{\partial t_n} \cdot \int_0^{\infty} \frac{1}{\sqrt{x}} e^{-zx} x^{n+1} dx$$

$$= \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) \Gamma\left(-n + \frac{1}{2}\right) z^{n-1/2} + \sum_{n \geq 0} \left(\frac{\partial F_0^{2D}}{\partial t_n}(\mathbf{t}) + (-1)^n t_{-n-1} \right) \cdot \Gamma\left(n + \frac{3}{2}\right) z^{-n-3/2}$$

$$= \sum_{n \in \mathbb{Z}} (t_n - \delta_{n,1})(-1)^n \Gamma\left(-n + \frac{1}{2}\right) z^{n-1/2} + \sum_{n \geq 0} \left(\frac{\partial F_0^{2D}}{\partial t_n}(\mathbf{t}) + 2(-1)^n t_{-n-1} \right) \cdot \Gamma\left(n + \frac{3}{2}\right) z^{-n-3/2}. \tag{5.6}$$

After setting $t_{-n} = 0$ for $n \geq 1$, the right-hand side of the last equality gives us w^{2D}, up to a factor of $\sqrt{\pi}$.

5.2 Renormalized ghost variables

From the above discussions we are lead to consider:

$$\sum_{n\in\mathbb{Z}}(-1)^n t_n \Gamma\left(-n+\frac{1}{2}\right) z^{n-\frac{1}{2}}. \tag{5.7}$$

This seems to play the role of $S(z)$ in 1D topological gravity. Recall the renomralization of the coupling constants in $S(z)$ leads us to the following identity

$$\sum_{n=0}^{\infty}(t_n - \delta_{n,1})\frac{x^{n+1}}{(n+1)!} = I_{-1} - \frac{1}{2}I_0^2 + \sum_{n=2}^{\infty}(I_{n-1} - \delta_{n,2})\frac{(x-I_0)^n}{n!}, \tag{5.8}$$

where I_{-1} is defined by:

$$I_{-1} = \sum_{n=0}^{\infty} t_n \frac{I_0^{n+1}}{(n+1)!}.$$

Also recall for $k \geq 0$,

$$I_k = \sum_{n\geq 0} t_{n+k}\frac{I_0^n}{n!}.$$

Therefore, after the introduction of the ghost variables $\{t_{-n}\}_{n\in\mathbb{Z}_+}$, we can introduce the renormalized ghost variables $\{I_{-k}\}_{k\in\mathbb{Z}_+}$ as follows:

$$I_{-k} = \sum_{n=0}^{\infty} t_{n-k}\frac{I_0^n}{n!}. \tag{5.9}$$

Recall for $n \geq 0$ we have [28, Prop. 2.4]:

$$t_n = \sum_{k=0}^{\infty} I_{n+k}\frac{(-1)^k I_0^k}{k!}. \tag{5.10}$$

Proposition 5.2.1. *The identity* (5.10) *holds for all* $n \in \mathbb{Z}$.

Next we present an analogue of (5.8) in 2D topological gravity:

Proposition 5.2.2. *The following identity holds:*

$$\sum_{n\in\mathbb{Z}}(-1)^n t_n \Gamma\left(-n+\frac{1}{2}\right) z^{n-\frac{1}{2}} = \sum_{p\in\mathbb{Z}}(-1)^p I_p \Gamma\left(n+\frac{3}{2}\right)(z-I_0)^{p-\frac{1}{2}}, \tag{5.11}$$

where we use the following convention for expanding $(z-I_0)^{p-\frac{1}{2}}$:

$$(z-I_0)^{p-\frac{1}{2}} = z^{p-\frac{1}{2}}\sum_{k=0}^{\infty}\binom{p-\frac{1}{2}}{k}\left(\frac{-I_0}{z}\right)^k. \tag{5.12}$$

Proof. By (5.10),

$$\sum_{n\in\mathbb{Z}}(-1)^n t_n \Gamma\left(-n+\frac{1}{2}\right)z^{n-\frac{1}{2}} = \sum_{n\in\mathbb{Z}}(-1)^n \sum_{k=0}^{\infty}\frac{(-1)^k I_0^k}{k!}I_{n+k}\Gamma\left(-n+\frac{1}{2}\right)z^{n-\frac{1}{2}}$$

$$= \sum_{p\in\mathbb{Z}}(-1)^p I_p \Gamma\left(n+\frac{3}{2}\right)\sum_{k=0}^{\infty}\frac{\Gamma(-p+k+\frac{1}{2})}{k!\Gamma\left(n+\frac{3}{2}\right)}I_0^k z^{p-k-\frac{1}{2}}$$

