

Chern-Simons theory on supermanifolds

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ABSTRACT: We consider quantum field theories on supermanifolds using integral forms. The latter are used to define a geometric theory of integration and they are essential for a consistent action principle. The construction relies on Picture Changing Operators, analogous to the one introduced in String Theory. As an application, we construct a geometric action principle for $N = 1$ $D = 3$ super-Chern-Simons theory.

KEYWORDS: Chern-Simons Theories, Superspaces, Supersymmetric gauge theory

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1 Introduction

One of the main differences between the geometry of supermanifolds and that of conventional manifolds is the distinction between differential forms and integral forms [1, 2]. The latter are essential to provide a geometric integration theory for supermanifolds.

Since the differentials $d\theta$'s (associated to anticommuting coordinates θ 's) are commuting variables,¹ there is no natural integrable top-differential form; then, one introduces distribution-like anti-commuting quantities, such as $\delta(d\theta)$, that can provide a suitable integral top-form (those which can be integrated) and for which the usual Cartan calculus can be extended (see here fore a non-exhaustive reference list [3–9]). The complex of the differential forms together with the complex of the integral forms are the highest and the lowest line of the interesting double complex of the *pseudo-forms*.

This complex, whose elements are denoted by $\Omega^{(p|q)}$, is filtered by two integer numbers: the *form number* p , which represents the usual form degree (which can also be negative as will be discussed in the text) and q , the *picture number*, which counts the number of delta functions and it ranges between 0 and m , with m the fermionic dimension of the supermanifold. It is customary to denote by *superforms* those with vanishing picture: $\Omega^{(p|0)}$ with unbound form number; while the *integral forms* are those in $\Omega^{(p|m)}$ with maximal picture. An integral form of top degree can be integrated on a supermanifold and it produces a number like a usual differential top-form does on a manifold. The differential d , suitably extended to the entire complex, increases the form number without touching

¹We denote by $\mathcal{M}^{(n|m)}$ a supermanifold which is locally homeomorphic to $\mathbb{R}^{(n|m)}$, the flat superspace, described in terms of the coordinates (x^a, θ^α) . We denote by (V^a, ψ^α) the supervielbeins and, in the case of the flat space, they corresponds to $V^a = dx^a + \bar{\theta} \gamma^a d\theta$ and $\psi^\alpha = d\theta^\alpha$ where γ^a are the Dirac matrices in the suitable representation.

the picture number. The latter can be modified by increasing and lowering the number of delta functions, and for that one needs new operators known as *picture changing operators* (PCO's) originally introduced in RNS string theory [10]. In string theory, the role of the supermanifold is played by the worldsheet super-Riemann surface or, more precisely, by the associated super-moduli space and super-conformal Killing group manifold, as discussed in [9], and integral forms are essential to define the amplitudes to all orders in perturbation theory. In higher dimensional spacetime theories, but based on worldsheet two-dimensional models, they were introduced in [11] and further discussed in [12].

In the present paper, we discuss the role of PCO in the context of spacetime QFT and the relation between different superspace formalisms. All of them are related by a choice of suitable PCO with different properties, but belonging to the same cohomology class. As a playground, we choose 3D, $N = 1$ super-Chern-Simons theory (see [13] and the reference therein).

The conventional bosonic Chern-Simons theory is described by the geometrical action

$$S_{\text{CS}} = \int_{\mathcal{M}} \text{Tr} \left(A^{(1)} \wedge dA^{(1)} + \frac{2}{3} A^{(1)} \wedge A^{(1)} \wedge A^{(1)} \right), \quad (1.1)$$

where $A^{(1)}$ is the 1-form gauge connection with values in the adjoint representation of the gauge group \mathcal{G} , the trace is taken over the same representation and the integral integrates a 3-form Lagrangian over a three dimensional manifold \mathcal{M} . As is well known, it provides a meaningful integral, independent of the parametrization of \mathcal{M} and of its metric. The 3-form Lagrangian is closed by construction and its gauge variation is exact.

For the corresponding *super* Chern-Simons action on a supermanifold $\mathcal{M}^{(3|2)}$, one needs a (3|2)-integral form that, however, cannot be built only by connections as $A^{(1|0)}$. The latter are differential 1-superforms with zero picture (as been explained in [1, 2]), leading to a (3|0) superform Lagrangian as (1.1) that cannot be integrated. Nonetheless, it can be converted to a (3|2)-integral form by multiplying it by a PCO belonging to $\Omega^{(0|2)}$ for example

$$\mathbb{Y}_{\text{susy}}^{(0|2)} = V^a \wedge V^b (\gamma_{ab})^{\alpha\beta} \iota_{\alpha} \iota_{\beta} \delta^2(\psi), \quad (1.2)$$

where ($V^a = dx^a + \theta^\alpha \gamma_{\alpha\beta}^a d\theta^\beta$, $\psi^\alpha = d\theta^\alpha$). $\gamma_{\alpha\beta}^a, \gamma_{\alpha\beta}^{ab}$ are the Dirac gamma matrices and ι_{α} is the usual contraction operators along the odd vector $D_{\alpha} = \partial_{\alpha} - (\theta\gamma^a)_{\beta} \partial_a$. The operator $\mathbb{Y}_{\text{susy}}^{(0|2)}$ is closed, supersymmetric and not exact, then it belongs to $H^{(0|2)}$.

Consequently the super Chern-Simons action reads

$$S_{\text{SCS}} = \int_{\mathcal{M}^{(3|2)}} \mathbb{Y}_{\text{susy}}^{(0|2)} \wedge \text{Tr} \left(A^{(1|0)} \wedge dA^{(1|0)} + \frac{2}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} \right). \quad (1.3)$$

The integration is extended to the entire supermanifold \mathcal{M} . As will be checked in the main text, the result is gauge invariant, supersymmetric and leads to the well-known super Chern-Simons action in superspace. An obvious question is whether one can change the PCO $\mathbb{Y}_{\text{susy}}^{(0|2)}$ without changing the action. Since $\mathbb{Y}_{\text{susy}}^{(0|2)}$ belongs to a cohomology class, it implies a choice of a representative inside the same class. This means that the invariance of the action w.r.t. to a change of $\mathbb{Y}_{\text{susy}}^{(0|2)}$ is achievable only if the (3|0)-Lagrangian is closed

by integration by parts in absence of non-trivial boundaries. That request, for a $(3|0)$ superform in the supermanifold $\mathcal{M}^{(3|2)}$, is non-trivial and indeed the action given in (1.3) has to be modified accordingly. It is easy to show that there is a missing term in the action and the closure implies the usual conventional constraints. Then, after that modification, we can change the PCO for getting new forms of the action with the same physical content, but displaying different properties.

In the present context, we provide a new geometrical perspective on QFT's superspace and on supermanifolds. We are able to prove that the *Rheonomic* action (see [14]) formulation of $N = 1$ $D = 3$ super Chern-Simons theory with rigid supersymmetry (the local supersymmetric case will be discussed separately) can be considered as a "mother" action which has built-in all possible superspace realizations for that theory. In particular, we will show that, using a suitable PCO, the action reduces to the usual action in terms of component fields and by another choice we get the superspace action written in terms of superfields. However, only for the choice (1.2), we are able to derive the conventional constraint by varying the action and without resorting to the rheonomic parametrization.

