

# Symmetric-group decomposition of $SU(N)$ group-theory constraints on four-, five-, and six-point color-ordered amplitudes at all loop orders

---

Alexander C. Edison and Stephen G. Naculich<sup>1</sup>

*Department of Physics, Bowdoin College,  
Brunswick, ME 04011, U.S.A.*

*E-mail:* [aedison@bowdoin.edu](mailto:aedison@bowdoin.edu), [naculich@bowdoin.edu](mailto:naculich@bowdoin.edu)

ABSTRACT: Color-ordered amplitudes for the scattering of  $n$  particles in the adjoint representation of  $SU(N)$  gauge theory satisfy constraints that arise from group theory alone. These constraints break into subsets associated with irreducible representations of the symmetric group  $S_n$ , which allows them to be presented in a compact and natural way. Using an iterative approach, we derive the constraints for six-point amplitudes at all loop orders, extending earlier results for  $n = 4$  and  $n = 5$ . We then decompose the four-, five-, and six-point group-theory constraints into their irreducible  $S_n$  subspaces. We comment briefly on higher-point two-loop amplitudes.

KEYWORDS: Scattering Amplitudes, Duality in Gauge Field Theories, Gauge Symmetry,  $1/N$  Expansion

ARXIV EPRINT: [1207.5511](https://arxiv.org/abs/1207.5511)

---

<sup>1</sup>Research supported in part by the National Science Foundation under Grant No. PHY10-67961.

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Trace bases for gauge-theory amplitudes</b>	<b>5</b>
<b>3</b>	<b>Projection operators for representations of <math>S_n</math></b>	<b>6</b>
3.1	Regular representation	7
3.2	Induced representations	8
<b>4</b>	<b>Constraints on four-point amplitudes</b>	<b>11</b>
4.1	$S_4$ decomposition of the four-point trace basis	11
4.2	Four-point null spaces	12
<b>5</b>	<b>Constraints on five-point amplitudes</b>	<b>14</b>
5.1	$S_5$ decomposition of the five-point trace basis	15
5.2	Five-point null spaces	16
<b>6</b>	<b>Constraints on six-point amplitudes</b>	<b>17</b>
6.1	$S_6$ decomposition of the six-point trace basis	18
6.2	Six-point null spaces	19
<b>7</b>	<b>Kleiss-Kuijf relations at higher loops?</b>	<b>29</b>
<b>8</b>	<b>Concluding remarks</b>	<b>30</b>
<b>A</b>	<b>Decomposition of trace representations</b>	<b>31</b>
<b>B</b>	<b>Five-point decomposition in an explicit trace basis</b>	<b>33</b>

---

## 1 Introduction

Scattering amplitudes in gauge theory may be given a gauge-invariant decomposition in terms of color-ordered (or partial) amplitudes [1, 2]. Color-ordered amplitudes are not independent but satisfy a number of constraints, some of which follow from the recently-discovered color-kinematic duality [3–5] and were proven in refs. [6–9]. Other constraints among the color-ordered amplitudes, however, follow directly from group theory, and have been known for over two decades at tree-level [10] and at one loop [11]. Four-point color-ordered amplitudes were also known to obey group-theory relations at two loops [12] and these were recently generalized to all loop orders [13]. All-loop-order group-theory constraints were also recently derived for five-point amplitudes [14]. Other recent work on loop-order relations among color-ordered amplitudes includes refs. [15–20].

In this paper, we derive the  $SU(N)$  group-theory constraints for six-point color-ordered amplitudes at all loop orders, following the iterative approach of refs. [13, 14] and generalizing the known relations at tree level [10] and at one loop [11]. The other major goal of this paper is to describe how group-theory constraints on  $n$ -point color-ordered amplitudes naturally fall into sets associated with irreducible representations of the symmetric group  $S_n$ . We decompose the constraints for  $n = 4, 5$ , and  $6$  into their irreducible  $S_n$  subspaces, thus presenting them in a compact way.

The derivation of group-theory constraints can be cast as a straightforward problem in linear algebra<sup>1</sup> that emerges from two alternative ways of expressing the color structure of a gauge theory amplitude. One way is to decompose the amplitude into a color basis [22, 23]

$$\mathcal{A} = \sum_i a_i c_i \tag{1.1}$$

where  $a_i$  carries the momentum- and polarization-dependence of the amplitude, and the color factors  $c_i$  are obtained by sewing together the gauge theory factors from all the vertices of the contributing Feynman diagrams. In a theory that contains only fields in the adjoint representation of  $SU(N)$ , such as pure or supersymmetric Yang-Mills theory, each cubic vertex contributes a factor of the  $SU(N)$  structure constants  $f^{abc}$ , and each quartic vertex contributes factors of  $f^{abe} f^{cde}$ ,  $f^{ace} f^{bde}$ , and  $f^{ade} f^{bce}$ , which are equivalent from a purely color point-of-view to a pair of cubic vertices sewn along one leg. Hence a complete set of color factors  $\{c_i\}$  can be constructed from diagrams with cubic vertices only. An independent basis of color factors<sup>2</sup> for tree-level and one-loop  $n$ -point amplitudes was described in refs. [22, 23].

The alternative trace decomposition [1, 2]

$$\mathcal{A} = \sum_\lambda A_\lambda t_\lambda \tag{1.2}$$

expresses the amplitude in terms of gauge-invariant color-ordered amplitudes  $A_\lambda$  with respect to a basis  $\{t_\lambda\}$  of single and (at loop level) multiple traces of gauge group generators  $T^a$  in the defining representation of  $SU(N)$ . We will characterize this basis more explicitly in section 2. The dimension of the trace basis, however, is larger than that of the independent color basis,<sup>3</sup> so there is redundancy among the color-ordered amplitudes.

The color (1.1) and trace (1.2) decompositions can be related by using<sup>4</sup>

$$\tilde{f}^{abc} = i\sqrt{2}f^{abc} = \text{Tr}([T^a, T^b]T^c), \tag{1.3}$$

---

<sup>1</sup>A similar approach was taken to the BCJ relations [3] in ref. [21].

<sup>2</sup>The color factors constructed from the set of all cubic diagrams are generally not independent but are related by Jacobi relations. Such an overcomplete set is usually required to make color-kinematic duality manifest [3, 24].

<sup>3</sup>For example, the number of independent color factors for tree-level  $n$ -point amplitudes is  $(n-2)!$  while the dimension of the trace basis is  $\frac{1}{2}(n-1)!$ .

<sup>4</sup>The generators are normalized using  $\text{Tr}(T^a T^b) = \delta^{ab}$ .

together with the  $SU(N)$  relations given later in eq. (6.20), to write each color factor  $c_i$  as a linear combination of trace factors

$$c_i = \sum_{\lambda} M_{i\lambda} t_{\lambda}. \tag{1.4}$$

Since the dimension of the trace space is larger than the number of independent color factors, the linear combinations given by eq. (1.4) span a proper subspace (which we will refer to as the *color space*) of the trace space. Consequently, the transformation matrix  $M_{i\lambda}$  possesses a set of independent null eigenvectors

$$\sum_{\lambda} M_{i\lambda} r_{\lambda m} = 0, \quad m = 1, \dots, n_{\text{null}} \tag{1.5}$$

whose number  $n_{\text{null}}$  is the difference between the dimensions of the trace space and the color space. The null vectors  $r_m = \sum_{\lambda} r_{\lambda m} t_{\lambda}$  are orthogonal to the color factors  $c_i$  with respect to the inner product

$$(t_{\lambda}, t_{\lambda'}) = \delta_{\lambda\lambda'} \tag{1.6}$$

and hence span the orthogonal complement of the color space, which we will refer to as the *null space*. One combines eq. (1.4) with eqs. (1.1) and (1.2) to express the color-ordered amplitudes as

$$A_{\lambda} = \sum_i a_i M_{i\lambda}. \tag{1.7}$$

The existence of the null eigenvectors (1.5) implies a set of constraints

$$\sum_{\lambda} A_{\lambda} r_{\lambda m} = 0, \quad m = 1, \dots, n_{\text{null}} \tag{1.8}$$

which we refer to as group-theory relations. Hence, specifying the null space is equivalent to specifying the complete set of group-theory relations satisfied by the color-ordered amplitudes.

For tree-level  $n$ -point amplitudes, the relations (1.8), which number  $\frac{1}{2}(n-3)(n-2)!$ , are the Kleiss-Kuijff [10, 23] relations. For one-loop  $n$ -point amplitudes, the group-theory relations were given in refs. [11, 23]. At two loops and above, the full set of constraints on  $n$ -point color-ordered amplitudes has yet to be identified,<sup>5</sup> although all-loop group-theory relations for  $n = 4$  and  $n = 5$  have been derived using an iterative approach in refs. [13, 14].

For *four-point amplitudes*, there are 4 group-theory relations at each loop order  $L \geq 2$  (with 1 at tree level, and 3 at one loop). The form of the constraints begins to repeat after three loops, so that the even-loop constraints for  $L \geq 4$  are essentially equivalent to the two-loop constraints, and the odd-loop constraints for  $L \geq 5$  are equivalent to the three-loop constraints [13].

For *five-point amplitudes*, there are 10 group-theory relations for odd  $L$ , and 12 for even  $L \geq 2$  (with 6 at tree level). Again the form of the constraints repeats after two loops, so that odd-loop constraints for  $L \geq 3$  are equivalent to the one-loop constraints, and even-loop constraints for  $L \geq 4$  are equivalent to the two-loop constraints [14].

---

<sup>5</sup>See refs. [12, 16] for partial results at two loops.

In this paper we will derive the group-theory relations for *six-point amplitudes* at all loop orders using the same iterative approach. We will find that there are 76 group-theory relations for  $L \geq 3$ , with 36 at tree level, 65 at one loop, and 80 at two loops. Once again the form of the constraints presents a repeating pattern, which only begins, however, after five loops, so that even-loop constraints for  $L \geq 6$  are equivalent to the four-loop constraints, and odd-loop constraints for  $L \geq 7$  are equivalent to the five-loop constraints.

A subset of the group-theory relations obeyed by color-ordered amplitudes can be derived as U(1) decoupling relations [2, 25–27]. To obtain these, one enlarges the gauge group from  $SU(N)$  to  $U(N)$ . The trace decomposition is expanded to include terms with factors of  $\text{Tr}(T^a)$  (which automatically vanish in the  $SU(N)$  trace decomposition). Next, one observes that an  $n$ -point amplitude containing any number of external U(1) gauge bosons vanishes because the associated structure constants are zero. By setting the corresponding  $U(N)$  generators in the full  $U(N)$  trace decomposition equal to the unit matrix, one obtains a series of equations that relate the remaining  $SU(N)$  amplitudes. These are the U(1) decoupling relations. Throughout the paper, we will indicate which of the group-theory relations can be obtained from U(1) decoupling relations and which cannot.

The group-theory relations at four and five points were presented in refs. [13, 14] by specifying a complete set of  $L$ -loop null vectors  $\{r_m^{(L)}\}$  in terms of their components  $r_{\lambda m}^{(L)}$  with respect to an explicit  $L$ -loop trace basis  $\{t_\lambda^{(L)}\}$ . While this approach has the advantage of explicitness, it quickly becomes unwieldy as the dimensions of both the null space and the trace space grow with  $n$ . For example, listing the 200 coefficients of the 80 null vectors for two-loop six-point amplitudes would hardly be enlightening. Clearly, a more streamlined approach is called for.

Such a compact approach for presenting the null vectors of  $n$ -point amplitudes is provided by the representation theory of the symmetric group  $S_n$ . Since a permutation of the arguments of an element of the trace basis  $t_\lambda$  for  $n$ -point amplitudes yields another element of the trace basis, the trace space constitutes a (reducible) representation of the symmetric group  $S_n$ . Similarly, a permutation of the external legs of a diagram corresponding to a color factor  $c_i$  yields a diagram corresponding to another color factor, hence the color space constitutes an invariant subspace (with respect to permutations) of the trace space. The orthogonal complement of the color space, namely the null space, is therefore also an invariant subspace, since the inner product (1.6) is invariant under permutations. Thus the null space forms a (reducible) representation of  $S_n$ , which can be decomposed into a direct sum of irreducible representations. Indeed, the null vectors for five-point amplitudes found in ref. [14] naturally broke into sets of 6 and 4, and correspond to irreducible representations of  $S_5$ . Thus, a compact and efficient way to characterize the null space is to describe its constituent irreducible representations.

In this paper, we will present the null spaces of four-, five-, and six-point amplitudes at all loop orders in terms of their irreducible subspaces with respect to  $S_n$ . We will find, for example, that the 80 vectors spanning the null space for two-loop six-point amplitudes can be classified into 13 irreducible representations whose dimensions vary from 1 to 16. In addition to compactness, this method has the advantage that the results are presented in a form independent of the specific choice of trace basis.

We also briefly examine whether the tree-level Kleiss-Kuijf relations also apply to subleading-color single-trace  $n$ -point amplitudes at loop level. We confirm the results of ref. [16] that at two loops, the KK relations hold for  $n \leq 7$ , but fail for eight-point amplitudes. We check that they fail for nine-point amplitudes as well.

We begin in section 2 with an explicit look at the trace basis for  $n$ -point amplitudes through two loops. In section 3, we review some representation theory of the symmetric group, showing how Young tableaux can be used to construct projection operators onto irreducible representations for regular and induced representations. In sections 4), (5, and 6, we describe the decomposition of the null spaces of four-, five-, and six-point amplitudes into irreducible representations of  $S_n$ . Section 7 discusses Kleiss-Kuijf relations at loop level, and section 8 contains our conclusions. Various additional details can be found in two appendices.

## 2 Trace bases for gauge-theory amplitudes

In this section, we spell out the explicit form of the trace basis for an  $n$ -point amplitude through two loops as a prelude to examining its decomposition into irreducible representations of  $S_n$  later in the paper.

