

Asymptotic fragility, near AdS₂ holography and $T\bar{T}$

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ABSTRACT: We present the exact solution for the scattering problem in the flat space Jackiw-Teitelboim (JT) gravity coupled to an arbitrary quantum field theory. JT gravity results in a gravitational dressing of field theoretical scattering amplitudes. The exact expression for the dressed S -matrix was previously known as a solvable example of a novel UV asymptotic behavior, dubbed asymptotic fragility. This dressing is equivalent to the $T\bar{T}$ deformation of the initial quantum field theory. JT gravity coupled to a single massless boson provides a promising action formulation for an integrable approximation to the worldsheet theory of confining strings in 3D gluodynamics. We also derive the dressed S -matrix as a flat space limit of the near AdS₂ holography. We show that in order to preserve the flat space unitarity the conventional Schwarzian dressing of boundary correlators needs to be slightly extended. Finally, we propose a new simple expression for flat space amplitudes of massive particles in terms of correlators of holographic CFT's.

KEYWORDS: 2D Gravity, AdS-CFT Correspondence, Field Theories in Lower Dimensions, Integrable Field Theories

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1 Introduction and summary

Physics in two space-time dimensions provides a useful playground to test ideas about strongly coupled quantum field theory and gravity in a safe child proof environment. In spite of a long history, studies of two-dimensional systems continue to provide new insights. The current study is triggered by the following three recent developments in the two-dimensional world. The earliest one is the gravitational dressing introduced in [1, 2], which is an explicit S -matrix construction of UV complete quantum theories exhibiting a novel type of high energy behavior, dubbed asymptotic fragility. The second is the recent progress in resolving the puzzles of AdS₂/CFT₁ correspondence, resulting in the Schwarzian description of the NAdS₂ holography (where N stands for *Near*) [3–10]. Last but not least, a solvable *irrelevant* deformation by the $T\bar{T}$ operator, present in any local two-dimensional quantum field theory, has been identified recently in [11, 12].

The main goal of the present paper is to argue that these three developments are very closely related. Namely, the gravitational dressing procedure is the S -matrix definition of the $T\bar{T}$ deformation. The resulting theory can be obtained as a flat space limit of NAdS₂ holography and provides a solution to the flat space Jackiw-Teitelboim (JT) gravity [13, 14].

In the remainder of this introductory section we present a lightning review of gravitational dressing, NAdS₂ holography and $T\bar{T}$ deformation and present heuristic arguments indicating that the three are closely related. In the rest of the paper we present more detailed and solid (to the extent we succeeded) arguments for the above equivalence.

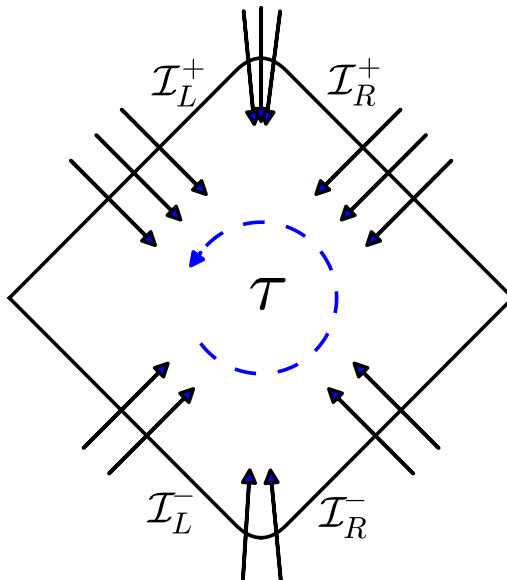


Figure 1. Particles at the asymptotic boundary of Minkowski space. Arrows correspond to on-shell two-momenta. τ is the coordinate parametrizing the boundary and ordering is given by the rapidities as described in the text.

The gravitational dressing operates as follows. Let us start with an arbitrary quantum field theory in two dimensions, which can be defined by its S -matrix elements $S(\{p_i\})$, where all momenta are taken to be incoming. Then the gravitationally dressed S -matrix is defined as

$$\hat{S}(\{p_i\}) = S(\{p_i\}) e^{\frac{i\ell^2}{4} \sum_{i < j} p_i * p_j}, \tag{1.1}$$

where ℓ^2 is a parameter characterizing the dressing,

$$p_i * p_j = \epsilon_{\alpha\beta} p_i^\alpha p_j^\beta, \quad \epsilon^{01} = -\epsilon^{10} = 1,$$

and the momenta are ordered according to the corresponding rapidities β_i : $i < j$ if i and j are both incoming particles and $\beta_i > \beta_j$, or if they are both outgoing and $\beta_i < \beta_j$, or else if i is incoming and j outgoing, see figure 1. The claim is that dressed amplitudes satisfy all the requirements expected for a healthy S -matrix. Their high energy behavior, however, is incompatible with the existence of a UV fixed point. Instead, a dressed theory exhibits many features expected from a gravitational theory, rather than from a conventional quantum field theory. In particular, as discussed in detail in [1], one does not expect to find sharply defined local observables in a dressed theory. The first and, perhaps, the most important example of dressing is provided by the worldsheet theory of an infinitely long critical (super)string, which in the Polyakov formalism indeed takes the form of a two-dimensional gravity.

The dressing formula (1.1) was obtained as a result of a guesswork. One of our results here is to provide a systematic derivation for the dressing. It is useful to have in mind the

following path integral presentation for the dressed S -matrix,

$$\hat{S}(\{p_i\}) = \int \mathcal{D}X^\alpha S(\{p_i\}) e^{iS_{\text{CS}}[X^\alpha] + i\sum_i p_{i\alpha} X^\alpha(\tau_i)}, \quad (1.2)$$

where

$$S_{\text{CS}} = \ell^{-2} \oint d\tau \epsilon_{\alpha\beta} X^\alpha \partial_\tau X^\beta \quad (1.3)$$

is the action of a Chern-Simons quantum mechanics, which can be thought of as living at the boundary of the Minkowski space-time. This presentation is inspired by the derivation of the eikonal scattering in higher-dimensional gravity presented in [15] and by the similarity between the dressing phases in (1.1) and the gravitational eikonal phase (for massless scattering they are exactly the same).

Let us turn now to the NAdS₂ holography. The results of [7–9], relevant for the present discussion, can be summarized as follows. Let us start with a quantum field theory in the rigid AdS₂ space. Then one may define a set of boundary correlators. Then the effect of introducing dynamical gravity in the bulk with the action of the JT form,

$$S_{\text{JT}} = \int d^2\sigma \sqrt{-g} \left(\phi \left(R + \frac{2}{L^2} \right) - \Lambda \right) + 2\phi_b \oint_{\mathcal{C}} du \sqrt{g_{uu}} K \quad (1.4)$$

can be described by the following dressing formula

$$\langle \mathcal{V}_1(u_1) \dots \mathcal{V}_n(u_n) \rangle_{\text{JT}} = \int \mathcal{D}t e^{(\Lambda L^2 - 2\phi_b) \oint_{\mathcal{C}} du \text{Sch}(t(u))} \prod_i (t'(u_i)^{\Delta_i}) \langle \mathcal{V}_1(t(u_1)) \dots \mathcal{V}_n(t(u_n)) \rangle, \quad (1.5)$$

where the Schwarzian $\text{Sch}(t(u))$ is defined as

$$\text{Sch}(t(u)) = \left(\frac{t''}{t'} \right)' - \frac{1}{2} \left(\frac{t''}{t'} \right)^2. \quad (1.6)$$

Note, that our notations here are different from the rest of NAdS₂ literature. Conventionally, the vacuum energy Λ is set to zero, and the dressing is parametrized by the boundary value of the dilaton ϕ_b . As explained in more detail in section 3.2, the two descriptions are equivalent, because at finite AdS₂ length L the vacuum energy can be traded for ϕ_b by performing a constant field shift $\phi \rightarrow \phi + \text{const}$. We keep Λ in (1.5) and will set

$$\phi_b = 0, \quad (1.7)$$

because eventually we are interested in the flat limit $L \rightarrow \infty$, where such a redefinition is impossible. An interesting aspect of the expression (1.5) is that the dressed generating functional Z_{JT} is not conformally invariant. This is the reason one refers to this setup as NAdS₂ holography, rather than AdS₂/CFT₁ correspondence.

Finally, the $T\bar{T}$ deformation operates as follows [11, 12] (a 3D holographic picture of this deformation was proposed in [16]). Based on the special remarkable properties [17] of the operator¹

$$T\bar{T} \equiv \frac{1}{2} \left(T_{\alpha\beta} T^{\alpha\beta} - T_\alpha^{\alpha 2} \right)$$

¹For a conformal theory this operator reduces to the product of holomorphic and antiholomorphic components of the energy-momentum tensor $T\bar{T}$, which explains its name. Away from a fixed point the operator is equal to $T\bar{T} - \frac{1}{4}\Theta^2$, where Θ is the trace of the energy-momentum tensor.

it was observed that an *arbitrary* quantum field theory deformed by $T\bar{T}$ gives rise to an RG trajectory, which is exactly solvable in the following sense. If $E_0(n, P, R)$ is the energy spectrum of the undeformed theory put on a circle of size R , with P being a spatial momentum, then the finite volume spectrum of the deformed theory is given by the solution of the following partial differential equation

$$\partial_\lambda E_\lambda(n, P, R) = E_\lambda(n, P, R) \partial_R E_\lambda(n, PR) + \frac{P(R)^2}{R}. \tag{1.8}$$

Here λ is the parameter of the deformation, so that the undeformed spectrum serves as the initial condition at $\lambda = 0$. Note that the unusual property of this deformation is that $T\bar{T}$ is an irrelevant operator, so the RG trajectory shoots from IR into UV.

As already said, the principal claim of the current paper is that the gravitational dressing and the $T\bar{T}$ deformation are two different descriptions of the same RG flow, which can be described also by coupling the undeformed theory to the flat space (i.e., $L \rightarrow \infty$) JT gravity. The parameters of the three constructions are identified as

$$\ell^{-2} = -\frac{\Lambda}{2} = \lambda.$$

Let us explain why this result is natural to expect. The similarity between the flat space gravitational dressing and the Schwarzian dressing in NAdS₂ holography is manifest. In both cases all the effects of gravity can be described by introducing a coupling between the non-gravitational asymptotic observables — be it S -matrix elements or boundary correlators — and a boundary quantum mechanics. Furthermore, it is easy to see that the boundary Chern-Simons theory (1.3) arises naturally in the JT gravity. Indeed, in the bulk the JT dilaton ϕ plays the role of a Lagrange multiplier, ensuring that the metric is flat (at $L = \infty$). Hence, at $L = \infty$ the path integral describing a matter system coupled to the JT gravity can be written schematically as

$$Z = \int \mathcal{D}X^a \mathcal{D}\psi e^{i \int d^2\sigma \sqrt{-g_f} (-\Lambda + \mathcal{L}_m(\psi, g_f))}, \tag{1.9}$$

where \mathcal{L}_m is the matter Lagrangian and g_f is a general flat metric, which can be presented as

$$g_{f\alpha\beta} = \partial_\alpha X^a \partial_\beta X^b \eta_{ab}.$$

Hence, the vacuum energy term turns into the action of a topological theory; when integrated by parts the latter is exactly the boundary Chern-Simons quantum mechanics (1.3) with

$$\ell^{-2} = -\frac{\Lambda}{2}. \tag{1.10}$$

Of course, this heuristic argument falls short of the derivation that the flat space JT gravity results in the gravitationally dressed amplitudes. In section 2 we will present an actual proof that this is the case. We will show how the dressing phase shift arises from the JT action perturbatively at the leading order in ℓ^2 expansion, as well as present a non-perturbative derivation of the full dressing phase.

Let us turn now to the $T\bar{T}$ deformation. There are several pieces of evidence strongly indicating that it is equivalent to the gravitational dressing. In the very first example of dressing — the critical string — the finite volume spectrum is known exactly and coincides with the one given by the $T\bar{T}$ deformation, as first pointed out in [18]. This agreement extends to deformations of conformal field theories (CFT) and of integrable models, where the effect of dressing on the finite volume spectrum can be described through the Thermodynamic Bethe Ansatz (TBA), see [11, 12] for the explicit discussion in the sine-Gordon case. In addition, as we will see in section 2, the $T\bar{T}$ operator arises at the leading order in perturbation theory based on the JT action for an arbitrary matter sector.

It appears that the JT gravity provides a tractable path integral formulation for the deformation, which should allow to directly calculate the deformed finite volume spectrum beyond the perturbation theory. Unfortunately, due to several (hopefully) technical difficulties, we did not manage to complete this calculation so far. We hope to report it later in a separate publication. In section 4 we present an analogous calculation for the critical string both to illustrate the idea and because we find this calculation instructive on its own. This calculation leads us to an integral which exhibits the localization property, i.e. it is 1-loop exact. As we explain in section 4 there are several reasons to expect that there is a localization story underlying also the general case of gravitational dressing/ $T\bar{T}$ deformation/JT gravity.

It will be very satisfactory to have a derivation of this sort. However, already at this point it would be very puzzling if gravitational dressing and $T\bar{T}$ turned out to be different deformations. Not surprisingly, identification of the two sheds light on both of them. On the one hand, the $T\bar{T}$ description provides the *raison d'être* for the gravitational dressing of an arbitrary quantum field theory — existence of the $T\bar{T}$ operator and its factorization properties proven in [17] rely neither on the conformal invariance nor on integrability. On the other hand, the S -matrix description of gravitational dressing serves as a rigorous definition of what $T\bar{T}$ deformation is. Indeed, the operator approach of [11, 12] is not rigorously justified in a situation when the RG flow is defined by an irrelevant deformation and the resulting theory admits neither a UV fixed point nor well-defined local operators. In fact, this description leaves one wonder whether the resulting theory is UV complete. The S -matrix definition eliminates these doubts by providing well-defined and healthy amplitudes at all collision energies.