The paper is organised as follows: section 2 deals with background material, the definition of integral forms and integration on supermanifolds. In section 3, we introduce PCO's for spacetime quantum field theory. In section 4, we discuss the action of super-Chern-Simons theory in $3d$. The relation between different types of PCO's and actions are given in section 5.

Integral forms, integration on supermanifolds, the role of picture changing operators in QFT and applications to gauge theories was one of the last discussions with Raymond Stora during the last extended period spent by one of the authors at CERN, for that reason this note is dedicated to him.

2 Background material

2.1 $3d, N = 1$

We recall that in $3d$ $N = 1$, the supermanifold $\mathcal{M}^{(3|2)}$ (homeomorphic to $\mathbb{R}^{3|2}$) is described locally by the coordinates (x^a, θ^α) , and in terms of these coordinates, we have the following two differential operators

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - (\gamma^a \theta)_\alpha \partial_a, \quad Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + (\gamma^a \theta)_\alpha \partial_a, \quad (2.1)$$

known as superderivative and supersymmetry generators, respectively. They have the properties

$$\{D_\alpha, D_\beta\} = -2\gamma_{\alpha\beta}^a \partial_a, \quad \{Q_\alpha, Q_\beta\} = 2\gamma_{\alpha\beta}^a \partial_a, \quad \{D_\alpha, Q_\beta\} = 0, \quad (2.2)$$

In $3d$, with $\eta_{ab} = (-, +, +)$, we use real and symmetric Dirac matrices $\gamma_{\alpha\beta}^a$ defined as

$$\begin{aligned} \gamma_{\alpha\beta}^0 &= (C\Gamma^0) = -\mathbf{1}, & \gamma_{\alpha\beta}^1 &= (C\Gamma^1) = \sigma^3, \\ \gamma_{\alpha\beta}^2 &= (C\Gamma^2) = -\sigma^1, & C_{\alpha\beta} &= i\sigma^2 = \epsilon_{\alpha\beta}. \end{aligned} \quad (2.3)$$

Numerically, we have $\hat{\gamma}_a^{\alpha\beta} = \gamma_{\alpha\beta}^a$ and $\hat{\gamma}_a^{\alpha\beta} = \eta_{ab}(C\gamma^b C)^{\alpha\beta} = C^{\alpha\gamma}\gamma_{a,\gamma\delta}C^{\delta\beta}$. The conjugation matrix is $\epsilon^{\alpha\beta}$ and a bi-spinor $R_{\alpha\beta}$ is decomposed as $R_{\alpha\beta} = R\epsilon_{\alpha\beta} + R_a\gamma_{\alpha\beta}^a$ where $R = -\frac{1}{2}\epsilon^{\alpha\beta}R_{\alpha\beta}$ and $R_a = \text{tr}(\gamma_a R)$ are a scalar and a vector, respectively. In addition, it is easy to show that $\gamma_{\alpha\beta}^{ab} \equiv \frac{1}{2}[\gamma^a, \gamma^b] = \epsilon^{abc}\gamma_{c\alpha\beta}$.

For computing the differential of $\Phi^{(0|0)}$, we can use the basis of $(1|0)$ -forms defined as follows

$$\begin{aligned} d\Phi^{(0|0)} &= dx^a\partial_a\Phi^{(0|0)} + d\theta^\alpha\partial_\alpha\Phi^{(0|0)} \\ &= \left(dx^a + \theta\gamma^a d\theta\right)\partial_a\Phi^{(0|0)} + d\theta^\alpha D_\alpha\Phi^{(0|0)} \equiv V^a\partial_a\Phi^{(0|0)} + \psi^\alpha D_\alpha\Phi^{(0|0)}, \end{aligned} \quad (2.4)$$

where $V^a = dx^a + \theta\gamma^a d\theta$ and $\psi^\alpha = d\theta^\alpha$ (for flat supermanifolds) which satisfy the Maurer-Cartan equations

$$dV^a = \psi\gamma^a\psi, \quad d\psi^\alpha = 0. \quad (2.5)$$

Given a $(0|0)$ -form $\Phi^{(0|0)}$, we can compute its supersymmetry variation (viewed as a super translation) as a Lie derivative \mathcal{L}_ϵ with $\epsilon = \epsilon^\alpha Q_\alpha + \epsilon^a\partial_a$ (ϵ^a are the infinitesimal parameters of the translations and ϵ^α are the supersymmetry parameters) and we have

$$\begin{aligned} \delta_\epsilon\Phi^{(0|0)} &= \mathcal{L}_\epsilon\Phi^{(0|0)} = \iota_\epsilon d\Phi^{(0|0)} = \iota_\epsilon \left(dx^a\partial_a\Phi^{(0|0)} + d\theta^\alpha\partial_\alpha\Phi^{(0|0)}\right) \\ &= (\epsilon^a + \epsilon\gamma^a\theta)\partial_a\Phi^{(0|0)} + \epsilon^\alpha\partial_\alpha\Phi^{(0|0)} = \epsilon^a\partial_a\Phi^{(0|0)} + \epsilon^\alpha Q_\alpha\Phi^{(0|0)}, \end{aligned} \quad (2.6)$$

In the same way, acting on $(p|q)$ forms, where p is the form number and q is the picture number, we use the usual Cartan formula $\mathcal{L}_\epsilon = \iota_\epsilon d + d\iota_\epsilon$. It follows easily that $\delta_\epsilon V^a = \delta_\epsilon\psi^\alpha = 0$ and $\delta_\epsilon d\Phi^{(0|0)} = d\delta_\epsilon\Phi^{(0|0)}$.

The top form is represented by the expression

$$\omega^{(3|2)} = \epsilon_{abc}V^a \wedge V^b \wedge V^c \wedge \epsilon_{\alpha\beta}\delta(\psi^\alpha) \wedge \delta(\psi^\beta), \quad (2.7)$$

which has the properties

$$d\omega^{(3|2)} = 0, \quad \mathcal{L}_\epsilon\omega^{(3|2)} = 0. \quad (2.8)$$

It is important to point out the transformation properties of $\omega^{(3|2)}$ under a Lorentz transformation $\text{SO}(2,1)$. Considering V^a , which transforms in the vector representation of $\text{SO}(2,1)$, the combination $\epsilon_{abc}V^a \wedge V^b \wedge V^c$ is clearly invariant. On the other hand, $d\theta^\alpha$ transforms under the spinorial representation of $\text{SO}(2,1)$, say $\Lambda_\alpha^\beta = (\gamma^{ab})_\alpha^\beta\Lambda_{ab}$ with $\Lambda_{ab} \in \text{SO}(2,1)$, and thus an expression like $\delta(\psi^\alpha)$ is not covariant. Nonetheless, the combination $\epsilon^{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta) = 2\delta(\psi^1)\delta(\psi^2)$ is invariant using the formal mathematical properties of distributions. We recall for instance $\psi\delta(\psi) = 0$ and $\psi\delta'(\psi) = -\delta(\psi)$. We recall that $\delta(\psi^\alpha)\wedge\delta(\psi^\beta) = -\delta(\psi^\beta)\wedge\delta(\psi^\alpha)$. In addition, $\omega^{(3|2)}$ has a bigger symmetry group: we can transform the variables (V^a, ψ^α) under an element of the supergroup $\text{SL}(3|2)$. The form $\omega^{(3|2)}$ is a representative of the Berezinian bundle, the equivalent for supermanifolds of the canonical bundle on bosonic manifolds.

