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On the rotating and oscillating strings in $(AdS_3 \times S^3)_\varkappa$

Aritra Banerjee[a] **and Kamal L. Panigrahi**[a,b]

[a] *Department of Physics, Indian Institute of Technology Kharagpur, Kharagpur-721 302, India*

[b] *Department of Physics, CERN Theory Division, CH-1211, Geneva 23, Switzerland*

E-mail: aritra@phy.iitkgp.ernet.in, panigrahi@phy.iitkgp.ernet.in

Abstract: We study rigidly rotating strings in the $\varkappa$-deformed $AdS_3 \times S^3$ background. We find out two classes of solutions corresponding to the giant magnon and single spike solutions of the string rotating in two $S^2_\varkappa$ subspace of rotations reduced along two different isometries. We verify that the dispersion relations reduce to the well known relation in the $\varkappa \to 0$ limit. We further study some oscillating string solutions in the $S^3_\varkappa$ subspace.

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Contents

1 Introduction

Integrability has played an important role in understanding the spectrum of the superstring theory on $AdS_5 \times S^5$ [1]. This fact has been one of the key ingredient in exploring the conjectured AdS/CFT duality [2] because of the fact that in planar limit both sides of the duality, namely the gauge theory and string theory, are more tractable [3–7]. This stems from the basic fact that bosonic rigid spinning strings in the $AdS_5 \times S^5$ space-time are naturally described as periodic solutions of the finite dimensional integrable system. It was further noticed that the string theory is also integrable in the semiclassical limit and the anomalous dimension of the $\mathcal{N} = 4$ Super Yang Mills (SYM) can be derived from the relation between conserved charges of the rotating strings in $AdS_5 \times S^5$.

There have been many ideas to introduce various classes of integrable deformations to the string sigma model on $AdS_5 \times S^5$. Typically they are constructed by applying T-duality on any given integrable model(e.g. [11–16]). Contrary to the usual wisdom, however more recently, a novel example of a one-parameter integrable deformation of the $AdS_5 \times S^5$ supercoset model was found in [8], following earlier proposals.[1] This model is parameterized by a real deformed parameter $\eta \in [0, 1)$ and string tension $T \equiv g = \frac{\sqrt{\lambda}}{2\pi}$. This model appears to be quite complicated and involved due to the presence of fermionic degrees of freedom. As a first step to understand the background better in [17] a Lagrangian corresponding to the bosonic degrees of freedom has been studied in detail. Infact, the tree-level bosonic S-matrix was also computed and quite successfully matched with the semi-classical limit of the q-deformed S-matrix computed in [18, 19]. In a related development, the bosonic

[1] Also in [9, 10] non-standard q-deformed model based on the classical r-matrix satisfying classical Yang-Baxter equations was explored.

subsectors of the q-deformed $AdS_5 \times S^5$ superstring action was studied and the classical integrable structure of anisotropic Landau-Lifshitz sigma models was derived by taking fast-moving limits in [20]. Recently in [21] the bosonic spinning strings in the η-deformed $(AdS_5 \times S^5)$ have been studied, and shown that they are naturally described by the periodic solution of a new finite dimensional integrable system, which in turn could be viewed as a deformed Neumann model. Restricting the motion of the string to the deformed sphere, the Lax representation for the deformed Neumann model has been presented. In [22] various deformation limits are discussed. In particular, in the so called maximal deformation $(\eta = 1)$, the $AdS_5 \times S^5$ is transformed to the T dual of the double wick rotated version of iteself, i.e. $dS_5 \times H^5$, where dS_5 is the five dimensional de-Sitter and the H^5 is the five dimensional hyperboloid. The corresponding worldsheet theory is non-unitary, nevertheless the later is a solution to the type IIB supergravity equations of motion with a imaginary five form flux. Infact more recently this problem of non-unitarity was resolved in [23] in a background known as mirror space [24] which is formally related to $dS_5 \times H^5$ by a double T-duality. The mirror sigma model, which is derived by taking a double Wick rotation in the worldsheet theory of light cone $AdS_5 \times S^5$, inherits the symmetries, and the integrability of the light cone $AdS_5 \times S^5$ sigma model, and is well-behaved. The corresponding metric admits a lift to a full solution of the standard type IIB supergravity as well. There exists also the so called "imaginary deformation" $\eta = i$, in which the total 10d metric transforms into a pp-wave like background having a curved transverse part. In [22], using a related parameter $\varkappa$,[2] the $6d$ and $4d$ reductions of the total 10d metric have been performed and $(AdS_3 \times S^3)_\varkappa$ and $(AdS_2 \times S^2)_\varkappa$ backgrounds have been proposed. The integrability of the $6d$ and $4d$ stems from the fact that the original $10d$ spacetime is integrable. Knowing the integrability of the sigma model in the limiting background, it is interesting to explore the dual gauge theories and the corresponding stringy states in the string theory side. It is notable that the deformed 10d background breaks the $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$ symmetry of $AdS_5 \times S^5$ into $[\mathrm{U}(1)]^6$, making the dual field theory description obscure. Of course the role of this deformation parameter $\varkappa$ in the dual CFT is not clear at all. In order to know more about the boundary field theory, it is imperative to investigate various rotating and pulsating strings in the gravity side and then look for the relevant operators. In this connection, in a subspace of the deformed $AdS_5 \times S^5$, the so called giant magnon solution was proposed and the magnon excitation energy was computed in [25]. In the deformed parameter $\varkappa \to 0$ limit, it reduces to the form of usual giant magnon dispersion relation proposed by Hofman-Maldacena (HM) [26]. The HM giant magnon dispersion relation was derived in the string theory side by looking at the rigidly rotating strings in the $AdS_5 \times S^5$.

General rotating and pulsating string solutions in the semiclassical limit have been very useful in understanding the AdS/CFT like dualities in various backgrounds. The semiclassical calculations in the string theory side has shown that the multi spin rotating and pulsating string solutions beyond their BPS limit with large charges are in perfect agreement with the ones calculated in dual gauge theory. The pulsating strings in general correspond to the highly excited sigma model operators. In this connection a large number

[2] $\varkappa = \frac{2\eta}{1-\eta^2}$ [17], where $\varkappa \in [0, \infty)$.