The rest of the paper is organized as follows. In section 2 we solve the JT gravity directly in the flat space. We start with identifying the proper 2d Poincaré symmetry of the resulting S -matrix. This allows to identify also the corresponding dynamical physical coordinates — “clocks and rods” — of the JT gravity. Then we show how the exact dressing formula follows from the transformation explicitly implementing these clocks and rods as the physical coordinates. Finally we demonstrate how the $T\bar{T}$ operator arises at the leading order in perturbation theory.

In section 3 we provide two alternative derivations of the dressed scattering, both motivated by the NAdS₂ holography. In subsection 3.1 we provide another flat space derivation, which is strongly guided by the NAdS₂ calculations. In subsection 3.2 we derive the dressing by taking the flat limit of Schwarzian correlators. Both derivations are

somewhat subtle and exhibit certain ambiguities, which would be a bit challenging to fix without knowing the expected answer. Perhaps, they should be considered as an exercise on how to take the flat space limit of boundary correlators, rather than bona fide independent derivations. In fact, we did learn from these derivations two somewhat surprising lessons.

First, in order to reproduce the unitary S -matrix (1.1) as a flat space limit of the boundary NAdS₂ correlators, the Schwarzian action needs to be slightly modified. Namely, one needs to perform an additional integration over the total length of the boundary, which results in averaging over the Schwarzian coupling constant with a certain weight. Without this averaging unitarity is lost in the flat space limit. It will be interesting to understand implications of this for the NAdS₂ holography itself.

Second, as a byproduct, in subsection 3.2.2 we obtained a surprisingly simple and direct relation (3.39) between the flat space S -matrix and boundary AdS correlators for the case of scattering of massive particles. Results of section 3 may be considered as a consistency check of this relation. Once the validity of (3.39) is firmly established, it will be interesting to study the resulting implications of the boundary OPE for the flat space S -matrix.

Section 4 is somewhat disjoint from the rest of the paper. We present there the calculation of the finite volume spectrum in the critical string case, which we hope to extend to the general JT case in the future. We present our conclusions and comment on the implications of the presented results for the worldsheet theory of the QCD string in section 5.

2 Solving the flat space JT gravity

2.1 The exact S -matrix from dynamical coordinates

The goal of the present section is to describe the solution of the flat space (i.e., $L = \infty$) JT gravity minimally coupled to an arbitrary matter sector. The action of this theory is given by

$$S_{\text{JT+matter}} = \int d^2\sigma \sqrt{-g} (\phi R - \Lambda) + S_{\text{matter}}[\psi, g_{\alpha\beta}]. \tag{2.1}$$

We stress that the matter fields, schematically denoted by ψ , do not directly couple to the dilaton. This is a different coupling than, for example, in [19] where matter is coupled to $\phi^{-1}g_{\alpha\beta}$ instead. By solution we mean the exact S -matrix describing scattering around the vacuum of the JT gravity given by

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2.2}$$

$$\phi = -\frac{\Lambda}{4} \eta_{\alpha\beta} \sigma^\alpha \sigma^\beta + c = \frac{\Lambda}{2} \sigma^+ \sigma^- + c, \tag{2.3}$$

where we introduced the light cone coordinates

$$\sigma^\pm = \frac{1}{\sqrt{2}} (\sigma^0 \pm \sigma^1).$$

The zero mode c does not affect the present discussion, but has to be kept track of in the path integral approach of the next section. To avoid confusion, note that, of course, it is

impossible to calculate the exact S -matrix of an arbitrary matter sector. Instead, our goal is to calculate how it gets modified due to turning on the JT gravity. The JT gravity does not bring in any new local degrees of freedom — the JT dilaton plays the role of a Lagrange multiplier, which kills the only candidate propagating mode of the metric. As we will see now the only role of the JT gravity is to provide a dynamical system of coordinates.

The first indication for this comes from the following consideration. Why does one expect to find a Poincaré invariant S -matrix for the scattering around the vacuum (2.2), (2.3), given that the dilaton field in this vacuum has non-trivial space-time dependence? To see the answer it is convenient to fix the conformal gauge,

$$g_{\alpha\beta} = e^{2\Omega} \eta_{\alpha\beta}.$$

Then the JT action reduces to

$$S_{\text{JT}} = \int d\sigma^+ d\sigma^- (4\phi \partial_+ \partial_- \Omega - \Lambda e^{2\Omega}).$$

This action is invariant under arbitrary holomorphic and antiholomorphic shifts of ϕ ,

$$\phi \rightarrow \phi + f(\sigma^+) + g(\sigma^-).$$

We see now that the vacuum (2.2), (2.3) is invariant under the combination of the coordinate translations with the Galilean shifts of the dilaton

$$\begin{aligned} \sigma^\pm &\rightarrow \sigma^\pm + a^\pm \\ \phi &\rightarrow \phi - \frac{\Lambda}{2}(a^+ \sigma^- + a^- \sigma^+). \end{aligned} \tag{2.4}$$

Note that, at least for the purpose of the scattering problem, this prescription is well-defined, because the conformal gauge fixing on a plane does not leave any residual gauge freedom if one also imposes $g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$ at infinity. The only conformal transformations preserving this property are those from the Poincaré subgroup. The latter is a physical global (rather than gauge) symmetry of the scattering.²

We do not expect to find an analogue of (2.4), which would make it possible to restore a symmetry of the JT vacuum in the AdS₂ case. Indeed, the important difference between the $L \rightarrow \infty$ JT model considered here and the NAdS₂ story is that in the latter case the boundary observables are not invariant under the AdS₂ isometries. However, given that in section 3 we find that the Schwarzian prescription needs to be extended to obtain a unitary flat space S -matrix, it is important to carefully check this expectation.

Returning to the flat space analysis, the situation is analogous to the worldsheet scattering for the critical string. There one starts with a long string background

$$X^0 = \sigma^0, \quad X^1 = \sigma^1,$$

which is invariant under the combined shift of the worldsheet and target space coordinates. Then, using X^0 and X^1 as physical coordinates, one obtains the worldsheet S -matrix

²It will be interesting to study what is the analogue of the BMS symmetry in this case.

described by the dressing formula (1.1), applied to a system of 24 massless free bosons representing the transverse string coordinates [1]. In the Polyakov description this setup is especially similar to the one encountered in the JT gravity: in addition to transverse fields X^i , representing propagating physical modes, one finds an additional topological sector consisting of the Polyakov metric and $X^{0,1}$, whose only role is to provide a dynamical set of coordinates. A detailed derivation of the worldsheet S -matrix in the Polyakov formalism³ is presented in [20]. Let us see what happens if one treats the JT gravity in a similar way.

In the conformal gauge the JT field equations for the metric and the dilaton take the following form

$$\partial_+^2 \phi = -\frac{1}{2} T_{++} \tag{2.5}$$

$$\partial_-^2 \phi = -\frac{1}{2} T_{--} \tag{2.6}$$

$$\partial_+ \partial_- \phi = \frac{1}{2} (\Lambda e^{2\Omega} + T_{+-}) \tag{2.7}$$

$$\partial_+ \partial_- \Omega = 0, \tag{2.8}$$

where the energy-momentum tensor is defined here as

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\alpha\beta}}.$$

With the above boundary conditions for the metric one finds $\Omega = 0$ everywhere. One treats the system (2.5)–(2.8) as a set of operator equations. The matter dynamics in σ^\pm coordinates remains exactly the same as in the absence of gravity. Just as in the Polyakov case, dressing arises as a consequence of a coordinate change. Namely, we introduce new dynamical coordinates defined in the conformal gauge as

$$X^\pm = 2 \frac{\partial_\mp \phi}{\Lambda} \equiv \sigma^\pm + Y^\pm,$$

where at the last step we separated the vacuum contribution. The motivation for this choice is that, just as the target space coordinates of a string, X^\pm shift by a constant under the physical Poincaré translations (2.4) and reduce to σ^\pm at the vacuum.

Then eqs. (2.5), (2.6), (2.7) turn into

$$\partial_+ Y^- = -\frac{T_{++}}{\Lambda}, \tag{2.9}$$

$$\partial_- Y^+ = -\frac{T_{--}}{\Lambda}, \tag{2.10}$$

$$\partial_+ Y^+ = \partial_- Y^- = \frac{T_{+-}}{\Lambda}. \tag{2.11}$$

Here eqs. (2.11) ensure that given Y^\pm one can always find the corresponding dilaton field ϕ . As a consequence of the energy-momentum conservation the system (2.9), (2.10), (2.11)

³Following the idea by Juan Maldacena.

always admits a solution. This solution is uniquely defined up to a constant shift. It is natural to pick this constant in a parity symmetric way, so that

$$Y^\pm(\sigma^1 \rightarrow -\infty) = \mp \frac{P^\pm}{2\Lambda},$$

where

$$P^\pm = \int_{-\infty}^{\infty} d\sigma^\mp T_{\mp\mp}$$

are the light cone components of the total 2-momentum carried by the matter. Then, as a consequence of (2.9), (2.10),

$$Y^\pm(\sigma^1 \rightarrow \infty) = \pm \frac{P^\pm}{2\Lambda}.$$

Consider now a general scattering process taking a set of on-shell momenta $\{p_i\}$ into a set $\{q_j\}$,⁴ see figure 2. Let us focus on incoming particles, the argument for outgoing ones is identical. Let us order the incoming momenta by the corresponding rapidities β_i 's, so that $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$. For in-states this order is equivalent to the order of particles in space. Then integrating eqs. (2.9), (2.10) at $\sigma^0 \rightarrow -\infty$ we find that at early times

$$Y^\pm(\sigma^0 \rightarrow -\infty, \sigma^1) = \frac{1}{2\Lambda} (\mp P^\pm \pm 2P_L^\pm(\sigma)).$$

Here $P_L^\pm(\sigma)$ is the total momentum of all particles on the left of σ^1 at a given σ^0 . This definition is ambiguous when σ coincides with a position of one of the particles, $\sigma = \sigma_i$. It is natural to define it there following the central value prescription

$$P_L^\pm(\sigma_i) = \frac{p_i^\pm}{2} + \sum_{j < i} p_j^\pm.$$

With this prescription $Y^\pm(\sigma_i)$ are independent of p_i . As we will see momentarily, this is very reasonable physically, because it eliminates contributions into the phase shift which would correspond to particles acting upon themselves. Explicitly, at the position of the i -th particle these operators are given by

$$Y^-(p_i) = \frac{1}{2\Lambda} (\mathcal{P}_{<}^-(p_i) - \mathcal{P}_{>}^-(p_i)) \tag{2.12}$$

$$Y^+(p_i) = \frac{1}{2\Lambda} (\mathcal{P}_{>}^+(p_i) - \mathcal{P}_{<}^+(p_i)), \tag{2.13}$$

where $\mathcal{P}_{<}^\alpha(p_i)$ ($\mathcal{P}_{>}^\alpha(p_i)$) is an operator which calculates the total momentum of all particles with smaller (larger) rapidities as compared to β_i . Note that we replaced the dependence of the operators Y^\pm on σ_i coordinates with the dependence on a particle momentum p_i . This is justified because Y^\pm depend only on the spatial order of particles, which is completely determined by the rapidities for the in-states.

⁴Unlike previously, here we do not treat all momenta as incoming. $\{q_j\}$ is a set of physical momenta without a sign flip.

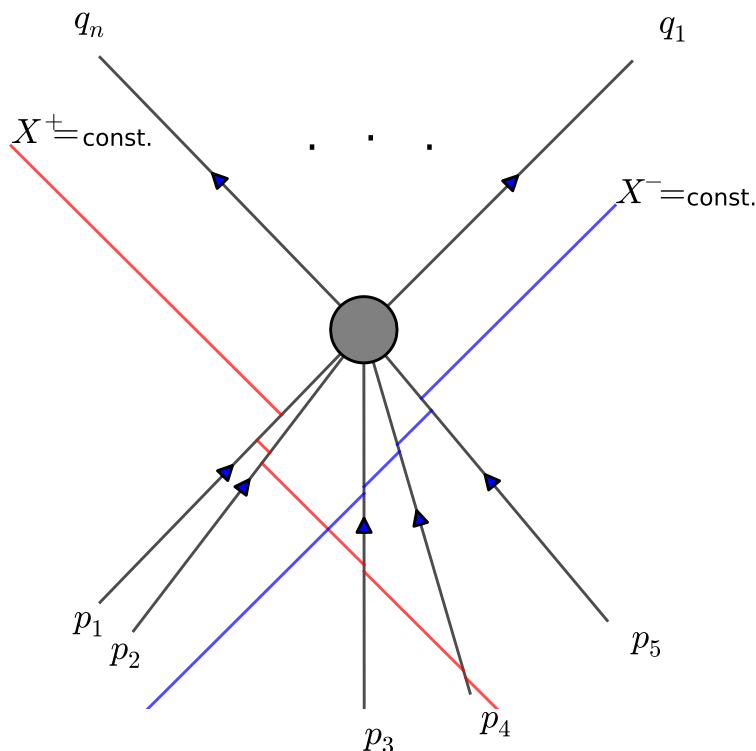


Figure 2. Lines of constant X^+ and X^- as seen in the σ coordinates.

Before introducing gravity any matter field ψ can be decomposed as

$$\psi = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \frac{1}{\sqrt{2E}} \left(a_{\text{in}}^\dagger(p) e^{-ip_\alpha \sigma^\alpha} + \text{h.c.} \right) \tag{2.14}$$

in the asymptotic region $\sigma^0 \rightarrow -\infty$.