2.2 Integral forms

Consider the generalized form multiplication as

$$\wedge : \Omega^{(p|r)}(\mathcal{M}) \times \Omega^{(q|s)}(\mathcal{M}) \longrightarrow \Omega^{(p+q|r+s)}(\mathcal{M}). \quad (2.9)$$

where $0 \leq p, q \leq n$ and $0 \leq r, s \leq m$ with $(n|m)$ are the bosonic and fermionic dimensions of the supermanifold \mathcal{M} . Due to the anticommuting properties of the delta forms this product is by definition equal to zero if the forms to be multiplied contain delta forms localized in the same variables $d\theta$. Being the present section more mathematically oriented, we use the non-supersymmetric differential $(dx^a, d\theta^\alpha)$ instead of (V^a, ψ^α) .

Given the space of pseudo forms $\Omega^{(p|r)}$, a $(p|r)$ -form ω formally reads

$$\omega = \sum_{l,h,r} \omega_{[a_1 \dots a_l](\alpha_1 \dots \alpha_h)[\beta_1 \dots \beta_r]} dx^{a_1} \dots dx^{a_l} d\theta^{\alpha_1} \dots d\theta^{\alpha_h} \delta^{g(\beta_1)}(d\theta^{\beta_1}) \dots \wedge \delta^{g(\beta_r)}(d\theta^{\beta_r}), \quad (2.10)$$

where $g(t)$ denotes the differentiation degree of the Dirac delta function corresponding to the 1-form $d\theta^t$. If $g(t) = 0$ it means that the Dirac delta function has no derivative. The three indices l, h and r satisfy the relation

$$l + h - \sum_{k=1}^r g(\beta_k) = p, \quad \alpha_l \neq \{\beta_1, \dots, \beta_r\} \quad \forall l = 1, \dots, h, \quad (2.11)$$

where the last equation means that each α_l in the above summation should be different from any β_k , otherwise the degree of the differentiation of the Dirac delta function can be reduced and the corresponding 1-form $d\theta^{\alpha_k}$ is removed from the basis. The components $\omega_{[a_1 \dots a_l](\alpha_1 \dots \alpha_m)[\beta_1 \dots \beta_r]}$ of ω are superfields.

In figure 1, we display the complete complex of pseudo-forms. We notice that the first line and the last line are bounded from below and from above, respectively. This is due to the fact that in the first line, being absent any delta functions, the form number cannot be negative, and in the last line, having saturated the number of delta functions we cannot admit any power of $d\theta$ (because of the distributional law $d\theta\delta(d\theta) = 0$).

Before discussing the Chern-Simons action, we analyze the dimension of each space $\Omega^{(p|r)}$. The dimension of $\Omega^{(p|0)}$ is given by the power of the dx^a 1-forms and by the power of the $d\theta$ 1-form

$$dx^{a_1} \dots dx^{a_l} d\theta^{\alpha_1} \dots d\theta^{\alpha_h}, \quad (2.12)$$

where we have decomposed the form degree p into $l + h$ where the degree l is carried by dx and the degree h is carried by $d\theta$. For that decomposition, we have $n(n-1) \dots (n-l+1)/l!$ components coming from $dx^{a_1} \dots dx^{a_l}$ plus $(m+h-1)(m+h-2) \dots m/h!$ coming from $d\theta^{\alpha_1} \dots d\theta^{\alpha_h}$. In the same way, if we consider the integral forms $\Omega^{(n-p|m)}$ of the last line, we see that we can have powers of dx and derivatives on the Dirac delta functions as

$$dx^{a_1} \dots dx^{a_l} \delta^{g(\alpha_1)}(d\theta^{\alpha_1}) \dots \delta^{g(\alpha_m)}(d\theta^{\alpha_m}), \quad (2.13)$$

where $g(t)$ is the order of the derivative on $\delta(t)$. The form degree is $l - \sum_{k=1}^m g(\alpha_k)$.

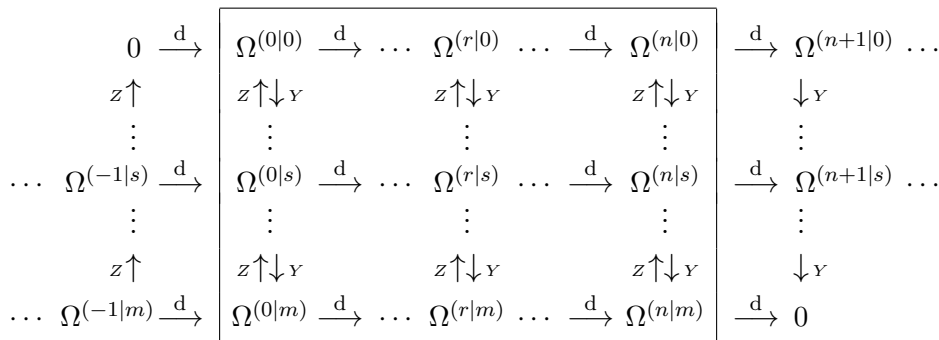


Figure 1. Structure of the supercomplex of forms on a supermanifold of dimension $(m|n)$. The form degree r changes going from left to right while the picture degree s changes going from up to down. The rectangle contains the subset of the supercomplex where the various pictures are isomorphic. In particular the de Rham cohomology is contained in square-box and each line is isomorphic to the other.

For example, for $n = 3, m = 2$ the supermanifold is $\mathcal{M}^{(3|2)}$ and there are three complexes: $\Omega^{(p|0)}, \Omega^{(p|1)}$ and $\Omega^{(p|2)}$. The first one is bounded from below being $\Omega^{(0|0)}$ the lowest space generated by constant functions, the last one is bounded from above with $\Omega^{(3|2)}$ the highest space spanned by the top form and finally, the middle one is unbounded. In addition, the dimension of each space of the first and of the last one is finite, while for the middle one each $\Omega^{(p|1)}$ is infinite dimensional.

The space $\Omega^{(1|0)}$, spanned by $(dx^a, d\theta^\alpha)$, has dimensions $(3|2)$ (which means 3 bosonic generators — instead of dx^a , one can use the supersymmetric variables $V^a = dx^a + \theta\gamma^a d\theta$ — and 2 fermionic generators ψ^α). The space $\Omega^{(2|2)}$ is spanned by

$$\left\{ \epsilon_{abc} dx^b dx^c \delta^2(d\theta), \epsilon_{abc} dx^a dx^b dx^c \iota_\alpha \delta^2(d\theta) \right\},$$

where $\iota_\alpha \delta^2(d\theta)$ denote the derivative of $\delta^2(d\theta)$ with respect $d\theta^\alpha$. It has dimensions $(3|2)$ and therefore there should be an isomorphism between the two spaces. The construction of that isomorphism, which is the generalization of the conventional Hodge dual to supermanifolds, has been provided in [15].

Let us consider another example: the space $\Omega^{(2|0)}$, spanned by

$$\left\{ \epsilon_{abc} dx^b dx^c, dx^a d\theta^\alpha, d\theta^{(\alpha_1} d\theta^{\alpha_2)} \right\},$$

with dimension $(6|6)$. The dual space is $\Omega^{(1|2)}$ and it is spanned by

$$\left\{ dx^a \delta^2(d\theta), \epsilon_{abc} dx^b dx^c \iota_\alpha \delta^2(d\theta), \epsilon_{abc} dx^a dx^b dx^c \iota_{(\alpha_1} \iota_{\alpha_2)} \delta^2(d\theta) \right\},$$

which has again $(6|6)$ dimensions. The last example is the one-dimensional space $\Omega^{(0|0)}$ of 0-forms and its dual $\Omega^{(3|2)}$, a one-dimensional space generated by $d^3x \delta^2(d\theta)$, the top form of the supermanifold $\mathcal{M}^{(3|2)}$.

