# Anomaly matching, (axial) Schwinger models, and high-T super Yang-Mills domain walls

**Mohamed M. Anber**[a] **and Erich Poppitz**[b]

[a]*Department of Physics, Lewis & Clark College,*
*Portland, OR 97219, U.S.A.*

[b]*Department of Physics, University of Toronto,*
*Toronto, ON M5S 1A7, Canada*

*E-mail:* manber@lclark.edu, poppitz@physics.utoronto.ca

ABSTRACT: We study the discrete chiral- and center-symmetry 't Hooft anomaly matching in the charge-$q$ two-dimensional Schwinger model. We show that the algebra of the discrete symmetry operators involves a central extension, implying the existence of $q$ vacua, and that the chiral and center symmetries are spontaneously broken. We then argue that an axial version of the $q = 2$ model appears in the worldvolume theory on domain walls between center-symmetry breaking vacua in the high-temperature SU(2) $\mathcal{N} = 1$ super-Yang-Mills theory and that it inherits the discrete 't Hooft anomalies of the four-dimensional bulk. The Schwinger model results suggest that the high-temperature domain wall exhibits a surprisingly rich structure: it supports a non-vanishing fermion condensate and perimeter law for spacelike Wilson loops, thus mirroring many properties of the strongly coupled four-dimensional low-temperature theory. We also discuss generalizations to theories with multiple adjoint fermions and possible lattice tests.

# Contents

## 1 Introduction

Quantum field theory (QFT) is a universal paradigm for writing down the fundamental laws of nature. In many situations, however, QFTs are strongly coupled and learning about their nonperturbative behavior becomes a daunting task. One of the powerful tools that sheds light on the nonperturbative structure of QFT is 't Hooft's anomaly matching [1]. Given a QFT with a continuous or discrete global symmetry $G$, one may try to introduce a background gauge field of $G$. If the theory doesn't maintain its gauge invariance, we say that it has a 't Hooft anomaly. The anomaly is renormalization group invariant and must be matched between the infrared (IR) and ultraviolet (UV) dynamics. This matching is especially powerful in asymptotically free theories: one computes the anomaly coefficient upon gauging $G$ in the UV, where the theory is amenable to perturbative analysis. Then, this coefficient has to be matched in the IR, which puts constraints on the strongly coupled IR spectrum of the theory, see [2, 3]. If $G = G_1 \times G_2$, then it might happen that $G_1$ and $G_2$ have no 't Hooft anomalies, but the product $G_1 \times G_2$ is anomalous. In this case, we say that the theory has a mixed 't Hooft anomaly.

A global symmetry $G$ is said to be a 0-form symmetry if it acts on local operators. If $G$ acts on operators of spacetime dimension $q$, then $G$ is a $q$-form symmetry [4]. A famous example is SU($N$) Yang-Mills theory, which enjoys a 1-form $\mathbb{Z}_N^C$ center symmetry that acts on Wilson line operators. Recently, it has been realized that gauging the 1-form discrete symmetries can also be obstructed due to the existence of 't Hooft anomalies, which can provide more handles to study the phases of gauge theories [4, 5, 7–10]. In particular, non trivial constraints can be imposed on the vacua of gauge theories (including their number) that enjoy both 0- and 1-form discrete symmetries upon gauging the latter.

With the help of the recently discovered mixed 't Hooft anomaly, we scrutinize the domain walls in hot super Yang-Mills theory and its multi-adjoint nonsupersymmetric

generalizations, describe their worldvolume effective field theory, and examine the bulk-domain wall anomaly inflow. We begin the paper with a warm-up exercise: we study the charge-$q$ two-dimensional (2d) vector Schwinger model and uncover the rich structure of its vacuum. This theory has a 1-form $\mathbb{Z}_q^C$ (for $q \geq 2$) center symmetry along with a 0-form $\mathbb{Z}_{2q}^{d\chi}$ anomaly free discrete chiral symmetry. Performing a global discrete chiral transformation in the background of a $\mathbb{Z}_q^C$ 2-form gauge field multiplies the partition function by a non-trivial phase $e^{i\frac{2\pi}{q}}$, indicating the existence of a mixed $\mathbb{Z}_q^C$-$\mathbb{Z}_{2q}^{d\chi}$ 't Hooft anomaly. A consequence of this anomaly is that the ground state of the system is far from trivial. We exploit the fact that the Schwinger model is exactly solvable and explicitly construct its Hilbert space, generalizing [11, 12] to $q \geq 2$. We show that, in the quantum theory, the algebra of operators representing the $\mathbb{Z}_q^C$ and $\mathbb{Z}_{2q}^{d\chi}$ discrete symmetries is modified by a central extension, signaling the presence of a mixed anomaly, as in [5]. Further, we find that the charge-$q$ model admits $q$ distinct vacua, which saturate the mixed 't Hooft anomaly. The theory has a non vanishing fermion bilinear condensate and screens arbitrary strength electric charges [12], implying that both $\mathbb{Z}_{2q}^{d\chi}$ and $\mathbb{Z}_q^C$ are broken in the IR.

In the second part of the paper, we argue that the axial version of the charge-2 Schwinger model appears in the worldvolume theory of the domain wall (DW) in the high temperature 4d SU(2) $\mathcal{N} = 1$ super Yang-Mills (SYM) theory. The 4d theory has an anomaly-free 0-form $\mathbb{Z}_4^{d\chi}$ discrete chiral (or $R$-) symmetry and a 1-form $\mathbb{Z}_2^C$ center symmetry, with a mixed $\mathbb{Z}_4^{d\chi}$-$[\mathbb{Z}_2^C]^2$ 't Hooft anomaly. Anomaly-inflow arguments imply that this anomaly appears on the DW worldvolume as a mixed $\mathbb{Z}_4^{d\chi}$-$\mathbb{Z}_2^C$ anomaly [4, 5, 13]. Owing to the 2d nature of Schwinger model, its axial version can be mapped to the charge-2 vector model studied in the first part of the paper. The results there show that the DW inherits the bulk anomaly, which is saturated by the presence of two degenerate vacua with broken chiral and center symmetries. Thus, the $q$=2 Schwinger model results suggest that the 2d theory on the high-$T$ DW mirrors many of the properties of the strongly-coupled 4d theory at low-$T$:

1. The broken $\mathbb{Z}_4^{d\chi}$ chiral symmetry on the wall implies that the fermion bilinear condensate should be nonzero on the DW in the high-$T$ chirally restored phase, something that should be in principle measurable on the lattice. In some sense, as far as the chiral symmetry is concerned, the high-$T$ dynamics on the DW resembles the low-$T$ dynamics of the bulk, where SYM theory is known to have two vacua with a broken discrete chiral (or $R$-) $\mathbb{Z}_4^{d\chi}$ symmetry.

2. The broken $\mathbb{Z}_2^C$ one-form center symmetry on the high-$T$ DW implies a perimeter law for a fundamental Wilson loop taken to lie in the DW worldvolume. In contrast, Wilson loops in the $\mathbb{R}^3$ bulk away from the DW exhibit area law (or unbroken 1-form center symmetry). Here, we see again that the DW theory reflects properties of the low-$T$ phase: the different behavior of the Wilson loop in the bulk and on the DW mirrors the deconfinement of quarks on the DWs (i.e. perimeter law for the Wilson loop along the DW) between chiral-breaking vacua in the confined low-$T$ phase (i.e. area law in the bulk) as observed in [14], see also [15].

We find these correspondences between high-$T$ DW physics and low-$T$ bulk and DW physics quite fascinating. The matching of various anomalies and the rich DW physics uncovered make these properties worth pointing out and pursuing further.[1]

This paper is organized as follows. In section 2, we study the charge-$q$ Schwinger model, its discrete symmetries, its 't Hooft anomalies, and the anomaly saturation. In section 3, we review the DW solution in the high temperature SU(2) SYM theory and show that the worldvolume of the DW is a charge-2 axial Schwinger model. We also discuss the anomaly inflow and the manifestation of the anomaly on the DW. We conclude, in section 4, by a discussion of the generalizations to QCD(adj) with a larger number of adjoint fermions and a proposal to study the high-$T$ domain walls on the lattice.