To describe the effect of the JT gravity we need to define the creation operators $A_{\text{in}}^\dagger(p)$ using the dynamical coordinates X^\pm rather than σ^\pm . This amounts to

$$A_{\text{in}}^\dagger(p) = a_{\text{in}}^\dagger(p) e^{ip_\alpha Y^\alpha(p)} = a_{\text{in}}^\dagger(p) e^{-i(p^+ Y^-(p) + p^- Y^+(p))}. \tag{2.15}$$

It is straightforward to check that these operators commute

$$[A_{\text{in}}^\dagger(p), A_{\text{in}}^\dagger(p')] = 0,$$

as it should be for creation operators. Hence, when creating an in-state, we can put them in an arbitrary order. It is convenient to order them according to the rapidities, so that

$$|\{p_i\}, \text{in}\rangle_{\text{dressed}} = \prod_{i=1}^{n_{\text{in}}} A_{\text{in}}^\dagger(p_i) |0\rangle = e^{-\frac{i}{2\lambda} \sum_{i < j} p_i^* p_j} |\{p_i\}, \text{in}\rangle.$$

The argument for out-states proceeds in exactly the same way, but results in an opposite sign in the final answer for Y^\pm (i.e., in the analogues of (2.12), (2.13)). Indeed, out-going

particles are *antiordered* in space according to their rapidities, which translates into a sign flip in the dressing phase,

$$|\{q_i\}, \text{out}\rangle_{\text{dressed}} = \prod_{i=1}^{n_{\text{out}}} A_{\text{out}}^\dagger(q_i)|0\rangle = e^{\frac{i}{2\lambda} \sum_{i<j} q_i^* q_j} |\{q_i\}, \text{out}\rangle.$$

Finally for the dressed S -matrix we get

$$\hat{S} \equiv {}_{\text{dressed}}\langle \text{out}, \{q_i\} | \{p_i\}, \text{in} \rangle_{\text{dressed}} = \langle \text{out}, \{q_i\} | \{p_i\}, \text{in} \rangle e^{-\frac{i}{2\lambda} \sum_{i<j} q_i^* q_j} e^{-\frac{i}{2\lambda} \sum_{i<j} p_i^* p_j}$$

or, equivalently, it can be written as the operator product

$$\hat{S} = USU, \tag{2.16}$$

where

$$U|\{p_i\}\rangle = e^{-\frac{i}{2\lambda} \sum_{i<j} p_i^* p_j} |\{p_i\}\rangle.$$

As discussed in [2], as a consequence of the momentum conservation, (2.16) is equivalent to (1.1) provided the relation (1.10) holds. Note that in the form (2.16) the unitarity of the dressed S -matrix is manifest, while the initial expression (1.1) is explicitly crossing symmetric.

2.2 Perturbative scattering and the $T\bar{T}$

The presented derivation of the S -matrix proceeds in a somewhat unconventional way. It is instructive to see how the gravitational scattering arises perturbatively in a more familiar language. Namely, let us consider the quadratic action for small metric $h_{\alpha\beta}$ and dilaton perturbations φ around the vacuum (2). It takes the following form

$$\begin{aligned} S_{\text{JT}}^{(2)} = & \int \varphi (\partial_+^2 h_{--} + \partial_-^2 h_{++} - 2\partial_+ \partial_- h_{+-}) + \frac{\Lambda}{4} (h_{++} h_{--} - 2h_{+-}^2) \\ & + \frac{\Lambda}{4} (\sigma^+ h_{++} (2\partial_- h_{+-} - \partial_+ h_{--}) + \sigma^- h_{--} (2\partial_+ h_{+-} - \partial_- h_{++})) \\ & + \frac{1}{2} h_{++} T_{--} + \frac{1}{2} h_{--} T_{++} + h_{+-} T_{+-}. \end{aligned} \tag{2.17}$$

Metric and dilaton do not contain any propagating degrees of freedom. Hence, we can exclude them using their field equations and this will lead to a local interaction for matter fields. The field equations following from the quadratic action (2.17) are

$$\partial_+^2 h_{--} + \partial_-^2 h_{++} - 2\partial_+ \partial_- h_{+-} = 0 \tag{2.18}$$

$$\partial_+^2 \varphi + \frac{\Lambda}{4} ((2 + \sigma^+ \partial_+ - \sigma^- \partial_-) h_{++} + 2\sigma^- \partial_+ h_{+-}) = -\frac{1}{2} T_{++} \tag{2.19}$$

$$\partial_-^2 \varphi + \frac{\Lambda}{4} ((2 + \sigma^- \partial_- - \sigma^+ \partial_+) h_{--} + 2\sigma^+ \partial_- h_{+-}) = -\frac{1}{2} T_{--} \tag{2.20}$$

$$\partial_+ \partial_- \varphi + \frac{\Lambda}{4} (2h_{+-} - \sigma^+ \partial_- h_{++} - \sigma^- \partial_+ h_{--}) = \frac{1}{2} T_{+-}. \tag{2.21}$$

A solution to these equations is not unique due to gauge ambiguity. In particular, one can fix the conformal gauge $h_{++} = h_{--} = 0$, which also implies $h_{+-} = 0$ and brings us to

the same situation as before in section 2.1, where the scattering arises from introducing dynamical coordinates. Instead, here we are looking for an analogue of the static gauge for the Nambu-Goto string, where the worldsheet scattering comes out directly. The metric solution, which accomplishes this, is

$$h_{\alpha\beta} = -\frac{2}{\Lambda}(T_{\alpha\beta} - \eta_{\alpha\beta}T_{\gamma}^{\gamma}). \tag{2.22}$$

To see that this provides a solution, note that as a consequence of the energy-momentum conservation

$$\partial_+h_{--} - \partial_-h_{+-} = \partial_-h_{++} - \partial_+h_{+-},$$

which implies also the flatness condition (2.18). Furthermore, after plugging (2.22) in (2.19), (2.20), (2.21) one arrives at the following set of equations for the dilaton,

$$\partial_{\alpha}\partial_{\beta}\varphi = \frac{1}{2}(1 + \sigma^{\gamma}\partial_{\gamma})T_{\alpha\beta}. \tag{2.23}$$

These equations are solved by

$$\partial_+\varphi = \frac{1}{2}(\sigma^+T_{++} - \sigma^-T_{+-}) \tag{2.24}$$

$$\partial_-\varphi = \frac{1}{2}(\sigma^+T_{--} - \sigma^+T_{+-}). \tag{2.25}$$

It is straightforward to check that (2.24), (2.25) satisfy the integrability condition

$$\partial_+\partial_-\varphi = \partial_-\partial_+\varphi.$$

By plugging (2.22) back into the action (2.17) we find that (at the leading order in $1/\Lambda$) the effect of the JT gravity is equivalent to deforming the matter action by

$$S_{T\bar{T}} = -\frac{1}{2\Lambda} \int (T_{\alpha\beta}T^{\alpha\beta} - T_{\gamma}^{\gamma 2}), \tag{2.26}$$

which is nothing but the $T\bar{T}$ deformation.

One can check, using the quadratic part of the stress-energy tensor, that this deformation reproduces the first nontrivial term in the dressing formula (1.1). The agreement with the non-perturbative derivation of section 2.1 can be understood from the fact that the solution (2.22) is what one would obtain by starting from the conformal gauge and performing a diffeomorphism with parameter Y^{α} of the previous section.

3 Gravitational dressing as a flat limit of NAdS₂ holography

In this section we establish a link between the NAdS₂ holography for the JT gravity and the gravitational dressing of 2D S -matrices. We start with a more detailed version of the argument stated in the Introduction that says that the effect of the flat space JT gravity on the S -matrix can be expressed as a functional integral similar to (1.2) and present a more careful treatment of this formula. In section 3.2 we turn to the holographic formulation of the JT gravity in AdS₂ given by the Schwarzian dressing (1.5). At this point it is natural

to expect that flat space limit of AdS boundary correlators will reproduce dressing of the S -matrix obtained directly in flat space. There are several obstacles that complicate the limit, for example conformal symmetry is broken by the Schwarzian contribution. However, with some guidance from the flat space “holographic” derivation of section 3.1, we present a procedure that indeed leads to the anticipated result. Interestingly, this requires a slight modification of the conventional Schwarzian dressing.

3.1 “Holographic” derivation of gravitational dressing

S -matrix is a natural asymptotic observable in flat space. However, so far there is no well-established procedure to express it purely in terms of a boundary theory. In order to proceed we will consider our theory placed in a finite region of the Minkowski space. The boundary of this region can be roughly thought of as the position of an observer’s lab which prepares initial states and measures final ones. To define the S -matrix elements we first calculate boundary Greens functions, then take the boundary to infinity and apply the LSZ formula.

It is convenient to perform the Wick rotation and to work in the Euclidean flat space.⁵ This will prepare us for the AdS₂ case, where the Euclidean formulation is almost a necessity, because massive particles do not reach the AdS₂ boundary.

After setting the metric to

$$g_{\alpha\beta} = \partial_\alpha X^a \partial_\beta X^b \eta_{ab}$$

and rewriting the area term as a boundary contribution we arrive at the following Euclidean action,

$$iS_{JT} = -\bar{S}_{JT} = -\frac{\Lambda}{2} \oint du \left(\dot{\bar{X}}^0 \bar{X}^1 - \dot{\bar{X}}^1 \bar{X}^0 \right), \tag{3.1}$$

where \bar{X} ’s are the coordinates of the boundary, which parametrize large diffeomorphisms. The boundary curve is fixed by the metric equation of motion to be at $\phi = \phi_b$.

One puzzling aspect of the heuristic argument presented in the Introduction is that the result (1.2) is coming from the vicinity of a trivial saddle point $\bar{X} = 0$, which does not have any transparent physical interpretation. On physical grounds one expects the path integral to be dominated by semiclassical configurations corresponding to curves with unit winding. However, the action (3.1) does not have any saddle points in sectors with windings: integration over the constant mode of ϕ in (2.3) changes the size of the curve and the area bounded by any non-trivial curve always decreases as the curve shrinks to zero.

This shortcoming can be fixed by adding to the boundary action a counterterm proportional to the length of the boundary,

$$\bar{S}_{ct} = -\Lambda R_0 \oint du \sqrt{g_{uu}}, \tag{3.2}$$

where g_{uu} is the induced metric on the boundary, and R_0 is a free parameter. It is convenient to fix reparametrizations along the boundary by requiring the induced metric to be

⁵We denote all Euclidean variables with a bar and employ the standard convention for the Wick rotation, $\bar{X}^0 = iX^0$, $\bar{X}^1 = X^1$.

constant,

$$g_{uu} = \delta_{ab} \dot{X}^a \dot{X}^b = R^2, \tag{3.3}$$

where R is a modulus, which determines the total length of the boundary. It is directly related to the constant mode of ϕ . The background solution for the boundary position in the absence of any matter sources is a circle of radius R_0 . This setup is inspired by the holographic prescription employed in the NAdS₂ case in [8].

This “holographic” formulation with a dynamical boundary makes it relatively straightforward to understand how asymptotic observables of the original $2d$ QFT are modified in the presence of the JT gravity. Here it is convenient to use the path integral formulation of the theory, rather than the operator language of section 2.1. The calculation can be done in two steps. We first notice that for any fixed boundary curve the boundary correlators, resulting from the path integral over all matter fields, are identical to those of the original QFT. Next, we integrate over the boundary fluctuations using the action (3.1) — together with the tadpole canceling counterterm (3.2) — to take into account the response of the dilaton to matter fluctuations. This gives rise to the gravitational dressing.

It is convenient to introduce radial coordinates

$$\bar{X}^0 = r \cos \theta, \quad \bar{X}^1 = r \sin(-\theta), \tag{3.4}$$

so that the Chern-Simons action (3.1) turns into

$$\bar{S}_{\text{JT}} = \frac{\Lambda}{2} \oint du r^2 \theta', \tag{3.5}$$

and the condition on the induced boundary metric (3.3) implies

$$r = \frac{1}{\theta'} \sqrt{R^2 - r'^2}. \tag{3.6}$$

Consider small perturbations around the vacuum solution $r = R = R_0$, $\theta = u$,

$$\theta = u + \varepsilon(u), \tag{3.7}$$

where

$$\varepsilon(0) = \varepsilon(2\pi).$$

Using (3.6) to express perturbations of r through ε and expanding to second order in perturbations one finds the following expression for the total boundary action

$$\bar{S}_b = -\pi\Lambda R_0^2 + \pi\Lambda(R - R_0)^2 + \frac{\Lambda R^2}{2} \oint du (\varepsilon'^2 - \varepsilon''^2 + O(\varepsilon^3)). \tag{3.8}$$

For the purpose of calculating the S -matrix we are interested in the $R_0 \rightarrow \infty$ limit. In this limit $(\Lambda R_0^2)^{-1}$ acts as a weak coupling parameter and the action (3.8) turns free.

