

## Tinkertoys for the $E_6$ theory

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**ABSTRACT:** Compactifying the 6-dimensional (2,0) superconformal field theory, of type ADE, on a Riemann surface,  $C$ , with codimension-2 defect operators at points on  $C$ , yields a 4-dimensional  $\mathcal{N} = 2$  superconformal field theory. An outstanding problem is to classify the 4D theories one obtains, in this way, and to understand their properties. In this paper, we turn our attention to the  $E_6$  (2,0) theory, which (unlike the A- and D-series) has no realization in terms of M5-branes. Classifying the 4D theories amounts to classifying all of the 3-punctured spheres (“fixtures”), and the cylinders that connect them, that can occur in a pants-decomposition of  $C$ . We find 904 fixtures: 19 corresponding to free hypermultiplets, 825 corresponding to isolated interacting SCFTs (with no known Lagrangian description) and 60 “mixed fixtures”, corresponding to a combination of free hypermultiplets and an interacting SCFT. Of the 825 interacting fixtures, we list only the 139 “interesting” ones. As an application, we study the strong coupling limits of the Lagrangian field theories:  $E_6$  with 4 hypermultiplets in the 27 and  $F_4$  with 3 hypermultiplets in the 26.

**KEYWORDS:** Supersymmetric gauge theory, Supersymmetry and Duality, Extended Supersymmetry, Duality in Gauge Field Theories

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## 1 Introduction

In [1, 2], Gaiotto and collaborators introduced a construction for a class of 4D  $\mathcal{N} = 2$  superconformal theories, realizing them as compactifications of the 6D (2,0) theories on a punctured Riemann surface. Because of its six-dimensional origin, this class of theories (sometimes called class “S”) is endowed with a powerful set of tools [1–11] for studying its physical properties. For a theory in this class, it is trivial to write its low-energy Seiberg-Witten solution and study  $\mathcal{N} = 2$  S-duality [12], as well as compute central charges, the global symmetry group, graded dimensions of the Coulomb branch, etc.

Equally interesting, the understanding of the theories in this class expands our knowledge of what 4D  $\mathcal{N} = 2$  SCFTs actually exist. The set of  $\mathcal{N} = 2$  SCFTs includes ordinary gauge theories (recently classified in [13]), some of which are realized from six dimensions, but also isolated interacting fixed points of the renormalization group with no known Lagrangian description. Gaiotto’s construction generates infinitely many new examples of these interacting theories. Thus, while it seems unlikely that this construction covers the full gamut of 4D  $\mathcal{N} = 2$  SCFTs, it becomes interesting to ask what subset it *does* cover.

In [3], we started a systematic classification of the 4D  $\mathcal{N} = 2$  SCFTs, arising from the  $A_{N-1}$  series of (2,0) theories. The classification is possible because the various pieces of the construction can themselves be classified. The (2,0) theories are classified by a choice of simply-laced Lie algebra  $\mathfrak{j}$  [14]; (a class of) punctures on the Riemann surface are labeled by nilpotent orbits in  $\mathfrak{j}$  [15, 16], and degenerating Riemann surfaces can be decomposed into a collection of three-punctured spheres connected by cylinders. The set of theories becomes larger if we allow for the (2,0) theory to be “twisted” by an outer-automorphism of  $\mathfrak{j}$ , when traversing a nontrivial cycle of the punctured Riemann surface,  $\mathcal{C}$ . In particular, a twist when circling a puncture introduces a new class of defects, called “twisted punctures,” which are classified by nilpotent orbits in non-simply-laced Lie algebras obtained by dividing  $\mathfrak{j}$  by the action of the outer automorphism [9, 17]. In [4], we extended the classification of [3] to the (untwisted)  $D_N$  series, and in [10] and [11] we incorporated  $\mathbb{Z}_2$  outer automorphism twists in the  $A_{2N-1}$  series and in the  $D_N$  series, respectively.

In this paper, we extend our classification program to the (2,0) theory of type  $E_6$ . We leave the study of this theory in the presence of  $\mathbb{Z}_2$  outer automorphism twists for another publication. There is no known construction of the  $E_6$  theory as a low-energy theory of a stack of M5 branes, as was the case for the A- and D-series. Rather, the only known construction is as a compactification of IIB string theory on a K3 manifold at an  $E_6$  singularity [14]. Still, computations are possible because the the 4D  $\mathcal{N} = 2$  compactification of the  $E_6$  theory is controlled by a Hitchin system [2] with gauge group  $E_6$ .

As a byproduct, we realize  $E_6$  gauge theory with matter in the 4(27), as well as  $F_4$  gauge theory with matter in the 3(26), as compactifications of the  $E_6$  (2,0) theory on a

4-punctured sphere. The Seiberg-Witten solution to the  $E_6$  gauge theory, with  $N_f \leq 3$  27s, appeared first in [18]. Our solution to the superconformal  $F_4$  gauge theory is new.

## 2 The $E_6$ theory

### 2.1 The Hitchin system

The Coulomb branch of the 4D  $\mathcal{N} = 2$  theories obtained from the compactification of the 6D (2,0) theory of type  $E_6$  on a Riemann surface  $C$  is described by the Hitchin equations on  $C$  with complexified gauge group  $E_6$  [2]. We may also include codimension-two defects of the (2,0) theory localized at points on  $C$ ; we refer to these as “punctures”. A class of punctures is classified by nilpotent orbits (or, equivalently, by embeddings of  $\mathfrak{sl}(2)$ ) in the complexified Lie algebra  $\mathfrak{e}_6$  [9]. One of the main points of the construction is that a number of physical properties of the 4D theories can be computed directly from geometric properties of the nilpotent orbits that label the punctures on  $C$ , without any detailed knowledge of the (2,0) theory.

A puncture labeled by a nilpotent orbit  $\mathcal{O}$ , and located at  $z = 0$  on  $C$ , corresponds to a local boundary condition for the Higgs field,

$$\Phi(z) = \frac{X}{z} + \dots \tag{2.1}$$

where  $\Phi$  is a holomorphic 1-form on  $C$  that takes values in  $\mathfrak{e}_6$  and transforms in the adjoint representation of the gauge group,  $X$  is a representative of the nilpotent orbit  $d(\mathcal{O})$  in  $\mathfrak{e}_6$ , and  $\dots$  represents a generic regular function of  $z$  taking values in  $\mathfrak{e}_6$ . Here,  $d(\mathcal{O})$  is the image of  $\mathcal{O}$  under the Lusztig-Spaltenstein map  $d$  [4, 7, 9]. Representatives of all nilpotent orbits in  $\mathfrak{e}_6$  can be found in [19], and a diagram specifying the action of  $d$ , as well as other properties of the  $\mathfrak{e}_6$  orbits, are collected in appendix C of [9] (taken from [20, 21]) When  $d$  is not injective, we distinguish different punctures with the same  $d(\mathcal{O})$  by their Sommers-Achar group  $\mathcal{C}(\mathcal{O})$  [9], which is a discrete subgroup of  $E_6$ , imposing gauge invariance of  $\Phi$  under the action of  $\mathcal{C}(\mathcal{O})$ .

As in our previous papers, we call  $\mathcal{O}$ , which labels the puncture, the *Nahm pole*, and  $d(\mathcal{O})$ , which appears in the Hitchin system boundary condition, the *Hitchin pole*. The physical properties of a puncture labeled by  $\mathcal{O}$  will be directly related to geometric properties of the orbits  $\mathcal{O}$  and  $d(\mathcal{O})$ , and the discrete group  $\mathcal{C}(\mathcal{O})$ .

Unlike classical Lie algebras, there is no natural parameterization of the nilpotent orbits of exceptional Lie algebras in terms of partitions or Young diagrams. Instead, the notation due to Bala and Carter is standard in the representation theory literature. This notation has been briefly discussed in previous works [9, 16, 22], but, for completeness, we review it in appendix A, and discuss how to extract relevant information from it.

### 2.2 $k$ -differentials

The low-energy solution of the 4D  $\mathcal{N} = 2$  theory is encoded in the Seiberg-Witten curve, which is given by the spectral curve of the Hitchin system, i.e., by the characteristic polynomial for the Higgs field  $\Phi$ , in representation  $R$  of  $\mathfrak{e}_6$ :

$$\Sigma_R : \det_R(\Phi - \lambda I) = \lambda^d + \lambda^{d-2}s_2 + \lambda^{d-3}s_3 + \dots + s_d = 0,$$

where  $d = \dim R$  and the  $\lambda^{d-1}$  is zero because  $\text{Tr}(\Phi) = 0$ . Different choices of  $R$  will yield different curves  $\Sigma_R$ . However, as discussed in [23], the physical information that one can extract from them is the same.

For a choice of  $R$ , let  $s_k$  be the coefficient of  $\lambda^{\dim(R)-k}$ , for  $k = 0, 1, 2, \dots, \dim(R)$ . ( $s_0$  and  $s_1$  are trivial — they are 1 and 0, respectively.) The  $s_k(z)$  are holomorphic  $k$ -differentials on  $C$  (with poles at the punctures), and can be expressed as polynomials in the trace invariants  $P_k = \text{Tr}(\Phi^k)$ . Notice that both the  $s_k$  and the  $P_k$  are dependent on the representation  $R$ .

On the other hand, we are actually interested in the Casimirs of  $\Phi$ , which are the independent  $k$ -differentials providing the gauge-invariant information contained in  $\Phi$ . For a Lie algebra  $\mathfrak{g}$ , the number of Casimirs is equal to the rank of  $\mathfrak{g}$ , and their scaling dimensions are the exponents (minus 1) of  $\mathfrak{g}$ . Unlike the  $s_k$  or the  $P_k$ , the Casimirs encode the non-redundant gauge-invariant information in  $\Phi$ .

In our previous papers [3, 4, 10, 11], the Lie algebra was of classical type, and  $R$  was always chosen to be the smallest non-trivial representation (the fundamental for  $A_{N-1}$ , or the vector for  $D_N$ ). In such cases, the coefficients  $s_k$  directly provide a basis for the Casimirs of  $\Phi$ . For example, for  $\mathfrak{g} = A_{N-1}$ , we have  $N - 1$  Casimirs, of dimensions  $2, 3, 4, \dots, N$ . These dimensions match precisely those of the non-trivial coefficients  $s_k$  if  $R$  is chosen to be the fundamental representation. Thus, the  $s_k$  can be taken to be the Casimirs of  $A_{N-1}$ . Similarly, for  $\mathfrak{g} = D_N$ , the  $N$  Casimirs have degrees  $2, 4, 6, \dots, 2N - 2; N$ . If  $R$  is the vector representation, the  $s_k$  with  $k$  odd vanish, and the non-trivial coefficients are  $s_2, s_4, \dots, s_{2N-2}, s_{2N}$ . Here,  $s_{2N}$  is the square of the Pfaffian,  $s_{2N} = \tilde{s}^2$ , and so  $\tilde{s}$  has dimension  $N$ . Thus, as before, the  $s_2, s_4, \dots, s_{2N-2}; \tilde{s}$  provide a basis of Casimirs of  $D_N$ .

But if for  $A_{N-1}$  and  $D_N$  we had chosen  $R$  to be, say, the adjoint representation, then, for large enough  $N$ , the  $s_k$  would not have given directly the Casimirs, but instead a lot of redundant information. For example, for  $A_5$ , there are five Casimirs, with dimensions  $2, 3, 4, 5, 6$ . However, we have 34 non-trivial coefficients  $s_k$ , with dimensions  $2, 3, 4, \dots, 35$ . These  $s_k$  are polynomials in the five Casimirs.

For  $\mathfrak{j} = \mathfrak{e}_6$ , the Casimirs have degrees  $2, 5, 6, 8, 9, 12$ . In our computations, we have chosen  $R$  to be the *adjoint* representation of  $\mathfrak{e}_6$ , as it is readily available in the form of structure constants; we used those from the computer algebra system GAP 4 [24]. Instead of trying to compute the 78 coefficients  $s_k$ , we focus directly on the trace invariants  $P_k$  for values of  $k$  only as large as needed to extract the Casimirs. For the adjoint representation of  $\mathfrak{e}_6$ , the  $P_k$  vanish for  $k$  odd, and are non-trivial for  $k$  even, except for  $P_4 = \frac{1}{32}(P_2)^2$ . Also, as we will see below, to extract the Casimirs, we only need to consider the  $P_k$  for  $k = 2, 6, 8, 10, 12, 14$ . From the  $P_k$ , one can construct a less-redundant basis,

$$\begin{aligned} \phi_2 &= \frac{1}{48} P_2 \\ \phi_6 &= \frac{1}{24} \left( P_6 - \frac{7}{4608} (P_2)^3 \right) \\ \phi_8 &= \frac{1}{30} \left( P_8 - \frac{2}{9} P_6 P_2 + \frac{155}{663552} (P_2)^4 \right) \end{aligned}$$

$$\begin{aligned}
\phi_{10} &= -\frac{1}{105} \left( P_{10} - \frac{17}{96} P_8 P_2 + \frac{77}{6912} P_6 (P_2)^2 - \frac{427}{63700992} (P_2)^5 \right) \\
\phi_{12} &= \frac{1}{155} \left( P_{12} - \frac{107}{504} P_{10} P_2 + \frac{515}{32256} P_8 (P_2)^2 - \frac{41}{108} (P_6)^2 + \frac{295}{497664} P_6 (P_2)^3 - \frac{5669}{9172942848} (P_2)^6 \right) \\
\phi_{14} &= \frac{1}{4389} \left( P_{14} - \frac{3479}{14880} P_{12} P_2 + \frac{61391}{3214080} P_{10} (P_2)^2 - \frac{539}{2160} P_8 P_6 - \frac{139733}{617103360} P_8 (P_2)^3 \right. \\
&\quad \left. + \frac{165781}{4821120} (P_6)^2 P_2 - \frac{3488947}{44431441920} P_6 (P_2)^4 + \frac{19596907}{409480168734720} (P_2)^7 \right)
\end{aligned}$$

This basis was constructed so that it reduces the constraints in our punctures to a minimum. In particular, the pole coefficients for the minimal puncture have no redundancies; that is, the  $\phi_k$  are such that it not be possible to reduce their pole orders in  $z$  further by a change of basis, for  $z$  a local coordinate centered at the minimal puncture. The  $\phi_k$  basis also makes apparent how the Casimirs of degree 5 and 9 appear. Specifically,  $\phi_{10}$  and  $\phi_{14}$  factor,

$$\begin{aligned}
\phi_{10} &\equiv (\phi_5)^2, \\
\phi_{14} &\equiv \phi_5 \phi_9
\end{aligned}$$

These relations define the odd-degree differentials  $\phi_5$  and  $\phi_9$  (up to a sign, which flips under the  $\mathbb{Z}_2$  outer automorphism of  $E_6$ ). So, we can declare the  $k$ -differentials  $\{\phi_2, \phi_5, \phi_6, \phi_8, \phi_9, \phi_{12}\}$  to be our basis of  $\mathfrak{e}_6$  Casimirs. In the following, by the  $\phi_k$ , we will refer to the Casimirs, and ignore the auxiliary differentials  $\phi_{10}$  and  $\phi_{14}$ .

As for the Seiberg-Witten curve, to write it explicitly, we need to know how the 78 coefficients  $s_k$  depend on the six Casimirs  $\phi_k$ . Instead, it is much simpler to write down the (representation independent) Seiberg-Witten geometry, given by an ALE fibration over  $C$ , and which equivalently describes the low-energy solution of 4D  $\mathcal{N} = 2$  theories, but directly using the Casimirs [25, 26]. Let us briefly review that construction.

### 2.3 ALE geometry

The 4D  $\mathcal{N} = 2$  SCFT constructed from the compactification of a 6D (2,0) theory of type  $J$  (where  $J$  is of A-D-E type) on the Riemann surface  $C$  can also be obtained, in a dual manner, from IIB string theory on a non-compact Calabi-Yau threefold, locally given by an ALE fibration over  $C$  of type  $J$  [25, 26]. For  $\mathfrak{e}_6$ , the threefold is realized as the hypersurface

$$\begin{aligned}
X_{\bar{u}} &= \{0 = w^2 + x^3 + y^4 + \epsilon_2(z)xy^2 + \epsilon_5(z)xy + \epsilon_6(z)y^2 + \epsilon_8(z)x + \epsilon_9(z)y + \epsilon_{12}(z)\} \\
&\subset \text{tot}(K_C^6 \oplus K_C^4 \oplus K_C^3)
\end{aligned}$$

where the  $\epsilon_k(z)$  are  $k$ -differentials on  $C$  [22] (in the “Katz-Morrison basis” [27]), related to our  $\phi_k(z)$  by

$$\begin{aligned} \epsilon_2 &= \frac{1}{2}\phi_2 \\ \epsilon_5 &= \frac{1}{6}\phi_5 \\ \epsilon_6 &= \frac{1}{72}(-3\phi_2^3 + 2\phi_6) \\ \epsilon_8 &= \frac{1}{144}(-3\phi_2^4 + 4\phi_2\phi_6 - \phi_8) \\ \epsilon_9 &= \frac{1}{72}(-\phi_2^2\phi_5 + 4\phi_9) \\ \epsilon_{12} &= \frac{1}{5184}(4\phi_{12} + 6\phi_2^6 - 12\phi_2^3\phi_6 + 4\phi_6^2 + 3\phi_2^2\phi_8) \end{aligned}$$

The  $\phi_k(z)$ , in turn, depend on the Coulomb branch parameters,  $\vec{u}$ , as we determine below.

The Seiberg-Witten solution is obtained by computing the periods of the holomorphic 3-form,  $\Omega$ , over a symplectic basis of (rational) 3-cycles on  $X_{\vec{u}}$  which are locally of the form of a 2-sphere in the fiber times a curve on  $C$ . In the conformal case (which will be our focus in this paper), many of these cycles will necessarily be noncompact (the curve on  $C$  being an open curve, stretching between punctures). But, precisely for the parabolic case (where the Higgs field  $\Phi(z)$  has simple poles at the punctures, with nilpotent residues), the singularity is integrable, and the requisite periods of  $\Omega$  are finite.

In our realization of  $F_4$  gauge theory in section 5.1.2, the differentials  $\phi_5(z)$  and  $\phi_9(z)$  vanish identically. In this case, the Calabi-Yau,  $X_{\vec{u}}$ , has a holomorphic involution,  $y \rightarrow -y$ , under which  $\Omega \rightarrow -\Omega$ . The 3-cycles which give the Seiberg-Witten solution are the anti-invariant cycles and the periods of  $\Omega$  over those cycles are finite, despite the slightly singular nature of  $X_{\vec{u}}$  itself.

## 2.4 Puncture properties

We describe below how to compute the properties of a puncture. There is a systematic way to compute every property of the puncture, except for the constraints, so it is easiest to compute the other properties first, and use them to guess the constraints. Below, let  $\mathcal{O}$  be the Nahm nilpotent orbit that labels a given puncture, and  $\mathfrak{su}(2)_{\mathcal{O}}$  the associated  $\mathfrak{su}(2)$  embedding in  $\mathfrak{e}_6$ .

### 2.4.1 Flavour groups

Associated to each puncture  $\mathcal{O}$ , there is a flavour group  $F = F_{\mathcal{O}}$ , whose Lie algebra  $\mathfrak{f}$  is the centralizer of  $\mathfrak{su}(2)_{\mathcal{O}}$  in  $\mathfrak{e}_6$ . A list of the centralizers for each  $\mathcal{O}$  can be found in table 14 of [9], taken originally from [28].

Let  $\mathfrak{f}_i$  be the simple, nonabelian factors in  $\mathfrak{f}$ . The *level*  $k_i$  of  $\mathfrak{f}_i$  is the central charge of the  $\mathfrak{f}_i$  current algebra. (See, e.g., [12] for a review.) Following [9], the  $k_i$  can be computed from the decomposition of the adjoint, 78, of  $\mathfrak{e}_6$  under the subalgebra  $\mathfrak{su}(2)_{\mathcal{O}} \times \mathfrak{f}$ . These decompositions can, in turn, be deduced from the Bala-Carter label for  $\mathcal{O}$ , and are summarized in the [table](#) in appendix A.

Let the decomposition of the 78 be

$$\mathfrak{e}_6 = \bigoplus_n V_n \otimes R_{n,i},$$

where  $V_n$  is the  $n$ -dimensional irrep of  $\mathfrak{su}(2)$  (denoted by “ $n$ ” in the [table](#)) and  $R_{n,i}$  is the corresponding (reducible) representation of  $\mathfrak{f}_i$ . Let  $l_{n,i}$  be the index of  $R_{n,i}$ . Then, the level of  $\mathfrak{f}_i$  is  $k_i = \sum_n l_{n,i}$ .

For example, consider the  $3A_1$  puncture, which has  $\mathfrak{f} = \mathfrak{su}(3) \times \mathfrak{su}(2)$ . From the [table](#) in appendix [A](#), we have, for  $\mathfrak{f}_1 = \mathfrak{su}(3)$ ,

$$R_1 = 8 + 3(1), \quad R_2 = 2(8), \quad R_3 = 8 + 1, \quad R_4 = 2(1),$$

and so the level is  $k_{\mathfrak{su}(3)} = 4l_8 = 24$ . Similarly, for  $\mathfrak{f}_2 = \mathfrak{su}(2)$ , we have

$$R_1 = 3 + 8(1), \quad R_2 = 8(2), \quad R_3 = 9(1), \quad R_4 = 2,$$

and thus the level is  $k_{\mathfrak{su}(2)} = l_3 + 9l_2 = 4 + 9 \times 1 = 13$ .

### 2.4.2 $\delta n_h$ and $\delta n_v$

The conformal anomaly central charges of the 4D theory,  $a$  and  $c$ , can be conveniently written in terms of effective numbers of hyper- and vector multiplets,  $n_h$  and  $n_v$ , as  $a = (5n_v + n_h)/24$  and  $c = (2n_v + n_h)/12$ . To each puncture,  $\mathcal{O}$ , we associate contributions  $\delta n_h$  and  $\delta n_v$  to  $n_h$  and  $n_v$ . The full  $n_h$  and  $n_v$  of the 4D theory are obtained by adding the contributions from the punctures plus certain global terms. The formulas in eq. (3.19) of [\[9\]](#) tell us that  $\delta n_h$  and  $\delta n_v$  can be computed from the decomposition of  $\mathfrak{e}_6$  into eigenspaces of the Cartan element of  $\mathfrak{su}(2)_{\mathcal{O}}$ .

Here, let us recast those formulas in terms of the *weighted Dynkin diagram* for  $\mathcal{O}$ , which can be found in table 14 of [\[9\]](#). Let  $\vec{x}$  be the six-dimensional vector consisting of the labels of the weighted Dynkin diagram for  $\mathcal{O}$ . Now, for each root  $\alpha$  of  $E_6$ , let  $\vec{k}$  be a six-dimensional vector consisting of the (integer) components of  $\alpha$  in any basis of simple roots. The “Weyl vector” is  $\vec{W} = \frac{1}{2} \sum_{\vec{k} \geq 0} \vec{k}$ , where the sum is over positive roots. Let  $n_0$  and  $n_{1/2}$  be the number of roots  $\alpha$  that satisfy  $(\vec{x}/2) \cdot \vec{k} = 0$  and  $(\vec{x}/2) \cdot \vec{k} = 1/2$ , respectively. (The dot product is Euclidean.) In this notation, the formulas in eq. (3.19) of [\[9\]](#) are:

$$\begin{aligned} \delta n_h(\vec{x}) &= 8 \left( \frac{1}{12} h^\vee(E_6) \dim(E_6) - \frac{1}{2} \vec{W} \cdot \vec{x} \right) + \frac{1}{2} n_{1/2} \\ \delta n_v(\vec{x}) &= 8 \left( \frac{1}{12} h^\vee(E_6) \dim(E_6) - \frac{1}{2} \vec{W} \cdot \vec{x} \right) - \frac{1}{2} n_0 \end{aligned} \tag{2.2}$$

where  $h^\vee(E_6) = 12$  denotes the dual Coxeter number of  $E_6$ .