$$= \sum_{p\in\mathbb{Z}}(-1)^p I_p \Gamma\left(n+\frac{3}{2}\right)z^{p-\frac{1}{2}}\sum_{k=0}^{\infty}\binom{p-\frac{1}{2}}{k}\left(\frac{-I_0}{z}\right)^k$$

$$= \sum_{p\in\mathbb{Z}}(-1)^p I_p \Gamma\left(n+\frac{3}{2}\right)z^{p-\frac{1}{2}}\left(1-\frac{I_0}{z}\right)^{p-\frac{1}{2}}$$

$$= \sum_{p\in\mathbb{Z}}(-1)^p I_p \Gamma\left(n+\frac{3}{2}\right)(z-I_0)^{p-\frac{1}{2}}.$$

□

5.3 F_0^{2D} in I-coordinates with renormalized ghost variables

When we impose the condition $t_{-n} = 0$ for $n > 0$, the ghost variables I_{-n} can be expressed as formal series in $\{t_m\}_{m\geq 0}$ by the following equation:

$$I_{-n}|_{t_{-m}=0,m\geq 1} = \sum_{k=0}^{\infty} t_k \frac{I_0^{k+n}}{(k+n)!}. \tag{5.13}$$

The case of I_{-1} is first introduced in [28, (26)], and can be also expressed as:

$$I_{-1}|_{t_{-m}=0,m\geq 1} = \sum_{k=0}^{\infty} I_k \frac{(-1)^k I_0^{k+1}}{(k+1)!}. \tag{5.14}$$

The following result generalizes this identity:

Proposition 5.3.1. *Under the same condition, one can also rewrite I_{-n} in terms of $\{I_m\}_{m\geq 0}$ as follows:*

$$I_{-n}|_{t_{-m}=0,m\geq 1} = \sum_{k=0}^{\infty} I_k \frac{(-1)^k I_0^{k+n}}{k!(n-1)!(k+n)}. \tag{5.15}$$

Proof. By (5.10) again, we get:

$$I_{-n}|_{t_{-m}=0,m\geq 1} = \sum_{m=0}^{\infty} t_m \frac{I_0^{m+n}}{(m+n)!}$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} I_{m+l}\frac{(-1)^l I_0^l}{l!}\frac{I_0^{m+n}}{(m+n)!}$$

$$= \sum_{k=0}^{\infty} I_k I_0^{k+n}\sum_{m+l=k}\frac{(-1)^l}{l!(m+n)!}$$

$$= \sum_{k=0}^{\infty} I_k \frac{(-1)^k I_0^{k+n}}{k!(n-1)!(k+n)}.$$

We have used the following identity:

$$\sum_{m+l=k} \frac{(-1)^l}{l!(m+n+1)!} = \frac{(-1)^k}{n!k!(n+k+1)}.$$ (5.16)

This can be proved as follows. By the following well known identity:

$$\binom{n+k+1}{l} = \binom{n+k}{l-1} + \binom{n+k}{l},$$ (5.17)

we have

$$\sum_{l=0}^{k} \binom{n+k+1}{l}(-1)^l = \sum_{l=1}^{k} \binom{n+k}{l-1}(-1)^l + \sum_{l=0}^{k} \binom{n+k}{l}(-1)^l$$

$$= \sum_{l=0}^{k-1} \binom{n+k}{l}(-1)^{l+1} + \sum_{l=0}^{k} \binom{n+k}{l}(-1)^l$$

$$= \binom{n+k}{k}(-1)^k.$$

Then, divide $(n+k+1)!$ on both side of this equation, we get (5.16). □

We also denote the shifted I-coordinates by $\tilde{I}_n$, the definitions are as follows:

$$\tilde{I}_n := \sum_{k \geqslant 0} \frac{I_0^k}{k!} \tilde{t}_{k+n} = I_n - \frac{I_0^{1-n}}{(1-n)!},$$ (5.18)

where $\tilde{t}_k = t_k - \delta_{k,1}$. Then one can see by above proposition,

$$\tilde{I}_{-n}|_{t_{-m}=0,m\geq 1} = I_{-n}|_{t_{-m}=0,m\geq 1} - \frac{I_0^{1+n}}{(1+n)!} = \sum_{k=1}^{\infty} \tilde{I}_k \frac{(-1)^k I_0^{k+n}}{k!(n-1)!(k+n)}.$$ (5.19)

We can express the free energy of genus 0 with these notations as below:

Theorem 5.3.1. *The following formula for F_0^{2D} holds:*

$$F_0^{2D} = \frac{1}{2} \sum_{n \geqslant 1} (-1)^n \tilde{I}_n \tilde{I}_{-n-1}|_{t_{-m}=0,m\geq 1}.$$ (5.20)

Proof. Recall in theorem 4.3.1 we have proved the following expression of F_0^{2D}:

$$F_0^{2D} = \frac{1}{6} I_0^3 - \sum_{n \geqslant 0} \frac{(-1)^n I_0^{n+2}}{(n+2)!} I_n + \frac{1}{2} \sum_{n,k \geqslant 0} \frac{(-1)^{n+k} I_0^{n+k+1}}{k!n!(n+k+1)} I_n I_k.$$ (5.21)

One can rewrite this formula using $\tilde{I}_n$ as follows:

$$F_0^{2D} = \frac{1}{2} \sum_{n,k \geqslant 1} \frac{(-1)^{n+k} I_0^{n+k+1}}{k!n!(n+k+1)} \tilde{I}_n \tilde{I}_k.$$ (5.22)

Together with equation (5.19), the proof is completed. □

Remark 5.3.1. Note we can rewrite (2.11) as follows:

$$F_0^{1D} = \tilde{I}_{-1}|_{t_{-n}=0, n \geq 1}.\tag{5.23}$$

For the thin expansion of Hermitian one-matrix model of size $N \times N$, because we have $F_0^N = N F_0^{1D}$, we also have

$$F_0^N = N \cdot \tilde{I}_{-1}|_{t_{-n}=0, n \geq 1}.\tag{5.24}$$

As a corollary, we now recover the following formula in [17, (5.12)]:

Corollary 5.3.1. *The genus zero free energy of 2D gravity satisfies the following equality:*

$$F_0^{2D} = \frac{I_0^3}{6} - \sum_{k \geq 0} \frac{I_0^{k+2}}{k!(k+2)} t_k + \frac{1}{2} \sum_{n \geq 0} \sum_{k \geq 0} \frac{I_0^{n+k+1}}{n!k!(n+k+1)} t_n t_k.\tag{5.25}$$

Proof. We have,

$$\sum_{p \geq 1} (-1)^p \tilde{I}_p \tilde{I}_{-p-1} = \sum_{p \geq 1} (-1)^p \sum_{i \geq 0} \tilde{t}_{i+p} \frac{I_0^i}{i!} \sum_{j \geq 0} \tilde{t}_{j-p-1} \frac{I_0^j}{j!}$$

$$= \sum_{n \in \mathbb{Z}} \tilde{t}_n \sum_{j-p-1=n} \sum_{i \geq 0} (-1)^p \tilde{t}_{i+p} \frac{I_0^{i+j}}{j!i!}$$

$$= \sum_{n \geq 0} \tilde{t}_n \sum_{p \geq 1} \sum_{i \geq 0} \tilde{t}_{i+p} \frac{(-1)^p I_0^{n+p+i+1}}{i!(n+p+1)!} + \sum_{n < 0} \tilde{t}_n \sum_{j \geq 0} \sum_{i \geq 0} (-1)^{j-1-n} \tilde{t}_{i+j-1-n} \frac{I_0^{i+j}}{i!j!}$$

$$= \sum_{n,k \geq 0} \tilde{t}_n \tilde{t}_k \sum_{p+i=k} \frac{(-1)^p I_0^{n+k+1}}{i!(n+p+1)!} + \sum_{n,k \geq 0} \tilde{t}_{-n-1} \tilde{t}_{k+n} \sum_{i+j=k} (-1)^{j+n} \frac{I_0^k}{i!j!}$$

$$= \sum_{n \geq 0} \sum_{k \geq 0} \frac{I_0^{n+k+1}}{n!k!(n+k+1)} \tilde{t}_n \tilde{t}_k + \sum_{n \geq 0} (-1)^n \tilde{t}_n \tilde{t}_{-n-1}.$$

Hence,

$$\frac{1}{2} \sum_{n \geq 1} (-1)^n \tilde{I}_n \tilde{I}_{-n-1}\tag{5.26}$$

$$= \frac{1}{2} \sum_{n \geq 0} \sum_{k \geq 0} \frac{I_0^{n+k+1}}{n!k!(n+k+1)} \tilde{t}_n \tilde{t}_k + \frac{1}{2} \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) t_{-n-1}$$

$$= \frac{I_0^3}{6} - \sum_{k \geq 0} \frac{I_0^{k+2}}{k!(k+2)} t_k + \frac{1}{2} \sum_{n \geq 0} \sum_{k \geq 0} \frac{I_0^{n+k+1}}{n!k!(n+k+1)} t_n t_k + \frac{1}{2} \sum_{n \geq 0} (-1)^n (t_n - \delta_{n,1}) t_{-n-1}.$$

The proof is completed by invoking the condition that $t_{-n-1} = 0$ for $n \geq 0$ to (5.20). $\qquad\square$

It is a remarkable coincidence that the right-hand side of (5.26) is almost the same as $\tilde{F}_0^{2D}$ in (5.3) used to interpret the special deformation of the Airy curve using ghost variables. Inspired by this coincidence, we have the following result.