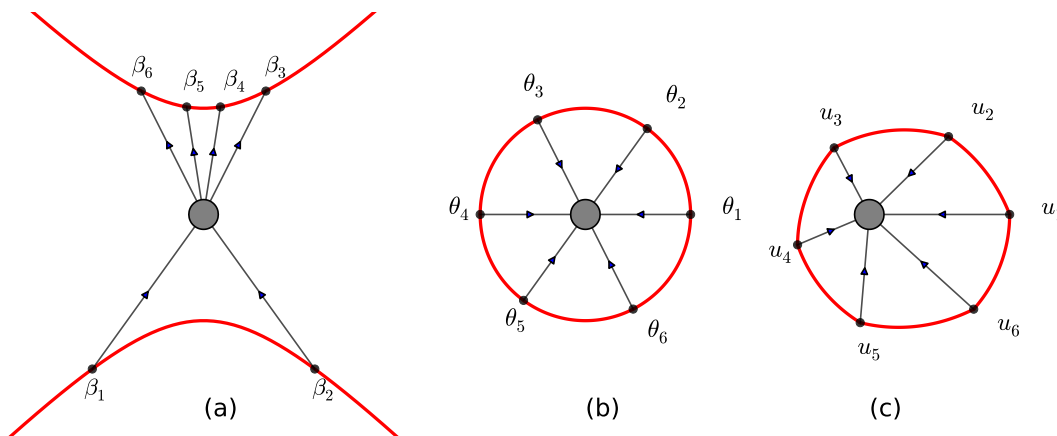


Figure 3. (a) Scattering process in the Lorentzian space with LSZ surfaces taken to be hyperbolas, the particles intersect the surfaces at hyperbolic angles determined by their rapidities. (b) The same process but rotated to the Euclidean space. Positions of the particles on the boundary circle are determined by their Euclidean rapidities $\theta_i = \bar{\beta}_i + \pi$. (c) Coupling to JT gravity results into the deformation of the boundary surface. Positions on the boundary are given by $u_i = \bar{\beta}_i + \pi$. The grey bulb on all three figures represents the interaction region \mathcal{M}_{int} .

3.1.1 LSZ recap

Before proceeding further, let us recall some details of the standard flat space LSZ formalism. It relates the S -matrix elements to the time-ordered correlation functions. In its final covariant form, the LSZ prescription requires knowledge of correlators over the entire space-time. However, to derive the S -matrix it is actually enough to know correlation functions of fields inserted on two Cauchy slices in the limit of infinite time separation. Following e.g. [21], for the n -particle scattering amplitude $S_n(\{p_i\})$ we can write⁶

$$S_n(\{p_i\}) = \lim_{T \rightarrow \infty} \int \prod_i^n dv_i e^{i \sum_i p_i \cdot X(v_i)} \prod_i^n (n_i \cdot \overleftrightarrow{\partial}_{X_i}) G_n(\{X(v_i)\}), \tag{3.9}$$

where p_i are on-shell momenta, assumed to have negative energies for out-states. $X^\alpha(v)$ belong to the surfaces where in- and out-states are created, the surfaces are parametrized by v and n_i are the corresponding unit normal vectors at points v_i , and T is the minimal time separation between in- and out-surfaces. Since both in- and out-fields as well as external wave functions $e^{ip_i \cdot X_i}$ satisfy the Klein-Gordon equation the above expression is independent of the choice of these surfaces. Conventionally, they are taken as two flat constant time hyperplanes. Instead, for our purposes it is convenient to consider hyperbolic spacelike surfaces, which give a connected Euclidean circle of a size $R_0 = T$ after the Wick rotation (see figure 3).

Let us parametrize the momenta via rapidities,

$$p^0 = m \cosh \beta, \quad p^1 = m \sinh \beta, \tag{3.10}$$

⁶To keep the equations short, we assume that all wave function normalization constants are set to unity.

where for out-states $\text{Im } \beta = \pi$. To keep the Euclidean path integral real, let us analytically continue rapidities to purely imaginary values $\beta = -i\bar{\beta}$, which corresponds to $\bar{p}^0 = p^0$ and $\bar{p}^1 = ip^1$. This continuation is distinct from the conventional Wick rotation. It is however a convenient analytic continuation of external kinematical data. Since S -matrix is an analytic function of rapidities, we are perfectly allowed to calculate it for Euclidean rapidities $\bar{\beta}$ in the $(0, 2\pi)$ interval and then analytically continue back into the physical region. We thus arrive at the ‘‘Euclidean’’ LSZ formula,

$$S_n(\{\bar{\beta}_i\}) = \lim_{R_0 \rightarrow \infty} \int \prod_i^n (m_i R_0 d\theta_i) e^{-R_0 m_i \cos(\theta_i - \bar{\beta}_i)} \prod_i^n (n_i \cdot \overleftrightarrow{\partial}_{X_i}) G_n(\{R_0, \theta_i\}), \quad (3.11)$$

where $G_n(\{\bar{X}\})$ is the n -point function of sources located on the circle of radius R_0 .

Note that, as always, in expressions like (3.11) one really has in mind the in- and out-wave functions to be well localized wave packets rather than true plane waves. These wave packets can be arranged in such a way that all interactions take place in a region \mathcal{M}_{int} of a finite characteristic size L_{int} , such that

$$p_i^{-1} \ll L_{\text{int}} \ll R_0.$$

Then for large R_0 the Green’s functions can be approximated by expressions of the form

$$G_n(\bar{X}_i) \simeq \int_{\mathcal{M}_{\text{int}}} d\bar{x} \prod_i^n G_2(\bar{x} - \bar{X}_i) \mathcal{K}_n(\{\partial_{\bar{X}_i}\}). \quad (3.12)$$

The meaning of this schematic formula is that quasilocal kernels $\mathcal{K}_n(\{\partial_{\bar{X}_i}\})$ determine the momentum structure of the scattering amplitudes, when convoluted with the external wave packets. For large distances the external propagators take the following asymptotic form

$$G_2(\bar{x} - \bar{X}_i) = K_0(m_i |\bar{x} - \bar{X}_i|) \sim \frac{e^{-m_i |\bar{x} - \bar{X}_i|}}{\sqrt{R_0}} \sim \frac{e^{-m_i (R_0 - \bar{X}_i \cdot \bar{x} / R_0)}}{\sqrt{R_0}}. \quad (3.13)$$

Keeping only the leading piece in the exponent in (3.13) one finds that the integral over θ_i in (3.11) localizes around $\theta_i = \bar{\beta}_i + \pi$. In the Lorentzian signature this localization corresponds to the fact that particles enter the asymptotic region from hyperbolic angles opposite to their rapidities, see figure 3(a).

The subleading pieces in the exponent combine to form

$$\exp \left(\sum_i - (m_i \cos \bar{\beta}_i \bar{x}^0 - m_i \sin \bar{\beta}_i \bar{x}^1) \right),$$

which in the Lorentzian signature turns into $\exp(\sum_i ip_i \cdot x)$, ensuring the energy-momentum conservation after integration over x . In the Euclidean signature this factor is equal to unity by definition, because the analytic continuation is always performed over a conserved set of momenta. Also, at large R_0 all the derivatives in the $\prod_i n_i \cdot \overleftrightarrow{\partial}_{X_i}$ factor in (3.11) can be taken to act on the external wave functions, turning this factor into $\prod_i -\bar{n}_i \cdot \bar{p}_i$. With this, the LSZ formula becomes holographic in the sense that it only depends on correlators restricted to the boundary.

3.1.2 Gravitational dressing

Let us come back now to our gravitational problem. Given the previous discussion the effect of the JT gravity amounts to averaging the Green's functions over the positions of the source insertions with the weight determined by the boundary action (3.8). Hence the expression for the dressed S -matrix reads

$$\hat{S}_n(\{\bar{\beta}_i\}) = \lim_{R_0 \rightarrow \infty} \int d\bar{x} dR \mathcal{D}\varepsilon e^{-\bar{S}_b(\varepsilon, R)} \prod_i \left(-R_0 du_i e^{-R_0 m_i \cos(u_i - \bar{\beta}_i)} \bar{n}_i \cdot \bar{p}_i G_2(r(u_i), \theta(u_i), \bar{x}) \right) \mathcal{K}_n(\{\partial_{\bar{X}_i}\}). \quad (3.14)$$

Note, that $r(u_i), \theta(u_i)$ now depend on the dynamical variables ε, R through (3.6), (3.7). This rather lengthy expression gets significantly simplified in the large R_0 limit. Indeed, as we discussed after (3.8), in this limit $(\Lambda R_0^2)^{-1}$ acts as a small coupling constant, and also the integral over R gets localized in the vicinity of R_0 . Hence, in this limit the dependence on ε only survives in the terms where ε is multiplied either by R_0 or R . Then, using (3.13), the leading part of the external two-point Greens functions reduce to

$$G_2 \sim \frac{e^{-mr}}{\sqrt{R}} \sim \frac{e^{-mR+mR\varepsilon'(u)}}{\sqrt{R}}, \quad (3.15)$$

which gives rise to a source for both ε and R . As a result we arrive at the following compact expression for the dressed S -matrix

$$\hat{S}(\{\bar{\beta}_i\}) = \mathcal{N} \int dR e^{-\pi\Lambda(R-R_0)^2} \int \mathcal{D}\varepsilon e^{-\frac{\Lambda R_0^2}{2} \oint du (\varepsilon'^2 - \varepsilon''^2) + \oint du \sum_i m_i (R_0 \varepsilon' - R + R_0) \delta(u - \bar{\beta}_i - \pi)} S(\{\bar{\beta}_i\}). \quad (3.16)$$

Here, as in the flat case, we took into account that the $\prod du_i$ integral localizes at $u_i = \bar{\beta}_i + \pi$, and \mathcal{N} is an overall constant normalization factor. We see that the integral over ε is Gaussian and is factored out. We evaluate the path integral over ε in the appendix A and after analytic continuation back to real β obtain that the dressed S -matrix is given by the expression (1.1), up to an additional factor,

$$F = e^{-\frac{(\sum m_k)^2}{4\pi\Lambda}} \int dR e^{-\pi\Lambda(R-R_0)^2 - \sum m_i (R-R_0)}. \quad (3.17)$$

Note that this factor potentially depends on particle masses, so it cannot be set to unity by adjusting the overall constant normalization \mathcal{N} , which should be process independent. Fortunately, all the dependence on masses cancels out after evaluating the integral in (3.17) and we recover the correct dressing formula (1.1).

3.2 Flat limit of Schwarzian correlators

Boundary correlation functions are asymptotic observables in the AdS gravity. They are thought to contain all information about a gravitational theory in the bulk. In particular it should be possible to extract from them the scattering matrix for states that are localized within a region much smaller than the AdS radius. By taking a family of theories with

increasing AdS radii one then expects to extract the full flat space S -matrix. This procedure is called the flat space limit of boundary correlators.

In the AdS₂ case the (Euclidean) action of the JT gravity takes the following form

$$\bar{S}_{\text{JT}} = - \int d^2x \sqrt{g} \phi \left(R + \frac{2}{L^2} \right) - 2\phi_b \oint_{\mathcal{C}} du \sqrt{g_{uu}} K + 2c_0 \oint_{\mathcal{C}} du \sqrt{g_{uu}}. \quad (3.18)$$

Here we assumed that the space is cut off by some closed boundary \mathcal{C} with K being its geodesic curvature. Variation of the geodesic curvature term w.r.t. the metric together with a boundary variation coming from the bulk JT action imply that at the boundary

$$\phi|_{\mathcal{C}} = \phi_b.$$

The last term in (3.18) is needed to stabilize the position of the boundary, just like in the flat case. The cutoff boundary is a usual necessity in the holographic renormalization procedure. However, in a conventional higher dimensional holography no additional boundary dynamics is introduced after the cutoff is removed. The situation is different in the JT NAdS₂ case, where gravity affects correlation functions exclusively through the boundary dynamics [3]–[9].

Analogously to the flat case the gravitational action reduces to a boundary term and the effect of gravity can be described as dressing of QFT correlators put in the rigid AdS₂ space. The most straightforward way to take the flat space limit of the theory (3.18) is to move in the boundary \mathcal{C} deep inside the AdS₂ radius. In this way one is guaranteed to eventually reproduce the results of section 3.1.

On the other hand, if the boundary AdS₂ correlators indeed contain all the information about the bulk dynamics, one should be able to make the calculation at the AdS₂ boundary and then perform a transformation of correlators that produces the flat space S -matrix. A priori it is not obvious that the two limits commute.

Before proceeding, let us note that we presented the action in the form (3.18) to agree with the rest of NAdS₂ literature. However, one should have in mind that with this choice the flat space limit becomes somewhat obscure. In particular, one may wonder what happened to the vacuum energy Λ , which used to be the only parameter in section 2. As pointed out in the Introduction, the answer is related to the possibility to perform a field redefinition,

$$\phi \rightarrow \phi + \phi_0, \quad (3.19)$$

with a constant ϕ_0 . This results in a simultaneous shift of Λ and ϕ_b ,

$$\Lambda \rightarrow \Lambda - \frac{2\phi_0}{L^2}, \quad \phi_b \rightarrow \phi_b - \phi_0,$$

so that the actual physical parameter is the invariant combination

$$\bar{\Lambda} = \Lambda - \frac{2\phi_b}{L^2}. \quad (3.20)$$

The flat limit $L \rightarrow \infty$ is most straightforward if one sets $\phi_b = 0$. Nevertheless, in the rest of this section we continue with the choice (3.18). In the appendix B we discuss some further details of the flat space limit, which become more transparent with the choice $\phi_b = 0$.

Within the conventional AdS/CFT it is well established how to take the $L \rightarrow \infty$ limit resulting in the scattering of massless particles [22, 23]. This procedure can be performed directly in the Lorentzian signature and in the global AdS coordinates. However, here we would like to check (1.1) also in the massive case. Furthermore, as we discuss in the very end of this section, massless scattering exhibits additional subtleties in the present setup.