For example, for  $\vec{x} = \vec{0}$ , corresponding to the maximal puncture, one has that the adjoint of  $E_6$  decomposes trivially into singlets of  $\mathfrak{su}(2)$ ,  $78 \rightarrow 78(1)$ , so  $n_0(\vec{0}) = \dim(E_6) - \text{rank}(E_6)$ , and  $n_{1/2}(\vec{0}) = 0$ . Thus,

$$\begin{aligned} \delta n_h(\vec{0}) &= \frac{2}{3} h^\vee(E_6) \dim(E_6) = 624 \\ \delta n_v(\vec{0}) &= \frac{2}{3} h^\vee(E_6) \dim(E_6) - \frac{1}{2} (\dim(E_6) - \text{rank}(E_6)) = 588 \end{aligned}$$

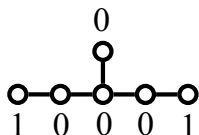


As a consistency check, we have the following relation between  $(\delta n_h, \delta n_v)$  and the complex (manifold) dimension  $\dim_{\mathbb{C}}(\mathcal{O})$  of  $\mathcal{O}$ :

$$\delta n_h - \delta n_v = \frac{1}{2}(\dim(E_6) - \text{rank}(E_6)) - \frac{1}{2} \dim_{\mathbb{C}}(\mathcal{O}). \tag{2.3}$$

The dimensions of the nilpotent orbits of  $E_6$  are listed in table 14 of [9].

For a non-trivial example, consider the puncture  $2A_1$ , which has weighted Dynkin diagram



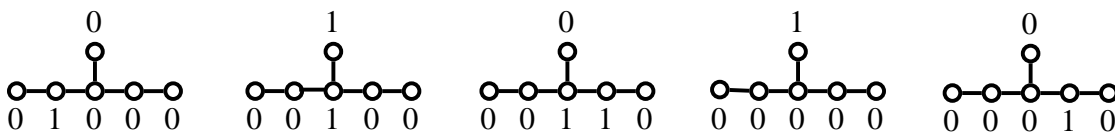
that is,  $\vec{x} = (1, 0, 0, 0, 1; 0)$ . One finds  $\vec{W} = (8, 15, 21, 15, 8; 11)$ ,  $n_{1/2} = 16$ ,  $n_0 = 24$ . Thus,  $n_h(2A_1) = 568$  and  $n_v(2A_1) = 548$ , and one indeed checks (2.3) for  $\dim_{\mathbb{C}}(2A_1) = 32$ .

### 2.4.3 Pole structures

The “pole structure” is the set of leading pole orders  $\{p_2, p_5, p_6, p_8, p_9, p_{12}\}$  in the expansion of the Casimirs  $\phi_k$  in a coordinate  $z$  centered at the puncture,  $\phi_k(z) \sim 1/z^{p_k}$ .

To compute the pole structure, we need a representative of the Hitchin nilpotent orbit  $d(\mathcal{O})$ . A table of representatives of all nilpotent orbits of  $E_6$  can be found in table 2 of [19]. In this table, a nilpotent representative is given by a sum of weighted Dynkin diagrams, and each weighted Dynkin diagram represents an element in the root vector space of  $\mathfrak{e}_6$  for a positive root  $\alpha$ , where  $\alpha$  is such that its components in a basis of simple roots are given by the labels of the Dynkin diagram. The nilpotent representative is the sum of these root-vector space elements. This procedure is most easily understood in terms of an example.

Take, for instance,  $\mathcal{O} = D_4(a_1)$ . The Hitchin orbit, given by the Spaltenstein dual, is the same,  $d(\mathcal{O}) = \mathcal{O} = D_4(a_1)$ . This orbit has a nilpotent representative  $X$  given by a sum of five elements [19],



The five summands above represent arbitrary non-zero elements  $X_{\alpha_i}$  ( $i = 1, \dots, 5$ ) in the root vector spaces for the positive roots

$$\begin{aligned} \alpha_1 &= s_2, & \alpha_4 &= s_6, \\ \alpha_2 &= s_3 + s_6, & \alpha_5 &= s_4, \\ \alpha_3 &= s_3 + s_4, \end{aligned}$$

respectively, where  $\{s_1, \dots, s_5; s_6\}$  is a basis of simple roots of  $E_6$ . So,  $X = X_{\alpha_1} + \dots + X_{\alpha_5}$ . Fortunately, GAP4 provides a Chevalley basis for the adjoint representation of  $\mathfrak{e}_6$ , so it is

trivial to find elements  $X_{\alpha_i}$ . Once we know  $X$ , we compute  $\Phi(z)$  using  $X$  as the residue in (2.1), then the Casimir  $k$ -differentials  $\phi_k$  as in section 2.2, and we finally find the pole structure  $\{1, 3, 4, 6, 6, 9\}$  for the  $D_4(a_1)$  puncture. (Actually, there are *three* orbits,  $D_4(a_1)$ ,  $A_3 + A_1$  and  $2A_2 + A_1$ , that map under Spaltenstein to  $D_4(a_1)$ , so we have three punctures with the same pole structure. However, the other properties of these punctures are different.)

### 2.4.4 Constraints

The constraints for some  $E_6$  punctures are, in some cases, much less obvious than those in the  $A_{N-1}$  and  $D_N$  series. The guiding quantities to find constraints are  $\delta n_v$  and the (complex) dimension of the Hitchin nilpotent orbit,  $d$ . These are, respectively, the graded and ungraded local contributions to the Coulomb branch.

Let us be specific. Let  $z$  be a local coordinate on  $C$  centered at the puncture, and let  $c_l^{(k)}$  be the coefficient of  $z^{-l}$  in the expansion of  $\phi_k = \phi_k(z)$  in  $z$ . Recall that, in the notation of our previous papers, a ‘‘c-constraint’’ is a polynomial relation among coefficients  $c_l^{(k)}$  (of homogeneous bi-degree in both  $k$  and  $l$ ). On the other hand, an ‘‘a-constraint’’ is a relation that defines a new quantity,  $a_l^{(k)}$ , of dimension  $k$ , in terms of the  $c_l^{(k)}$ . Only the  $c_l^{(k)}$  with  $l > 0$  parameterize the Hitchin nilpotent orbit [16]. In the absence of constraints, all the  $c_l^{(k)}$  with  $0 < l \leq p_k$  are independent, so their total number,  $\sum p_k$ , should be equal to the dimension of the Hitchin nilpotent orbit. Thus, if there are no constraints,  $\sum p_k = d$ . A c-constraint reduces the total number of independent parameters by one, whereas an a-constraint does not affect this number. So, one should have:

$$\sum p_k - (\text{number of c-constraints}) = d$$

Hence,  $d$  tells us how many c-constraints exist. On the other hand, the graded sum of the parameters, that is, the result of adding  $(2k - 1)$  for each parameter of degree  $k$  (in the presence of ‘‘a’’-constraints,  $k$  is not restricted to the degrees of the Casimirs), should be equal to  $n_v$ . An a-constraint replaces a parameter of a certain degree  $k$  by another one of a different degree  $k' < k$ . So, to get precisely  $n_v$ , one must take into account all a-constraints and c-constraints.

For example, consider the Higgs field near a  $D_5$  puncture, and the corresponding  $k$ -differentials in the  $\{\phi_k\}$  basis. We find the pole structure  $\{1, 2, 3, 4, 4, 6\}$ , and thus,  $\sum p_k = 20$ . However, the  $2A_1$  (Hitchin) nilpotent orbit, dual to  $D_5$ , has dimension  $d = 16$ . So, the difference, 4, is the number of c-constraints. Inspection of the  $k$ -differentials reveals that the leading coefficients of  $\phi_8$  and  $\phi_{12}$  are proportional to, respectively, the square and cube of a single quantity, which we call  $a_2^{(4)}$ , of scaling dimension 4 and pole order 2:

$$\begin{aligned} \phi_8 &= \frac{c_4^{(8)}}{z^4} + \dots, & c_4^{(8)} &= 3 \left( a_2^{(4)} \right)^2, \\ \phi_{12} &= \frac{c_6^{(12)}}{z^4} + \dots, & c_6^{(12)} &= \frac{3}{2} \left( a_2^{(4)} \right)^3. \end{aligned}$$

Since there is no Casimir of  $E_6$  of dimension four,  $a_2^{(4)}$  cannot be a coefficient in one of the  $\phi_k$  (or a polynomial in such coefficients), so we regard it as a basic, independent

parameter. It is not hard to discover additional relations by inspection, and we see that they, too, involve the  $a_2^{(4)}$  parameter:

$$c_3^{(6)} = \frac{3}{2}c_1^{(2)}a_2^{(4)}, \quad c_4^{(9)} = -\frac{1}{4}a_2^{(4)}c_2^{(5)}, \quad c_5^{(12)} = \frac{3}{4}c_3^{(8)}a_2^{(4)}$$

We can take one of the relations linear in  $a_2^{(4)}$ , say the one for  $c_3^{(6)}$ , as the a-constraint (that is, the relation that defines  $a_2^{(4)}$  unambiguously in terms of the  $c_l^{(k)}$ ), and the other 4 relations as the c-constraints. Assuming there are no more constraints, the independent graded Coulomb branch dimensions are  $(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12}) = (1, 0, 1, 2, 2, 3, 3, 4)$ . Finally, we compute  $\delta n_v = \sum_k (2k-1)n_k = 238$ , which agrees with the result obtained from the formula in section 2.4.2. We conclude that we have found the correct set of constraints for this puncture.

As we will see next, when two punctures collide, we get a new puncture. Thus, working with the  $k$ -differentials, we must be able to reproduce the pole structure and constraints of the new puncture from those of the colliding punctures. We can regard this expectation as another test of the correctness of our expressions.

### 2.4.5 Puncture collisions

Suppose we have two punctures on a plane, so the Higgs field has two simple poles with residues  $X_1$  and  $X_2$ . Near each puncture, the Higgs field  $\Phi$  looks like eq. (2.1). In the limit where the two punctures collide, the Higgs field has one simple pole with residue  $X = X_1 + X_2$  (by the residue theorem applied to the sphere that bubbles off), which corresponds to a new puncture. Generically,  $X$  will be mass deformed. The mass deformations are interpreted as VEVs of the scalars in the gauge multiplet associate to the factor in the gauge group which becomes weakly coupled in the collision limit. One can also study this degeneration by computing the Casimirs  $\phi_k$  from the Higgs field before taking the collision limit.

Alternatively, one can bypass the Higgs field, and study the collision directly with the  $\phi_k$ , by writing a generic  $k$ -differential with poles at the positions of the two punctures (given by their pole structures), and imposing at each pole the constraints of the corresponding puncture. Then, taking the collision limit, the pole structure and constraints of the resulting puncture on the plane arise naturally.

As an example, let us see that the collision of two  $D_5$  punctures on a plane produces an  $Sp(2)$  gauge group, gauged off an  $A_3$  puncture. Let us write generic Casimirs for the collision, taking the  $D_5$  punctures to be at  $z = 0$  and  $z = x$ :

$$\begin{aligned} \phi_2(z) &= \frac{u_2 + zv_2 + z(z-x)P_2(z)}{z(z-x)} \\ \phi_5(z) &= \frac{u_5 + zv_5 + z(z-x)w_5 + z^2(z-x)P_5(z)}{z^2(z-x)^2} \\ \phi_6(z) &= \frac{u_6 + zv_6 + z(z-x)w_6 + z^2(z-x)P_6(z)}{z^3(z-x)^3} \\ \phi_8(z) &= \frac{u_8 + zv_8 + z(z-x)w_8 + z^2(z-x)y_8 + z^2(z-x)^2P_8(z)}{z^4(z-x)^4} \end{aligned}$$

$$\begin{aligned}\phi_9(z) &= \frac{u_9 + zv_9 + z(z-x)w_9 + z^2(z-x)P_9(z)}{z^4(z-x)^4} \\ \phi_{12}(z) &= \frac{u_{12} + zv_{12} + z(z-x)w_{12} + z^2(z-x)y_{12} + z^2(z-x)^2P_{12}(z)}{z^6(z-x)^6},\end{aligned}$$

where  $P_2(z), P_5(z), \dots, P_{12}(z)$  denote regular functions in  $z$ . To solve the constraints at each  $D_5$  puncture, we introduce new parameters  $s_4$  and  $t_4$  of dimension four, and write:

$$\begin{aligned}u_6 &= \frac{3s_4u_2}{4}, & v_6 &= \frac{3}{2}(t_4u_2 + s_4v_2 + t_4v_2x), \\ u_8 &= 3s_4^2, & v_8 &= 3(2s_4t_4 + t_4^2x), \\ u_9 &= -\frac{s_4u_5}{4}, & v_9 &= -\frac{1}{4}(t_4u_5 + s_4v_5 + t_4v_5x), \\ u_{12} &= \frac{3s_4^3}{2}, & v_{12} &= \frac{3}{2}t_4(3s_4^2 + 3s_4t_4x + t_4^2x^2), \\ w_{12} &= \frac{3}{4}(3s_4t_4^2 + s_4w_8 + 2t_4^3x), & y_{12} &= -\frac{3}{4}(t_4^3 - t_4w_8 - s_4y_8 - t_4y_8x)\end{aligned}$$

In the collision limit,  $x \rightarrow 0$ , the new puncture appears at  $z = 0$ . The expansion in  $z$  of the Casimirs in this limit is:

$$\begin{aligned}\phi_2(z) &= \frac{u_2}{z^2} + \frac{v_2}{z} + \dots \\ \phi_5(z) &= \frac{u_5}{z^4} + \dots \\ \phi_6(z) &= \frac{3s_4u_2}{2z^6} + \frac{3(t_4u_2 + s_4v_2)}{2z^5} + \frac{w_6}{z^4} + \dots \\ \phi_8(z) &= \frac{3s_4^2}{z^8} + \frac{6s_4t_4}{z^7} + \frac{w_8}{z^6} + \dots \\ \phi_9(z) &= -\frac{s_4u_5}{4z^8} - \frac{(t_4u_5 + s_4v_5)}{4z^7} + \frac{w_9}{z^6} + \dots \\ \phi_{12}(z) &= \frac{3s_4^3}{2z^{12}} + \frac{9s_4^2t_4}{2z^{11}} + \frac{3(3s_4t_4^2 + s_4w_8)}{4z^{10}} - \frac{3(t_4^3 - t_4w_8 - s_4y_8)}{4z^9} + \dots,\end{aligned}$$

where the  $\dots$  indicate less singular terms in  $z$ . So,  $u_2$  and  $s_4$  can be interpreted as the VEVs of Coulomb branch parameters (of degree two and four) of the gauge group (which, with a little more work, can be checked to be  $\text{Sp}(2)$ ). In the limit  $u_2, s_4 \rightarrow 0$ , we obtain the Casimirs for the massless puncture, with pole orders  $\{1, 4, 4, 6, 7, 9\}$ , and with constraints

$$\begin{aligned}c_7^{(9)} &= \frac{1}{2}\tilde{t}_4u_5 \\ c_9^{(12)} &= 6\tilde{t}_4^3 - \frac{3}{2}w_8\tilde{t}_4,\end{aligned}$$

where  $\tilde{t}_4 \equiv -t_4/2$ . Thus, we get precisely the pole structure and constraints of the  $A_3$  puncture.

## 2.5 Global symmetries and the superconformal index

### 2.5.1 Cataloguing fixtures using the superconformal index

For the  $E_6$  theory, we find 880 fixtures with three regular punctures which correspond to interacting SCFTs, possibly with additional decoupled hypermultiplets. Each of these SCFTs has a manifest global symmetry group, which is given by the product of the flavor symmetry groups of the three punctures. This global symmetry group may, in general, become enhanced to a larger group.

To determine the global symmetry group and number of free hypermultiplets for each of these fixtures, we use the superconformal index [29–33]. The superconformal index of  $E$ -type class  $\mathcal{S}$  theories has not yet been systematically studied. However, since the methods used for  $A$ - and  $D$ -type theories generalize to any root system, we assume the superconformal index<sup>1</sup> for a fixture in the  $E_6$  theory takes the usual form

$$\mathcal{I}(\mathbf{a}_i, \tau) = \mathcal{A}(\tau) \sum_{\lambda} \frac{\prod_{i=1}^3 \mathcal{K}(\mathbf{a}_i) P^{\lambda}(\mathbf{a}_i|\tau)}{P^{\lambda}(\mathbf{a}_{\text{triv}}|\tau)} \tag{2.4}$$

where

- The sum is over  $\lambda$  labeling the highest weights of finite-dimensional irreducible representations of  $\mathfrak{e}_6$ .
- The  $P^{\lambda}(\mathbf{a}_i|\tau)$  are Hall-Littlewood polynomials, defined for a general root system by

$$P^{\lambda} = W^{-1}(\tau) \sum_{w \in W} w \left( e^{\lambda} \prod_{\alpha \in R^+} \frac{1 - \tau^2 e^{-\alpha}}{1 - e^{-\alpha}} \right),$$

$$W(\tau) = \sqrt{\sum_{\substack{w \in W \\ w\lambda = \lambda}} \tau^{2\ell(w)}}$$

where  $R^+$  denotes the set of positive roots,  $W$  the Weyl group, and  $\ell(w)$  the length of the Weyl group element  $w$ .

- $\mathbf{a}_i \equiv \{e^{\alpha}\}_{\alpha \in R^+}$  denotes a set of flavor fugacities dual to the Cartan subalgebra of the flavor symmetry of the  $i^{\text{th}}$  puncture.  $\mathbf{a}_{\text{triv}}$  denotes the set of fugacities dual to the Cartan of the embedded  $\mathfrak{su}(2)$  of the trivial puncture.
- The  $\mathcal{K}$ -factors are discussed in [32–35]. We will not need their detailed form for our purposes.
- $\mathcal{A}(\tau)$  is an overall, flavor fugacity independent normalization.

Consider a fixture corresponding to an interacting SCFT, with global symmetry  $G_{\text{global}}$ , plus free hypermultiplets transforming in a representation  $R$  of a flavor symmetry  $F$ . Let  $G_{\text{fixt}} \equiv G_{\text{global}} \times F$  denote the global symmetry of the fixture. As discussed in [34], the

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<sup>1</sup>In what follows we will consider the Hall-Littlewood limit of the index [32], which depends on one superconformal fugacity,  $\tau$ .

number of free hypers in the fixture and the global symmetry of the fixture can be read off from the first two non-trivial terms in the Taylor expansion of the index. Schematically, this is given by

$$\mathcal{I} = 1 + \chi_{\mathbb{F}}^R \tau + \chi_{G_{\text{fixt}}}^{\text{adj}} \tau^2 + \dots \tag{2.5}$$

where  $\chi_{\mathbb{F}}^R$  is the Weyl character of  $R$  and  $\chi_{G_{\text{fixt}}}^{\text{adj}}$  is the character of the adjoint representation of  $G_{\text{fixt}}$ , where both of these representations are viewed as reducible representations of the manifest symmetry algebra. By Taylor expanding  $\mathcal{I}_{\text{free}} = PE[\tau \chi_{\mathbb{F}}^R]$  (where  $PE$  denotes the Plethystic exponential) and removing the contribution of the free hypermultiplets in (2.5), we arrive at

$$\begin{aligned} \mathcal{I}_{\text{SCFT}} &= \mathcal{I} / \mathcal{I}_{\text{free}} \\ &= 1 + \chi_{G_{\text{global}}}^{\text{adj}} \tau^2 + \dots \end{aligned}$$

from which we can read off the global symmetry of the interacting SCFT.

### 2.5.2 Computing the expansion of the index

In (2.4) the term in the sum coming from the trivial representation of  $\mathfrak{e}_6$  gives, to second order in  $\tau$ , [34]

$$\mathcal{I} = 1 + \chi_{G_{\text{manifest}}}^{\text{adj}} \tau^2 + \dots$$

encoding the manifest global symmetry group. The global symmetry group of the fixture is enhanced if there are terms contributing at order  $\tau^2$  coming from the sum over  $\lambda > 0$ .

To order  $\tau^2$ , (2.4) simplifies to<sup>2</sup>

$$\mathcal{I} = 1 + \chi_{G_{\text{manifest}}}^{\text{adj}} \tau^2 + \left[ \sum_{\lambda > 0} \frac{\prod_{i=1}^3 \chi^{\lambda}(\mathbf{a}_i | \tau)}{\chi^{\lambda}(\mathbf{a}_{\text{triv}} | \tau)} \right]_{\mathcal{O}(\tau^2)} \tag{2.6}$$

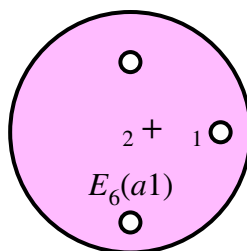
To compute (2.6), we consider each  $\mathfrak{e}_6$  representation in the sum to be a reducible representation of  $\mathfrak{su}(2) \times \mathfrak{f}$  and plug in the corresponding character expansion, where the embedded  $\mathfrak{su}(2)$  has fugacity  $\tau$ . The decomposition of any  $\mathfrak{e}_6$  representation in terms of  $\mathfrak{su}(2) \times \mathfrak{f}$  representations can be obtained using the projection matrices listed in appendix B.

Of the 881 fixtures involving three regular punctures, we find that 1 is a free-field fixture, 60 are mixed fixtures and another 134 are interacting fixtures with an enhanced global symmetry group. We list these in the tables below. For the remaining 686 interacting fixtures, the global symmetry group is the manifest one.

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<sup>2</sup>Since the theories considered here are all “good” or “ugly” (in the sense of [8]), the lowest possible contribution from the sum over  $\lambda > 0$  is at order  $\tau$  (see [34] for a discussion of the superconformal index in the context of the good/ugly/bad trichotomy of 4d  $\mathcal{N} = 2$  theories). From (2.6), we see that  $\mathcal{A}(\tau)$  and  $\mathcal{K}(\mathbf{a}_i)$  are both  $1 + \mathcal{O}(\tau^2)$ , so we can set them both to one in the order  $\tau^2$  approximation. We have also used the fact that  $P^{\lambda} = \chi^{\lambda} + \mathcal{O}(\tau^2)$ .

As an example, consider the fixture



The manifest global symmetry is  $(E_6)_{24} \times SU(3)_{12} \times U(1)$ . The contributions at order  $\tau^2$  come from the sum over the 27,  $\overline{27}$ , 78, 351,  $\overline{351}$ ,  $351'$ ,  $\overline{351}'$ , and 650 of  $\mathfrak{e}_6$ . The expansion of the superconformal index is given by<sup>3</sup>

$$\begin{aligned} \mathcal{I} = & 1 + \{(27, 1)_1 + (\overline{27}, 1)_{-1}\}\tau + \{(1, 1)_0 + (78, 1)_0 + \\ & + (650, 1)_0 + (27, 1)_{-2} + (351', 1)_{-2} + (\overline{27}, 1)_2 + \\ & + (\overline{351}', 1)_2 + (78, 1)_0 + (1, 8)_0 + (27, \overline{3})_0 + (\overline{27}, 3)_0\}\tau^2 + \dots \end{aligned}$$

Due to the order  $\tau$  term, this is a mixed fixture, with 27 free hypermultiplets transforming in the fundamental representation of  $E_6$ . The index of these free hypers is given by

$$\begin{aligned} \mathcal{I}_{\text{free}} = & PE[\tau\{(27, 1)_1 + (\overline{27}, 1)_{-1}\}] \\ = & 1 + \{(27, 1)_1 + (\overline{27}, 1)_{-1}\}\tau + \\ & \{(1, 1)_0 + (78, 1)_0 + (650, 1)_0 + (27, 1)_{-2} + (351', 1)_{-2} + (\overline{27}, 1)_2 + (\overline{351}', 1)_2\}\tau^2 + \dots \end{aligned}$$

The index of the underlying SCFT is then

$$\begin{aligned} \mathcal{I}_{\text{SCFT}} = & \mathcal{I}/\mathcal{I}_{\text{free}} \\ = & 1 + \{(78, 1)_0 + (1, 8)_0 + (27, \overline{3})_0 + (\overline{27}, 3)_0\}\tau^2 \end{aligned}$$

We recognize the coefficient of  $\tau^2$  as the character of the adjoint representation of  $E_8$ . Computing the other numerical invariants of the fixture, we find that this is the  $(E_8)_{12}$  theory of Minahan and Nemeschansky [36] with 27 additional free hypermultiplets.