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of rotating and pulsating string solutions have been studied in various string theory backgrounds, see for example, [27–39]. Motivated by the recent surge of interest in the string spectrum of deformed $AdS_5 \times S^5$ background that in general preserves the integrability, here we find a class of rotating and pulsating strings in this background [22]. We solve for the most general ansatz for the rotating strings in various subspaces of $\varkappa$-deformed $AdS_3 \times S^3$ and look for classical solutions. We construct two classes of solutions corresponding to the giant magnon [25, 40] and single spike solutions from the equations of motion of a fundamental string in this deformed background.[3] Single Spike strings are a general class of solutions corresponding to the higher twist operators in gauge theory [27]. Furthermore it was also shown that the giant magnon can be thought of as a subclass of more general solutions in the short wavelength limit. Infact in [35] it was found out that both the giant magnon and single spike solutions of the string on the sphere can be thought of as two limiting solutions.

Rest of the paper is organized as follows. In section-2, we will study the rotating strings in the $\varkappa$-deformed $AdS_3 \times S^3$ background. As explained in [22] this background reduced from the $\varkappa$-deformed $AdS_5 \times S^5$ [17] does not contain a NS-NS B-field even though the original space has it. We find two classes of solutions corresponding to the known giant magnon and new single spike solutions of the F-string equations of motion in two different subspaces of the $\varkappa$-deformed $AdS_3 \times S^3$. We write the relevant dispersion relations and check that in $\varkappa \to 0$ limit they do reduce to the known dispersion relations. We also provide some additional comments here. Section-3 is devoted to the study of string solutions which is pulsating in $S^3_\varkappa$. In section-4 we conclude with some remarks.

2 Rigidly rotating strings in $\varkappa$-deformed $AdS_3 \times S^3$

We are interested in the deformed $AdS_3 \times S^3$ metric proposed in [22] (a consistent reduction from the deformed $AdS_5 \times S^5$ [17]). It is given by

$$ds^2 = -h(\rho)dt^2 + f(\rho)d\rho^2 + \rho^2 d\psi^2 + \tilde{h}(r)d\varphi^2 + \tilde{f}(r)dr^2 + r^2 d\phi^2\,, \tag{2.1}$$

where

$$\begin{aligned} h(\rho) &= \frac{1+\rho^2}{1-\varkappa^2\rho^2}, \quad f(\rho) = \frac{1}{(1+\rho^2)(1-\varkappa^2\rho^2)} \\ \tilde{h}(r) &= \frac{1-r^2}{1+\varkappa^2 r^2}, \quad \tilde{f}(r) = \frac{1}{(1-r^2)(1+\varkappa^2 r^2)}\,, \end{aligned} \tag{2.2}$$

and the NS-NS two form B field vanishes. We are interested in studying rigidly rotating and pulsating strings in this background in two different planes of rotation when the motion is restricted to $R_t \times S^3_\varkappa$ only which can be obtained by putting $\rho = 0$.[4] The relevant metric is written as

$$ds^2 = -dt^2 + \frac{1-r^2}{1+\varkappa^2 r^2}d\varphi^2 + \frac{1}{(1-r^2)(1+\varkappa^2 r^2)}dr^2 + r^2 d\phi^2\,. \tag{2.3}$$

[3] Finite size giant magnon solution in η deformed $AdS_5 \times S^5$ has been investigated in [41].

[4] this has been argued by solving the equations of motion of a fundamental string in the given background in [40].

2.1 String rotating in (θ, φ) plane

We start by putting ϕ = constant and $r = \cos\theta$ in (2.3) to get the 2d subspace which is just the 1-parameter $\varkappa$-deformed S^2 solution presented in [22] as

$$ds^2 = -dt^2 + \frac{1}{1+\varkappa^2\cos^2\theta}d\theta^2 + \frac{\sin^2\theta}{1+\varkappa^2\cos^2\theta}d\varphi^2 \tag{2.4}$$

Our starting point is the Polyakov action of the string in this $R_t \times S_\varkappa$-deformed background

$$S = -\frac{\hat{T}}{2}\int d\sigma d\tau[\sqrt{-\gamma}\gamma^{\alpha\beta}g_{MN}\partial_\alpha X^M\partial_\beta X^N], \tag{2.5}$$

where $\gamma^{\alpha\beta}$ is the world-sheet metric and $\hat{T} = T\sqrt{1+\varkappa^2}$ is the effective string tension [17]. Under conformal gauge (i.e. $\sqrt{-\gamma}\gamma^{\alpha\beta} = \eta^{\alpha\beta}$) with $\eta^{\tau\tau} = -1$, $\eta^{\sigma\sigma} = 1$ and $\eta^{\tau\sigma} = \eta^{\sigma\tau} = 0$. Here we have to mention that when the deformation parameter $\varkappa$ has a general value, it is not known whether the total 10d background satisfies full type IIB supergravity field equations. Specifically the RR fluxes and the nontrivial dilaton are not exactly known in this case. Indeed, it was shown in [22] that the full $(AdS_3 \times S^3)_\varkappa$ admits a dilaton and RR 3-form fluxes in the $\varkappa \to i$ limit. But since we are using the conformal gauge, the worldsheet scalar curvature $R^{(2)} = 0$, so that the dilaton will not affect the worldsheet action in general.