Hence, to start with we consider scattering of massive particles only. This implies that at $L \rightarrow \infty$ dimensions Δ_a of all operators that create scattering states are also taken large, so that the corresponding masses $m_a = \Delta_a/L$ all stay finite. The flat space limit for massive scattering is subtle even within conventional AdS/CFT. The main problem is that in the Lorentzian signature massive particles never reach the AdS boundary. This makes it somewhat challenging to construct scattering states in the boundary theory. Hence, similarly to section 3.1, we work in the Euclidean signature. Our prescription is to a large extent guided by the flat space LSZ formalism. Morally the resulting procedure is similar to the one implemented in [24], although some details are different. In particular, we do not require conformal invariance of boundary correlators. Once the massive S -matrix is obtained, one can take masses to zero to study the dressing of massless S -matrices. In this sense mass can be thought of as an IR regulator.

3.2.1 Boundary action

In flat space all background solutions for the dilaton are equivalent. This is no longer the case in AdS₂. To describe the AdS₂ dilaton solutions it is convenient to introduce embedding coordinates \bar{X}^A , satisfying

$$-\bar{X}_{-1}^2 + \bar{X}_0^2 + \bar{X}_1^2 = -L^2. \tag{3.21}$$

Here, as before, we use bars for Euclidean coordinates. Then the general dilaton solution reads as

$$\phi = Z \cdot \bar{X}, \tag{3.22}$$

where Z^A are integration constants. Solutions with time-like, space-like or null vectors Z are qualitatively different. Only the time-like ones survive in the flat limit (see appendix B for details). Hence, in what follows we will focus on the solution with Z_0 and Z_1 equal to zero. It is convenient to parametrize the embedding coordinates as

$$\bar{X}_{-1} = L \cosh \frac{r}{L}, \quad \bar{X}_0 = L \sinh \frac{r}{L} \cos \theta, \quad \bar{X}_1 = L \sinh \frac{r}{L} \sin(-\theta). \tag{3.23}$$

In these coordinates the metric reads

$$ds^2 = dr^2 + L^2 \sinh^2(r/L) d\theta^2 \tag{3.24}$$

while the unperturbed boundary will be located at constant $r = R_0$. To get the boundary action in these coordinates we follow exactly the same path as in section 3.1. We keep the metric in the bulk equal to (3.24) and assume that the boundary is dynamical and parametrized by two functions $r(u)$ and $\theta(u)$. To gauge fix the reparametrizations of u we set a condition analogous to (3.3),

$$g_{uu} = \frac{L^2}{4} e^{\frac{2R}{L}}. \tag{3.25}$$

As before, using this condition one can express $r(u)$ through $\theta(u)$. Perturbatively for large R one gets

$$r(u) = R - L \log \theta'(u) + L e^{-\frac{2R}{L}} \left(\theta'(u)^2 - 2 \frac{\theta''(u)^2}{\theta'(u)^2} \right) + O(e^{-\frac{4R}{L}}). \quad (3.26)$$

In section 3.1 it was important to keep the zero mode of the metric R dynamical in order to obtain a unitary S -matrix. We will see that the same is true for the $L \rightarrow \infty$ limit of boundary correlators. Consequently, we allow R to fluctuate around its central value R_0 . Note that this is different from the conventional NAdS₂ prescription, where g_{uu} is kept fixed. The coupling constant ϕ_b should scale as $e^{\frac{R_0}{L}}$ in the $R_0 \rightarrow \infty$ limit to keep the action finite. So we set it to

$$\phi_b = -\frac{\Lambda L^2}{2} \cosh \frac{R_0}{L}. \quad (3.27)$$

In addition to reproducing the correct $R_0 \rightarrow \infty$ asymptotics at fixed L , with this choice by taking the $L \rightarrow \infty$ limit at fixed R_0 one gets

$$\phi_b|_{L \rightarrow \infty} = -\frac{\Lambda L^2}{2} + O(L^0),$$

independently of R_0 . Hence, in view of (3.20), we expect to match the earlier flat space results. The role of the last term in the action (3.18) is to stabilize the value of R . The solution $R = R_0$ is obtained with the choice

$$c_0 = -\frac{\Lambda L}{2} \sinh \frac{R_0}{L}.$$

Then in the large R_0 limit the boundary action reads

$$\bar{S}_b^{\text{AdS}} = \pi \Lambda L^2 e^{\frac{R-R_0}{L}} + \Lambda L^2 e^{\frac{R_0-R}{L}} \oint du \left(\frac{1}{2} \theta'(u)^2 - \frac{3}{2} \left(\frac{\theta''(u)}{\theta'(u)} \right)^2 + \frac{\theta'''(u)}{\theta'(u)} \right). \quad (3.28)$$

In the second piece we recognize the Schwarzian action (1.6) with $t = \tan(\theta/2)$. Unlike in previous works, however, the coupling constant in front of the Schwarzian is dynamical and is integrated over. As before, one can expand around the classical saddle point by writing

$$\theta(u) = u + \varepsilon(u).$$

At finite L the boundary theory (3.28) is highly non-linear. Also, one may worry that integrating over R makes it highly non-local.⁷ However, somewhat surprisingly, one may completely get rid of this non-locality by changing the gauge condition. Indeed, instead of the proper time gauge defined by (3.25) one may choose to work in the “static” gauge, defined by fixing

$$\theta = u.$$

In this case the boundary action is manifestly local and at large R_0 takes the following form,

$$\bar{S}_b^{(s)} = \Lambda L^2 \oint du \left(\cosh \left(\frac{r - R_0}{L} \right) - \frac{1}{2} e^{\frac{R_0-r}{L}} r'^2 \right), \quad (3.29)$$

⁷Interestingly, this type of non-locality — a promotion of a coupling constant into a fluctuating global dynamical variable — is analogous to the effect expected from Euclidean wormholes [25].

where we now kept $r(u)$ as a dynamical degree of freedom. In this form the prescription of [7]–[9] appears non-local, implying the following integral constraint

$$\oint du e^{\frac{r}{L}} = 2\pi e^{\frac{R_0}{L}}.$$

We checked that, at least as far as the $L \rightarrow \infty$ S -matrix goes, both gauge conditions give rise to the same results, so we will continue with the initial choice (3.25).

In the $L \rightarrow \infty$ limit nonlinearities of the action (3.28) are suppressed and only the Gaussian piece survives. The resulting quadratic action is

$$\bar{S}_b^{(2)} = \pi\Lambda(R - R_0)^2 + \frac{\Lambda L^2}{2} \oint du (\varepsilon'^2 - \varepsilon''^2). \quad (3.30)$$

This action does look very similar, up to some normalizations, to the dressing action (3.8), which we previously obtained directly in the flat space calculation. To check whether we indeed reproduce the same flat space dressing from the AdS calculation, we need to find the form of sources, which need to be added to (3.30) in the presence of scattering particles.

Hence, the next step in our analysis is to define the proper smearing of boundary correlators that will produce the S -matrix in the $L \rightarrow \infty$ limit. The resulting prescription is analogous to the LSZ formula, which computes the S -matrix from the asymptotics of the flat space Green's functions. It is universal and with straightforward modifications can be applied also in higher dimensional theories.

3.2.2 Single-particle states

Let us consider a massive quantum field theory in a rigid AdS₂. We start with constructing one-particle states that after a proper analytic continuation correspond to plane waves in the center of AdS₂. In order to make similarity with the LSZ formula manifest we first use the language of bulk fields. Consider the following state of the bulk theory

$$|V(p)\rangle_{\text{AdS}} = \oint d\theta (-P \cdot \bar{X}(\theta))^{\Delta_V} V(\bar{X}(\theta)) |0\rangle_{\text{AdS}}, \quad (3.31)$$

where the integral is taken over a constant r circle close to the boundary, $V(\bar{X})$ is a bulk field of a mass m dual to a boundary primary operator V with a dimension

$$\Delta_V = 1 + \sqrt{1 + (mL)^2} \simeq mL.$$

Here P is a vector of the form

$$P = (0, L \cos \bar{\beta}, -L \sin \bar{\beta}). \quad (3.32)$$

and $\bar{\beta}$ is a rapidity corresponding to an on-shell Euclidean momentum of the form

$$\bar{p}^\alpha = (m \cos \bar{\beta}, m \sin \bar{\beta}).$$

As $\bar{X}(u)$ approaches the boundary of AdS, $e^{r\Delta_V} V(\bar{X})$ corresponds to a boundary operator $\mathcal{V}(u)$ dual to V , and the bulk state (3.31) turns into the following smeared state of the boundary theory,

$$|V(p)\rangle = \oint du (-\cos(u - \bar{\beta}))^{\Delta_V} \mathcal{V}(u) |0\rangle. \quad (3.33)$$

To see the reason for considering this particular state it is instructive to inspect the corresponding bulk wavefunction

$$\Psi(\bar{Y}) = \langle 0|V(\bar{Y})|V(p)\rangle, \tag{3.34}$$

in the flat patch close to the center of the AdS, i.e. for

$$\bar{Y} = (L, y \cos \alpha, -y \sin \alpha)$$

with $y \ll L$. It is straightforward to check that upon analytic continuation $\bar{\beta} \rightarrow i\beta$ and $\bar{Y}^0 \rightarrow iY^0$ it takes the form of a plane wave. To verify this, we use the asymptotic behavior of bulk correlation functions, which also determines the form of the bulk-to-boundary correlators,

$$\langle V(\bar{Y})V(\bar{X}) \rangle = \frac{1}{(-2\bar{X} \cdot \bar{Y})^{\Delta_V}}. \tag{3.35}$$

Using (3.23) at $r \rightarrow \infty$ we get

$$\Psi(\bar{y}) = \oint d\theta \exp \left[\Delta_V \log - \frac{\cos(\theta - \bar{\beta})}{1 - \frac{y}{L} \cos(\theta - \alpha)} \right]. \tag{3.36}$$

For the scattering of massive particles, dimensions Δ_V all grow linearly with L . Then at $\Delta_V \rightarrow \infty$ the integral (3.36) can be evaluated by a saddle point approximation and gets localized at $\theta = \bar{\beta} + \pi$. An analytic continuation of the saddle point answer gives

$$\Psi(y) = \sqrt{\frac{2\pi}{\Delta_V}} e^{ip \cdot y}. \tag{3.37}$$

The localization becomes exact in the limit $\Delta_V \rightarrow \infty$.

We see that the situation is very similar to the flat space LSZ formula. Hence, we follow the same logic. We assume that the wave functions are convoluted with some external wave packets in P , that focuses plane waves in some region small compared to the AdS radius. We keep these external wave packets implicit as they stay smooth in the flat space limit. On the other hand the smearing function in (3.33) is important both for localization of boundary integrals in the flat space limit and for creating the plane wave states in the center of AdS.

As a cross-check, we can verify that applying the above construction to a Euclidean correlation function obtained from a Witten diagram with a local bulk vertex reproduces the S -matrix which would be generated by the corresponding Feynman diagram. The Witten diagram gives

$$G(\{\bar{X}_n\}) = \int_{\text{AdS}} d\bar{Y} \prod_n \frac{1}{(-2\bar{X}_n \cdot \bar{Y})^{\Delta_{V_n}}} \lambda, \tag{3.38}$$

where λ is a coupling constant of the bulk interaction. Attaching the state (3.31) to each external leg and continuing to the Minkowski momenta and coordinates leads to

$$S(\{p_n\}) = i\lambda \int d^d y e^{i \sum_n p_n \cdot y} = i\lambda \delta^d \left(\sum_n p_n \right), \tag{3.39}$$

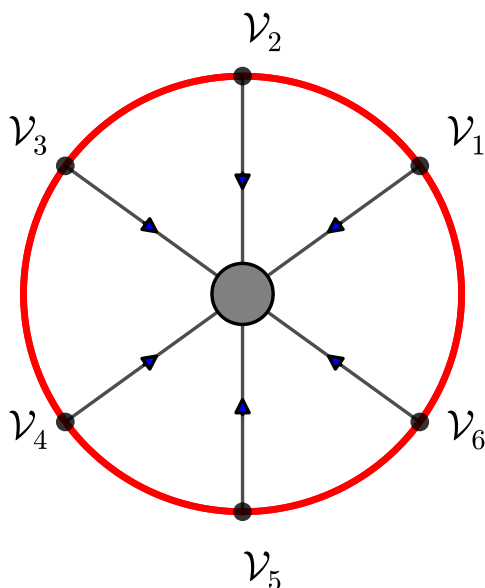


Figure 4. The Euclidean S -matrix is obtained from the boundary correlators by placing the source corresponding to the i -th particle at $\theta_i = \bar{\beta}_i + \pi$.

where we assume analogously to the flat space case that the y integral is taken over some region of AdS space small compared to L and consequently the delta-function is understood to be smoothed over the width of the external wave-packets. More generally, we arrived at the following remarkably simple expression for the Euclidean S -matrix⁸ (see figure 4),

$$S_n(\{\bar{\beta}_i\}) = \lim_{\Delta_i \rightarrow \infty} \langle \Delta_1^{1/2} \mathcal{V}_1(\theta_1) \dots \Delta_n^{1/2} \mathcal{V}_n(\theta_n) \rangle, \tag{3.40}$$

where $\theta_i = \bar{\beta}_i + \pi$ and factors of $\Delta_i^{1/2}$ are introduced to cancel the corresponding prefactor in (3.37). It is straightforward to generalize this prescription to higher dimension, where Euclidean rapidities β_i should be promoted into spherical coordinates of dimensionless Euclidean momenta \bar{p}^α/m . It will be interesting to perform further consistency checks on this formula and to derive it from Mellin amplitudes. Here we proceed to calculating the effect of the JT dressing. In fact, this can be considered as one of the consistency checks of (3.40).