## 2.6 Levels of enhanced global symmetry groups

Since the superconformal index gives the branching rule for the adjoint representation of  $G_{\text{global}}$  under the subgroup  $G_{\text{manifest}}$ , in most cases it is straightforward to determine the level of each factor in  $G_{\text{global}}$  from those of  $G_{\text{manifest}}$ : if  $H_{k'}$  is a subgroup of  $G_k$ , then  $k$  is given by [12]

$$k = \frac{k'}{I_{H \hookrightarrow G}}$$

where  $I_{H \hookrightarrow G}$  is the index of the embedding of  $H$  in  $G$ .

There are two cases which require a little more work. The first is when a manifest  $U(1)$  becomes enhanced to  $SU(2)$ . Since we do not know how to assign a level to a  $U(1)$  flavor

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<sup>3</sup>For simplicity, we write the dimension to stand for the character of the corresponding representation. The subscript is the  $U(1)$  weight.

symmetry (which would require a precise understanding of how the generator is normalized), we cannot immediately determine the level of the enhanced  $SU(2)$  from the index.

The second case is when some factor  $H_k$  in  $G_{\text{manifest}}$  is embedded diagonally as

$$H_k \hookrightarrow H_{k_1} \times H_{k_2}.$$

Since the only embedding of  $H$  in itself has index one, in this case, all we know is that  $k_1 + k_2 = k$ .

If any of these remain as factors in  $G_{\text{global}}$  (that is, if they do not combine with some other factor, with known level, to enhance  $G_{\text{global}}$ ), we cannot determine their levels from the index, and must determine them using an  $S$ -duality. To do so, we look for a 4-punctured sphere for which the SCFT appears in some degeneration, with  $H_{k_i}$  in the centralizer of subgroup of  $G_{\text{global}}$  being weakly gauged.

Unfortunately, there are a few such fixtures for which no puncture can be gauged (some of these can still be gauged in the twisted sector, which will be discussed in section 4). For these, we do not have a way to determine the levels. In the end, there are two interacting fixtures whose levels we cannot completely determine.

### 3 Tinkertoys

#### 3.1 Regular punctures

The pole structure  $\{p_2, p_5, p_6, p_8, p_9, p_{12}\}$  of a puncture at  $z = 0$  will be the leading pole orders in  $z$  of the differentials  $\phi_k(z)$  for  $k = 2, 5, 6, 8, 9, 12$ . Notice that in some cases there are constraints, not just on the coefficient of this leading singularity, but also on *subleading* terms in the Laurent expansion of the  $k$ -differentials.

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
0	$E_6$	$\{1, 4, 5, 7, 8, 11\}$	-	$(E_6)_{24}$	$(624, 588)$
$A_1$	$E_6(a_1)$	$\{1, 4, 5, 7, 8, 10\}$	-	$SU(6)_{18}$	$(590, 565)$
$2A_1$	$D_5$	$\{1, 4, 5, 7, 7, 10\}$	-	$Spin(7)_{16} \times U(1)$	$(568, 548)$
$3A_1$ ( <u>ns</u> )	$(E_6(a_3), \mathbb{Z}_2)$	$\{1, 4, 5, 6, 7, 10\}$	-	$SU(3)_{24} \times SU(2)_{13}$	$(549, 533)$
$A_2$	$E_6(a_3)$	$\{1, 4, 5, 6, 7, 10\}$	$c_{10}^{(12)} = -(c_5^{(6)})^2 + (a_5^{(6)})^2$	$SU(3)_{12}^2$	$(536, 521)$
$A_2 + A_1$	$D_5(a_1)$	$\{1, 4, 5, 6, 7, 9\}$	-	$SU(3)_{12} \times U(1)$	$(523, 510)$
$2A_2$	$D_4$	$\{1, 3, 5, 6, 6, 9\}$	-	$(G_2)_{12}$	$(496, 484)$
$A_2 + 2A_1$	$A_4 + A_1$	$\{1, 4, 4, 6, 7, 9\}$	-	$SU(2)_{54} \times U(1)$	$(510, 499)$
$A_3$	$A_4$	$\{1, 4, 4, 6, 7, 9\}$	$c_7^{(9)} = \frac{1}{2} c_4^{(5)} a_3^{(4)}$ $c_9^{(12)} = 6(a_3^{(4)})^3 - \frac{3}{2} c_6^{(8)} a_3^{(4)}$	$Sp(2)_{10} \times U(1)$	$(476, 466)$



Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$2A_2 + A_1$ ( $\underline{ns}$ )	$(D_4(a_1), S_3)$	$\{1, 3, 4, 6, 6, 9\}$	-	$SU(2)_{26}$	(482, 473)
$A_3 + A_1$ ( $\underline{ns}$ )	$(D_4(a_1), \mathbb{Z}_2)$	$\{1, 3, 4, 6, 6, 9\}$	$c_9^{(12)} = a_3^{(4)} \left( \frac{16}{9} (a_3^{(4)})^2 - c_6^{(8)} \right)$	$SU(2)_9 \times U(1)$	(465, 457)
$D_4(a_1)$	$D_4(a_1)$	$\{1, 3, 4, 6, 6, 9\}$	$c_6^{(8)} = \frac{4}{3} \left( (a_3^{(4)})^2 + 3(a_3'^{(4)})^2 \right)$ $c_9^{(12)} = \frac{4}{9} a_3^{(4)} \left( (a_3^{(4)})^2 - 9(a_3'^{(4)})^2 \right)$	$U(1)^2$	(456, 449)
$A_4$	$A_3$	$\{1, 3, 4, 6, 6, 9\}$	$c_6^{(8)} = 3(a_3^{(4)})^2$ $c_6^{(9)} = \frac{1}{4} c_3^{(5)} a_3^{(4)}$ $c_9^{(12)} = -\frac{3}{2} (a_3^{(4)})^3$ $c_8^{(12)} = -\frac{3}{4} a_3^{(4)} c_5^{(8)}$	$SU(2)_8 \times U(1)$	(408, 402)
$D_4$	$2A_2$	$\{1, 3, 4, 5, 6, 8\}$	$c_5^{(8)} = -4c_4^{(6)} c_1^{(2)} + 4c_3^{(5)} a_2^{(3)} - 2(a_2^{(3)})^2 c_1^{(2)}$ $c_6^{(9)} = -\frac{1}{12} a_2^{(3)} \left( c_4^{(6)} + \frac{1}{2} (a_2^{(3)})^2 \right)$ $c_8^{(12)} = -\left( c_4^{(6)} + \frac{1}{2} (a_2^{(3)})^2 \right) \cdot \left( c_4^{(6)} - \frac{3}{2} (a_2^{(3)})^2 \right)$ $c_7^{(12)} = -12c_5^{(9)} a_2^{(3)} - 2c_4^{(6)} c_3^{(6)} - c_3^{(6)} (a_2^{(3)})^2$	$SU(3)_{12}$	(368, 362)
$A_4 + A_1$	$A_2 + 2A_1$	$\{1, 3, 4, 5, 5, 7\}$	-	$U(1)$	(400, 395)
$D_5(a_1)$	$A_2 + A_1$	$\{1, 3, 4, 5, 5, 7\}$	$c_4^{(6)} = -\frac{1}{8} (a_2^{(3)})^2$ $c_5^{(8)} = 2c_3^{(5)} a_2^{(3)}$ $c_7^{(12)} = -6c_5^{(9)} a_2^{(3)}$	$U(1)$	(355, 351)
$A_5$ ( $\underline{ns}$ )	$(A_2, \mathbb{Z}_2)$	$\{1, 2, 4, 4, 4, 6\}$	-	$SU(2)_7$	(335, 331)
$E_6(a_3)$	$A_2$	$\{1, 2, 4, 4, 4, 6\}$	$c_4^{(6)} = (a_2^{(3)})^2$	none	(328, 325)

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$D_5$	$2A_1$	$\{1, 2, 3, 4, 4, 6\}$	$c_3^{(6)} = \frac{3}{2}c_1^{(2)}a_2^{(4)}$ $c_4^{(8)} = 3(a_2^{(4)})^2$ $c_4^{(9)} = -\frac{1}{4}a_2^{(4)}c_2^{(5)}$ $c_6^{(12)} = \frac{3}{2}(a_2^{(4)})^3$ $c_5^{(12)} = \frac{3}{4}c_3^{(8)}a_2^{(4)}$	U(1)	(240, 238)
$E_6(a_1)$	$A_1$	$\{1, 1, 2, 2, 2, 3\}$	-	none	(168, 167)

Note that there is a special piece, consisting of three punctures:  $2A_2 + A_1$ ,  $A_3 + A_1$  and the special puncture  $D_4(a_1)$ . For  $2A_2 + A_1$ , the Sommers-Achar group is the nonabelian group,  $S_3$ . It acts on  $a^{(4)}, a'^{(4)}$  as

$$\begin{pmatrix} a^{(4)} \\ a'^{(4)} \end{pmatrix} \rightarrow \gamma \begin{pmatrix} a^{(4)} \\ a'^{(4)} \end{pmatrix}$$

for

$$\gamma \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & -3 \\ 1 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & 3 \\ -1 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \right\}$$

For  $A_3 + A_1$ , the Sommers-Achar group is the  $\mathbb{Z}_2$  subgroup of  $S_3$ , generated by  $a'^{(4)} \rightarrow -a'^{(4)}$ . For  $D_4(a_1)$ , the Sommers-Achar group is of course trivial, so that both  $a^{(4)}, a'^{(4)}$  survive as Coulomb branch parameters.

### 3.2 Free-field fixtures

We denote a 3-punctured sphere, in the tables below, by listing the Bala-Carter labels of the three punctures. For the free-field fixtures, one of the punctures is an irregular puncture<sup>4</sup> (in the sense used in our previous papers), which we denote<sup>5</sup> by the pair,  $(\mathcal{O}, G_k)$ , where  $\mathcal{O}$  is the regular puncture obtained as the OPE of the two regular punctures which collide, and this fixture is attached to the rest of the surface via a cylinder

$$(\mathcal{O}, G_k) \xleftarrow{G} \mathcal{O}$$

with gauge group  $G \subset F \subset E_6$ . Here,  $F$  is the flavour symmetry group of the puncture,  $\mathcal{O}$ , and the levels are such that  $G$  has vanishing  $\beta$ -function.

#	Fixture	$n_h$	Representation
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<sup>4</sup>Or, in the case of fixture 13, a full puncture, corresponding to the trivial orbit, 0.

<sup>5</sup>For brevity, we will often omit the level,  $k$ , when denoting an irregular puncture.

#	Fixture	$n_h$	Representation
1	$E_6(a_1)$ $E_6(a_1)$ ( $A_5, \text{SU}(2)_1$ )	1	$\frac{1}{2}(2)$
2	$E_6(a_1)$ $D_5$ ( $A_4, \text{SU}(2)_0$ )	0	empty
3	$E_6(a_1)$ $E_6(a_3)$ ( $2A_2, \text{SU}(3)_0$ )	0	empty
4	$E_6(a_1)$ $A_5$ ( $2A_2, (G_2)_4$ )	7	$\frac{1}{2}(2, 7)$
5	$E_6(a_1)$ $D_5(a_1)$ ( $A_2 + A_1, \text{SU}(3)_0$ )	0	empty
6	$E_6(a_1)$ $A_4 + A_1$ ( $2A_1, (G_2)_0$ )	0	empty
7	$E_6(a_1)$ $D_4$ ( $A_2, \text{SU}(3)_0$ )	0	empty
8	$E_6(a_1)$ $A_4$ ( $2A_1, \text{Spin}(7)_4$ )	8	$\frac{1}{2}(2, 8)$
9	$E_6(a_1)$ $D_4(a_1)$ ( $0, \text{Spin}(8)_0$ )	0	empty
10	$E_6(a_1)$ $A_3 + A_1$ ( $0, \text{Spin}(9)_4$ )	9	$\frac{1}{2}(2, 9)$
11	$E_6(a_1)$ $2A_2 + A_1$ ( $0, (F_4)_{12}$ )	26	$\frac{1}{2}(2, 26)$
12	$E_6(a_1)$ $A_3$ ( $0, \text{Spin}(10)_8$ )	20	$\frac{1}{2}(4, 10)$
13	$E_6(a_1)$ $A_2 + 2A_1$ 0	54	(2, 27)
14	$D_5$ $D_5$ ( $A_3, \text{Sp}(2)_2$ )	4	1(4)
15	$D_5$ $E_6(a_3)$ ( $2A_1, \text{SU}(4)_0$ )	0	empty
16	$D_5$ $A_5$ ( $2A_1, \text{Spin}(7)_4$ )	7	$\frac{1}{2}(2, 7)$

#	Fixture	$n_h$	Representation
17	$D_5$ $D_5(a_1)$ $(A_1, \text{SU}(5)_2)$	5	1(5)
18	$D_5$ $A_4 + A_1$ $(0, \text{Spin}(10)_8)$	16	1(16)
19	$D_5$ $D_4$ $(A_1, \text{SU}(6)_6)$	18	3(6)

### 3.3 Interacting fixtures with one irregular puncture

In the tables below,  $n_d$  is the number of Coulomb branch parameters of degree  $d$ . The total Coulomb branch dimension is  $\sum_d n_d$  and the effective number of vector multiplets is  $n_v = \sum_d (2d - 1)n_d$ .

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
$E_6(a_1)$ $2A_2$ $(0, (F_4)_{12})$	$(0, 0, 0, 0, 1, 0, 0, 0)$	$(40, 11)$	$(E_8)_{12}$ SCFT
$D_5$ $A_4$ $(0, \text{Spin}(10)_8)$	$(0, 0, 1, 0, 0, 0, 0, 0)$	$(24, 7)$	$(E_7)_8$ SCFT
$E_6(a_3)$ $E_6(a_3)$ $(0, (F_4)_{12})$	$(0, 2, 0, 0, 0, 0, 0, 0)$	$(32, 10)$	$[(E_6)_6 \text{ SCFT}]^2$
$E_6(a_3)$ $A_5$ $(0, (F_4)_{12})$	$(0, 1, 0, 0, 1, 0, 0, 0)$	$(39, 16)$	$(E_6)_{12} \times \text{SU}(2)_7$ SCFT
$A_5$ $A_5$ $(0, (F_4)_{12})$	$(0, 0, 0, 0, 2, 0, 0, 0)$	$(46, 22)$	$(F_4)_{12} \times \text{SU}(2)_7^2$ SCFT

The  $(E_6)_{12} \times \text{SU}(2)_7$  and  $(F_4)_{12} \times \text{SU}(2)_7^2$  first appeared in [4], as fixtures in the untwisted  $D_4$  theory.