## 2 Discrete 't Hooft anomalies in the charge-$q$ Schwinger model

Consider the charge-$q$ vector massless Schwinger model with Lagrangian[2]

$$L = -\frac{1}{4e^2}\, f_{kl} f^{kl} + i\bar{\psi}_+(\partial_- + iqA_-)\psi_+ + i\bar{\psi}_-(\partial_+ + iqA_+)\psi_- \,, \tag{2.1}$$

where $k, l = 0, 1$ are spacetime indices, $\partial_\pm \equiv \partial_t \pm \partial_x$, $A_\pm \equiv A_t \pm A_x$, $t$ and $x$ are the two-dimensional Minkowski space coordinates, $q \geq 2$ is an integer and $e$ is the gauge coupling. The spacetime metric is $g^{kl} = \text{diag}(+, -)$, and we further assume that space is compactified on a circle of circumference $L$, with $x \equiv x + L$. The fields $\psi_+$ ($\psi_-$) are the left (right) moving components of the Dirac fermion and $\bar{\psi}_\pm$ are the hermitean conjugate fields. Our notation follows from that of [12] and, as in that reference, we impose antiperiodic boundary conditions on $\psi_\pm$ around the spatial circle.[3]

The major difference of our discussion from that in [11, 12]—where the model (2.1) with $q = 1$ was solved exactly in Hamiltonian language for arbitrary values of $L$ (see also the textbook [16] which emphasizes the $eL \ll 1$ limit)—is in the assumption that $q > 1$ and in the corresponding global issues and discrete anomalies that arise.[4] Understanding the symmetry structure and anomalies of (2.1) is of interest from multiple points of view:

1. On its own, the charge-$q$ vectorlike Schwinger model (2.1) is an interesting example that provides an exactly solvable setting to study the manifestation of the recently discovered mixed discrete 0-form/1-form 't Hooft anomalies [4, 5].

2. Two-dimensional models closely related to (2.1) also appear within the framework of four-dimensional gauge theories. We show in section 3 that the axial version of

---

[1]The spirit of the correspondences outlined resembles those found in the high-$T$ DWs of pure Yang-Mills theory at $\theta = \pi$ [5] but the dynamics here appears richer.

[2]The charge-$q$ Schwinger model was also discussed in [6], but with no reference to anomalies.

[3]We note that we could also follow [11] and take the fermions periodic, with no change in the results regarding symmetry realizations and anomalies; also, the utility of Weyl fermion notation will become clear further below.

[4]We caution the reader against concluding that the value of $q$ is irrelevant: we are considering a compact U(1) theory with (light) dynamical charges with quantized charge $q > 1$. The theory can be probed with nondynamical $q = 1$ charges. One can think of the latter as of very (infinitely) massive dynamical charges.

the Schwinger model (2.1) with $q = 2$ arises as a worldvolume theory on domain walls (DWs) between center-symmetry breaking states in high-temperature SU(2) super-Yang-Mills theory, i.e. Yang-Mills theory with $n_f = 1$ adjoint Weyl fermions.

Similarly, related multi-flavor axial generalizations of (2.1) appear as worldvolume theories on hot DWs in SU(2) gauge theories with $n_f > 1$ adjoint Weyl fermions.

3. It turns out that, in all cases mentioned above, the 0-form/1-form 't Hooft anomalies lead to a rich structure of the DWs that is in principle amenable to lattice studies. As opposed to the study of $\theta = \pi$ pure Yang-Mills theories, where related anomalies arise [5, 13, 17], the sign problem does not hinder the lattice studies of these theories (in the continuum limit [18]), at least for real values of the fermion mass (of course, here the chiral limit will have to be approached). A proposal for such studies will be discussed in section 4.

## 2.1 Symmetries and mixed 't Hooft anomaly

Thus armed with reasons to study the symmetries and dynamics of (2.1), we proceed to the salient points. We begin with a discussion of the symmetries of the model (2.1). In addition to the gauged vectorlike symmetry U(1)$_V$, under which $\psi_\pm \to e^{iq\alpha}\psi_\pm$, the model has an anomalous global axial U(1)$_A$ symmetry:

$$\text{U(1)}_A: \ \psi_\pm \to e^{\pm i\chi}\psi_\pm, \text{ with anomaly free subgroup } \mathbb{Z}_{2q}^{d\chi}: \ \psi_\pm \to e^{\pm i\frac{\pi}{q}}\psi_\pm \ . \quad (2.2)$$

Under a U(1)$_A$ transformation, the fermion measure changes by a factor of $e^{i2q\chi T}$, where $T = \frac{1}{2\pi}\int f_{12}d^2x \in \mathbb{Z}$ is the integer topological charge of the gauge field; recall that we allow probes with $q = 1$ and note that we temporarily adopted Euclidean notation. Thus, for $q \geq 2$, a discrete $\mathbb{Z}_{2q}^{d\chi}$ subgroup of the U(1)$_A$ axial transformations, the anomaly free discrete chiral symmetry, survives. Under the discrete chiral symmetry $\psi_\pm$ transform with $\chi = \frac{2\pi}{2q}$, as also indicated on the r.h.s. of (2.2). Notice that for $q = 1$ there is only a fermion number symmetry and no nontrivial chiral symmetry. The $\mathbb{Z}_{2q}^{d\chi}$ symmetry (2.2) is a 0-form symmetry as it acts on the local degrees of freedom.

A further global symmetry of the $q \geq 2$ theory is the 1-form $\mathbb{Z}_q^C$ center symmetry. It does not act on any local degrees of freedom, but only on line operators, as its name suggests.[5] The $\mathbb{Z}_q^C$ 1-form center symmetry action on the Wilson loop around the spatial circle, $W \equiv e^{i\oint A_x dx}$, is to multiply it by a $\mathbb{Z}_q$ phase factor

$$\mathbb{Z}_q^C: \ e^{i\oint A_x dx} \ \to \ \omega_q \, e^{i\oint A_x dx}, \ \omega_q \equiv e^{i\frac{2\pi}{q}}. \quad (2.3)$$

Both the chiral 0-form and center 1-form discrete symmetries, (2.2) and (2.3), are exact symmetries of the quantum theory. However, they suffer a 't Hooft anomaly: gauging one of the symmetries explicitly breaks the other so that they can not be simultaneously

---

[5]This is easiest to understand on the lattice, where the global $\mathbb{Z}_q^C$ center symmetry acts by multiplying the unitary links representing the gauge field component in the $\hat{\mu}$-direction by a $\hat{\mu}$-dependent $\mathbb{Z}_q$ phase factor, very much as in (2.3). Thus, the symmetry parameter itself is a $\mathbb{Z}_q$ valued link, or a 1-form; see [4, 19, 20] for a variety of perspectives.

gauged. Gauging the 1-form $\mathbb{Z}_q^C$ center symmetry is most straightforward on the lattice: one introduces a 2-form (plaquette-based) $\mathbb{Z}_q^C$ gauge field to make the 1-form symmetry (acting on links) local.[6] In continuum language, introducing a 2-form $\mathbb{Z}_q^C$ gauge field background is equivalent, see discussion in [19], to turning on nontrivial 't Hooft fluxes, known to carry fractional topological charge $T = \frac{k}{q}$ ($k \in \mathbb{Z}$) (see [21, 22], dimensionally reduced).

Now, as argued in the paragraph after eq. (2.2), under a discrete chiral $\mathbb{Z}_{2q}^{d\chi}$ transformation, the fermion measure changes by a phase factor $e^{i2\pi T}$. This factor is unity for integer $T$, but equals $\omega_q = e^{i\frac{2\pi}{q}}$ when a fractional topological charge (a nontrivial 2-form center gauge background with $k = 1$) is introduced. The phase in the chiral transformation of the partition function in the 't Hooft flux background is the manifestation of the mixed $\mathbb{Z}_{2q}^{d\chi}$-$\mathbb{Z}_q^C$ 't Hooft anomaly. This phase is renormalization group invariant — it is independent of the volume of the spacetime torus and can also be viewed as the variation of a bulk 3d term [5, 17]. Refs. [4, 5] argued that this anomaly has to be matched by the infrared (IR) dynamics of the theory and outlined various options for the way the matching can happen.