3.2.3 Dressing

Calculation of the dressed S -matrix proceeds in the same way as in the Minkowski space. We first place the QFT in question in a rigid AdS₂ space which produces the set of boundary correlators. After smearing with the help of formula (3.33) they produce the S -matrix (3.40) of the QFT in the $L \rightarrow \infty$ limit. We then use the relation (3.40) between the S -matrix and the correlators, take the path integral over boundary fluctuations weighted by the action (3.30) and analytically continue in rapidities. More explicitly, the dressing prescription

⁸We thank Shota Komatsu, Joao Penedones and Pedro Vieira for a discussion on this and for informing us that Shota has independently arrived at this representation [26].

instructs us to replace the scattering states (3.31) with

$$|\widehat{V(p)}\rangle_{\text{AdS}} = \oint du (P \cdot \bar{X}(u))^{\Delta_V} V(\bar{X}(\theta(u))) |0\rangle_{\text{AdS}}, \quad (3.41)$$

where $\bar{X}(u)$ is given by (3.23) with $\theta = u$ and $r = R_0$. The analog of the formula (3.14) is then

$$\hat{S}_n(\{\bar{\beta}_i\}) = \lim_{\substack{L \rightarrow \infty \\ \Delta_i/L \rightarrow m_i}} \int dR \mathcal{D}\varepsilon e^{-\bar{S}_b^{(2)}(\varepsilon, R)} \prod_i^n \left(\left(e^{\frac{R_0-R}{L}} \theta'(u_i) \right)^{\Delta_i} \Delta_i^{1/2} \right) \langle \mathcal{V}_1(\theta(u_1)) \dots \mathcal{V}_n(\theta(u_n)) \rangle, \quad (3.42)$$

where $\theta(u) = u + \varepsilon(u)$ and $u_i \equiv \bar{\beta}_i + \pi$. As explained in [8] the rescaling factors

$$\mathcal{N}_i = \left(e^{\frac{R_0-R}{L}} \theta'(u_i) \right)^{\Delta_i}$$

in front of the correlation function appear from the relation between the bulk fields and operators (or, equivalently, from the mismatch between the boundary asymptotics of $V(\bar{X}(\theta))$ and of $(P \cdot \bar{X}(u))^{\Delta_V}$ in (3.41)) after the use of (3.26). These factors match the conformal transformation rule for boundary primary operators. Hence, in the intrinsic CFT terms dressing operates by averaging over the conformal transformations with the Schwarzian weight (or rather with its extension (3.28)). In the large Δ_i limit \mathcal{N}_i turn into the following source terms

$$\lim_{\substack{L \rightarrow \infty \\ \Delta_i/L \rightarrow m_i}} \left(e^{\frac{R_0-R}{L}} \theta'(u_i) \right)^{\Delta_i} = e^{m_i(R_0-R) + L m_i \varepsilon'(u_i)}. \quad (3.43)$$

Analogously to the flat case, $\varepsilon(u)$ -dependence of the correlator $\langle \mathcal{V}_1(\theta(u_1)) \dots \mathcal{V}_n(\theta(u_n)) \rangle$ can be neglected at $L \rightarrow \infty$, as a consequence of the bulk locality. Namely, the largest sensitivity to boundary fluctuations is in the bulk-to-boundary propagators that are already factored out in (3.42). Then after performing a rescaling

$$\varepsilon \rightarrow \frac{L}{R_0} \varepsilon$$

the $L \rightarrow \infty$ AdS dressing action, given by the sum of (3.28) and (3.43), exactly matches the flat action in (3.8). The Gaussian integral over ε is then the one we computed in the appendix A so the flat space limit of the Schwarzian dressing indeed reproduces the gravitational dressing (1.1). Note, in particular, that if we were to freeze the zero mode of the metric and do not take the integral over R in (3.42) we do not get a unitary S -matrix.

The above derivation relies on the fact that dimensions of operators go to infinity together with the AdS radius. In particular, the source term (3.43) becomes trivial if one keeps Δ_i 's finite in the flat space limit. So, as it stands, this derivation cannot be applied to massless particles. Of course from the results of section 2 we know that massless particles in principle can be treated on an equal footing with the massive ones, and that massless scattering can be obtained as a limit of massive amplitudes.

Massless flat space S -matrix is usually associated with the so-called ‘‘bulk-point’’ singularity present in the correlators of holographic CFTs in the Lorentzian signature [22].

However, there is an additional subtlety in the present setup. Similarly to the flat space calculation we would like to place a boundary along the lines of constant dilaton. Background dilaton solutions which provide constant dilaton slices sufficient to support massless scattering states in the Lorentzian signature correspond to space-like vectors Z in (3.22). Unfortunately, it is straightforward to check that those backgrounds do not have a smooth flat space limit. This is different from the standard situation where the entire AdS boundary is available. Nevertheless, massless scattering can be obtained in the limit when energies of particles are much larger than their masses. In this limit the dressing phase (1.1) takes a simpler form

$$\hat{S} = e^{i\ell^2 E_L E_R} S, \tag{3.44}$$

where E_L and E_R are total energies of left- and right-moving particles correspondingly. In particular, for the case of two-to-two massless scattering the dressing factor is equal to $e^{i\ell^2 s/4}$ which is a familiar eikonal gravitational phase shift.

4 Towards finite volume spectrum and localization

So far we established the equivalence between the gravitational dressing and coupling to the JT gravity. We also presented several reasons to expect that this construction is equivalent to the $T\bar{T}$ deformation. Of course, it will be very satisfactory to have a direct derivation of the latter equivalence as well. The JT gravity appears to provide a tractable path integral description for the gravitational dressing. Hence one may hope to derive the generalized Burger’s equation (1.8), describing the $T\bar{T}$ -deformed finite volume spectrum, by calculating the toric partition function of the JT gravity coupled to a matter system. Due to several technical difficulties, related mainly to the integration over the overall size of the torus, we did not complete this calculation yet, and hope to present it in a separate publication. Here, instead, we report a prototype calculation, which applies when the matter sector is described by a $c = 24$ CFT. Besides illustrating the general idea, this calculation has an independent interest because it gives rise to a localizable (*i.e.*, one loop exact) integral. We will argue that it is natural to expect the general JT gravity to exhibit the localization property as well.

The $c = 24$ conformal matter is special because in this case one can arrive at the dressed S -matrix (1.1) also by using the conventional Polyakov formalism, similarly to how it happens for the critical bosonic string. In fact the whole calculation is almost identical to the classic evaluation of the one loop string path integral [27]. Namely, instead of introducing a JT sector one may couple a $c = 24$ theory to a metric $g_{\alpha\beta}$ and two massless bosons X^μ , $\mu = 0, 1$ so that the dressed Euclidean partition function is

$$Z_{\text{dressed}} = \int \frac{\mathcal{D}g_{\alpha\beta}}{\text{Diff} \times \text{Weyl}} \mathcal{D}X^\mu e^{-\int \sqrt{g} \frac{1}{2\ell^2} (\partial_\alpha X^\mu)^2} Z_{\text{CFT}}(g_{\alpha\beta}), \tag{4.1}$$

where $Z_{\text{CFT}}(g_{\alpha\beta})$ is the CFT partition function in the external metric $g_{\alpha\beta}$, and the integral over metrics is moded out by the action of diffeomorphisms and the Weyl group. Note

that for a general interacting CFT the S -matrix is an ill-defined object, and it is better to define dressing directly in terms of the finite volume spectrum.⁹

For simplicity, to avoid subtleties associated with zero modes, we restrict this discussion to compact CFT's (so that it would need to be modified to cover the first and the simplest example of dressing — the critical bosonic string). The states of the undressed theory put on a circle of size R have energies of the form

$$E_n(R) = \frac{2\pi}{R} \left(h + \tilde{h} + N + \tilde{N} - \frac{c}{12} \right) \equiv \frac{\mathcal{E}_n}{R}, \quad (4.2)$$

where h, \tilde{h} are conformal weights of the corresponding primaries, N, \tilde{N} are the Virasoro levels, and $n = (h, \tilde{h}, N, \tilde{N})$ is a collective label for all of these. The finite volume spectrum of the dressed theory can be extracted from the toric partition function. To define the latter, recall (see section 2.1) that the fields X^μ 's play the role of physical coordinates of the dressed theory in this case. Hence, to calculate the partition function of the dressed theory on a rectangular torus with sides R_0 and R_1 one needs to integrate over X^μ field configurations of the form

$$X^\mu = R_\mu \sigma^\mu + \delta X^\mu, \quad (4.3)$$

where no summation is implied in the first term, and δX^μ 's are periodic w.r.t. the world-sheet coordinates σ^α with a unit periodicity,

$$\sigma^\alpha \sim \sigma^\alpha + 1.$$

All matter fields also have the same periodicity, and the metric can be chosen in the form

$$g_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = |d\sigma^0 + \tau d\sigma^1|^2,$$

where

$$\tau \equiv \tau_1 + i\tau_2$$

is the modular parameter. The only difference with the conventional one loop string calculations [27] is that as a consequence of windings for both X^0 and X^1 in (4.3) the region of integration over τ extends to the whole upper half-plane $\tau_2 > 0$. This is similar to how in [27] the region of integration extends from the fundamental domain of the modular group to the stripe $|\tau_1| < \frac{1}{2}$, when one moves from the situation with no windings to the one with a winding of X^0 only. Hence, the result in [27] translates into the following expression for the dressed toric partition function

$$Z_{\text{dressed}}(R_0, R_1) = \frac{R_0 R_1}{\ell^2} \int_0^\infty \frac{d\tau_2}{2\pi\tau_2^2} \int_{-\infty}^\infty d\tau_1 e^{-Y} Z_{\text{CFT}}(\tau), \quad (4.4)$$

where

$$Y = \frac{1}{2\ell^2} \left(\frac{R_0^2}{\tau_2} + R_1^2 \left(\tau_2 + \frac{\tau_1^2}{\tau_2} \right) \right)$$

⁹For some CFTs it makes sense to talk about S -matrices, understood mostly as a formal object, appearing in the TBA equations which determine the finite volume spectrum, see e.g. [28].

and $Z_{\text{CFT}}(\tau)$ is a toric partition function of the initial CFT. For a compact CFT the latter is given by

$$Z_{\text{CFT}}(\tau) = \sum_n e^{-\mathcal{E}_n \tau_2 - i\tau_1 \mathcal{P}_n}, \tag{4.5}$$

where \mathcal{E}_n are energies of CFT states on a unit circle, (4.2), and

$$\mathcal{P}_n = 2\pi(N - \tilde{N})$$

are the corresponding spatial momenta. At any value of n we end up with an integral of the same form,

$$I_n = \frac{R_0 R_1}{\ell^2} \int_0^\infty \frac{d\tau_2}{2\pi\tau_2^2} \int_{-\infty}^\infty d\tau_1 e^{-Y} e^{-\mathcal{E}_n \tau_2 - i\tau_1 \mathcal{P}_n}. \tag{4.6}$$

This integral is straightforward to evaluate. The integration over τ_1 is Gaussian and leads to

$$I_n = \frac{R_0}{\ell} \int \frac{d\tau_2}{\sqrt{2\pi\tau_2^3}} e^{-\frac{R_0^2}{2\ell^2\tau_2} - \tau_2 \left(\frac{R_1^2}{2\ell^2} + \frac{\ell^2 \mathcal{P}_n^2}{2R_1^2} + \mathcal{E}_n \right)}. \tag{4.7}$$

The remaining integration over τ_2 gives

$$I_n = e^{-\frac{R_0}{\ell} \sqrt{\frac{R_1^2}{\ell^2} + \frac{\ell^2 \mathcal{P}_n^2}{R_1^2} + 2\mathcal{E}_n}}.$$

Hence, the dressed partition function is given by

$$Z_{\text{dressed}} = \sum_n I_n,$$

and the corresponding energy spectrum is

$$E_{\text{dressed},n} = \frac{1}{\ell} \sqrt{\frac{R_1^2}{\ell^2} + \frac{\ell^2 \mathcal{P}_n^2}{R_1^2} + 2\mathcal{E}_n}.$$

This is exactly the same result as predicted by the generalized Burger's equation (1.8). Note that this result reproduces correctly also the spectrum of the standard bosonic string, even though the derivation itself needs to proceed more carefully in that case, to account for the presence of zero modes.

As we said, we expect that this derivation can be extended to non-critical CFT's and general matter theories by replacing the Polyakov sector with the JT gravity. In the standard approach to non-critical strings one needs to deal with peculiarities of the Liouville dynamics for the conformal mode Ω of the metric, which so far successfully resisted any treatment for $c > 1$ matter. The heuristic argument presented in the Introduction strongly suggests that these problems are mostly ameliorated in the JT case. Indeed, locally the coupling to the JT sector is equivalent to introducing a $\delta(\partial^2 \Omega)$ factor in the path integral, which kills most of the Liouville dynamics.

In the critical case, this factor can be thought of as an alternative way to gauge fix the Weyl symmetry. The presence of ∂^2 in the argument of the δ -function provides the determinant, which in the standard Polyakov treatment comes from integrating out $X^{0,1}$

modes. In a non-critical/massive case one still needs to deal properly with the integral over the global part of the conformal mode. We leave this task for the future. Here, instead, let us discuss the following interesting aspect of the presented derivation in the critical case.

The brute force derivation above lead us to the integral (4.7), which after rescaling of τ_2 and changing an overall constant factor, can be written as

$$\frac{1}{2} \int_0^\infty \frac{d\tau_2}{\tau_2^{\alpha+1}} e^{-\frac{1}{2\hbar}(\tau_2 + \tau_2^{-1})} = K_\alpha(\hbar^{-1}) \tag{4.8}$$

where K_α is the modified Bessel function, and in our case $\alpha = 1/2$, so that we get

$$K_{1/2}(\hbar^{-1}) = \sqrt{\frac{2\pi}{\hbar}} e^{-\frac{1}{\hbar}}.$$

At small \hbar the integral (4.8) can be approximated semiclassically, and dominated by the contribution from the vicinity of a saddle point at $\tau_2 = 1$. The interesting feature of the $\alpha = 1/2$ integral is that semiclassics is exact in this case, as if the integral were Gaussian. The answer is exactly given by the semiclassical exponent, multiplied by the corresponding determinant.