Theorem 5.3.2. *The extension $\tilde{F}_0^{2D}$ of the free energy F_0^{2D} satisfies the following equality:*

$$\tilde{F}_0^{2D} = \frac{1}{6}I_0^3 - \frac{1}{2}\sum_{n,k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{k!n!(n+k+1)}I_nI_k + \sum_{n\geqslant 1}(-1)^n(I_n - \delta_{n,1})I_{-n-1}. \tag{5.27}$$

Proof. We have,

$$\sum_{p\geqslant 0}(-1)^p t_p t_{-p-1}$$

$$= \sum_{p\geqslant 0}(-1)^p \sum_{i\geqslant 0}\frac{(-1)^i I_0^i}{i!}I_{i+p}\sum_{j\geqslant 0}\frac{(-1)^j I_0^j}{j!}I_{j-p-1}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{j-p-1=n}(-1)^p\frac{(-1)^j I_0^j}{j!}I_{j-p-1}\sum_{i\geqslant 0}\frac{(-1)^i I_0^i}{i!}I_{i+p}$$

$$= \sum_{n\geqslant 0}I_n\sum_{p\geqslant 0}\sum_{i\geqslant 0}\frac{(-1)^{n+i+1}I_0^{n+p+i+1}}{i!(n+p+1)!}I_{i+p} + \sum_{n<0}I_n\sum_{j\geqslant 0}\sum_{i\geqslant 0}(-1)^{i-1-n}\frac{I_0^{i+j}}{i!j!}I_{i+j-1-n}$$

$$= \sum_{n\geqslant 0}I_n\sum_{k\geqslant 0}I_k\sum_{p+i=k}\frac{(-1)^{p+n+k+1}I_0^{n+k+1}}{i!(n+p+1)!} + \sum_{n\geqslant 0}I_{-n-1}\sum_{k\geqslant 0}I_{k+n}\sum_{i+j=k}(-1)^{i+n}\frac{I_0^k}{i!j!}$$

$$= \sum_{n\geqslant 0}\sum_{k\geqslant 0}\frac{(-1)^{n+k+1}I_0^{n+k+1}}{n!k!(n+k+1)}I_nI_k + \sum_{n\geqslant 0}(-1)^n I_n I_{-n-1},$$

and

$$t_{-2} = \sum_{n\geqslant 0}\frac{(-1)^n I_0^n}{n!}I_{n-2}. \tag{5.28}$$

By theorem [4.3.1](#),

$$\tilde{F}_0^{2D} = \frac{1}{6}I_0^3 - \sum_{n\geqslant 0}\frac{(-1)^n I_0^{n+2}}{(n+2)!}I_n + \frac{1}{2}\sum_{n,k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{(n+k+1)k!n!}I_nI_k + t_{-2} + \sum_{n\geqslant 0}(-1)^n t_n t_{-n-1}$$

$$= \frac{1}{6}I_0^3 - \sum_{n\geqslant 0}\frac{(-1)^n I_0^{n+2}}{(n+2)!}I_n + \frac{1}{2}\sum_{n,k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{(n+k+1)k!n!}I_nI_k$$

$$+ \sum_{n\geqslant 0}\frac{(-1)^n I_0^n}{n!}I_{n-2} + \sum_{n\geqslant 0}\sum_{k\geqslant 0}\frac{(-1)^{n+k+1}I_0^{n+k+1}}{n!k!(n+k+1)}I_nI_k + \sum_{n\geqslant 0}(-1)^n I_n I_{-n-1}$$

$$= \frac{1}{6}I_0^3 - \frac{1}{2}\sum_{n,k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{(n+k+1)k!n!}I_nI_k + I_{-2} - I_0 I_{-1} + \sum_{n\geqslant 0}(-1)^n I_n I_{-n-1}$$

$$= \frac{1}{6}I_0^3 - \frac{1}{2}\sum_{n,k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{(n+k+1)k!n!}I_nI_k + \sum_{n\geqslant 1}(-1)^n(I_n - \delta_{n,1})I_{-n-1}.$$