### 3.4 Interacting fixtures with enhanced global symmetry

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
1	$E_6(a_1)$ $A_2$ 0	$(0, 0, 0, 0, 2, 0, 0, 0)$	$(80, 22)$	$[(E_8)_{12} \text{ SCFT}]^2$
2	$E_6(a_1)$ $3A_1$ 0	$(0, 0, 0, 0, 1, 0, 0, 1)$	$(93, 34)$	$(E_8)_{24} \times \text{SU}(2)_{13}$
3	$E_6(a_1)$ $2A_1$ 0	$(0, 0, 0, 0, 1, 1, 0, 1)$	$(112, 49)$	$(E_7)_{24} \times \text{Spin}(7)_{16}$
4	$E_6(a_1)$ $A_1$ $A_1$	$(0, 0, 0, 0, 1, 1, 1, 0)$	$(100, 43)$	$\text{SU}(12)_{18}$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
5	$\begin{matrix} D_5 \\ D_4(a_1) \end{matrix} \quad 0$	(0, 0, 3, 0, 0, 0, 0, 0)	(72, 21)	$[(E_7)_8 \text{ SCFT}]^3$
6	$\begin{matrix} D_5 \\ A_3 + A_1 \end{matrix} \quad 0$	(0, 0, 2, 0, 0, 1, 0, 0)	(81, 29)	$[(E_7)_8 \text{ SCFT}]$ $\times [(E_7)_{16} \times \text{SU}(2)_9 \text{ SCFT}]$
7	$\begin{matrix} D_5 \\ 2A_2 + A_1 \end{matrix} \quad 0$	(0, 0, 1, 0, 0, 1, 0, 1)	(98, 45)	$(E_7)_{24} \times \text{SU}(2)_{26}$
8	$\begin{matrix} D_5 \\ A_3 \end{matrix} \quad 0$	(0, 0, 2, 1, 0, 1, 0, 0)	(92, 38)	$[(E_7)_8 \text{ SCFT}]$ $\times [(E_6)_{16} \times \text{Sp}(2)_{10} \times \text{U}(1) \text{ SCFT}]$
9	$\begin{matrix} D_5 \\ 2A_2 \end{matrix} \quad 0$	(0, 0, 1, 0, 1, 1, 0, 1)	(112, 56)	$(E_7)_{24} \times (G_2)_{12}$
10	$\begin{matrix} D_5 \\ A_2 + 2A_1 \end{matrix} \quad A_1$	(0, 0, 1, 1, 0, 1, 1, 0)	(92, 48)	$\text{SU}(8)_{18} \times \text{SU}(2)_{36} \times \text{U}(1)$
11	$\begin{matrix} D_5 \\ A_2 + A_1 \end{matrix} \quad A_1$	(0, 0, 1, 1, 1, 1, 1, 0)	(105, 59)	$\text{SU}(7)_{18} \times \text{SU}(3)_{12} \times \text{U}(1)^2$
12	$\begin{matrix} D_5 \\ A_2 \end{matrix} \quad 2A_1$	(0, 0, 1, 1, 2, 1, 0, 0)	(96, 53)	$\text{Spin}(8)_{16} \times \text{SU}(4)_{12}^2 \times \text{U}(1)$
13	$\begin{matrix} D_5 \\ A_2 \end{matrix} \quad A_1$	(0, 0, 1, 1, 2, 1, 1, 0)	(118, 70)	$\text{SU}(6)_{18} \times \text{SU}(3)_{12}^2 \times \text{U}(1)^2$
14	$\begin{matrix} D_5 \\ 3A_1 \end{matrix} \quad 3A_1$	(0, 0, 1, 1, 1, 0, 0, 1)	(90, 50)	$\text{SU}(6)_{24} \times \text{Sp}(2)_{13}$
15	$\begin{matrix} D_5 \\ 3A_1 \end{matrix} \quad 2A_1$	(0, 0, 1, 1, 1, 1, 0, 1)	(109, 65)	$\text{Spin}(7)_{16} \times \text{SU}(4)_{24} \times \text{SU}(2)_{13} \times \text{U}(1)$
16	$\begin{matrix} D_5 \\ 2A_1 \end{matrix} \quad 2A_1$	(0, 0, 1, 1, 1, 2, 0, 1)	(128, 80)	$\text{Spin}(7)_{16}^2 \times \text{SU}(2)_{24} \times \text{U}(1)^2$
17	$\begin{matrix} E_6(a_3) \\ A_4 + A_1 \end{matrix} \quad 0$	(0, 1, 0, 0, 1, 1, 0, 1)	(104, 54)	$(E_7)_{24}$
18	$\begin{matrix} E_6(a_3) \\ D_4 \end{matrix} \quad 0$	(0, 2, 0, 0, 1, 0, 0, 0)	(72, 21)	$[(E_8)_{12} \text{ SCFT}] \times [(E_6)_6 \text{ SCFT}]^2$
19	$\begin{matrix} E_6(a_3) \\ A_4 \end{matrix} \quad 0$	(0, 1, 1, 0, 1, 1, 0, 1)	(112, 61)	$(E_7)_{24} \times \text{SU}(2)_8$
20	$\begin{matrix} E_6(a_3) \\ D_4(a_1) \end{matrix} \quad A_2$	(0, 1, 2, 0, 2, 0, 0, 0)	(72, 41)	$\text{Spin}(8)_{12}^2 \times \text{U}(1)^2$
21	$\begin{matrix} E_6(a_3) \\ D_4(a_1) \end{matrix} \quad 3A_1$	(0, 1, 2, 0, 1, 0, 0, 1)	(85, 53)	$\text{Spin}(8)_{24} \times \text{SU}(2)_{13}$
22	$\begin{matrix} E_6(a_3) \\ D_4(a_1) \end{matrix} \quad 2A_1$	(0, 1, 2, 0, 1, 1, 0, 1)	(104, 68)	$\text{Spin}(7)_{16} \times \text{SU}(2)_{24}^3$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
23	$E_6(a_3)$ $A_3 + A_1$ $A_2$	(0, 1, 1, 0, 2, 1, 0, 0)	(81, 49)	$\text{Spin}(7)_{12}^2 \times \text{SU}(2)_9 \times \text{U}(1)$
24	$E_6(a_3)$ $A_3 + A_1$ $3A_1$	(0, 1, 1, 0, 1, 1, 0, 1)	(94, 61)	$\text{Spin}(7)_{24} \times \text{SU}(2)_{13} \times \text{SU}(2)_9$
25	$E_6(a_3)$ $A_3 + A_1$ $2A_1$	(0, 1, 1, 0, 1, 2, 0, 1)	(113, 76)	$\text{Spin}(7)_{16} \times \text{SU}(2)_{48} \times \text{SU}(2)_{24} \times \text{SU}(2)_9$
26	$E_6(a_3)$ $2A_2 + A_1$ $A_2$	(0, 1, 0, 0, 2, 1, 0, 1)	(98, 65)	$(G_2)_{12}^2 \times \text{SU}(2)_{26}$
27	$E_6(a_3)$ $2A_2 + A_1$ $3A_1$	(0, 1, 0, 0, 1, 1, 0, 2)	(111, 77)	$(G_2)_{24} \times \text{SU}(2)_{26} \times \text{SU}(2)_{13}$
28	$E_6(a_3)$ $2A_2 + A_1$ $2A_1$	(0, 1, 0, 0, 1, 2, 0, 2)	(130, 92)	$\text{Spin}(7)_{16} \times \text{SU}(2)_{26} \times \text{SU}(2)_{72}$
29	$E_6(a_3)$ $A_3$ $A_2$	(0, 1, 1, 1, 2, 1, 0, 0)	(92, 58)	$\text{SU}(4)_{12}^2 \times \text{Sp}(2)_{10} \times \text{U}(1)$
30	$E_6(a_3)$ $A_3$ $3A_1$	(0, 1, 1, 1, 1, 1, 0, 1)	(105, 70)	$\text{SU}(4)_{24} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{13}$
31	$E_6(a_3)$ $A_3$ $2A_1$	(0, 1, 1, 1, 1, 2, 0, 1)	(124, 85)	$\text{Spin}(7)_{16} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{24} \times \text{U}(1)$
32	$E_6(a_3)$ $A_2 + 2A_1$ $A_2 + 2A_1$	(0, 1, 0, 1, 0, 1, 1, 1)	(100, 69)	$\text{SU}(4)_{54} \times \text{U}(1)$
33	$E_6(a_3)$ $A_2 + 2A_1$ $A_2 + A_1$	(0, 1, 0, 1, 1, 1, 1, 1)	(113, 80)	$\text{SU}(3)_{54} \times \text{SU}(3)_{12} \times \text{U}(1)$
34	$E_6(a_3)$ $2A_2$ $A_2$	(0, 1, 0, 0, 3, 1, 0, 1)	(112, 76)	$(G_2)_{12}^3$
35	$E_6(a_3)$ $2A_2$ $3A_1$	(0, 1, 0, 0, 2, 1, 0, 2)	(125, 88)	$(G_2)_{24} \times (G_2)_{12} \times \text{SU}(2)_{13}$
36	$E_6(a_3)$ $2A_2$ $2A_1$	(0, 1, 0, 0, 2, 2, 0, 2)	(144, 103)	$\text{Spin}(7)_{16} \times (G_2)_{12} \times \text{SU}(2)_{72}$
37	$E_6(a_3)$ $A_2 + A_1$ $A_2 + A_1$	(0, 1, 0, 1, 2, 1, 1, 1)	(126, 91)	$\text{SU}(3)_{12}^2 \times \text{SU}(2)_{24} \times \text{U}(1)$
38	$A_5$ $A_4 + A_1$ 0	(0, 0, 0, 0, 2, 1, 0, 1)	(111, 60)	$(E_7)_{24} \times \text{SU}(2)_7$
39	$A_5$ $D_4$ 0	(0, 1, 0, 0, 2, 0, 0, 0)	(79, 27)	$[(E_8)_{12} \text{ SCFT}] \times$ $[(E_6)_{12} \times \text{SU}(2)_7 \text{ SCFT}]$
40	$A_5$ $A_4$ 0	(0, 0, 1, 0, 2, 1, 0, 1)	(119, 67)	$(E_7)_{24} \times \text{SU}(2)_8 \times \text{SU}(2)_7$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
41	$A_5$ $D_4(a_1)$ $A_2$	(0, 0, 2, 0, 3, 0, 0, 0)	(79, 47)	$\text{Spin}(8)_{12}^2 \times \text{SU}(2)_7$
42	$A_5$ $D_4(a_1)$ $3A_1$	(0, 0, 2, 0, 2, 0, 0, 1)	(92, 59)	$\text{Spin}(8)_{24} \times \text{SU}(2)_{13} \times \text{SU}(2)_7$
43	$A_5$ $D_4(a_1)$ $2A_1$	(0, 0, 2, 0, 2, 1, 0, 1)	(111, 74)	$\text{Spin}(7)_{16} \times \text{SU}(2)_{24}^3 \times \text{SU}(2)_7$
44	$A_5$ $A_3 + A_1$ $A_2$	(0, 0, 1, 0, 3, 1, 0, 0)	(88, 55)	$\text{Spin}(7)_{12}^2 \times \text{SU}(2)_9 \times \text{SU}(2)_7$
45	$A_5$ $A_3 + A_1$ $3A_1$	(0, 0, 1, 0, 2, 1, 0, 1)	(101, 67)	$\text{Spin}(7)_{24} \times \text{SU}(2)_{13} \times \text{SU}(2)_9 \times \text{SU}(2)_7$
46	$A_5$ $A_3 + A_1$ $2A_1$	(0, 0, 1, 0, 2, 2, 0, 1)	(120, 82)	$\text{Spin}(7)_{16} \times \text{SU}(2)_{48} \times \text{SU}(2)_{24} \times \text{SU}(2)_9 \times \text{SU}(2)_7$
47	$A_5$ $2A_2 + A_1$ $A_2$	(0, 0, 0, 0, 3, 1, 0, 1)	(105, 71)	$(G_2)_{12}^2 \times \text{SU}(2)_{26} \times \text{SU}(2)_7$
48	$A_5$ $2A_2 + A_1$ $3A_1$	(0, 0, 0, 0, 2, 1, 0, 2)	(118, 83)	$(G_2)_{24} \times \text{SU}(2)_{26} \times \text{SU}(2)_{13} \times \text{SU}(2)_7$
49	$A_5$ $2A_2 + A_1$ $2A_1$	(0, 0, 0, 0, 2, 2, 0, 2)	(137, 98)	$\text{Spin}(7)_{16} \times \text{SU}(2)_{72} \times \text{SU}(2)_{26} \times \text{SU}(2)_7$
50	$A_5$ $A_3$ $A_2$	(0, 0, 1, 1, 3, 1, 0, 0)	(99, 64)	$\text{SU}(4)_{12}^2 \times \text{Sp}(2)_{10} \times \text{SU}(2)_7$
51	$A_5$ $A_3$ $3A_1$	(0, 0, 1, 1, 2, 1, 0, 1)	(112, 76)	$\text{SU}(4)_{24} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{13} \times \text{SU}(2)_7$
52	$A_5$ $A_3$ $2A_1$	(0, 0, 1, 1, 2, 2, 0, 1)	(131, 91)	$\text{Spin}(7)_{16} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{24} \times \text{SU}(2)_7 \times \text{U}(1)$
53	$A_5$ $A_2 + 2A_1$ $A_2 + 2A_1$	(0, 0, 0, 1, 1, 1, 1, 1)	(107, 75)	$\text{SU}(4)_{54} \times \text{SU}(2)_7 \times \text{U}(1)$
54	$A_5$ $A_2 + 2A_1$ $A_2 + A_1$	(0, 0, 0, 1, 2, 1, 1, 1)	(120, 86)	$\text{SU}(3)_{54} \times \text{SU}(3)_{12} \times \text{SU}(2)_7 \times \text{U}(1)$
55	$A_5$ $2A_2$ $A_2$	(0, 0, 0, 0, 4, 1, 0, 1)	(119, 82)	$(G_2)_{12}^3 \times \text{SU}(2)_7$
56	$A_5$ $2A_2$ $3A_1$	(0, 0, 0, 0, 3, 1, 0, 2)	(132, 94)	$(G_2)_{24} \times (G_2)_{12} \times \text{SU}(2)_{13} \times \text{SU}(2)_7$
57	$A_5$ $2A_2$ $2A_1$	(0, 0, 0, 0, 3, 2, 0, 2)	(151, 109)	$\text{Spin}(7)_{16} \times (G_2)_{12} \times \text{SU}(2)_{72} \times \text{SU}(2)_7$
58	$A_5$ $A_2 + A_1$ $A_2 + A_1$	(0, 0, 0, 1, 3, 1, 1, 1)	(133, 97)	$\text{SU}(3)_{12}^2 \times \text{SU}(2)_{24} \times \text{SU}(2)_7 \times \text{U}(1)$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
59	$D_5(a_1)$ $D_5(a_1)$ 0	(0, 2, 0, 1, 0, 0, 1, 0)	(86, 36)	$(E_7)_{18} \times (E_6)_6 \times U(1)$
60	$D_5(a_1)$ $A_4 + A_1$ $A_1$	(0, 1, 0, 1, 1, 1, 1, 0)	(97, 57)	$SU(7)_{18} \times U(1)^2$
61	$D_5(a_1)$ $D_4$ 0	(0, 2, 0, 1, 1, 0, 1, 0)	(99, 47)	$(E_6)_{18} \times (E_6)_6 \times SU(3)_{12} \times U(1)$
62	$D_5(a_1)$ $A_4$ $A_1$	(0, 1, 1, 1, 1, 1, 1, 0)	(105, 64)	$SU(7)_{18} \times SU(2)_8 \times U(1)^2$
63	$D_5(a_1)$ $D_4(a_1)$ $A_2 + 2A_1$	(0, 1, 2, 1, 0, 0, 1, 0)	(73, 45)	$SU(3)_{54-k-k'} \times SU(3)_k \times SU(3)_{k'} \times U(1)$
64	$D_5(a_1)$ $D_4(a_1)$ $A_2 + A_1$	(0, 1, 2, 1, 1, 0, 1, 0)	(86, 56)	$SU(3)_{12} \times SU(2)_{18}^3 \times U(1)^3$
65	$D_5(a_1)$ $D_4(a_1)$ $A_2$	(0, 1, 2, 1, 2, 0, 1, 0)	(99, 67)	$SU(3)_{12}^2 \times U(1)^5$
66	$D_5(a_1)$ $A_3 + A_1$ $A_2 + 2A_1$	(0, 1, 1, 1, 0, 1, 1, 0)	(82, 53)	$SU(3)_{54-k} \times SU(3)_k \times SU(2)_9 \times U(1)$
67	$D_5(a_1)$ $A_3 + A_1$ $A_2 + A_1$	(0, 1, 1, 1, 1, 1, 1, 0)	(95, 64)	$SU(3)_{12} \times SU(2)_{36} \times SU(2)_{18} \times SU(2)_9 \times U(1)^2$
68	$D_5(a_1)$ $A_3 + A_1$ $A_2$	(0, 1, 1, 1, 2, 1, 1, 0)	(108, 75)	$SU(3)_{12}^2 \times SU(2)_9 \times U(1)^3$
69	$D_5(a_1)$ $2A_2 + A_1$ $A_2 + 2A_1$	(0, 1, 0, 1, 0, 1, 1, 1)	(99, 69)	$SU(3)_{54} \times SU(2)_{26} \times U(1)$
70	$D_5(a_1)$ $2A_2 + A_1$ $A_2 + A_1$	(0, 1, 0, 1, 1, 1, 1, 1)	(112, 80)	$SU(3)_{12} \times SU(2)_{54} \times SU(2)_{26} \times U(1)$
71	$D_5(a_1)$ $A_3$ $A_2 + 2A_1$	(0, 1, 1, 2, 0, 1, 1, 0)	(93, 62)	$SU(3)_{18} \times SU(2)_{36} \times Sp(2)_{10} \times U(1)^2$
72	$D_5(a_1)$ $A_3$ $A_2 + A_1$	(0, 1, 1, 2, 1, 1, 1, 0)	(106, 73)	$SU(3)_{12} \times Sp(2)_{10} \times SU(2)_{18} \times U(1)^3$
73	$D_5(a_1)$ $A_3$ $A_2$	(0, 1, 1, 2, 2, 1, 1, 0)	(119, 84)	$SU(3)_{12}^2 \times Sp(2)_{10} \times U(1)^3$
74	$D_5(a_1)$ $A_2 + 2A_1$ $2A_2$	(0, 1, 0, 1, 1, 1, 1, 1)	(113, 80)	$(G_2)_{12} \times SU(3)_{54} \times U(1)$
75	$D_5(a_1)$ $2A_2$ $A_2 + A_1$	(0, 1, 0, 1, 2, 1, 1, 1)	(126, 91)	$(G_2)_{12} \times SU(3)_{12} \times SU(2)_{54} \times U(1)$
76	$A_4 + A_1$ $A_4 + A_1$ $A_2$	(0, 0, 0, 1, 3, 1, 0, 0)	(88, 57)	$SU(4)_{12}^2 \times U(1)$



#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
77	$A_4 + A_1$ $A_4 + A_1$ $3A_1$	(0, 0, 0, 1, 2, 1, 0, 1)	(101, 69)	$SU(4)_{24} \times SU(2)_{13} \times U(1)$
78	$A_4 + A_1$ $A_4 + A_1$ $2A_1$	(0, 0, 0, 1, 2, 2, 0, 1)	(120, 84)	$Spin(7)_{16} \times SU(2)_{24} \times U(1)^2$
79	$A_4 + A_1$ $D_4$ $2A_1$	(0, 1, 0, 1, 2, 1, 0, 0)	(88, 51)	$Spin(8)_{16} \times SU(4)_{12} \times U(1)^2$
80	$A_4 + A_1$ $D_4$ $A_1$	(0, 1, 0, 1, 2, 1, 1, 0)	(110, 68)	$SU(6)_{18} \times SU(3)_{12} \times U(1)^2$
81	$A_4 + A_1$ $A_4$ $A_2$	(0, 0, 1, 1, 3, 1, 0, 0)	(96, 64)	$SU(4)_{12}^2 \times SU(2)_8 \times U(1)$
82	$A_4 + A_1$ $A_4$ $3A_1$	(0, 0, 1, 1, 2, 1, 0, 1)	(109, 76)	$SU(4)_{24} \times SU(2)_{13} \times SU(2)_8 \times U(1)$
83	$A_4 + A_1$ $A_4$ $2A_1$	(0, 0, 1, 1, 2, 2, 0, 1)	(128, 91)	$Spin(7)_{16} \times SU(2)_8 \times SU(2)_{24} \times U(1)^2$
84	$A_4 + A_1$ $D_4(a_1)$ $D_4(a_1)$	(0, 0, 4, 0, 1, 0, 0, 0)	(64, 39)	$SU(2)_8^9$
85	$A_4 + A_1$ $D_4(a_1)$ $A_3 + A_1$	(0, 0, 3, 0, 1, 1, 0, 0)	(73, 47)	$SU(2)_{16}^3 \times SU(2)_9 \times SU(2)_8^3$
86	$A_4 + A_1$ $D_4(a_1)$ $2A_2 + A_1$	(0, 0, 2, 0, 1, 1, 0, 1)	(90, 63)	$SU(2)_{26} \times SU(2)_{24}^3$
87	$A_4 + A_1$ $D_4(a_1)$ $A_3$	(0, 0, 3, 1, 1, 1, 0, 0)	(84, 56)	$Sp(2)_{10} \times SU(2)_8^3 \times U(1)^3$
88	$A_4 + A_1$ $D_4(a_1)$ $2A_2$	(0, 0, 2, 0, 2, 1, 0, 1)	(104, 74)	$(G_2)_{12} \times SU(2)_{24}^3$
89	$A_4 + A_1$ $A_3 + A_1$ $A_3 + A_1$	(0, 0, 2, 0, 1, 2, 0, 0)	(82, 55)	$SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_9^2 \times SU(2)_8^2$
90	$A_4 + A_1$ $A_3 + A_1$ $2A_2 + A_1$	(0, 0, 1, 0, 1, 2, 0, 1)	(99, 71)	$SU(2)_{48} \times SU(2)_{26} \times SU(2)_{24} \times SU(2)_9$
91	$A_4 + A_1$ $A_3 + A_1$ $A_3$	(0, 0, 2, 1, 1, 2, 0, 0)	(93, 64)	$Sp(2)_{10} \times SU(2)_{16} \times SU(2)_9$ $\times SU(2)_8 \times U(1)^2$
92	$A_4 + A_1$ $A_3 + A_1$ $2A_2$	(0, 0, 1, 0, 2, 2, 0, 1)	(113, 82)	$(G_2)_{12} \times SU(2)_{48} \times SU(2)_{24} \times SU(2)_9$
93	$A_4 + A_1$ $2A_2 + A_1$ $2A_2 + A_1$	(0, 0, 0, 0, 1, 2, 0, 2)	(116, 87)	$SU(2)_{72} \times SU(2)_{26}^2$
94	$A_4 + A_1$ $2A_2 + A_1$ $A_3$	(0, 0, 1, 1, 1, 2, 0, 1)	(110, 80)	$Sp(2)_{10} \times SU(2)_{26} \times SU(2)_{24} \times U(1)$
95	$A_4 + A_1$ $2A_2 + A_1$ $2A_2$	(0, 0, 0, 0, 2, 2, 0, 2)	(130, 98)	$(G_2)_{12} \times SU(2)_{72} \times SU(2)_{26}$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
96	$A_4 + A_1$ $A_3$ $A_3$	(0, 0, 2, 2, 1, 2, 0, 0)	(104, 73)	$\text{Sp}(2)_{10}^2 \times \text{SU}(2)_8 \times \text{U}(1)^3$
97	$A_4 + A_1$ $A_3$ $2A_2$	(0, 0, 1, 1, 2, 2, 0, 1)	(124, 91)	$(G_2)_{12} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{24} \times \text{U}(1)$
98	$A_4 + A_1$ $2A_2$ $2A_2$	(0, 0, 0, 0, 3, 2, 0, 2)	(144, 109)	$(G_2)_{12}^2 \times \text{SU}(2)_{72}$
99	$D_4$ 0 $D_4$	(0, 2, 0, 1, 2, 0, 1, 0)	(112, 58)	$(E_6)_{18} \times (E_6)_6 \times \text{SU}(3)_{12}^2$
100	$D_4$ $2A_1$ $A_4$	(0, 1, 1, 1, 2, 1, 0, 0)	(96, 58)	$\text{Spin}(8)_{16} \times \text{SU}(4)_{12} \times \text{SU}(2)_8 \times \text{U}(1)^2$
101	$D_4$ $A_1$ $A_4$	(0, 1, 1, 1, 2, 1, 1, 0)	(118, 75)	$\text{SU}(6)_{18} \times \text{SU}(3)_{12} \times \text{SU}(2)_8 \times \text{U}(1)^2$
102	$D_4$ $A_2 + 2A_1$ $D_4(a_1)$	(0, 1, 2, 1, 1, 0, 1, 0)	(86, 56)	$\text{SU}(3)_{12} \times \text{SU}(2)_{18}^3 \times \text{U}(1)^3$
103	$D_4$ $2A_2$ $D_4(a_1)$	(0, 1, 2, 0, 2, 0, 0, 0)	(72, 41)	$\text{Spin}(8)_{12}^2 \times \text{U}(1)^2$
104	$D_4$ $A_2 + A_1$ $D_4(a_1)$	(0, 1, 2, 1, 2, 0, 1, 0)	(99, 67)	$\text{SU}(3)_{12}^2 \times \text{U}(1)^5$
105	$D_4$ $A_2$ $D_4(a_1)$	(0, 1, 2, 1, 3, 0, 1, 0)	(112, 78)	$\text{SU}(3)_{12}^3 \times \text{U}(1)^4$
106	$D_4$ $A_2 + 2A_1$ $A_3 + A_1$	(0, 1, 1, 1, 1, 1, 1, 0)	(95, 64)	$\text{SU}(3)_{12} \times \text{SU}(2)_{36} \times \text{SU}(2)_{18} \times \text{SU}(2)_9 \times \text{U}(1)^2$
107	$D_4$ $2A_2$ $A_3 + A_1$	(0, 1, 1, 0, 2, 1, 0, 0)	(81, 49)	$\text{Spin}(7)_{12}^2 \times \text{SU}(2)_9 \times \text{U}(1)$
108	$D_4$ $A_2 + A_1$ $A_3 + A_1$	(0, 1, 1, 1, 2, 1, 1, 0)	(108, 75)	$\text{SU}(3)_{12}^2 \times \text{SU}(2)_9 \times \text{U}(1)^3$
109	$D_4$ $A_2$ $A_3 + A_1$	(0, 1, 1, 1, 3, 1, 1, 0)	(121, 86)	$\text{SU}(3)_{12}^3 \times \text{SU}(2)_9 \times \text{U}(1)^2$
110	$D_4$ $2A_2 + A_1$ $2A_2 + A_1$	(0, 1, 0, 0, 1, 1, 0, 1)	(84, 54)	$(G_2)_{12} \times \text{Sp}(2)_{26}$
111	$D_4$ $2A_2$ $2A_2 + A_1$	(0, 1, 0, 0, 2, 1, 0, 1)	(98, 65)	$(G_2)_{12}^2 \times \text{SU}(2)_{26}$
112	$D_4$ $A_2 + 2A_1$ $A_3$	(0, 1, 1, 2, 1, 1, 1, 0)	(106, 73)	$\text{Sp}(2)_{10} \times \text{SU}(3)_{12} \times \text{SU}(2)_{36} \times \text{SU}(2)_{18} \times \text{U}(1)^2$
113	$D_4$ $2A_2$ $A_3$	(0, 1, 1, 1, 2, 1, 0, 0)	(92, 58)	$\text{Spin}(7)_{12} \times \text{SU}(4)_{12} \times \text{Sp}(2)_{10} \times \text{U}(1)$
114	$D_4$ $A_2 + A_1$ $A_3$	(0, 1, 1, 2, 2, 1, 1, 0)	(119, 84)	$\text{SU}(3)_{12}^2 \times \text{Sp}(2)_{10} \times \text{U}(1)^3$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
115	$D_4$ $A_2$ $A_3$	(0, 1, 1, 2, 3, 1, 1, 0)	(132, 95)	$SU(3)_{12}^3 \times Sp(2)_{10} \times U(1)^2$
116	$D_4$ $2A_2$ $2A_2$	(0, 1, 0, 0, 3, 1, 0, 1)	(112, 76)	$(G_2)_{12}^3$
117	$A_4$ $A_2$ $A_4$	(0, 0, 2, 1, 3, 1, 0, 0)	(104, 71)	$SU(4)_{12}^2 \times SU(2)_8^2 \times U(1)$
118	$A_4$ $3A_1$ $A_4$	(0, 0, 2, 1, 2, 1, 0, 1)	(117, 83)	$SU(4)_{24} \times SU(2)_{13} \times SU(2)_8^2 \times U(1)$
119	$A_4$ $2A_1$ $A_4$	(0, 0, 2, 1, 2, 2, 0, 1)	(136, 98)	$Spin(7)_{16} \times SU(2)_8^2 \times SU(2)_{24} \times U(1)^2$
120	$A_4$ $D_4(a_1)$ $D_4(a_1)$	(0, 0, 5, 0, 1, 0, 0, 0)	(72, 46)	$SU(2)_8^{10}$
121	$A_4$ $A_3 + A_1$ $D_4(a_1)$	(0, 0, 4, 0, 1, 1, 0, 0)	(81, 54)	$SU(2)_{16}^3 \times SU(2)_9 \times SU(2)_8^4$
122	$A_4$ $2A_2 + A_1$ $D_4(a_1)$	(0, 0, 3, 0, 1, 1, 0, 1)	(98, 70)	$SU(2)_{26} \times SU(2)_{24}^3 \times SU(2)_8$
123	$A_4$ $A_3$ $D_4(a_1)$	(0, 0, 4, 1, 1, 1, 0, 0)	(92, 63)	$Sp(2)_{10} \times SU(2)_8^4 \times U(1)^3$
124	$A_4$ $2A_2$ $D_4(a_1)$	(0, 0, 3, 0, 2, 1, 0, 1)	(112, 81)	$(G_2)_{12} \times SU(2)_8 \times SU(2)_{24}^3$
125	$A_4$ $A_3 + A_1$ $A_3 + A_1$	(0, 0, 3, 0, 1, 2, 0, 0)	(90, 62)	$SU(2)_{32} \times SU(2)_{16}^2 \times SU(2)_9^2 \times SU(2)_8^2$
126	$A_4$ $2A_2 + A_1$ $A_3 + A_1$	(0, 0, 2, 0, 1, 2, 0, 1)	(107, 78)	$SU(2)_{48} \times SU(2)_{26} \times SU(2)_{24} \times SU(2)_9 \times SU(2)_8$
127	$A_4$ $A_3$ $A_3 + A_1$	(0, 0, 3, 1, 1, 2, 0, 0)	(101, 71)	$Sp(2)_{10} \times SU(2)_{16} \times SU(2)_9 \times SU(2)_8^2 \times U(1)^2$
128	$A_4$ $2A_2$ $A_3 + A_1$	(0, 0, 2, 0, 2, 2, 0, 1)	(121, 89)	$(G_2)_{12} \times SU(2)_{48} \times SU(2)_{24} \times SU(2)_9 \times SU(2)_8$
129	$A_4$ $2A_2 + A_1$ $2A_2 + A_1$	(0, 0, 1, 0, 1, 2, 0, 2)	(124, 94)	$SU(2)_{72} \times SU(2)_{26}^2 \times SU(2)_8$
130	$A_4$ $A_3$ $2A_2 + A_1$	(0, 0, 2, 1, 1, 2, 0, 1)	(118, 87)	$Sp(2)_{10} \times SU(2)_{26} \times SU(2)_{24} \times SU(2)_8 \times U(1)$
131	$A_4$ $2A_2$ $2A_2 + A_1$	(0, 0, 1, 0, 2, 2, 0, 2)	(138, 105)	$(G_2)_{12} \times SU(2)_{72} \times SU(2)_{26} \times SU(2)_8$
132	$A_4$ $A_3$ $A_3$	(0, 0, 3, 2, 1, 2, 0, 0)	(112, 80)	$Sp(2)_{10}^2 \times SU(2)_8^2 \times U(1)^3$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	$G_k$
133	$A_4$ $2A_2$ $A_3$	(0, 0, 2, 1, 2, 2, 0, 1)	(132, 98)	$(G_2)_{12} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{24}$ $\times \text{SU}(2)_8 \times \text{U}(1)$
134	$A_4$ $2A_2$ $2A_2$	(0, 0, 1, 0, 3, 2, 0, 2)	(152, 116)	$(G_2)_{12}^2 \times \text{SU}(2)_{72} \times \text{SU}(2)_8$

We were unable to determine the  $\text{SU}(3)$  levels in fixtures 63 and 66.