Now, let consider the middle complex $\Omega^{(1|1)}$ spanned (in the sense of formal series) by the following psuedo-forms

$$\Omega^{(1|1)} = \text{span} \left\{ (d\theta^\alpha)^{n+1} \delta^{(n)}(d\theta^\beta), dx^a (d\theta^\alpha)^n \delta^{(n)}(d\theta^\beta), \right. \\ \left. \epsilon_{abc} dx^b dx^c (d\theta^\alpha)^n \delta^{(n+1)}(d\theta^\beta), \epsilon_{abc} dx^a dx^b dx^c (d\theta^\alpha)^n \delta^{(n+2)}(d\theta^\beta) \right\}_{n \geq 0}, \quad (2.14)$$

where the number n is not fixed and it must be a non-negative integer. Due to the bosonic 1-forms dx^a and due to the fact that the index α must be different from β for a non-vanishing integral form (we recall that $d\theta^\alpha \delta^{(n)}(d\theta^\alpha) = -n\delta^{(n-1)}(d\theta^\alpha)$, and $\delta^{(0)}(d\theta^\alpha) = \delta(d\theta^\alpha)$), the number of generators (monomial forms) at a given n is $(8|8)$, but the total number of monomial generators in $\Omega^{(1|1)}$ is infinite. The dual of $\Omega^{(1|1)}$ is itself, but the isomorphism is realised by an infinite matrix whose entries are $(8|8) \times (8|8)$ supermatrices.

In the same way, for a general supermanifold $\mathcal{M}^{(n|m)}$ any form belonging to the middle complex $\Omega^{(p|r)}$ with $0 < r < m$ is decomposed into an infinite number of components as in (2.14).

In general, if ω is a poly-form in $\Omega^\bullet(\mathcal{M})$ this can be written as direct sum of $(p|q)$ pseudo forms

$$\omega = \sum_{p,q} \omega^{(p|q)}, \quad (2.15)$$

and its integral on the supermanifold is defined as follows: (in analogy with the Berezin integral for bosonic forms):

$$\int_{\mathcal{M}} \omega \equiv \int_M \epsilon^{\alpha_1 \dots \alpha_n} \epsilon^{\beta_1 \dots \beta_m} \omega_{[a_1 \dots a_n][\beta_1 \dots \beta_m]}(x, \theta) [d^n x d^m \theta], \quad (2.16)$$

where the last integral over M is the usual Riemann-Lebesgue integral over the coordinates x^a (if it exists) and the Berezin integral over the coordinates θ^α . The superfields $\omega_{[a_1 \dots a_n][\beta_1 \dots \beta_m]}(x, \theta)$ are the components of the integral form and the symbol $[d^n x d^m \theta]$ denotes the integration variables.

3 Picture raising operator

In the present section, we discuss a class of PCO's relevant to the study of differential forms in $\Omega^{(p|q)}$. In particular we define a new operator that increases the number of delta's (then, increases the picture number), the *Picture Raising Operator*.² It acts vertically mapping superforms into integral forms.

To start with, given a constant commuting vector v^α , consider the following object

$$Y_v = v \cdot \theta \delta(v \cdot \psi), \quad (3.1)$$

²We warn the reader the meaning of raising and lowering is opposite to that used in string theory literature. In that case the picture is carried by the delta of the superghost $\delta(\gamma) = e^{-\phi}$ and it is conventionally taken to be negative, and identified with the ϕ charge.

which has the properties

$$dY_v = 0, \quad Y_v \neq d\eta^{(-1|1)}, \quad Y_{v+\delta v} = Y_v + d\left(\delta v \cdot \theta v \cdot \theta \delta'(v \cdot \psi)\right), \quad (3.2)$$

where $\eta^{(-1|1)}$ is a pseudo-form. Notice that Y_v belongs to $H^{(0|1)}$ (which is the de-Rham cohomology class in $\Omega^{(0|1)}$) and by choosing two independent vectors $v_{(\alpha)}$, we set

$$\mathbb{Y}^{(0|2)} = \prod_{\alpha=1}^2 Y_{v_{(\alpha)}} = \theta^2 \delta^2(\psi), \quad (3.3)$$

The result is independent of v^α . We can apply the PCO operator to a given integral form by taking the wedge product of forms. For example, given ω in $\Omega^{(p|0)}$ we have

$$\omega \longrightarrow \omega \wedge \mathbb{Y}^{(0|2)} = \mathbb{Y}^{(0|2)} \wedge \omega \in \Omega^{(p|2)}. \quad (3.4)$$

If $d\omega = 0$ then $d(\omega \wedge \mathbb{Y}^{(0|2)}) = 0$ (by applying the Leibniz rule), and if $\omega \neq d\eta$ then it follows that also $\omega \wedge \mathbb{Y}^{(0|2)} \neq dU$ where U is an integral form of $\Omega^{(p-1|2)}$. In [1], it has been proved that $\mathbb{Y}^{(0|2)}$ is an element of the de Rham cohomology and that they are also globally defined. So, given an element of the cohomogy $H_d^{(p|0)}$, the new integral form $\omega \wedge \mathbb{Y}^{(0|2)}$ is an element of $H_d^{(p|2)}$.

Let us consider again the example of $\mathcal{M}^{(3|2)}$ and the 2-form $F^{(2|0)} = dA^{(1|0)} \in \Omega^{(2|0)}$ where $A^{(1|0)} = A_a V^a + A_\alpha \psi^\alpha \in \Omega^{(1|0)}$ is an abelian connection. Then, we have

$$F^{(2|0)} \longrightarrow \tilde{F}^{(2|2)} = F^{(2|0)} \wedge \mathbb{Y}^{(0|2)}, \quad (3.5)$$

which satisfies the Bianchi identity $d\tilde{F}^{(2|2)} = 0$.

Since the curvature $\tilde{F}^{(2|2)} = F^{(2|0)} \wedge \mathbb{Y}$ can be also written as $dA^{(1|0)} \wedge \mathbb{Y}^{(0|2)}$, using $d\mathbb{Y}^{(0|2)} = 0$, we have

$$\tilde{F}^{(2|2)} = d\left(A^{(1|0)} \wedge \mathbb{Y}^{(0|2)}\right) = d\tilde{A}^{(1|2)},$$

where $\tilde{A}^{(1|2)}$ is the gauge connection at picture number 2.³ Notice that performing a gauge transformation on $A^{(1|0)}$, we have

$$\delta\tilde{A}^{(1|2)} = d\left(\lambda^{(0|0)} \wedge \mathbb{Y}^{(0|2)}\right),$$

and we can consider $\tilde{\lambda}^{(0|2)} = \lambda^{(0|0)} \wedge \mathbb{Y}^{(0|2)}$ as the gauge parameter at picture number 2.