$\square$

5.4 Special deformation of Airy curve in I-coordinates

Recall the special deformation of Airy curve:

$$w^{2D} = z^{\frac{1}{2}} - \sum_{n=0}^{\infty}\frac{t_n}{(2n-1)!!}z^{n-\frac{1}{2}} - \sum_{n=0}^{\infty}(2n+1)!!\frac{\partial F_0^{2D}}{\partial t_n}z^{-n-\frac{3}{2}}. \tag{5.29}$$

Now we consider a related deformation in a slightly different normalization as follows:

$$y^{2D} := z^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n t_n \Gamma\left(-n+\frac{1}{2}\right) z^{n-\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\partial F_0^{2D}}{\partial t_n} \Gamma\left(n+\frac{3}{2}\right) z^{-n-\frac{3}{2}}. \quad (5.30)$$

Now we rewrite it in I-coordinates, we get the following result:

Theorem 5.4.1. *In I-coordinates, we have:*

$$y^{2D} = (z - I_0)^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=1}^{\infty} (-1)^n I_n \Gamma\left(-n + \frac{1}{2}\right) (z - I_0)^{n-\frac{1}{2}}, \quad (5.31)$$

and conversely, this property determined F_0^{2D} uniquely.

Proof. One can easily check that,

$$\frac{\partial \tilde{I}_n}{\partial t_k} = \frac{I_0^{k-n}}{(k-n)!} + \frac{I_0^k}{k!} \frac{\tilde{I}_{n+1}}{1 - I_1}, \quad (5.32)$$

for all $n, k \in \mathbb{Z}$. Therefore, by (5.20)

$$\begin{aligned}
\frac{\partial F_0^{2D}}{\partial t_k} &= \frac{1}{2} \sum_{n \geqslant 1} (-1)^n \left(\left(\frac{I_0^{k-n}}{(k-n)!} + \frac{I_0^k}{k!} \frac{\tilde{I}_{n+1}}{1-I_1} \right) \tilde{I}'_{-n-1} + \tilde{I}_n \left(\frac{I_0^{k+n+1}}{(k+n+1)!} + \frac{I_0^k}{k!} \frac{\tilde{I}'_{-n}}{1-I_1} \right) \right) \\
&= \frac{1}{2} \sum_{n \geqslant 0} (-1)^n \frac{I_0^{k-n}}{(k-n)!} \tilde{I}'_{-n-1} + \frac{1}{2} \sum_{n \geqslant 1} (-1)^n \tilde{I}_n \frac{I_0^{k+n+1}}{(k+n+1)!} \\
&= \sum_{n \geqslant 0} (-1)^n \frac{I_0^{k-n}}{(k-n)!} \tilde{I}'_{-n-1}, \quad (5.33)
\end{aligned}$$

where $\tilde{I}'_n = I_n|_{t_{-m}=0, \, m \geq 1}$. Hence,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\partial F_0^{2D}}{\partial t_k} \Gamma\left(k + \frac{3}{2}\right) z^{-k-\frac{3}{2}} &= \sum_{k,n \geqslant 0} (-1)^n \frac{I_0^{k-n}}{(k-n)!} \tilde{I}'_{-n-1} \Gamma\left(k + \frac{3}{2}\right) z^{-k-\frac{3}{2}} \\
&= \sum_{n \geqslant 0} (-1)^n \tilde{I}'_{-n-1} \sum_{k \geqslant 0} \frac{I_0^k}{k!} \Gamma\left(k + n + \frac{3}{2}\right) z^{-k-n-\frac{3}{2}} \\
&= \sum_{n \geqslant 0} (-1)^n \tilde{I}'_{-n-1} \frac{\Gamma\left(n + \frac{3}{2}\right)}{(z - I_0)^{n+\frac{3}{2}}}.
\end{aligned}$$

Together with proposition 5.2.2, one has,

$$\begin{aligned}
y^{2D} &= z^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n t_n \Gamma\left(-n+\frac{1}{2}\right) z^{n-\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\partial F_0^{2D}}{\partial t_n} \Gamma\left(n+\frac{3}{2}\right) z^{-n-\frac{3}{2}} \\
&= z^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n \in \mathbb{Z}} (-1)^n I'_n \Gamma\left(-n+\frac{1}{2}\right) (z-I_0)^{n-\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n \geqslant 0} (-1)^n \tilde{I}'_{-n-1} \frac{\Gamma\left(n+\frac{3}{2}\right)}{(z-I_0)^{n+\frac{3}{2}}} \\
&= z^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n \geqslant 0} (-1)^n I'_n \Gamma\left(-n+\frac{1}{2}\right) (z-I_0)^{n-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}} \sum_{n \geqslant 0} (-1)^n \frac{I_0^{n+2}}{(n+2)!} \frac{\Gamma\left(n+\frac{3}{2}\right)}{(z-I_0)^{n+\frac{3}{2}}} \\
&= (z-I_0)^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n \geqslant 1} (-1)^n I_n \Gamma\left(-n+\frac{1}{2}\right) (z-I_0)^{n-\frac{1}{2}}.
\end{aligned}$$

The proof is completed.