We show below that the $\mathbb{Z}_{2q}^{d\chi}$-$\mathbb{Z}_q^C$ mixed 't Hooft anomaly in the $q \geq 2$ Schwinger model is reproduced by the IR theory in the "Goldstone" mode such that both the discrete chiral and center symmetries are spontaneously broken. We also explicitly show that the mixed anomaly in the $q \geq 2$ Schwinger model appears as a "central extension" of the algebra of the operators generating the discrete chiral $\mathbb{Z}_{2q}^{d\chi}$ and center $\mathbb{Z}_q^C$ transformations, see eq. (2.17) in the next section.[7]

## 2.2 The realization of the symmetries and their algebra

In this section, we study the realization of the discrete symmetries and their 't Hooft anomaly in the charge-$q$ Schwinger model (2.1), by borrowing the results of [11, 12]. As our focus is on the symmetry realization, we shall be mostly concerned with the properties of the ground state. Briefly, the strategy behind the first steps of the Hamiltonian solution of (2.1) in $A_t = 0$ gauge is to explicitly solve the Weyl equation in the $A_x$ background (this is possible in one space dimension) and use its eigenfunctions and eigenvalues to construct Dirac sea states. To find the physical ground state, one then imposes Gauss' law, i.e., invariance under infinitesimal gauge transformations. Finally, one demands that the vacuum states be eigenstates of the large gauge transformations $G : A_x \to e^{ig(x)}(A_x + i\partial_x)e^{-ig(x)}$, where $e^{ig(x)} \equiv e^{i\frac{2\pi x}{L}}$ is the unit winding number large gauge transformation.

To introduce some of the notation of [12], the holonomy of the gauge field around the spatial circle is $\oint A_x dx \equiv cL$, with $cL$ shifted by $2\pi$ under large gauge transformations $G$. The action of the center symmetry (2.3) on the holonomy $cL$ is

$$\mathbb{Z}_q^C : \ cL \to cL + \frac{2\pi}{q} \ . \tag{2.4}$$

---

[6] In two spacetime dimensions, there is no 3-form field strength associated to the 2-form $\mathbb{Z}_q^C$ gauge field, thus any background is necessarily topological, see e.g. [20].

[7] This is similar to the appearance of the CP/center anomaly in the quantum mechanical and field theory models of [5, 17, 23]. Enhancement of the discrete symmetry group in Yang-Mills theory at $\theta = \pi$ due to discrete 't Hooft anomaly considerations was also discussed in [24].

The Dirac sea states obeying Gauss' law can be found as was briefly outlined above. The end result is that the states are labeled by an integer $n$ and we shall simply denote them by $|n\rangle$, not displaying their dependence on $cL$; the explicit form is in [12]. The Dirac sea state $|n\rangle$ is the one where the states of all left moving particles of (gauge non-invariant) momenta $\leq \frac{2\pi(n-1)}{L}$ are occupied and the rest are empty, and, simultaneously, all states of the right moving particles of momenta $\geq \frac{2\pi n}{L}$ are occupied. This left vs. right moving "Fermi level" matching ensures validity of the Gauss' law [11, 12].

We now list the properties of the Dirac sea states $|n\rangle$ that matter to us. See [12] for precise definitions and derivations. We notice that $q > 1$ is easily incorporated and is seen to lead to important new points, see items 3, 5, and 6 below:

1. The different $|n\rangle$ states are orthogonal; their norm can be defined as unity, $\langle n|m\rangle = \delta_{mn}$.

2. Their $U(1)_V$ charge vanishes, but the chiral (or axial $U(1)_A$, recall (2.2)) charge $Q_5$, is nonzero and depends on the holonomy of the gauge field

$$Q_5|n\rangle = |n\rangle \left( 2n - \frac{qcL}{\pi} \right) . \tag{2.5}$$

The gauge field-dependence of the axial charge $Q_5$ is a reflection of the chiral anomaly. One can define a gauge-field independent $\tilde{Q}_5$ with integer eigenvalues

$$\tilde{Q}_5 \equiv Q_5 + \frac{qcL}{\pi}, \tag{2.6}$$

but this operator shifts under large gauge transformations

$$G : \tilde{Q}_5 \to \tilde{Q}_5 + 2q . \tag{2.7}$$

3. It is clear, however, that the operator

$$X_{2q} \equiv e^{i\frac{2\pi}{2q}\tilde{Q}_5} \tag{2.8}$$

is invariant under large gauge transformations. It generates the $\mathbb{Z}_{2q}^{d\chi}$ anomaly free subgroup of the chiral transformations (2.2) and acts on the $|n\rangle$ states as

$$X_{2q}|n\rangle = |n\rangle\, \omega_q^n \qquad (\omega_q \equiv e^{i\frac{2\pi}{q}}). \tag{2.9}$$

4. The Dirac sea states $|n\rangle$ are eigenstates of the fermion Hamiltonian $H^F$ in the $A_x$ background and their energies are

$$E_n^F = \frac{2\pi}{L} \left[ \frac{Q_5^2}{4} - \frac{1}{12} \right] = \frac{2\pi}{L} \left[ \frac{1}{4} \left( 2n - \frac{qcL}{\pi} \right)^2 - \frac{1}{12} \right] . \tag{2.10}$$

(Here and elsewhere we take the liberty to denote operators and eigenvalues with the same letter, hoping that this does not cause undue confusion.)

As eigenstates of the total Hamiltonian, however, the $|n\rangle$ states, supplemented by a holonomy wave function [11, 12], are degenerate; one way to see this is by noting that the holonomy fluctuations $cL$ obtain the same "mass" from the fermion vacuum energy (2.10) in all $|n\rangle$ Dirac sea states.[8]

5. Under large gauge transformations $G$, shifting $cL$ by $2\pi$, the $|n\rangle$ states are not invariant but transform into each other as

$$G|n\rangle = |n + q\rangle .\qquad(2.11)$$

6. The center symmetry $\mathbb{Z}_q^C$, a $\frac{2\pi}{q}$ shift of $cL$ (2.4), acts on the $|n\rangle$ states as

$$Y_q|n\rangle = |n + 1\rangle ,\qquad(2.12)$$

where we introduced the $Y_q$ operator, representing the center-symmetry action on the gauge field holonomy.

We are now ready, as in [11, 12], to construct states that are eigenstates of the large gauge transformations $G$. Since the $|n\rangle$ states transform as (2.11), in the $q > 1$ theory we can define $q$ different linear combinations of the $|n\rangle$ states that are eigenstates of $G$. For convenience, we introduce a $\theta$ parameter (it is unobservable in the massless theory [11]) and define the linear combinations $|\theta, k\rangle$ of the Dirac sea states as

$$|\theta, k\rangle \equiv \sum_{n \in \mathbb{Z}} e^{i(k+qn)\theta}|k + qn\rangle, \; k = 0, 1, \ldots, q - 1 .\qquad(2.13)$$

As follows from (2.11), all $|\theta, k\rangle$ states are eigenstates of $G$ with eigenvalue $e^{-iq\theta}$. We note also that $\langle\theta', k'|\theta, k\rangle = \delta_{k,k'(\mathrm{mod}\,q)} \; \delta(\theta - \theta'(\mathrm{mod}\frac{2\pi}{q}))$, with $\delta(\theta - \theta'(\mathrm{mod}\frac{2\pi}{q})) = \sum_{m \in \mathbb{Z}} e^{iqm(\theta-\theta')}$.