Variation of the action (2.5) with respect to X^M gives us the following equation of motion

$$2\partial_\alpha(\eta^{\alpha\beta}\partial_\beta X^N g_{KN}) - \eta^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\partial_K g_{MN} = 0, \tag{2.6}$$

and variation with respect to the metric gives the two Virasoro constraints,

$$\begin{aligned} g_{MN}(\partial_\tau X^M\partial_\tau X^N + \partial_\sigma X^M\partial_\sigma X^N) &= 0, \\ g_{MN}(\partial_\tau X^M\partial_\sigma X^N) &= 0. \end{aligned} \tag{2.7}$$

To study rotating strings in this background we use the following ansatz

$$t = \mu\tau,\ \ \theta = \theta(y),\ \ \varphi = \omega(\tau + h(y)),\ \ \ y = \sigma - v\tau. \tag{2.8}$$

The equation of motion for φ gives

$$\partial_y h = \frac{1}{1-v^2}\left[\frac{A_1(1+\varkappa^2\cos^2\theta)}{\sin^2\theta} - v\right]. \tag{2.9}$$

A_1 here is an integration constant. Putting the above equation into the equation of motion for θ we get

$$\begin{aligned} (1-v^2)^2(\frac{\partial\theta}{\partial y})^2 = &- \omega^2\left[\frac{A_1^2(1+\varkappa^2\cos^2\theta)^2}{\sin^2\theta} + \sin^2\theta\right] \\ &+ A_2(1+\varkappa^2\cos^2\theta). \end{aligned} \tag{2.10}$$

Since we are interested in the infinite J magnon, we put the boundary condition when $\theta = \theta_{\rm max} = \pi/2$, $\frac{\partial\theta}{\partial y} \to 0$. This means $A_2 = \omega^2(A_1^2+1)$. Now putting these values back into the θ equation of motion, we recover

$$(1-v^2)^2\left(\frac{\partial\theta}{\partial y}\right)^2 = \omega^2(1+\varkappa^2)\cot^2\theta[\sin^2\theta(1+A_1^2\varkappa^2) - A_1^2(1+\varkappa^2)]\,, \tag{2.11}$$

which leads to the equation

$$\frac{\partial\theta}{\partial y} = \frac{\omega\sqrt{(1+\varkappa^2)(1+A_1^2\varkappa^2)}\cot\theta}{(1-v^2)}\sqrt{\sin^2\theta - \sin^2\theta_0}\,, \tag{2.12}$$

where

$$\sin^2\theta_0 = \frac{A_1^2(1+\varkappa^2)}{1+A_1^2\varkappa^2}\ . \tag{2.13}$$

Again subtracting the two Virasoro constraints we get a relation between the constants

$$\mu^2 - \frac{\omega^2 A_1}{v} = 0\ . \tag{2.14}$$

Also by equating the θ equation with the first Virasoro constraint we get the equation for A_1 as

$$A_1^2 - A_1\frac{1+v^2}{v} + 1 = 0, \tag{2.15}$$

the solutions of which gives the values of A_1 consistent with the Virasoro constraints. We can see the roots of the above equation correspond to two different limiting solutions which we identify as

$$\begin{aligned} A_1 &= v \qquad \text{magnon case} \\ &= \frac{1}{v} \qquad \text{single spike case} \end{aligned} \tag{2.16}$$

Now the symmetry of the background gives rise to the following conserved charges

$$\begin{aligned} E &= -\int\frac{\partial\mathcal{L}}{\partial\dot{t}}d\sigma \\ &= \frac{\hat{T}}{\sqrt{(1+\varkappa^2)(1+A_1^2\varkappa^2)}}\frac{\mu(1-v^2)}{\omega}\int\frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_0)}} \\ J_\varphi &= \int\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}d\sigma = \hat{T}\int\dot{\varphi}\ g_{\varphi\varphi}\ d\sigma \\ &= \frac{\hat{T}}{\sqrt{(1+\varkappa^2)(1+A_1^2\varkappa^2)}}(1-vA_1)\int\frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_0)}} \\ &\quad - \hat{T}\sqrt{\frac{(1+\varkappa^2)}{(1+A_1^2\varkappa^2)}}\int\frac{\sin\theta\cos\theta\ d\theta}{(1+\varkappa^2\cos^2\theta)\sqrt{(\sin^2\theta-\sin^2\theta_0)}}. \end{aligned} \tag{2.17}$$

The angle deficit in this case can be defined by

$$\begin{aligned} \Delta\varphi &= \omega\int\frac{\partial h}{\partial y}dy \\ &= \frac{1}{\sqrt{(1+\varkappa^2)(1+A_1^2\varkappa^2)}}\int\left(\frac{A_1(1+\varkappa^2\cos^2\theta)}{\sin^2\theta} - v\right)\frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_0)}}\ . \end{aligned} \tag{2.18}$$

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Now we wish to find the relation between the above Noether charges in the limits mentioned in (2.16) using some particular regularization scheme to avoid divergences.

2.1.1 Giant magnon solution

Using the limit $A_1 = v$ we evaluate the charges as explained above. The expression for energy

$$E = 2\frac{\hat{T}}{\sqrt{(1+\varkappa^2)(1+v^2\varkappa^2)}}\frac{\mu(1-v^2)}{\omega}\int_{\theta_0}^{\pi/2}\frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_0)}} \tag{2.19}$$

diverges in the upper limit. While the angular momenta looks like

$$\begin{aligned} J_\varphi =& \frac{2\hat{T}}{\sqrt{(1+\varkappa^2)(1+v^2\varkappa^2)}}(1-v^2)\int_{\theta_0}^{\pi/2}\frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_0)}} \\ &- 2\hat{T}\sqrt{\frac{(1+\varkappa^2)}{(1+v^2\varkappa^2)}}\int_{\theta_0}^{\pi/2}\frac{\sin\theta\cos\theta\ d\theta}{(1+\varkappa^2\cos^2\theta)\sqrt{(\sin^2\theta-\sin^2\theta_0)}}, \end{aligned} \tag{2.20}$$

which is also divergent evidently due to the first integral. However the difference $\frac{\omega E}{\mu} - J_\varphi$ remains finite. This can be easily seen by writing explicitly,

$$\begin{aligned} \tilde{E} - J_\varphi =& 2\hat{T}\sqrt{\frac{(1+\varkappa^2)}{(1+v^2\varkappa^2)}}\int_{\theta_0}^{\pi/2}\frac{\sin\theta\cos\theta\ d\theta}{(1+\varkappa^2\cos^2\theta)\sqrt{(\sin^2\theta-\sin^2\theta_0)}} \\ =& \frac{2\hat{T}}{\varkappa}\sqrt{\frac{(1+\varkappa^2)}{(1+v^2\varkappa^2)(1+\varkappa^2\cos^2\theta_0)}}\tanh^{-1}\left(\frac{\varkappa\cos\theta_0}{\sqrt{1+\varkappa^2\cos^2\theta_0}}\right). \end{aligned} \tag{2.21}$$