First we consider tree-level  $n$ -point amplitudes  $\mathcal{A}_n^{(0)}$ . One can use eq. (1.3) together with  $\tilde{f}^{abc}T^a = [T^b, T^c]$  to decompose the color factor  $c_i$  built from any tree-level color diagram into a linear combination of single-trace terms, so that the amplitude may be expressed as [1]

$$\mathcal{A}_n^{(0)} = \sum_{\sigma \in S_n / \mathbb{Z}_n} A(\sigma(1), \dots, \sigma(n)) \text{Tr}(\sigma(1) \cdots \sigma(n)) \tag{2.1}$$

where  $\mathbb{Z}_n$  denotes the subgroup of cyclic permutations. For easier readability we write  $\text{Tr}(12 \cdots n)$  for  $\text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ . In addition to being invariant under cyclic permutations of their arguments (due to the cyclicity of the trace), color-ordered tree amplitudes are invariant (up to sign) under reversal of the arguments

$$A(\sigma(n), \dots, \sigma(2), \sigma(1)) = (-1)^n A(\sigma(1), \sigma(2), \dots, \sigma(n)) \tag{2.2}$$

as can be seen by using the antisymmetry of  $\tilde{f}^{abc}$ . Hence, the tree-level  $n$ -point amplitude can be written

$$\mathcal{A}_n^{(0)} = \sum_{\lambda=1}^{n_s} A_\lambda^{(0)} T_\lambda \tag{2.3}$$

in terms of a basis  $\{T_\lambda\}$  of dimension  $n_s = \frac{1}{2}(n-1)!$ , consisting of sums/differences of the form

$$\text{Tr}(12 \cdots n) + (-1)^n \text{Tr}(n \cdots 21) \tag{2.4}$$

and permutations thereof.

At one loop and above, the amplitude also contains double-trace terms

$$A(\sigma(1), \dots, \sigma(p); \sigma(p+1), \dots, \sigma(n)) \text{Tr}(\sigma(1) \cdots \sigma(p)) \text{Tr}(\sigma(p+1) \cdots \sigma(n)), \quad \sigma \in S_n \tag{2.5}$$

with  $2 \leq p \leq \lfloor n/2 \rfloor$ . The color-ordered amplitudes again satisfy a reflection property

$$A(\sigma(p), \dots, \sigma(1); \sigma(n), \dots, \sigma(p+1)) = (-1)^n A(\sigma(1), \dots, \sigma(p); \sigma(p+1), \dots, \sigma(n)) \quad (2.6)$$

allowing the amplitude to be written in terms of a basis containing terms of the form

$$\text{Tr}(1 \cdots p) \text{Tr}((p+1) \cdots n) + (-1)^n \text{Tr}(p \cdots 1) \text{Tr}(n \cdots (p+1)) \quad (2.7)$$

and permutations thereof, which we label as  $\{T_\lambda\}$  with  $\lambda = n_s + 1, \dots, n_s + n_d$ . The number of independent terms of the form (2.7) is  $n_d = \frac{1}{4} \sum_{p=2}^{n-2} n! / p(n-p)$ , except for  $n = 4$ , in which case there are 3 such terms. The full one-loop  $n$ -point amplitude can then be expressed as [2]

$$\mathcal{A}_n^{(1)} = \sum_{\lambda=1}^{n_s+n_d} A_\lambda^{(1)} t_\lambda^{(1)}, \quad t_\lambda^{(1)} = \begin{cases} NT_\lambda, & \lambda = 1, \dots, n_s \\ T_\lambda, & \lambda = n_s + 1, \dots, n_s + n_d. \end{cases} \quad (2.8)$$

Beginning at two loops (and  $n \geq 6$ ), the trace basis also requires triple-trace terms of the form

$$\text{Tr}(1 \cdots p) \text{Tr}((p+1) \cdots q) \text{Tr}((q+1) \cdots n) + (-1)^n \text{Tr}(p \cdots 1) \text{Tr}(q \cdots (p+1)) \text{Tr}(n \cdots (q+1)) \quad (2.9)$$

and permutations thereof, which we label as  $\{T_\lambda\}$  with  $\lambda = n_s + n_d + 1, \dots, n_s + n_d + n_t$ . The full two-loop  $n$ -point amplitude can be written

$$\mathcal{A}_n^{(2)} = \sum_{\lambda=1}^{2n_s+n_d+n_t} A_\lambda^{(2)} t_\lambda^{(2)}, \quad t_\lambda^{(2)} = \begin{cases} N^2 T_\lambda, & \lambda = 1, \dots, n_s \\ NT_\lambda, & \lambda = n_s + 1, \dots, n_s + n_d \\ T_\lambda, & \lambda = n_s + n_d + 1, \dots, n_s + n_d + n_t \\ T_{\lambda-n_s-n_d-n_t}, & \lambda = n_s + n_d + n_t + 1, \dots, 2n_s + n_d + n_t \end{cases} \quad (2.10)$$

which includes subleading-color single-trace amplitudes, as well as leading-color single-trace amplitudes and double- and triple-trace amplitudes.

One can continue this procedure at higher loops, where the  $L$ -loop trace basis will include (for general  $n$ ) up to  $(L+1)$ -trace terms, as well as additional subleading-color terms. In sections 4–6, we will characterize the general  $L$ -loop basis for  $n = 4, 5$ , and 6, which only require up to triple-trace terms.

We now turn to the decomposition of the single- and multiple-trace bases into irreducible representation of  $S_n$ .

### 3 Projection operators for representations of $S_n$

In this section, we review some standard results from the representation theory of the symmetric group  $S_n$ , which can be found in many group theory textbooks. We found ref. [28] particularly useful. First we will describe a method to project the reducible regular representation onto its irreducible subspaces. Then we will turn our attention to the induced representations spanned by the trace bases described in section 2.

### 3.1 Regular representation

We begin by recalling that irreducible representations of  $S_n$  are labeled by  $\tau \in Y_n$ , where  $Y_n$  denotes the set of Young tableaux with  $n$  boxes. The dimension of the representation labeled by  $\tau$  is given by  $d_\tau = n!/H$ , where  $H$  is the product of all the hook lengths associated with  $\tau$ .

The  $n!$ -dimensional regular representation of  $S_n$  is reducible, and contains all of the irreducible representations  $\tau$  of  $S_n$ , each with a multiplicity equal to  $d_\tau$ , which implies that  $\sum_{\tau \in Y_n} d_\tau^2 = n!$ . For example, the regular representation of  $S_4$  reduces into

$$\begin{aligned} R_{S_4}^{\text{reg}} &= \square\square\square \oplus 3 \square\square \oplus 2 \square \oplus 3 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \\ \dim = 24 &= 1 + 3 \cdot 3 + 2 \cdot 2 + 3 \cdot 3 + 1 \end{aligned} \tag{3.1}$$

A basis  $\{f_\sigma\}$  for the regular representation of  $S_n$  can be constructed from an arbitrary function of  $n$  variables  $f(x_1, x_2, \dots, x_n)$  by permuting the arguments

$$f_\sigma \equiv f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}), \quad \sigma \in S_n. \tag{3.2}$$

Then a permutation  $\rho \in S_n$  acts on  $f_\sigma$  by<sup>6</sup>

$$\rho \cdot f_\sigma \equiv f_{\rho \cdot \sigma} = \sum_{\sigma' \in S_n} f_{\sigma'} D_{\sigma' \sigma}^{\text{reg}}(\rho) \tag{3.3}$$

where  $D_{\sigma' \sigma}^{\text{reg}}(\rho)$  are the matrices of the regular representation. We can put  $D_{\sigma' \sigma}^{\text{reg}}(\rho)$  into block diagonal form by choosing a different basis  $\{f_{ij}^\tau\}$  defined by

$$f_{ij}^\tau = e_{ij}^\tau \cdot f(x_1, x_2, \dots, x_n), \quad \tau \in Y_n, \quad i, j = 1, \dots, d_\tau \tag{3.4}$$

where  $\{e_{ij}^\tau\}$  are a set of  $n!$  elements of the group algebra (i.e., linear combinations of permutations) that satisfy (for each  $\tau$ ) the simple matrix algebra

$$e_{kl}^\tau e_{ij}^{\tau'} = \delta^{\tau \tau'} \delta_{li} e_{kj}^\tau. \tag{3.5}$$

A permutation  $\rho$  acts on this basis as<sup>7</sup>

$$\rho \cdot f_{ij}^\tau = \sum_{k=1}^{d_\tau} f_{kj}^\tau D_{ki}^\tau(\rho) \tag{3.6}$$

showing that each subset of functions

$$\{f_{ij}^\tau \mid i = 1, \dots, d_\tau\} \tag{3.7}$$

spans an irreducible subspace of the regular representation. Hence,  $e_{ij}^\tau$  act as projection operators from the regular representation onto irreducible representations of  $S_n$ . The

---

<sup>6</sup>Our convention is that  $\rho \cdot \sigma$  denotes the permutation obtained by acting first with  $\sigma$  and then with  $\rho$ . Hence if  $\sigma = (12)$  and  $\rho = (23)$  then  $\rho \cdot \sigma = (132)$ , where we use the cycle notation for permutations. Note that the permutations act on the labels rather than the positions of the arguments, so that  $(123) \cdot f(x_4, x_3, x_2, x_1) = f(x_4, x_1, x_3, x_2)$  and not  $f(x_2, x_4, x_3, x_1)$ .

<sup>7</sup>Equation (3.6) follows from eq. (3.5) and the fact that any  $\rho$  is a linear combination of the  $e_{kl}^\tau$ .



index  $j = 1, \dots, d_\tau$  labels distinct copies of the representation  $\tau$ , each of which transforms according to the same set of matrices  $D_{ki}^\tau(\rho)$ .

Explicit expressions for  $e_{ij}^\tau$  may be constructed from Young tableaux as follows [28, 29]. For any representation  $\tau$ , define the set of standard tableaux  $\tau_i$ ,  $i = 1, \dots, d_\tau$ , using the ordering given on p. 139 of ref. [28]. Then for each standard tableau  $\tau_i$ , define the row symmetrizer  $P_i$  as the sum of permutations that exchange only symbols within the same row, and the column antisymmetrizer  $Q_i$  as the (signed) sum of permutations that exchange only symbols within the same column, where the sign is given by the parity of the permutation. Finally, define  $s_{ij}$  as the permutation of symbols that converts tableau  $\tau_j$  into  $\tau_i$ . For example, the standard tableaux for the  $\boxplus$  representation of  $S_4$  are given by

$$\tau_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \tau_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad (3.8)$$

so that

$$\begin{aligned} P_1 &= 1 + (12) + (34) + (12)(34), & Q_1 &= 1 - (13) - (24) + (13)(24), & s_{12} &= (23), \\ P_2 &= 1 + (13) + (24) + (13)(24), & Q_2 &= 1 - (12) - (34) + (12)(34), & s_{21} &= (23). \end{aligned} \quad (3.9)$$

With these definitions, one can construct projection operators

$$e_{ij}^\tau = \left( \frac{d_\tau}{n!} \right) Q_i s_{ij} P_j M_j, \quad M_j = 1 - \sum_{k=1}^{j-1} e_{kk}^\tau \quad (3.10)$$

that obey the simple matrix algebra (3.5). The diagonal elements  $e_{ii}^\tau$  are mutually annihilating idempotents

$$e_{kk}^\tau e_{ii}^{\tau'} = \delta^{\tau\tau'} \delta_{ki} e_{ii}^\tau \quad (3.11)$$

known as Young operators.

In the next subsection, we show that  $e_{ij}^\tau$  can also be used to project single- and multiple-trace spaces onto irreducible representations of  $S_n$ .

### 3.2 Induced representations

As discussed in section 2, gauge theory amplitudes can be expressed in terms of a basis of single and multiple traces of  $SU(N)$  generators. Since these sets of traces are closed under permutations of their arguments, they form representations of the symmetric group. These single- and multiple-trace representations are induced by one-dimensional representations of various subgroups of the symmetric group. They are in general reducible, and in this section we describe how to determine their decomposition into irreducible representations, and how to use the projection operators introduced in section 3.1 to find their irreducible subspaces.

First consider an element of the single-trace basis introduced in section 2

$$\text{Tr}(12 \cdots n) + (-1)^n \text{Tr}(n \cdots 21). \quad (3.12)$$

Permutations act on eq. (3.12) in the same way as on  $f(x_1, \dots, x_n)$ , namely

$$\rho \cdot \left[ \text{Tr}(12 \cdots n) + (-1)^n \text{Tr}(n \cdots 21) \right] = \text{Tr}(\rho(1)\rho(2) \cdots \rho(n)) + (-1)^n \text{Tr}(\rho(n) \cdots \rho(2)\rho(1)) \quad (3.13)$$

thus generating a (reducible) representation of  $S_n$ , which we will refer to as the single-trace representation  $R_n^{(\text{ind})}$ . Equation (3.12) is invariant<sup>8</sup> under the dihedral subgroup  $D_n \subset S_n$  of order  $2n$  (generated by cyclic permutations and reversal of indices), so the dimension of  $R_n^{(\text{ind})}$  is  $\frac{1}{2}(n-1)!$ . The superscript (ind) denotes that  $R_n^{(\text{ind})}$  is the representation induced by a one-dimensional representation of the dihedral group. When  $n$  is even,  $R_n^{(\text{ind})}$  is induced by the trivial representation of the dihedral subgroup; when  $n$  is odd, it is induced by the representation of the dihedral subgroup that assigns  $+1$  to cyclic permutations and  $-1$  to reversals.

The single-trace representation  $R_n^{(\text{ind})}$  is a subset of the regular representation, and contains  $m_\tau \leq d_\tau$  copies of the irreducible representation  $\tau$  of  $S_n$ , where the multiplicities  $m_\tau$  may be determined by character analysis (see appendix A). As in section 3.1, we can explicitly construct the irreducible subspaces of  $R_n^{(\text{ind})}$  using the projection operators (3.10). For each  $\tau \in Y_n$ , we construct a basis of the associated irreducible representation(s) via

$$v_i(n | \tau, j) \equiv e_{ij}^\tau [\text{Tr}(1 \cdots n) + (-1)^n \text{Tr}(n \cdots 1)], \quad i = 1, \dots, d_\tau. \quad (3.14)$$

For each  $\tau$ , there are several sets of basis elements, labeled by  $j = 1, \dots, d_\tau$ , but if  $\tau$  appears  $m_\tau$  times in  $R_n^{(\text{ind})}$ , then only  $m_\tau$  of these sets will be linearly independent, and we will use only the first  $m_\tau$  (independent) values of  $j$ . (If  $m_\tau = 0$ , then all of the associated  $v_i(n | \tau, j)$  vanish.) If  $m_\tau = 1$ , then we will use the lowest value of  $j$  for which  $v_i(n | \tau, j)$  does not vanish, generally  $j = 1$ , and will omit  $j$  from the argument, i.e.  $v_i(n | \tau)$ .

