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Nonlinear gauge invariance and WZW-like action for NS-NS superstring field theory

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Abstract: We complete the construction of a gauge-invariant action for NS-NS superstring field theory in the large Hilbert space begun in arXiv:1305.3893 by giving a closed-form expression for the action and nonlinear gauge transformations. The action has the WZW-like form and vertices are given by a pure-gauge solution of NS heterotic string field theory in the small Hilbert space of right movers.

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Contents

1 Introduction

While bosonic string field theories have been well-understood [1–8], superstring field theories remain mysterious. A formulation of supersymmetric theories in the early days [9], which is a natural extension of bosonic theory, has some disadvantages caused by picture-changing operators inserted into string products: singularities and broken gauge invariances [10]. To remedy these, various approaches have been proposed within the same Hilbert space of (β, γ) [11–17].

There exists an alternative formulation of superstring field theory: large space theory [18–26]. Large space theories are formulated by utilizing the extended Hilbert space of (ξ, η, ϕ) [27] and the WZW-like action including no explicit insertions of picture-changing operators. One can check the variation of the action, the equation of motion, and gauge invariance without taking account of these operators. Of course, the action implicitly includes picture-changing operators, which appear when we concretely compute scattering amplitudes after gauge fixing. The singular behavior of them is, however, completely regulated and there is no divergence [28, 29].

The cancellation of singularities can also occur in the small Hilbert space. Recently, by the brilliant works of [30, 31], it is revealed how to obtain gauge-invariant insertions of picture-changing operators into (super-) string products in the small Hilbert space: the NS and NS-NS sectors of superstring field theories in the small Hilbert space is completely formulated. In this paper, we find that using the elegant technique of [31], one can construct the WZW-like action for NS-NS superstring field theory in the large Hilbert space.

A pure-gauge solution of small-space theory is the key concept of WZW-like formulation of NS superstring field theory in the large Hilbert space, which determines the vertices of theory, and we expect that it goes in the case of the NS-NS sector. There is an attempt to construct non-vanishing interaction terms of NS-NS string fields utilizing a pure-gauge solution $\mathcal{G}_{\mathcal{B}}$ of *bosonic* closed string field theory [22]. However, the construction is not complete: the nonlinear gauge invariance is not clear and the defining equation of $\mathcal{G}_{\mathcal{B}}$ is ambiguous. To obtain nonlinear gauge invariances, we have to add appropriate terms to these interaction terms defined by $\mathcal{G}_{\mathcal{B}}$ at each order. Then, the ambiguities of vertices are removed and we obtain the defining equation of a suitable pure-gauge solution $\mathcal{G}_L$, which we explain in the following sections.

In this paper, we complete this construction begun in [22] by determining these additional terms which are necessitated for the nonlinear gauge invariance and by giving closed-form expressions for the action and nonlinear gauge transformations in the NS-NS sector of closed superstring field theory. We propose the action

$$S = \int_0^1 dt\, \langle \eta \Psi_t,\, \mathcal{G}_L(t) \rangle, \tag{1.1}$$

where Ψ_t is an NS-NS string field Ψ plus $\tilde{\eta}$-exact terms and $\mathcal{G}_L$ is a pure-gauge solution to the NS *heterotic* string equation of motion in *the small Hilbert space of right movers*. The action has the WZW-like form and the almost same algebraic properties as the large-space action for NS open and NS closed (heterotic) string field theory [20, 21].

This paper is organized as follows. In section 2 we show that cubic and quartic actions can be determined by adding appropriate terms and imposing gauge invariance. In section 3, we briefly review the method of gauge-invariant insertions of picture-changing operators [30, 31] and provide some useful properties of the $(-,\text{NS})$ closed superstring products. In section 4, first, we give the defining equation of $\mathcal{G}_L$ and associated fields which are necessitated to construct the NS-NS action. Then, we derive the WZW-like expression for the action and nonlinear gauge transformations and show that $\eta\,\mathcal{G}_L = 0$ gives the NS-NS superstring equation of motion just as other large-space theories. We end with some conclusions.

2 Nonlinear gauge invariance

Let κ be the coupling constant of closed string fields. We expand an action S for NS-NS string field theory in powers of κ: $S = \frac{2}{\alpha'} \sum_n \kappa^n S_{n+2}$. In the large Hilbert space, which includes the ξ- and $\tilde{\xi}$-zero modes coming from bosonization of the $\beta\gamma$- and $\tilde{\beta}\tilde{\gamma}$-systems, the NS-NS string field Ψ is a Grassmann even, (total) ghost number 0, left-mover picture number 0, and right-mover picture number 0 state. The free action S_2 is given by

$$S_2 = \frac{1}{2}\langle \eta\Psi, Q\,\tilde{\eta}\Psi\rangle, \tag{2.1}$$

where Q is the BRST operator, η is the zero mode of the left-moving current $\eta(z)$, and $\tilde{\eta}$ is the zero mode of the right-moving current $\tilde{\eta}(\tilde{z})$ [19]. The bilinear is the c_0^--inserted BPZ inner product: $\langle A, B\rangle \equiv \langle \mathrm{bpz}(A)|c_0^-|B\rangle$, where $c_0^- = \frac{1}{2}(c_0 - \tilde{c}_0)$. For brevity, we use the symbol $(\mathbf{G}|\mathbf{p}, \tilde{\mathbf{p}})$ which denotes that the total ghost number is $\mathbf{G}$, the left-mover picture number is $\mathbf{p}$, and the right-mover picture number is $\tilde{\mathbf{p}}$. Then, ghost-and-picture numbers of Ψ, Q, η, and $\tilde{\eta}$ are $(0|0,0)$, $(1|0,0)$, $(1|-1,0)$, and $(1|0,-1)$ respectively. Note that the inner product $\langle A, B\rangle$ gives a nonzero value if and only if the sum of A's and B's ghost-and-picture numbers is equal to $(\mathbf{G}|\mathbf{p}, \tilde{\mathbf{p}}) = (3|-1,-1)$. Computing the variation of this action $\delta S_2 = \langle \delta\Psi, \eta Q\tilde{\eta}\Psi\rangle$, we obtain the equation of motion $Q\eta\tilde{\eta}\Psi = 0$ and find that S_2 is invariant under the gauge transformation

$$\delta_1\Psi = Q\,\Lambda + \eta\,\Omega + \tilde{\eta}\,\widetilde{\Omega}, \tag{2.2}$$

where Λ, Ω, and $\widetilde{\Omega}$ are gauge parameters of Q-, η, and $\tilde{\eta}$-gauge transformations respectively.

We would like to construct nonzero and nonlinear gauge-invariant interacting terms S_3, S_4, S_5, $\ldots$ using string fields Ψ belonging to the large Hilbert space. In the kinetic term S_2, there exist three generators of gauge transformations: Q, η, and $\tilde{\eta}$. However, as we will see, only Q- and η-gauge invariances are extended to be nonlinear and $\tilde{\eta}$-gauge invariance remains to be linear in our interacting theory. Consequently, with two nonlinear and one trivial gauge invariances, the theory is Wess-Zumino-Witten-likely formulated.

In section 2, starting with the action proposed in [22] and adding appropriate terms at each order of κ, we construct cubic and quartic terms S_3, S_4 of the action, whose nonlinear gauge transformations of Q and η take WZW-like forms just as other large-space theories.

2.1 Cubic vertex

Let ξ and $\tilde{\xi}$ be the zero modes of $\xi(z)$- and $\tilde{\xi}(\tilde{z})$-ghosts respectively, and X and $\widetilde{X}$ be the zero modes of left- and right-moving picture-changing operators respectively. The $(n+2)$-point interaction term S_{n+2} proposed in [22] includes $\langle \eta\Psi, \big[(Q\widetilde{Q}\Psi)^n, \tilde{\eta}\Psi\big]\rangle$ to correspond to the result of first quantization, where $[A_1, \ldots, A_n]$ is the bosonic string n-product and $\widetilde{Q} := \tilde{\eta}\tilde{\xi}\cdot Q\cdot\tilde{\xi}\tilde{\eta}$ is a projected BRST operator. This term becomes $\langle \xi\mathcal{V}, \big[(X\widetilde{X}\mathcal{V})^n, \tilde{\xi}\mathcal{V}\big]\rangle$ under partial gauge fixing $\Psi = \xi\tilde{\xi}\,\mathcal{V}_{(2|-1,-1)}$. However, naive use of this term makes nonlinear gauge invariance not clear. In this subsection, as the simplest example, we show that a gauge-invariant cubic term S_3 can be obtained by adding appropriate terms to

$$\langle \eta\Psi, [Q\widetilde{Q}\Psi, \tilde{\eta}\Psi]\rangle = \langle \eta\Psi, [\widetilde{X}Q\,\tilde{\eta}\Psi, \tilde{\eta}\Psi]\rangle. \tag{2.3}$$

Cyclicity. It would be helpful to consider the cyclicity of vertices. A $(n+1)$-point vertex V_n is called a (BPZ-) cyclic vertex if V_n satisfies

$$\langle A_0, V_n(A_1, \ldots, A_n)\rangle = (-)^{A_0(A_1+\cdots+A_n)}\langle A_1, V_n(A_2, \ldots, A_n, A_0)\rangle. \tag{2.4}$$

The upper index of $(-1)^A$ means the ghost number of A. When the $(n+1)$-point action S_{n+1} is given by $S_{n+1} = \frac{1}{(n+1)!}\langle\Psi, V_n(\Psi^n)\rangle$ using a cyclic vertex V_n, its variation becomes

$$\delta S_{n+1} = \frac{1}{n!}\langle\delta\Psi, V_n(\Psi^n)\rangle. \tag{2.5}$$

Then, if there exist a zero divisor of the state $V_n(\Psi^n)$, it generates gauge transformations. For example, bosonic string field theories and superstring field theories in the small Hilbert space are the case and their gauge transformations are determined by L_∞- or A_∞-algebras.

Next, we consider the following case: a vertex V'_n is not cyclic but has the following property

$$\delta S_{n+1} = \frac{1}{n!}\langle\delta\Psi, V'_n(\Psi^n) + W_n(\Psi^n)\rangle. \tag{2.6}$$

Nonzero W_n implies that the cyclicity of V_n is broken. For instance, this relation holds when V'_n consists of a cyclic vertex with operator insertions: $V'_n(\Psi^n) = V_n(Q\Psi^k, \Psi^{n-k})$ or V'_n consists of some combination of cyclic vertices $V'_n(\Psi^n) = V_{m+1}(V_{n-m}(\Psi^{n-m}), \Psi^m)$. In this case, in general, a zero divisor of $V'_n + W_n$ gives the generator of gauge transformations. However, when W_n consists of lower $V_{k<n}$, there exists a special case that the zero divisor of V_n gives the generator of gauge transformations just as WZW-like formulations of superstring field theories in the large Hilbert space, which is the subject of this paper.

Adding terms. We know that *naive* insertions of operators which do not work as derivations of string products, such as ξ, $\widetilde{\xi}$, X, $\widetilde{X}$, and $\widetilde{Q}$, makes nonlinear gauge invariances not clear. We show that a cubic vertex satisfying the special case of (2.6) can be constructed by adding appropriate terms and by imposing gauge invariances. Computing the variation of (2.3), we obtain

$$\begin{aligned}\delta\Big(\langle\eta\Psi, [Q\widetilde{Q}\Psi, \tilde{\eta}\Psi]\rangle\Big) &= \langle\delta\Psi, \eta[Q\widetilde{Q}\Psi, \tilde{\eta}\Psi] + \widetilde{Q}[Q\Psi, \tilde{\eta}\Psi] - [Q\tilde{\eta}\Psi, \widetilde{Q}\Psi]\rangle \\ &= \langle\delta\Psi, \eta\Big\{2[Q\widetilde{Q}\Psi, \tilde{\eta}\Psi] + \widetilde{Q}[Q\Psi, \tilde{\eta}\Psi]\Big\} + \Big\{2\widetilde{Q}[\Psi, Q\tilde{\eta}\eta\Psi] - [\tilde{\eta}\Psi, \eta Q\widetilde{Q}\Psi\Big\}\rangle.\end{aligned} \tag{2.7}$$

We therefore add the term $\langle\eta\Psi, \widetilde{Q}[Q\Psi, \tilde{\eta}\Psi]\rangle$ for (2.6). The variation of this term becomes

$$\begin{aligned}\delta\Big(\langle\eta\Psi, \widetilde{Q}[Q\Psi, \tilde{\eta}\Psi]\rangle\Big) &= \langle\delta\Psi, \eta\widetilde{Q}[Q\Psi, \tilde{\eta}\Psi] + Q[\eta\widetilde{Q}\Psi, \tilde{\eta}\Psi] - \tilde{\eta}[\eta\widetilde{Q}\Psi, Q\Psi]\rangle \\ &= \langle\delta\Psi, \eta\Big\{\widetilde{Q}[Q\Psi, \tilde{\eta}\Psi] + 2[Q\tilde{\eta}\Psi, \widetilde{Q}\Psi]\Big\} + \Big\{2[Q\tilde{\eta}\eta\Psi, \widetilde{Q}\Psi] - [\tilde{\eta}\Psi, \eta Q\widetilde{Q}\Psi\Big\}\rangle.\end{aligned} \tag{2.8}$$