Here $\tilde{E} = \frac{\omega E}{\mu}$ is the re-scaled energy.[5] We can also evaluate the deficit angle, which is a finite quantity in this limit, as

$$\begin{aligned} \Delta\varphi =& \frac{2v\sqrt{1+\varkappa^2}}{\sqrt{1+v^2\varkappa^2}}\int_{\theta_0}^{\pi/2}\frac{\cos\theta\ d\theta}{\sin\theta\sqrt{(\sin^2\theta-\sin^2\theta_0)}} \\ =& \frac{2v\sqrt{1+\varkappa^2}}{\sqrt{1+v^2\varkappa^2}}\frac{\cos^{-1}(\sin\theta_0)}{\sin\theta_0} \\ =& 2\cos^{-1}(\sin\theta_0). \end{aligned} \tag{2.22}$$

Here, we have used the value of $\sin\theta_0$ from (2.13). So using the above expression, we can readily see that the finite difference (2.21) takes the form

$$\tilde{E} - J_\varphi = \frac{2\hat{T}}{\varkappa}\tanh^{-1}\left(\frac{\varkappa|\sin(\frac{\Delta\varphi}{2})|}{\sqrt{1+\varkappa^2\sin^2(\frac{\Delta\varphi}{2})}}\right), \tag{2.23}$$

Which is the giant magnon dispersion relation presented in [40]. It can be shown that under a $\varkappa \to 0$ limit the above expression reduces to the form of HM giant magnon on $\mathbf{R}\times S^2$

$$\lim_{\varkappa\to 0}\ \tilde{E} - J_\varphi = 2T\sin\left(\frac{\Delta\varphi}{2}\right) = \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{\Delta\varphi}{2}\right). \tag{2.24}$$

[5]Here one can note that from (2.14) the scaling factor $\frac{\omega}{\mu}$ can be safely put to be 1 in this case.

Further it has also been noted in [40] that the giant magnon dispersion relation (2.23) can be shown to agree with the magnon excitation energy calculated in [25], by using the relation between η and $\varkappa$ defined in [17].

2.1.2 Single spike solution

In the opposite limit solution we put $A_1 = \frac{1}{v}$, so the expression for energy remains the same

$$E = \frac{2\hat{T}}{\sqrt{(1+\varkappa^2)(1+\frac{\varkappa^2}{v^2})}} \frac{\mu(1-v^2)}{\omega} \int_{\pi/2}^{\theta_0} \frac{\sin\theta \; d\theta}{\cos\theta\sqrt{(\sin^2\theta - \sin^2\theta_0)}}. \tag{2.25}$$

The angular momentum J_φ now becomes finite as

$$\begin{aligned} J_\varphi &= -2\hat{T}\sqrt{\frac{(1+\varkappa^2)}{(1+\frac{\varkappa^2}{v^2})}} \int_{\pi/2}^{\theta_0} \frac{\sin\theta\cos\theta \; d\theta}{(1+\varkappa^2\cos^2\theta)\sqrt{(\sin^2\theta - \sin^2\theta_0)}} \\ &= \frac{2\hat{T}}{\varkappa}\tanh^{-1}\left(\frac{\varkappa\cos\theta_0}{\sqrt{1+\varkappa^2\cos^2\theta_0}}\right). \end{aligned} \tag{2.26}$$

We can show that in the required limit the above reduces to the known value as

$$\lim_{\varkappa\to 0} J_\varphi = 2T\cos\theta_0. \tag{2.27}$$

In this case we can also write the expression for deficit angle as

$$\begin{aligned} \Delta\varphi = &\frac{2}{\sqrt{(1+\varkappa^2)(1+\frac{\varkappa^2}{v^2})}}\left[\left(\frac{1}{v} - v\right)\int_{\pi/2}^{\theta_0} \frac{\sin\theta \; d\theta}{\cos\theta\sqrt{(\sin^2\theta - \sin^2\theta_0)}}\right. \\ &\left. + \frac{1+\varkappa^2}{v}\int_{\pi/2}^{\theta_2} \frac{\cos\theta \; d\theta}{\sin\theta\sqrt{(\sin^2\theta - \sin^2\theta_0)}}\right]. \end{aligned} \tag{2.28}$$

This is now a divergent quantity, but we can subtract the divergent integral by combining it with E as follows

$$\begin{aligned} \bar{E} - \hat{T}\Delta\varphi &= \frac{\omega E}{\mu v} - \hat{T}\Delta\varphi \\ &= -\frac{2\hat{T}}{v}\frac{\sqrt{1+\varkappa^2}}{\sqrt{1+\frac{\varkappa^2}{v^2}}}\int_{\pi/2}^{\theta_0} \frac{\cos\theta \; d\theta}{\sin\theta\sqrt{(\sin^2\theta - \sin^2\theta_0)}} \\ &= \frac{2\hat{T}}{v}\frac{\sqrt{1+\varkappa^2}}{\sqrt{1+\frac{\varkappa^2}{v^2}}}\frac{\cos^{-1}(\sin\theta_0)}{\sin\theta_0} \\ &= 2\hat{T}\left(\frac{\pi}{2} - \theta_0\right), \end{aligned} \tag{2.29}$$

which is analogous to the usual spike height relation, where $\bar{\theta} = (\frac{\pi}{2} - \theta_0)$ denotes the height of the spike.[6] This relation evidently reduces to the usual spike height relation [35] in the $\varkappa \to 0$ limit.

[6] Again we note that the scaling factor $\frac{\omega}{\mu v}$ can be put to be unity from (2.14) in this case.