Let us illustrate this with an example. Consider the four-point single-trace representation  $R_4^{(\text{ind})}$  spanned by

$$\{\text{Tr}(1234) + \text{Tr}(1432), \quad \text{Tr}(1243) + \text{Tr}(1342), \quad \text{Tr}(1324) + \text{Tr}(1423)\}. \quad (3.15)$$

Using character analysis, as described in appendix A, we find that this reduces into

$$\begin{aligned} R_4^{(\text{ind})} &= \square\square\square \oplus \boxplus, \\ 3 &= 1 + 2 \end{aligned} \quad (3.16)$$

Since  $e^{\square\square\square} = \frac{1}{24} \sum_{\sigma \in S_4} \sigma$ , the one-dimensional subspace corresponding to  $\square\square\square$  consists of the completely symmetric sum of single-trace terms

$$\begin{aligned} v(4 | \square\square\square) &= e^{\square\square\square} [\text{Tr}(1234) + \text{Tr}(4321)] \\ &= \frac{1}{3} [\text{Tr}(1234) + \text{Tr}(1432) + \text{Tr}(1243) + \text{Tr}(1342) + \text{Tr}(1324) + \text{Tr}(1423)] \end{aligned} \quad (3.17)$$

while the two-dimensional subspace corresponding to  $\boxplus$  is spanned by

$$\begin{aligned} v_i(4 | \boxplus) &= e_{i1}^{\boxplus} [\text{Tr}(1234) + \text{Tr}(4321)] \\ &= \begin{cases} \frac{1}{3} [\text{Tr}(1243) + \text{Tr}(1342) - \text{Tr}(1324) - \text{Tr}(1423)], & i = 1, \\ \frac{1}{3} [\text{Tr}(1243) + \text{Tr}(1342) - \text{Tr}(1234) - \text{Tr}(1432)], & i = 2. \end{cases} \end{aligned} \quad (3.18)$$

---

<sup>8</sup>Up to a sign when  $n$  is odd.

As explained above, since  $\boxplus$  appears only once in  $R_4^{(\text{ind})}$ ,  $v_i(4 | \boxplus, 2)$  is proportional to  $v_i(4 | \boxplus, 1)$ .

Next we consider double-trace representations. An element of the form

$$\text{Tr}(1 \cdots p) \text{Tr}((p+1) \cdots n) + (-1)^n \text{Tr}(p \cdots 1) \text{Tr}(n \cdots (p+1)) \quad (3.19)$$

is invariant (up to sign when  $n$  is odd) under the subgroup of  $S_n$  generated by cyclic permutations of the arguments of each trace and by simultaneous reversal of the arguments within each trace (and, if  $p = n/2$ , by an exchange of the two traces). We will denote by  $R_{n;p-n}^{(\text{ind})}$  the  $S_n$  representation induced by the representation of this subgroup that leaves eq. (3.19) invariant.

The multiplicities of the irreducible representations  $\tau$  contained in  $R_{n;p-n}^{(\text{ind})}$  may again be obtained by character analysis, and the irreducible subspace(s) corresponding to  $\tau$  are spanned by

$$v_i(p; n-p | \tau, j) \equiv e_{ij}^\tau \left[ \text{Tr}(1 \cdots p) \text{Tr}((p+1) \cdots n) + (-1)^n \text{Tr}(p \cdots 1) \text{Tr}(n \cdots (p+1)) \right]. \quad (3.20)$$

For example, the four-point double-trace representation  $R_{2;2}^{(\text{ind})}$  is spanned by

$$\{2 \text{Tr}(12) \text{Tr}(34), \quad 2 \text{Tr}(13) \text{Tr}(24), \quad 2 \text{Tr}(14) \text{Tr}(23)\} \quad (3.21)$$

and reduces into

$$R_{2;2}^{(\text{ind})} = \begin{array}{c} \square\square \\ 3 \end{array} \oplus \begin{array}{c} \boxplus \\ 2 \end{array}. \quad (3.22)$$

The one-dimensional subspace corresponding to  $\square\square$  consists of the symmetric sum of double-trace terms

$$\begin{aligned} v(2; 2 | \square\square) &= e^{\square\square} [2 \text{Tr}(12) \text{Tr}(34)] \\ &= \frac{1}{3} [2 \text{Tr}(12) \text{Tr}(34) + 2 \text{Tr}(13) \text{Tr}(24) + 2 \text{Tr}(14) \text{Tr}(23)] \end{aligned} \quad (3.23)$$

while the two-dimensional subspace corresponding to  $\boxplus$  is spanned by

$$v_i(2; 2 | \boxplus) = e_{i1}^{\boxplus} [2 \text{Tr}(12) \text{Tr}(34)] = \begin{cases} \frac{2}{3} [2 \text{Tr}(12) \text{Tr}(34) - 2 \text{Tr}(14) \text{Tr}(23)], & i = 1, \\ \frac{2}{3} [2 \text{Tr}(13) \text{Tr}(24) - 2 \text{Tr}(14) \text{Tr}(23)], & i = 2. \end{cases} \quad (3.24)$$

As before, since  $\boxplus$  appears only once in  $R_{2;2}^{(\text{ind})}$ ,  $v_i(2; 2 | \boxplus, 2)$  is proportional to  $v_i(2; 2 | \boxplus)$ .

We apply the same approach to multiple-trace representations. For example the irreducible subspaces of the induced triple-trace representation  $R_{p;q-p;n-q}^{(\text{ind})}$  are spanned by

$$\begin{aligned} v_i(p; q-p; n-q | \tau, j) &\equiv e_{ij}^\tau \left[ \text{Tr}(1 \cdots p) \text{Tr}((p+1) \cdots q) \text{Tr}((q+1) \cdots n) \right. \\ &\quad \left. + (-1)^n \text{Tr}(p \cdots 1) \text{Tr}(q \cdots (p+1)) \text{Tr}(n \cdots (q+1)) \right] \end{aligned} \quad (3.25)$$

and so forth.

We will use the vectors  $v_i(- | \tau, j)$  defined in this section to characterize the null spaces of four-, five-, and six-point amplitudes in the remainder of this paper.

## 4 Constraints on four-point amplitudes

The group-theory constraints on four-point color-ordered amplitudes at all loop orders were obtained in ref. [13] via an iterative approach. We will not rederive these results, but will simply restate the  $L$ -loop null vectors (using the normalization conventions of the present paper) and then re-express them in terms of irreducible representations of  $S_4$ , as a warm-up for the more complicated cases to follow.

### 4.1 $S_4$ decomposition of the four-point trace basis

The four-point amplitude at any loop order may be expanded in terms of single and double traces of  $SU(N)$  generators. We express these in terms of the explicit basis<sup>9</sup>

$$\begin{aligned} T_1 &= \text{Tr}(1234) + \text{Tr}(1432), & T_4 &= 2 \text{Tr}(13) \text{Tr}(24), \\ T_2 &= \text{Tr}(1243) + \text{Tr}(1342), & T_5 &= 2 \text{Tr}(14) \text{Tr}(23), \\ T_3 &= \text{Tr}(1324) + \text{Tr}(1423), & T_6 &= 2 \text{Tr}(12) \text{Tr}(34). \end{aligned} \tag{4.1}$$

The  $L$ -loop four-point amplitude may then be expressed as

$$\mathcal{A}_4^{(L)} = \sum_{\lambda=1}^{d(L)} A_\lambda^{(L)} t_\lambda^{(L)} \tag{4.2}$$

where the  $L$ -loop four-point trace basis

$$t_{\lambda+6k}^{(L)} = \begin{cases} N^{L-2k} T_\lambda, & \lambda = 1, 2, 3, \quad k = 0, \dots, \left\lfloor \frac{L}{2} \right\rfloor \\ N^{L-2k-1} T_\lambda, & \lambda = 4, 5, 6, \quad k = 0, \dots, \left\lfloor \frac{L-1}{2} \right\rfloor \end{cases} \tag{4.3}$$

has dimension  $d(L) = 3L + 3$ .

We now decompose the four-point trace basis into irreducible representations of  $S_4$ . The single- and double-trace terms (4.1) span the representations  $R_4^{(\text{ind})}$  and  $R_{2;2}^{(\text{ind})}$  respectively, which we recall from section 3.2 have the following decompositions

$$\begin{aligned} R_4^{(\text{ind})} &= \square\square \oplus \boxplus, & R_{2;2}^{(\text{ind})} &= \square\square \oplus \boxplus. \\ 3 &= 1 + 2 & 3 &= 1 + 2 \end{aligned} \tag{4.4}$$

The  $L$ -loop trace basis (4.3) therefore contains  $(L + 1)$  copies each of  $\square\square$  and  $\boxplus$ .

We can characterize the vectors  $v_i(4 | \tau)$  and  $v_i(2; 2 | \tau)$  spanning the irreducible subspaces of  $R_4^{(\text{ind})}$  and  $R_{2;2}^{(\text{ind})}$  by specifying their components  $v_{\lambda i}$  in the explicit basis (4.1)

$$v_i(4 | \tau) = \sum_{\lambda=1}^3 T_\lambda v_{\lambda i}(4 | \tau), \quad v_i(2; 2 | \tau) = \sum_{\lambda=4}^6 T_\lambda v_{\lambda i}(2; 2 | \tau). \tag{4.5}$$

---

<sup>9</sup>For consistency with the general  $n$ -point case (2.7), we include a factor of two in the double-trace terms, which differs from the definition in ref. [13].

By comparing eqs. (3.17) and (3.18) and eqs. (3.23) and (3.24) with eqs. (4.1) and (4.5) we obtain

$$v_\lambda(4 | \square\square\square) = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\lambda_i}(4 | \boxplus) = \frac{1}{3} \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.6)$$

and

$$v_\lambda(2; 2 | \square\square) = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_{\lambda_i}(2; 2 | \boxplus) = \frac{2}{3} \begin{pmatrix} 0 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.7)$$

We will use these in the next section to decompose the four-point null spaces into irreducible representations of  $S_4$ .

### 4.2 Four-point null spaces

As described in the introduction, the  $L$ -loop color space is smaller than the  $L$ -loop trace space, and the null space, defined as the orthogonal complement of the color space, is spanned by a set of null eigenvectors

$$r_m^{(L)} = \sum_{\lambda=1}^{d(L)} r_{\lambda m}^{(L)} t_\lambda^{(L)}, \quad m = 1, \dots, n_{\text{null}}. \quad (4.8)$$

These imply that the color-ordered amplitudes (4.2) obey a set of  $n_{\text{null}}$  group-theory relations

$$\sum_\lambda A_\lambda^{(L)} r_{\lambda m}^{(L)} = 0, \quad m = 1, \dots, n_{\text{null}}. \quad (4.9)$$

**Even-loop four-point null space ( $L \geq 2$ ).** In ref. [13], it was shown that at even-loop level (with  $L \geq 2$ ) the null space is four-dimensional and spanned by<sup>10</sup>

$$r^{(2\ell)} = \begin{pmatrix} \vdots \\ 0 \\ 3u \\ -u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 2x \\ x \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 2y \\ y \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ u \end{pmatrix}, \quad \ell \geq 1 \quad (4.10)$$

where

$$u \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x \equiv \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad y \equiv \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \quad (4.11)$$

Comparing these with eqs. (4.6) and (4.7), one sees that  $u$  corresponds to the  $\square\square\square$  representation and  $x$  and  $y$  to the  $\boxplus$  representation. The even-loop null space decomposes into

<sup>10</sup>The factor of two discrepancies with ref. [13] are due to a change in normalization; see footnote 9.

three irreducible representations of  $S_4$  spanned by

$$N^2 v(4 | \square\square\square) - \frac{1}{3} N v(2; 2 | \square\square) \tag{4.12a}$$

$$N v_i(2; 2 | \boxplus) - v_i(4 | \boxplus) \quad i = 1, 2 \tag{4.12b}$$

$$v(4 | \square\square\square) \tag{4.12c}$$

Equation (4.12c) contains only the irreducible representation belonging to the single-trace representation  $R_4^{(\text{ind})}$  and so the group-theory relation (4.9) that corresponds to it involves only the most-subleading-color<sup>11</sup> single-trace amplitudes, namely

$$A^{(2\ell, 2\ell)}(1, 2, 3, 4) + A^{(2\ell, 2\ell)}(1, 3, 4, 2) + A^{(2\ell, 2\ell)}(1, 4, 2, 3) = 0 \tag{4.13}$$

where  $A^{(L, 2k)}(1, 2, 3, 4)$  is the coefficient of  $N^{L-2k} [\text{Tr}(1234) + \text{Tr}(4321)]$  in  $\mathcal{A}_4^{(L)}$ . In general, however, the irreducible subspaces of the null space are linear combinations of irreducible representations belonging to single- and double-trace representations. Thus, the corresponding group-theory constraints relate single- and double-trace amplitudes to one another. For example, eqs. (4.12b) and (4.12c) can be combined to express the most-subleading-color single-trace amplitudes in terms of the double-trace amplitudes

$$A^{(2\ell, 2\ell)}(1, 2, 3, 4) = \frac{4}{3} A^{(2\ell, 2\ell-1)}(1, 4; 2, 3) - \frac{2}{3} A^{(2\ell, 2\ell-1)}(1, 2; 3, 4) - \frac{2}{3} A^{(2\ell, 2\ell-1)}(1, 3; 2, 4) \tag{4.14}$$

where  $A^{(L, 2k+1)}(1, 2; 3, 4)$  is the coefficient of  $N^{L-2k-1} [2 \text{Tr}(12) \text{Tr}(34)]$  in  $\mathcal{A}_4^{(L)}$ . Finally, eq. (4.12a) corresponds to the constraint

$$A^{(2\ell, 2\ell-2)}(1, 2, 3, 4) + A^{(2\ell, 2\ell-2)}(1, 3, 4, 2) + A^{(2\ell, 2\ell-2)}(1, 4, 2, 3) - \frac{1}{3} \left[ A^{(2\ell, 2\ell-1)}(1, 2; 3, 4) + A^{(2\ell, 2\ell-1)}(1, 3; 4, 2) + A^{(2\ell, 2\ell-1)}(1, 4; 2, 3) \right] = 0 \tag{4.15}$$

relating leading-color single-trace amplitudes and double-trace amplitudes. We note that eqs. (4.13) and (4.15), but not eq. (4.14), can be derived as U(1) decoupling relations (for all  $\ell$ ).