Hence, the term $\langle\eta\Psi, [Q\tilde{\eta}\Psi, \widetilde{Q}\Psi]\rangle$ is necessitated for the property (2.6). The variation of this term is given by

$$\begin{aligned}\delta\Big(\langle\eta\Psi, [Q\tilde{\eta}\Psi, \widetilde{Q}\Psi]\rangle\Big) &= \langle\delta\Psi, \eta[Q\tilde{\eta}\Psi, \widetilde{Q}\Psi] + Q\tilde{\eta}[\eta\Psi, \widetilde{Q}\Psi] - \widetilde{Q}[\eta\Psi, Q\tilde{\eta}\Psi]\rangle \\ &= \langle\delta\Psi, \eta\Big\{[Q\widetilde{Q}\Psi, \tilde{\eta}\Psi] + \widetilde{Q}[Q\Psi, \tilde{\eta}\Psi] + [Q\tilde{\eta}\Psi, \widetilde{Q}\Psi]\Big\} \\ &\qquad + \Big\{\widetilde{Q}[\Psi, Q\tilde{\eta}\eta\Psi] - [\widetilde{Q}\Psi, Q\tilde{\eta}\eta\Psi] - [\tilde{\eta}\Psi, \eta Q\widetilde{Q}\Psi\Big\}\rangle.\end{aligned} \tag{2.9}$$

Averaging these three terms, we obtain the cubic action satisfying (2.6)

$$S_3 = \frac{1}{3!}\langle\eta\Psi, \frac{1}{3}\Big(\widetilde{Q}[Q\Psi, \tilde{\eta}\Psi] + [Q\widetilde{Q}\Psi, \tilde{\eta}\Psi] + [Q\tilde{\eta}\Psi, \widetilde{Q}\Psi]\Big)\rangle.$$

The variation of this cubic action is given by

$$\delta S_3 = \frac{1}{2}\langle \delta\Psi, \frac{\eta}{3}\Big(\widetilde{Q}[Q\Psi,\tilde{\eta}\Psi] + [Q\widetilde{Q}\Psi,\tilde{\eta}\Psi] + [Q\tilde{\eta}\Psi,\widetilde{Q}\Psi]\Big)\rangle$$
$$+\frac{1}{2}\langle \delta\Psi, \frac{1}{3}\Big(\widetilde{Q}[\Psi,\eta Q\tilde{\eta}\Psi] - [\widetilde{Q}\Psi,\eta Q\tilde{\eta}\Psi] - [\tilde{\eta}\Psi,\eta Q\widetilde{Q}\Psi]\Big)\rangle$$
$$= \frac{1}{2}\langle \delta\Psi, V_3(\Psi^3)\rangle - \frac{1}{2}\langle \delta\Psi, \frac{1}{3}\Big(\widetilde{X}[\tilde{\eta}\Psi, V_1] + [\widetilde{X}\tilde{\eta}\Psi, V_1] + [\tilde{\eta}\Psi,\widetilde{X}V_1]\Big)\rangle \tag{2.10}$$

Note that $V_1 = Q\tilde{\eta}\eta\Psi$ appears in $W_2 \equiv \frac{1}{3}\Big(\widetilde{Q}[\Psi,V_1] - [\widetilde{Q}\Psi,V_1] - [\tilde{\eta}\Psi,\widetilde{X}V_1]\Big)$ if one only if we use $\widetilde{X}\tilde{\eta} = [\![Q,\tilde{\xi}]\!]\tilde{\eta}$ instead of $\widetilde{Q}$, where $[\![Q,\xi]\!] := Q\xi - (-)^{Q\xi}\xi Q$ is the graded commutator.

Gauge invariance $\boldsymbol{\delta_2 S_2 + \delta_1(\kappa S_3)}$. Let us determine second order gauge transformation $\delta_2\Psi$ satisfying $\delta_2 S_2 + \delta_1(\kappa S_3) = 0$. For this purpose, it is rather suitable to use a pair of fundamental operators $(Q,\eta,\tilde{\eta},\tilde{\xi})$ than to use a pair of composite operators such as $(Q,\eta,\tilde{\eta},\widetilde{Q})$. For example, $V_1(\Psi) = \eta Q\tilde{\eta}\Psi$ appears in $W_2(\Psi^2)$ if and only if we use $(Q,\eta,\tilde{\eta},\tilde{\xi})$ and furthermore, while $\langle \widetilde{Q}[Q\Psi,\Lambda],\eta Q\tilde{\eta}\Psi\rangle = 0$, $\langle \widetilde{X}[Q\tilde{\eta}\Psi,\Lambda],\eta Q\tilde{\eta}\Psi\rangle \neq 0$. The first order Q-gauge transformation of S_3 is given by

$$\delta_{1,\Lambda} S_3 = \frac{1}{2}\langle Q\Lambda, \frac{1}{3}\Big(\widetilde{X}[Q\tilde{\eta}\Psi,\tilde{\eta}\eta\Psi] + [\widetilde{X}Q\tilde{\eta}\Psi,\tilde{\eta}\eta\Psi] + [Q\tilde{\eta}\Psi,\widetilde{X}\tilde{\eta}\eta\Psi]\Big)\rangle. \tag{2.11}$$

Note that the zero mode $\widetilde{X}$ of the right-mover picture-changing operator is inserted cyclicly. We find that under the following second order gauge transformation

$$\delta_{2,\Lambda}\Psi = \frac{\kappa}{3}\Big\{\widetilde{X}[Q\tilde{\eta}\Psi,\Lambda] + [\widetilde{X}Q\tilde{\eta}\Psi,\Lambda] + [Q\tilde{\eta}\Psi,\widetilde{X}\Lambda]\Big\}$$
$$-\frac{\kappa}{6}\Big\{\widetilde{X}[\tilde{\eta}\Psi,Q\Lambda] + [\widetilde{X}\tilde{\eta}\Psi,Q\Lambda] + [\tilde{\eta}\Psi,\widetilde{X}Q\Lambda]\Big\}, \tag{2.12}$$

the cubic term S_3 of the action is gauge invariant: $\delta_{2,\Lambda}S_2 + \delta_{1,\Lambda}(\kappa S_3) = 0$. For brevity, we define the following new string product which includes the zero mode $\widetilde{X}$ of the right-mover picture-changing operator

$$[A,B]^L := \frac{1}{3}\Big(\widetilde{X}[A,B] + [\widetilde{X}A,B] + [A,\widetilde{X}B]\Big). \tag{2.13}$$

This new product $[A,B]^L$ satisfies the same properties as original product $[A,B]$, namely, symmetric property $[A,B]^L = (-)^{AB}[B,A]^L$ and derivation properties of Q, η, and $\tilde{\eta}$. Note that when we use this new product, the cubic term S_3 of the action is given by

$$S_3 = \frac{1}{3!}\langle \eta\Psi, [Q\tilde{\eta}\Psi,\tilde{\eta}\Psi]^L\rangle, \tag{2.14}$$

and the variation δS_3 becomes

$$\delta S_3 = \frac{1}{2}\langle \delta\Psi, \eta[Q\tilde{\eta}\Psi,\tilde{\eta}\Psi]^L + [\eta Q\tilde{\eta}\Psi,\tilde{\eta}\Psi]^L\rangle. \tag{2.15}$$

Then, we can quickly show that $S_2 + \kappa S_3$ is gauge invariant up to $O(\kappa^2)$

$$\delta_{2,\Lambda}S_2 + \delta_{1,\Lambda}(\kappa S_3) = \langle \delta_{2,\Lambda}\Psi, \eta Q\tilde{\eta}\Psi\rangle + \frac{\kappa}{2}\langle \delta_{1,\Lambda}\Psi, [Q\tilde{\eta}\Psi,\tilde{\eta}\eta\Psi]^L\rangle = 0, \tag{2.16}$$

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under the following Q-gauge transformations

$$\delta_{1,\Lambda}\Psi = Q\Lambda, \qquad \delta_{2,\Lambda}\Psi = \kappa[Q\tilde{\eta}\Psi, \Lambda]^L - \frac{\kappa}{2}[\tilde{\eta}\Psi, Q\Lambda]^L. \tag{2.17}$$

Similarly, we find that $S_2+\kappa S_3$ is gauge invariant under η- and $\tilde{\eta}$-gauge transformations

$$\delta_{1,\Omega}\Psi = \eta\,\Omega, \qquad \delta_{2,\Omega}\Psi = -\frac{\kappa}{2}[\tilde{\eta}\Psi, \eta\Omega]^L, \tag{2.18}$$

$$\delta_{1,\widetilde{\Omega}}\Psi = \tilde{\eta}\,\widetilde{\Omega}, \qquad \delta_{2,\widetilde{\Omega}}\Psi = -\frac{\kappa}{2}[\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}]^L, \tag{2.19}$$

for example, as follows

$$\delta_{2,\Omega}S_2 + \delta_{1,\Omega}(\kappa S_3) = \langle \delta_{2,\Omega}\Psi, \eta Q\tilde{\eta}\Psi\rangle + \frac{\kappa}{2}\langle \delta_{1,\Omega}\Psi, [Q\tilde{\eta}\Psi, \tilde{\eta}\eta\Psi]^L\rangle = 0. \tag{2.20}$$

Linear gauge invariance. In the same way as (2.20), the action $S_2+\kappa S_3$ is invariant under

$$\delta_{1,\widetilde{\Omega}}\Psi = \tilde{\eta}\,\widetilde{\Omega}, \qquad \delta_{2,\widetilde{\Omega}}\Psi = -\frac{\kappa}{2}[\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}]^L. \tag{2.21}$$

Although it naively looks like nonlinear $\tilde{\eta}$-gauge invariance of the action, it is essentially *linear*. Note that the second order $\tilde{\eta}$-gauge transformation $\delta_{2,\widetilde{\Omega}}\Psi = \tilde{\eta}\big(-\frac{\kappa}{2}[\tilde{\eta}\Psi,\widetilde{\Omega}]^L\big)$ as well as the first order one $\delta_{1,\widetilde{\Omega}}\Psi = \tilde{\eta}\widetilde{\Omega}$ is $\tilde{\eta}$-exact. As a result, after the redefinition of the $\tilde{\eta}$-gauge parameter

$$\widetilde{\Omega}' \equiv \widetilde{\Omega} - \frac{\kappa}{2}[\tilde{\eta}\Psi, \widetilde{\Omega}]^L + \dots, \tag{2.22}$$

the $\tilde{\eta}$-gauge transformation $\delta_{\widetilde{\Omega}'}\Psi \equiv \delta_{1,\widetilde{\Omega}}\Psi + \delta_{2,\widetilde{\Omega}}\Psi + \dots$ becomes *linear*

$$\delta_{\widetilde{\Omega}'}\Psi = \tilde{\eta}\,\widetilde{\Omega}'. \tag{2.23}$$

One can quickly check that $\delta_{\widetilde{\Omega}}S_2 = \delta_{\widetilde{\Omega}}S_3 = 0$ holds, or equivalently, the second input of (2.15) is an $\tilde{\eta}$-exact state. Hence, $S_2+\kappa S_3$ still remains invariant under the *linear* $\tilde{\eta}$-gauge transformation.

2.2 Quartic vertex

We can construct the quartic term S_4 and, in principle, higher interaction terms $S_{n>4}$ by repeating the same procedure. More precisely, starting with $\langle \eta\Psi, [(\widetilde{X}Q\tilde{\eta}\Psi)^2, \tilde{\eta}\Psi]\rangle$, adding appropriate terms for (2.6), and imposing the gauge invariance $\delta_3 S_2 + \delta_2(\kappa S_3) + \delta_1(\kappa^2 S_4) = 0$, we obtain the quartic term S_4. First, we consider the gauge invariance $\delta_3 S_2 + \delta_2(\kappa S_3) + \delta_1(\kappa^2 S_4) = 0$.