Before we close this section, let us make some remarks about the fate of our solution in the two different limits of the background discussed in [22] i.e. i) $\varkappa = i$ and ii) $\varkappa = \infty$. As explained in [22], in the first limit, together with scaling of the coordinates, transforms the background to a pp-wave geometry with the transverse part having a non zero curvature. However the limit has to be taken cautiously, because naively taking $\varkappa \to i$ will make the effective string tension $\hat{T}$ vanish. For this purpose we can follow the recipe pointed out in [22] and take the limit in non trivial way as

$$\varkappa^2 = -1 + s\epsilon^2, \;\; t = \frac{x^+}{\epsilon} - \epsilon x^-, \;\; \varphi = \frac{x^+}{\epsilon} + \epsilon x^- \tag{2.30}$$

where $\epsilon \to 0$ and s can be put to 1 after taking the limit without loss of generality. For the simple deformed $(\mathbf{R} \times S^2)$ case $((\theta, \varphi)$ plane) we can find the pp-wave type metric as

$$ds^2 = 4dx_+ dx_- - \sinh^2\beta (dx_+)^2 + d\beta^2, \tag{2.31}$$

where we have used the transformation $r = \tanh\beta$. Evidently when $\sinh\beta = \beta$, this reduces to the usual pp-wave metric for AdS backgrounds, which have been extensively studied. However, studying string solutions in this type of pp-wave backgrounds with general rotating string ansatz as proposed in (2.8) may not be useful. However, one can study simpler solutions like the "straight strings" in the AdS-pp wave background as studied in, for example, [43].

On the other hand $\varkappa = \infty$ limit relates the $AdS_5 \times S^5$ metric to the T-dual of double wick rotated version of it, i.e. $dS_5 \times H^5$. As mentioned earlier, this background is a solution of IIB supergravity equations supported by imaginary 5-form flux. Again, naively taking the $\varkappa \to \infty$ limit on the dispersion relations will not be meaningful. To proceed further, we start with the limiting background derived from $AdS_2 \times S^2$ [22]

$$\varkappa^2 ds^2 = -\frac{d\widetilde{\rho}^2}{1+\widetilde{\rho}^2} + (1+\widetilde{\rho}^2)dt^2 + \frac{d\widetilde{r}^2}{\widetilde{r}^2 - 1} + (\widetilde{r}^2 - 1)d\varphi^2, \tag{2.32}$$

where $\widetilde{r} = \frac{1}{r}$ and $\widetilde{\rho} = \frac{1}{\rho}$. This geometry corresponds to that of $dS_2 \times H^2$ without any need of T-duality, albeit upto an overall $\varkappa^2$ factor. Here, $\widetilde{\rho}$ acts like the new time coordinate. String solutions in this kind of geometry has to be understood in a better way.

2.2 String rotating in (θ, ϕ) plane

We start with putting $r = \sin\theta$ in the metric (2.1) and using the general ansatz

$$t = \mu\tau, \quad \theta = \theta(y), \quad \phi = \omega(\tau + g(y)), \quad \varphi = constant, \quad y = \sigma - v\tau \tag{2.33}$$

The choice can be justified by the equation of motion for φ

$$\partial_\alpha [g_{\varphi\varphi}\, \eta^{\alpha\beta} \partial_\beta \varphi] = 0\,, \tag{2.34}$$

which is satisfied by a constant value of φ. For this geometry the string equation of motion are written as

$$\partial_y g = \frac{1}{1-v^2}\left[\frac{C_1}{\sin^2\theta} - v\right]\,. \tag{2.35}$$

Here C_1 is the integration constant. Using these two above equations and putting them into the θ equation of motion we get,

$$(1-v^2)^2\left(\frac{\partial\theta}{\partial y}\right)^2 = -\,\omega^2\left[\frac{C_1^2(1+\varkappa^2\sin^2\theta)}{\sin^2\theta}+\sin^2\theta(1+\varkappa^2\sin^2\theta)\right] + C_2(1+\varkappa^2\sin^2\theta) \tag{2.36}$$

Again we would like to investigate the infinite J string states, so we can put the boundary condition when $\theta=\theta_{\max}=\pi/2$ $\quad\frac{\partial\theta}{\partial y}\to 0$. This means $C_2=\omega^2(C_1^2+1)$. Now putting these values into the θ equation of motion, we recover

$$(1-v^2)^2\left(\frac{\partial\theta}{\partial y}\right)^2 = \cot^2\theta[(\omega^2-\omega^2C_2^2\varkappa^2)\sin^2\theta+\omega^2\varkappa^2\sin^4\theta-\omega^2C_1^2]\,, \tag{2.37}$$

which leads to the equation

$$\frac{\partial\theta}{\partial y}=\frac{\omega\varkappa\cot\theta}{(1-v^2)}\sqrt{[(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)]}\,. \tag{2.38}$$

Here, we can note that the roots are $\sin^2\theta_1=-\frac{1}{\varkappa^2}<0$ and $\sin^2\theta_2=C_1^2>0$. In this case the existence of a negative root can argued as in [42] for a large J expansion. Since we have already put one of the roots to be $\pi/2$ we would expect the positive solution $\sin^2\theta_2\in(0,1)$ which will be justified by the string state solutions in question. The Virasoro constraints on the other hand will again lead to two limiting values of C_1 as before, which will lead to the two independent solutions, namely giant magnon and single spiky string solution. Now looking at the isometries of the metric (2.3) we can write the conserved charges in this background as follows

$$\begin{aligned}
E &= -\int\frac{\partial\mathcal{L}}{\partial\dot t}d\sigma\\
&= \frac{\hat T}{\varkappa}\frac{\mu(1-v^2)}{\omega}\int\frac{\sin\theta\;d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}\,,\\
J_\phi &= \int\frac{\partial\mathcal{L}}{\partial\dot\phi}d\sigma=\hat T\int\dot\phi\sin^2\theta\;d\sigma\,,\\
&= \frac{\hat T}{\varkappa}[(1-vC_1)\int\frac{\sin\theta\;d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}\,,\\
&\quad -\int\frac{\sin\theta\cos\theta\;d\theta}{\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}].
\end{aligned} \tag{2.39}$$

We can also define the angle deficit as

$$\begin{aligned}
\Delta\phi &= \omega\int\frac{\partial g}{\partial y}dy\\
&= \frac{1}{\varkappa}\int\left(\frac{C_1}{\sin^2\theta}-v\right)\frac{\sin\theta\;d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}.
\end{aligned} \tag{2.40}$$

Using these conserved quantities we can investigate the relationship between them for the two limits mentioned before.