**Tree-level four-point null space.** At tree level, there is only one null vector, namely  $r^{(0)} = u$ , and so the tree-level null space is

$$v(4 | \square\square\square) \tag{4.16}$$

corresponding to the single group-theory relation (4.13) with  $\ell = 0$ .

---

<sup>11</sup>We refer to the amplitudes  $A^{(2\ell, 2\ell)}$  as most-subleading-color because they have the lowest power of  $N$ .

**Odd-loop four-point null space** ( $L \geq 3$ ). At odd-loop level (with  $L \geq 3$ ), the null space is again four-dimensional and spanned by [13]

$$r^{(2\ell+1)} = \begin{pmatrix} \vdots \\ 0 \\ 3u \\ -u \\ u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 3u \\ -u \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ y \end{pmatrix}, \quad \ell \geq 1. \quad (4.17)$$

The odd-loop null space decomposes into three irreducible representations of  $S_4$ , spanned by

$$N^3 v(4 | \square\square\square) - \frac{1}{3} N^2 v(2; 2 | \square\square\square) + \frac{1}{3} N v(4 | \square\square\square) \quad (4.18a)$$

$$N v(4 | \square\square\square) - \frac{1}{3} v(2; 2 | \square\square\square) \quad (4.18b)$$

$$v_i(2; 2 | \boxplus) \quad i = 1, 2 \quad (4.18c)$$

Equations (4.18b) and (4.18c) can be combined to express the most-subleading-color double-trace amplitudes in terms of single trace amplitudes

$$A^{(2\ell+1, 2\ell+1)}(1, 2; 3, 4) = A^{(2\ell+1, 2\ell)}(1, 2, 3, 4) + A^{(2\ell+1, 2\ell)}(1, 3, 4, 2) + A^{(2\ell+1, 2\ell)}(1, 4, 2, 3). \quad (4.19)$$

Finally, eq. (4.18a) gives rise to

$$0 = A^{(2\ell+1, 2\ell-2)}(1, 2, 3, 4) + A^{(2\ell+1, 2\ell-2)}(1, 3, 4, 2) + A^{(2\ell+1, 2\ell-2)}(1, 4, 2, 3) \\ - \frac{1}{3} \left[ A^{(2\ell+1, 2\ell-1)}(1, 2; 3, 4) + A^{(2\ell+1, 2\ell-1)}(1, 3; 2, 4) + A^{(2\ell+1, 2\ell-1)}(1, 4; 2, 3) \right] \\ + \frac{1}{3} \left[ A^{(2\ell+1, 2\ell-2)}(1, 2, 3, 4) + A^{(2\ell+1, 2\ell-2)}(1, 3, 4, 2) + A^{(2\ell+1, 2\ell-2)}(1, 4, 2, 3) \right]. \quad (4.20)$$

We note that eq. (4.19) follows from U(1) decoupling (for all  $\ell$ ) but eq. (4.20) does not.

**One-loop four-point null space.** The one-loop null space is spanned by the last three vectors of eq. (4.17) and breaks into two irreducible subspaces spanned by

$$N v(4 | \square\square\square) - \frac{1}{3} v(2; 2 | \square\square\square) \quad (4.21a)$$

$$v_i(2; 2 | \boxplus) \quad i = 1, 2 \quad (4.21b)$$

These give rise to the constraint (4.19) with  $\ell = 0$ , which follows from U(1) decoupling.

## 5 Constraints on five-point amplitudes

The group-theory constraints on five-point color-ordered amplitudes at all loop orders were obtained in ref. [14]. These were expressed in terms of a complete set of null vectors by giving their components with respect to an explicit basis. In this section, we show how these results can be expressed more compactly by writing the null space in terms of irreducible representations of  $S_5$ .

### 5.1 $S_5$ decomposition of the five-point trace basis

Five-point amplitudes at any loop order can be expanded in a basis including single and double traces of  $SU(N)$  generators. As we described in section 2, the twelve independent single-trace terms of the form

$$\text{Tr}(12345) - \text{Tr}(54321) \quad (5.1)$$

are labeled  $T_\lambda$  with  $\lambda = 1, \dots, 12$ , and the ten independent double-trace terms of the form

$$\text{Tr}(12) \text{Tr}(345) - \text{Tr}(21) \text{Tr}(543) \quad (5.2)$$

are labeled  $T_\lambda$  with  $\lambda = 13, \dots, 22$ . An explicit ordering for the basis  $\{T_\lambda\}$  is given in appendix B, but the results presented in this section will be independent of the specific choice of basis.

The  $L$ -loop five-point amplitude can be expressed as

$$\mathcal{A}_5^{(L)} = \sum_{\lambda=1}^{d(L)} A_\lambda^{(L)} t_\lambda^{(L)} \quad (5.3)$$

where the  $L$ -loop five-point trace basis

$$t_{\lambda+22k}^{(L)} = \begin{cases} N^{L-2k} T_\lambda, & \lambda = 1, \dots, 12, & k = 0, \dots, \left\lfloor \frac{L}{2} \right\rfloor \\ N^{L-2k-1} T_\lambda, & \lambda = 13, \dots, 22, & k = 0, \dots, \left\lfloor \frac{L-1}{2} \right\rfloor \end{cases} \quad (5.4)$$

has dimension

$$d(L) = \begin{cases} 12 + 11L, & \text{for } L \text{ even,} \\ 11 + 11L, & \text{for } L \text{ odd.} \end{cases} \quad (5.5)$$

Now we decompose the trace basis into irreducible representations of  $S_5$ . The induced representations  $R_5^{(\text{ind})}$  and  $R_{2,3}^{(\text{ind})}$ , spanned by the single-trace (5.1) and double-trace (5.2) terms respectively, have the following decompositions

$$\begin{aligned} R_5^{(\text{ind})} &= 2 \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} & R_{2,3}^{(\text{ind})} &= \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline 5 & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \\ 12 &= 2 \cdot 6 & 10 &= 6 + 4 \end{aligned} \quad (5.6)$$

As described in section 3.1, to define vectors in the trace space that span the irreducible subspaces  $\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array}$  and  $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array}$ , we define standard tableaux  $\tau_i$  for each of these representations

$$\begin{aligned} \tau_1 &= \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & & \\ \hline 5 & & \\ \hline \end{array} & \tau_2 &= \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & & \\ \hline 5 & & \\ \hline \end{array} & \tau_3 &= \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & & \\ \hline 4 & & \\ \hline \end{array} & \tau_4 &= \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & & \\ \hline 5 & & \\ \hline \end{array} & \tau_5 &= \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & & \\ \hline 4 & & \\ \hline \end{array} & \tau_6 &= \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & & \\ \hline 3 & & \\ \hline \end{array} \\ & & \tau_1 &= \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} & \tau_2 &= \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} & \tau_3 &= \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & \\ \hline 3 & \\ \hline 5 & \\ \hline \end{array} & \tau_4 &= \begin{array}{|c|c|} \hline 1 & 5 \\ \hline 2 & \\ \hline 3 & \\ \hline 4 & \\ \hline \end{array} \end{aligned} \quad (5.7)$$



From these, we construct the projection operators  $e_{ij}^\tau$  using eq. (3.10). Acting with the projection operators on single- and double-trace representations, we obtain the vectors

$$v_i ( 5 | \tau , j ) = e_{ij}^\tau [\text{Tr}(12345) - \text{Tr}(54321)] , \tag{5.8}$$

$$v_i ( 2; 3 | \tau , j ) = e_{ij}^\tau [\text{Tr}(12) \text{Tr}(345) - \text{Tr}(21) \text{Tr}(543)] \tag{5.9}$$

that span the irreducible subspaces. Needless to say, all of this is done using a symbolic manipulation program. In appendix B, we write down the components of  $v_i ( 5 | \tau , j )$  and  $v_i ( 2; 3 | \tau , j )$  in an explicit basis.

## 5.2 Five-point null spaces

In ref. [14], it was shown that the five-point  $L$ -loop null space is spanned by a set of null vectors

$$r_m^{(L)} = \sum_{\lambda=1}^{d(L)} r_{\lambda m}^{(L)} t_\lambda^{(L)} , \quad m = 1, \dots , \begin{cases} 6, & L = 0 \\ 10, & \text{odd } L \\ 12, & \text{even } L \geq 2. \end{cases} \tag{5.10}$$

We will not rederive these null vectors, but simply restate them in terms of irreducible spaces.

**Tree-level five-point null space.** The 6 tree-level null vectors belong to a representation of  $S_5$  contained within the tree-level trace space  $R_5^{(\text{ind})}$  and so must correspond to some linear combination of the two copies of  $\mathbb{F}^\square$  contained therein (cf. eq. (5.6)):

$$\kappa_1 v_i ( 5 | \mathbb{F}^\square , 1 ) + \kappa_2 v_i ( 5 | \mathbb{F}^\square , 2 ) . \tag{5.11}$$

By writing these vectors in an explicit basis using eq. (B.3) and then acting with  $M^{(0)}$  (cf. eq. (3.9) of ref. [14]), we find a result proportional to  $\kappa_1 - 3\kappa_2$ . Hence the tree-level five-point null space is spanned by

$$3v_i ( 5 | \mathbb{F}^\square , 1 ) + v_i ( 5 | \mathbb{F}^\square , 2 ) \quad i = 1, \dots , 6 \tag{5.12}$$

The 6 null vectors (5.12) give rise to the relations among tree-level amplitudes [10, 23]

$$A^{(0,0)}(1, \{\alpha\}, 5, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{(0,0)}(1, \sigma, 5) \tag{5.13}$$

where  $A^{(L,2k)}(1, 2, \dots, n)$  is the coefficient of  $N^{L-2k} [\text{Tr}(12 \dots n) + (-1)^n \text{Tr}(n \dots 21)]$  in  $\mathcal{A}_n^{(L)}$ , and  $\{\alpha\}$  and  $\{\beta\}$  denote complementary subsets of  $\{2, 3, \dots, n-1\}$ , with  $n_\beta$  the number of elements of  $\{\beta\}$ . Here  $OP\{\alpha\}\{\beta^T\}$  denotes the set of *ordered permutations* (mergings) of  $\{\alpha\}$  and  $\{\beta^T\}$ , those that preserve the order of  $\{\alpha\}$  and  $\{\beta^T\}$  inside  $\sigma$ , where  $\{\beta^T\}$  is  $\{\beta\}$  with the ordering reversed. Equation (5.13) can be derived from U(1) decoupling.

**Odd-loop five-point null space.** At odd-loop level, the five-point null space is 10-dimensional, and breaks into two irreducible subspaces, spanned by

$$3Nv_i(5 | \boxplus, 1) + Nv_i(5 | \boxplus, 2) - \frac{1}{2}v_i(2; 3 | \boxplus) \quad i = 1, \dots, 6 \quad (5.14a)$$

$$v_i(2; 3 | \boxminus) \quad i = 1, \dots, 4 \quad (5.14b)$$

Since the null space involves both irreducible representations  $\boxplus$  and  $\boxminus$  of  $R_{2;3}^{(\text{ind})}$ , the null vectors are sufficient to express the most-subleading-color double-trace amplitudes in terms of single-trace amplitudes

$$A^{(2\ell+1, 2\ell+1)}(1, 2; 3, 4, 5) = \sum_{\sigma \in \text{COP}\{2,1\}\{3,4,5\}} A^{(2\ell+1, 2\ell)}(\sigma) \quad (5.15)$$

where  $A^{(L, 2k+1)}(1, \dots, p; p+1, \dots, n)$  is the coefficient of  $N^{L-2k-1}[\text{Tr}(1 \dots p) \text{Tr}((p+1) \dots n) + (-1)^n \text{Tr}(p \dots 1) \text{Tr}(n \dots (p+1))]$  in  $\mathcal{A}_n^{(L)}$ , and  $\text{COP}\{\alpha\}\{\beta\}$  is the set of *cyclically-ordered permutations*, i.e., permutations of  $\{1, \dots, n\}$  (with  $n$  held fixed) that preserve the cyclic ordering of  $\{\alpha\}$  and  $\{\beta\}$ . Equation (5.15) can be derived from U(1) decoupling (for all  $\ell$ ), and extends the one-loop results of ref. [2] to all odd-loop orders.

**Even-loop five-point null space ( $L \geq 2$ ).** At even-loop level (with  $L \geq 2$ ) the null space is 12-dimensional and decomposes into two irreducible subspaces spanned by

$$3N^2v_i(5 | \boxplus, 1) + N^2v_i(5 | \boxplus, 2) - \frac{1}{2}Nv_i(2; 3 | \boxplus) + v_i(5 | \boxplus, 1) \quad i = 1, \dots, 6 \quad (5.16a)$$

$$3v_i(5 | \boxplus, 1) + v_i(5 | \boxplus, 2) \quad i = 1, \dots, 6 \quad (5.16b)$$

Equation (5.16b) gives rise to 6 relations among most-subleading-color single-trace amplitudes

$$A^{(2\ell, 2\ell)}(1, \{\alpha\}, 5, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in \text{OP}\{\alpha\}\{\beta^T\}} A^{(2\ell, 2\ell)}(1, \sigma, 5). \quad (5.17)$$

These relations can be derived (for all  $\ell$ ) from U(1) decoupling.

Equations (5.16a) and (5.16b) can be combined to express each of the most-subleading-color single-trace amplitudes  $A^{(2\ell, 2\ell)}$  in terms of double-trace amplitudes  $A^{(2\ell, 2\ell-1)}$  and single-trace amplitudes  $A^{(2\ell, 2\ell-2)}$ . Explicit expressions for these were given in ref. [14]. These relations, however, cannot be derived using U(1) decoupling.

## 6 Constraints on six-point amplitudes

In this section, we derive the group-theory relations obeyed by six-point color-ordered amplitudes to all loop orders. First we describe the decomposition of the six-point trace basis into irreducible representations of  $S_6$ . Then we derive the  $L$ -loop six-point null space using the iterative approach of refs. [13, 14], and express the results in terms of irreducible subspaces of  $S_6$ .