For further use (cluster decomposition, see below), we also define the $\mathbb{Z}_q$ Fourier transform of the basis (2.13). We denote the states of this basis by $|P, \theta\rangle$ (to not confuse them with the $|\theta, k\rangle$ states):

$$|P, \theta\rangle \equiv \frac{1}{\sqrt{q}} \sum_{k=0}^{q-1} \omega_q^{kP}|\theta, k\rangle, \; P = 0, \ldots, q - 1,$$
$$\langle P', \theta'|P, \theta\rangle = \delta_{P,P'(\mathrm{mod}\,q)} \, \delta\left(\theta - \theta'\left(\mathrm{mod}\frac{2\pi}{q}\right)\right).\qquad(2.14)$$

Clearly, the $|P, \theta\rangle$ states are also eigenstates of $G$ with the same eigenvalue $e^{-iq\theta}$. Further, (2.14), (2.13) and (2.9) imply that under the discrete chiral symmetry $\mathbb{Z}_{2q}^{d\chi}$ the $|P\rangle$ states transform cyclically into each other

$$X_{2q}\,|P, \theta\rangle = |P + 1(\mathrm{mod}\,q), \theta\rangle ,\qquad(2.15)$$

---

[8]Diagonalizing the full Hamiltonian, including fermion excitations above the Dirac sea, in the language used here involves a Bogolyubov transformation [12], but the details will not be relevant for us.

while (2.12) implies that they are eigenstates of the $\mathbb{Z}_q^C$ center symmetry

$$Y_q \, |P, \theta\rangle = |P, \theta\rangle \, \omega_q^{-P} e^{-i\theta} \, . \tag{2.16}$$

Further, following the discussion after (2.10), the $|P, \theta\rangle$ states are degenerate. The action of $X_{2q}$ and $Y_q$ found above, (2.15), (2.16), implies that, when acting on the $|P, \theta\rangle$ states,[9] they do not commute but obey the algebra

$$X_{2q} \, Y_q = \omega_q \, Y_q \, X_{2q} \quad (\omega_q = e^{i\frac{2\pi}{q}}). \tag{2.17}$$

This algebra is familiar from the 't Hooft commutation relation between Wilson and 't Hooft loop operators in $\mathrm{SU}(q)$ gauge theories [25] (the $q$-dimensional representation on the $|P, \theta\rangle$ states, (2.15), (2.16), was also found there). Here, however, one of the operators $Y_q$, being a center-symmetry generator, is indeed a (lower dimensional version of a) 't Hooft loop operator, but the other, $X_{2q}$, is not a Wilson loop but a generator of discrete chiral transformations.

The 't Hooft algebra (2.17) implies that even though the symmetries generated by $X_{2q}$ and $Y_q$ commute classically, the discrete chiral and center symmetries $\mathbb{Z}_{2q}^{d\chi}$ and $\mathbb{Z}_q^C$ do not commute in the quantum theory but instead obey (2.17). Their noncommutativity in the quantum theory signals the presence of the mixed 't Hooft anomaly.[10]

Finally, let us argue that (2.17) implies that both symmetries are spontaneously broken. The $|P, \theta\rangle$ ground states obey the cluster decomposition principle, as opposed to the $|\theta, k\rangle$ ground states. This is because the latter are mixed by local operators, the gauge invariant fermion bilinear $\phi(x) \equiv \bar{\psi}_+(x)\psi_-(x)$. The fermion bilinear has charge $-2$ under the $\mathbb{Z}_{2q}^{d\chi}$ discrete chiral symmetry (2.2) and nonzero matrix elements between the $|n\rangle$ states:

$$\langle n'|\phi(x)|n\rangle = \delta_{n',n+1} \, C' \, e^{-i\frac{2\pi x}{L}} \, , \text{ where } \phi(x) \equiv \bar{\psi}_+(x)\psi_-(x). \tag{2.18}$$

The constant $C'$ was computed in [12] in the Hamiltonian formalism for any $L$ and was shown to not vanish, including as $L \to \infty$, where $C' \sim e$. It is also clear that (2.18) is consistent with the nature of the $|n\rangle$ states explained earlier. Using the matrix elements (2.18) it is straightforward to show that $\phi(x)$ has nonzero matrix elements between different $|\theta, k\rangle$ states, $\langle \theta, k+1|\phi|\theta, k\rangle \neq 0$, but is diagonal in the $|P, \theta\rangle$ basis

$$\langle P', \theta|\phi(x)|P, \theta\rangle = e^{-i\theta} \, \omega_q^{-P} \, \delta_{P,P'} C', \tag{2.19}$$

where we took the infinite-$L$ limit on the r.h.s. Furthermore, one can use eq. (2.19) to show that correlation functions factorize

$$\langle P', \theta|\phi^\dagger(x)\phi(0)|P, \theta\rangle\big|_{x \to \infty} \to \text{const.} \ \delta_{P,P'} \langle P, \theta|\phi^\dagger(x)|P, \theta\rangle\langle P, \theta|\phi(0)|P, \theta\rangle \, , \tag{2.20}$$

and that connected correlators in the $|P, \theta\rangle$ vacua vanish as $x \to \infty$, as required by cluster decomposition.

---

[9] A slightly more careful study of the definitions of the operators from [12] shows that the algebra (2.17) holds in the entire Hilbert space.

[10] Following [5], we call the appearance of $\omega_q$ in (2.17) a "central extension" of the algebra of symmetry operators, as the new element $\omega_q$ commutes with $X_{2q}$ and $Y_q$.

To conclude, we have shown that the charge-$q$ Schwinger model has $q$ ground states, $|P, \theta\rangle$, $P = 0, 1, \ldots q - 1$, cyclically permuted by the discrete chiral $\mathbb{Z}_{2q}^{d\chi}$ symmetry, which is spontaneously broken to $\mathbb{Z}_2$, see (2.19).

To see that the 1-form $\mathbb{Z}_q^C$ symmetry is also broken in the large-$L$ limit, one can study the expectation value of a $q = 1$ Wilson loop by taking two static $q = 1$ charges some distance apart and studying the potential between them. The calculation of [12] showed that arbitrary charges are screened in the massless Schwinger model, due to vacuum polarization effects, with the screening length of order $1/e$ (in the $L \to \infty$ limit). Hence the Wilson loop obeys perimeter law, signaling the breaking of the 1-form symmetry [4].

# 3 The high-T domain wall in SU(2) super-Yang-Mills: the axial Schwinger model and symmetry realizations

In this section, we show that the Schwinger model studied above and its generalizations appear as the worldvolume theory on high-temperature DWs[11] in 4d SU(2) gauge theories with $n_f \leq 5$ massless adjoint Weyl fermions, QCD(adj).

QCD(adj) with SU(2) gauge group has a discrete anomaly free $\mathbb{Z}_{4n_f}^{d\chi}$ 0-form chiral symmetry and a $\mathbb{Z}_2^C$ 1-form center symmetry, as well as a continuous SU($n_f$) flavor symmetry. The discrete symmetries have a $\mathbb{Z}_{4n_f}^{d\chi}$-$\left[\mathbb{Z}_2^C\right]^2$ mixed 't Hooft anomaly [4, 9, 15], much like the one in the Schwinger model of the previous section. All 't Hooft anomalies have to be matched by the IR dynamics. At zero temperature, the discrete anomaly matching, combined with continuous and mixed discrete-continuous anomalies, can be used to suggest the possible existence of interesting new phases [30], see the remarks in [31]. Furthermore, anomaly inflow arguments for the discrete 't Hooft anomalies imply that there is nontrivial physics on the DWs separating vacua with broken discrete symmetries [5, 13]. For example, at zero temperature, the discrete chiral symmetry is broken, at least for small $n_f$, and DWs connecting different vacua are found to exhibit rich worldvolume physics (for example quark deconfinement) [14] dictated by the anomaly [4, 15].

Here, we focus on the high-temperature regime of SU(2) QCD(adj), in the deconfined phase, where the $\mathbb{Z}_2^C$ center symmetry associated with the Euclidean time direction is broken.[12] Here, anomaly inflow arguments also suggest that the DW between the center broken vacua inherits the bulk symmetries and discrete 't Hooft anomalies [5, 13]. Our goal is to study this in some detail and see how the 't Hooft anomalies are saturated on the DW worldvolume. We uncover a rich structure of the DWs and find an explicit connection to the Schwinger model of the previous section (for $n_f = 1$) or its generalizations (for $n_f > 1$). We begin with the Euclidean action of SU(2) Yang-Mills theory endowed with $n_f$ adjoint

---

[11]These are really Euclidean objects, see [26–29].