### 3.5 Mixed fixtures

We find many “new” SCFTs in our list of mixed fixtures. For each fixture in the table below, we list the global symmetry group, the graded Coulomb branch dimensions, and the effective number of vector and hypermultiplets of the SCFT. The effective number of hypermultiplets, for the fixture as a whole, is the sum of the  $n_h$  listed in the table and the number of free hypermultiplets in the last column. When the hypermultiplets transform under the nonabelian part of the “manifest” global symmetry of the fixture, we list that representation. Otherwise, we just give their number.

All SCFTs in the list below are “new”, except for the  $(E_6)_6$  SCFT, the  $(E_6)_{12} \times \text{SU}(2)_7$  SCFT, the  $\text{SU}(4)_8^3$  SCFT, and the  $(E_8)_{12}$  SCFT, which have previously appeared in the classification of the A- and D-series fixtures, and the  $(E_7)_{16} \times \text{SU}(2)_9$  and  $(G_2)_{12} \times \text{Sp}(2)_{26}$  SCFTs, which appeared above.

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
1	$E_6(a_1)$ 0 $A_2 + A_1$	(0, 0, 0, 0, 1, 0, 0, 0)	(40, 11)	$(E_8)_{12}$ SCFT + 1(1, 27)
2	$E_6(a_1)$ $A_1$ $3A_1$	(0, 0, 0, 0, 1, 0, 0, 0)	(40, 11)	$(E_8)_{12}$ SCFT + $\frac{1}{2}(1, 2, 1) + 1(3, 1, 6)$
3	$E_6(a_1)$ $A_1$ $2A_1$	(0, 0, 0, 0, 1, 1, 0, 0)	(72, 26)	$\text{Spin}(20)_{16}$ SCFT + 1(6, 1)
4	$D_5$ $A_1$ $2A_2 + A_1$	(0, 0, 1, 0, 0, 1, 0, 0)	(57, 22)	$(E_7)_{16} \times \text{SU}(2)_9 + \frac{1}{2}(2, 1) + 1(1, 6)$
5	$D_5$ $3A_1$ $A_2 + 2A_1$	(0, 0, 1, 1, 0, 0, 0, 0)	(42, 16)	$\text{SU}(8)_{10} \times \text{SU}(3)_{12} + 1(2; 3, 1) + \frac{1}{2}(3; 1, 2)$
6	$D_5$ $2A_1$ $A_2 + 2A_1$	(0, 0, 1, 1, 0, 1, 0, 0)	(68, 31)	$(E_6)_{16} \times \text{Sp}(2)_{10} \times \text{U}(1) + 1(2, 1)$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
7	$D_5$ $A_1$ $2A_2$	$(0, 0, 1, 0, 1, 1, 0, 0)$	$(72, 33)$	$\text{Spin}(7)_{12} \times \text{Spin}(12)_{16} \text{ SCFT} + 1(1, 6)$
8	$D_5$ $A_2 + A_1$ $3A_1$	$(0, 0, 1, 1, 1, 0, 0, 0)$	$(60, 27)$	$\text{SU}(8)_{12} \times \text{SU}(4)_{10} + \frac{1}{2}(1; 1, 2) + 1(1; 3, 1)$
9	$D_5$ $A_2 + A_1$ $2A_1$	$(0, 0, 1, 1, 1, 1, 0, 0)$	$(82, 42)$	$\text{Spin}(10)_{16} \times \text{SU}(4)_{12} \times \text{SU}(2)_{10} \times \text{U}(1) + 1 \text{ free hyper}$
10	$D_5$ $A_2$ $3A_1$	$(0, 0, 1, 1, 2, 0, 0, 0)$	$(76, 38)$	$\text{SU}(6)_{12}^2 \times \text{SU}(2)_{12} + \frac{1}{2}(1, 1; 1, 2)$
11	$E_6(a_3)$ $D_5(a_1)$ 0	$(0, 2, 0, 0, 0, 0, 0, 0)$	$(32, 10)$	$[(E_6)_6 \text{ SCFT}]^2 + 1(27)$
12	$E_6(a_3)$ $A_4 + A_1$ $A_1$	$(0, 1, 0, 0, 1, 1, 0, 0)$	$(64, 31)$	$\text{Spin}(13)_{16} \times \text{U}(1) + 1(6)$
13	$E_6(a_3)$ $A_4$ $A_1$	$(0, 1, 1, 0, 1, 1, 0, 0)$	$(72, 38)$	$\text{Spin}(12)_{16} \times \text{SU}(2)_8 \times \text{U}(1) \text{ SCFT} + 1(1, 6)$
14	$E_6(a_3)$ $D_4(a_1)$ $A_2 + 2A_1$	$(0, 1, 2, 0, 0, 0, 0, 0)$	$(40, 19)$	$\text{SU}(4)_8^3 \text{ SCFT} + 3(2)$
15	$E_6(a_3)$ $D_4(a_1)$ $A_2 + A_1$	$(0, 1, 2, 0, 1, 0, 0, 0)$	$(56, 30)$	$\text{Spin}(8)_{12} \times \text{SU}(2)_8^3 \times \text{U}(1)^2 + 3 \text{ free hyps}$
16	$E_6(a_3)$ $A_3 + A_1$ $A_2 + 2A_1$	$(0, 1, 1, 0, 0, 1, 0, 0)$	$(51, 27)$	$\text{Sp}(3)_9 \times \text{SU}(4)_{16} + 2(1, 2)$
17	$E_6(a_3)$ $A_3 + A_1$ $A_2 + A_1$	$(0, 1, 1, 0, 1, 1, 0, 0)$	$(66, 38)$	$\text{Spin}(7)_{12} \times \text{Sp}(2)_9 \times \text{SU}(2)_{32} \times \text{U}(1) + 2 \text{ free hyps}$
18	$E_6(a_3)$ $2A_2 + A_1$ $A_2 + 2A_1$	$(0, 1, 0, 0, 0, 1, 0, 1)$	$(70, 43)$	$\text{Sp}(3)_{26} + 1(1, 2)$
19	$E_6(a_3)$ $2A_2 + A_1$ $A_2 + A_1$	$(0, 1, 0, 0, 1, 1, 0, 1)$	$(84, 54)$	$(G_2)_{12} \times \text{Sp}(2)_{26} + 1 \text{ free hyper}$

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
20	$E_6(a_3)$ $A_3$ $A_2 + 2A_1$	$(0, 1, 1, 1, 0, 1, 0, 0)$	$(64, 36)$	$\text{Sp}(4)_{10} \times \text{SU}(2)_{16}^2 \times \text{U}(1)^2 + 1(1, 2)$
21	$E_6(a_3)$ $A_3$ $A_2 + A_1$	$(0, 1, 1, 1, 1, 1, 0, 0)$	$(78, 47)$	$\text{SU}(4)_{12} \times \text{Sp}(3)_{10} \times \text{U}(1)^2 + 1$ free hyper
22	$E_6(a_3)$ $A_2 + 2A_1$ $2A_2$	$(0, 1, 0, 0, 1, 1, 0, 1)$	$(84, 54)$	$(G_2)_{12} \times \text{Sp}(2)_{26} + 1(2, 1)$
23	$E_6(a_3)$ $A_2 + A_1$ $2A_2$	$(0, 1, 0, 0, 2, 1, 0, 1)$	$(98, 65)$	$(G_2)_{12}^2 \times \text{SU}(2)_{26} + 1$ free hyper
24	$A_5$ $D_5(a_1)$ $0$	$(0, 1, 0, 0, 1, 0, 0, 0)$	$(39, 16)$	$(E_6)_{12} \times \text{SU}(2)_7$ SCFT + $1(1, 27)$
25	$A_5$ $A_4 + A_1$ $A_1$	$(0, 0, 0, 0, 2, 1, 0, 0)$	$(71, 37)$	$\text{Spin}(13)_{16} \times \text{SU}(2)_7 + 1(1, 6)$
26	$A_5$ $A_4$ $A_1$	$(0, 0, 1, 0, 2, 1, 0, 0)$	$(79, 44)$	$\text{Spin}(12)_{16} \times \text{SU}(2)_8 \times \text{SU}(2)_7$ SCFT + $1(1, 1, 6)$
27	$A_5$ $D_4(a_1)$ $A_2 + 2A_1$	$(0, 0, 2, 0, 1, 0, 0, 0)$	$(47, 25)$	$\text{Sp}(2)_8^3 \times \text{SU}(2)_7 + 3(1, 2)$
28	$A_5$ $D_4(a_1)$ $A_2 + A_1$	$(0, 0, 2, 0, 2, 0, 0, 0)$	$(63, 36)$	$\text{Spin}(8)_{12} \times \text{SU}(2)_8^3 \times \text{SU}(2)_7$ + 3 free hypers
29	$A_5$ $A_3 + A_1$ $A_2 + 2A_1$	$(0, 0, 1, 0, 1, 1, 0, 0)$	$(58, 33)$	$\text{Sp}(3)_9 \times \text{Sp}(2)_{16} \times \text{SU}(2)_7 + 2(1, 1, 2)$
30	$A_5$ $A_3 + A_1$ $A_2 + A_1$	$(0, 0, 1, 0, 2, 1, 0, 0)$	$(73, 44)$	$\text{Spin}(7)_{12} \times \text{Sp}(2)_9 \times \text{SU}(2)_{32}$ $\times \text{SU}(2)_7 + 2$ free hypers
31	$A_5$ $2A_2 + A_1$ $A_2 + 2A_1$	$(0, 0, 0, 0, 1, 1, 0, 1)$	$(77, 49)$	$\text{Sp}(3)_{26} \times \text{SU}(2)_7 + 1(1, 1, 2)$
32	$A_5$ $2A_2 + A_1$ $A_2 + A_1$	$(0, 0, 0, 0, 2, 1, 0, 1)$	$(91, 60)$	$(G_2)_{12} \times \text{Sp}(2)_{26} \times \text{SU}(2)_7 + 1$ free hyper

#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
33	$A_5$ $A_2 + 2A_1$ $A_3$	$(0, 0, 1, 1, 1, 1, 0, 0)$	$(71, 42)$	$\text{Sp}(4)_{10} \times \text{SU}(2)_{32} \times \text{SU}(2)_7 \times \text{U}(1)$ $+1(1, 1, 2)$
34	$A_5$ $A_2 + A_1$ $A_3$	$(0, 0, 1, 1, 2, 1, 0, 0)$	$(85, 53)$	$\text{SU}(4)_{12} \times \text{Sp}(3)_{10} \times \text{SU}(2)_7 \times \text{U}(1)$ $+1$ free hyper
35	$A_5$ $2A_2$ $A_2 + 2A_1$	$(0, 0, 0, 0, 2, 1, 0, 1)$	$(91, 60)$	$(G_2)_{12} \times \text{Sp}(2)_{26} \times \text{SU}(2)_7 + 1(1, 2, 1)$
36	$A_5$ $A_2 + A_1$ $2A_2$	$(0, 0, 0, 0, 3, 1, 0, 1)$	$(105, 71)$	$(G_2)_{12}^2 \times \text{SU}(2)_{26} \times \text{SU}(2)_7 +$ $1$ free hyper
37	$D_5(a1)$ $3A_1$ $A_4 + A_1$	$(0, 1, 0, 1, 1, 0, 0, 0)$	$(52, 25)$	$\text{SU}(6)_{12} \times \text{Spin}(7)_{10} + \frac{1}{2}(1, 2) + 1(3, 1)$
38	$D_5(a1)$ $2A_1$ $A_4 + A_1$	$(0, 1, 0, 1, 1, 1, 0, 0)$	$(74, 40)$	$\text{Spin}(10)_{16} \times \text{SU}(2)_{10} \times \text{SU}(2)_{32}$ $\times \text{U}(1) + 1$ free hyper
39	$D_5(a1)$ $3A_1$ $A_4$	$(0, 1, 1, 1, 1, 0, 0, 0)$	$(60, 32)$	$\text{SU}(5)_{12} \times \text{SU}(4)_{10} \times \text{SU}(2)_8 \times \text{U}(1)$ $+\frac{1}{2}(1; 1, 2) + 1(1; 3, 1)$
40	$D_5(a1)$ $2A_1$ $A_4$	$(0, 1, 1, 1, 1, 1, 0, 0)$	$(82, 47)$	$\text{Spin}(10)_{16} \times \text{SU}(2)_8 \times \text{SU}(2)_{10}$ $\times \text{U}(1)^2 + 1$ free hyper
41	$D_5(a1)$ $2A_2 + A_1$ $D_4(a1)$	$(0, 1, 2, 0, 0, 0, 0, 0)$	$(40, 19)$	$\text{SU}(4)_8^3 \text{SCFT} + 1(2) + 3$ free hypers
42	$D_5(a1)$ $2A_2$ $D_4(a1)$	$(0, 1, 2, 0, 1, 0, 0, 0)$	$(56, 30)$	$\text{Spin}(8)_{12} \times \text{SU}(2)_8^3 \times \text{U}(1)^2$ $+3$ free hypers
43	$D_5(a1)$ $2A_2 + A_1$ $A_3 + A_1$	$(0, 1, 1, 0, 0, 1, 0, 0)$	$(51, 27)$	$\text{SU}(4)_{16} \times \text{Sp}(3)_9 + \frac{1}{2}(2, 1) +$ $2$ free hypers
44	$D_5(a1)$ $2A_2$ $A_3 + A_1$	$(0, 1, 1, 0, 1, 1, 0, 0)$	$(66, 38)$	$\text{Spin}(7)_{12} \times \text{Sp}(2)_9 \times \text{SU}(2)_{32} \times \text{U}(1)$ $+2$ free hypers
45	$D_5(a1)$ $2A_2 + A_1$ $2A_2 + A_1$	$(0, 1, 0, 0, 0, 1, 0, 1)$	$(70, 43)$	$\text{Sp}(3)_{26} + 1$ free hyper

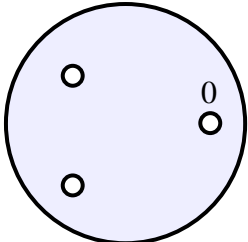
#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
46	$D_5(a_1)$ $2A_2 + A_1$ $A_3$	$(0, 1, 1, 1, 0, 1, 0, 0)$	$(63, 36)$	$\text{Sp}(3)_{10} \times \text{SU}(3)_{16} \times \text{SU}(2)_9 \times \text{U}(1)$ $+ \frac{1}{2}(2, 1) + 1$ free hyper
47	$D_5(a_1)$ $2A_2 + A_1$ $2A_2$	$(0, 1, 0, 0, 1, 1, 0, 1)$	$(84, 54)$	$(G_2)_{12} \times \text{Sp}(2)_{26} + 1$ free hyper
48	$D_5(a_1)$ $A_3$ $2A_2$	$(0, 1, 1, 1, 1, 1, 0, 0)$	$(78, 47)$	$\text{Spin}(7)_{12} \times \text{Sp}(3)_{10} \times \text{U}(1)^2 + 1$ free hyper
49	$D_5(a_1)$ $2A_2$ $2A_2$	$(0, 1, 0, 0, 2, 1, 0, 1)$	$(98, 65)$	$(G_2)_{12}^2 \times \text{SU}(2)_{26} + 1$ free hyper
50	$A_4 + A_1$ $A_4 + A_1$ $A_2 + 2A_1$	$(0, 0, 0, 1, 1, 1, 0, 0)$	$(60, 35)$	$\text{SU}(3)_{32} \times \text{Sp}(3)_{10} + 1(2)$
51	$A_4 + A_1$ $A_4 + A_1$ $A_2 + A_1$	$(0, 0, 0, 1, 2, 1, 0, 0)$	$(74, 46)$	$\text{SU}(4)_{12} \times \text{SU}(2)_{10} \times \text{SU}(2)_{32}^2 + 1$ free hyper
52	$A_4 + A_1$ $A_4$ $A_2 + 2A_1$	$(0, 0, 1, 1, 1, 1, 0, 0)$	$(68, 42)$	$\text{SU}(3)_{32} \times \text{Sp}(2)_{10} \times \text{SU}(2)_8 \times \text{U}(1) + 1(1, 2)$
53	$A_4 + A_1$ $A_4$ $A_2 + A_1$	$(0, 0, 1, 1, 2, 1, 0, 0)$	$(82, 53)$	$\text{SU}(4)_{12} \times \text{SU}(2)_{32} \times \text{SU}(2)_{10} \times \text{SU}(2)_8 \times \text{U}(1) + 1$ free hyper
54	$D_4$ $A_4 + A_1$ $3A_1$	$(0, 1, 0, 1, 2, 0, 0, 0)$	$(68, 36)$	$\text{SU}(6)_{12} \times \text{SU}(3)_{12}^2 + \frac{1}{2}(1; 1, 2)$
55	$D_4$ $A_4$ $3A_1$	$(0, 1, 1, 1, 2, 0, 0, 0)$	$(76, 43)$	$\text{SU}(6)_{12} \times \text{SU}(3)_{12} \times \text{SU}(2)_8 \times \text{U}(1) + \frac{1}{2}(1, 1; 1, 2)$
56	$D_4$ $D_4(a_1)$ $2A_2 + A_1$	$(0, 1, 2, 0, 1, 0, 0, 0)$	$(56, 30)$	$\text{Spin}(8)_{12} \times \text{SU}(2)_8^3 + 1(1, 2)$
57	$D_4$ $A_3 + A_1$ $2A_2 + A_1$	$(0, 1, 1, 0, 1, 1, 0, 0)$	$(66, 38)$	$\text{Spin}(7)_{12} \times \text{Sp}(2)_9 \times \text{SU}(2)_{16} \times \text{U}(1) + \frac{1}{2}(1, 1, 2)$
58	$D_4$ $2A_2 + A_1$ $A_3$	$(0, 1, 1, 1, 1, 1, 0, 0)$	$(77, 47)$	$\text{SU}(4)_{12} \times \text{Sp}(2)_{10} \times \text{SU}(2)_{16} \times \text{SU}(2)_9 \times \text{U}(1) + \frac{1}{2}(1, 2, 1)$



#	Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Theory
59	$A_4$ $A_2 + 2A_1$ $A_4$	$(0, 0, 2, 1, 1, 1, 0, 0)$	$(76, 49)$	$\text{Sp}(2)_{10} \times \text{SU}(2)_{32} \times \text{SU}(2)_8^2 \times \text{U}(1)^2$ $+1(1, 1, 2)$
60	$A_4$ $A_2 + A_1$ $A_4$	$(0, 0, 2, 1, 2, 1, 0, 0)$	$(90, 60)$	$\text{SU}(4)_{12} \times \text{SU}(2)_{10} \times \text{SU}(2)_8^2 \times \text{U}(1)^2$ $+1 \text{ free hyper}$

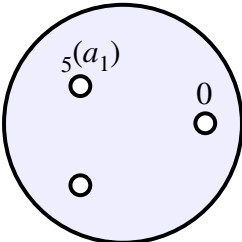
#### 4 A detour through the twisted sector

There are several fixtures on our list, where the levels of the enhanced flavour symmetry group cannot be determined by considerations from the untwisted sector alone. For instance, consider the pair of fixtures,



$(E_6)_{18} \times (E_6)_6 \times \text{SU}(3)_{12}^2$  SCFT

and

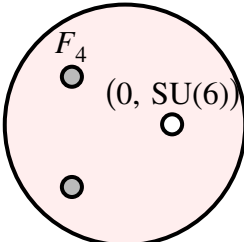


$(E_6)_{18} \times (E_6)_6 \times \text{SU}(3)_{12} \times \text{U}(1)$  SCFT

In each case, only the diagonal  $(E_6)_{24} \subset (E_6)_{24-k} \times (E_6)_k$  is manifest. Moreover, the only gaugings, available in the untwisted sector, have Abelian centralizers in  $(E_6)_{24-k} \times (E_6)_k$ , which makes determining the individual levels (as opposed to their sum) difficult.

To fix the ambiguity, we need to make recourse to the  $\mathbb{Z}_2$ -twisted sector. While a full discussion of the  $\mathbb{Z}_2$ -twisted sector is beyond the scope of this paper, we will borrow a few results of that analysis, deferring a full discussion to a future paper.

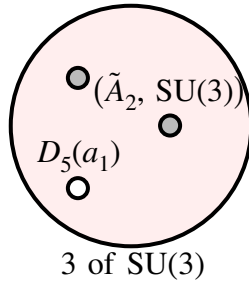
The twisted punctures are labeled by nilpotent orbits in  $F_4$ . We will denote them by their Bala-Carter labels, and colour them grey. The empty fixture



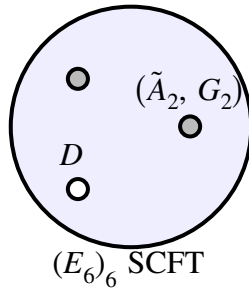
empty

will allow us to gauge an  $\text{SU}(6)_{24} \subset (E_6)_{24-k} \times (E_6)_k$ . The centralizer is  $\text{SU}(2)_{24-k} \times \text{SU}(2)_k$ , from which we can read off the “missing” information about the levels.