$\square$

Remark 5.4.1. When we impose the condition $t_{-n} = 0$ for $n > 0$, we have $F_0^{2D} = \tilde{F}_0^{2D}$. By theorem 5.3.2

$$F_0^{2D} = \frac{1}{6}I_0^3 - \frac{1}{2}\sum_{n,k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{(n+k+1)k!n!}I_nI_k + \sum_{n\geqslant 1}(-1)^n(I_n - \delta_{n,1})I_{-n-1},$$

if we take derivatives of F_0^{2D} with respect to $\{I_n\}_{n\in\mathbb{Z}}$ formally, we have,

$$\frac{\partial F_0^{2D}}{\partial I_n}\Big|_{t_{-m}=0,\ m\geq 1} = -\sum_{k\geqslant 0}\frac{(-1)^{n+k}I_0^{n+k+1}}{(n+k+1)k!n!}I_k + (-1)^nI_{-n-1}\Big|_{t_{-m}=0,\ m\geq 1} = 0, \qquad (5.34)$$

$$\frac{\partial F_0^{2D}}{\partial I_{-n-1}} = (-1)^nI_n, \qquad (5.35)$$

for $n \geqslant 1$. Consider the generating series

$$\sum_{n\in\mathbb{Z}-\{0\}}\frac{\partial F_0^{2D}}{\partial I_n}\Big|_{t_{-m}=0,\ m\geq 1}\cdot x^{n+1} = \sum_{n\geq 1}(-1)^n(I_n - \delta_{n,1})x^{-n}. \qquad (5.36)$$

and its Laplace transform:

$$\sum_{n\in\mathbb{Z}-\{0\}}\frac{\partial F_0^{2D}}{\partial I_n}\cdot\int_0^\infty\frac{1}{\sqrt{x}}e^{-\tilde{z}x}x^{n+1}dx = \sum_{n\geq 1}(-1)^n(I_n - \delta_{n,1})\Gamma\left(-n + \frac{1}{2}\right)\tilde{z}^{n-1/2}, \qquad (5.37)$$

we get y^{2D} by taking $\tilde{z} = z - I_0$.

6 Applications: constitutive relations

In this section we will use our results to rederive the following constitutive relations in 2D topological gravity:

$$\langle\sigma_i\sigma_j\rangle = \frac{1}{i+j+1}u^{i+j+1}. \qquad (6.1)$$

This is (2.34) derived by Dijkgraaf and Witten [9] as an example of constitutive relations derived from topological recursion relations in genus zero. We will also derive the analogous relations for F_0^{1D} and F_0^N.

6.1 Constitutive relations in 2D topological gravity

In our notations, (6.1) is just:

$$\frac{\partial^2 F_0^{2D}}{\partial t_j\partial t_k} = \frac{I_0^{k+j+1}}{k!j!(k+j+1)}. \qquad (6.2)$$

This can be proved by our results as follows. We have proved the following equation in the proof of theorem 5.4.1:

$$\frac{\partial F_0^{2D}}{\partial t_k} = \sum_{n\geqslant 0}(-1)^n\frac{I_0^{k-n}}{(k-n)!}\tilde{I}'_{-n-1}, \qquad (6.3)$$

where $\tilde{I}'_n = I_n|_{t_{-m}=0,\ m\geq 1}$. By this equation and (5.32), one has,