To be $\delta_3 S_2 + \delta_2(\kappa S_3) + \delta_1(\kappa^2 S_4) = 0$. To be gauge invariant, the first order gauge transformation of $\kappa^2 S_4$ has to cancel $\delta_3 S_2 + \delta_2(\kappa S_3)$. Note that $\delta_2(\kappa S_3)$ is given by

$$\delta_2(\kappa S_3) = \frac{\kappa^2}{4}\langle \Lambda, 2\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\eta\Psi]^L, Q\tilde{\eta}\Psi\big]^L + Q\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\eta\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle. \tag{2.24}$$

Therefore, we have to consider the following terms

$$\big[[A,B]^L, C\big]^L = \frac{1}{3\cdot 3}\Big(\widetilde{X}\big[[\widetilde{X}A,B] + [A,\widetilde{X}B], C\big] + \big[[\widetilde{X}A,B] + [A,\widetilde{X}B], \widetilde{X}C\big]\Big) \tag{2.25}$$

$$+\frac{1}{3\cdot 3}\Big(\big[\widetilde{X}[\widetilde{X}A,B]+\widetilde{X}[A,\widetilde{X}B],C\big]+\big[\widetilde{X}[A,B],\widetilde{X}C\big]+\widetilde{X}\big[\widetilde{X}[A,B],C\big]+\big[\widetilde{X}^2[A,B],C\big]\Big),$$

where $A,B,C=\tilde{\eta}\Psi$, $Q\tilde{\eta}\Psi$, $\eta\tilde{\eta}\Psi$, or $Q\eta\tilde{\eta}\Psi$. To cancel the second line by acting Q, the quartic term S_4 has to include the following types of terms

$$\begin{aligned}
&\tilde{\xi}\big[\widetilde{X}[A,B],C\big], \quad \big[\widetilde{X}[\tilde{\xi}A,B],C\big], \quad \big[\widetilde{X}[A,\tilde{\xi}B],C\big], \quad \big[\widetilde{X}[A,B],\tilde{\xi}C\big], \quad \big[\tilde{\xi}\widetilde{X}[A,B],C\big],\\
&\widetilde{X}\big[\tilde{\xi}[A,B],C\big], \quad \big[\tilde{\xi}[\widetilde{X}A,B],C\big], \quad \big[\tilde{\xi}[A,\widetilde{X}B],C\big], \quad \big[\tilde{\xi}[A,B],\widetilde{X}C\big].
\end{aligned}$$

Of course, we can repeat similar computations of above terms as we did in subsection 2.1. However, there exists a reasonable shortcut. Note that, for example, the following relation holds

$$\begin{aligned}
Q\Big(\widetilde{X}\big[\tilde{\xi}[A,B],C\big]\Big)+\widetilde{X}\big[\tilde{\xi}[QA,B],C\big]+(-)^A\widetilde{X}\big[\tilde{\xi}[A,QB],C\big]&\\
+(-)^{A+B}\widetilde{X}\big[\tilde{\xi}[A,B],QC\big]+\Big(\widetilde{X}\big[\widetilde{X}[A,B],C\big]\Big)&=0.
\end{aligned}$$

Hence, the three product $L_{2+2}(A,B,C)$ defined by

$$\begin{aligned}
L_{2+2}(A,B,C):=\frac{1}{9\cdot 2}\Big\{&2\big[\tilde{\xi}\widetilde{X}[A,B],C\big]-\tilde{\xi}\big[\widetilde{X}[A,B]+[\widetilde{X}A,B]+[A,\widetilde{X}B],C\big]\\
&+\widetilde{X}\big[\tilde{\xi}[A,B],C\big]+\big[\tilde{\xi}\big([\widetilde{X}A,B]+[A,\widetilde{X}B]\big),C\big]+\big[\tilde{\xi}[A,B],\widetilde{X}C\big]\\
&-\widetilde{X}\big[[\tilde{\xi}A,B],C\big]-\big[\widetilde{X}[\tilde{\xi}A,B],C\big]-\big[[\tilde{\xi}A,B],\widetilde{X}C\big]\\
&-(-)^A\Big(\big[\widetilde{X}[A,\tilde{\xi}B],C\big]+\widetilde{X}\big[[A,\tilde{\xi}B],C\big]+\big[[A,\tilde{\xi}B],\widetilde{X}C\big]\Big)\\
&-(-)^{A+B}\Big(\big[\widetilde{X}[A,B]+[\widetilde{X}A,B]+[A,\widetilde{X}B],\tilde{\xi}C\big]\Big)\Big\}\\
&+\big\{(B,C,A)\text{-terms}\big\}+\big\{(C,A,B)\text{-terms}\big\}
\end{aligned}\tag{2.26}$$

and the two product $[A,B]^L$ defined by (2.13) satisfy an L_∞-relation up to $\mathcal{O}(\Psi^4)$:

$$\begin{aligned}
QL_{2+2}(A,B,C)+L_{2+2}(QA,B,C)+(-)^AL_{2+2}(A,QB,C)+(-)^{A+B}L_{2+2}(A,B,QC)&\\
+\big[[A,B]^L,C\big]^L+(-)^{A(B+C)}\big[[B,C]^L,A\big]^L+(-)^{C(A+B)}\big[[C,A]^L,B\big]^L&=0.
\end{aligned}\tag{2.27}$$

This new three product $L_{2+2}(A,B,C)$ possesses the symmetric property and the derivation property of Q. Note that, however, the operator η does not act as a derivation of $L_{2+2}(A,B,C)$. To see this fact, we introduce the derivation-testing operation $\Delta_{\mathbb{X}}$ for $\mathbb{X}=Q,\eta,\tilde{\eta}$

$$\begin{aligned}
\Delta_{\mathbb{X}}L_{2+2}(A,B,C):=\mathbb{X}L_{2+2}(A,B,C)+(-)^{\mathbb{X}}L_{2+2}(\mathbb{X}A,B,C)&\\
+(-)^{\mathbb{X}(1+A)}L_{2+2}(A,\mathbb{X}B,C)+(-)^{\mathbb{X}(1+A+B)}L_{2+2}(A,B,\mathbb{X}C)&.
\end{aligned}\tag{2.28}$$

For example, $\Delta_{\mathbb{X}}[A,B]^L=\mathbb{X}[A,B]^L+(-)^{\mathbb{X}}[\mathbb{X}A,B]^L+(-)^{\mathbb{X}(1+A)}[A,\mathbb{X}B]^L=0$ holds for $\mathbb{X}=Q$, η, and $\tilde{\eta}$. Computing $\Delta_{\eta/\tilde{\eta}}L_{2+2}(A,B,C)$, we find $\Delta_\eta L_{2+2}(A,B,C)=0$ but

$$\Delta_{\tilde{\eta}}L_{2+2}(A,B,C)=\frac{1}{3\cdot 2}\Big\{\widetilde{X}\big[[A,B],C\big]+\big[\tilde{\xi}\big([\widetilde{X}A,B]+[A,\widetilde{X}B]\big),C\big]+\big[[A,B],\widetilde{X}C\big]\Big\}$$

$$-\frac{1}{3}\big[\widetilde{X}[A,B],C\big]+\big\{(B,C,A)\text{-terms}\big\}+\big\{(C,A,B)\text{-terms}\big\}. \quad (2.29)$$

$\Delta_{\tilde{\eta}} L_{2+2}(A,B,C) \neq 0$ means that the $\tilde{\eta}$-derivation property is broken. However, by adding the appropriate term L_{1+3} satisfying $\Delta_{Q/\eta} L_{1+3}(A,B,C)=0$, we can construct the three product satisfying the L_∞-relation (2.27) and η- and $\tilde{\eta}$-derivation properties $\Delta_{\eta/\tilde{\eta}}(L_{2+2}+L_{1+3})=0$.

Note that the following types of products have the Q-derivation property

$$\begin{aligned} M(A,B,C) &= \big[\widetilde{X}A,\widetilde{X}B,C\big]+\big[[\tilde{\xi}A,\widetilde{X}B],C\big]+(-)^{(A+1)(B+C)}\big[[\widetilde{X}B,C],\tilde{\xi}A\big] \\ &\qquad +(-)^{C(A+B+1)}\big[[C,\tilde{\xi}A],\widetilde{X}B\big], \\ N(A,B,C) &= \widetilde{X}\big[\tilde{\xi}[A,B],C\big]+\tilde{\xi}\big[\widetilde{X}[A,B],C\big], \end{aligned}$$

namely, $\Delta_Q M = \Delta_Q N = 0$. Therefore, L_{1+3} is given by a linear combination of these M- and N-types of products, whose coefficients are fixed by the cancellation of the second line of (2.29) and the sum of N-type products:

$$\begin{aligned} L_{1+3}(A,B,C) := \frac{1}{8\cdot 2}&\Big\{\widetilde{X}^2\big[A,B,C\big]+\big[\widetilde{X}^2A,B,C\big]+\big[A,\widetilde{X}^2B,C\big]+\big[A,B,\widetilde{X}^2C\big]\Big\} \\ +\frac{1}{8\cdot 2}&\Big\{\tilde{\xi}\widetilde{X}\big[[A,B],C\big]+\big[[\tilde{\xi}\widetilde{X}A,B],C\big] \\ &\qquad +(-)^A\big[[A,\tilde{\xi}\widetilde{X}B],C\big]+(-)^{A+B}\big[[A,B],\tilde{\xi}\widetilde{X}C\big]\Big\} \\ +\frac{1}{8}&\Big\{\widetilde{X}\big[\widetilde{X}A,B,C\big]+\widetilde{X}\big[A,\widetilde{X}B,C\big]+\widetilde{X}\big[A,B,\widetilde{X}C\big] \\ &\qquad +\big[\widetilde{X}A,\widetilde{X}B,C\big]+\big[\widetilde{X}A,B,\widetilde{X}C\big]+\big[A,\widetilde{X}B,\widetilde{X}C\big]\Big\} \\ -\frac{1}{8\cdot 2}&\Big\{\tilde{\xi}\big[[\widetilde{X}A,B]+[A,\widetilde{X}B],C\big]+\tilde{\xi}\big[[A,B],\widetilde{X}C\big] \\ &\quad +\widetilde{X}\big[[\tilde{\xi}A,B],C\big]-\big[[\tilde{\xi}A,\widetilde{X}B],C\big]-\big[[\tilde{\xi}A,B],\widetilde{X}C\big] \\ &\quad +(-)^A\Big(\widetilde{X}\big[[A,\tilde{\xi}B],C\big]-\big[[\widetilde{X}A,\tilde{\xi}B],C\big]-\big[[A,\tilde{\xi}B],\widetilde{X}C\big]\Big) \\ &\quad +(-)^{A+B}\Big(\widetilde{X}\big[[A,B],\tilde{\xi}C\big]-\big[[\widetilde{X}A,B],\tilde{\xi}C\big]-\big[[A,\widetilde{X}B],\tilde{\xi}C\big]\Big)\Big\} \\ +\frac{1}{4\cdot 3}&\Big\{\widetilde{X}\big[\tilde{\xi}[A,B],C\big]+\big[\tilde{\xi}[\widetilde{X}A,B]+\tilde{\xi}[A,\widetilde{X}B],C\big]+\big[\tilde{\xi}[A,B],\widetilde{X}C\big] \\ &+\tilde{\xi}\big[\widetilde{X}[A,B],C\big]+\big[\widetilde{X}\big([\tilde{\xi}A,B]+(-)^A[A,\tilde{\xi}B]\big),C\big]+(-)^{A+B}\big[\widetilde{X}[A,B],\tilde{\xi}C\big]\Big\} \\ &\qquad +\big\{(B,C,A)\text{-terms}\big\}+\big\{(C,A,B)\text{-terms}\big\}. \end{aligned} \quad (2.30)$$

$L_{1+3}(A,B,C)$ satisfies $\Delta_Q L_{1+3}(A,B,C)=0$ because of $\Delta_Q M = \Delta_Q N = 0$. Hence, we define the following new three string product

$$[A,B,C]^L := L_{2+2}(A,B,C)+L_{1+3}(A,B,C), \quad (2.31)$$

which satisfies the same properties as the original three product $[A,B,C]$, namely, the L_∞-relation and the η- and $\tilde{\eta}$-derivation properties

$$\Delta_Q[A,B,C]^L+\big[[A,B]^L,C\big]^L+(-)^{A(B+C)}\big[[B,C]^L,A\big]^L+(-)^{C(A+B)}\big[[C,A]^L,B\big]^L=0, \quad (2.32)$$

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$$\Delta_\eta[A,B,C]^L = \Delta_{\tilde{\eta}}[A,B,C]^L = 0. \tag{2.33}$$

Note also that the new product $[A,B,C]^L$ includes $\langle \eta\Psi, [\widetilde{X}Q\tilde{\eta}\Psi, \widetilde{X}Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]\rangle$ (for $A,B = Q\tilde{\eta}\Psi$, $C = \tilde{\eta}\Psi$) and this term becomes $\langle \xi\mathcal{V}, [X\widetilde{X}\mathcal{V}, X\widetilde{X}\mathcal{V}, \tilde{\xi}\mathcal{V}]\rangle$ under the partial gauge fixing $\Psi = \xi\tilde{\xi}\mathcal{V}$, which is necessitated for the correspondence to the result of first quantization.