[12]In the terminology of [4] the symmetry broken at high-$T$ is called a 0-form center symmetry, from the point of view of the dimensionally reduced 3d theory. In addition, there is a 1-form center symmetry in the dimensionally reduced 3d theory, unbroken at high-$T$ (a fundamental Wilson loop in the spatial $\mathbb{R}^3$ exhibits area law in the deconfined phase).

Weyl fermions at finite temperature $T$:

$$S = \int_{\mathbb{R}^3 \times \mathbb{S}^1_\beta} \frac{1}{2g^2} \operatorname{tr}(F_{\mu\nu} F_{\mu\nu}) + 2i\operatorname{tr}\left(\bar\lambda \bar\sigma^\mu D_\mu \lambda\right) , \tag{3.1}$$

where $\mu, \nu = 1, 2, 3, 4$ and the trace is normalized as $\operatorname{tr}\left(t^a t^b\right) = \frac{\delta_{a,b}}{2}$ such that $t^a = \frac{\tau^a}{2}$ and $\tau^a$ are the color-space Pauli matrices. $\mathbb{S}^1_\beta$ is the thermal circle, which is taken along the $x_4$-direction and has circumference $\beta = 1/T$. The covariant derivative is given by $D_\mu \lambda = \partial_\mu \lambda - i[A_\mu, \lambda]$ and $\bar\sigma = (-i, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ are the spacetime Pauli matrices. In addition, the fermion field $\lambda$ carries an implicit flavor index; we set $n_f = 1$ for the rest of this section.[13]

At temperatures much smaller than the strong coupling scale $\Lambda_{QCD}$, the theory preserves its $\mathbb{Z}_2^C$ symmetry, the trace of the Polyakov loop vanishes $\operatorname{Tr}_F \exp\left[i \oint_{\mathbb{S}^1_\beta} A_4\right] = 0$, static charges are confined, and a Wilson loop wrapped in the time direction obeys the area law. At temperatures larger than $\Lambda_{QCD}$, many aspects of the theory become amenable to semiclassical treatment owing to asymptotic freedom. In this regime, we can dimensionally reduce the action (3.1) to 3d after integrating out a tower of heavy Matsubara excitations of the gauge and fermion fields along $\mathbb{S}^\beta_1$. To one-loop order, the resulting bosonic part of the action reads

$$S_{3D}^{\text{boson}} = \frac{\beta}{g^2} \int_{\mathbb{R}^3} \left(\frac{1}{2}\operatorname{tr}(F_{ij} F_{ij}) + \operatorname{tr}(D_i A_4)^2 + g^2 V(A_4) + \mathcal{O}\left(g^4\right)\right) , \tag{3.2}$$

where $i, j = 1, 2, 3$, $g$ is the gauge coupling at the scale $T$, and $V(A_4)$ is the one-loop effective potential for the Matsubara zero mode of the $x^4$-component of the gauge field. The potential, written below in terms of the Cartan subalgebra component $A_4^3$ and for $n_f = 1$, is given (see e.g. [32]), up to a constant, by

$$V(A_4) = -\frac{1}{12\pi\beta^4}\left[-6\pi \left(\beta A_4^3\right)^2 + 4\left(\beta A_4^3\right)^3\right] , \quad \text{for } \beta A_4^3 \in [0, \pi] , \tag{3.3}$$

where the extension to the interval $[\pi, 2\pi]$ is given by replacing $\beta A_4^3 \to 2\pi - \beta A_4^3$ in (3.3). The two minima of the potential are at $\beta A_4^3 = 0, 2\pi$, so that at $T \gg \Lambda_{QCD}$ the $\mathbb{Z}_2^C$ center symmetry along the $x^4$ direction is broken and the theory admits DWs [26, 27]. The two center-symmetry breaking vacua are characterized by nonvanishing expectation values of the trace of the Polyakov loop, $\frac{1}{2}\langle \operatorname{Tr}_F \exp\left[i \oint_{\mathbb{S}^1_\beta} A_4\right]\rangle = \pm 1$.

The DW is a solution of the equations of motion of (3.2). A DW perpendicular to $x^3$ is parameterized as $A_\mu^{\text{DW}}(x_3) = \delta_{\mu 4} T \Phi(x_3) \frac{\tau^3}{2}$, where $\frac{\tau^3}{2}$ is SU(2) Cartan generator and the profile function $\Phi(x_3)$ interpolates between 0 and $2\pi$, the two 0-form center-symmetry breaking vacua, as $x_3 \to \mp\infty$. The inverse width of the DW is of order $gT$ [26, 27]. At the DW core, we find $\Phi(x_3 = 0) = \pi$ and the trace of the Polyakov loop vanishes, $\operatorname{Tr}_F \exp\left[i \oint_{\mathbb{S}^1_\beta} A_4(x_3 = 0)\right] = 0$, restoring the 0-form center symmetry on the DW. Furthermore, the center-symmetric expectation value $\Phi(x_3 = 0) = \pi$ spontaneously breaks the

---

[13]The analysis of the zero modes in this section goes verbatim for any $n_f$, simply increasing the number of zero modes. The analysis of the DW world-volume theory, however, differs for $n_f > 1$, see section 4.

SU(2) gauge symmetry to U(1) and the off-diagonal $W$-bosons have mass $\sim T$. Thus, the DW worldvolume supports massless abelian fields. In the $\mathbb{R}^3$ bulk, on the other hand, the gauge sector (3.2) has a nonperturbative gap, of order $g^2 T$, while on the DW the $W$-bosons have a larger gap of order $T$, due to the adjoint Higgsing. Thus, in the presence of a DW, we expect that at sufficiently low energy scales (presumably below the bulk gap) the high-$T$ 3d theory dynamically compactifies to an abelian theory on the 2d worldvolume, in a manner resembling [33]. Having a 2d abelian gauge field on the worldvolume is not very interesting in pure YM theory, except at $\theta = \pi$, where it was shown to have interesting consequences [5]. Here, we show that in theories with adjoints the dynamics is even richer.

To this end, we study the fermions in the DW background and show that the DW supports two fermionic zero-modes (for $n_f = 1$). We expand the gauge fluctuations and fermion fields in Fourier modes, taking into account that the gauge field (fermions) satisfy periodic (anti-periodic) boundary conditions along $\mathbb{S}^1_\beta$:

$$A_\mu = A_\mu^{\mathrm{DW}}(x_3) + \sum_{p \in \mathbb{Z}} \left( a_{\mu, p} \frac{\tau_3}{2} + W^+_{\mu, p} \tau^+ + W^-_{\mu, p} \tau^- \right) e^{i2\pi p \frac{x_4}{\beta}},$$

$$\lambda = \sum_{p \in \mathbb{Z}} \left( \lambda^0_p \frac{\tau^3}{2} + \lambda^+_p \tau^+ + \lambda^-_p \tau^- \right) e^{i2\pi p' \frac{x_4}{\beta}}, \tag{3.4}$$

where $p' = p + \frac{1}{2}$ and $\tau^\pm = (\tau^1 \pm i\tau^2)/2$. The photon $a_{\mu, p=0}$ is the only massless mode on the DW. All other gauge modes — the W-bosons and their Kaluza-Klein excitations $W^\pm_{\mu, p}$ as well as the photons $a_{\mu, p \neq 0}$ — are massive and we neglect them in our treatment. The DW-worldvolume scalar $a_{3, p=0}$ is also expected to be massive as there is no symmetry protecting it, is uncharged under the 2d U(1), and we ignore it in what follows. Substituting (3.4) into the covariant derivative and varying the Lagrangian (3.1) with respect to $\bar{\lambda}$ we obtain the equation of motion

$$\left(2\pi p' T + \sigma^i \partial_i\right) \lambda^0_p \frac{\tau_3}{2} + \left[\left(2\pi p' - \Phi(x_3)\right) T + \sigma^i \left(\partial_i - ia_i\right)\right] \lambda^+_p \tau^+$$
$$+ \left[\left(2\pi p' + \Phi(x_3)\right) T + \sigma^i \left(\partial_i + ia_i\right)\right] \lambda^-_p \tau^- = 0, \tag{3.5}$$

where $i, j = 1, 2, 3$ and we used the notation $a_i \equiv a_{i, p=0}$ for the Matsubara zero mode of the Cartan gauge field (the U(1) photon). One can immediately see that the lightest $\lambda^0_p$ has a mass $\sim T$. A careful examination of the (charged under the unbroken U(1)) components $\lambda^\pm_p$, however, reveals that there are two zero modes on the DW. Setting $a_i = 0$ and bearing in mind that $\lambda^\pm_p$ is a two-component spinor, i.e., $\lambda^\pm_p = \begin{bmatrix} \lambda^\pm_{p, 1} \\ \lambda^\pm_{p, 2} \end{bmatrix}$, the $x^3$-dependent solution of the equation of motion of $\lambda^\pm_p$ reads

$$\lambda^\pm_{p, 1}(x_3) = \exp\left[\left(-2\pi p' x^3 \pm \int^{x_3} dz \Phi(z)\right) T\right] \lambda^\pm_{p, 1}(0),$$

$$\lambda^\pm_{p, 2}(x_3) = \exp\left[\left(2\pi p' x^3 \mp \int^{x_3} dz \Phi(z)\right) T\right] \lambda^\pm_{p, 2}(0). \tag{3.6}$$