We will also need the free-field fixture



and the interacting fixture

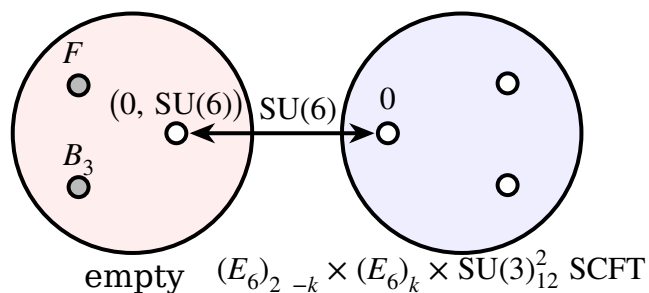


which is a realization of the  $(E_6)_6$  SCFT. Finally, we will also need two “new” interacting fixtures

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	$(n_h, n_v)$	Global Symmetry
	$(0, 2, 1, 2, 1, 0, 1, 0)$	$(83, 63)$	$(G_2)_{10} \times \text{SU}(2)_{18} \times \text{SU}(2)_6 \times \text{U}(1)$
	$(0, 2, 1, 2, 2, 0, 1, 0)$	$(96, 74)$	$(G_2)_{10} \times \text{SU}(3)_{12} \times \text{SU}(2)_{18} \times \text{SU}(2)_6$

In both cases, all of the global symmetry *except* the  $\text{SU}(2)_{18}$  is manifest (in particular,

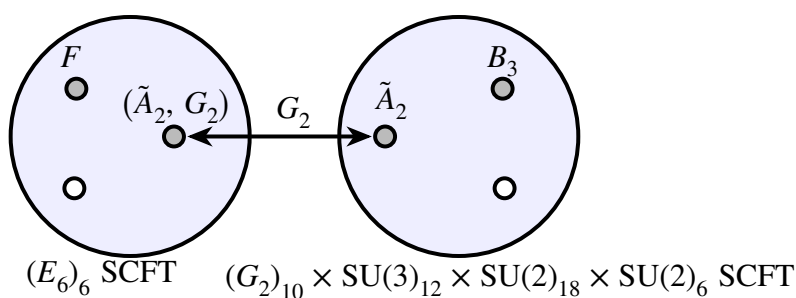
the  $SU(2)_6$  is manifest). The 4-punctured sphere



has global symmetry

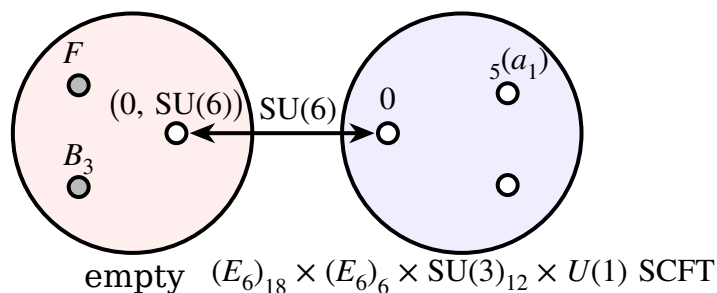
$$F = SU(3)_{12}^2 \times SU(2)_{24-k} \times SU(2)_k$$

The S-dual



manifestly has one of the  $SU(2)$  levels as  $k = 6$ , which determines the other level to be 18.

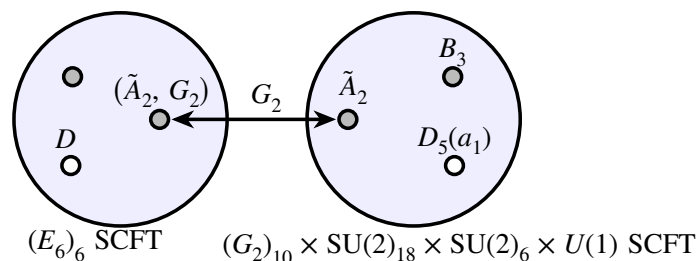
Similarly, for



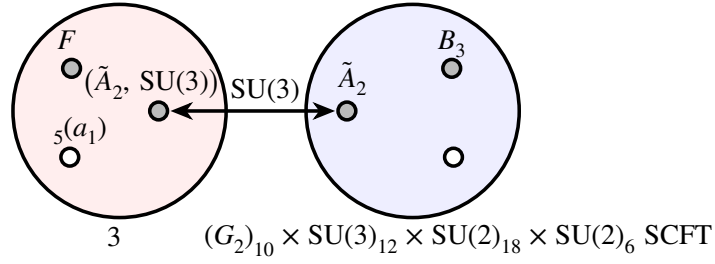
the global symmetry group is

$$F = SU(3)_{12} \times SU(2)_{24-k} \times SU(2)_k \times U(1)$$

Now there are two S-dual presentations of the theory:



and



Again, the fact that one of the  $SU(2)$  levels is manifest suffices to determine the other.

As another example, consider the pair of fixtures

(4.1)

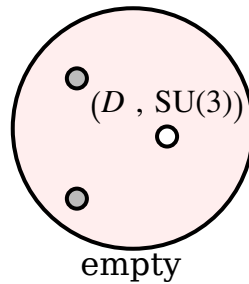
$SU(3)_{12} \times SU(2)_{5-k-k'} \times SU(2)_k \times SU(2)_{k'} \times U(1)^3$  SCFT

and

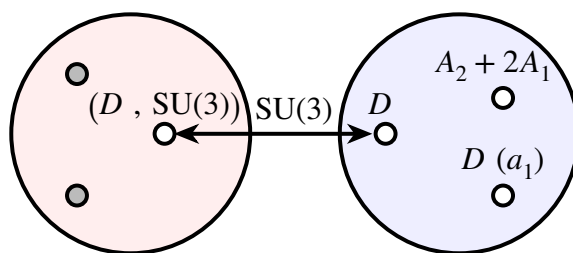
(4.2)

$SU(3)_{12} \times SU(2)_{5-k} \times SU(2)_k \times SU(2)_9 \times U(1)^2$  SCFT

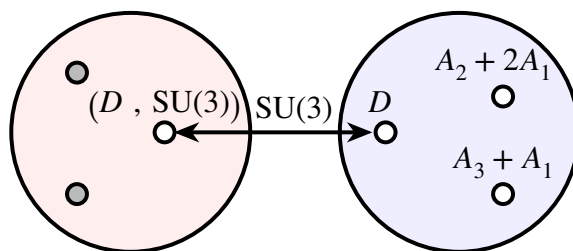
In each case, only the diagonal  $SU(2)_{54}$  subgroup, of the indicated  $SU(2)$ s, is manifest. Moreover, these fixtures are not gaugeable within the untwisted theory. So there is no obvious way to determine the individual  $SU(2)$  levels. Fortunately, the twisted sector provides the empty fixture



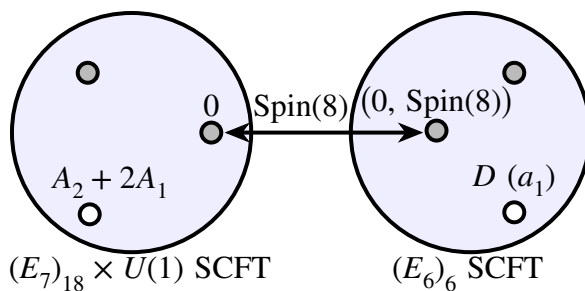
which allows us to gauge the  $SU(3)_{12}$  symmetry of each of these fixtures:



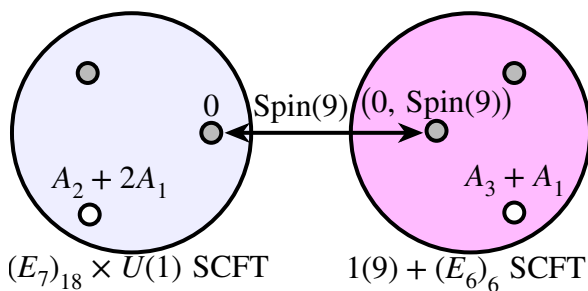
and



From the S-duals



and

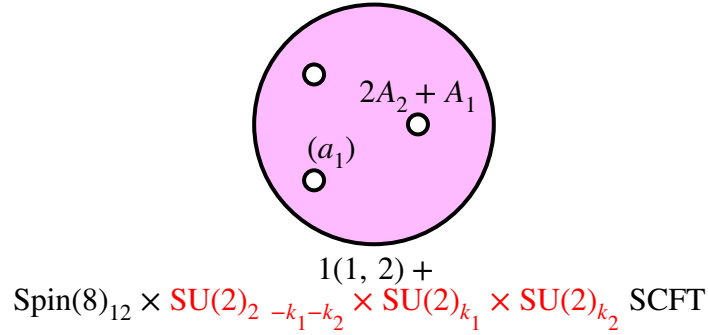


and the Lie-algebra embeddings

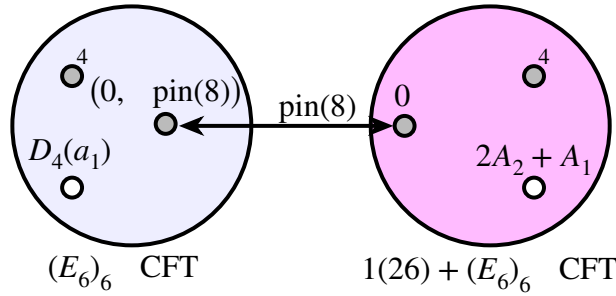
$$\begin{aligned}
 (\mathfrak{e}_7)_k &\supset (\mathfrak{f}_4)_k \oplus \mathfrak{su}(2)_{3k} \\
 (\mathfrak{e}_7)_k &\supset \mathfrak{so}(9)_k \oplus \mathfrak{su}(2)_{2k} \oplus \mathfrak{su}(2)_k \\
 (\mathfrak{e}_7)_k &\supset \mathfrak{so}(8)_k \oplus \mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_k
 \end{aligned}$$

we determine the levels in (4.1) and (4.2) to be  $k = k' = 18$ .

Finally, let us turn to the mixed fixture



Only the diagonal  $\text{SU}(2)_{24} \subset \text{SU}(2)_{24-k_1-k_2} \times \text{SU}(2)_{k_2} \times \text{SU}(2)_{k_2}$  is manifest. Gauging the  $\text{SU}(3)_{12}$  symmetry of the  $D_4$  puncture, as before, we find that the S-dual is a  $\text{Spin}(8)$  gauge theory, with matter in the  $1(8_v) + 1(8_s) + 1(8_c) + 2(1)$ , coupled to two copies of the  $(E_6)_6$  SCFT.



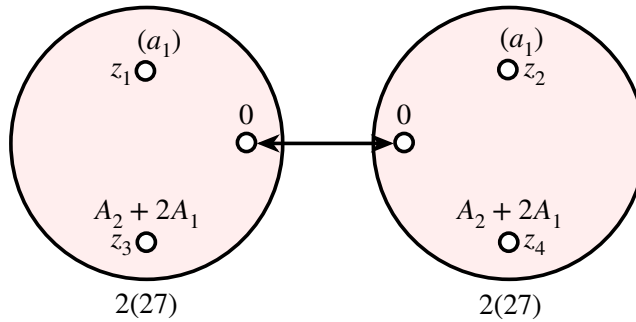
From this, we read off the levels of the three  $\text{SU}(2)$ s:  $k_1 = k_2 = 24 - k_1 - k_2 = 8$ .

## 5 Applications

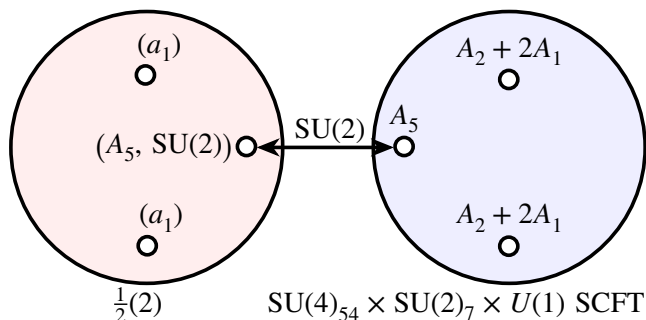
### 5.1 $E_6$ and $F_4$ gauge theory

#### 5.1.1 $E_6 + 4(27)$

$E_6$  gauge theory, with four fundamental hypermultiplets, is superconformal. It is realized as the 4-punctured sphere



The S-dual theory is an  $SU(2)$  gauging of the  $SU(4)_{54} \times SU(2)_7 \times U(1)$  SCFT, with an additional half-hypermultiplet in the fundamental.



The  $k$ -differentials, which determine the Seiberg-Witten solution, are

$$\begin{aligned}
 \phi_2(z) &= \frac{u_2 z_{12} z_{34} (dz)^2}{(z - z_1)(z - z_2)(z - z_3)(z - z_4)} \\
 \phi_5(z) &= \frac{u_5 z_{12} z_{34}^4 (dz)^5}{(z - z_1)(z - z_2)(z - z_3)^4(z - z_4)^4} \\
 \phi_6(z) &= \frac{u_6 z_{12}^2 z_{34}^4 (dz)^6}{(z - z_1)^2(z - z_2)^2(z - z_3)^4(z - z_4)^4} \\
 \phi_8(z) &= \frac{u_8 z_{12}^2 z_{34}^6 (dz)^8}{(z - z_1)^2(z - z_2)^2(z - z_3)^6(z - z_4)^6} \\
 \phi_9(z) &= \frac{u_9 z_{12}^2 z_{34}^7 (dz)^9}{(z - z_1)^2(z - z_2)^2(z - z_3)^7(z - z_4)^7} \\
 \phi_{12}(z) &= \frac{u_{12} z_{12}^3 z_{34}^9 (dz)^{12}}{(z - z_1)^3(z - z_2)^3(z - z_3)^9(z - z_4)^9}
 \end{aligned} \tag{5.1}$$

The gauge coupling,  $\tau = \frac{\theta}{\pi} + \frac{8\pi i}{g^2}$ , is determined by the  $SL(2, \mathbb{C})$ -invariant cross-ratio

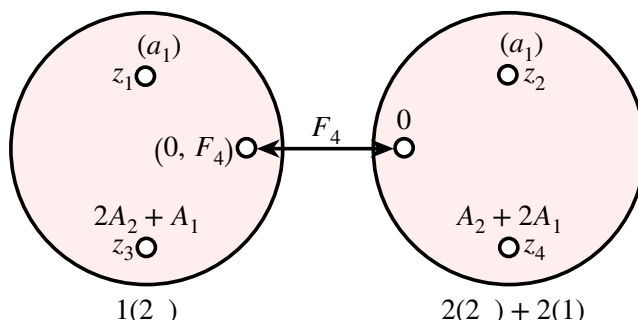
$$f(\tau) \equiv -\frac{\theta_2^4(0, \tau)}{\theta_4^4(0, \tau)} = \frac{z_{13} z_{24}}{z_{14} z_{23}} \tag{5.2}$$

and, for calculational purposes, it is usually convenient to use  $SL(2, \mathbb{C})$  to fix  $(z_1, z_2, z_3, z_4) = (0, \infty, f(\tau), 1)$  in (5.1).

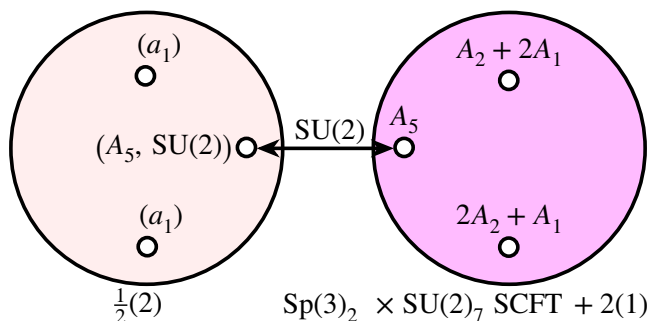
The solution to  $E_6$  gauge theory with  $N_f \leq 3$  fundamental hypermultiplets was first found in [18].

### 5.1.2 $F_4 + 3(26)$

$F_4$  gauge theory, with three fundamentals, is also superconformal. It is realized as



The S-dual theory is an  $SU(2)$  gauging of the  $Sp(3)_{26} \times SU(2)_7$  SCFT, with additional matter in the  $\frac{1}{2}(2) + 2(1)$ .



The nonzero  $k$ -differentials, which determine the Seiberg-Witten solution, are the same as in (5.1) but with  $\phi_5(z) \equiv 0 \equiv \phi_9(z)$ . The gauge coupling is again given by (5.2). Physically, this theory is obtained by Higgsing  $E_6 \rightarrow F_4$ , using one of the hypermultiplets in the 27.

In practice, given the solution to  $E_6 + 4(27)$ , the solution to  $F_4 + 3(26) + 2(1)$  is obtained by noting that

- There is a  $\mathbb{Z}_2$  symmetry,  $\sigma : (u_5, u_9) \mapsto (-u_5, -u_9)$ , acting on the Coulomb branch of the  $E_6 + 4(27)$ .
- The Coulomb branch geometry of  $F_4 + 3(26) + 2(1)$  is the geometry of the fixed-locus of  $\sigma$ .

### 5.2 Adding $(E_8)_{12}$ SCFTs

Starting with the  $E_6 + 4(27)$  Lagrangian field theory, we can start replacing hypermultiplets in the 27 with copies of the  $(E_8)_{12}$  SCFT. For  $n$  27s and  $4 - n$  copies of the  $(E_8)_{12}$  SCFT, the flavour symmetry group of the theory is

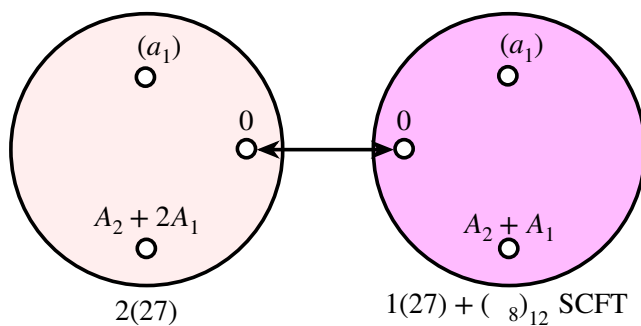
$$F = SU(3)_{12}^{4-n} \times U(n)_{54}$$

In each of these cases, the S-dual theory is an  $SU(2)$  gauging of the  $SU(3)_{12}^{4-n} \times SU(n)_{54} \times SU(2)_7 \times U(1)$  SCFT, with an additional half-hypermultiplet in the fundamental (the  $U(1)$  is absent for  $n = 0$ ).

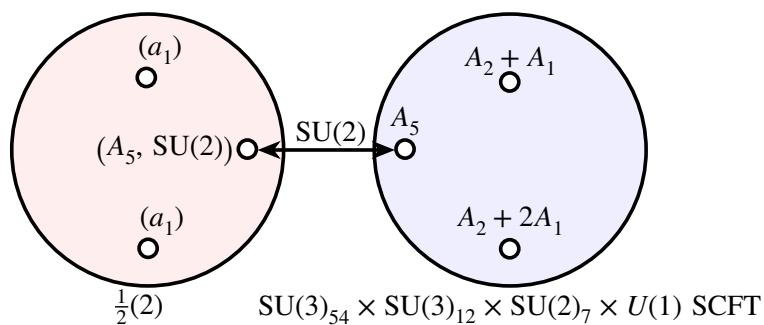


**5.2.1  $n = 3$**

With one copy of the  $(E_8)_{12}$  SCFT,

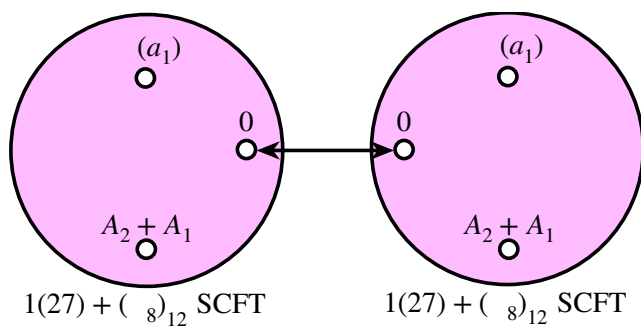


is dual to

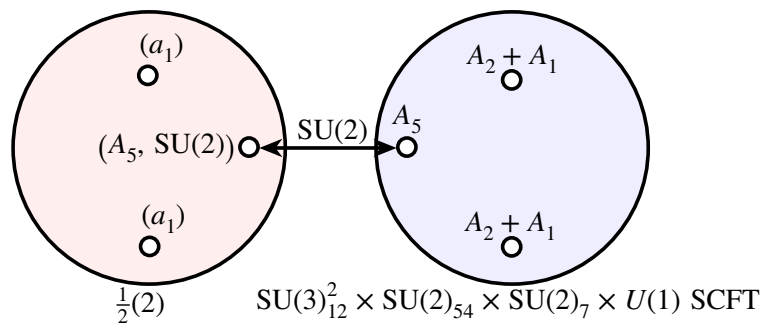


**5.2.2  $n = 2$**

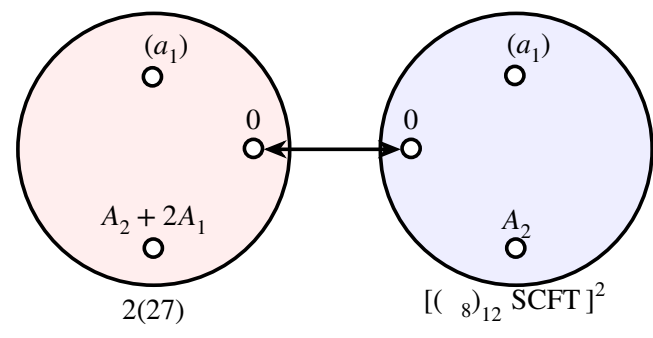
With two copies of the  $(E_8)_{12}$  SCFT, there are two possible realizations. Either



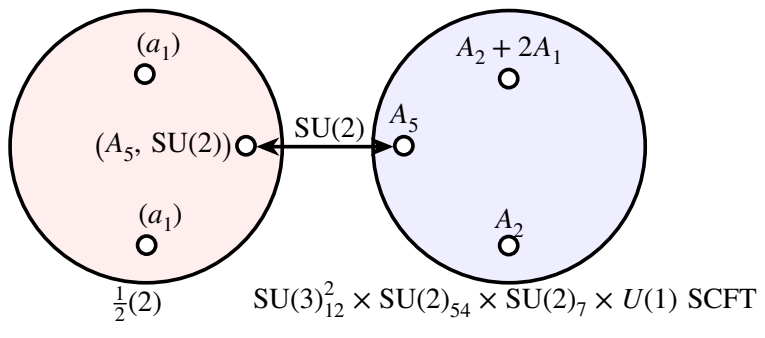
dual to



or



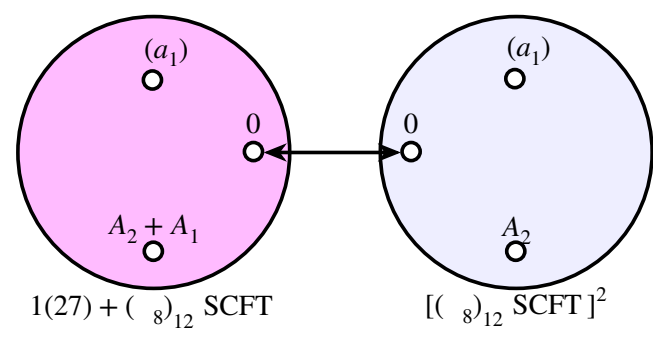
dual to



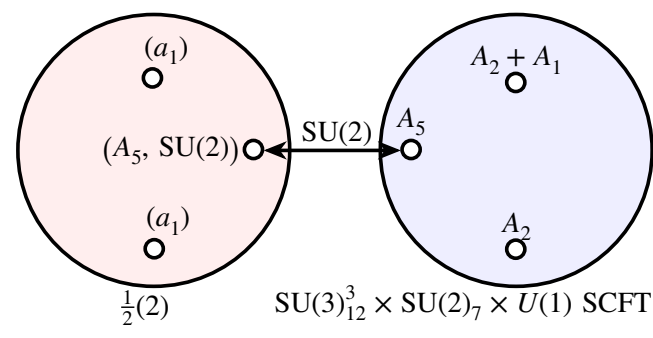
These give two, apparently distinct, realizations of the  $SU(3)_{12}^2 \times SU(2)_{54} \times SU(2)_7 \times U(1)$  SCFT.