Quartic vertex S_4. Let us now consider the quartic vertex having the property (2.6) and determine the third order gauge transformation $\delta_3\Psi$. To obtain the gauge invariance $\delta_3 S_2 + \delta_2(\kappa S_3) + \delta_1(\kappa^2 S_4) = 0$, the quartic term S_4 has to include the term $\langle \eta\Psi, [Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L\rangle$ because the L_∞-relation (2.32) is the only way to eliminate the term $\big[[A,B]^L, C\big]^L$ appearing in $\delta_3 S_2 + \delta_2(\kappa S_3)$. We therefore start with the following computation

$$\begin{aligned}\delta\Big(\langle \eta\Psi, [Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L\rangle\Big) &= 4\langle \delta\Psi, \eta\Big\{[Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L + \frac{1}{2}\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\Big\}\rangle \\ &\quad + 4\langle \delta\Psi, 2[\eta Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L - \frac{1}{2}\big[[\eta Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle \\ &\quad + 4\langle \delta\Psi, \big[[\tilde{\eta}\eta\Psi, Q\tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle.\end{aligned} \tag{2.34}$$

To obtain the quartic vertex having the property (2.6), the term $\langle \Psi, \eta\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle$ is necessitated. The variation of this term becomes

$$\delta\Big(\langle \Psi, \eta\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle\Big) = 2\langle \delta\Psi, \eta\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L + \big[[\tilde{\eta}\eta\Psi, Q\tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle. \tag{2.35}$$

Hence, the quartic term S_4 defined by

$$S_4 = \frac{1}{4!}\langle \eta\Psi, [Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L + \big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle, \tag{2.36}$$

has the property (2.6) and its variation is given by

$$\begin{aligned}\delta S_4 &= \frac{1}{3!}\langle \delta\Psi, \eta\Big(\big[Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi\big]^L + \big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\Big)\rangle + \frac{1}{4}\langle \delta\Psi, \big[[Q\tilde{\eta}\Psi, \tilde{\eta}\eta\Psi]^L, \tilde{\eta}\Psi\big]^L\rangle \\ &\quad + \frac{2}{4!}\langle \delta\Psi, \big[\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \eta Q\tilde{\eta}\Psi]^L\big]^L\rangle + \frac{1}{3}\langle \delta\Psi, \big[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi, \eta Q\tilde{\eta}\Psi\big]^L\rangle.\end{aligned} \tag{2.37}$$

Note that $W_3 = \frac{3}{2}[V_2, \tilde{\eta}\Psi]^L - [W_2, \tilde{\eta}\Psi]^L - 2[Q\tilde{\eta}\Psi, V_1, \tilde{\eta}\Psi]^L$ (the second term and the second line), where $V_1 = \eta Q\tilde{\eta}\Psi$, $V_2 = \eta[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L$ and $W_2 = [V_1, \tilde{\eta}\Psi]^L$. Using this δS_4, we can determine the third order gauge transformation $\delta_3\Psi$ satisfying $\delta_3 S_2 + \delta_2(\kappa S_3) + \delta_1(\kappa^2 S_4) = 0$. Since the Q-gauge transformation $\delta_{2,\Lambda}(\kappa S_3) + \delta_{1,\Lambda}(\kappa^2 S_4)$ is given by

$$\begin{aligned}&\frac{\kappa^2}{2}\langle \Lambda, \big[Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \eta Q\tilde{\eta}\Psi\big]^L\rangle + \frac{\kappa^2}{2}\langle \Lambda, \big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \eta Q\tilde{\eta}\Psi\big]^L\rangle + \frac{\kappa^2}{2}\langle \Lambda, \big[Q\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \eta Q\tilde{\eta}\Psi]^L\big]^L\rangle \\ &\quad + \frac{\kappa^2}{12}\langle Q\Lambda, \big[\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \eta Q\tilde{\eta}\Psi]^L\big]^L\rangle + \frac{\kappa^2}{3}\langle Q\Lambda, \big[\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \eta Q\tilde{\eta}\Psi\big]^L\rangle,\end{aligned} \tag{2.38}$$

we obtain the third order Q-gauge transformation

$$\delta_{3,\Lambda}\Psi = \frac{\kappa^2}{2}[Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \Lambda]^L + \frac{\kappa^2}{2}\big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \Lambda\big]^L - \frac{\kappa^2}{2}\big[\tilde{\eta}\Psi, [Q\tilde{\eta}\Psi, \Lambda]^L\big]^L$$

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$$-\frac{\kappa^2}{12}\big[\tilde{\eta}\Psi,[\tilde{\eta}\Psi,Q\Lambda]^L\big]^L-\frac{\kappa^2}{3}[\tilde{\eta}\Psi,Q\tilde{\eta}\Psi,Q\Lambda]^L. \tag{2.39}$$

Similarly, the third order η- and $\tilde{\eta}$-gauge transformations are given by

$$\delta_{3,\Omega}\Psi=-\frac{\kappa^2}{12}\big[\tilde{\eta}\Psi,[\tilde{\eta}\Psi,\eta\Omega]^L\big]^L-\frac{\kappa^2}{3}\big[\eta\Omega,Q\tilde{\eta}\Psi,\tilde{\eta}\Psi\big]^L, \tag{2.40}$$

$$\delta_{3,\widetilde{\Omega}}\Psi=-\frac{\kappa^2}{12}\big[\tilde{\eta}\Psi,[\tilde{\eta}\Psi,\tilde{\eta}\widetilde{\Omega}]^L\big]^L-\frac{\kappa^2}{3}\big[\tilde{\eta}\widetilde{\Omega},Q\tilde{\eta}\Psi,\tilde{\eta}\Psi\big]^L. \tag{2.41}$$

Note, however, that since $\delta_{3,\widetilde{\Omega}}\Psi$ as well as $\delta_{2,\widetilde{\Omega}}\Psi$ is $\tilde{\eta}$-exact, redefining $\tilde{\eta}$-gauge parameters as

$$\widetilde{\Omega}'\equiv\widetilde{\Omega}-\frac{\kappa}{2}[\tilde{\eta}\Psi,\widetilde{\Omega}]^L-\frac{\kappa^2}{3}\bigg(\frac{1}{4}\big[\tilde{\eta}\Psi,[\tilde{\eta}\Psi,\widetilde{\Omega}]^L\big]^L+\big[Q\tilde{\eta}\Psi,\tilde{\eta}\Psi,\widetilde{\Omega}\big]^L\bigg)+\ldots, \tag{2.42}$$

the $\tilde{\eta}$-gauge transformation $\delta_{\widetilde{\Omega}'}\Psi\equiv\delta_{1,\widetilde{\Omega}}\Psi+\delta_{2,\widetilde{\Omega}}\Psi+\delta_{3,\widetilde{\Omega}}\Psi+\ldots$ becomes linear $\delta_{\widetilde{\Omega}'}\Psi=\tilde{\eta}\widetilde{\Omega}'$.

In principle, we can construct higher vertices $S_5, S_6, \ldots$ by repeating these steps at each order of κ. However, it is not easy to read a closed form expression by hand calculation because higher order vertices consist of a lot of terms and each term has complicated operator insertions. To obtain a closed form expression of all vertices in an elegant way, we necessitate another point of view, which we explain in section 3.

3 Gauge-invariant insertions of picture-changing operators

In this section, we briefly review the coalgebraic description of string vertices [32–34] and how to construct NS superstring vertices [31]. Since the NS string products satisfies L_∞-relations, the shifted NS string products satisfies L_∞-relations up to the equation of motion.

3.1 Coalgebraic description of vertices

Let $T(\mathcal{H})$ be a tensor algebra of the graded vector space $\mathcal{H}$. As the quotient algebra of $T(\mathcal{H})$ by the ideal generated by all differences of products $A_1\otimes A_2-(-1)^{\deg(A_1)\deg(A_2)}A_2\otimes A_1$ for $A_1,A_2\in\mathcal{H}$, we can construct the symmetric algebra $S(\mathcal{H})$. *The product of states* in $S(\mathcal{H})$ is graded commutative and associative as follows

$$A_1A_2=(-1)^{\deg(A_1)\deg(A_2)}A_2A_1,\qquad A_1(A_2A_3)=(A_1A_2)A_3, \tag{3.1}$$

where $A_1,A_2,A_3\in S(\mathcal{H})$. For us, $\mathcal{H}$ is the closed superstring state space, $S(\mathcal{H})$ is the Fock space of superstrings, and the $\mathbb{Z}_2$ grading, called degree, is just Grassmann parity. *The product of two multilinear maps* $L:\mathcal{H}^n\to\mathcal{H}^l$, $M:\mathcal{H}^m\to\mathcal{H}^k$ also becomes a map $L\cdot M:\mathcal{H}^{n+m}\to\mathcal{H}^{k+l}$ which acts as

$$L\cdot M(A_1A_2\cdots A_{n+m})=\sum_\sigma(-)^{\sigma(n,m)}L(A_{\sigma(1)}\cdots A_{\sigma(n)})\cdot M(A_{\sigma(n+1)}\cdots A_{\sigma(n+m)}). \tag{3.2}$$

Note that the n-product of the identity map $\mathbb{I}:\mathcal{H}\to\mathcal{H}$ on symmetric algebras is different from the n-tensor product or the identity $\mathbb{I}_n$ on $\mathcal{H}^n$:

$$\mathbb{I}_n\equiv\frac{1}{n!}\overbrace{\mathbb{I}\cdots\mathbb{I}}^{n}=\overbrace{\mathbb{I}\otimes\cdots\otimes\mathbb{I}}^{n}. \tag{3.3}$$

In other words, $\mathbb{I}^n$ is the sum of all permutations, $\mathbb{I}_n$ gives the sum of equivalent permutations, and $\mathbb{I}_k \cdot \mathbb{I}_l$ is equivalent to the sum of different (k,l)-partitions of $k+l$.

The n string product $L_n(A_1 A_2 \ldots A_n) \equiv [A_1, \ldots, A_n]$ defines a n-fold linear map $L_n : \mathcal{H}^n \to \mathcal{H}$. We can naturally define a coderivation $\mathbf{L}_n : S(\mathcal{H}) \to S(\mathcal{H})$ from the map $L_n : \mathcal{H}^n \to \mathcal{H}$. Specifically, the coderivation $\mathbf{L}_n$ acts on the $\mathcal{H}^N \subset S(\mathcal{H})$ as

$$\mathbf{L}_n A = (L_n \cdot \mathbb{I}_{N-n}) A, \qquad \mathbf{L}_n B = 0, \tag{3.4}$$

where $A \in \mathcal{H}^{N \geq n}$ and $B \in \mathcal{H}^{N<n}$. Note that the commutator $[\![\mathbf{L}_m, \mathbf{L}'_n]\!]$ of two coderivations $\mathbf{L}_m$ and $\mathbf{L}'_n$ also becomes a coderivation of the $(m+n-1)$-product

$$[\![\mathbf{L}_m, \mathbf{L}'_n]\!] := L_m(L'_n \cdot \mathbb{I}_{m-1}) - (-1)^{\deg(L_m)\deg(L'_n)} L'_n(L_m \cdot \mathbb{I}_{n-1}). \tag{3.5}$$

Hence, in the coalgebraic description, we can write L_∞-relations of closed string products as

$$[\![\mathbf{L}_1, \mathbf{L}_n]\!] + [\![\mathbf{L}_2, \mathbf{L}_{n-1}]\!] + \cdots + [\![\mathbf{L}_n, \mathbf{L}_1]\!] = 0, \tag{3.6}$$

or, more simply,

$$[\![\mathbf{L}(s), \mathbf{L}(s)]\!] = 0, \tag{3.7}$$

where s is a real parameter and $\mathbf{L}(s)$ is the generating function for the bosonic string products

$$\mathbf{L}(s) = \sum_{n=0}^{\infty} s^n \mathbf{L}_{n+1}. \tag{3.8}$$

3.2 Gauge-invariant insertions

Let $L^{(n)}_{N+1}$ be a $(N+1)$-product including n-insertions of picture-changing operators. We consider a series of these inserted products

$$\mathbf{L}^{[m]}(t) := \sum_{n=0}^{\infty} t^n \mathbf{L}^{(n)}_{m+n+1}, \tag{3.9}$$

where t is a real parameter and $\mathbf{L}^{(n)}_{m+n+1}$ acts on $S(\mathcal{H})$ as (3.4). Note that the upper index $[m]$ on $\mathbf{L}^{[m]}(t)$ indicates the deficit in picture number of the products relative to what is needed for superstrings: $\mathbf{L}^{[0]}(t)$ is the sum of all superstring products and $\mathbf{L}^{[m]}(0)$ is the $(m+1)$-product of bosonic strings. To associate the generating functions $\mathbf{L}^{[0]}(t)$ of the NS superstring products with (3.8), we define the following generating function

$$\mathbf{L}(s,t) := \sum_{s=0}^{\infty} s^m \mathbf{L}^{[m]}(t) = \sum_{m,n=0}^{\infty} s^m t^n \mathbf{L}^{(n)}_{m+n+1}, \tag{3.10}$$

where s is a real parameter. Note that powers of t count the picture number, and powers of s count the deficit in picture number. These two parameters t and s connect the generating function $\mathbf{L}(0,t) = \mathbf{L}^{[0]}(t)$ for the NS superstring products and the generating function $\mathbf{L}(s,0)$ for the bosonic string products.

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The L_∞-relations of bosonic products and derivation properties of η can be represented by

$$[\![\mathbf{L}(s,0),\mathbf{L}(s,0)]\!]=0, \tag{3.11}$$
$$[\![\boldsymbol{\eta},\mathbf{L}(s,0)]\!]=0. \tag{3.12}$$

Starting with these relations, we can construct the NS superstring products $\mathbf{L}(0,t)$ satisfying the L_∞-relations and derivation properties of η, which we explain in this subsection.

Gauge-invariant insertions. To construct $\mathbf{L}(0,t)$ satisfying the L_∞-relations and η-derivation conditions

$$[\![\mathbf{L}(0,t),\mathbf{L}(0,t)]\!]=0, \tag{3.13}$$
$$[\![\boldsymbol{\eta},\mathbf{L}(0,t)]\!]=0, \tag{3.14}$$

we have to solve the following pair of differential equations

$$\frac{\partial}{\partial t}\mathbf{L}(s,t)=[\![\mathbf{L}(s,t),\boldsymbol{\Xi}(s,t)]\!], \tag{3.15}$$
$$\frac{\partial}{\partial s}\mathbf{L}(s,t)=[\![\boldsymbol{\eta},\boldsymbol{\Xi}(s,t)]\!], \tag{3.16}$$

with the initial conditions (3.11) and (3.12) at $t=0$. As well as $\mathbf{L}(s,t)$, we define a generating function $\boldsymbol{\Xi}(s,t)$ for gauge parameters

$$\boldsymbol{\Xi}(s,t):=\sum_{m=0}^{\infty}s^m\boldsymbol{\Xi}^{[m]}(t)=\sum_{m,n=0}^{\infty}s^m t^n\boldsymbol{\Xi}^{(n+1)}_{m+n+2}. \tag{3.17}$$

The solution of this pair of differential equations generates all products including appropriate insertions of picture-changing operators.