It is easy to check that only two of these solutions, $\lambda^+_{p=0, 2}(x_3)$ and $\lambda^-_{p=-1, 1}(x_3)$, are normalizable. It is crucial for our purposes to note that these two zero modes have opposite charges under the U(1) field $a_i$ and also have opposite 2d chirality, as can be seen from (3.5).

In what follows, when writing the DW-volume theory of the zero modes, we drop the Matsubara and 4d spinor indices, and denote the two dimensional fields corresponding to the above zero modes by $\lambda_+$ and $\lambda_-$, respectively. Also, to emphasize the fact that $\lambda$ are adjoint fermions, and therefore, carry twice the fundamental charge, we make the change of variables $A_{1,2} = \frac{a_{1,2}}{2}$. Then, the effective 2d Lagrangian on the DW worldvolume is given by

$$\mathcal{L}_{\mathrm{DW}}^{\mathrm{axial}} = \frac{1}{4e^2} F_{kl} F_{kl} + i\bar{\lambda}_+ \left[\partial_1 - i\partial_2 - i2(a_1 - ia_2)\right] \lambda_+$$
$$+ i\bar{\lambda}_- \left[\partial_1 + i\partial_2 + i2(a_1 + ia_2)\right] \lambda_- . \tag{3.7}$$

where $F_{kl} = \partial_k a_l - \partial_l a_k$, $k, l = 1, 2$, and $e^2$ is the two dimensional gauge coupling.[14] The Lagrangian (3.7) describes the Euclidean axial Schwinger model of charge 2 and, from a 4d perspective, the high-$T$ DW worldvolume theory in SU(2) super-Yang-Mills (SYM) theory (QCD(adj) with $n_f = 1$).

It is interesting to note that the DW worldvolume theory (3.7) inherits the symmetries and anomalies of the bulk SYM theory. The U(1)$_A$, under which $\lambda_\pm$ transform with opposite charges, is gauged in the axial charge-2 model.[15] The U(1)$_V$, under which $\lambda_\pm$ have the same charge is anomalous, instead. There is a $\mathbb{Z}_4^{d\chi}$ discrete "chiral" (from the bulk point of view) symmetry remaining anomaly free. In addition there is a $\mathbb{Z}_2^C$ center symmetry due to the fact that the adjoint fermions carry twice the fundamental charge (this worldvolume $\mathbb{Z}_2^C$ symmetry originates from the 1-form center symmetry in the $\mathbb{R}^3$ bulk and should not be confused with the zero-form center symmetry along $x^4$). There is also a $\mathbb{Z}_4^{d\chi}$-$\mathbb{Z}_2^C$ mixed 't Hooft anomaly on the DW worldvolume, as predicted by anomaly inflow [5, 13], and as follows directly by repeating the arguments of section 2.1 for the axial model (3.7).

To more explicitly see the relation of (3.7) to the Euclidean version of the vector Schwinger model (2.1), the Lagrangian (3.7) can be brought to the vector form of Schwinger model via the change of variables $\bar{\psi}_+ = \lambda_+$, $\psi_+ = \bar{\lambda}_+$ in the + sector and a relabeling $\psi_- = \lambda_-$, $\bar{\psi}_- = \bar{\lambda}_-$ in the − sector. Performing this in (3.7), after integration by parts in the + sector we find

$$\mathcal{L}_{\mathrm{DW}}^{\mathrm{vector}} = \frac{1}{4e^2} F_{kl} F_{kl} + i\bar{\psi}_+ \left[\partial_1 - i\partial_2 + i2(a_1 - ia_2)\right] \psi_+$$
$$+ i\bar{\psi}_- \left[\partial_1 + i\partial_2 + i2(a_1 + ia_2)\right] \psi_- \tag{3.8}$$

---

[14]The localization of the abelian fields on the DW is due to nonperturbative effects in the bulk that generate a mass gap for the gauge fluctuations (in the absence of a bulk gap, the abelian gauge field in the DW background would propagate in the $\mathbb{R}^3$ bulk). Thus, we can only estimate the value of the 2d coupling $e^2$: we take $e^2 = g^2 T/\delta$, where $\delta \sim 1/g^2 T$ is the bulk confining scale, much larger than the DW width, leading to $e^2 \sim g^4 T^2$. This estimate may raise the issue of scale separation between DW and bulk dynamics: from the above estimate, nonperturbative effects in the 2d Schwinger model occur at scales $e \sim g^2 T$ which is parametrically the same as the nonperturbative bulk gap. These estimates equally apply to the $\theta = \pi$ YM case of [5, 13]. In what follows, we assume that the results from section 2 apply to the DW theory and offer the heuristic justification that the only light charged states near the DW are the $\lambda_\pm$ zero modes, charged $W^\pm$-bosons and fermions have mass of order $T$ on the DW, while the bulk confined states are uncharged.

[15]We remind the reader that gauging U(1)$_A$ is possible in 2d, due to the vector-axial duality ($\epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma^5$).

which is the Euclidean version of (2.1) with $q = 2$. Now the anomaly free $\mathbb{Z}_4^{d\chi}$ is a subgroup of the $U(1)_A$ acting on $\psi_+$ and $\psi_-$ with opposite charges, as in (2.1). We see that at the level of the Lagrangian (3.8) (where the roles of $U(1)_V$ and $U(1)_A$ are interchanged w.r.t. (3.7)) the symmetries are realized exactly as in the model (2.1).

For completeness, we should note that the worldvolume theory,[16] in addition to the terms in (3.8), allows for a single classically (and quantum mechanically, as the results of [34] imply) marginal coupling of the form

$$\mathcal{L}_{\mathrm{DW}}^{4-\mathrm{Fermi}} \sim g^2 \, \bar{\psi}_+ \psi_+ \bar{\psi}_- \psi_-. \tag{3.9}$$

This gauge invariant four-fermi coupling preserves the anomalous $U(1)_A$ symmetry, and is hence expected to be induced by perturbative loop effects in the bulk theory; we have indicated this by including the $g^2$ bulk coupling in its definition. Thus, in the high-$T$ limit, we expect that the coefficient of this term is small, but have not calculated it precisely. The results of [34] for the gauged single-flavor Thirring model, with Lagrangian given by (3.8) and (3.9), show that it has the spectrum and chiral condensate of the massless Schwinger model, with the 4-fermi coupling inducing renormalization of the boson mass and condensate. These effects are small in the limit of small four-fermi coupling and we assume that, in our DW theory, they are negligible as $\beta \to 0$.

In addition to $U(1)_A$-preserving perturbative effects leading to (3.9), bulk nonperturbative effects are known to generate 't Hooft vertices. These preserve only the $\mathbb{Z}_4^{d\chi}$ subgroup of $U(1)_A$. However, when projected on the DW worldvolume, they only induce higher-dimension non-renormalizable terms that are irrelevant from 2d perspective, in addition to being exponentially suppressed in the high-$T$ phase. This follows already from the fact that (3.9) is the only gauge invariant and 2d Euclidean invariant local four-fermi coupling in the theory (3.8).