2.2.1 Giant magnon solution

Using $C_1 = v$, we evaluate the conserved charges using the integrals mentioned in the appendix. The expression for energy

$$\tilde{E} = E\frac{\omega}{\mu} = \frac{2\hat{T}}{\varkappa}(1-v^2)\int_{\theta_2}^{\pi/2} \frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta - \sin^2\theta_1)(\sin^2\theta - \sin^2\theta_2)}}, \tag{2.41}$$

which diverges. Also in this limit the angular momenta

$$\begin{aligned} J_\phi = \frac{2\hat{T}}{\varkappa}\Bigg[&(1-v^2)\int_{\theta_2}^{\pi/2} \frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta - \sin^2\theta_1)(\sin^2\theta - \sin^2\theta_2)}} \\ &- \int_{\theta_2}^{\pi/2} \frac{\sin\theta\cos\theta\ d\theta}{\sqrt{(\sin^2\theta - \sin^2\theta_1)(\sin^2\theta - \sin^2\theta_2)}}\Bigg] \end{aligned} \tag{2.42}$$

diverge due to the first integral. However the difference $\tilde{E} - J_\phi$ remains finite and can be evaluated to be

$$\tilde{E} - J_\phi = \frac{2\hat{T}}{\varkappa}\tanh^{-1}(\frac{\cos\theta_2}{\cos\theta_1}). \tag{2.43}$$

However the angular deficit is finite in this case and can be written as

$$\begin{aligned} \Delta\phi &= \frac{2v}{\varkappa}\int_{\theta_2}^{\pi/2} \frac{\cos\theta\ d\theta}{\sin\theta\sqrt{(\sin^2\theta - \sin^2\theta_1)(\sin^2\theta - \sin^2\theta_2)}} \\ &= \frac{2v}{\varkappa\sin\theta_1\sin\theta_2}\tanh^{-1}\left(\frac{\sin\theta_1\cos\theta_2}{\cos\theta_1\sin\theta_2}\right). \end{aligned} \tag{2.44}$$

Putting the values of $\sin\theta_1$ and $\sin\theta_2$ into the above equation, we can see it translates to

$$\sin\left(\frac{\Delta\phi}{2}\right) = \frac{\sqrt{1-v^2}}{\sqrt{1+v^2\varkappa^2}}. \tag{2.45}$$

Now it can be easily proved that the conserved charges in this case obey the dispersion relation as

$$\tilde{E} - J_\phi = \frac{2\hat{T}}{\varkappa}\tanh^{-1}\left(\frac{\varkappa|\sin(\frac{\Delta\phi}{2})|}{\sqrt{1+\varkappa^2\sin(\frac{\Delta\phi}{2})}}\right), \tag{2.46}$$

which is the same dispersion relation as mentioned in [40]. We can take a $\varkappa \to 0$ limit to get

$$\tilde{E} - J_\phi = 2T\sin\left(\frac{\Delta\phi}{2}\right) = \frac{\sqrt{\lambda}}{\pi}\sin\left(\frac{\Delta\phi}{2}\right), \tag{2.47}$$

which is indeed the giant magnon dispersion relation in $\mathbf{R}\times S^2$ as mentioned in [26].

2.2.2 Single spike solution

Using the other viable limit $C_1 = \frac{1}{v}$, we again evaluate the conserved charges as before. Here also the $\tilde{E}$ remains divergent as

$$\tilde{E} = \frac{2\hat{T}}{\varkappa}(1-v^2)\int_{\pi/2}^{\theta_2} \frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta - \sin^2\theta_1)(\sin^2\theta - \sin^2\theta_2)}}. \tag{2.48}$$

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While J_ϕ in this case is finite as

$$J_\phi = -\frac{2\hat{T}}{\varkappa}\int_{\pi/2}^{\theta_2}\frac{\sin\theta\cos\theta\ d\theta}{\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}$$
$$= \frac{2\hat{T}}{\varkappa}\tanh^{-1}\left(\frac{\cos\theta_2}{\cos\theta_1}\right). \tag{2.49}$$

On the other hand the angle deficit

$$\Delta\phi = \frac{2}{\varkappa}\Bigg[\left(\frac{1}{v}-v\right)\int_{\pi/2}^{\theta_2}\frac{\sin\theta\ d\theta}{\cos\theta\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}$$
$$+\frac{1}{v}\int_{\pi/2}^{\theta_2}\frac{\cos\theta\ d\theta}{\sin\theta\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}\Bigg] \tag{2.50}$$

diverges due to the first integral present here. Although it is easy to see that by combining, we get

$$\bar{E}-\hat{T}\Delta\phi = E\frac{\omega}{\mu v}-\hat{T}\Delta\phi$$
$$= -\frac{2\tilde{T}}{v\varkappa}\int_{\pi/2}^{\theta_2}\frac{\cos\theta\ d\theta}{\sin\theta\sqrt{(\sin^2\theta-\sin^2\theta_1)(\sin^2\theta-\sin^2\theta_2)}}$$
$$= \frac{2\tilde{T}}{v\varkappa\sin\theta_1\sin\theta_2}\tanh^{-1}\left(\frac{\sin\theta_1\cos\theta_2}{\cos\theta_1\sin\theta_2}\right). \tag{2.51}$$