We can obtain the superstring L_∞-relations (3.13) as a solution of the differential equation

$$\frac{\partial}{\partial t}[\![\mathbf{L}(s,t),\mathbf{L}(s,t)]\!]=\Big[\!\!\Big[[\![\mathbf{L}(s,t),\mathbf{L}(s,t)]\!],\boldsymbol{\Xi}(s,t)\Big]\!\!\Big], \tag{3.18}$$

where $\boldsymbol{\Xi}(s,t)$ is a generating function for gauge parameters, if we impose the initial condition (3.11) at $t=0$. Provided that $\mathbf{L}(s,t)$ satisfies (3.15), the equation (3.18) automatically holds because of Jacobi identities $\Big[\!\!\Big[[\![\mathbf{L}(s,t),\mathbf{L}(s,t)]\!],\boldsymbol{\Xi}(s,t)\Big]\!\!\Big]=2\Big[\!\!\Big[[\![\mathbf{L}(s,t),\boldsymbol{\Xi}(s,t)]\!],\mathbf{L}(s,t)\Big]\!\!\Big]$.

Further, the equation (3.15) leads to the equation

$$\frac{\partial}{\partial t}[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!]=\Big[\!\!\Big[[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!],\boldsymbol{\Xi}(s,t)\Big]\!\!\Big]+\Big[\!\!\Big[[\![\boldsymbol{\eta},\boldsymbol{\Xi}(s,t)]\!],\mathbf{L}(s,t)\Big]\!\!\Big]. \tag{3.19}$$

Therefore, when $\mathbf{L}(s,t)$ satisfies the equation (3.16), we obtain

$$\frac{\partial}{\partial t}[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!]=\Big[\!\!\Big[[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!],\boldsymbol{\Xi}(s,t)\Big]\!\!\Big]-\frac{1}{2}\frac{\partial}{\partial s}[\![\mathbf{L}(s,t),\mathbf{L}(s,t)]\!] \tag{3.20}$$

and the differential equation $\frac{\partial}{\partial t}[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!]=\Big[\!\!\Big[[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!],\boldsymbol{\Xi}(s,t)\Big]\!\!\Big]$ (up to $[\![\mathbf{L}(s,t),\mathbf{L}(s,t)]\!]$) with the initial condition (3.12) indicates the η-dirivative conditions (3.14) of the superstring products. As a result, the pair of equations (3.15) and (3.16) generates inserted

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products $\mathbf{L}$ satisfying the L_∞-relations $[\![\mathbf{L}(s,t),\mathbf{L}(s,t)]\!]=0$ and η-derivative conditions $[\![\boldsymbol{\eta},\mathbf{L}(s,t)]\!]=0$.

Expanding (3.15) and (3.16) in powers of (s,t), we obtain the following formulae

$$\mathbf{L}^{(n+1)}_{m+n+2}=\frac{1}{n+1}\sum_{k=0}^{n}\sum_{l=0}^{m}[\![\mathbf{L}^{(k)}_{k+l+a},\boldsymbol{\Xi}^{(n-k+1)}_{m+n+2-k-l}]\!],\tag{3.21}$$

$$[\![\boldsymbol{\eta},\boldsymbol{\Xi}^{(n+1)}_{m+n+2}]\!]=(m+1)\mathbf{L}^{(n)}_{m+n+2},\tag{3.22}$$

at each order of $s^m t^n$ and these formulae determine $L^{(n)}_{m+n+2}$ and $\Xi^{(n+1)}_{m+n+2}$ recursively.

Note that $\boldsymbol{\Xi}(s,t)$ is not unique, however, there exists the one including symmetric insertions

$$\boldsymbol{\Xi}^{(n+1)}_{m+n+2}=\frac{m+1}{m+n+3}\Big(\xi\mathbf{L}^{(n)}_{m+n+2}-\mathbf{L}^{(n)}_{m+n+2}\big(\xi\cdot\mathbb{I}^{m+n+1}\big)\Big).\tag{3.23}$$

Therefore, we can always derive explicit forms of these inserted products as follows:

$$\begin{aligned}
\mathbf{L}^{(0)}_{n+1}&=\text{given},\\
\Xi^{(1)}_{n+1}&=\frac{n}{n+2}\Big(\xi\mathbf{L}^{(0)}_{n+1}-\mathbf{L}^{(0)}_{n+1}\big(\xi\cdot\mathbb{I}^{n}\big)\Big),\\
\mathbf{L}^{(1)}_{n+1}&=[\![\mathbf{Q},\boldsymbol{\Xi}^{(1)}_{n+1}]\!]+[\![\mathbf{L}^{(0)}_{2},\boldsymbol{\Xi}^{(1)}_{n}]\!]+\cdots+[\![\mathbf{L}^{(0)}_{n},\boldsymbol{\Xi}^{(1)}_{2}]\!],\\
\Xi^{(2)}_{n+1}&=\frac{n-1}{n+2}\Big(\xi\mathbf{L}^{(1)}_{n+1}-\mathbf{L}^{(1)}_{n+1}\big(\xi\cdot\mathbb{I}^{n}\big)\Big),\\
\mathbf{L}^{(2)}_{n+1}&=\frac{1}{2}\Big([\![\mathbf{Q},\boldsymbol{\Xi}^{(2)}_{n+1}]\!]+[\![\mathbf{L}^{(0)}_{2},\boldsymbol{\Xi}^{(2)}_{n}]\!]+[\![\mathbf{L}^{(1)}_{2},\boldsymbol{\Xi}^{(1)}_{n}]\!]+\dots,\\
&\qquad\qquad+[\![\mathbf{L}^{(0)}_{n-1},\boldsymbol{\Xi}^{(2)}_{3}]\!]+[\![\mathbf{L}^{(1)}_{n-1},\boldsymbol{\Xi}^{(1)}_{3}]\!]+[\![\mathbf{L}^{(1)}_{n},\boldsymbol{\Xi}^{(1)}_{2}]\!]\Big),\\
&\ \vdots\\
\Xi^{(n)}_{n+1}&=\frac{1}{n+2}\Big(\xi\mathbf{L}^{(n)}_{n+1}-\mathbf{L}^{(n)}_{n+1}\big(\xi\cdot\mathbb{I}^{n}\big)\Big),\\
\mathbf{L}^{(n)}_{n+1}&=\frac{1}{n}\Big([\![\mathbf{Q},\boldsymbol{\Xi}^{(n)}_{n+1}]\!]+[\![\mathbf{L}^{(1)}_{2},\boldsymbol{\Xi}^{(n-1)}_{n}]\!]+\cdots+[\![\mathbf{L}^{(n-1)}_{n},\boldsymbol{\Xi}^{(1)}_{2}]\!]\Big).
\end{aligned}\tag{3.24}$$

For example, we find that the lowest inserted product $\mathbf{L}^{(1)}_2$ is given by

$$\begin{aligned}
\mathbf{L}^{(1)}_{2}(A,B)&=[\![\mathbf{Q},\boldsymbol{\Xi}^{(1)}_{2}]\!](A,B)\\
&=\frac{1}{3}\Big(X[A,B]+[XA,B]+[A,XB]\Big),
\end{aligned}\tag{3.25}$$

where $L^{(0)}_2(A,B)=[A,B]$, and the second lowest product $\mathbf{L}^{(2)}_3$ is given by

$$\mathbf{L}^{(2)}_{3}(A,B,C)=\frac{1}{2}\Big([\![\mathbf{Q},\boldsymbol{\Xi}^{(2)}_{3}]\!]+[\![\mathbf{L}^{(1)}_{3},\boldsymbol{\Xi}^{(1)}_{3}]\!]\Big)(A,B,C),\tag{3.26}$$

where $L^{(0)}_3(A,B,C)=[A,B,C]$ and

$$\begin{aligned}
\frac{1}{2}[\![\mathbf{Q},\boldsymbol{\Xi}^{(2)}_{3}]\!](A,B,C)=&\frac{1}{8\cdot 2}\Big\{X^2[A,B,C]+[X^2A,B,C,]+[A,X^2B,C,]+[A,B,X^2C]\Big\}\\
&+\frac{1}{8\cdot 2}\Big\{\xi X[[A,B],C]+[[\xi XA,B],C]
\end{aligned}$$

$$
\begin{aligned}
&+(-)^{A}[[A,\xi XB],C]+(-)^{A+B}[[A,B],\xi XC]\Big\} \\
&+\frac{1}{8}\Big\{X[XA,B,C]+X[A,XB,C]+X[A,B,XC] \\
&\qquad +[XA,XB,C]+[XA,B,XC]+[A,XB,XC]\Big\} \\
&-\frac{1}{8\cdot 2}\Big\{\xi[[XA,B]+[A,XB],C]+\xi[[A,B],XC] \\
&\qquad +X[[\xi A,B],C]-[[\xi A,XB],C]-[[\xi A,B],XC] \\
&\qquad +(-)^{A}\Big(X[[A,\xi B],C]-[[XA,\xi B],C]-[[A,\xi B],XC]\Big) \\
&\qquad +(-)^{A+B}\Big(X[[A,B],\xi C]-[[XA,B],\xi C]-[[A,XB],\xi C]\Big)\Big\} \\
&+\frac{1}{4\cdot 3}\Big\{X[\xi[A,B],C]+[\xi[XA,B]+\xi[A,XB],C]+[\xi[A,B],XC] \\
&\qquad +\xi\big[X[A,B],C\big]+\big[X[\xi A,B],C\big] \\
&\qquad +(-)^{A}\big[X[A,\xi B],C\big]+(-)^{A+B}\big[X[A,B],\xi C\big]\Big\} \\
&+\big\{(B,C,A)\text{-terms}\big\}+\big\{(C,A,B)\text{-terms}\big\}. \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}[\![\mathbf{L}_2^{(1)},\boldsymbol{\Xi}_2^{(1)}]\!](A,B,C)=\frac{1}{9\cdot 2}\Big\{&2\big[\xi X[A,B],C\big]-\xi\big[X[A,B]+[XA,B]+[A,XB],C\big] \\
&+X\big[\xi[A,B],C\big]+\big[\xi\big([XA,B]+[A,XB]\big),C\big]+\big[\xi[A,B],XC\big] \\
&-X\big[[\xi A,B],C\big]-\big[X[\xi A,B],C\big]-\big[[\xi A,B],XC\big] \\
&-(-)^{A}\Big(\big[X[A,\xi B],C\big]+X\big[[A,\xi B],C\big]+\big[[A,\xi B],XC\big]\Big) \\
&-(-)^{A+B}\Big(\big[X[A,B]+[XA,B]+[A,XB],\xi C\big]\Big)\Big\} \\
&+\big\{(B,C,A)\text{-terms}\big\}+\big\{(C,A,B)\text{-terms}\big\} \qquad (3.28)
\end{aligned}
$$

3.3 NS string products

The generating function $\mathbf{L}(0,t)$ of the superstring products, as well as that of bosonic ones $\mathbf{L}(s,0)$, has nice properties, which we explain in this subsection. Note that the above $\mathbf{L}_{n+1}^{(n)}$ obtained from (3.21) and (3.23) carries ghost-and-picture number $(1-2n|n,0)$. In this sense, we write $\mathbf{L}_{n+1}^{(n,0)}$ for this $\mathbf{L}_{n+1}^{(n)}$, an NS superstring product with insertions of *left*-moving picture-changing operators. $\mathbf{L}_{n+1}^{(n,0)}$ gives the $(n+1)$-product of NS (heterotic) superstring field theory in the *small* Hilbert space of *left* movers [31].

By construction, we can also obtain an NS superstring product $\mathbf{L}_{n+1}^{(0,n)}$ with insertions of *right*-moving picture-changing operators $\widetilde{X}$ by replacing (η,ξ,X) with $(\tilde{\eta},\tilde{\xi},\widetilde{X})$ in (3.21), (3.22), and (3.23). In the rest, we consider these *right*-mover NS products $\{\mathbf{L}_{n+1}^{(0,n)}\}_{n=0}^{\infty}$ and write

$$
[A_0,A_1,\dots,A_n]^L:=\mathbf{L}_{n+1}^{(0,n)}(A_0,A_1,\dots,A_n). \qquad (3.29)
$$

The right-mover NS products also satisfies L_∞-relations

$$
\sum_{\sigma}\sum_{k}(-1)^{|\sigma(A)|}\big[[A_{\sigma(1)},\dots,A_{\sigma(k)}]^L,A_{\sigma(k+1)},\dots,A_{\sigma(n)}\big]^L=0. \qquad (3.30)
$$

Note also that the n-product $[A_1,\dots,A_n]^L$ has ghost-and-picture number $(3-2n|0,n-1)$.