**5.2.3  $n = 1$**

With three copies of the  $(E_8)_{12}$  SCFT, we have

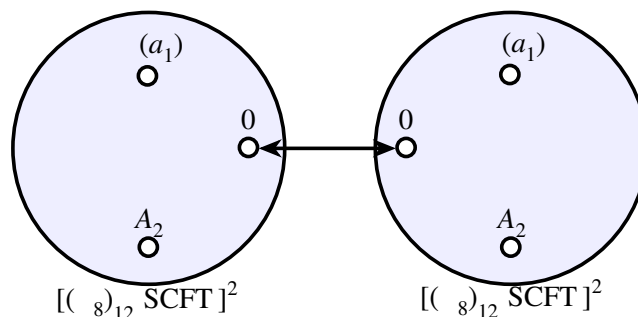


dual to

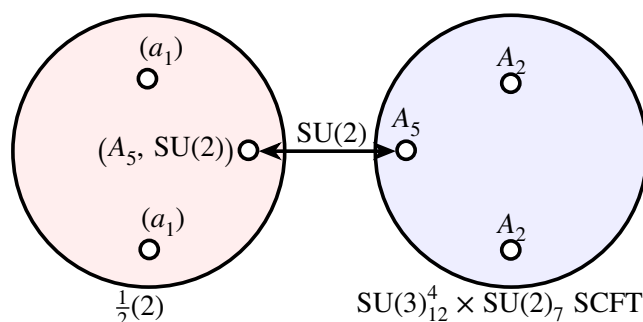


### 5.2.4 $n = 0$

Finally, the  $E_6$  gauging of four copies of the  $(E_8)_{12}$  SCFT,



is dual to



### 5.3 Connections with F-theory

Placing  $n$  D3-branes at a  $IV^*$ ,  $III^*$  or  $II^*$  singularity in F-Theory yields an  $\mathcal{N} = 2$  superconformal field theory on the world-volume of the D3-branes [37, 38]. For  $n = 1$  these are, respectively, the  $(E_6)_6$ ,  $(E_7)_8$  and  $(E_8)_{12}$  superconformal field theories of Minahan and Nemeschansky [36]. For higher  $n$ , the properties of these SCFTs were computed in [39]. The results may be summarized as follows

F-Theory singularity	Flavour symmetry	Graded Coulomb branch dimensions	$(n_h, n_v)$
$IV^*$	$(E_6)_{6n} \times SU(2)_{(n-1)(3n+1)}$	$n_{3l} = 1, \quad l = 1, 2, \dots, n$	$(3n^2 + 14n - 1, n(3n + 2))$
$III^*$	$(E_7)_{8n} \times SU(2)_{(n-1)(4n+1)}$	$n_{4l} = 1, \quad l = 1, 2, \dots, n$	$(4n^2 + 21n - 1, n(4n + 3))$
$II^*$	$(E_8)_{12n} \times SU(2)_{(n-1)(6n+1)}$	$n_{6l} = 1, \quad l = 1, 2, \dots, n$	$(6n^2 + 35n - 1, n(6n + 5))$

In [34], Gaiotto and Razamat proposed a realization of these ( $n \geq 2$ ) SCFTs as a mixed fixture, with one free hypermultiplet, in the  $A_{N-1}$  theory, for  $N = 3n, 4n$  and  $6n$ , respectively.

Theory	Fixture	Manifest flavour symmetry	Enhanced to
--------	---------	---------------------------	-------------

IV*		$SU(3)_{6n}^2 \times SU(2)_{6n} \times U(1)^2$	$(E_6)_{6n} \times SU(2)_k + \frac{1}{2}(2)$
III*		$SU(2)_{8n} \times SU(4)_{8n} \times SU(3)_{8n} \times U(1)^2$	$(E_7)_{8n} \times SU(2)_k + \frac{1}{2}(2)$
II*		$SU(2)_{12n} \times SU(3)_{12n} \times SU(5)_{12n} \times U(1)^2$	$(E_8)_{12n} \times SU(2)_k + \frac{1}{2}(2)$

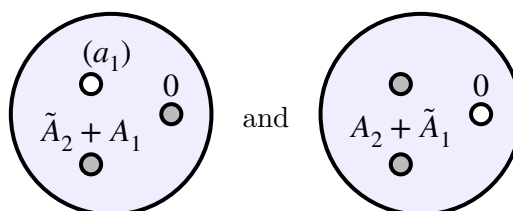
For  $n = 2$ , the  $SU(2)$  flavour symmetry is manifest, and one readily verifies that it has the predicted level (given that the hypermultiplet transforms as  $\frac{1}{2}(2)$  under the  $SU(2)$ ). But, for  $n \geq 3$ , only the  $U(1)$  Cartan is manifest and it is not easy to determine the level of the  $SU(2)$ .

We have, of course, numerous realizations of the  $n = 1$  theories. But we also find examples of the higher- $n$  theories

- We find the  $n = 2$  IV\* SCFT as one of our fixtures in section 3.3 and as part of a product SCFT in fixture 39 of section 3.4. It also appeared as an interacting fixture in the  $D_4$  theory in [4].
- We find the  $n = 2$  III\* SCFT as mixed fixture 4 in section 3.5 and as part of a product SCFT in fixture 6 of section 3.4.
- We find the  $n = 2$  II\* SCFT as interacting fixture 2 in section 3.4.
- We find the  $n = 3$  III\* SCFT as interacting fixture 7 in section 3.4.

In particular, the latter gives a nice check of the  $SU(2)$  level for  $n = 3$ .

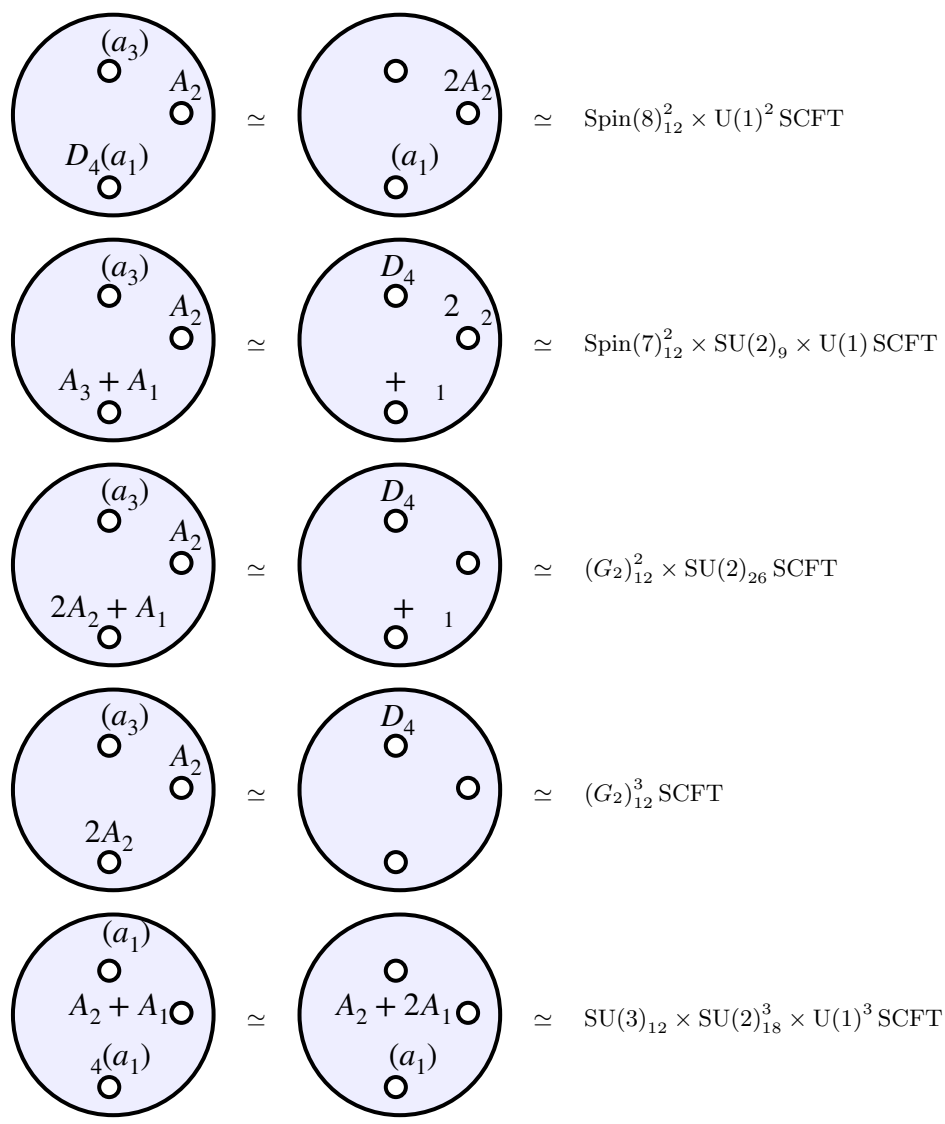
Further examples can be found in the  $\mathbb{Z}_2$ -twisted sector. Notably, the fixtures

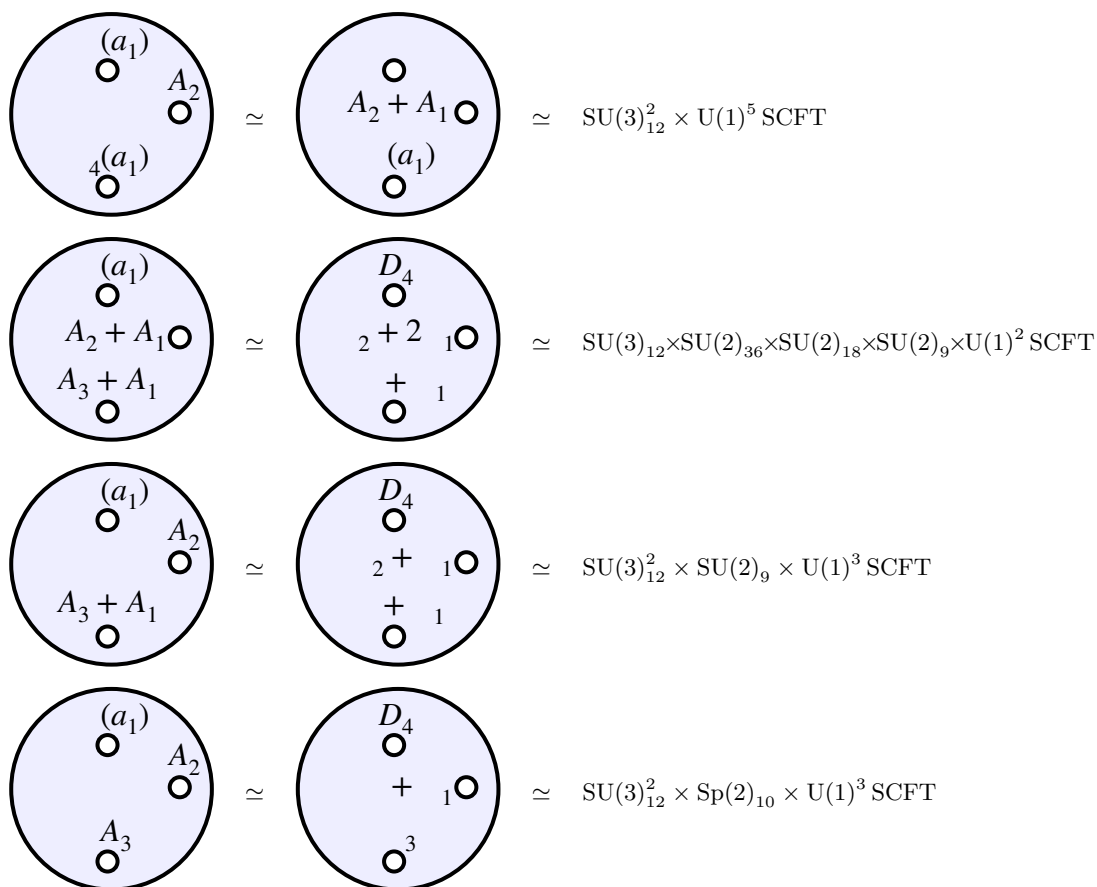


provide realizations, respectively, of the  $n = 3, 4$  IV\* SCFTs. Again, the  $SU(2)$  levels agree with the predictions of [39]. Together with the above examples, these exhaust all the IV\*, III\* and II\* theories with nonzero graded Coulomb branch dimensions in degrees  $\leq 12$ .

### 6 Isomorphic theories

In our table of interacting fixtures with enhanced global symmetry in section 3.4, we find several SCFTs which seem to be realized in more than one way. Most of these isomorphisms can be checked by various dualities. Some, however, cannot and we list them below.





It would be nice to check these conjectured isomorphisms by comparing the expansions of the superconformal indices for these pairs of fixtures to higher order in  $\tau$ .

### Acknowledgments

We would like to thank S. Katz, D. Morrison, A. Neitzke, R. Plesser and Y. Tachikawa for helpful discussions. J. D. and O. C. would like to thank the Aspen Center for Physics (supported, in part, by the National Science Foundation under Grant PHY-1066293) for their hospitality when this work was initiated. O. C. would further like to thank the Simons Foundation for partial support in Aspen. The work of J. D. and A. T. was supported in part by the National Science Foundation under Grant PHY-1316033. The work of O. C. was supported in part by the INCT-Matemática and the ICTP-SAIFR in Brazil through a Capes postdoctoral fellowship.

### A Bala-Carter labels

In the twisted and untwisted sectors of the  $A$  and  $D$  series, punctures were in one-to-one correspondence with certain classes of partitions [1, 9, 10, 40]. The partition denotes

how the fundamental representation (vector representation, in the case of  $\mathfrak{so}(N)$ ) of  $\mathfrak{g}$ <sup>6</sup> decomposes into representations of the corresponding (Nahm)  $\mathfrak{su}(2)$ . Moreover, one can also read off the centralizer,  $\mathfrak{f}$ , of  $\mathfrak{su}(2)$  inside  $\mathfrak{g}$ , as well as the decomposition of the fundamental representation of  $\mathfrak{g}$  under  $\mathfrak{su}(2) \times \mathfrak{f}$ , from the partition (see (2.7) in [9]). The decomposition under  $\mathfrak{su}(2) \times \mathfrak{f}$  for each puncture is precisely the information needed to compute the flavour group levels in section 2.4.1, as well as the expansion of the superconformal index in section 2.5. In what follows, we will explain how these decompositions are obtained for the punctures in the  $\mathfrak{e}_6$  theory.

In contrast to classical  $\mathfrak{g}$ , nilpotent orbits in the exceptional Lie algebras, which label our punctures, are not naturally classified by partitions. Here, we recall the classification of Bala and Carter [41, 42], following the exposition in [43]. Their theorem states that there is a one-to-one correspondence between nilpotent orbits in  $\mathfrak{g}$  and (conjugacy classes of) pairs  $(\mathfrak{l}, O^{\mathfrak{l}})$  where  $\mathfrak{l}$  is a Levi subalgebra<sup>7</sup> of  $\mathfrak{g}$  and  $O^{\mathfrak{l}}$  is a *distinguished*<sup>8</sup> nilpotent orbit in  $\mathfrak{l}$ . By the Jacobson-Morozov theorem, any representative  $X$  of  $O^{\mathfrak{l}}$  embeds in a standard triple<sup>9</sup>  $\{H, X, Y\} \subset \mathfrak{l}$ , where  $H \in \mathfrak{h}$ .  $\mathfrak{l}$  then has a decomposition into  $ad_H$ -eigenspaces

$$\mathfrak{l} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{l}_k$$

where  $\mathfrak{l}_k = \{x \in \mathfrak{l} \mid [H, x] = kx\}$ . Let  $\mathfrak{l}' \equiv \mathfrak{l}_0$  and  $\mathfrak{u}' \equiv \bigoplus_{0 < k \in \mathbb{Z}} \mathfrak{l}_k$ . Then,  $\mathfrak{p} = \mathfrak{l}' + \mathfrak{u}'$  is a parabolic subalgebra of  $\mathfrak{l}$ , with explicit Levi decomposition into a Levi subalgebra  $\mathfrak{l}'$  and the nilradical  $\mathfrak{u}'$  of  $\mathfrak{p}$ . (Notice that the Cartan of  $\mathfrak{l}$  is contained in  $\mathfrak{l}'$ , so  $\text{rank}(\mathfrak{l}') = \text{rank}(\mathfrak{l})$ .)

A nilpotent orbit in  $\mathfrak{g}$  is then given the label  $X_N(a_i)$ , called the *Bala-Carter label*, where  $X_N$  is the Cartan type of the semisimple part of  $\mathfrak{l}$ , and  $i$  is the number of simple roots in  $\mathfrak{l}'$ . The case  $i = 0$  is denoted just by  $X_N$ , and corresponds to the principal orbit in  $\mathfrak{l}$ , which is always distinguished.

There are 16 conjugacy classes of Levi subalgebras of  $E_6$ . These are specified by their semisimple parts:  $0, A_1, 2A_1, 3A_1, A_2, A_2 + A_1, 2A_2, A_2, A_2 + 2A_1, A_3 + A_1, D_4, A_4, A_4 + A_1, A_5, D_5$ , and  $E_6$ . Here,  $kA_N$  denotes the direct sum of  $k$  copies of  $A_N$ . The label  $0$  denotes the Cartan subalgebra, for which the only distinguished orbit is the zero orbit. For  $\mathfrak{l}$  of classical type, distinguished orbits in  $\mathfrak{l}$  are easily specified in terms of their partition: for  $\mathfrak{l}$  of type  $A$ , the only distinguished orbit is the principal orbit (which, for  $A_{N-1}$ , has partition  $[N]$ ), while for  $\mathfrak{l}$  of type  $B, C, D$ , distinguished orbits are those for which the partition has no repeated parts. It was found by Bala and Carter that, for  $\mathfrak{l}$  of type  $G_2, F_4, E_6, E_7$ , and  $E_8$ , there are 2, 4, 3, 6, and 11 distinguished orbits, respectively.

The distinguished orbits in the Levi subalgebras listed above give rise to 21 nilpotent orbits in  $\mathfrak{e}_6$ . We list these in the [table](#) below, along with the centralizer,  $\mathfrak{f}$ , and the de-

<sup>6</sup>For untwisted (twisted) punctures in the  $A$  and  $D$  series,  $\mathfrak{g}$  is of type  $A$  ( $B$ ) and  $D$  ( $C$ ), respectively.

<sup>7</sup>A Levi subalgebra  $\mathfrak{h} \subset \mathfrak{l} \subset \mathfrak{g}$  is a reductive subalgebra,  $\mathfrak{l}$ , containing the Cartan subalgebra,  $\mathfrak{h}$ , of  $\mathfrak{g}$ . See section 3.8 of [43] for an introduction.

<sup>8</sup>A nilpotent orbit,  $\mathcal{O}$ , in  $\mathfrak{g}$  is *distinguished* if and only if the only Levi subalgebra of  $\mathfrak{g}$ , containing  $\mathcal{O}$ , is  $\mathfrak{g}$  itself.

<sup>9</sup>Any  $\mathfrak{su}(2)$  subalgebra of  $\mathfrak{g}$  is spanned by a *standard triple*  $\{H, X, Y\}$  of nonzero elements of  $\mathfrak{g}$  satisfying the bracket relations  $[H, X] = 2X, [H, Y] = -2Y$ , and  $[X, Y] = H$ .

composition of the 27 and 78 of  $\mathfrak{e}_6$  under  $\mathfrak{su}(2) \times \mathfrak{f}$ .<sup>10</sup> But, before that, let us give a few examples of how to obtain the decomposition of the 27 for various embeddings.

First, consider  $\mathfrak{l} = D_4$ . In this case there are two distinguished orbits, with partitions [7,1] and [5,3], corresponding to nilpotent orbits  $D_4$  and  $D_4(a_1)$ , respectively, in  $\mathfrak{e}_6$ . The first has centralizer  $\mathfrak{su}(3)$  and the second,  $\mathfrak{u}(1)^2$ . We can obtain the decomposition of the 27 for each of these by embedding  $\mathfrak{su}(2)$  in the  $\mathfrak{so}(8)$  factor in  $\mathfrak{so}(8) \times \mathfrak{u}(1)^2 \subset \mathfrak{so}(10) \times \mathfrak{u}(1) \subset \mathfrak{e}_6$ . The 27 of  $\mathfrak{e}_6$  decomposes under  $\mathfrak{so}(10) \times \mathfrak{u}(1)$  as

$$\begin{aligned}\mathfrak{e}_6 &\supset \mathfrak{so}(10) \times \mathfrak{u}(1) \\ 27 &= 1_{-4} + 10_2 + 16_{-1}\end{aligned}$$

The 10 and 16 of  $\mathfrak{so}(10)$  decompose under  $\mathfrak{so}(8) \times \mathfrak{u}(1)$  as

$$\begin{aligned}\mathfrak{so}(10) &\supset \mathfrak{so}(8) \times \mathfrak{u}(1) \\ 10 &= 1_2 + 1_{-2} + (8_v)_0 \\ 16 &= (8_s)_1 + (8_c)_{-1}\end{aligned}$$

so we have

$$\begin{aligned}\mathfrak{e}_6 &\supset \mathfrak{so}(8) \times \mathfrak{u}(1) \times \mathfrak{u}(1) \\ 27 &= 1_{0,-4} + 1_{2,2} + 1_{-2,2} + (8_v)_{0,2} + (8_s)_{1,-1} + (8_c)_{-1,-1}\end{aligned}$$

For  $D_4(a_1)$ , we embed  $\mathfrak{su}(2)$  in  $\mathfrak{so}(8)$  by taking

$$\begin{aligned}\mathfrak{so}(8) &\supset \mathfrak{su}(2) \\ 8_{v,s,c} &= 5 + 3\end{aligned}$$

which gives

$$\begin{aligned}\mathfrak{e}_6 &\supset \mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1) \\ 27 &= 1_{0,-4} + 1_{2,2} + 1_{-2,2} + 3_{0,2} + 3_{1,-1} + 3_{-1,-1} + 5_{0,2} + 5_{1,-1} + 5_{-1,-1}\end{aligned}$$

For  $D_4$ , we embed  $\mathfrak{su}(2)$  in  $\mathfrak{so}(8)$  by taking

$$\begin{aligned}\mathfrak{so}(8) &\supset \mathfrak{su}(2) \\ 8_{v,s,c} &= 7 + 1\end{aligned}$$

which gives

$$\begin{aligned}\mathfrak{e}_6 &\supset \mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1) \\ 27 &= 1_{0,-4} + 1_{2,2} + 1_{-2,2} + 1_{0,2} + 1_{1,-1} + 1_{-1,-1} + 7_{0,2} + 7_{1,-1} + 7_{-1,-1}\end{aligned}$$

---

<sup>10</sup>The decomposition of the 27 determines a projection matrix, which can be used to obtain the decompositions of higher-dimensional representations. We list a projection matrix for each puncture in appendix B. The decomposition of the 78 determines the levels of the flavor groups, as described in section 2.4.1.



For this embedding, the  $\mathfrak{u}(1)^2$  centralizer enhances to  $\mathfrak{su}(3)$ . To see this, we can make a change of basis so that the two  $\mathfrak{u}(1)$  charges are given in terms of the old ones by

$$\begin{aligned} q'_1 &= \frac{1}{2}(q_1 + q_2) \\ q'_2 &= \frac{1}{2}(q_1 - q_2) \end{aligned}$$

Then the decomposition becomes

$$\begin{aligned} \mathfrak{e}_6 &\supset \mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1) \\ 27 &= 1_{-2,2} + 1_{2,0} + 1_{0,-2} + 1_{1,-1} + 1_{0,1} + 1_{-1,0} + 7_{1,-1} + 7_{0,1} + 7_{-1,0} \end{aligned}$$

where we recognize these  $\mathfrak{u}(1)^2$  charges as the weights (in the Dynkin basis) of the 6 and  $\bar{3}$  of  $\mathfrak{su}(3)$ . Thus, the decomposition of the 27 is given by

$$\begin{aligned} \mathfrak{e}_6 &\supset \mathfrak{su}(2) \times \mathfrak{su}(3) \\ 27 &= (1, 6) + (7, \bar{3}) \end{aligned}$$

Now, consider  $\mathfrak{l} = E_6$ . There are three distinguished orbits in  $\mathfrak{e}_6$ , giving rise to nilpotent orbits  $E_6, E_6(a_1)$ , and  $E_6(a_3)$ . The decomposition of the 27 for each of these can be obtained by taking the inner product of the Cartan element  $H$  of the embedded  $\mathfrak{su}(2)$  (which can be read off from the weighted Dynkin diagram) into the weight vectors of the 27.