Integrals with this property are called localizable. Localization techniques are becoming more and more common in physical applications (see, e.g., [29–32]). Following [29] many of these applications operate in the context covered by the Duistermaat and Heckman (DH) formula [33], i.e. the corresponding integral is of the form

$$\int_{\mathcal{M}} \omega^n e^{-\beta H}, \tag{4.9}$$

where \mathcal{M} is a symplectic manifold with a symplectic form ω , and the symplectic flow generated by the Hamiltonian H corresponds to the $U(1)$ action on \mathcal{M} .

Let us see that the localization property of the integral (4.8) with $\alpha = 1/2$ can be understood almost in the same way. Let us first set $\mathcal{E}_n = \mathcal{P}_n = 0$. Then the modular integral we started with, namely

$$I = \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-\infty}^\infty d\tau_1 e^{-Y} \tag{4.10}$$

indeed has a form of the integral over a symplectic manifold — the Poincaré half-plane, so that $n = 1$ and the symplectic form ω is associated with the standard Hermitian metric

$$h_P = \frac{d\tau \otimes d\bar{\tau}}{\tau^2}.$$

The Hamiltonian flow generated by Y can be written as

$$\dot{\tau} = \frac{1}{2\ell^2} (R_1^2 \tau^2 + R_0^2). \tag{4.11}$$

This is not a $U(1)$ flow. However, compactness of the flow in the DH localization is needed only to establish the existence of a Riemannian metric invariant under the flow (see,

e.g., [32]). The Poincaré half-plane is a Kähler manifold, and the flow (4.11) generates an isometry of the corresponding Riemannian metric. Hence the DH localization applies even though the flow is non-compact. This explains why semiclassics is exact for (4.8). Finally, the integral (4.6) for general \mathcal{E}_n, P_n can be reduced to (4.10) by rescalings and shifts of the form $\tau_2 \rightarrow a\tau_2$, and $\tau_1 \rightarrow \tau_1 + b\tau_2$.

We believe it is hardly a coincidence that the $T\bar{T}$ deformation lead us to a localizable integral for a critical matter, and expect the localization to persist in the general case as well. There are several indications that the exact answer (1.1) is semiclassical in its nature. First, the main physical consequence of the dressing phase shift is that it introduces a time delay, linearly growing with the collision energy. In the Nambu-Goto case this time delay has a very transparent geometrical meaning — the physical length of a string segment is proportional to its energy, hence the time delay. The exact value of the delay can be obtained by solving the classical Nambu-Goto field equations [1]. Clearly, also in the general JT case one may calculate the exact time delay by downgrading the derivation presented in section 2.1 from the quantum operator language to the classical one. Also the factorization property of the $T\bar{T}$ operator [17], providing the basis for the $T\bar{T}$ deformation, is highly reminiscent of the large N factorization, which is semiclassical in its nature. A further independent support for the semiclassical nature of the JT gravity comes from the recent observation [32] that even before taking the flat limit the Schwarzian path integral is one loop exact. Finally, yet another surprising empirical evidence for the localization property of the JT gravity is coming from the QCD strings, as will be explained in the next section.

5 Future directions and the QCD string

In this concluding section let us describe a number of open questions raised by the analysis above, and discuss some future directions. To start with, let us explain the promising and somewhat surprising implications of the results presented here for the dynamics of confining strings (flux tubes) in the planar gluodynamics.

As we already discussed, gravitational dressing of 24 free massless bosons gives rise to a theory on the worldsheet of a free critical bosonic string. In particular, this theory is invariant under the non-linearly realized target space Poincaré symmetry $\text{ISO}(1, 25)$. Of course, in this case a convenient manifestly covariant path integral description is provided by the Polyakov formalism. However, as proven in [34], gravitational dressing of a single massless boson is also invariant under the non-linearly realized target space Poincaré symmetry $\text{ISO}(1, 2)$. Hence, one may wonder whether this system may serve as a starting point for a new type of non-critical strings. Furthermore this setup allows also a natural generalization to $D = 4$, which requires introducing an additional pseudoscalar field on the worldsheet.

Intriguingly, the TBA analysis [35, 36] of the available lattice data in $D = 3$ and $D = 4$ gluodynamics [37–39] suggests that confining strings may be described as deformations of these simple integrable models — leading to the Axionic String Ansatz (ASA) [20, 34]. Up until now a theoretical development of the ASA was largely impeded by the absence of a tractable path integral formalism to describe gravitational dressing. Such a formalism is

especially called-for in the description of the short string (glueball) sector. The reformulation of the gravitational dressing via coupling to the JT gravity gives rise to a hope to resolve this problem. In particular, restricting to $D = 3$, we expect that the theory

$$S_{3D} = \int \sqrt{-g} \left(\phi R + 2\ell_s^{-2} - \frac{1}{2}(\partial X)^2 \right) \tag{5.1}$$

enjoys a non-linearly realized $\text{ISO}(1, 2)$ symmetry.¹⁰ It is likely that to identify this symmetry a first order reformulation of the JT gravity as a Poisson sigma model [40, 41] should be useful. Earlier discussions of the JT gravity in relation to non-critical strings can be found in [42, 43].

After the full symmetry is identified it should be possible to evaluate the partition function in the long string sector (in analogy to section 4) and also to extend this analysis to short strings¹¹ (similarly to the critical string case [27]). This may open the path to put on a solid footing the heuristic analysis of the $D = 3$ glueball spectrum presented in [20]. A rather surprising outcome of [20] is that glueball quantum numbers apparently all can be determined by the semiclassical ansatz, which a priori is expected to work only in the large spin limit. If confirmed, this surprising effectiveness of semiclassics may be yet another indication that the partition function for (5.1) exhibits the localization property, similarly to section 4.

Naively, one could expect the string tension to be equal to the cosmological constant in the action (5.1). This expectation does not hold — one finds an extra factor of 2 in the action (5.1). Also the sign of the cosmological constant in (5.1) is opposite (negative) to what one may expect for a positive tension string. A possible explanation for this apparent discrepancy is that to reproduce the correct result for the partition function the action (5.1) needs to be supplemented with additional topological (total derivative) terms which do not affect the scattering, but play a role in the partition function calculation. An example of such a term is

$$\int \sqrt{-g} \square \phi.$$

In the analogue of (4.4) this term will contribute a factor of e^{-Y} , which in the Polyakov case originated from the $(\partial X^\mu)^2$ term in the presence of $X^{0,1}$ windings.

Finally, extending the ASA to $D = 2$ confining strings suggests that the corresponding worldsheet theory may be given by the pure JT gravity without matter, perhaps with an additional deformation and/or extension. Confining strings in $D = 2$ are topological and understood much better than in higher dimensions [44–46]. However, the local worldsheet description is not completely understood yet. Interestingly, the candidate action described in [46] indeed takes the form of the JT gravity coupled to a topological σ -model.

Coming back to the present setup as a theory of quantum gravity, one may wonder whether obtaining the exact solution of this simple model may shed light on numerous

¹⁰In fact, at the classical level one expects to find the non-linearly realized $\text{ISO}(1, D-1)$ for $D-2$ massless bosons coupled to the JT gravity. This symmetry survives at the quantum level only for $D = 3, 26$.

¹¹This might require adding global degrees of freedom on the worldsheet, similar to NSR sectors in superstrings.

profound puzzles present in quantum gravity. Indeed, there is a vast body of literature exploring black holes (see, e.g., [19, 47]), traversable wormholes [48], baby universes [49] and other beasts in models which appear as rather minimal deformations of the setup studied here.

The exact S -matrix (1.1) does exhibit a number of unconventional features specific for gravitational theories. These include the novel UV asymptotics, precluding construction of local observables; a time delay proportional to the collision energy, as expected for Hawking evaporation in two dimensions; coordinate uncertainty principle [1]; as well as the saturation of the bound on chaos [50] for a Rindler observer. However, it falls short of delivering the black hole creation and evaporation. For instance, there is no particle production associated with gravitational interactions in this model.

It remains unclear at the moment whether resolving this deficiency requires a drastic modification of the whole setup or a minor deformation/change of interpretation may be enough. It appears that the required deformation is likely to introduce a direct coupling between the JT dilaton and matter fields. Indeed, the most straightforward way to introduce black holes into the JT gravity is to perform the dilaton dependent Weyl rescaling of the JT metric (see, e.g., [51] for a recent study of the resulting black holes).

The relation to the QCD strings is likely to be useful for clarifying these issues as well. Indeed, lattice simulations of the large N gluodynamics provide an operational and fully non-perturbative definition of the confining strings. If the expectation that their worldsheet theory is described by a non-integrable deformation of the JT gravity is correct, this opens a unique and exciting experimental window into the world of quantum gravity.¹² Also from a purely theoretical viewpoint many of the subtleties of quantum gravity (and of two-dimensional dilaton gravity in particular) are related to the proper identification of the physical observables. To a large extent this is the central theme of the current paper — the only effect of the JT sector is to introduce a dynamical system of physical clocks and rods into the system. A relation to the bulk gluodynamics makes it easier to stay on a solid ground and to properly identify the relevant physical observables, such as the glueball spectra and scattering amplitudes.

Let us conclude the paper with a wild speculation. As emphasized in [2], a puzzling aspect of the dressing formula (1.1) is that it challenges the conventional notion of naturalness. Namely, (1.1) provides a simple recipe to introduce a new UV scale ℓ into an arbitrary quantum field theory without destabilizing any unprotected relevant operators. The Lagrangian description for this construction via JT gravity adds another surprising twist to the story. Namely, the JT gravity may be thought of as an extreme example of “degravitation” — the vacuum energy Λ can take an arbitrary value without curving the space-time. Instead, here it manifests itself as the UV deformation of the scattering amplitudes. We are unable to refrain from reporting a hallucination that extending this story to higher dimension might open a path towards explaining the $M_{\text{weak}} \sim \sqrt{M_{\text{Pl}} E_{\text{vac}}}$ coincidence.

¹²With the obvious caveat that by an experiment we understand computer simulations here.

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A Evaluation of the boundary path integral

Let us evaluate the path integral over ε , which enters in the dressing formula (3.16),

$$I = \int \mathcal{D}\varepsilon e^{-\frac{\Lambda R_0^2}{2} \int du (\varepsilon'^2 - \varepsilon''^2) + \int du \sum_i m_i R_0 \varepsilon' \delta(u - \bar{\beta}_i - \pi)}. \quad (\text{A.1})$$

This integral is Gaussian, so it can be evaluated exactly by the saddle point approximation. We do not keep track of the overall constant normalization, which will be fixed at the very end by unitarity. To evaluate the saddle point action we use the equation of motion for ε

$$\Lambda R_0 (\varepsilon'''' + \varepsilon'') = \sum_n m_n \delta'(u_n), \quad u_n = \bar{\beta}_n + \pi. \quad (\text{A.2})$$

Expanding ε in terms of solutions of the homogeneous equation in between the sources we find

$$\varepsilon(u_{n-1} < u < u_n) = a_n + b_n u + c_n e^{iu} + d_n e^{-iu}. \quad (\text{A.3})$$

At the sources ε'''' should have a δ' contribution, implying that ε'' has a discontinuity given by

$$\varepsilon''(u_n^+) - \varepsilon''(u_n^-) = \frac{m_n}{\Lambda R_0}. \quad (\text{A.4})$$

Then we can fix the coefficients by demanding the continuity and 2π periodicity of ε and its first and third derivatives. Let us set $c_1 = d_1 = 0$, then one finds

$$a_n = \sum_{k < n} \frac{m_k}{\Lambda R_0}, \quad b_n = -\sum_{k=1}^{n_{\text{tot}}} \frac{m_k}{2\pi \Lambda R_0}, \quad c_n = \sum_{k < n} -\frac{m_k}{2\Lambda R_0} e^{-iu_k}, \quad d_n = \sum_{k < n} -\frac{m_k}{2\Lambda R_0} e^{iu_k}, \quad (\text{A.5})$$

where n_{tot} is the total number of particles. Note that as a consequence of energy and momentum conservation the $c_1 = d_1 = 0$ condition is consistent with (A.5).

Plugging the solution back in the action and using that $u_k - u_n = \bar{\beta}_k - \bar{\beta}_n$, we obtain for the integral (A.1)

$$I = D e^{-\frac{(\sum m_k)^2}{4\pi\Lambda}},$$

where

$$D = \exp\left(-\frac{\sum_{\bar{\beta}_k > \bar{\beta}_n} m_n m_k \sin(\bar{\beta}_k - \bar{\beta}_n)}{2\Lambda}\right). \quad (\text{A.6})$$

After substituting $\bar{\beta}_n = i\beta_n$ we get an analytic expression for the dressing factor, valid on the entire physical sheet,

$$D = \exp\left(-i\frac{\sum_{\text{Im}\beta_k < \text{Im}\beta_n} m_n m_k \sinh(\beta_k - \beta_n)}{2\Lambda}\right). \quad (\text{A.7})$$

This is just another way of rewriting the dressing factor (1.1). To understand the analytic continuation to the physical domain of real β 's, let us recall that on the physical sheet the imaginary part of rapidity differences doesn't change the sign [2] and consequently the ordering of rapidities stays fixed. In terms of the conventional Mandelstam variables $s_{nk} = m_n^2 + m_k^2 + 2m_n m_k \cosh(\beta_n - \beta_k)$ physical domain is approached from above. Consequently, when both particles are in or both are out, $\beta_k - \beta_n > 0$ implies $\text{Im}\beta_k > \text{Im}\beta_n$. Due to the properties of our phase shift, cross terms containing both in- and out-particles cancel out and can be ignored.