L_∞-properties of right-mover NS string products. Let $\mathcal{G}$ be a ghost-and-picture number $(2|0,-1)$ state and $A, A_1, \ldots, A_n$ be arbitrary states. We can define a shifted BRST operator $Q_\mathcal{G}$ and shifted right-mover NS string products

$$Q_\mathcal{G} A \equiv [A]^L_\mathcal{G} := \; QA + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} [\mathcal{G}^n, \, A]^L, \tag{3.31}$$

$$[A_1, \ldots, A_n]^L_\mathcal{G} := \sum_{k=1}^{\infty} \frac{\kappa^k}{k!} [\mathcal{G}^k, A_1, \ldots, A_n]^L, \tag{3.32}$$

in the same manner as shifted bosonic string products. Provided that the state $\mathcal{G}$ shifting these products satisfies the equation of motion $\mathcal{F}(G) = 0$ of NS (heterotic) string field theory in the small Hilbert space of right movers

$$\mathcal{F}(\mathcal{G}) \equiv Q\mathcal{G} + \sum_{n=1}^{\infty} \frac{\kappa^n}{(n+1)!} \big[\mathcal{G}^n, \mathcal{G}\big]^L = 0, \tag{3.33}$$

these shifted products satisfy (strong) L_∞-relations:[1]

$$\sum_\sigma \sum_k (-1)^{|\sigma|} \big[[A_{\sigma(1)}, \ldots, A_{\sigma(k)}]^L_\mathcal{G}, A_{\sigma(k+1)}, \ldots, A_{\sigma(n)}\big]^L_\mathcal{G} = 0. \tag{3.34}$$

Then, $Q_\mathcal{G}$ becomes a nilpotent operator.

4 WZW-like expression

In this section, first, we gives the defining equations of a formal pure-gauge $\mathcal{G}_L$ and associated fields $\Psi_t, \Psi_\eta, \Psi_\delta$. These are functions of NS-NS string fields Ψ and become key ingredients of our construction. Then, we present a closed form expression of WZW-like action for NS-NS string field theory, the equation of motion, and the gauge invariance of the action.

4.1 Pure-gauge $\mathcal{G}_L$ and 'large' associated field $\Psi_\mathbb{X}$

The NS-NS string field Ψ is a Grassmann-even and ghost-and-picture number $(0|0,0)$ state living in the large Hilbert space of left-and-right movers: $\eta\Psi \neq 0$ and $\tilde{\eta}\Psi \neq 0$.

A pure-gauge $\mathcal{G}_L$ of right-mover NS theory. We can build a formal pure-gauge solution $\mathcal{G}_L$ of NS heterotic string field theory in the small Hilbert space of right-movers with a finite gauge parameter $\tilde{\eta}\Psi$ living in the left-mover large and right-mover small Hilbert space by successive infinitesimal gauge transformations. The pure-gauge field $\mathcal{G}_L = \mathcal{G}_L[\tilde{\eta}\Psi]$ is a function of $\tilde{\eta}\Psi$, defined by the $\tau = 1$ solution of the differential equation

$$\begin{aligned} \frac{\partial}{\partial\tau} \mathcal{G}_L[\tau\tilde{\eta}\Psi] &= Q\,\tilde{\eta}\Psi + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \big[\big(\mathcal{G}_L[\tau\tilde{\eta}\Psi]\big)^n, \tilde{\eta}\Psi\big]^L \\ &\equiv Q_{\mathcal{G}_L[\tau\tilde{\eta}\Psi]} \tilde{\eta}\Psi \end{aligned} \tag{4.1}$$

[1]For general $\mathcal{G}$, weak L_∞-relations hold, which are equivalent to (strong) L_∞-relations up to $\mathcal{F}(\mathcal{G})$.

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with the initial condition $\mathcal{G}_L[0]=0$, where $\tau\in[0,1]$ is a real parameter connecting 0 and $\mathcal{G}_L[\tilde{\eta}\Psi]$. (See also [6, 20, 21], appendix A, and appendix B.) Solving the defining equation (4.1) and setting $\tau=1$, we obtain the explicit form of the pure-gauge $\mathcal{G}_L\equiv\mathcal{G}_L[\tau=1]$ as follows

$$\mathcal{G}_L=Q\tilde{\eta}\Psi+\frac{\kappa}{2}[Q\tilde{\eta}\Psi,\tilde{\eta}\Psi]^L+\frac{\kappa^2}{3!}\Big(\big[Q\tilde{\eta}\Psi,Q\tilde{\eta}\Psi,\tilde{\eta}\Psi\big]^L+\big[[Q\tilde{\eta}\Psi,\tilde{\eta}\Psi]^L\tilde{\eta}\Psi\big]^L\Big)+\dots\tag{4.2}$$

Note that $\mathcal{G}_L$ is a Grassmann even and ghost-and-picture number $(2|0,-1)$ state satisfying $\eta\mathcal{G}_L\neq 0$ and $\tilde{\eta}\mathcal{G}_L=0$, the field $\tilde{\eta}\Psi$ is a Grassmann odd and ghost-and-picture number $(1|0,-1)$ state satisfying $\eta(\tilde{\eta}\Psi)\neq 0$ and $\tilde{\eta}(\tilde{\eta}\Psi)=0$, and the n-product $[A_1,\dots,A_n]^L$ is a Grassmann odd and ghost-and-picture number $(3-2n|0,n-1)$ product satisfying $\Delta_{\eta/\tilde{\eta}}[A_1,\dots,A_n]^L=0$.

An associated field $\psi_{\mathbb{X}}$ with derivation $\mathbb{X}$. In the rest, we simply write $\mathcal{G}_L[\tau]$ for the intermediate pure-gauge field rather than $\mathcal{G}_L[\tau\tilde{\eta}\Psi]$. There exists a special string field $\psi_{\mathbb{X}}$, so-called *an associated field*, satisfying

$$(-1)^{\mathbb{X}}\mathbb{X}\,\mathcal{G}_L=Q_{\mathcal{G}_L}\psi_{\mathbb{X}},\tag{4.3}$$

where $\mathbb{X}$ is a derivation of our right-mover $(-,\mathrm{NS})$ string products $[A_1,\dots,A_n]^L$:

$$(-1)^{\mathbb{X}}\mathbb{X}[A_1,\dots,A_n]^L+\sum_{i=1}^{n}(-1)^{\mathbb{X}(A_1+\cdots+A_{i-1})}[A_1,\dots,\mathbb{X}A_i,\dots A_n]^L=0.\tag{4.4}$$

Utilizing the $\mathcal{G}_L[\tau]$-shifted two-product $[A_1,A_2]^L_{\mathcal{G}_L[\tau]}$, the defining equation of $\psi_{\mathbb{X}}$ is given by

$$\frac{\partial}{\partial\tau}\psi_{\mathbb{X}}[\tau]=\mathbb{X}\,\tilde{\eta}\Psi+\kappa\big[\tilde{\eta}\Psi,\psi_{\mathbb{X}}[\tau]\big]^L_{\mathcal{G}_L[\tau]}\tag{4.5}$$

with the initial condition $\psi_{\mathbb{X}}[0]=0$. Note that $\psi_{\mathbb{X}}[\tau]$, as well as $\mathcal{G}_L[\tau]$, is a function of $\tau\tilde{\eta}\Psi$ and τ is a real parameter connecting 0 and $\psi_{\mathbb{X}}:=\psi_{\mathbb{X}}[1]$. The associated filed $\psi_{\mathbb{X}}$ carries ghost-and-picture number $(\mathbf{G}_{\mathbb{X}}+1|\mathbf{p}_{\mathbb{X}},\tilde{\mathbf{p}}_{\mathbb{X}}-1)$, where $(\mathbf{G}_{\mathbb{X}}|\mathbf{p}_{\mathbb{X}},\tilde{\mathbf{p}}_{\mathbb{X}})$ is that of $\mathbb{X}$.

A 'large' associated field $\Psi_{\mathbb{X}}$. These pure-gauge field $\mathcal{G}_L$ and associated field $\psi_{\mathbb{X}}$ belong to the left-mover large and right-mover small Hilbert space: $\eta\,\mathcal{G}_L\neq 0$, $\eta\,\psi_{\mathbb{X}}\neq 0$, and $\tilde{\eta}\,\mathcal{G}_L=\tilde{\eta}\,\psi_{\mathbb{X}}=0$. Since $\tilde{\eta}$-cohomology is trivial in the large Hilbert space of left-and-right movers, there exist large fields G_L and $\Psi_{\mathbb{X}}$ such that

$$\mathcal{G}_L=\tilde{\eta}\,G_L,\qquad\psi_{\mathbb{X}}=\tilde{\eta}\,\Psi_{\mathbb{X}}.\tag{4.6}$$

Note that the relation $\tilde{\eta}\,\mathbb{X}G_L=-\tilde{\eta}\,Q_{\mathcal{G}_L}\Psi_{\mathbb{X}}$ holds because of $(-)^{\mathbb{X}}\mathbb{X}(\eta G_L)=Q_{\mathcal{G}_L}(\tilde{\eta}\Psi_{\mathbb{X}})$ and $\tilde{\eta}\mathcal{G}_L=0$. Hence, up to $Q_{\mathcal{G}_L}$- and $\tilde{\eta}$-exact terms, these large-space fields G_L and $\Psi_{\mathbb{X}}$ satisfy

$$\mathbb{X}\,G_L=-Q_{\mathcal{G}_L}\Psi_{\mathbb{X}},\tag{4.7}$$

and the defining equations (up to $Q_{\mathcal{G}_L}$- and $\tilde{\eta}$-exact terms) of G_L and $\Psi_{\mathbb{X}}$ are given by

$$\frac{\partial}{\partial\tau}G_L[\tau]=-Q_{\mathcal{G}_L[\tau]}\Psi,\tag{4.8}$$

$$\frac{\partial}{\partial\tau}\Psi_{\mathbb{X}}[\tau] = (-1)^{\mathbb{X}}\mathbb{X}\Psi + \kappa\big[\tilde{\eta}\Psi, \Psi_{\mathbb{X}}[\tau]\big]^{L}_{\mathcal{G}_L[\tau]}, \tag{4.9}$$

with the initial conditions $G_L[\tau = 0] = 0$ and $\Psi_{\mathbb{X}}[\tau = 0] = 0$. As well as $\mathcal{G}_L$ and $\psi_{\mathbb{X}}$, large fields G_L and $\Psi_{\mathbb{X}}$ are also functions of $(\Psi, \tilde{\eta}\Psi)$. Here τ is a real parameter connecting 0 and $\tilde{\eta}\Psi$. With the initial condition $\Psi_{\mathbb{X}}[\tau = 0] = 0$, we find that the first few terms in $\Psi_{\mathbb{X}}$ are given by

$$(-1)^{\mathbb{X}}\Psi_{\mathbb{X}} = \mathbb{X}\Psi + \frac{\kappa}{2}[\tilde{\eta}\Psi, \mathbb{X}\Psi]^L + \frac{\kappa^2}{3!}\Big(2[\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \mathbb{X}\Psi]^L + [\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \mathbb{X}\Psi]^L]^L\Big) + \dots. \tag{4.10}$$

Note that the large associated field $\Psi_{\mathbb{X}}$ has the same ghost-and-picture number as $\mathbb{X}$.

A t-parametrized large field Ψ_t. Let $\Psi(t)$ be a t-parametrized path connecting $\Psi(0) = 0$ and $\Psi(1) = \Psi$. The above defining equations of $\mathcal{G}_L$, $\psi_{\mathbb{X}}$, G_L, and $\Psi_{\mathbb{X}}$ hold not only for the field Ψ but for the t-parametrized field $\Psi(t)$. Hence, we can built t-parametrized ones $\mathcal{G}_L(t)$, $\psi_{\mathbb{X}}(t)$, $G_L(t)$, and $\Psi_{\mathbb{X}}(t)$ by replacing Ψ with $\Psi(t)$ in the defining equations of $\mathcal{G}_L$, $\psi_{\mathbb{X}}$, G_L, and $\Psi_{\mathbb{X}}$. For example, for $\mathbb{X} = \partial_t$, solving (4.9) with replacement of Ψ and setting $\tau = 1$, we obtain the t-parametrized field $\Psi_t \equiv \Psi_{\partial_t}(t)$

$$\begin{aligned}\Psi_t = \partial_t\Psi(t) + \frac{\kappa}{2}[\tilde{\eta}\Psi(t), \partial_t\Psi(t)]^L + \frac{\kappa^2}{3!}\Big(&2[\tilde{\eta}\Psi(t), Q\tilde{\eta}\Psi(t), \partial_t\Psi(t)]^L \\ &+ [\tilde{\eta}\Psi(t), [\tilde{\eta}\Psi(t), \partial_t\Psi(t)]^L]^L\Big) + \dots,\end{aligned} \tag{4.11}$$

which appears in the action for NS-NS string fields with general t-parametrization. Note that this Ψ_t has the same ghost-and-picture number $(0|0,0)$ as NS-NS string field Ψ, and the equation $\tilde{\eta}\Psi_t = \tilde{\eta}\Psi$ holds for the linear path $\Psi(t) = t\Psi$.

4.2 Wess-Zumino-Witten-like action

Let $\mathcal{G}_L = \sum_{n=0}^{\infty}\kappa^n\mathcal{G}_L^{(n)}$ be the expansion of the pure-gauge $\mathcal{G}_L$ in powers of κ. Here, we propose a large-space WZW-like action utilizing the pure-gauge $\mathcal{G}_L(t)$ and the large associated field Ψ_t.