Finally, we have

$$\begin{aligned} F^{(2|0)} \wedge \mathbb{Y}^{(0|2)} &= \left(\partial_a A_b V^a V^b + \dots + (D_\alpha A_\beta + \gamma_{\alpha\beta}^a A_a) \psi^\alpha \psi^\beta\right) \wedge \mathbb{Y}^{(0|2)} \\ &= (\partial_a A_b \theta^2) V^a V^b \delta^2(\psi) = \partial_{[a} (A_{b]}(x, 0) \theta^2) V^a V^b \delta^2(\psi), \end{aligned} \quad (3.6)$$

where $A_a(x, 0)$ is the lowest component of the superfield A_a appearing in the superconnection $A^{(1|0)}$. This seems puzzling since we have “killed” the complete superfield dependence

³Notice that besides the cases $A^{(1|0)}$ and $A^{(1|2)}$, we can also consider the case with one picture $A^{(1|1)}$, that would be the natural way to distribute the picture for CS theory. This shares similarities with open super string field theory in the A_∞ formulation [16] and it would be interesting to explore this further.

of $A_a(x, \theta)$ leaving aside the first component $A_a(x, 0)$. This happens because $\mathbb{Y}^{(0|2)}$ as defined in (3.3) has an obvious non-trivial kernel.

However, we can modify the PCO given in (3.3) with a more general construction. If we consider a set of anticommuting superfields $\Sigma^\alpha(x, \theta)$ such that $\Sigma^\alpha(x, 0) = 0$. They can be normalised as $\Sigma^\alpha(x, \theta) = \theta^\alpha + K^\alpha(x, \theta)$ with $K^\alpha \approx \mathcal{O}(\theta^2)$. Then, we define a generic PCO as follows

$$\begin{aligned} \mathbb{Y}_{\text{gen}}^{(0|2)} &= \prod_{i=1}^2 \Sigma^{\alpha_i} \delta(d\Sigma^{\alpha_i}) = \prod_{i=1}^2 \Sigma^{\alpha_i} \delta\left((\delta_\beta^{\alpha_i} + D_\beta K^{\alpha_i})\psi^\beta + V^a \partial_a K^{\alpha_i}\right), \\ &= \prod_{i=1}^2 \Sigma^{\alpha_i} \delta\left[(\delta_\beta^{\alpha_i} + D_\beta K^{\alpha_i})\left(\psi^\beta + V^a \frac{\partial_a K^\beta}{(1 + DK)}\right)\right], \end{aligned} \quad (3.7)$$

where $(1 + DK)$ is a $m \times m$ invertible matrix and it should be obvious from the above formula how the indices are contracted. Expanding the Dirac delta function and recalling that the bosonic dimension of the space is 3, we get the formula

$$\begin{aligned} \mathbb{Y}_{\text{gen}}^{(0|2)} &= H(x, \theta) \delta^2(\psi) + K_a^\alpha(x, \theta) V^a \iota_\alpha \delta^2(\psi) \\ &\quad + L_{ab}^{(\alpha\beta)}(x, \theta) V^a V^b \iota_\alpha \iota_\beta \delta^2(\psi) + M_{abc}^{(\alpha\beta\gamma)}(x, \theta) V^a V^b V^c \iota_\alpha \iota_\beta \iota_\gamma \delta^2(\psi), \end{aligned} \quad (3.8)$$

where the superfields $H, K_a^\alpha, L_{ab}^{(\alpha\beta)}$ and $M_{abc}^{(\alpha\beta\gamma)}$ are easily computed in terms of Σ^α and its derivatives. Even if it is not obvious from the above expression, $\mathbb{Y}_{\text{gen}}^{(0|2)}$ is closed and not exact. It belongs to $H^{(0|2)}$ and it is globally defined; this can be checked by decomposing the supermanifold in patches and checking that $\mathbb{Y}_{\text{gen}}^{(0|2)}$ is an element of the Čech cohomology, as carefully done in [1]. Now, if we compute the new field strength $\tilde{F}^{(2|2)}$ by (3.5), one sees that the different pieces in (3.8) from $\mathbb{Y}_{\text{gen}}^{(0|2)}$ are going to pick up different contributions from $F^{(2|0)}$. For instance, the $\psi^\alpha \wedge \psi^\beta$ is soaked up from the third piece in (3.8) with the two derivatives acting on Dirac delta function.

The choice of $\mathbb{Y}_{\text{gen}}^{(0|2)}$ is the key of the present work, since the arbitrariness of the choice of the PCO allows us to relate different superspace formulations. For example, the manifestly supersymmetric invariant PCO

$$\mathbb{Y}_{\text{susy}}^{(0|2)} = V^a \wedge V^b (\gamma_{ab})^{\alpha\beta} \iota_\alpha \iota_\beta \delta^2(\psi). \quad (3.9)$$

is closed as can be easily verified

$$d\mathbb{Y}_{\text{susy}}^{(0|2)} = 2\psi \gamma^a \psi V^b (\gamma_{ab})^{\alpha\beta} \iota_\alpha \iota_\beta \delta^2(\psi) = \text{tr}(\gamma^a \gamma_{ab}) V^b \delta^2(\psi) = 0. \quad (3.10)$$

by using $dV^a = \psi \gamma^a \psi$ and $d\psi^\alpha = 0$. It is not exact, it is invariant under rigid supersymmetry and it differs from $\mathbb{Y}^{(0|2)}$ by exact terms. This PCO can be expanded in different pieces by decomposing V^a and by taking the derivatives ι_α from $\delta^2(\psi)$ to V 's:

$$\mathbb{Y}_{\text{susy}}^{(0|2)} = a_1 dx^a \wedge dx^b (\gamma_{ab})^{\alpha\beta} \iota_\alpha \iota_\beta \delta^2(\psi) + a_2 dx^a \wedge (\gamma^a \theta)^\beta \iota_\beta \delta^2(\psi) + a_3 \theta^2 \delta^2(\psi), \quad (3.11)$$

where the coefficients a_i are fixed by simple Dirac matrix algebra. We notice that all pieces have zero form degree and picture number +2. Another property of $\mathbb{Y}_{\text{susy}}^{(0|2)}$ is its duality

with $\omega^{(3|0)} = \psi\gamma_a\psi V^a$. The latter is an element of the Chevalley-Eilenberg cohomology (see [14] for a complete discussion and references) and therefore it is closed (by using the Fierz identities $\gamma^a\psi(\psi\gamma_a\psi) = 0$) and is not exact. The duality with $\mathbb{Y}_{\text{susy}}^{(0|2)}$ means

$$\omega^{(3|0)} \wedge \mathbb{Y}_{\text{susy}}^{(0|2)} = \epsilon_{abc}V^a \wedge V^b \wedge V^c \delta^2(\psi), \quad (3.12)$$

where $\epsilon_{abc}V^a \wedge V^b \wedge V^c \delta^2(\psi)$ is the volume form belonging to $\Omega^{(3|2)}$.

If the gauge group is non-abelian, the field strength $F^{(2|0)}$ has to be modified in

$$F^{(2|0)} = dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)}, \quad (3.13)$$

where the wedge product of two superform (at picture zero) gives a superform again at picture zero. However, to define a field strength at picture number 2, we immediately see that the product of $A^{(1|2)} \wedge A^{(1|2)} = 0$, independently of the non-abelianity of the gauge group, but because $\delta^3(\psi) = 0$.