As we can see the spike height (and hence the energy) in this case is modified non trivially in contrast to the previous case (2.29). Putting all the values and doing little algebra we can see that under the limit $\varkappa\to 0$ this relation again becomes

$$\bar{E}-T\Delta\phi = 2T\tan^{-1}\left(\frac{\cos\theta_2}{\sin\theta_2}\right)$$
$$= 2T\left(\frac{\pi}{2}-\theta_2\right), \tag{2.52}$$

which is the usual spike-height relation as mentioned in [35]. Even it can also be proved that under $\varkappa\to 0$ limit the expression for angular momentum becomes

$$J_\phi = 2T\cos\theta_2, \tag{2.53}$$

as we get usually for the $\mathbf{R}\times S^2$ case. Now, few comments are in order about the solutions derived above, mainly in the light of the recent work [22]. The background (2.3) has an unique property in the sense the ϕ and φ are related by a discrete Z_2 symmetry [22] which is manifested as

$$\phi\to\varphi;\ \ r\to\sqrt{\frac{1-r^2}{1+\varkappa^2 r^2}}. \tag{2.54}$$

Surely, We can see that these symmetries are quite evident by the form of equations (2.23) and (2.46) as these two giant magnon dispersion relations are simply connected by the change $\phi\to\varphi$. For the single spike solutions, from equations (2.29) and (2.52), it can be

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seen that the height of the spike solutions depend on the two angles θ_0 and θ_2. Further, keeping in mind that we have put $r = \sin\theta$ for the solutions in (θ, ϕ) plane and $r = \cos\theta$ for the solutions in (θ, φ) plane, we can relate the angles θ_0 and θ_2 as

$$r_2 = \sin\theta_2 \to r_0 = \cos\theta_0 = \sqrt{\frac{1 - r_2^2}{1 + \varkappa^2 r_2^2}}, \tag{2.55}$$

since $\sin\theta_2 = \frac{1}{v}$ and $\cos\theta_0 = \sqrt{\frac{v^2-1}{v^2+\varkappa^2}}$ with the appropriate choice of constants for the spiky string cases as expalined in the previous subsections. This, coupled with $\phi \to \varphi$ relates (2.29) and (2.52) explicitly. These symmetries have been used extensively in [22] to relate the $\varkappa$ deformed model to other classically integrable models.

3 Pulsating strings in deformed R × S^3

In this section, we wish to study a class of strings pulsating in full deformed S^3 with an extra angular momentum. Beginning with the metric (2.1) and putting $r = \cos\theta$ we get the full deformed $\mathbf{R} \times S^3$ metric as

$$ds^2 = -dt^2 + \frac{1}{1 + \varkappa^2 \cos^2\theta} d\theta^2 + \frac{\sin^2\theta}{1 + \varkappa^2 \cos^2\theta} d\varphi^2 + \cos^2\theta d\phi^2. \tag{3.1}$$

The Polyakov action for the string in the above background is given by

$$S = \frac{\hat{T}}{2}\int d\tau d\sigma \Big[-(\dot{t}^2 - t'^2) + \frac{1}{1 + \varkappa^2\cos^2\theta}(\dot{\theta}^2 - \theta'^2) + \frac{\sin^2\theta}{1 + \varkappa^2\cos^2\theta}(\dot{\varphi}^2 - \varphi'^2) + \cos^2\theta(\dot{\phi}^2 - \phi'^2)\Big]. \tag{3.2}$$

We chose the following simple ansatz for studying the pulsating and rotating string in the above mentioned space

$$t = t(\tau), \qquad \theta = \theta(\tau), \qquad \varphi = m\sigma, \qquad \phi = \phi(\tau). \tag{3.3}$$

Instead of solving the equation of motion, we concentrate on the Virasoro constraint

$$g_{MN}(\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N) = 0, \tag{3.4}$$

which gives us the following equation for the evolution of θ as

$$-\dot{t}^2 + \frac{1}{1 + \varkappa^2\cos^2\theta}\dot{\theta}^2 + \frac{\sin^2\theta}{1 + \varkappa^2\cos^2\theta} m^2 + \cos^2\theta\dot{\phi}^2 = 0 \tag{3.5}$$

The conserved Noether charges are given by

$$\mathcal{E} = \dot{t}, \qquad \mathcal{J} = \cos^2\theta\dot{\phi}. \tag{3.6}$$

Putting the above in 3.5 we get the equation

$$\dot{\theta}^2 = (1 + \varkappa^2\cos^2\theta)\mathcal{E}^2 - m^2\sin^2\theta - \frac{(1 + \varkappa^2\cos^2\theta)\mathcal{J}^2}{\cos^2\theta}. \tag{3.7}$$

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The above equation looks like the equation of motion of a particle in a classical potential. We note that the effective potential is finite at $\theta = 0$ while it diverges at $\theta = \pi/2$. So in a classical perspective this is the equation of a particle oscillating between $\theta = 0$ to some extremum value. We then define the oscillation number, which is an adiabatic invariant for the system, as

$$\mathcal{N} = \frac{1}{2\pi} \oint d\theta \sqrt{(1+\varkappa^2\cos^2\theta)\mathcal{E}^2 - m^2\sin^2\theta - \frac{(1+\varkappa^2\cos^2\theta)\mathcal{J}^2}{\cos^2\theta}}. \tag{3.8}$$

Putting $\sin\theta = x$ in the above equation (3.8), we get

$$\mathcal{N} = \frac{1}{\pi}\int_0^{\sqrt{R}} \frac{dx}{1-x^2}\sqrt{\mathcal{E}^2(1-x^2)(1+\varkappa^2(1-x^2)) - m^2x^2(1-x^2) - \mathcal{J}^2(1+\varkappa^2(1-x^2))}, \tag{3.9}$$

where R is an appropriate root of the polynomial

$$g(x) = x^4(\varkappa^2\mathcal{E}^2 + m^2) + x^2(-2\varkappa^2\mathcal{E}^2 - m^2 - \mathcal{E}^2 + \varkappa^2\mathcal{J}^2) + (1+\varkappa^2)(\mathcal{E}^2 - \mathcal{J}^2). \tag{3.10}$$