The generating function for $V_n(\Psi^n)$. Recall that the kinetic term $S_2 = \frac{1}{2}\langle\Psi, V_1(\Psi)\rangle$ is given by $V_1(\Psi) = \eta Q\tilde{\eta}\Psi$, which is equivalent to $\eta\mathcal{G}_L^{(0)}$. In section 2, we derived the gauge-invariant cubic vertex V_2 of $S_3 = \frac{1}{3!}\langle\Psi, V_2(\Psi^2)\rangle$

$$V_2(\Psi^2) = \eta[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L = \frac{\eta}{3!}\Big(\widetilde{X}[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi] + [\widetilde{X}Q\tilde{\eta}\Psi, \tilde{\eta}\Psi] + [Q\tilde{\eta}\Psi, \widetilde{X}\tilde{\eta}\Psi]\Big), \tag{4.12}$$

and the gauge-invariant quartic vertex V_3 of $S_4 = \frac{1}{4!}\langle\Psi, V_3(\Psi^3)\rangle$

$$V_3(\Psi^3) = \eta\Big(\big[Q\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi\big]^L + \big[[Q\tilde{\eta}\Psi, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L\Big), \tag{4.13}$$

which are equivalent to $2 \cdot \eta\,\mathcal{G}_L^{(1)}$ and $3! \cdot \eta\,\mathcal{G}_L^{(2)}$ respectively. Note that quintic vertex

$$V_4(\Psi^4) = \eta\Big(\big[(Q\tilde{\eta}\Psi)^3, \tilde{\eta}\Psi\big]^L + \big[[(Q\tilde{\eta}\Psi)^2, \tilde{\eta}\Psi]^L, \tilde{\eta}\Psi\big]^L$$

$$+\,3\big[[Q\tilde\eta\Psi,\tilde\eta\Psi]^L,Q\tilde\eta\Psi,\tilde\eta\Psi\big]^L+\big[[[Q\tilde\eta\Psi,\tilde\eta\Psi]^L,\tilde\eta\Psi]^L\big]^L\Big), \tag{4.14}$$

is also given by $4!\cdot\eta\,\mathcal{G}_L^{(3)}$. Similarly, the relation $V_n(\Psi^n)=n!\cdot\eta\,\mathcal{G}_L^{(n-1)}$ holds for the $(n+1)$-point vertex V_n. Therefore, the pure-gauge solution $\mathcal{G}_L$

$$\mathcal{G}_L=Q\tilde\eta\Psi+\frac{\kappa}{2}[Q\tilde\eta\Psi,\tilde\eta\Psi]^L+\frac{\kappa^2}{3!}\Big([Q\tilde\eta\Psi,Q\tilde\eta\Psi,\tilde\eta\Psi]^L+\big[[Q\tilde\eta\Psi,\tilde\eta\Psi]^L,\tilde\eta\Psi\big]^L\Big)+\dots, \tag{4.15}$$

defined by $\partial_\tau\mathcal{G}_L=Q_{\mathcal{G}_L}(\tilde\eta\Psi)$ in (4.1) gives a generating function for vertices. Provided that the t-parametrization of $\Psi(t)$ is linear: $\Psi(t)=t\Psi$, we find

$$\sum_{n=1}^{\infty}\frac{\kappa^{n-1}}{(n+1)!}\langle\Psi,V_n(\Psi^n)\rangle=\int_0^1 dt\,\langle\Psi,\eta\,\mathcal{G}_L(t)\rangle. \tag{4.16}$$

Note that all coefficients of V_{n+1} and $\mathcal{G}_L^{(n)}$ match by the t-integral.

The generating function for $W_n(\Psi^n)$. Let $\Psi_\delta=\sum_{n=0}^\infty\kappa^n\Psi_\delta^{(n)}$ be the expansion of the associated field Ψ_δ in powers of κ, where 'δ' is the variation operator. Recall that the variation of S_2 is given by $\delta S_2=\langle\delta\Psi,V_1(\Psi)+W_1(\Psi)\rangle=\langle\delta\Psi,\eta Q\tilde\eta\Psi\rangle$, which means $W_1(\Psi)=0$. In section 2, we also determined W_2 and W_3, as well as V_2 and V_3, appearing in the calculation of the variation δS_3 and δS_4. Recall that W_2 is given by

$$\frac{\kappa}{2}\langle\delta\Psi,W_2(\Psi^2)\rangle=-\frac{\kappa}{2}\langle\delta\Psi,[\tilde\eta\Psi,V_1(\Psi)]^L\rangle, \tag{4.17}$$

which is equivalent to $\langle\Psi_\delta^{(1)},V_1(\Psi)\rangle$, and W_3 is given by

$$\begin{aligned}\frac{\kappa^2}{3!}\langle\delta\Psi,W_3(\Psi^3)\rangle=&-\frac{\kappa^2}{3!}\langle\delta\Psi,[[V_1(\Psi),\tilde\eta\Psi]^L,\tilde\eta\Psi]^L+2[V_1(\Psi),Q\tilde\eta\Psi,\tilde\eta\Psi]^L\rangle\\&+\frac{\kappa^2}{2}\langle\delta\Psi,\frac{1}{2}[V_2(\Psi^2),\tilde\eta\Psi]^L\rangle,\end{aligned} \tag{4.18}$$

which is equivalent to $\langle\Psi_\delta^{(1)},\frac{\kappa}{2!}V_2(\Psi^2)\rangle+\langle\Psi_\delta^{(2)},V_1(\Psi)\rangle$. Similarly, the following relation holds

$$\frac{\kappa^n}{n!}\langle\delta\Psi,W_n(\Psi^n)\rangle=\sum_{k=1}^{n-1}\frac{\kappa^{k-1}}{k!}\langle\Psi_\delta^{(n-k)},V_k(\Psi^k)\rangle. \tag{4.19}$$

Hence, the associated field Ψ_δ

$$\Psi_\delta=\delta\Psi+\frac{\kappa}{2}[\tilde\eta\Psi,\delta\Psi]^L+\frac{\kappa^2}{3!}\Big([\tilde\eta\Psi,Q\tilde\eta\Psi,\delta\Psi]^L+2\big[\tilde\eta\Psi,[\tilde\eta\Psi,\delta\Psi]^L\big]^L\Big)+\dots, \tag{4.20}$$

defined by $\partial_\tau\Psi_\delta=\delta\Psi+\kappa[\tilde\eta\Psi,\Psi_\delta]^L_{\mathcal{G}_L}$ in (4.9) determines W_n-terms.

The WZW-like action. Let $\Psi(t)$ be a t-parametrized NS-NS string field satisfying $\Psi(0)=0$ and $\Psi(1)=\Psi$. Replacing NS-NS string fields Ψ with t-parametrized NS-NS string fields $\Psi(t)$ in (4.1) and (4.9), we obtain t-parametrized pure-gauge and large associated fields: $\mathcal{G}_L(t)$, Ψ_t, and $\Psi_\eta(t)$. WZW-like NS-NS action consists of these fields, which we explain in the rest. Since the relation

$$\eta\,\mathcal{G}_L(t)=-Q_{\mathcal{G}_L(t)}\big(\tilde\eta\Psi_\eta(t)\big)=\tilde\eta\,Q_{\mathcal{G}_L(t)}\Psi_\eta(t) \tag{4.21}$$

holds and $Q_{\mathcal{G}_L}$, η, and $\tilde{\eta}$ are nilpotent operators, the state $\eta\,\mathcal{G}_L$ is a $Q_{\mathcal{G}_L}$-, η-, and $\tilde{\eta}$-exact state. The equation $[\![\partial_t, \delta]\!]\mathcal{G}_L(t) = 0$ implies $\partial_t(Q_{\mathcal{G}_L(t)}\Psi_\delta(t)) = \delta(Q_{\mathcal{G}_L(t)}\Psi_t)$.

Thus, we propose the following WZW-like action for NS-NS string field theory

$$S = \frac{2}{\alpha'}\int_0^1 dt\,\langle \eta\Psi_t,\, \mathcal{G}_L(t)\rangle, \tag{4.22}$$

which reduces to (4.16) or the familiar WZW form (see appendix B)

$$S|_{\Psi(t)=t\Psi} = -\frac{1}{\alpha'}\Big(\langle \Psi_\eta, \mathcal{G}_L\rangle + \kappa\int_0^1 dt\langle \Psi_t, [\Psi_\eta(t), \mathcal{G}_L(t)]^L_{\mathcal{G}_L(t)}\rangle\Big), \tag{4.23}$$

if we set $\Psi(t) = t\Psi$. Note that the $(n+1)$-point vertex includes n insertions of $\tilde{\eta}$ and the action S is invariant under the linear[2] gauge transformation $\delta_{\widetilde{\Omega}'}\Psi = \tilde{\eta}\,\widetilde{\Omega}'$.

The equation of motion is given by

$$\eta\,\mathcal{G}_L = \int_0^1 d\tau\Big(\tilde{\eta}\eta\, Q_{\mathcal{G}_L[\tau]}\Psi\Big) = 0, \tag{4.24}$$

which is derived in subsection 4.3. Although the action includes the integral over a real parameter t, the action S, the variation δS, the equation of motion $\eta\,\mathcal{G}_L = 0$, and gauge transformations are independent of the t-parametrization or t-parametrized path $\Psi(t)$.

4.3 Nonlinear gauge invariance

Here, we derive the equation of motion and the closed form expression of nonlinear gauge transformations. Note that, for example, $\mathcal{G}_L(t=0) = 0$, $\mathcal{G}_L(t=1) = \mathcal{G}_L$, $\Psi_\delta(t=0) = 0$, and $\Psi_\delta(t=1) = \Psi_\delta$ hold.

For this purpose, we prove that the variation δS does not includes t and is given by

$$\delta S = \langle \Psi_\delta, \eta\,\mathcal{G}_L\rangle. \tag{4.25}$$

Using the relation $\tilde{\eta}\,Q_{\mathcal{G}_L}(\partial_t\Psi_\delta - \delta\Psi_t + \kappa[\Psi_t, \psi_\delta]^L_{\mathcal{G}_L}) = 0$, which is equivalent to $\partial_t(\delta\mathcal{G}_L) = \delta(\partial_t\mathcal{G}_L)$ with $\tilde{\eta}\Psi_{\mathbb{X}} = \psi_{\mathbb{X}}$, we find that the following equation holds for any t

$$\begin{aligned}\langle \delta\Psi_t,\, \eta\,\mathcal{G}_L(t)\rangle &= -\langle \delta\Psi_t,\, Q_{\mathcal{G}_L(t)}\psi_\eta(t)\rangle\\ &= -\langle \partial_t\Psi_\delta(t) - \kappa[\Psi_\delta(t), \psi_t]^L_{\mathcal{G}_L(t)},\, Q_{\mathcal{G}_L(t)}\psi_\eta(t)\rangle\\ &= \langle \partial_t\Psi_\delta(t),\, \eta\,\mathcal{G}_L(t)\rangle - \kappa\langle \eta\,\mathcal{G}_L(t),\, [\Psi_\delta(t), \psi_t]^L_{\mathcal{G}_L(t)}\rangle.\end{aligned} \tag{4.26}$$

Similarly, since $\eta\,\mathcal{G}_L = 0$, $[\![\eta, Q_{\mathcal{G}_L}]\!] = 0$, and $\psi_{\mathbb{X}} = \eta\,\Psi_{\mathbb{X}}$, we obtain

$$\begin{aligned}\langle \Psi_t,\, \delta\big(\eta\,\mathcal{G}_L(t)\big)\rangle &= \langle \Psi_t,\, \eta\big(Q_{\mathcal{G}_L(t)}\psi_\delta(t)\big)\rangle = -\langle Q_{\mathcal{G}_L(t)}(\eta\Psi_t),\, \psi_\delta(t)\rangle\\ &= -\langle \psi_\delta(t),\, Q_{\mathcal{G}_L(t)}(\eta\Psi_t)\rangle = -\langle \Psi_\delta(t),\, Q_{\mathcal{G}_L(t)}(\eta\psi_t)\rangle\\ &= \langle \Psi_\delta(t),\, \eta(Q_{\mathcal{G}_L(t)}\psi_t) + \kappa[\eta\,\mathcal{G}_L(t), \psi_t]^L_{\mathcal{G}_L(t)}\rangle\\ &= \langle \Psi_\delta(t),\, \partial_t\big(\eta\,\mathcal{G}_L(t)\big)\rangle + \kappa\langle \eta\,\mathcal{G}_L(t),\, [\Psi_\delta(t), \psi_t]^L_{\mathcal{G}_L(t)}\rangle.\end{aligned} \tag{4.27}$$

[2] Note that, however, $\mathcal{G}_L$ and Ψ_t include a lot of $\tilde{\eta}\Psi$ and as seen in 4.3, it does not mean that there are no gauge transformations including nonlinear terms of $\tilde{\eta}\Psi$.