4 Super Chern-Simons action

Let's begin by reviewing the standard superspace construction for Chern-Simons. We start from a 1-super form $A^{(1|0)} = A_a V^a + A_\alpha \psi^\alpha$, (where the superfields $A_a(x, \theta)$ and $A_\alpha(x, \theta)$ take value in the adjoint representation of the gauge group) and we define the field strength

$$F^{(2|0)} = dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)} = F_{ab}V^a \wedge V^b + F_{a\alpha}V^a \wedge \psi^\alpha + F_{\alpha\beta}\psi^\alpha \wedge \psi^\beta, \quad (4.1)$$

where

$$\begin{aligned} F_{ab} &= \partial_a A_b - \partial_b A_a + [A_a, A_b], \\ F_{a\alpha} &= \partial_a A_\alpha - D_\alpha A_a + [A_\alpha, A_a], \\ F_{\alpha\beta} &= D_{(\alpha} A_{\beta)} + \gamma_{\alpha\beta}^a A_a + \{A_\alpha, A_\beta\}, \end{aligned} \quad (4.2)$$

In order to reduce the redundancy of degrees of freedom because of the two components A_a and A_α of the $(1|0)$ connection, one imposes (by hand) the conventional constraint

$$\iota_{\alpha\beta} F^{(2|0)} = 0 \quad \iff \quad F_{\alpha\beta} = D_{(\alpha} A_{\beta)} + \gamma_{\alpha\beta}^a A_a + \{A_\alpha, A_\beta\} = 0, \quad (4.3)$$

from which it follows that $F_{a\alpha} = \gamma_{a,\alpha\beta} W^\beta$ and it defines W^α such that $\nabla_\alpha W^\alpha = 0$ (∇^a is the covariant derivative in the adjoint representation). The gaugino field strength W^α is gauge invariant under the non-abelian transformations $\delta A_\alpha = \nabla_\alpha \Lambda$. These gauge transformations descend from the gauge transformations $\delta A = \nabla \Lambda$ where Λ is a $(0|0)$ -form.

The field strengths satisfy the following Bianchi's identities

$$\begin{aligned} \nabla_{[a} F_{bc]} &= 0, \\ \nabla_\alpha F_{ab} + (\gamma_{[a} \nabla_{b]} W)_\alpha &= 0, \\ F_{ab} + \frac{1}{2} (\gamma_{ab})^\alpha{}_\beta \nabla_\alpha W^\beta &= 0, \\ \nabla_\alpha W^\alpha &= 0. \end{aligned} \quad (4.4)$$

and by expanding the superfields A_a, A_α and W^α at the first components we have

$$A_\alpha = (\gamma^a \theta)_\alpha a_a + \lambda_\alpha \frac{\theta^2}{2}, \quad A_a = a_a + \lambda \gamma_a \theta + \dots, \quad W^\alpha = \lambda^\alpha + f_\beta^\alpha \theta^\beta + \dots, \quad (4.5)$$

where $a_a(x)$ is the gauge field, $\lambda_\alpha(x)$ is the gaugino and $f_{\alpha\beta} = \gamma_{\alpha\beta}^{ab} f_{ab}$ is the gauge field strength with $f_{ab} = \partial_a a_b - \partial_b a_a$.

In terms of those fields, the super-Chern-Simons lagrangian becomes

$$S_{\text{SCS}} = \int \text{Tr} \left[A_\alpha \left(W^\alpha - \frac{1}{6} [A_\beta, A^a] \gamma_a^{\alpha\beta} \right) \right] [d^3 x d^2 \theta], \quad (4.6)$$

which in component reads

$$S_{\text{SCS}} = \int d^3 x \text{Tr} \left[\epsilon^{abc} \left(a_a \partial_b a_c + \frac{2}{3} a_a a_b a_c \right) + \lambda_\alpha \epsilon^{\alpha\beta} \lambda_\beta \right]. \quad (4.7)$$

That coincides with the bosonic Chern-Simons action with free non-propagating fermions.

In order to obtain an action principle by integration on supermanifolds we consider the natural candidates for the super-Chern-Simons lagrangian

$$\mathcal{L}^{(3|0)} = \text{Tr} \left[A^{(1|0)} \wedge dA^{(1|0)} + \frac{2}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} \right], \quad (4.8)$$

where $A^{(1|0)}$ is the superconnection and d is the differential on the superspace, and then we multiply it by a PCO, for example by $\mathbb{Y}^{(0|2)} = \theta^2 \delta^2(\psi)$ discussed in (3.3). That leads to (3|2) integral form that can be integrated on the supermanifold, that is

$$S_{\text{SCS}} = \int_{\mathcal{M}^{(3|2)}} \mathbb{Y}^{(0|2)} \wedge \text{Tr} \left[A^{(1|0)} \wedge dA^{(1|0)} + \frac{2}{3} A^{(1|0)} \wedge A^{(1|0)} \wedge A^{(1|0)} \right]. \quad (4.9)$$

However, this action fails to give the correct answer yielding only the bosonic part of the action of S_{SCS} . The reason is that the supersymmetry transformations of the PCO is

$$\delta_\epsilon \mathbb{Y} = d \left[\theta^2 \iota_\epsilon \delta^2(\psi) \right], \quad (4.10)$$

and by integrating by parts, we find that the action is not supersymmetric invariant. On the other hand, as we observed in the previous section, we can use the new operator

$$\mathbb{Y}_{\text{susy}}^{(0|2)} = V^a \wedge V^b (\gamma_{ab})^{\alpha\beta} \iota_{\alpha\beta} \delta^2(\psi), \quad (4.11)$$

which is manifestly supersymmetric. Computing the expression in the integral, we see that $\mathbb{Y}_{\text{susy}}^{(0|2)}$ picks up at least two powers of ψ 's and one power of V^a and that forces us to expand $\mathcal{L}^{(3|0)}$ as 3-form selecting the monomial $\psi \gamma_a \psi V^a$ dual to $\mathbb{Y}_{\text{susy}}^{(0|2)}$. Explicitly we find

$$S_{\text{SCS}} = \int \text{Tr} \left[A_\alpha (\partial_b A_\gamma - D_\alpha A_b) \gamma_{ac}^{\alpha\gamma} \epsilon^{abc} \right. \\ \left. + A_a (D_\beta A_\gamma + D_\beta A_\gamma) \gamma_{bc}^{\beta\gamma} \epsilon^{abc} - \frac{1}{6} A_\alpha [A_\beta, A^a] \gamma_a^{\alpha\beta} \right] [d^3 x d^2 \theta]. \quad (4.12)$$

That finally gives the supersymmetric action described in (4.6), together with the conventional constraint $F_{\alpha\beta} = 0$.

Some observations are in order.

1. The equations of motion derived from the new action (1.3) are

$$\begin{aligned} \mathbb{Y}_{\text{susy}}^{(0|2)}(dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)}) = 0 &\implies \\ V^3(\gamma^a \iota)^{\alpha} \delta^2(\psi) F_{a\alpha} + (V^a \wedge V^b) \epsilon_{abc} (\gamma^c)^{\alpha\beta} F_{\alpha\beta} = 0. &\quad (4.13) \end{aligned}$$

The equations of motion correctly imply $F_{\alpha\beta} = 0$ (which is the conventional constraint) and $W^\alpha = 0$ which are the super-Chern-Simons equations of motion. The second condition follows from $F_{\alpha\beta} = 0$ and by the Bianchi identities which implies that $F_{a\alpha} = \gamma_{a\alpha\beta} W^\beta$.