$$
\begin{aligned}
\frac{\partial^2 F_0^{2D}}{\partial t_j \partial t_k} &= \frac{\partial I_0}{\partial t_j} \sum_{n\geqslant 0} (-1)^n \frac{I_0^{k-1-n}}{(k-1-n)!} \tilde{I}'_{-n-1} + \sum_{n\geqslant 0} (-1)^n \frac{I_0^{k-n}}{(k-n)!} \frac{\partial \tilde{I}'_{-n-1}}{\partial t_j} \\
&= \frac{1}{1-I_1} \frac{I_0^j}{j!} \sum_{n\geqslant 0} (-1)^n \frac{I_0^{k-1-n}}{(k-1-n)!} \tilde{I}'_{-n-1} + \sum_{n\geqslant 0} (-1)^n \frac{I_0^{k-n}}{(k-n)!} \left(\frac{I_0^{j+n+1}}{(j+n+1)!} + \frac{I_0^j}{j!} \frac{\tilde{I}'_{-n}}{1-I_1} \right) \\
&= \sum_{n\geqslant 0} (-1)^n \frac{I_0^{k-n}}{(k-n)!} \frac{I_0^{j+n+1}}{(j+n+1)!} \\
&= \frac{I_0^{k+j+1}}{k!j!(k+j+1)}.
\end{aligned}
$$

This gives a proof of Constitutive Relations (6.1).

We can now interpret the Itzykson-Zuber Ansatz as given by formulas (4.11)–(4.13) from the point of view of constitutive relations. The formula for F_0^{2D} can be obtained from the genus zero n-point functions on the small phase space by changing t_0^n to $(-1)^{n-1} I_0^n$, $t_k \to I_k$ for $k \geq 1$ as follows. Because

$$
\langle \tau_0^n \rangle_0 = \delta_{n,3}, \tag{6.4}
$$

the 0-point function in genus zero on the small phase space is $\frac{t_0^3}{6}$, it gives the term $\frac{I_0^3}{6}$. The one-point function can be computed by

$$
\langle \tau_n \tau_0^m \rangle_0 = \delta_{m,n+2}, \tag{6.5}
$$

it is $\sum_{n\geq 0} \frac{t_0^{n+2}}{(n+2)!} t_n$, and so it gives the term $-\sum_{n\geq 0} \frac{(-1)^n I_0^{n+2}}{(n+2)!} I_n$. The two-point function is computed by

$$
\langle \tau_j \tau_k \tau_0^m \rangle_0 = \frac{(j+k)!}{j!k!} \delta_{m,j+k+1}, \tag{6.6}
$$

it is $\frac{1}{2} \sum_{j,k\geq 0} \frac{t_0^{j+k+1}}{j!k!(j+k+1)} t_j t_k$, and so it gives us the term $\frac{1}{2} \sum_{j,k\geq 0} \frac{(-1)^{j+k} I_0^{j+k+1}}{j!k!(j+k+1)} I_j I_k$. Since for $g \geq 1$, F_g does not involve I_0, one can further restrict to the origin $t_0 = 0$ in the small phase space, compute a few n-point functions and change t_n to I_n for $n \geq 1$. Such formulas generalize the constitutive relations in the mean field theory considerations of Dijkgraaf-Witten [9]. So our discussions suggest that renormalization naturally leads to constitutive relations.

6.2 Analogues for F_0^{1D} and F_0^N

By (5.23) and (5.32), one has:

$$
\frac{\partial F_0^{1D}}{\partial t_k} = \frac{I_0^{k+1}}{(k+1)!}. \tag{6.7}
$$

Since $F_0^N = N F_0^{1D}$, one also has,

$$
\frac{\partial F_0^N}{\partial t_k} = N \frac{I_0^{k+1}}{(k+1)!}. \tag{6.8}
$$

We regard them as the analogues of (6.1). Furthermore, the Hessians are given by:

$$\frac{\partial^2 F_0^{1D}}{\partial t_j \partial t_k} = \frac{I_0^k}{k!} \frac{\partial I_0}{\partial t_j} = \frac{1}{1 - I_1} \frac{I_0^{k+j}}{k! j!}, \tag{6.9}$$

$$\frac{\partial^2 F_0^N}{\partial t_j \partial t_k} = N \frac{I_0^k}{k!} \frac{\partial I_0}{\partial t_j} = \frac{N}{1 - I_1} \frac{I_0^{k+j}}{k! j!}. \tag{6.10}$$

Note the appearance of I_1 on the right-hand sides means F_0^{1D} and F_0^N do not satisfy the topological recursion relations, hence they do not give us topological field theories in the sense of [9].

7 Concluding remarks

In this paper, we have further studied the I-coordinates. By rewriting L_0^{2D} in I-coordinates, we have proved the Itzykson-Zuber ansatz. Furthermore, we have developed the techniques of rewriting all the Virasoro constraints for free energies in I-coordinates and solving free energies recursively.

As pointed out by the second named author in [28], we understand the I-variables as new coordinates on the big phase space. In this paper, we have checked that at least in the cases of 1D topological gravity, Hermitian one-matrix model and 2D topological gravity, the free energies have good properties in these new coordinates. We believe this is a general phenomenon and hope to make generalizations in subsequent work. We have also seen that the use of renormalized coupling constants shed some lights on the mean field theory approach to the original theories.