Hence, the variation δS of the WZW-like action S is given by

$$\delta S = \int_0^1 dt \Big(\langle \delta\Psi_t,\, \eta\, \mathcal{G}_L(t)\rangle + \langle \Psi_t,\, \delta(\eta\, \mathcal{G}_L(t))\rangle \Big)$$
$$= \int_0^1 dt\, \frac{\partial}{\partial t} \langle \Psi_\delta(t),\, \eta\, \mathcal{G}_L(t)\rangle = \langle \Psi_\delta,\, \eta\, \mathcal{G}_L\rangle, \tag{4.28}$$

which does not include t-parametrized fields. The equation of motion is, therefore, given by (4.24) and it is independent of t-parametrization of fields.

Since $\eta\, \mathcal{G}_L$ is a $Q_{\mathcal{G}_L}$-, η-, and $\tilde{\eta}$-exact state, we find that the action is invariant under the following nonlinear Q- and η-gauge transformations and linear $\tilde{\eta}$-gauge transformation

$$\Psi_\delta = Q_{\mathcal{G}_L} \Lambda + \eta\, \Omega + \tilde{\eta}\, \widetilde{\Omega}, \tag{4.29}$$

where Λ, Ω, and $\widetilde{\Omega}$ are gauge parameter fields whose ghost-and-picture numbers are $(-1|0,0)$, $(-1|1,0)$, and $(-1|0,1)$ respectively. Note that Ψ_δ is an invertible function of $\delta\Psi$, at least in the expansion in powers of κ as follows

$$\delta\Psi = \Psi_\delta - \frac{\kappa}{2}[\tilde{\eta}\Psi, \Psi_\delta]^L - \frac{\kappa^2}{3!}\left(\frac{1}{2}\big[\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \Psi_\delta]^L\big]^L + 2\big[\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \Psi_\delta\big]^L \right) + O(\kappa^3). \tag{4.30}$$

For instance, an explicit expression for Q-gauge transformation $\delta_\Lambda \Psi$ and η-gauge transformation $\delta_\Omega \Psi$ are given by

$$\delta_\Lambda \Psi = Q\Lambda + \kappa[Q\tilde{\eta}\Psi, \Lambda]^L - \frac{\kappa}{2}[\tilde{\eta}\Psi, Q\Lambda]^L + O(\kappa^2) \tag{4.31}$$

$$\delta_\Omega \Psi = \eta\Omega - \frac{\kappa}{2}[\tilde{\eta}\Psi, \eta\Omega]^L - \frac{\kappa^2}{3}\big[\eta\Omega, Q\tilde{\eta}\Psi, \tilde{\eta}\Psi\big]^L - \frac{\kappa^2}{12}\big[[\eta\Omega, \tilde{\eta}\Psi], \tilde{\eta}\Psi\big]^L + O(\kappa^3). \tag{4.32}$$

These gauge transformations are nonlinear. Note, however, that since $\tilde{\eta}$-gauge transformation

$$\delta_{\widetilde{\Omega}} \Psi = \tilde{\eta}\widetilde{\Omega} - \frac{\kappa}{2}[\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}]^L - \frac{\kappa^2}{3}[\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}]^L - \frac{\kappa^2}{12}[\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}]^L]^L + O(\kappa^3) \tag{4.33}$$

obtained from $\Psi_{\delta_{\widetilde{\Omega}}} = \tilde{\eta}\, \widetilde{\Omega}$ consists of $\tilde{\eta}$-exact terms, it is equivalent to the linear $\tilde{\eta}$-gauge transformation

$$\delta_{\widetilde{\Omega}'} \Psi = \tilde{\eta}\, \widetilde{\Omega}', \tag{4.34}$$

where $\widetilde{\Omega}'$ is a redefined $\tilde{\eta}$-gauge parameter

$$\widetilde{\Omega}' \equiv \widetilde{\Omega} - \frac{\kappa}{2}[\tilde{\eta}\Psi, \widetilde{\Omega}]^L - \frac{\kappa^2}{3!}\left(2\big[\tilde{\eta}\Psi, Q\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}\big]^L + \frac{1}{2}\big[\tilde{\eta}\Psi, [\tilde{\eta}\Psi, \tilde{\eta}\widetilde{\Omega}]^L\big]^L \right) + O(\kappa^3). \tag{4.35}$$

As a result, although the action has three generators of gauge transformations, since one of these gauge invariances reduces to trivial, the resulting theory is Wess-Zumino-Witten-likely formulated with two nonlinear gauge invariances.

5 Conclusion

In this paper, we proposed WZW-like expressions for the action and nonlinear gauge transformations in the NS-NS sector of superstring field theory in the large Hilbert space. Although the action uses t-parametrized large fields $\Psi(t)$ satisfying $\Psi(0) = 0$ and $\Psi(1) = \Psi$, it does not depend on t-parametrization. Vertices are determined by a pure-gauge solution of NS (heterotic) string field theory in the small Hilbert space of right movers, which is constructed by NS closed superstring products (except for the BRST operator) including insertions of right-moving picture-changing operators [31].

Gauge equivalent vertices. We used the $(-, \mathrm{NS})$ string products, namely, the right edge points at the diamonds of products in figure 5.1 of [31]. It would be possible to write the large-space NS-NS action utilizing another but gauge-equivalent products in [31] instead of the $(-, \mathrm{NS})$ string products.

Ramond sectors. We have not analyzed how to incorporate the R sector(s). Our large-space NS-NS action has the almost same algebraic properties as the large-space action for NS closed string field theory. Thus, we can expect that the method proposed in [25, 26] also goes in the NS-NS case.

It is very important to obtain clear understandings of the geometrical meaning of theory, gauge fixing [35–37], the relation between two formulations: large- and small-space formulations. However, our large-space formulation is purely algebraic and these aspects remain mysterious.

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A Heterotic theory in the small Hilbert space

The action for heterotic string field theory in the small Hilbert space of right movers is given by

$$S = \frac{1}{2}\langle \Phi, Q\Phi \rangle + \sum_{n=1}^{\infty} \frac{\kappa^n}{(n+2)!} \langle \Phi, [\Phi^n, \Phi]^L \rangle, \tag{A.1}$$

where the NS heterotic string field Φ is a ghost-and-picture number $(2|0, -1)$ state in the small Hilbert space of right movers and right-moving picture-changing operators $\widetilde{X}$ inserted product $[A_1, \dots, A_n]^L$ given by [31] carries ghost-and-picture number $(3-2n|0, n-1)$. This action is invariant under the following gauge transformation [6, 7]

$$\delta\Phi = Q\lambda + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} [\Phi^n, \lambda]^L \equiv Q_\Phi \lambda, \tag{A.2}$$

where λ is a gauge parameter carrying ghost-and-picture number $(1|0, -1)$.

Just as bosonic theory [5, 7], the equation of motion is given by

$$Q\Phi + \sum_{n=1}^{\infty} \frac{\kappa^n}{(n+1)!} [\Phi^n, \Phi]^L = 0, \tag{A.3}$$

and a pure-gauge $\mathcal{G}_L$ is constructed by infinitisimal gauge transformations [6, 20, 21]. Therefore, $\mathcal{G}_L$ is defined by the $\tau = 1$ value solution $\mathcal{G}_L \equiv \mathcal{G}_L[\tau = 1]$ of the following differential equation

$$\frac{\partial}{\partial\tau}\mathcal{G}_L[\tau] = Q\lambda + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} \big[\mathcal{G}_L[\tau]^n, \lambda\big]^L = Q_{\mathcal{G}_L[\tau]}\lambda, \tag{A.4}$$

with the initial condition $\mathcal{G}_L[\tau = 0] = 0$.

B Some identities

BPZ-properties. The c_0^--inserted BPZ inner product $\langle A, B\rangle := \langle \mathrm{bpz}(A)|c_0^-|B\rangle$, bosonic or heterotic string products, and a derivation operator $\mathbb{X}$ satisfy

$$\langle A, B\rangle = (-)^{(A+1)(B+1)}\langle B, A\rangle, \tag{B.1}$$

$$\langle [A_0, \ldots, A_{n-1}], A_n\rangle = (-)^{A_0+\cdots+A_{n-1}}\langle A_0, [A_1, \ldots, A_n]\rangle, \tag{B.2}$$

$$\langle \mathbb{X}A, B\rangle = (-)^{A\mathbb{X}}\langle A, \mathbb{X}B\rangle, \tag{B.3}$$

where $c_0^- = \frac{1}{2}(c_0 - \tilde{c}_0)$ and $\mathbb{X} = Q, \eta, \tilde{\eta}$.

The Maurer-Cartan element. A pure-gauge solution $\mathcal{G}_L$ satisfies the equation of motion $\mathcal{F}(\mathcal{G}_L) = 0$ of NS heterotic string field theory in the small Hilbert space of right movers. Using the defining equation of $\mathcal{G}_L$, we find that

$$\begin{aligned}
\frac{\partial}{\partial\tau}\mathcal{F}(\mathcal{G}_L) &= \frac{\partial}{\partial\tau}\bigg(Q\,\mathcal{G}_L + \sum_{n=1}^{\infty} \frac{\kappa^n}{(n+1)!}\big[\mathcal{G}_L^n, \mathcal{G}_L\big]^L\bigg) \\
&= QQ_{\mathcal{G}_L}\tilde{\eta}\Psi + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!}\big[\mathcal{G}_L^n, Q_{\mathcal{G}_L}\tilde{\eta}\Psi\big]^L = Q_{\mathcal{G}_L}^2(\tilde{\eta}\Psi),
\end{aligned} \tag{B.4}$$

which leads to the differential equation $\partial_\tau\mathcal{F} = \big[\mathcal{F}, \tilde{\eta}\Psi\big]^L_{\mathcal{G}_L}$ with the initial condition $\mathcal{F}(0) = 0$. Hence, $\mathcal{G}_L$ satisfies $\mathcal{F}(\mathcal{G}_L) = 0$ and $Q_{\mathcal{G}_L}$ is a nilpotent operator. (See also [20, 21].)

The standard WZW form. Recall that when there exist higher sting products $[A_1, \ldots, A_n]^L$ $(n > 2)$, a field-strength-like object $f_{XY} \equiv X\psi_Y - (-)^{XY}Y\psi_X + (-)^X\kappa[\psi_X, \psi_Y]^L_{\mathcal{G}_L}$ is not zero $f_{XY} \neq 0$ but a $Q_{\mathcal{G}_L}$-exact state: $Q_{\mathcal{G}_L}f_{XY} = 0$, where X and Y are derivation operators satisfying $[\![X, Y]\!] = 0$. Let

$$F_{XY} \equiv X\Psi_Y + (-)^{(X+1)(Y+1)}Y\Psi_X + \kappa[\Psi_X, \psi_Y]^L_{\mathcal{G}_L} \tag{B.5}$$

be a *large* field-strength-like object satisfying $\tilde{\eta}F_{XY} = (-)^X f_{XY}$. Utilizing this $F_{\eta t}$ and the relation $\langle\Psi_t,\, Q_{\mathcal{G}_L}\psi_\eta\rangle = \langle\Psi_\eta,\, \partial_t\mathcal{G}_L\rangle$, our WZW-like action can be rewritten as

$$S = \frac{1}{\alpha'}\int_0^1 dt\,\Big(\langle\eta\,\Psi_t,\, \mathcal{G}_L\rangle + \langle\Psi_t,\, \eta\,\mathcal{G}_L\rangle\Big)$$

$$= \frac{1}{\alpha'}\int_0^1 dt\Big(\langle F_{\eta t} - \partial_t\Psi_\eta - \kappa[\Psi_t,\psi_\eta]^L_{\mathcal{G}_L},\,\mathcal{G}_L\rangle - \langle\Psi_t, Q_{\mathcal{G}_L}\psi_\eta\rangle\Big)$$

$$= \frac{1}{\alpha'}\int_0^1 dt\langle\mathcal{G}_L,\,F_{\eta t}\rangle - \frac{1}{\alpha'}\int_0^1 dt\Big[\Big(\langle\partial_t\Psi_\eta,\mathcal{G}_L\rangle + \langle\Psi_\eta,\partial_t\mathcal{G}_L\rangle\Big) + \kappa\langle\Psi_t,[\psi_\eta,\mathcal{G}_L]^L_{\mathcal{G}_L}\rangle\Big]. \quad (B.6)$$

Recall also that the linear t-parametrization $\Psi(t) = t\Psi$ gives $\Psi_t = \Psi$ up to $\tilde{\eta}$-exact terms. When we identify τ and t, the defining equation of ψ_X becomes $\partial_t\psi_X = X\tilde{\eta}\Psi + \kappa[\tilde{\eta}\Psi,\psi_X]^L_{\mathcal{G}_L}$, which implies $\tilde{\eta}\big(\partial_t\Psi_X - (-)^X X\Psi + \kappa[\Psi,\psi_X]^L_{\mathcal{G}_L}\big) = 0$. Hence, provided that $\Psi(t) = t\Psi$, we obtain $\tilde{\eta}F_{\eta t} = 0$ and the action reduces to the familiar WZW-form:

$$S|_{\Psi(t)=t\Psi} = -\frac{1}{\alpha'}\Big(\langle\Psi_\eta,\,\mathcal{G}_L\rangle + \kappa\int_0^1 dt\,\langle\Psi_t,\,[\psi_\eta(t),\mathcal{G}_L(t)]^L_{\mathcal{G}_L(t)}\rangle\Big). \quad (B.7)$$

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