Notice that this formulation allows us to get the conventional constraint as an equation of motion. In particular we find that the equation of motion, together with the Bianchi identity imply the vanishing of the full field-strength.

$$\begin{cases} \mathbb{Y}_{\text{susy}}^{(0|2)} F^{(2|0)} = 0, \\ dF^{(2|0)} + [A^{(1|0)}, F^{(2|0)}] = 0, \end{cases} \implies F^{(2|0)} = 0. \quad (4.14)$$

2. Consider instead of the flat superspace $R^{(3|2)}$, the group manifold with the underlying supergroup $\text{Osp}(1|2)$. The corresponding Maurer-Cartan equations are

$$dV^a + \epsilon^a_{bc} V^b \wedge V^c + \psi \gamma^a \psi = 0, \quad d\psi^\alpha + (\epsilon \gamma_a)^\alpha_{\beta} V^a \psi^\beta = 0. \quad (4.15)$$

Then, it is easy to show that

$$d\mathbb{Y}_{\text{susy}}^{(0|2)} = 0, \quad \delta_\epsilon \mathbb{Y}_{\text{susy}}^{(0|2)} = 0. \quad (4.16)$$

The second equation is obvious since it is expressed in terms of supersymmetric invariant quantities. The first equation follows from the MC equations and gamma matrix algebra. Chern-Simons theory on this group supermanifold share interesting similarities with a particular version of open super string field theory [17, 18]. The reason for this is that the supergroup $\text{Osp}(1|2)$ is infact the superconformal Killing group of an $N = 1$ SCFT on the disk. There is however an important difference w.r.t. to [17, 18]. Our choice of the picture changing operator \mathbb{Y} applied to the field strength $(dA^{(1|0)} + A^{(1|0)} \wedge A^{(1|0)})$ leads to equation (4.13) and it directly implies the vanishing of the full field strength. In particular the kernel of the picture-changing operator is harmless in our case. It would be interesting to search for an analogous object in the RNS string.

3. The PCO $\mathbb{Y}_{\text{susy}}^{(0|2)}$ is related to the product of two non-covariant operators, each shifting the picture by one unit.

$$Y_v = V^a v_\alpha \gamma_a^{\alpha\beta} \iota_\beta \delta(v \cdot \psi), \quad Y_w = V^a w_\alpha \gamma_a^{\alpha\beta} \iota_\beta \delta(w \cdot \psi), \quad (4.17)$$

with $v \cdot w \neq 0$ and by a little a bit of algebra, one gets

$$\mathbb{Y}_{\text{susy}}^{(0|2)} = Y_v Y_w + d\Omega. \quad (4.18)$$

The PCO's Y_v and Y_w are closed (in the case of flat superspace, while in the case of $\text{Osp}(1|2)$, they are invariant if v and w transform under the corresponding isometry transformations). They are also supersymmetric invariant because written in terms of invariant quantities.

The piece Ω is a $(-1|2)$ form which depends on v and w . The two PCO's are equivalent in the sense that they belong to the same cohomology class, but they behave differently off-shell. One can check by direct inspection that this PCO does not lead to the conventional constraint $F_{\alpha\beta} = 0$ and therefore the exact term in (4.18) relating the two actions is important to get the full-fledged action principle.

4. We study the kernel of the PCO $\mathbb{Y}^{(0|2)}$ and of the new PCO $\mathbb{Y}_{\text{susy}}^{(0|2)}$.

Acting on the complete set of differential form $\Omega^{(p|q)}$, with the PCO's, for $\omega^{(p|q)} \in \Omega^{(p|q)}$ with $q > 0$, we have $\mathbb{Y}^{(0|2)} \wedge \omega^{(p|q)} = 0$ due to the anticommuting properties of $\delta(d\theta)$. Therefore, we need to study only $\Omega^{(p|0)}$. We observe that $\mathbb{Y}^{(0|2)} \wedge \omega^{(0|0)} = 0$, this implies $\omega^{(0|0)} = f_{1,\alpha}(x)\theta^\alpha + f_2(x)\theta^2$. In the same way, given a 1-form of $\Omega^{(1|0)}$, we have $\omega^{(1|0)} = \omega_a(x, \theta)V^a + \omega_\alpha(x, \theta)\psi^\alpha$. Then, the kernel of $\mathbb{Y}^{(0|2)}$ on $\Omega^{(1|0)}$ is given by

$$\omega^{(1|0)} = \left(\omega_{1,a\alpha}(x)\theta^\alpha + \omega_{2,a}(x)\theta^2 \right) V^a + \omega_\alpha(x, \theta)\psi^\alpha. \quad (4.19)$$

For higher p -forms, we have similar kernels. For instance, in the case of 2-forms $\Omega^{(2|0)}$, we have

$$\begin{aligned} \omega^{(2|0)} = & \left(\omega_{1,ab\alpha}(x)\theta^\alpha + \omega_{2,ab}(x)\theta^2 \right) V^a \wedge V^b \\ & + \omega_{a\alpha}(x, \theta)V^a \wedge \psi^\alpha + \omega_{\alpha\beta}(x, \theta)\psi^\alpha \wedge \psi^\beta, \end{aligned} \quad (4.20)$$

Let us study the kernel of the new PCO $\mathbb{Y}_{\text{susy}}^{(0|2)}$. On $\Omega^{(0|0)}$, there is no kernel. Acting on $\omega^{(1|0)} = \omega_a(x, \theta)V^a + \omega_\alpha(x, \theta)\psi^\alpha$, we have

$$\begin{aligned} \mathbb{Y}_{\text{susy}}^{(0|2)} \wedge \omega^{(1|0)} = & V^3 \epsilon^{abc} \omega_c(x, \theta) (\gamma_{ab})^{\alpha\beta} \iota_\alpha \iota_\beta \delta^2(\psi) \\ & + 2V^a \wedge V^b (\gamma_{ab})^{\alpha\beta} \omega_\alpha(x, \theta) \iota_\beta \delta^2(\psi) = 0. \end{aligned} \quad (4.21)$$

Since the two forms $V^a \wedge V^b (\gamma_{ab})^{\alpha\beta} \iota_\alpha \iota_\beta \delta^2(\psi)$ and $V^3 \epsilon^{abc} (\gamma_{ab})^{\alpha\beta} \iota_\alpha \iota_\beta \delta^2(\psi)$ generate the space $\Omega^{(1|2)}$ (which has dimension $(3|2)$), the kernel of $\mathbb{Y}_{\text{susy}}^{(0|2)}$ is given by the solution of

$$\epsilon^{abc} \omega_c(x, \theta) (\gamma_{ab})^{\alpha\beta} = 0, \quad \epsilon^{abc} (\gamma_{ab})^{\alpha\beta} \omega_\alpha(x, \theta) = 0 \quad (4.22)$$

which imply that $\omega_c(x, \theta) = \omega_\alpha(x, \theta) = 0$. Thus, there is no kernel on $\Omega^{(1|0)}$. We move to the more important class: $\Omega^{(2|0)}$. For that we consider the generic 2-form, and the kernel equation gives

$$\gamma_{ab}^{\alpha\beta} \omega_{\alpha\beta}(x, \theta) = 0, \quad \gamma_{ab}^{\alpha\beta} \epsilon^{abc} \omega_{c\alpha}(x, \theta) = 0. \quad (4.23)$$

No condition imposed on $\omega_{ab}(x, \theta)$. The first equation implies that $\omega_{\alpha\beta}(x, \theta) = 0$, while, by decomposing $\omega_{c\alpha}(x, \theta) = (\gamma_c)^\beta \tilde{\omega}_{\alpha\beta\gamma} + (\gamma_c)_{\alpha\beta} \hat{\omega}^\beta$ where $\tilde{\omega}_{\alpha\beta\gamma}(x, \theta)$ is totally symmetric in the spinorial indices, we have $\hat{\omega}^\beta = 0$. The reason why $\mathbb{Y}_{\text{susy}}^{(0|2)}$ works in the construction of an action is that the $\tilde{\omega}_{\alpha\beta\gamma}(x, \theta)$ component of the field strength is independently set to zero by the Bianchi identity. In the same way, one can analyze further higher p -forms.