Furthermore, we extend the definitions of I_n for $n \geq -1$ to include I_n for all $n \in \mathbb{Z}$. These are inspired by the introduction of t_n for all $n \in \mathbb{Z}$ in [33]. They suggest to study an even larger phase space to include the ghost variables.

A surprising consequence is that we discover a connection between the emergent spectral curve of 2D topological gravity and the action of the 1D topological gravity. The special deformations of spectral curves of the three theories considered in this paper are:

$$y^{1D} = \frac{1}{\sqrt{2}} \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}}{z} + \sqrt{2} \sum_{n \geq 1} \frac{n!}{z^{n+1}} \frac{\partial F_0^{1D}}{\partial t_{n-1}}. \tag{7.1}$$

$$y^N = \frac{1}{\sqrt{2}} \sum_{n \geq 0} \frac{t_n - \delta_{n,1}}{n!} z^n + \frac{\sqrt{2}N}{z} + \sqrt{2} \sum_{n \geq 1} \frac{n!}{z^{n+1}} \frac{\partial F_0^N}{\partial t_{n-1}}. \tag{7.2}$$

$$y^{2D} = z^{\frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n t_n \Gamma\left(-n + \frac{1}{2}\right) z^{n - \frac{1}{2}} - \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\partial F_0^{2D}}{\partial t_n} \Gamma\left(n + \frac{3}{2}\right) z^{-n - \frac{3}{2}}. \tag{7.3}$$

When rewritten in the I-coordinates they take the following unified form:

$$y^{1D} = \frac{\sqrt{2}}{z - I_0} + \frac{1}{\sqrt{2}} \sum_{n \geq 1} \frac{I_n - \delta_{n,1}}{n!} (z - I_0)^n. \tag{7.4}$$

$$y^N = \frac{\sqrt{2}N}{z - I_0} + \frac{1}{\sqrt{2}} \sum_{n \geq 1} \frac{I_n - \delta_{n,1}}{n!} (z - I_0)^n. \tag{7.5}$$

$$y^{2D} = \qquad -\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{I_n - \delta_{n,1}}{\Gamma(n + \frac{1}{2})} (z - I_0)^{n - \frac{1}{2}}. \tag{7.6}$$

Furthermore, if we define their action functions by:

$$S^{1D} = \int y^{1D} dz = \sqrt{2} \log(z - I_0) + \frac{1}{\sqrt{2}} \sum_{n \geqslant 1} \frac{I_n - \delta_{n,1}}{(n+1)!} (z - I_0)^{n+1}, \tag{7.7}$$

$$S^N = \int y^N dz = \sqrt{2} N \log(z - I_0) + \frac{1}{\sqrt{2}} \sum_{n \geqslant 1} \frac{I_n - \delta_{n,1}}{(n+1)!} (z - I_0)^{n+1}, \tag{7.8}$$

$$S^{2D} = \int y^{2D} dz = -\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{I_n - \delta_{n,1}}{\Gamma\left(n + \frac{3}{2}\right)} (z - I_0)^{n+\frac{1}{2}}. \tag{7.9}$$

Then one has:

$$\frac{\partial S^{1D}}{\partial t_0} = -\frac{1}{1 - I_1} \frac{\sqrt{2}}{z - I_0} + \frac{z - I_0}{\sqrt{2}}, \tag{7.10}$$

$$\frac{\partial S^N}{\partial t_0} = -\frac{1}{1 - I_1} \frac{\sqrt{2} N}{z - I_0} + \frac{z - I_0}{\sqrt{2}}, \tag{7.11}$$

$$\frac{\partial S^{2D}}{\partial t_0} = -(z - I_0)^{\frac{1}{2}}. \tag{7.12}$$

These are deformations of spectral curves:

$$y^{1D} = \frac{\sqrt{2}}{z} - \frac{z}{\sqrt{2}}, \tag{7.13}$$

$$y^N = \frac{\sqrt{2} N}{z} - \frac{z}{\sqrt{2}}, \tag{7.14}$$

$$y^{2D} = z^{\frac{1}{2}}. \tag{7.15}$$

I.e., the spectral curves are related to the action functions defined above in the following way:

$$y^* = -\frac{\partial S^*}{\partial t_0}\bigg|_{I_0 = I_1 = 0}, \tag{7.16}$$

for $* = 1D$, N, and $2D$. We hope to understand and generalize this in future investigations.

Acknowledgments

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