Differentiating (3.9) w.r.t. m we can write

$$\frac{\partial\mathcal{N}}{\partial m} = -\frac{m}{\pi}\int_0^{\sqrt{R}} \frac{x^2}{\sqrt{(\mathcal{E}^2(1-x^2) - \mathcal{J}^2)(1+\varkappa^2(1-x^2)) - m^2x^2(1-x^2)}}dx. \tag{3.11}$$

Putting $x^2 = z$ we can write the above integral as follows

$$\frac{\partial\mathcal{N}}{\partial m} = -\frac{m}{\pi\sqrt{\varkappa^2\mathcal{E}^2 + m^2}}\int_0^{R_-} \frac{z\ dz}{\sqrt{z(z-R_+)(z-R_-)}}\,, \tag{3.12}$$

where the roots $R_\pm$ are the solutions of (3.10) with $z = x^2$. It can be worthily noted that the bigger root R_+ is greater than 1, hence not appropriate here. In this consideration, the above integral can be written as a combination of Elliptic integrals as follows,

$$\frac{\partial\mathcal{N}}{\partial m} = \frac{2m\sqrt{R_+}}{\pi\sqrt{\varkappa^2\mathcal{E}^2 + m^2}}\left[\mathbb{E}\Big(\frac{R_-}{R_+}\Big) - \mathbb{K}\Big(\frac{R_-}{R_+}\Big)\right] \tag{3.13}$$

$\mathbb{E}$ and $\mathbb{K}$ are the known Elliptic integral of first and second kind respectively. Now to find a relation between Energy and Angular momenta via the oscillation number, we expand the above equation in small $\mathcal{E}$ and $\mathcal{J}$ limit to get

$$\begin{aligned}\frac{\partial\mathcal{N}}{\partial m} =& \left[\frac{(1+\varkappa^2)\mathcal{J}^2}{2m^2} - \frac{3(1+\varkappa^2)(5+\varkappa^2)\mathcal{J}^4}{16m^4} + \mathcal{O}(\mathcal{J}^6)\right] \\ &+ \left[-\frac{(1+\varkappa^2)}{2m^2} + \frac{(9+6\varkappa^2-3\varkappa^4)\mathcal{J}^2}{8m^4} - \frac{15(\varkappa^6 - 9\varkappa^4 - 45\varkappa^4 - 35)\mathcal{J}^4}{128m^6} + \mathcal{O}(\mathcal{J}^6)\right]\mathcal{E}^2 \\ &+ \left[\frac{3(-1+2\varkappa^2+3\varkappa^4)}{16m^4} + \frac{45(5+3\varkappa^2-\varkappa^4+\varkappa^6)\mathcal{J}^2}{128m^6}\right. \\ &\left. - \frac{105(105+140\varkappa^2+30\varkappa^4-4\varkappa^6+\varkappa^8)\mathcal{J}^4}{1024m^8} + \mathcal{O}(\mathcal{J}^6)\right]\mathcal{E}^4 + \mathcal{O}(\mathcal{E}^6).\end{aligned} \tag{3.14}$$

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Integrating the above equation (3.14) with respect to m and reversing the series to that of $\mathcal{E}$, we get a very long and complicated expression. But this expression under the limit $\varkappa \to 0$ can be shown to reduce to the expression

$$\mathcal{E} = \sqrt{2m\mathcal{G}} \;\; A(\mathcal{J}) \left[1 - B(\mathcal{J}) \frac{\mathcal{G}}{8m} + \mathcal{O}[\mathcal{G}^2]\right], \tag{3.15}$$

$$\begin{aligned} \text{with} \quad \mathcal{G} &= \mathcal{N} + \frac{\mathcal{J}^2}{2m} - \frac{5\mathcal{J}^4}{16m^3} + \mathcal{O}[\mathcal{J}^6] \\ A(\mathcal{J}) &= \left[1 - \frac{3\mathcal{J}^2}{4m^2} + \frac{105\mathcal{J}^4}{64m^4} + \mathcal{O}[\mathcal{J}^6]\right]^{-1/2} \\ \text{also,} \quad B(\mathcal{J}) &= \left[1 - \frac{45\mathcal{J}^2}{8m^2} + \frac{1575\mathcal{J}^4}{64m^4} + \mathcal{O}[\mathcal{J}^6]\right] A^4(\mathcal{J}). \end{aligned} \tag{3.16}$$

Which is the exact same relation as obtained recently in [44] for rotating and pulsating strings in the undeformed $\mathbf{R} \times S^3$. This small $\mathcal{J}$ limit is called the short string limit. The above equation presents the energy-spin relation of the strings in this limit.

4 Conclusion

In this paper, we have studied rigidly rotating and oscillating string solutions in the $\varkappa$-deformed $AdS_3 \times S^3$ background. For the rigidly rotating string, we have restricted the motion to two subspaces of the $S^3_\varkappa$ by reducing along two isometries. We have found two limits corresponding to the known giant magnon [25, 40] limit and the new single spike solution of the string moving in this background. We have found out the most general form of the charges and derived the dispersion relation among the charges and the angular separation between the end points of the string. We also have explicitly checked that in the limit $\varkappa \to 0$ limit they reduce to the well known relation presented, for example, in [35]. Since it is not understood how the parameter $\varkappa$ deforms the dual CFT, the realization of the newly found dispersion relations in the CFT are not clear to us at present. It would further be interesting to generalize the results presented here to the case of two angular momenta along the two isometries of the $S^3_\varkappa$. Unfortunately, the analysis appears to be tricky and we hope to present the results in a future work. Also it would be interesting enough to study string dynamics in the two limiting backgrounds mentioned in [22]. Since these can be related to other integrable models, they might give us more options to gain insight into the dual field theory. Further we study the oscillating strings in the $S^3_\varkappa$ space and derive the energy of pulsating string, as function of the oscillation number and conserved angular momentum. We also check that the relation reduces to the known one in the $\varkappa \to 0$ limit. There are other problems which could be pursued further given the integrable nature of this background and its reductions. It will be useful to find more general rotating and wound strings in the full $(AdS_5 \times S^5)_\varkappa$ of [17] as the AdS part contains a singularity for particular value of $\varkappa$. We hope to report on these issues in future.

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