B Details of the flat space limit

Let us spell out here elementary geometrical details of taking the $L \rightarrow \infty$ limit of the AdS₂ vacuum (3.22), (3.24) in the JT gravity. As discussed in section 3.2, the $L \rightarrow \infty$ limit of the JT action becomes manifestly smooth when one makes use of the field redefinition (3.19) to set $\phi_b = 0$. Also, it is convenient to perform a shift of the embedding coordinate $\bar{X}_{-1} \rightarrow L + \bar{X}_{-1}$ so that the AdS₂ hyperboloid also has a smooth $L \rightarrow \infty$ limit. After these two redefinitions (3.21) is replaced by

$$\bar{X}_{-1} = \sqrt{L^2 + \bar{X}_0^2 + \bar{X}_1^2} - L \simeq \frac{\bar{X}_0^2 + \bar{X}_1^2}{2L}. \quad (\text{B.1})$$

At $L = \infty$ this turns into a plane

$$X_{-1} = 0.$$

The time-like dilaton solution (3.22) with $Z^0 = Z^1 = 0$, $Z^{-1} = Z$ turns into

$$\phi = Z(\bar{X}_{-1} + L) + \frac{\Lambda L^2}{2}. \quad (\text{B.2})$$

For this solution to be smooth in the $L \rightarrow \infty$ limit one chooses

$$Z = -\frac{\Lambda L}{2} + \frac{\phi_0}{L},$$

where the constant ϕ_0 is determined from requiring $\phi = 0$ at the boundary cutoff surface \mathcal{C} . Then at $L \rightarrow \infty$ the dilaton solution becomes

$$\phi = -\frac{\Lambda}{4} (\bar{X}_0^2 + \bar{X}_1^2) + \phi_0, \quad (\text{B.3})$$

reproducing the flat space result.

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References

- [1] S. Dubovsky, R. Flauger and V. Gorbenko, *Solving the Simplest Theory of Quantum Gravity*, *JHEP* **09** (2012) 133 [[arXiv:1205.6805](https://arxiv.org/abs/1205.6805)] [[INSPIRE](#)].
- [2] S. Dubovsky, V. Gorbenko and M. Mirbabayi, *Natural Tuning: Towards A Proof of Concept*, *JHEP* **09** (2013) 045 [[arXiv:1305.6939](https://arxiv.org/abs/1305.6939)] [[INSPIRE](#)].
- [3] A. Almheiri and J. Polchinski, *Models of AdS_2 backreaction and holography*, *JHEP* **11** (2015) 014 [[arXiv:1402.6334](https://arxiv.org/abs/1402.6334)] [[INSPIRE](#)].
- [4] A. Kitaev, *A simple model of quantum holography*, (2015) <http://online.kitp.ucsb.edu/online/entangled15/kitaev/>.
- [5] S. Sachdev and J. Ye, *Gapless spin fluid ground state in a random, quantum Heisenberg magnet*, *Phys. Rev. Lett.* **70** (1993) 3339 [[cond-mat/9212030](https://arxiv.org/abs/cond-mat/9212030)] [[INSPIRE](#)].
- [6] J. Maldacena and D. Stanford, *Remarks on the Sachdev-Ye-Kitaev model*, *Phys. Rev. D* **94** (2016) 106002 [[arXiv:1604.07818](https://arxiv.org/abs/1604.07818)] [[INSPIRE](#)].
- [7] K. Jensen, *Chaos in AdS_2 Holography*, *Phys. Rev. Lett.* **117** (2016) 111601 [[arXiv:1605.06098](https://arxiv.org/abs/1605.06098)] [[INSPIRE](#)].
- [8] J. Maldacena, D. Stanford and Z. Yang, *Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space*, *PTEP* **2016** (2016) 12C104 [[arXiv:1606.01857](https://arxiv.org/abs/1606.01857)] [[INSPIRE](#)].
- [9] J. Engelsöy, T.G. Mertens and H. Verlinde, *An investigation of AdS_2 backreaction and holography*, *JHEP* **07** (2016) 139 [[arXiv:1606.03438](https://arxiv.org/abs/1606.03438)] [[INSPIRE](#)].
- [10] M. Cvetič and I. Papadimitriou, *AdS_2 holographic dictionary*, *JHEP* **12** (2016) 008 [Erratum *ibid.* **01** (2017) 120] [[arXiv:1608.07018](https://arxiv.org/abs/1608.07018)] [[INSPIRE](#)].
- [11] F.A. Smirnov and A.B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys. B* **915** (2017) 363 [[arXiv:1608.05499](https://arxiv.org/abs/1608.05499)] [[INSPIRE](#)].
- [12] A. Cavaglià, S. Negro, I.M. Szécsényi and R. Tateo, *$T\bar{T}$ -deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [[arXiv:1608.05534](https://arxiv.org/abs/1608.05534)] [[INSPIRE](#)].
- [13] R. Jackiw, *Lower Dimensional Gravity*, *Nucl. Phys. B* **252** (1985) 343 [[INSPIRE](#)].
- [14] C. Teitelboim, *Gravitation and Hamiltonian Structure in Two Space-Time Dimensions*, *Phys. Lett. B* **126** (1983) 41 [[INSPIRE](#)].
- [15] H.L. Verlinde and E.P. Verlinde, *Scattering at Planckian energies*, *Nucl. Phys. B* **371** (1992) 246 [[hep-th/9110017](https://arxiv.org/abs/hep-th/9110017)] [[INSPIRE](#)].
- [16] L. McGough, M. Mezei and H. Verlinde, *Moving the CFT into the bulk with $T\bar{T}$* , [arXiv:1611.03470](https://arxiv.org/abs/1611.03470) [[INSPIRE](#)].
- [17] A.B. Zamolodchikov, *Expectation value of composite field $T\bar{T}$ in two-dimensional quantum field theory*, [hep-th/0401146](https://arxiv.org/abs/hep-th/0401146) [[INSPIRE](#)].
- [18] M. Caselle, D. Fioravanti, F. Gliozzi and R. Tateo, *Quantisation of the effective string with TBA*, *JHEP* **07** (2013) 071 [[arXiv:1305.1278](https://arxiv.org/abs/1305.1278)] [[INSPIRE](#)].

- [19] C.G. Callan Jr., S.B. Giddings, J.A. Harvey and A. Strominger, *Evanescent black holes*, *Phys. Rev. D* **45** (1992) R1005 [[hep-th/9111056](#)] [[INSPIRE](#)].
- [20] S. Dubovsky and G. Hernandez-Chifflet, *Yang-Mills Glueballs as Closed Bosonic Strings*, *JHEP* **02** (2017) 022 [[arXiv:1611.09796](#)] [[INSPIRE](#)].
- [21] C. Itzykson and J.B. Zuber, *Quantum Field Theory*, International Series In Pure and Applied Physics, **McGraw-Hill**, New York (1980).
- [22] M. Gary, S.B. Giddings and J. Penedones, *Local bulk S-matrix elements and CFT singularities*, *Phys. Rev. D* **80** (2009) 085005 [[arXiv:0903.4437](#)] [[INSPIRE](#)].
- [23] A.L. Fitzpatrick and J. Kaplan, *Scattering States in AdS/CFT*, [arXiv:1104.2597](#) [[INSPIRE](#)].
- [24] M.F. Paulos, J. Penedones, J. Toledo, B.C. van Rees and P. Vieira, *The S-matrix Bootstrap I: QFT in AdS*, [arXiv:1607.06109](#) [[INSPIRE](#)].
- [25] S.R. Coleman, *Black Holes as Red Herrings: Topological Fluctuations and the Loss of Quantum Coherence*, *Nucl. Phys. B* **307** (1988) 867 [[INSPIRE](#)].
- [26] S. Komatsu, to appear,
- [27] J. Polchinski, *Evaluation of the One Loop String Path Integral*, *Commun. Math. Phys.* **104** (1986) 37 [[INSPIRE](#)].
- [28] P. Fendley, H. Saleur and A.B. Zamolodchikov, *Massless flows, 2. The Exact S matrix approach*, *Int. J. Mod. Phys. A* **8** (1993) 5751 [[hep-th/9304051](#)] [[INSPIRE](#)].
- [29] E. Witten, *Two-dimensional gauge theories revisited*, *J. Geom. Phys.* **9** (1992) 303 [[hep-th/9204083](#)] [[INSPIRE](#)].
- [30] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [[arXiv:0712.2824](#)] [[INSPIRE](#)].
- [31] V. Pestun et al., *Localization techniques in quantum field theories*, [arXiv:1608.02952](#) [[INSPIRE](#)].
- [32] D. Stanford and E. Witten, *Fermionic Localization of the Schwarzian Theory*, [arXiv:1703.04612](#) [[INSPIRE](#)].
- [33] J.J. Duistermaat and G.J. Heckman, *On the Variation in the cohomology of the symplectic form of the reduced phase space*, *Invent. Math.* **69** (1982) 259 [[INSPIRE](#)].
- [34] S. Dubovsky and V. Gorbenko, *Towards a Theory of the QCD String*, *JHEP* **02** (2016) 022 [[arXiv:1511.01908](#)] [[INSPIRE](#)].
- [35] S. Dubovsky, R. Flauger and V. Gorbenko, *Evidence from Lattice Data for a New Particle on the Worldsheet of the QCD Flux Tube*, *Phys. Rev. Lett.* **111** (2013) 062006 [[arXiv:1301.2325](#)] [[INSPIRE](#)].
- [36] S. Dubovsky, R. Flauger and V. Gorbenko, *Flux Tube Spectra from Approximate Integrability at Low Energies*, *J. Exp. Theor. Phys.* **120** (2015) 399 [[arXiv:1404.0037](#)] [[INSPIRE](#)].
- [37] A. Athenodorou, B. Bringoltz and M. Teper, *Closed flux tubes and their string description in $D = 3 + 1$ $SU(N)$ gauge theories*, *JHEP* **02** (2011) 030 [[arXiv:1007.4720](#)] [[INSPIRE](#)].
- [38] A. Athenodorou and M. Teper, *$SU(N)$ gauge theories in $2+1$ dimensions: glueball spectra and k -string tensions*, *JHEP* **02** (2017) 015 [[arXiv:1609.03873](#)] [[INSPIRE](#)].
- [39] A. Athenodorou and M. Teper, *On the mass of the world-sheet ‘axion’ in $SU(N)$ gauge theories in $3 + 1$ dimensions*, *Phys. Lett. B* **771** (2017) 408 [[arXiv:1702.03717](#)] [[INSPIRE](#)].

- [40] H.L. Verlinde, *Black holes and strings in two-dimensions*, in *Recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories. Proceedings, 6th Marcel Grossmann Meeting*, Kyoto, Japan, June 23–29, 1991. Pts. A, B, pp. 178–207.
- [41] D. Cangemi and R. Jackiw, *Gauge invariant formulations of lineal gravity*, *Phys. Rev. Lett.* **69** (1992) 233 [[hep-th/9203056](#)] [[INSPIRE](#)].
- [42] E.S. Fradkin and A.A. Tseytlin, *Quantization of Two-Dimensional Supergravity and Critical Dimensions for String Models*, *Phys. Lett. B* **106** (1981) 63 [[INSPIRE](#)].
- [43] A.A. Tseytlin, *Finite σ -models and exact string solutions with Minkowski signature metric*, *Phys. Rev. D* **47** (1993) 3421 [[hep-th/9211061](#)] [[INSPIRE](#)].
- [44] D.J. Gross, *Two-dimensional QCD as a string theory*, *Nucl. Phys. B* **400** (1993) 161 [[hep-th/9212149](#)] [[INSPIRE](#)].
- [45] D.J. Gross and W. Taylor, *Two-dimensional QCD is a string theory*, *Nucl. Phys. B* **400** (1993) 181 [[hep-th/9301068](#)] [[INSPIRE](#)].
- [46] S. Cordes, G.W. Moore and S. Ramgoolam, *Lectures on 2-D Yang-Mills theory, equivariant cohomology and topological field theories*, *Nucl. Phys. Proc. Suppl.* **41** (1995) 184 [[hep-th/9411210](#)] [[INSPIRE](#)].
- [47] L. Susskind, L. Thorlacius and J. Uglum, *The Stretched horizon and black hole complementarity*, *Phys. Rev. D* **48** (1993) 3743 [[hep-th/9306069](#)] [[INSPIRE](#)].
- [48] J. Maldacena, D. Stanford and Z. Yang, *Diving into traversable wormholes*, *Fortsch. Phys.* **65** (2017) 1700034 [[arXiv:1704.05333](#)] [[INSPIRE](#)].
- [49] V.A. Rubakov, *Baby universes and energy (non)conservation in $(1+1)$ -dimensional dilaton gravity*, *Phys. Rev. D* **56** (1997) 3523 [[hep-th/9611171](#)] [[INSPIRE](#)].
- [50] J. Maldacena, S.H. Shenker and D. Stanford, *A bound on chaos*, *JHEP* **08** (2016) 106 [[arXiv:1503.01409](#)] [[INSPIRE](#)].
- [51] M. Fitkevich, D. Levkov and Y. Zenkevich, *Exact solutions and critical chaos in dilaton gravity with a boundary*, *JHEP* **04** (2017) 108 [[arXiv:1702.02576](#)] [[INSPIRE](#)].