Thus, borrowing the results of section 2, we conclude that the DW theory breaks both the worldvolume center symmetry (originating from the 1-form $\mathbb{R}^3$-bulk center symmetry) and the discrete chiral symmetry. This symmetry realization has some interesting implications:

1. The broken $\mathbb{Z}_4^{d\chi}$ chiral symmetry implies that the fermion bilinear condensate should be nonzero on the DW in the high-$T$ (chirally restored!) phase, something that should be in principle measurable on the lattice.

   As far as the chiral symmetry is concerned, the high-$T$ dynamics on the DW mirrors the low-$T$ dynamics of the bulk, where SYM theory is known to have two vacua with a broken discrete chiral (or $R$-) $\mathbb{Z}_4^{d\chi}$ symmetry.

2. The broken $\mathbb{Z}_2^C$ one-form center symmetry on the high-$T$ DW implies a perimeter law for a fundamental Wilson loop taken to lie in the DW worldvolume. In contrast, recall that a Wilson loop in the $\mathbb{R}^3$ bulk away from the DW exhibits area law.

---

[16]We stress again that the present discussion applies only to the $n_f = 1$ SYM case, see Footnote 13. We shall come back to the $n_f > 1$ case in the future.

Here, the DW theory again reflects properties of the low-$T$ phase: the different behavior of the Wilson loop in the bulk and on the DW mirrors the deconfinement of quarks on the DWs (i.e. perimeter law for the Wilson loop along the DW) between chiral-breaking vacua in the confined low-$T$ phase (i.e. area law in the bulk), as found in [14].

The above relations between low-$T$ bulk physics and high-$T$ DW physics are quite tantalizing. One can not help but wonder about their generality and speculate on their possible utility in constraining the features of difficult to study strongly-coupled low-$T$ phases from the often more easily tractable high-$T$ DW properties.

## 4  Outlook: generalizations and lattice studies

**Higher $n_f$:**  motivated by the last remark in section 3 we discuss the higher-$n_f$ generalization of our study. Theories with different number of adjoint flavors have been studied on the lattice with various motivations (a few recent references are [18, 35, 36]) and a conclusive picture of their phase structure as a function of $n_f$ in the chiral limit has not emerged yet.

As already alluded to, study of the $n_f > 1$ high-$T$ DW theories requires more work and we only give a brief qualitative discussion. The DW fermion zero modes now come in $n_f$ multiples of those in (3.8), which we label as $\psi_{+,a}$, $\psi_{-,j}$, with $a, b$ and $j, i$ denoting $\mathrm{SU}(n_f)_{L,R}$ indices, respectively. The multi flavor generalization of the minimal coupling DW Lagrangian (3.8) now has a $\mathrm{SU}(n_f)_L \times \mathrm{SU}(n_f)_R$ global symmetry, while the bulk QCD(adj) theory only has the $\mathrm{SU}(n_f)$ global chiral symmetry. In the absence of interactions other than those in (3.8), the 't Hooft anomalies of the additional nonabelian global symmetries on the worldvolume have to be matched and one expects that new massless degrees of freedom on the DW worldvolume would appear.[17]  However, similar to (3.9), four-fermi terms are allowed and will be induced by bulk loop effects on the DW worldvolume, reducing $\mathrm{SU}(n_f)_L \times \mathrm{SU}(n_f)_R$ to the diagonal subgroup, the bulk $\mathrm{SU}(n_f)$ chiral symmetry. These perturbatively induced classically marginal terms should preserve the Euclidean rotations, gauge symmetry, and anomalous chiral $\mathrm{U}(1)_A$, and hence are expected to have the form

$$\mathcal{L}_{\mathrm{DW}}^{4-\mathrm{Fermi}} \sim \bar\psi_+^a \psi_{+,b}\, \bar\psi_-^i \psi_{-,j} \left( \alpha_1\, \delta_a^b\, \delta_i^j + \alpha_2\, \delta_a^j\, \delta_i^b \right) \; . \tag{4.1}$$

The $\alpha_2$ term reduces the enhanced global symmetry of the free-fermion DW Lagrangian to the bulk $\mathrm{SU}(n_f)$ chiral symmetry.[18] Thus, as opposed to the $n_f = 1$ four-fermi term (3.9), we expect that (4.1) will affect the IR spectrum and possibly the phase structure and symmetry realization (as far as terms breaking $\mathrm{U}(1)_A$ but preserving $\mathbb{Z}_{4n_f}$, the same comment as in the $n_f = 1$ case applies — no such local terms relevant in 2d sense are allowed). The

---

[17]In fact, the solution of the multiflavor massless Schwinger models in [37] shows that the massive spectrum of the multiflavor model is universal while a nonuniversal massless sector matches the nonabelian flavor 't Hooft anomalies.

[18]We note that for $n_f=2$ one can also write an invariant using $\epsilon^{bj}\epsilon_{ai}$ and stress that we have neither calculated nor studied the effect of any of the bulk-induced four-fermi terms. We also note that recently ref. [31] pointed out subtleties related to additional 't Hooft anomalies for $n_f = 2$.

higher-$n_f$ DW worldvolume theories have, as for $n_f = 1$, a mixed $\mathbb{Z}_{4n_f}$-$\mathbb{Z}_2^C$ 't Hooft anomaly, exactly as in the bulk theory, while the $SU(n_f)$ global symmetry preserved by (4.1) has no (mixed) anomaly in 2d.

We expect that the multi flavor model with (4.1) added will break the discrete chiral and center symmetries to match the anomaly. In light of the observed (so far, for $n_f = 1$) tantalizing similarities between the behavior of the high-$T$ domain wall worldvolume theory and the low-$T$ bulk theory, it would be worthwhile to study this further. It would be especially interesting to see whether there is any $n_f$ dependence of the DW worldvolume symmetry realization mirroring that of the low-$T$ bulk theory.

An extension of our studies to higher numbers of colors would also be of interest.

**Possible lattice studies:** one of the more surprising indications of our analysis is that there should be a nonzero fermion condensate and perimeter law for a (necessarily spacelike) Wilson loop in the worldvolume of the high-$T$ DWs — all of this in the deconfined, chirally symmetric phase.

These effects should be, at least in principle, observable in lattice studies. As lattice simulations are always performed at finite fermion mass, a chiral limit would have to be approached. While this is a difficult task, at least there is no sign problem (in the continuum limit, as opposed to $\theta = \pi$ pure gauge theories) for real values of the fermion mass. DW backgrounds in the high-$T$ phase can be induced by imposing appropriate twisted boundary conditions. For example, inserting a $\mathbb{Z}_2^C$ phase in the action of a single gauge plaquette in the $x_1$-$x_4$ plane ($x_4 \equiv x_4 + \beta$), for all $x_2$ and $x_3$, would, in the center-broken phase, induce a DW with an $x_2$-$x_3$ worldvolume (such configurations are also known as high-$T$ center vortices [19]; in this language, our studies amount to saying that they have a rather rich structure in theories with adjoint fermions). Needless to say, we leave the judgment regarding the feasibility of such studies to lattice experts. We only note that our analytic studies can be generalized to account for a small nonzero fermion mass. For example, to leading order in the fermion mass $m$, the degeneracy of the $|P, \theta\rangle$ vacua (2.14) in the charge-$q$ Schwinger model is lifted, $E(P) \sim C'|m| \cos(\theta + \frac{2\pi P}{q})$, except for $\theta = \frac{\pi}{q}$, where $\theta$ now includes the phase of $m$.

**Other theories with center symmetry:** our final remark concerns the effects of mixed discrete chiral/center symmetry anomalies in other gauge theories. While theories with light or massless fundamental fermions have no (not even approximate) center symmetry, there has been recent interest in theories with $n_f$ two-index symmetric (or antisymmetric) tensor Dirac fermions (see [38] and references therein). Other theories with center symmetry are discussed in [32]. The two-index symmetric (antisymmetric) tensor theories, with an even number of colors, have a $\mathbb{Z}_2^C$ 1-form center symmetry and an anomaly free $\mathbb{Z}_{n_f(2N+4)}^{d\chi}$ ($\mathbb{Z}_{n_f(2N-4)}^{d\chi}$) 0-form discrete chiral symmetry with a mixed 't Hooft anomaly. It would be interesting to study the matching of this anomaly in the various phases of these theories.