### 6.1 $S_6$ decomposition of the six-point trace basis

Six-point amplitudes at any loop order can be expanded in a basis that includes single, double, and triple traces of  $SU(N)$  generators. We label the single-, double-, and triple-trace terms as  $T_\lambda$  with  $\lambda = 1, \dots, 140$ , as described in section 2. The sixty independent single-trace terms of the form

$$\text{Tr}(123456) + \text{Tr}(654321) \quad (6.1)$$

are labeled  $T_\lambda$  with  $\lambda = 1, \dots, 60$ . The precise ordering of the terms is unimportant since we will present our results in a form independent of it. These single-trace terms span the representation  $R_6^{(\text{ind})}$  which decomposes into

$$\begin{aligned} R_6^{(\text{ind})} &= \square\square\square\square \oplus 2 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus 2 \begin{array}{|c|} \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ 60 &= 1 + 2 \cdot 9 + 16 + 2 \cdot 5 + 10 + 5 \end{aligned} \quad (6.2)$$

At loop level, we also require forty-five independent double-trace terms of the form

$$\text{Tr}(12) \text{Tr}(3456) + \text{Tr}(21) \text{Tr}(6543) \quad (6.3)$$

which we label  $T_\lambda$  with  $\lambda = 61, \dots, 105$ . These span  $R_{2;4}^{(\text{ind})}$  which decomposes into

$$\begin{aligned} R_{2;4}^{(\text{ind})} &= \square\square\square\square \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus 2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ 45 &= 1 + 5 + 2 \cdot 9 + 16 + 5 \end{aligned} \quad (6.4)$$

There are also twenty independent double-trace terms of the form

$$\text{Tr}(123) \text{Tr}(456) + \text{Tr}(321) \text{Tr}(654) \quad (6.5)$$

labeled  $T_\lambda$  with  $\lambda = 106, \dots, 125$ . These span  $R_{3;3}^{(\text{ind})}$ , which decomposes into

$$\begin{aligned} R_{3;3}^{(\text{ind})} &= \square\square\square\square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \\ 20 &= 1 + 9 + 5 + 5 \end{aligned} \quad (6.6)$$

Finally, at two loops and above, there are fifteen independent triple-trace terms of the form

$$\text{Tr}(12) \text{Tr}(34) \text{Tr}(56) + \text{Tr}(21) \text{Tr}(43) \text{Tr}(65) = 2 \text{Tr}(12) \text{Tr}(34) \text{Tr}(56) \quad (6.7)$$

labeled  $T_\lambda$  with  $\lambda = 126, \dots, 140$ . These span  $R_{2;2;2}^{(\text{ind})}$ , which decomposes into

$$\begin{aligned} R_{2;2;2}^{(\text{ind})} &= \square\square\square\square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \end{array} \\ 15 &= 1 + 9 + 5 \end{aligned} \quad (6.8)$$

As before, by constructing projection operators  $e_{ij}^\tau$  for each of the tableaux appearing in the induced representations, one can project those representations onto the vectors

$$v_i(6 | \tau, j), \quad v_i(2; 4 | \tau, j), \quad v_i(3; 3 | \tau, j), \quad v_i(2; 2; 2 | \tau, j) \quad (6.9)$$

that span the irreducible subspaces of  $R_6^{(\text{ind})}$ ,  $R_{2,4}^{(\text{ind})}$ ,  $R_{3,3}^{(\text{ind})}$ , and  $R_{2,2;2}^{(\text{ind})}$  respectively. For obvious reasons we will not give explicit expressions for these. If the label  $j$  is omitted, one should presume  $j = 1$ , except in the following cases

$$\begin{aligned}
 v_i(6 | \square) &\equiv v_i(6 | \square, 2), & v_i(3;3 | \square) &\equiv v_i(3;3 | \square, 5), \\
 v_i(3;3 | \square) &\equiv v_i(3;3 | \square, 3),
 \end{aligned}
 \tag{6.10}$$

because the vectors for smaller values of  $j$  vanish in these cases.

The  $L$ -loop six-point amplitude can be expanded as

$$\mathcal{A}_6^{(L)} = \sum_{\lambda=1}^{d(L)} A_\lambda^{(L)} t_\lambda^{(L)}
 \tag{6.11}$$

where the  $L$ -loop six-point trace basis

$$t_{\lambda+140k}^{(L)} = \begin{cases} N^{L-2k} T_\lambda, & \lambda = 1, \dots, 60, & k = 0, \dots, \left\lfloor \frac{L}{2} \right\rfloor \\ N^{L-2k-1} T_\lambda, & \lambda = 61, \dots, 125, & k = 0, \dots, \left\lfloor \frac{L-1}{2} \right\rfloor \\ N^{L-2k-2} T_\lambda, & \lambda = 126, \dots, 140, & k = 0, \dots, \left\lfloor \frac{L-2}{2} \right\rfloor \end{cases}
 \tag{6.12}$$

has dimension

$$d(L) = \begin{cases} 60 + 70L, & \text{for } L \text{ even,} \\ 55 + 70L, & \text{for } L \text{ odd.} \end{cases}
 \tag{6.13}$$

## 6.2 Six-point null spaces

Having defined the  $L$ -loop six-point trace space, we now turn to the determination of a complete set of  $L$ -loop six-point color factors

$$c_i^{(L)} = \sum_{\lambda=1}^{d(L)} M_{i\lambda}^{(L)} t_\lambda^{(L)}
 \tag{6.14}$$

which span a subspace of the trace space (the color space). The null space, which is the orthogonal complement<sup>12</sup> of the color space, is spanned by the null vectors

$$r_m^{(L)} = \sum_{\lambda=1}^{d(L)} r_{\lambda m}^{(L)} t_\lambda^{(L)}, \quad \text{where} \quad \sum_{\lambda=1}^{d(L)} M_{i\lambda}^{(L)} r_{\lambda m}^{(L)} = 0.
 \tag{6.15}$$

A complete set of null vectors then determines the group-theory relations

$$\sum_{\lambda} A_\lambda^{(L)} r_{\lambda m}^{(L)} = 0, \quad m = 1, \dots, n_{\text{null}}
 \tag{6.16}$$

satisfied by the six-point color-ordered amplitudes defined in eq. (6.11).

<sup>12</sup>Orthogonality is defined with respect to the inner product  $(t_\lambda^{(L)}, t_{\lambda'}^{(L)}) = \delta_{\lambda\lambda'}$ .

**Tree-level six-point null space.** We begin with the tree-level six-point color space, spanned by the 24 independent color factors [22, 23]

$$\tilde{f}^{a_1 a_j b} \tilde{f}^{b a_k c} \tilde{f}^{c a_l d} \tilde{f}^{d a_m a_6} \quad (6.17)$$

where  $\{j, k, l, m\}$  runs over all permutations of  $\{2, 3, 4, 5\}$ . We expand these in the tree-level trace basis  $\{t_\lambda^{(0)}\}$  to find  $M_{i\lambda}^{(0)}$  and then solve for its 36 null eigenvectors  $r_{\lambda m}^{(0)}$ . They span a null space that decomposes into the following irreducible representations of  $S_6$ :

$$v(6 \mid \square\square\square\square) \quad (6.18a)$$

$$v_i(6 \mid \square\square, 1) + \frac{1}{4}v_i(6 \mid \square\square, 2) \quad i = 1, \dots, 9 \quad (6.18b)$$

$$v_i(6 \mid \square\square) \quad i = 1, \dots, 16 \quad (6.18c)$$

$$v_i(6 \mid \square, 1) + v_i(6 \mid \square, 2) \quad i = 1, \dots, 5 \quad (6.18d)$$

$$v_i\left(6 \mid \begin{array}{|c|} \hline \square \\ \hline \end{array}\right) \quad i = 1, \dots, 5 \quad (6.18e)$$

These null vectors give rise to the 36 relations among tree-level amplitudes [10, 23]

$$A^{(0,0)}(1, \{\alpha\}, 6, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{(0,0)}(1, \sigma, 6). \quad (6.19)$$

Equation (6.19) can be derived from U(1) decoupling [10].

Next we turn to the iterative construction of the loop-level color basis. An  $(L+1)$ -loop color diagram may be obtained from an  $L$ -loop color diagram by attaching a rung between two of its external legs,  $j$  and  $k$ . First, we describe the effect of adding a rung on the trace basis  $\{T_\lambda\}$  defined in the previous section. Choose an arbitrary trace term  $T_\lambda$ , contract it with  $\tilde{f}^{a_j a'_j b} \tilde{f}^{b a'_k a_k}$ , and simplify using eq. (1.3) together with the SU( $N$ ) relations

$$\begin{aligned} \text{Tr}(PT^a) \text{Tr}(QT^a) &= \text{Tr}(PQ) - \frac{1}{N} \text{Tr}(P) \text{Tr}(Q), \\ \text{Tr}(PT^a QT^a) &= \text{Tr}(P) \text{Tr}(Q) - \frac{1}{N} \text{Tr}(PQ). \end{aligned} \quad (6.20)$$

This procedure yields a linear combination of trace terms

$$T_\lambda \longrightarrow \sum_{\kappa=1}^{140} G_{\lambda\kappa} T_\kappa. \quad (6.21)$$

Attaching a rung to single-trace terms yields both single- and double-trace terms; attaching a rung to a triple-trace term yields double- and triple-trace terms, and attaching a rung to a double-trace terms yields all three. Hence the  $140 \times 140$  matrix  $G$  has the block form

$$G = \begin{pmatrix} Na & b & c & 0 \\ d & Ne & 0 & f \\ g & 0 & Nh & 0 \\ 0 & i & 0 & Nj \end{pmatrix} \quad (6.22)$$

with blocks of size 60, 45, 20, and 15 respectively, where the  $N$  dependence is made explicit. The matrix elements depend, of course, on the choice of legs  $j$  and  $k$  between which the rung extends.

Now using eq. (6.22), we can determine the effect of contracting an arbitrary element  $t_\lambda^{(L)}$  of the  $L$ -loop trace basis (6.12) with  $\tilde{f}^{a_j a'_j b} \tilde{f}^{b a'_k a_k}$ . The result is a linear combination of elements of the  $(L + 1)$ -loop trace basis

$$t_\lambda^{(L)} \rightarrow \sum_{\kappa=1}^{d(L+1)} g_{\lambda\kappa}^{(L)} t_\kappa^{(L+1)} \tag{6.23}$$

where  $g^{(L)}$  is the  $d(L) \times d(L + 1)$  matrix

$$g^{(L)} = \begin{pmatrix} a & b & c & 0 & 0 & 0 & 0 & \dots \\ 0 & e & 0 & f & d & 0 & 0 & \dots \\ 0 & 0 & h & 0 & g & 0 & 0 & \dots \\ 0 & 0 & 0 & j & 0 & i & 0 & \dots \\ 0 & 0 & 0 & 0 & a & b & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.24}$$

The pattern repeats starting from  $g^{(0)} = (a \ b \ c)$ , with each increment of  $L$  adding two extra rows and columns of blocks.

Now we make the assumption that we can construct the complete  $(L + 1)$ -loop color space by attaching rungs between the legs of  $L$ -loop color factors in all possible ways. We act on the  $L$ -loop color basis (6.14) with eq. (6.23) to obtain the  $(L + 1)$ -loop color basis

$$\sum_{\lambda=1}^{d(L)} \sum_{\kappa=1}^{d(L+1)} M_{i\lambda}^{(L)} g_{\lambda\kappa}^{(L)} t_\kappa^{(L+1)}. \tag{6.25}$$

The  $(L + 1)$ -loop null vectors  $r_m^{(L+1)}$  must then satisfy

$$\sum_{\lambda=1}^{d(L)} \sum_{\kappa=1}^{d(L+1)} M_{i\lambda}^{(L)} g_{\lambda\kappa}^{(L)} r_{\kappa m}^{(L+1)} = 0. \tag{6.26}$$

Equation (6.26) must be satisfied for all possible choices of  $j$  and  $k$  used to construct  $g^{(L)}$ . We solve eq. (6.26) to obtain a complete set of  $(L + 1)$ -loop null vectors  $\{r_m^{(L+1)}\}$ . Finally, we construct  $M_i^{(L+1)}$  as the set of vectors orthogonal to  $r_m^{(L+1)}$ , and begin the process again.

**One-loop six-point null space.** Using this procedure, we find 65 null vectors at the one-loop level. The null space can be decomposed into  $S_6$  subspaces spanned by

$$Nv(6 | \text{□□□□}) - \frac{1}{20} v(2; 4 | \text{□□□□}) \tag{6.27a}$$

$$Nv_i(6 | \text{□□□}, 1) + \frac{1}{4} Nv_i(6 | \text{□□□}, 2) - \frac{1}{8} v_i(2; 4 | \text{□□□}, 1) \quad i = 1, \dots, 9 \tag{6.27b}$$

$$Nv_i(6 | \text{□□□}) - \frac{1}{4} v_i(2; 4 | \text{□□□}) \quad i = 1, \dots, 16 \tag{6.27c}$$

$$Nv_i(6 | \boxplus, 1) + Nv_i(6 | \boxplus, 2) + \frac{1}{2}v_i(2; 4 | \boxplus) \quad i = 1, \dots, 5 \quad (6.27d)$$

$$Nv_i\left(6 | \boxplus\right) - \frac{1}{6}v_i\left(3; 3 | \boxplus\right) \quad i = 1, \dots, 5 \quad (6.27e)$$

$$v(2; 4 | \boxplus) + \frac{2}{3}v(3; 3 | \boxplus) \quad (6.27f)$$

$$v_i(2; 4 | \boxplus) \quad i = 1, \dots, 5 \quad (6.27g)$$

$$v_i(2; 4 | \boxplus, 1) + \frac{1}{2}v_i(2; 4 | \boxplus, 2) \quad i = 1, \dots, 9 \quad (6.27h)$$

$$v_i(2; 4 | \boxplus) + \frac{1}{6}v_i(3; 3 | \boxplus) \quad i = 1, \dots, 5 \quad (6.27i)$$

$$v_i(3; 3 | \boxplus) \quad i = 1, \dots, 9 \quad (6.27j)$$

Since all of the irreducible representations in the double-trace representations (6.4) and (6.6) appear in the null space, the corresponding constraints are sufficient to express all 65 of the double-trace amplitudes in terms of single-trace amplitudes. Specifically, eqs. (6.27a,b,c,d,g,h) can be combined to solve for the 45 independent 2;4 double-trace amplitudes

$$A^{(1,1)}(1, 2; 3, 4, 5, 6) = \sum_{\sigma \in COP\{2,1\}\{3,4,5,6\}} A^{(1,0)}(\sigma). \quad (6.28)$$

These relations were deduced in ref. [2] using U(1) decoupling. Equations (6.27e,f,i,j) together with eqs. (6.27a,d) yield the 20 independent 3;3 double-trace amplitudes

$$A^{(1,1)}(1, 2, 3; 4, 5, 6) = (-1) \sum_{\sigma \in COP\{3,2,1\}\{4,5,6\}} A^{(1,0)}(\sigma). \quad (6.29)$$

These relations were obtained in ref. [11] using string-theoretic methods, and later proved field-theoretically in ref. [23].