We conclude this appendix with a summary of the nilpotent orbits in  $\mathfrak{e}_6$  and the corresponding decompositions of the 27 and the 78 under  $\mathfrak{su}(2) \times \mathfrak{f}$ .

Bala-Carter	$\mathfrak{f}$	27	78
0	$\mathfrak{e}_6$	(1; 27)	(1; 78)
$A_1$	$\mathfrak{su}(6)$	(1; $\bar{15}$ ) + (2; 6)	(1; 35) + (2; 20) + (3; 1)
$2A_1$	$\mathfrak{so}(7) \times \mathfrak{u}(1)$	(1; $7_2 + 1_{-4}$ ) + (2; $8_{-1}$ ) + (3; $1_2$ )	(1; $1_0 + 21_0$ ) + (2; $8_3 + 8_{-3}$ ) + (3; $7_0 + 1_0$ )
$3A_1$	$\mathfrak{su}(3) \times \mathfrak{su}(2)$	(1; $\bar{6}$ , 1) + (2; 3, 2) + (3; 3, 1)	(1; 8, 1) + (1; 1, 3) + (2; 8, 2) + (3; 1, 1) + (3; 8, 1) + (4; 1, 2)
$A_2$	$\mathfrak{su}(3) \times \mathfrak{su}(3)$	(1; 3, $\bar{3}$ ) + (3; 1, 3) + (3; $\bar{3}$ , 1)	(1; 8, 1) + (1; 1, 8) + (3; 1, 1) + (3; 3, 3) + (3; $\bar{3}$ , $\bar{3}$ ) + (5; 1, 1)
$A_2 + A_1$	$\mathfrak{su}(3) \times \mathfrak{u}(1)$	(1; $3_2$ ) + (2; $3_{-1} + 1_1$ ) + (3; $\bar{3}_0 + 1_{-2}$ ) + (4; $1_1$ )	(1; $8_0 + 1_0$ ) + (2; $3_1 + \bar{3}_{-1} + 1_{-3} + 1_3$ ) + (3; $3_{-2} + \bar{3}_2 + 1_0 + 1_0$ ) + (4; $3_1 + \bar{3}_{-1}$ ) + (5; $1_0$ )
$2A_2$	$\mathfrak{g}_2$	(1; 1) + (3; 7) + (5; 1)	(1; 14) + (3; 7 + 1) + (5; 7 + 1)
$A_2 + 2A_1$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	(1; $1_2 + 1_{-4}$ ) + (2; $4_{-1}$ ) + (3; $3_2$ ) + (4; $2_{-1}$ )	(1; $1_0 + 3_0$ ) + (2; $4_3 + 4_{-3}$ ) + (3; $1_0 + 3_0 + 5_0$ ) + (4; $2_3 + 2_{-3}$ ) + (5; $3_0$ )
$A_3$	$\mathfrak{sp}(2) \times \mathfrak{u}(1)$	(1; $5_{-2} + 1_4$ ) + (4; $4_1$ ) + (5; $1_{-2}$ )	(1; $10_0 + 1_0$ ) + (3; $1_0$ ) + (4; $4_3 + 4_{-3}$ ) + (5; $5_0$ ) + (7; $1_0$ )
$2A_2 + A_1$	$\mathfrak{su}(2)$	(1; 1) + (2; 2) + (3; 3) + (4; 2) + (5; 1)	(1; 3) + (2; 4 + 2) + (3; 3 + 1 + 1) + (4; 2 + 2) + (5; 3 + 1) + (6; 2)

Bala-Carter	$\mathfrak{f}$	27	78
$A_3 + A_1$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$(1; 1_4 + 1_{-2}) + (2; 2_{-2})$ $+ (3; 1_1) + (4; 2_1)$ $+ (5; 1_1 + 1_{-2})$	$(1; 1_0 + 3_0) + (2; 2)_0$ $+ (3; 1_3 + 1_{-3} + 1_0 + 1_0)$ $+ (4; 2_3 + 2_{-3} + 2_0)$ $+ (5; 1_3 + 1_0 + 1_{-3})$ $+ (6; 2)_0 + (7; 1_0)$
$D_4(a1)$	$\mathfrak{u}(1) \times \mathfrak{u}(1)$	$1_{2,2} + 1_{0,-4} + 1_{-2,2}$ $+ 3_{1,-1} + 3_{0,2} + 3_{-1,-1}$ $+ 5_{1,-1} + 5_{0,2} + 5_{-1,-1}$	$1_{0,0} + 1_{0,0} + 3_{0,0}$ $+ 3_{2,0} + 3_{1,3} + 3_{1,-3} + 3_{0,0}$ $+ 3_{-2,0} + 3_{-1,-3} + 3_{-1,3} + 3_{0,0}$ $+ 5_{2,0} + 5_{1,3} + 5_{1,-3} + 5_{0,0}$ $+ 5_{-2,0} + 5_{-1,-3} + 5_{-1,3}$ $+ 7_{0,0} + 7_{0,0}$
$A_4$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$(1; 2_{-5}) + (3; 1_{-2})$ $+ (5; 2_1 + 1_4) + (7; 1_{-2})$	$(1; 3_0 + 1_0) + (3; 2_3 + 2_{-3} + 1_0)$ $+ (5; 1_6 + 1_0 + 1_{-6})$ $+ (7; 2_3 + 2_{-3} + 1_0) + (9; 1_0)$
$D_4$	$\mathfrak{su}(3)$	$(1; 6) + (7; \bar{3})$	$(1; 8) + (3; 1) + (7; 8) + (11; 1)$
$A_4 + A_1$	$\mathfrak{u}(1)$	$2_{-5} + 3_{-2} + 4_1$ $+ 5_4 + 6_1 + 7_{-2}$	$1_0 + 2_3 + 2_{-3} + 3_0 + 3_0 + 4_3 + 4_{-3}$ $+ 5_6 + 5_0 + 5_{-6} + 6_{-3} + 6_3$ $+ 7_0 + 8_{-3} + 8_3 + 9_0$
$D_5(a1)$	$\mathfrak{u}(1)$	$1_{-4} + 2_{-1} + 3_2$ $+ 6_{-1} + 7_2 + 8_{-1}$	$1_0 + 2_3 + 2_{-3} + 3_0 + 3_0 + 5_0$ $+ 6_3 + 6_{-3} + 7_0 + 7_0$ $+ 8_3 + 8_{-3} + 9_0 + 11_0$
$A_5$	$\mathfrak{su}(2)$	$(1; 1) + (5; 1) + (6; 2) + (9; 1)$	$(1; 3) + (3; 1) + (4; 2) + (5; 1) + (6; 2)$ $+ (7; 1) + (9; 1) + (10; 2) + (11; 1)$
$E_6(a3)$	–	$1 + 5 + 5 + 7 + 9$	$3 + 3 + 3 + 5 + 5 + 5$ $+ 7 + 7 + 9 + 9 + 11 + 11$
$D_5$	$\mathfrak{u}(1)$	$1_2 + 1_{-4} + 5_{-1} + 9_2 + 11_{-1}$	$1_0 + 3_0 + 5_3 + 5_{-3} + 7_0 + 9_0$ $+ 11_3 + 11_0 + 11_{-3} + 15_0$
$E_6(a1)$	–	$5 + 9 + 13$	$3 + 5 + 7 + 9 + 11 + 11 + 15 + 17$
$E_6$	–	$1 + 9 + 17$	$3 + 9 + 11 + 15 + 17 + 23$

## B Projection matrices

Our classification of interacting and mixed fixtures using the superconformal index, carried out in section 2.5, required that we know the decomposition of a number of higher-dimensional  $\mathfrak{e}_6$  representations (and not just the 27 and the 78) under  $\mathfrak{su}(2) \times \mathfrak{f}$ . These are trivial to obtain using LieART [44], provided we know a projection matrix for each embedding [44, 45].

From the decomposition of the 27, listed in the table above, one obtains a projection matrix simply by defining a  $6 \times \text{rk}(\mathfrak{su}(2) \times \mathfrak{f})$  matrix,  $M$ , such that the LieART command

```
In[1]= Project[M,WeightSystem[Irrep[E6][1,0,0,0,0,0]]]
```

gives the corresponding  $\mathfrak{su}(2) \times \mathfrak{f}$  weights. This projection matrix can then be used to obtain the decomposition of any  $\mathfrak{e}_6$  irrep under  $\mathfrak{su}(2) \times \mathfrak{f}$ .

Below, we list a projection matrix for each embedding, following the conventions of LieART.

Bala-Carter	f	Projection Matrix
$A_1$	$\mathfrak{su}(6)$	$\begin{pmatrix} -1 & -2 & -3 & -2 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
$2A_1$	$\mathfrak{so}(7) \times \mathfrak{u}(1)$	$\begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & -1 & -2 & 0 \end{pmatrix}$
$3A_1$	$\mathfrak{su}(3) \times \mathfrak{su}(2)$	$\begin{pmatrix} 2 & 3 & 4 & 3 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{pmatrix}$
$A_2$	$\mathfrak{su}(3) \times \mathfrak{su}(3)$	$\begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -2 & -1 & -2 \end{pmatrix}$
$A_2 + A_1$	$\mathfrak{su}(3) \times \mathfrak{u}(1)$	$\begin{pmatrix} 3 & 5 & 7 & 5 & 3 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}$
$2A_2$	$\mathfrak{g}_2$	$\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

Bala-Carter	f	Projection Matrix
$A_2 + 2A_1$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 3 & 4 & 6 & 4 & 3 & 4 \\ 1 & 4 & 6 & 4 & 1 & 2 \\ -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix}$
$A_3$	$\mathfrak{sp}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 4 & 7 & 10 & 7 & 4 & 6 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{pmatrix}$
$2A_2 + A_1$	$\mathfrak{su}(2)$	$\begin{pmatrix} 4 & 6 & 9 & 6 & 4 & 5 \\ 0 & 2 & 3 & 2 & 0 & 1 \end{pmatrix}$
$A_3 + A_1$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 4 & 8 & 11 & 8 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -2 & -1 & 0 & 1 & 2 & 0 \end{pmatrix}$
$D_4(a1)$	$\mathfrak{u}(1) \times \mathfrak{u}(1)$	$\begin{pmatrix} 4 & 8 & 10 & 8 & 4 & 6 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 & 1 & 0 \end{pmatrix}$
$A_4$	$\mathfrak{su}(2) \times \mathfrak{u}(1)$	$\begin{pmatrix} 6 & 10 & 12 & 10 & 6 & 6 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & -3 & -2 & 2 & 0 \end{pmatrix}$
$D_4$	$\mathfrak{su}(3)$	$\begin{pmatrix} 6 & 10 & 16 & 10 & 6 & 10 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \end{pmatrix}$

Bala-Carter	f	Projection Matrix
$A_4 + A_1$	$\mathfrak{u}(1)$	$\begin{pmatrix} 6 & 10 & 12 & 10 & 6 & 7 \\ -2 & -4 & -6 & -2 & 2 & -3 \end{pmatrix}$
$D_5(a1)$	$\mathfrak{u}(1)$	$\begin{pmatrix} 7 & 12 & 18 & 12 & 7 & 10 \\ -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix}$
$A_5$	$\mathfrak{su}(2)$	$\begin{pmatrix} 8 & 14 & 19 & 14 & 8 & 10 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
$E_6(a3)$	–	$(8 \ 14 \ 18 \ 14 \ 8 \ 8)$
$D_5$	$\mathfrak{u}(1)$	$\begin{pmatrix} 10 & 18 & 24 & 18 & 10 & 10 \\ -1 & -2 & 0 & 2 & 1 & 0 \end{pmatrix}$
$E_6(a1)$	–	$(12 \ 22 \ 30 \ 22 \ 12 \ 16)$
$E_6$	–	$(16 \ 30 \ 42 \ 30 \ 16 \ 22)$

As an example, let's work out the decomposition of the 51975 for the orbit  $2A_2$ . Running LieART, we obtain the decomposition with the following two lines of code:

```
In[1]= ProjectionMatrix[E6,ProductAlgebra[SU2,G2]]= $\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$  ;
```

```
In[2]= DecomposeIrrep[Irrep[E6][1,0,1,0,0,0],ProductAlgebra[SU2,G2]]
```

```
Out [2]= (1, 1) + 14(3, 1) + 10(5, 1) + 13(1, 7) + 13(7, 1) + 25(3, 7) + 5(9, 1) + 34(5, 7)
+ 4(11, 1) + 25(7, 7) + 9(1, 14) + 17(9, 7) + 16(3, 14) + 6(11, 7) + 22(5, 14)
+ 2(13, 7) + 15(7, 14) + 10(9, 14) + 3(11, 14) + (13, 14) + 6(1, 27) + 25(3, 27)
+ 23(5, 27) + 21(7, 27) + 9(9, 27) + 4(11, 27) + 5(1, 64) + 12(3, 64) + 13(5, 64)
+ 9(7, 64) + 4(9, 64) + (11, 64) + 4(1, 77) + 6(3, 77) + 2(3, 77') + 8(5, 77)
+ (5, 77') + 4(7, 77) + (7, 77') + 2(9, 77) + (3, 182) + (1, 189) + 2(3, 189)
+ 2(5, 189) + (7, 189)
```

This works for all of the orbits above, except for  $D_4(a_1)$ , as the LieART command “DecomposeIrrep” does not seem to work when the target subalgebra has more than one  $\mathfrak{u}(1)$  factor. In this case, getting the decomposition is only slightly more complicated. For example, we obtain the decomposition of the 27 of  $E_6$  as follows:

```
In [1]= ProjectionMatrix[D5,ProductAlgebra[D4,U1]] =  $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$  ;
```

```
In [2]= ProjectionMatrix[D4,ProductAlgebra[A1]] = (4 6 4 4) ;
```

```
In [3]= DecomposeIrrep[ DecomposeIrrep[ DecomposeIrrep[
Irrep[E6][1, 0, 0, 0, 0, 0], ProductAlgebra[D5, U1]],
ProductAlgebra[D4, U1], 1], ProductAlgebra[A1], 1]
```

```
Out [3]= (1)(2)(2)+(1)(0)(-4)+(1)(-2)(2)+(3)(1)(-1)+(3)(0)(2)
+(3)(-1)(-1)+(5)(1)(-1)+(5)(0)(2)+(5)(-1)(-1)
```

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## References

- [1] D. Gaiotto,  $N = 2$  dualities, *JHEP* **08** (2012) 034 [[arXiv:0904.2715](https://arxiv.org/abs/0904.2715)] [[INSPIRE](#)].
- [2] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin systems and the WKB approximation*, [arXiv:0907.3987](https://arxiv.org/abs/0907.3987) [[INSPIRE](#)].
- [3] O. Chacaltana and J. Distler, *Tinkertoys for Gaiotto duality*, *JHEP* **11** (2010) 099 [[arXiv:1008.5203](https://arxiv.org/abs/1008.5203)] [[INSPIRE](#)].
- [4] O. Chacaltana and J. Distler, *Tinkertoys for the  $D_N$  series*, *JHEP* **02** (2013) 110 [[arXiv:1106.5410](https://arxiv.org/abs/1106.5410)] [[INSPIRE](#)].
- [5] D. Nanopoulos and D. Xie, *Hitchin equation, singularity and  $N = 2$  superconformal field theories*, *JHEP* **03** (2010) 043 [[arXiv:0911.1990](https://arxiv.org/abs/0911.1990)] [[INSPIRE](#)].
- [6] D. Nanopoulos and D. Xie,  $N = 2$  generalized superconformal quiver gauge theory, *JHEP* **09** (2012) 127 [[arXiv:1006.3486](https://arxiv.org/abs/1006.3486)] [[INSPIRE](#)].

- [7] F. Benini, Y. Tachikawa and D. Xie, *Mirrors of 3d Sicilian theories*, *JHEP* **09** (2010) 063 [[arXiv:1007.0992](#)] [[INSPIRE](#)].
- [8] D. Gaiotto, G.W. Moore and Y. Tachikawa, *On 6d  $N = (2, 0)$  theory compactified on a Riemann surface with finite area*, *Prog. Theor. Exp. Phys.* **2013** (2013) 013B03 [[arXiv:1110.2657](#)] [[INSPIRE](#)].
- [9] O. Chacaltana, J. Distler and Y. Tachikawa, *Nilpotent orbits and codimension-two defects of 6d  $N = (2, 0)$  theories*, *Int. J. Mod. Phys. A* **28** (2013) 1340006 [[arXiv:1203.2930](#)] [[INSPIRE](#)].
- [10] O. Chacaltana, J. Distler and Y. Tachikawa, *Gaiotto duality for the twisted  $A_{2N-1}$  series*, *JHEP* **05** (2015) 075 [[arXiv:1212.3952](#)] [[INSPIRE](#)].
- [11] O. Chacaltana, J. Distler and A. Trimm, *Tinkertoys for the twisted  $D$ -series*, *JHEP* **04** (2015) 173 [[arXiv:1309.2299](#)] [[INSPIRE](#)].
- [12] P.C. Argyres and N. Seiberg,  *$S$ -duality in  $N = 2$  supersymmetric gauge theories*, *JHEP* **12** (2007) 088 [[arXiv:0711.0054](#)] [[INSPIRE](#)].
- [13] L. Bhardwaj and Y. Tachikawa, *Classification of 4d  $N = 2$  gauge theories*, *JHEP* **12** (2013) 100 [[arXiv:1309.5160](#)] [[INSPIRE](#)].
- [14] E. Witten, *Some comments on string dynamics*, [hep-th/9507121](#) [[INSPIRE](#)].
- [15] S. Gukov and E. Witten, *Gauge theory, ramification, and the geometric Langlands program*, [hep-th/0612073](#) [[INSPIRE](#)].
- [16] S. Gukov and E. Witten, *Rigid surface operators*, *Adv. Theor. Math. Phys.* **14** (2010) 87 [[arXiv:0804.1561](#)] [[INSPIRE](#)].
- [17] Y. Tachikawa,  *$N = 2$   $S$ -duality via outer-automorphism twists*, *J. Phys. A* **44** (2011) 182001 [[arXiv:1009.0339](#)] [[INSPIRE](#)].
- [18] S. Terashima and S.-K. Yang, *Exceptional Seiberg-Witten geometry with massive fundamental matters*, *Phys. Lett. B* **430** (1998) 102 [[hep-th/9803014](#)] [[INSPIRE](#)].
- [19] R. Lawther and D.M. Testerman, *Centres of centralizers of unipotent elements in simple algebraic groups*, *Mem. Amer. Math. Soc.* **210** (2011) no. 988, vi+188.
- [20] W.M. McGovern, *The adjoint representation and the adjoint action*, in *Algebraic Quotients. Torus Actions and Cohomology. The Adjoint Representation and the Adjoint Action*, *Encyclopaedia Math. Sci.* **131** (2002) 159, Springer, Germany (2002).
- [21] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, *Lect. Notes Math.* **946**, Springer, Germany (1982).
- [22] Y. Tachikawa and S. Terashima, *Seiberg-Witten geometries revisited*, *JHEP* **09** (2011) 010 [[arXiv:1108.2315](#)] [[INSPIRE](#)].
- [23] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, *Nucl. Phys. B* **459** (1996) 97 [[hep-th/9509161](#)] [[INSPIRE](#)].
- [24] GAP GROUP collaboration, *GAP — Groups, Algorithms, and Programming, Version 4.7.4*, <http://www.gap-system.org>, (2014).
- [25] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner, *Selfdual strings and  $N = 2$  supersymmetric field theory*, *Nucl. Phys. B* **477** (1996) 746 [[hep-th/9604034](#)] [[INSPIRE](#)].

- [26] W. Lerche and N.P. Warner, *Exceptional SW geometry from ALE fibrations*, *Phys. Lett. B* **423** (1998) 79 [[hep-th/9608183](#)] [[INSPIRE](#)].
- [27] S. Katz and D.R. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups*, *J. Alg. Geom.* **1** (1992) 449 [[alg-geom/9202002](#)].
- [28] R.W. Carter, *Finite groups of Lie type — conjugacy classes and complex characters*, Pure and Applied Mathematics, John Wiley & Sons, U.S.A. (1985).
- [29] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[INSPIRE](#)].
- [30] A. Gadde, E. Pomoni, L. Rastelli and S.S. Razamat, *S-duality and 2d topological QFT*, *JHEP* **03** (2010) 032 [[arXiv:0910.2225](#)] [[INSPIRE](#)].
- [31] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *The 4d superconformal index from  $q$ -deformed 2d Yang-Mills*, *Phys. Rev. Lett.* **106** (2011) 241602 [[arXiv:1104.3850](#)] [[INSPIRE](#)].
- [32] A. Gadde, L. Rastelli, S.S. Razamat and W. Yan, *Gauge theories and Macdonald polynomials*, *Commun. Math. Phys.* **319** (2013) 147 [[arXiv:1110.3740](#)] [[INSPIRE](#)].
- [33] M. Lemos, W. Peelaers and L. Rastelli, *The superconformal index of class S theories of type D*, *JHEP* **05** (2014) 120 [[arXiv:1212.1271](#)] [[INSPIRE](#)].
- [34] D. Gaiotto and S.S. Razamat, *Exceptional indices*, *JHEP* **05** (2012) 145 [[arXiv:1203.5517](#)] [[INSPIRE](#)].
- [35] D. Gaiotto, L. Rastelli and S.S. Razamat, *Bootstrapping the superconformal index with surface defects*, *JHEP* **01** (2013) 022 [[arXiv:1207.3577](#)] [[INSPIRE](#)].
- [36] J.A. Minahan and D. Nemeschansky, *Superconformal fixed points with  $E_N$  global symmetry*, *Nucl. Phys. B* **489** (1997) 24 [[hep-th/9610076](#)] [[INSPIRE](#)].
- [37] K. Dasgupta and S. Mukhi, *F theory at constant coupling*, *Phys. Lett. B* **385** (1996) 125 [[hep-th/9606044](#)] [[INSPIRE](#)].
- [38] T. Banks, M.R. Douglas and N. Seiberg, *Probing F-theory with branes*, *Phys. Lett. B* **387** (1996) 278 [[hep-th/9605199](#)] [[INSPIRE](#)].
- [39] O. Aharony and Y. Tachikawa, *A holographic computation of the central charges of  $D = 4$ ,  $N = 2$  SCFTs*, *JHEP* **01** (2008) 037 [[arXiv:0711.4532](#)] [[INSPIRE](#)].
- [40] Y. Tachikawa, *Six-dimensional  $D_N$  theory and four-dimensional SO-USp quivers*, *JHEP* **07** (2009) 067 [[arXiv:0905.4074](#)] [[INSPIRE](#)].
- [41] P. Bala and R. Carter, *Classes of unipotent elements in simple algebraic groups I*, *Math. Proc. Camb. Phil. Soc.* **79** (1976) 401.
- [42] P. Bala and R. Carter, *Classes of unipotent elements in simple algebraic groups II*, *Math. Proc. Camb. Phil. Soc.* **80** (1976) 1.
- [43] D.H. Collingwood and W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand, U.S.A. (1993).
- [44] R. Feger and T.W. Kephart, *LieART — a Mathematica application for Lie algebras and representation theory*, *Comput. Phys. Commun.* **192** (2015) 166 [[arXiv:1206.6379](#)] [[INSPIRE](#)].
- [45] R. Slansky, *Group theory for unified model building*, *Phys. Rept.* **79** (1981) 1 [[INSPIRE](#)].