5 Changing the PCO and the relation between different superspace formulations

During the last thirty years, we have seen two independent superspace formalisms taking place, aiming to describe supersymmetric theories from a geometrical point of view. They are known as *superspace* technology, whose basic ingredients are collected in series of books (see for example [19, 20]) and the *rheonomic* (also known as *group manifold*) formalism (see the main reference book [14]). They are based on a different approach and they have their own advantages and drawbacks. Without entering the details of those formalisms, we would like to illustrate some of their main features on the present example of super-Chern-Simons theories. A basic difference is that in the superspace few superfields contain the basic fields of the theory as components, while in the rheonomic approach any basic field of the theory is promoted to a superfield.

Let us start from the rheonomic action. This is given as follows

$$S_{\text{rheo}}[A, \mathcal{M}^3] = \int_{\mathcal{M}^3 \subset \mathcal{M}^{(3|2)}} \mathcal{L}^{(3)}(A), \quad (5.1)$$

where \mathcal{M}^3 is a three-dimensional surface immersed into the supermanifold $\mathcal{M}^{(3|2)}$ and $\mathcal{L}^{(3)}(A)$ is defined as a three-form Lagrangian constructed with the superform A , its derivatives without the Hodge dual operator (that is without any reference to a metric on the supermanifold $\mathcal{M}^{(3|2)}$). Notice that the fields A are indeed superforms whose components are superfields. We will give the explicit form of $\mathcal{L}^{(3)}(A)$ shortly.

The action $S_{\text{rheo}}[A, \mathcal{M}^3]$ is a functional of the superfields and of the embedding of \mathcal{M}^3 into $\mathcal{M}^{(3|2)}$. We can then consider the classical equations of motion by minimizing the action both respect to the variation of the fields and of the embedding. However, the variation of the immersion can be compensated by diffeomorphisms on the fields if the action $\mathcal{L}^{(3)}$ is a differential form. This implies that the complete set of equations associated to the action (5.1) are the usual equations obtained by varying the fields on a fixed surface \mathcal{M}^3 with the proviso that these equations hold not only on \mathcal{M}^3 , but on the whole supermanifold $\mathcal{M}^{(3|2)}$, namely the Lagrangian is a function of $(x^a, \theta^\alpha, V^a, \psi^\alpha)$.

The rules to build the action (5.1) are listed and discussed in the book [14] in detail. An important ingredient is the fact that for the action to be supersymmetric invariant, the Lagrangian must be invariant up to a d -exact term and, in addition, if the algebra of supersymmetry closes off-shell (either because there is no need of auxiliary fields or because there exists a formulation with auxiliary fields), the Lagrangian must be closed: $d\mathcal{L}^{(3)}(A) = 0$, upon using the rheonomic parametrization. One of the rules of the geometrical construction for supersymmetric theories given in [14] is that by setting to zero the coordinates θ^α and its differential $\psi^\alpha = d\theta^\alpha$, the action

$$S_{\text{rheo}}[A] = \int_{\mathcal{M}^3} \mathcal{L}^{(3)}(A) \Big|_{\theta=0, d\theta=0}, \quad (5.2)$$

reduces to the component action invariant under supersymmetry. Furthermore, the equations of motion in the full-fledged superspace imply the rheonomic constraints (which coincide with the conventional constraints of the superspace formalism).

In order to express the action (5.1) in a more geometrical way by including the dependence upon the embedding into the integrand, we refer to [21] and we introduce the Poincaré dual form $\mathbb{Y}^{(0|2)} = \theta^2 \delta^2(\psi)$. As already discussed in the previous section, $\mathbb{Y}^{(0|2)}$ is closed and its supersymmetry variation is d -exact. The action can be written on the full supermanifold as

$$S[A] = \int_{\mathcal{M}^{(3|2)}} \mathcal{L}^{(3|0)}(A) \wedge \mathbb{Y}^{(0|2)}, \tag{5.3}$$

Therefore the factor θ^2 projects the Lagrangian $\mathcal{L}^{(3|0)}(A)$ to $\mathcal{L}^{(3)}(A)|_{\theta=0}$ while the factor $\delta^2(\psi)$ projects the latter to $\mathcal{L}^{(3)}(A)|_{\theta=0, \psi=0}$ reducing $S[A]$ to the component action (4.7) and it also coincides with (5.2).

Any variation of the embedding is reproduced by d -exact variation of the PCO, namely $\delta \mathbb{Y}^{(0|2)} = d\Lambda^{(-1|2)}$, and it leaves the action invariant if the Lagrangian is closed. In the case of Chern-Simons discussed until now, the chosen action was identified only with the bosonic term $A \wedge dA$, but that turns out to be not closed. Therefore, that has to be modified it as follows: besides the gauge field a_μ , there is the gaugino λ_α which are the zero-order components of the supergauge field $A(x, \theta)$ and of the spinorial superfield $W^\alpha(x, \theta)$. Therefore the complete closed action reads

$$\begin{aligned} S[A] &= \int_{\mathcal{M}^{(3|2)}} \mathcal{L}^{(3|0)} \wedge \mathbb{Y}^{(0|2)}, \\ \mathcal{L}^{(3|0)} &= \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A + W^\alpha W_\alpha V^3 \right), \end{aligned} \tag{5.4}$$

where $\mathcal{L}^{(3|0)}$ is a (3|0) form.⁴ Imposing the closure of $\mathcal{L}^{(3|0)}$ we get the rheonomic parametrizations of the curvatures, or differently said, the conventional constraints. Once this is achieved, we are free to choose any PCO in the same cohomology class. If we choose the PCO $\mathbb{Y}^{(0|2)} = \theta^2 \delta^2(\psi)$ we get directly the component action (4.7) and the third term in the action is needed to get the mass term for the non-dynamical fermions. On the other hand, by choosing $\mathbb{Y}_{\text{susy}}^{(0|2)}$, (1.2) the last term drops out because of the anti-symmetrized powers of V^a and, by using the Bianchi identities (4.4), the final expression can be rewritten as the superspace action (4.6).

This is the most general action and the closure of $\mathcal{L}^{(3|0)}$ implies that any gauge invariant and supersymmetric action can be built by choosing $\mathbb{Y}^{(0|2)}$ inside of the same cohomology class. Therefore, starting from the rheonomic action, one can choose a different “gauge” — or better said a different embedding of the submanifold \mathcal{M}^3 inside the supermanifold $\mathcal{M}^{(3|2)}$ — leading to different forms of the action with the same physical content. It should be stressed, however, that the choice of $\mathbb{Y}_{\text{susy}}^{(0|2)}$, (1.2), is a convenient “gauge” choice, which imply the conventional constraints by varying the action without using the rheonomic parametrization.

⁴This (3|0) Lagrangian in (5.4) already appeared in [22] by reducing their formula from $N = 2$ to $N = 1$.

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