Because U(1) decoupling relations only involve the *symmetrized* 3;3 amplitudes

$$S_{123}^{(L,2k+1)} \equiv A^{(L,2k+1)}(1, 2, 3; 4, 5, 6) + A^{(L,2k+1)}(1, 2, 3; 4, 6, 5) \quad (6.30)$$

they are not sufficient to derive eq. (6.29), but can be used to obtain [2]

$$S_{123}^{(1,1)} = (-1) \sum_{\sigma \in COP\{3,2,1\}} A^{(1,0)}(\sigma). \quad (6.31)$$

Since the 10 symmetrized 3;3 amplitudes are contained in the irreducible representations  $v_i(3; 3 | \boxplus)$  and  $v_i(3; 3 | \boxplus)$ , eq. (6.31) follows from eqs. (6.27a,f,j) alone. As we will see below, the U(1) decoupling relations (6.28) and (6.31) continue to hold at higher (odd) loops, whereas eq. (6.29) is modified.

**Two-loop six-point null space.** At the two-loop level, we find 80 null vectors, which can be decomposed into irreducible subspaces spanned by

$$N^2 v(6 | \square\square\square\square) - \frac{1}{20} N v(2; 4 | \square\square\square\square) - \frac{1}{30} v(2; 2; 2 | \square\square\square\square) \quad (6.32a)$$

$$N^2 v_i(6 | \square\square\square, 1) + \frac{1}{4} N^2 v_i(6 | \square\square\square, 2) - \frac{1}{4} N v_i(2; 4 | \square\square\square, 1) - \frac{1}{16} N v_i(2; 4 | \square\square\square, 2) + \frac{1}{4} v_i(6 | \square\square\square, 1) \quad i = 1, \dots, 9 \quad (6.32b)$$

$$N^2 v_i(6 | \square\square, 1) + N^2 v_i(6 | \square\square, 2) + \frac{1}{10} N v_i(2; 4 | \square\square) - \frac{1}{15} N v_i(3; 3 | \square\square) - \frac{2}{15} v_i(6 | \square\square, 1) \quad i = 1, \dots, 5 \quad (6.32c)$$

$$N v(2; 4 | \square\square\square\square) + \frac{2}{3} N v(3; 3 | \square\square\square\square) - \frac{1}{3} v(2; 2; 2 | \square\square\square\square) \quad (6.32d)$$

$$N v_i(2; 4 | \square\square\square\square) \quad i = 1, \dots, 5 \quad (6.32e)$$

$$N v_i(2; 4 | \square\square\square, 1) + \frac{1}{2} N v_i(2; 4 | \square\square\square, 2) - \frac{1}{4} v_i(2; 2; 2 | \square\square\square) \quad i = 1, \dots, 9 \quad (6.32f)$$

$$N v_i(3; 3 | \square\square\square) + \frac{3}{8} v_i(2; 2; 2 | \square\square\square) \quad i = 1, \dots, 9 \quad (6.32g)$$

$$v_i(2; 2; 2 | \square\square) \quad i = 1, \dots, 5 \quad (6.32h)$$

$$v(6 | \square\square\square\square) \quad (6.32i)$$

$$v_i(6 | \square\square\square, 1) + \frac{1}{4} v_i(6 | \square\square\square, 2) \quad i = 1, \dots, 9 \quad (6.32j)$$

$$v_i(6 | \square\square\square) \quad i = 1, \dots, 16 \quad (6.32k)$$

$$v_i(6 | \square\square, 1) + v_i(6 | \square\square, 2) \quad i = 1, \dots, 5 \quad (6.32l)$$

$$v_i\left(6 | \begin{array}{|c|} \hline \square \\ \hline \end{array}\right) \quad i = 1, \dots, 5 \quad (6.32m)$$

The two-loop null vectors (6.32i–m) exactly parallel the tree-level null vectors (6.18) and therefore lead to 36 independent relations (previously obtained by Feng et al. [16])

$$A^{(2,2)}(1, \{\alpha\}, 6, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{(2,2)}(1, \sigma, 6) \quad (6.33)$$

among the subleading-color single-trace amplitudes. The relations (6.33) can be derived using U(1) decoupling. Equations (6.32b,c) can be used to obtain 14 additional constraints (not derivable from U(1) decoupling) relating the subleading-color single-trace amplitudes to leading-color single-trace amplitudes as well as double- and triple-trace amplitudes. Unlike the four- and five-point cases, however, one cannot use the set of null vectors (6.32) to solve for all 60 subleading-color single-trace amplitudes  $A^{(2,2)}$  in terms of other amplitudes. This is most easily seen by observing that  $v_i(6 | \begin{array}{|c|} \hline \square \\ \hline \end{array})$  is absent from the null space (6.32).

Since the two-loop null vectors (6.32a,f,h) contain all of the irreducible representations in the triple-trace representation (6.8), they are sufficient to solve for the 15 triple-trace



amplitudes in terms of double-trace amplitudes and leading-color single-trace amplitudes

$$A^{(2,2)}(1, 2; 3, 4; 5, 6) = \frac{1}{2} \left[ S_{12}^{(2,1)} + S_{34}^{(2,1)} + S_{56}^{(2,1)} \right] - \frac{1}{6} \sum_{i < j} S_{ij}^{(2,1)} + \sum_{\sigma \in S_6/\mathbb{Z}_6} A^{(2,0)}(\sigma) \quad (6.34)$$

where  $A^{(L,2k)}(1, 2; 3, 4; 5, 6)$  is the coefficient of  $N^{L-2k} [2 \text{Tr}(12) \text{Tr}(34) \text{Tr}(56)]$  in  $\mathcal{A}_6^{(L)}$ , and  $S_{ij} = S_{ji}$  are the symmetrized 2;4 double-trace amplitudes

$$S_{ij}^{(L,2k+1)} \equiv A^{(L,2k+1)}(i, j; l, p, q, r) + A^{(L,2k+1)}(i, j; l, r, p, q) + A^{(L,2k+1)}(i, j; l, q, r, p) \quad (6.35)$$

where  $\{i, j\} \cup \{l, p, q, r\} = \{1, 2, 3, 4, 5, 6\}$ . By virtue of eq. (2.6) and cyclicity of the trace,  $S_{ij}$  is completely symmetric in  $l, p, q$ , and  $r$ , which can therefore be suppressed. By virtue of eq. (6.32e), the fifteen  $S_{ij}$  are not independent but obey the five independent relations

$$S_{ij}^{(2,1)} + S_{il}^{(2,1)} + S_{jl}^{(2,1)} = S_{pq}^{(2,1)} + S_{pr}^{(2,1)} + S_{qr}^{(2,1)} \quad (6.36)$$

where  $\{i, j, l, p, q, r\}$  is any permutation of  $\{1, 2, 3, 4, 5, 6\}$ .

Because the irreducible representation  $Nv_i \left( 3; 3 \mid \begin{array}{|c|} \hline \square \\ \hline \end{array} \right)$  does not appear in the two-loop null space (6.32), it is not possible to solve for the 3;3 double-trace amplitude  $A^{(2,1)}(1, 2, 3; 4, 5, 6)$  in terms of other amplitudes. Equations (6.32d,g), however, contain the irreducible representations  $Nv_i \left( 3; 3 \mid \begin{array}{|c|} \hline \square\square\square\square \\ \hline \end{array} \right)$  and  $Nv_i \left( 3; 3 \mid \begin{array}{|c|} \hline \square\square \\ \hline \end{array} \right)$ , and are therefore sufficient to solve for the 10 symmetrized 3;3 double-trace amplitudes (6.30)

$$S_{123}^{(2,1)} = -\frac{1}{2} \left[ S_{12}^{(2,1)} + S_{23}^{(2,1)} + S_{13}^{(2,1)} \right] + \frac{1}{2} \sum_{\sigma \in S_6/\mathbb{Z}_6} A^{(2,0)}(\sigma). \quad (6.37)$$

All 66 of the relations (6.33), (6.34), (6.36), and (6.37), which follow from eqs. (6.32a,d-m), can alternatively be derived using U(1) decoupling.

Because the irreducible representation  $Nv_i \left( 2; 4 \mid \begin{array}{|c|} \hline \square\square \\ \hline \end{array} \right)$  does not appear in the two-loop null space (6.32), it is not possible to solve for the 2;4 double-trace amplitude  $A^{(2,1)}(1, 2; 3, 4, 5, 6)$  in terms of other amplitudes. It is possible, however, to express the symmetrized 2;4 double-trace amplitudes  $S_{ij}$ , which correspond to the irreducible representations  $Nv_i \left( 2; 4 \mid \begin{array}{|c|} \hline \square\square\square\square \\ \hline \end{array} \right)$ ,  $Nv_i \left( 2; 4 \mid \begin{array}{|c|} \hline \square\square\square \\ \hline \end{array} \right)$ , and  $Nv_i \left( 2; 4 \mid \begin{array}{|c|} \hline \square\square, 1 \\ \hline \end{array} \right) + \frac{1}{2} Nv_i \left( 2; 4 \mid \begin{array}{|c|} \hline \square\square, 2 \\ \hline \end{array} \right)$ , in terms of other amplitudes, namely,

$$S_{12}^{(2,1)} = -\frac{2}{3} \left[ S_{123}^{(2,1)} + S_{124}^{(2,1)} + S_{125}^{(2,1)} + S_{126}^{(2,1)} \right] + \frac{1}{3} \left[ S_{134}^{(2,1)} + S_{135}^{(2,1)} + S_{136}^{(2,1)} + S_{145}^{(2,1)} + S_{146}^{(2,1)} + S_{156}^{(2,1)} \right] + \frac{1}{3} \sum_{\sigma \in S_6/\mathbb{Z}_6} A^{(2,0)}(\sigma). \quad (6.38)$$

The 15 relations (6.38) are equivalent to eqs. (6.36) and (6.37), and therefore also follow from U(1) decoupling relations.

In total, there are 80 group-theory constraints among 200 color-ordered amplitudes at the two-loop level, 66 of them derivable from U(1) decoupling and 14 that are not.

**Three-loop six-point null space.** At the three-loop level, there are 76 null vectors, which decompose into irreducible subspaces spanned by

$$\begin{aligned}
 N^3 v(6 | \square\square\square\square) - \frac{1}{20} N^2 v(2; 4 | \square\square\square\square) \\
 - \frac{1}{30} N v(2; 2; 2 | \square\square\square\square) + \frac{1}{75} v(2; 4 | \square\square\square\square)
 \end{aligned} \tag{6.39a}$$

$$\begin{aligned}
 N^2 v(2; 4 | \square\square\square\square) + \frac{2}{3} N^2 v(3; 3 | \square\square\square\square) \\
 - \frac{1}{3} N v(2; 2; 2 | \square\square\square\square) + \frac{1}{30} v(2; 4 | \square\square\square\square)
 \end{aligned} \tag{6.39b}$$

$$\begin{aligned}
 N^2 v_i(2; 4 | \boxplus, 1) + \frac{1}{2} N^2 v_i(2; 4 | \boxplus, 2) \\
 + \frac{1}{3} N^2 v_i(3; 3 | \boxplus) - \frac{1}{8} N v_i(2; 2; 2 | \boxplus) \quad i = 1, \dots, 9
 \end{aligned} \tag{6.39c}$$

$$N v(6 | \square\square\square\square) - \frac{1}{20} v(2; 4 | \square\square\square\square) \tag{6.39d}$$

$$N v_i(6 | \boxplus, 1) + \frac{1}{4} N v_i(6 | \boxplus, 2) - \frac{1}{8} v_i(2; 4 | \boxplus, 1) \quad i = 1, \dots, 9 \tag{6.39e}$$

$$N v_i(6 | \boxplus) - \frac{1}{4} v_i(2; 4 | \boxplus) \quad i = 1, \dots, 16 \tag{6.39f}$$

$$N v_i(6 | \boxplus, 1) + N v_i(6 | \boxplus, 2) + \frac{1}{2} v_i(2; 4 | \boxplus) \quad i = 1, \dots, 5 \tag{6.39g}$$

$$N v_i\left(6 | \boxplus\right) - \frac{1}{6} v_i\left(3; 3 | \boxplus\right) \quad i = 1, \dots, 5 \tag{6.39h}$$

$$N v_i(2; 2; 2 | \boxplus) - 2 v_i(2; 4 | \boxplus) - \frac{1}{3} v_i(3; 3 | \boxplus) \quad i = 1, \dots, 5 \tag{6.39i}$$

$$v(2; 4 | \square\square\square\square) + \frac{2}{3} v(3; 3 | \square\square\square\square) \tag{6.39j}$$

$$v_i(2; 4 | \boxplus) \quad i = 1, \dots, 5 \tag{6.39k}$$

$$v_i(2; 4 | \boxplus, 1) + \frac{1}{2} v_i(2; 4 | \boxplus, 2) \quad i = 1, \dots, 9 \tag{6.39l}$$

$$v_i(3; 3 | \boxplus) \quad i = 1, \dots, 9 \tag{6.39m}$$

As we will see, eqs. (6.39d–m) continue to hold at all higher odd-loop levels, and the three-loop constraints derived from them also apply at all higher odd-loop levels. Equations (6.39d–m) are identical to the one-loop null vectors (6.27a–j), except for eq. (6.39i) which contains an extra triple-trace term  $N v_i(2; 2; 2 | \boxplus)$  relative to eq. (6.27i). Equations (6.39d,e,f,g,k,l) allow one to solve for the 45 most-subleading-color 2;4 double-trace amplitudes

$$A^{(2\ell+1, 2\ell+1)}(1, 2; 3, 4, 5, 6) = \sum_{\sigma \in COP\{2,1\}\{3,4,5,6\}} A^{(2\ell+1, 2\ell)}(\sigma) \tag{6.40}$$

but the expression for the 20 most-subleading-color 3;3 double-trace amplitudes at three- and higher loops differs from the one-loop expression (6.29) by the addition of triple-trace

terms

$$\begin{aligned}
 A^{(2\ell+1,2\ell+1)}(1,2,3; 4,5,6) = & (-1) \sum_{\sigma \in COP\{3,2,1\}\{4,5,6\}} A^{(2\ell+1,2\ell)}(\sigma) \\
 & + \frac{1}{2} \sum_{\sigma \in S_3} (-1)^\sigma A^{(2\ell+1,2\ell)}(\sigma(1),4; \sigma(2),5; \sigma(3),6). \quad (6.41)
 \end{aligned}$$

The extra terms in eq. (6.41) cancel from the symmetrized 3;3 double-trace amplitudes (6.30), which are thus given by

$$S_{123}^{(2\ell+1,2\ell+1)} = (-1) \sum_{\sigma \in COP\{3,2,1\}} A^{(2\ell+1,2\ell)}(\sigma) \quad (6.42)$$

and have the same form as eq. (6.31). Both eqs. (6.40) and (6.42) can be derived using U(1) decoupling but eq. (6.41) cannot. There are 11 additional relations that follow from eqs. (6.39a,b,c) that also cannot be derived using U(1) decoupling, yielding altogether 76 group-theory constraints on three-loop color-ordered amplitudes.