## Acknowledgments

## References

[1] G. 't Hooft in *Recent developments in gauge theories*, G. 't Hofft et al. eds., NATO Sci. Ser. B volume 59, Springer, Germany (1980).

[2] Y. Frishman, A. Schwimmer, T. Banks and S. Yankielowicz, *The axial anomaly and the bound state spectrum in confining theories*, *Nucl. Phys.* **B 177** (1981) 157 [INSPIRE].

[3] C. Csáki and H. Murayama, *Discrete anomaly matching*, *Nucl. Phys.* **B 515** (1998) 114 [hep-th/9710105] [INSPIRE].

[4] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized global symmetries*, *JHEP* **02** (2015) 172 [arXiv:1412.5148] [INSPIRE].

[5] D. Gaiotto, A. Kapustin, Z. Komargodski and N. Seiberg, *Theta, time reversal and temperature*, *JHEP* **05** (2017) 091 [arXiv:1703.00501] [INSPIRE].

[6] T.H. Hansson, H.B. Nielsen and I. Zahed, *QED with unequal charges: a study of spontaneous $Z_n$ symmetry breaking*, *Nucl. Phys.* **B 451** (1995) 162 [*Erratum ibid.* **B 456** (1995) 757] [hep-ph/9405324 [INSPIRE].

[7] T. Sulejmanpasic and Y. Tanizaki, *C-P-T anomaly matching in bosonic quantum field theory and spin chains*, *Phys. Rev.* **B 97** (2018) 144201 [arXiv:1802.02153] [INSPIRE].

[8] Y. Tanizaki and T. Sulejmanpasic, *Anomaly and global inconsistency matching: θ-angles, $SU(3)/U(1)^2$ nonlinear σ-model, SU(3) chains and its generalizations*, arXiv:1805.11423 [INSPIRE].

[9] H. Shimizu and K. Yonekura, *Anomaly constraints on deconfinement and chiral phase transition*, *Phys. Rev.* **D 97** (2018) 105011 [arXiv:1706.06104] [INSPIRE].

[10] Y. Tanizaki, T. Misumi and N. Sakai, *Circle compactification and 't Hooft anomaly*, *JHEP* **12** (2017) 056 [arXiv:1710.08923] [INSPIRE].

[11] N.S. Manton, *The Schwinger model and its axial anomaly*, *Annals Phys.* **159** (1985) 220 [INSPIRE].

[12] S. Iso and H. Murayama, *Hamiltonian formulation of the Schwinger model: nonconfinement and screening of the charge*, *Prog. Theor. Phys.* **84** (1990) 142 [INSPIRE].

[13] D. Gaiotto, Z. Komargodski and N. Seiberg, *Time-reversal breaking in QCD₄, walls and dualities in 2 + 1 dimensions*, *JHEP* **01** (2018) 110 [arXiv:1708.06806] [INSPIRE].

[14] M.M. Anber, E. Poppitz and T. Sulejmanpasic, *Strings from domain walls in supersymmetric Yang-Mills theory and adjoint QCD*, *Phys. Rev.* **D 92** (2015) 021701 [arXiv:1501.06773] [INSPIRE].

[15] Z. Komargodski, T. Sulejmanpasic and M. Unsal, *Walls, anomalies and deconfinement in quantum antiferromagnets*, *Phys. Rev.* **B 97** (2018) 054418 [arXiv:1706.05731] [INSPIRE].

[16] M. Shifman, *Advanced topics in quantum field theory*, Cambridge University Press, Cambridge, U.K. (2012).

[17] Z. Komargodski, A. Sharon, R. Thorngren and X. Zhou, *Comments on abelian Higgs models and persistent order*, arXiv:1705.04786 [INSPIRE].

[18] G. Bergner, S. Piemonte and M. Unsal, *Adiabatic continuity and confinement in supersymmetric Yang-Mills theory on the lattice*, arXiv:1806.10894 [INSPIRE].

[19] J. Greensite, *An introduction to the confinement problem*, *Lect. Notes Phys.* **821** (2011) 1 [INSPIRE].

[20] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and duality*, *JHEP* **04** (2014) 001 [arXiv:1401.0740] [INSPIRE].

[21] G. 't Hooft, *A property of electric and magnetic flux in nonabelian gauge theories*, *Nucl. Phys.* **B 153** (1979) 141 [INSPIRE].

[22] G. 't Hooft, *Aspects of quark confinement*, *Phys. Scripta* **24** (1981) 841 [INSPIRE].

[23] Y. Kikuchi and Y. Tanizaki, *Global inconsistency, 't Hooft anomaly and level crossing in quantum mechanics*, *PTEP* **2017** (2017) 113B05 [arXiv:1708.01962] [INSPIRE].

[24] K. Aitken, A. Cherman and M. Unsal, *Dihedral symmetry in* SU(N) *Yang-Mills theory*, arXiv:1804.05845 [INSPIRE].

[25] G. 't Hooft, *On the phase transition towards permanent quark confinement*, *Nucl. Phys.* **B 138** (1978) 1 [INSPIRE].

[26] T. Bhattacharya, A. Gocksch, C. Korthals Altes and R.D. Pisarski, *Interface tension in an* SU(N) *gauge theory at high temperature*, *Phys. Rev. Lett.* **66** (1991) 998 [INSPIRE].

[27] T. Bhattacharya, A. Gocksch, C. Korthals Altes and R.D. Pisarski, *Z(N) interface tension in a hot* SU(N) *gauge theory*, *Nucl. Phys.* **B 383** (1992) 497 [hep-ph/9205231] [INSPIRE].

[28] A.V. Smilga, *Are Z(N) bubbles really there?*, *Annals Phys.* **234** (1994) 1 [INSPIRE].

[29] C. Korthals-Altes, A. Kovner and M.A. Stephanov, *Spatial 't Hooft loop, hot QCD and Z(N) domain walls*, *Phys. Lett.* **B 469** (1999) 205 [hep-ph/9909516] [INSPIRE].

[30] M.M. Anber and E. Poppitz, *Two-flavor adjoint QCD*, *Phys. Rev.* **D 98** (2018) 034026 [arXiv:1805.12290] [INSPIRE].

[31] C. Cordova and T.T. Dumitrescu, *Candidate phases for* SU(2) *adjoint QCD$_4$ with two flavors from* $\mathcal{N} = 2$ *supersymmetric Yang-Mills theory*, arXiv:1806.09592 [INSPIRE].

[32] M.M. Anber and L. Vincent-Genod, *Classification of compactified su(N_c) gauge theories with fermions in all representations*, *JHEP* **12** (2017) 028 [arXiv:1704.08277] [INSPIRE].

[33] G.R. Dvali and M.A. Shifman, *Dynamical compactification as a mechanism of spontaneous supersymmetry breaking*, *Nucl. Phys.* **B 504** (1997) 127 [hep-th/9611213] [INSPIRE].

[34] I. Sachs and A. Wipf, *Generalized Thirring models*, *Annals Phys.* **249** (1996) 380 [hep-th/9508142] [INSPIRE].

[35] A. Athenodorou, E. Bennett, G. Bergner and B. Lucini, *Infrared regime of* SU(2) *with one adjoint Dirac flavor*, *Phys. Rev.* **D 91** (2015) 114508 [arXiv:1412.5994] [INSPIRE].

[36] S. Ali et al., *The light bound states of* $\mathcal{N} = 1$ *supersymmetric* SU(3) *Yang-Mills theory on the lattice*, *JHEP* **03** (2018) 113 [arXiv:1801.08062] [INSPIRE].

[37] D. Kutasov and A. Schwimmer, *Universality in two-dimensional gauge theory*, *Nucl. Phys.* **B 442** (1995) 447 [hep-th/9501024] [INSPIRE].

[38] Z. Fodor et al., *Status of a minimal composite Higgs theory*, *Int. J. Mod. Phys.* **A 32** (2017) 1747001 [INSPIRE].