**Even-loop six-point null space ( $L \geq 4$ ).** We find 76 null vectors at the four-loop level, which can be decomposed into irreducible subspaces spanned by

$$\begin{aligned}
 N^4 v(6 | \text{□□□□}) - \frac{9}{20} N^3 v(2;4 | \text{□□□□}) \\
 - \frac{4}{15} N^3 v(3;3 | \text{□□□□}) + \frac{1}{10} N^2 v(2;2;2 | \text{□□□□}) \quad (6.43a)
 \end{aligned}$$

$$\begin{aligned}
 N^3 v_i(2;4 | \text{□□□}, 1) + \frac{1}{2} N^3 v_i(2;4 | \text{□□□}, 2) + \frac{1}{3} N^3 v_i(3;3 | \text{□□□}) \\
 - \frac{1}{8} N^2 v_i(2;2;2 | \text{□□□}) - N^2 v_i(6 | \text{□□□}, 1) - \frac{1}{4} N^2 v_i(6 | \text{□□□}, 2) \\
 + \frac{1}{4} N v_i(2;4 | \text{□□□}, 1) + \frac{1}{16} N v_i(2;4 | \text{□□□}, 2) - \frac{1}{4} v_i(6 | \text{□□□}, 1) \quad i = 1, \dots, 9 \quad (6.43b)
 \end{aligned}$$

$$N^2 v(6 | \text{□□□□}) - \frac{1}{20} N v(2;4 | \text{□□□□}) - \frac{1}{30} v(2;2;2 | \text{□□□□}) \quad (6.43c)$$

$$N v(2;4 | \text{□□□□}) + \frac{2}{3} N v(3;3 | \text{□□□□}) - \frac{1}{3} v(2;2;2 | \text{□□□□}) \quad (6.43d)$$

$$N v_i(2;4 | \text{□□□□}) \quad i = 1, \dots, 5 \quad (6.43e)$$

$$N v_i(2;4 | \text{□□□}, 1) + \frac{1}{2} N v_i(2;4 | \text{□□□}, 2) - \frac{1}{4} v_i(2;2;2 | \text{□□□}) \quad i = 1, \dots, 9 \quad (6.43f)$$

$$N v_i(3;3 | \text{□□□}) + \frac{3}{8} v_i(2;2;2 | \text{□□□}) \quad i = 1, \dots, 9 \quad (6.43g)$$

$$v_i(2;2;2 | \text{□□}) \quad i = 1, \dots, 5 \quad (6.43h)$$

$$v(6 | \text{□□□□}) \quad (6.43i)$$

$$v_i(6 | \square\square\square, 1) + \frac{1}{4}v_i(6 | \square\square\square, 2) \quad i = 1, \dots, 9 \quad (6.43j)$$

$$v_i(6 | \square\square) \quad i = 1, \dots, 16 \quad (6.43k)$$

$$v_i(6 | \square\square, 1) + v_i(6 | \square\square, 2) \quad i = 1, \dots, 5 \quad (6.43l)$$

$$v_i\left(6 | \begin{array}{|c|} \hline \square \\ \hline \end{array}\right) \quad i = 1, \dots, 5 \quad (6.43m)$$

Here eqs. (6.43c–m) are identical to the two-loop null vectors (6.32a,d–m), and this pattern persists for all higher even-loop orders. Hence the 66 relations derived from these equations hold for all even-loop orders

$$A^{(2\ell, 2\ell)}(1, \{\alpha\}, 6, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{(2\ell, 2\ell)}(1, \sigma, 6), \quad (6.44)$$

$$A^{(2\ell, 2\ell)}(1, 2; 3, 4; 5, 6) = \frac{1}{2} \left[ S_{12}^{(2\ell, 2\ell-1)} + S_{34}^{(2\ell, 2\ell-1)} + S_{56}^{(2\ell, 2\ell-1)} \right] - \frac{1}{6} \sum_{i < j} S_{ij}^{(2\ell, 2\ell-1)} + \sum_{\sigma \in S_6/\mathbb{Z}_6} A^{(2\ell, 2\ell-2)}(\sigma), \quad (6.45)$$

$$S_{123}^{(2\ell, 2\ell-1)} = -\frac{1}{2} \left[ S_{12}^{(2\ell, 2\ell-1)} + S_{23}^{(2\ell, 2\ell-1)} + S_{13}^{(2\ell, 2\ell-1)} \right] + \frac{1}{2} \sum_{\sigma \in S_6/\mathbb{Z}_6} A^{(2\ell, 2\ell-2)}(\sigma), \quad (6.46)$$

$$S_{ij}^{(2\ell, 2\ell-1)} + S_{il}^{(2\ell, 2\ell-1)} + S_{jl}^{(2\ell, 2\ell-1)} = S_{pq}^{(2\ell, 2\ell-1)} + S_{pr}^{(2\ell, 2\ell-1)} + S_{qr}^{(2\ell, 2\ell-1)}. \quad (6.47)$$

Equations (6.44)–(6.47) can be derived using U(1) decoupling (for all  $\ell$ ). The 14 additional two-loop relations derived from eqs. (6.32b,c) are replaced at four- (and higher-) loops by 10 relations that follow from eqs. (6.43a,b), and that cannot be derived using U(1) decoupling. Altogether there are 76 constraints on even-loop color-ordered amplitudes for  $L \geq 4$ .

**Odd-loop six-point null space ( $L \geq 5$ ).** We find 76 null vectors at the five-loop level, which can be decomposed into the  $S_6$  representations,

$$\begin{aligned} & N^5 v(6 | \square\square\square\square) - \frac{9}{20} N^4 v(2; 4 | \square\square\square\square) - \frac{4}{15} N^4 v(3; 3 | \square\square\square\square) \\ & + \frac{1}{10} N^3 v(2; 2; 2 | \square\square\square\square) + \frac{3}{4} N^3 v(6 | \square\square\square\square) - \frac{3}{80} N^2 v(2; 4 | \square\square\square\square) \\ & - \frac{1}{40} N v(2; 2; 2 | \square\square\square\square) + \frac{1}{100} v(2; 4 | \square\square\square\square) \end{aligned} \quad (6.48a)$$

$$\begin{aligned} & N^3 v(6 | \square\square\square\square) - \frac{9}{20} N^2 v(2; 4 | \square\square\square\square) \\ & - \frac{4}{15} N^2 v(3; 3 | \square\square\square\square) + \frac{1}{10} N v(2; 2; 2 | \square\square\square\square) \end{aligned} \quad (6.48b)$$

$$\begin{aligned}
 N^2 v_i(2; 4 | \boxplus, 1) + \frac{1}{2} N^2 v_i(2; 4 | \boxplus, 2) \\
 + \frac{1}{3} N^2 v_i(3; 3 | \boxplus) - \frac{1}{8} N v_i(2; 2; 2 | \boxplus) \quad i = 1, \dots, 9 \quad (6.48c)
 \end{aligned}$$

$$N v(6 | \boxplus) - \frac{1}{20} v(2; 4 | \boxplus) \quad (6.48d)$$

$$N v_i(6 | \boxplus, 1) + \frac{1}{4} N v_i(6 | \boxplus, 2) - \frac{1}{8} v_i(2; 4 | \boxplus, 1) \quad i = 1, \dots, 9 \quad (6.48e)$$

$$N v_i(6 | \boxplus) - \frac{1}{4} v_i(2; 4 | \boxplus) \quad i = 1, \dots, 16 \quad (6.48f)$$

$$N v_i(6 | \boxplus, 1) + N v_i(6 | \boxplus, 2) + \frac{1}{2} v_i(2; 4 | \boxplus) \quad i = 1, \dots, 5 \quad (6.48g)$$

$$N v_i(6 | \boxplus) - \frac{1}{6} v_i(3; 3 | \boxplus) \quad i = 1, \dots, 5 \quad (6.48h)$$

$$N v_i(2; 2; 2 | \boxplus) - 2 v_i(2; 4 | \boxplus) - \frac{1}{3} v_i(3; 3 | \boxplus) \quad i = 1, \dots, 5 \quad (6.48i)$$

$$v(2; 4 | \boxplus) + \frac{2}{3} v(3; 3 | \boxplus) \quad (6.48j)$$

$$v_i(2; 4 | \boxplus) \quad i = 1, \dots, 5 \quad (6.48k)$$

$$v_i(2; 4 | \boxplus, 1) + \frac{1}{2} v_i(2; 4 | \boxplus, 2) \quad i = 1, \dots, 9 \quad (6.48l)$$

$$v_i(3; 3 | \boxplus) \quad i = 1, \dots, 9 \quad (6.48m)$$

The five-loop null vectors (6.48d–m) are precisely the same as the three-loop null vectors (6.39d–m), and hence give rise to the same 65 constraints (6.40) and (6.41). Of the 11 additional three-loop relations derived from eqs. (6.39a,b,c), 10 survive, but one is replaced by a relation that follows from eq. (6.48a). None of the additional relations follow from U(1) decoupling. Altogether there are 76 group-theory constraints on odd-level color-ordered amplitudes for  $L \geq 5$ .

After five loops, the pattern of null vectors begins to repeat. We find that the six-loop null space is equivalent to the four-loop null space, and the seven-loop null space is equivalent to the five-loop null space. By examining the structure of eq. (6.24), one can then prove that all even-loop null spaces with  $L \geq 4$  are equivalent to the four-loop null space, and that all odd-loop null spaces with  $L \geq 5$  are equivalent to the five-loop null space. Hence the group theory constraints at four and five loops are repeated at all higher even- and odd-loop levels respectively.

Finally, a word about how we determined the coefficients in the preceding equations. As we have seen, a given irreducible representation of  $S_6$  can appear multiple times in the induced representations given in eqs. (6.2), (6.4), (6.6), and (6.8); e.g., the  $\boxplus$  representation appears twice in the single-trace representation, and once in each of the multiple-trace representations. Null vectors belong to certain specific linear combinations of these irreducible representations. To determine these linear combinations, we construct the most general linear combination of a given tableau  $\tau$  that can appear in the  $L$ -loop trace space. For example, the most general five-dimensional representation  $\boxplus$  appearing in the two-loop

trace basis is

$$\begin{aligned}
 u_i = & \kappa_1 N^2 v_i(6 | \boxplus, 1) + \kappa_2 N^2 v_i(6 | \boxplus, 2) + \kappa_3 N v_i(2; 4 | \boxplus) + \kappa_4 N v_i(3; 3 | \boxplus) \\
 & + \kappa_5 v_i(2; 2; 2 | \boxplus) + \kappa_6 v_i(6 | \boxplus, 1) + \kappa_7 v_i(6 | \boxplus, 2). \tag{6.49}
 \end{aligned}$$

We expand  $u_i$  in the two-loop trace basis

$$u_i = \sum_{\lambda=1}^{200} t_{\lambda}^{(2)} u_{\lambda i}. \tag{6.50}$$

It is only necessary to carry out this procedure for a single element, e.g.,  $i = 1$ , because if  $u_1$  belongs to the null space, so does the rest of the irreducible representation. We then act on  $u_1$  with  $M^{(2)}$ , calculated as described earlier in this section, and set the result to zero to determine the conditions that must be satisfied by the coefficients  $\kappa$ . In this case we find

$$2\kappa_1 = 2\kappa_2 = 20\kappa_3 = -30\kappa_4 = 15(\kappa_7 - \kappa_6) \tag{6.51}$$

allowing for three independent solutions which are precisely those occurring in the two-loop null space, viz. eqs. (6.32c,h,l). Although we used a specific choice for the trace basis to determine the coefficients, the result is independent of this choice.

Another consistency check that we found useful is the following. If we take any  $L$ -loop null vector, drop the  $N$ -independent part, and divide the rest by  $N$ , the result must necessarily belong to the  $(L - 1)$ -loop null space. This follows from eq. (6.26) if we set  $G$  in eq. (6.22) equal to  $N$  times the unit matrix.

## 7 Kleiss-Kuijf relations at higher loops?

Throughout this paper, we have encountered the tree-level Kleiss-Kuijf (KK) relations [10, 23]

$$A^{(0,0)}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_{\beta}} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{(0,0)}(1, \sigma, n). \tag{7.1}$$

This set of  $\frac{1}{2}(n - 3)(n - 2)!$  relations follows from  $U(1)$  decoupling for  $n \leq 6$ , and  $U(1)$  decoupling arguments also apply to the most-subleading-color single-trace amplitudes at all even-loop orders

$$A^{(2\ell, 2\ell)}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_{\beta}} \sum_{\sigma \in OP\{\alpha\}\{\beta^T\}} A^{(2\ell, 2\ell)}(1, \sigma, n), \quad n = 4, 5, 6 \tag{7.2}$$

as we have seen in eqs. (4.13), (5.17), and (6.44).

For  $n \geq 7$ , however, the tree-level KK relations cannot be derived from  $U(1)$  decoupling [10], and so the question naturally arises whether eq. (7.2) holds for  $n \geq 7$ , i.e., whether the KK relations apply at loop level. Feng et. al [16] found that for  $n = 7$ , the relations (7.2) hold at two loops, but were not able to establish them for  $n = 8$ .

We have confirmed that the 240 group-theory relations obeyed by the two-loop subleading-color single-trace amplitudes are identical to those obeyed by the tree-level amplitudes for  $n = 7$ . However, we found that, while the tree-level eight-point amplitudes obey the 1800 independent relations given by eq. (7.1), the two-loop subleading-color single-trace eight-point amplitudes satisfy only 1786 group-theory relations among themselves, 14 shy of the number necessary to establish eq. (7.2) for  $\ell = 1$ . Hence the relations (7.2) are *not* valid for  $n = 8$  and  $\ell = 1$  (or at least cannot be derived from group theory alone).<sup>13</sup>

The eight-point single-trace representation decomposes into the following irreducible representations of  $S_8$

$$\begin{aligned}
 R_8^{(\text{ind})} &= \text{[diagram]} \oplus 3 \text{[diagram]} \oplus \text{[diagram]} \oplus 4 \text{[diagram]} \oplus 2 \text{[diagram]} \oplus 3 \text{[diagram]} \oplus 3 \text{[diagram]} \\
 2520 &= 1 + 3 \cdot 20 + 28 + 4 \cdot 64 + 2 \cdot 35 + 3 \cdot 14 + 3 \cdot 70 \\
 &\quad \oplus 7 \text{[diagram]} \oplus 4 \text{[diagram]} \oplus 4 \text{[diagram]} \oplus \text{[diagram]} \oplus 5 \text{[diagram]} \oplus 4 \text{[diagram]} \oplus 4 \text{[diagram]} \\
 &\quad + 7 \cdot 56 + 4 \cdot 90 + 4 \cdot 35 + 42 + 5 \cdot 56 + 4 \cdot 70 + 4 \cdot 64 \\
 &\quad \oplus \text{[diagram]} \oplus 3 \text{[diagram]} \oplus 2 \text{[diagram]} \\
 &\quad + 21 + 3 \cdot 14 + 2 \cdot 20
 \end{aligned} \tag{7.3}$$

which contains 6 copies of 14-dimensional representations. We conclude that one of these representations contained in the tree-level eight-point null space must be absent from the two-loop eight-point null space, although we have not tried to ascertain which.

In conclusion, for  $n \geq 8$ , the KK relations do not extend beyond tree level (or at least cannot be established using group theory alone).

## 8 Concluding remarks

In this paper, we have extended the iterative approach of refs. [13, 14] to obtain the complete set of relations (at all loop orders) obeyed by color-ordered six-point amplitudes derivable from  $SU(N)$  group theory alone. We have also shown how group-theory relations among  $n$ -point amplitudes can be decomposed into subsets associated with irreducible representations of the symmetric group  $S_n$  acting on the external legs. We used this to present the  $L$ -loop relations among four-, five-, and six-point amplitudes in a compact way by listing the irreducible subspaces of the  $L$ -loop,  $n$ -point null space.

At tree level, our six-point results reproduce the 36 Kleiss-Kuijff relations (6.19), which follow from  $U(1)$  decoupling, and at one loop, we reproduce the 65 relations (6.28), (6.29) expressing the double-trace amplitudes in terms of single-trace amplitudes originally found in refs. [2, 11]. At two loops, we obtained 80 relations, of which 66 are derivable from  $U(1)$  decoupling and 14 are not. The 66 two-loop relations include 36 Kleiss-Kuijff relations among the subleading-color single-trace amplitudes (6.33), 15 relations expressing the triple-trace amplitudes (6.34), and 15 relations among double-trace and leading-color single-trace amplitudes (6.36), (6.37). At three loops, we obtained 76 relations, of which

<sup>13</sup>We also checked that the group-theory relations for two-loop subleading-color single-trace nine-point amplitudes are not sufficient to establish eq. (7.2).

65 can be used to express the most-subleading-color double-trace amplitudes in terms of single- and triple-trace amplitudes. Of these 65 three-loop relations, 55 can be derived from U(1) decoupling (6.40), (6.42) and have a form analogous to the one-loop relations. The other three-loop relations (6.41) differ from the one-loop relations (6.29) by the addition of triple-trace amplitudes. At all higher loops, there are 76 group-theory relations among the color-ordered amplitudes, and the pattern of relations begins to repeat after five loops. Further details may be found in section 6.

We should mention one final caveat. In our iterative approach, we made the assumption that one can construct the complete  $(L + 1)$ -loop color space by attaching rungs between the external legs of  $L$ -loop color factors in all possible ways. There are no known counterexamples to this assumption, but neither has it been proved. If this assumption were incorrect, then some of the  $L$ -loop color spaces would be larger than we have surmised, and the complementary null spaces smaller. Consequently, some of the irreducible subspaces of  $S_n$  would be absent from the null spaces, and there would be fewer group-theory relations than we have stated. In no case, however, can there be any additional null vectors or new group-theory relations among color-ordered amplitudes.

## Acknowledgments

It is a pleasure to thank W. Barker, L. Dixon, and T. Pietraho for helpful discussions and remarks. We are especially grateful to C. Boucher-Veronneau for her involvement in the early stages of this project.

## A Decomposition of trace representations

To find the decomposition into irreducible representations of  $S_n$  of a representation invariant under a subgroup  $B \subset S_n$  we note that the number of times a given irreducible representation  $\tau$  appears in the decomposition is given by [28]

$$\gamma^\tau = \frac{1}{n_B} \sum_{\sigma \in B} \chi^\tau(\sigma), \tag{A.1}$$

where  $\chi^\tau(\sigma)$  is the character of  $\sigma$  in the irreducible representation  $\tau$  of  $S_n$ , and  $n_B$  is the order of  $B$ .

For  $n$  even, the single-trace basis

$$\text{Tr}(123 \cdots n) + (-1)^n \text{Tr}(n \cdots 321) \tag{A.2}$$

is invariant under the dihedral subgroup  $D_n \subset S_n$  of order  $2n$ , generated by cyclic and reflection permutations. Taking, for example, the four-point case, the elements of  $D_4$  are

$$\{1, (24), (12)(34), (1234), (13), (13)(24), (1432), (14)(23)\}. \tag{A.3}$$



Since the characters only depend on the conjugacy class of  $\sigma$ , we identify each element's conjugacy class, which is determined by the lengths and numbers of cycles

$$\begin{array}{cccccccc}
 1 & (24) & (12)(34) & (1234) & (13) & (13)(24) & (1432) & (14)(23) \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 [1^4] & [2\ 1^2] & [2^2] & [4] & [2\ 1^2] & [2^2] & [4] & [2^2]
 \end{array} \quad (\text{A.4})$$

We then use eq. (A.1) to compute

$$\gamma^\tau = \frac{1}{8} [\chi^\tau([1^4]) + 2\chi^\tau([2\ 1^2]) + 3\chi^\tau([2^2]) + 2\chi^\tau([4])], \quad (\text{A.5})$$

yielding

$$\gamma^{\square\square\square} = 1, \quad \gamma^{\square\square} = 0, \quad \gamma^{\square} = 1, \quad \gamma^{\square} = 0, \quad \gamma^{\square} = 0. \quad (\text{A.6})$$

Thus for the four-point case, the single-trace basis decomposes into

$$R_4^{(\text{ind})} = \square\square\square\square \oplus \square. \quad (\text{A.7})$$

For  $n$  odd, however, the trace basis (A.2) is only invariant up to a sign under the dihedral subgroup.  $D_n$  contains two different types of permutations:  $n$  cyclic permutations  $\sigma_{\text{cyc}}$  and  $n$  reversal permutations  $\sigma_{\text{rev}}$ . These act differently on the antisymmetric basis element

$$\begin{aligned}
 \sigma_{\text{cyc}}[\text{Tr}(123 \cdots n) - \text{Tr}(n \cdots 321)] &= \text{Tr}(123 \cdots n) - \text{Tr}(n \cdots 321) \\
 \sigma_{\text{rev}}[\text{Tr}(123 \cdots n) - \text{Tr}(n \cdots 321)] &= -[\text{Tr}(123 \cdots n) - \text{Tr}(n \cdots 321)].
 \end{aligned} \quad (\text{A.8})$$

To deal with this, we observe that  $D_n$  has (at least) two one-dimensional representations, the trivial representation with

$$\chi_{D_n}^T(\sigma) = 1 \quad (\text{A.9})$$

and the non-trivial one-dimensional representation with

$$\chi_{D_n}^{NT}(\sigma_{\text{cyc}}) = 1, \quad \chi_{D_n}^{NT}(\sigma_{\text{rev}}) = -1. \quad (\text{A.10})$$

Hence, when  $n$  is odd, the trace basis elements are invariant under the action of  $\chi_{D_n}^{NT}(\sigma)\sigma$ :

$$\chi_{D_n}^{NT}(\sigma)\sigma [\text{Tr}(123 \cdots n) - \text{Tr}(n \cdots 321)] = \text{Tr}(123 \cdots n) - \text{Tr}(n \cdots 321), \quad (\text{A.11})$$

and we can use this to modify (A.1)

$$\gamma^\tau = \frac{1}{n_B} \sum_{\sigma \in B} \chi_{D_n}^R(\sigma) \chi^\tau(\sigma), \quad R = \begin{cases} T, & \text{for } n \text{ even,} \\ NT, & \text{for } n \text{ odd.} \end{cases} \quad (\text{A.12})$$

Multiple-trace representations can be treated in a similar fashion.

## B Five-point decomposition in an explicit trace basis

In section 5, we presented the five-point null spaces in terms of irreducible  $S_5$  subspaces in a form independent of the choice of basis. In this appendix, we specify an explicit basis in order to facilitate comparison of the results of this paper with those of ref. [14].

The single-trace basis we use is

$$\begin{aligned}
 T_1 &= [\text{Tr}(12345) - \text{Tr}(15432)], & T_7 &= [\text{Tr}(12543) - \text{Tr}(13452)], \\
 T_2 &= [\text{Tr}(14325) - \text{Tr}(15234)], & T_8 &= [\text{Tr}(14523) - \text{Tr}(13254)], \\
 T_3 &= [\text{Tr}(13425) - \text{Tr}(15243)], & T_9 &= [\text{Tr}(13524) - \text{Tr}(14253)], \\
 T_4 &= [\text{Tr}(12435) - \text{Tr}(15342)], & T_{10} &= [\text{Tr}(12534) - \text{Tr}(14352)], \\
 T_5 &= [\text{Tr}(14235) - \text{Tr}(15324)], & T_{11} &= [\text{Tr}(14532) - \text{Tr}(12354)], \\
 T_6 &= [\text{Tr}(13245) - \text{Tr}(15423)], & T_{12} &= [\text{Tr}(13542) - \text{Tr}(12453)].
 \end{aligned} \tag{B.1}$$

The vectors (5.8) that span the irreducible representations of the single-trace representation  $R_5^{(\text{ind})}$  can be expanded in this basis as

$$v_i(5 | \tau, j) = \sum_{\lambda=1}^{12} T_\lambda v_{\lambda_i}(5 | \tau, j) \tag{B.2}$$

where

$$\begin{aligned}
 v_{\lambda_i}(5 | \mathbb{F}^{\text{pp}}, 1) &= \frac{1}{10} \begin{pmatrix} 2 & 1 & 1 & 2 & 1 & 2 \\ 2 & 1 & -2 & 2 & -1 & -1 \\ 1 & 2 & 2 & 2 & -1 & -1 \\ 1 & 2 & -1 & 2 & 2 & 1 \\ 2 & 2 & -1 & 1 & -2 & 1 \\ 2 & 2 & 1 & 1 & 1 & -2 \\ -1 & -1 & -2 & 2 & 1 & 2 \\ 2 & -1 & -2 & -1 & -1 & 2 \\ -1 & 2 & 2 & -1 & 2 & -1 \\ -1 & -1 & 2 & 2 & 2 & 1 \\ 2 & -1 & -1 & -1 & -2 & -2 \\ -1 & 2 & 1 & -1 & -2 & -2 \end{pmatrix}, \\
 v_{\lambda_i}(5 | \mathbb{F}^{\text{pp}}, 2) &= \frac{1}{10} \begin{pmatrix} -1 & 2 & 2 & -1 & 2 & -1 \\ -1 & 2 & 1 & -1 & -2 & -2 \\ 2 & -1 & -1 & -1 & -2 & -2 \\ 2 & -1 & -2 & -1 & -1 & 2 \\ -1 & -1 & -2 & 2 & 1 & 2 \\ -1 & -1 & 2 & 2 & 2 & 1 \\ -2 & -2 & 1 & -1 & 2 & -1 \\ -1 & -2 & 1 & -2 & -2 & -1 \\ -2 & -1 & -1 & -2 & -1 & -2 \\ -2 & -2 & -1 & -1 & -1 & 2 \\ -1 & -2 & -2 & -2 & 1 & 1 \\ -2 & -1 & 2 & -2 & 1 & 1 \end{pmatrix}.
 \end{aligned} \tag{B.3}$$

The linear combination of these representations that spans the tree-level null space (5.12) is

$$3v_{\lambda_i}(5 | \mathbb{F}^{\text{pp}}, 1) + v_{\lambda_i}(5 | \mathbb{F}^{\text{pp}}, 2) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 \end{pmatrix}. \tag{B.4}$$

These are linear combinations of the six column vectors of  $x^{(0)}$  given in eq. (3.11) of ref. [14], and so span the same space.

The double-trace basis we use is

$$\begin{aligned}
 T_{13} &= \text{Tr}(12) [\text{Tr}(345) - \text{Tr}(543)], & T_{18} &= \text{Tr}(13) [\text{Tr}(245) - \text{Tr}(542)], \\
 T_{14} &= \text{Tr}(23) [\text{Tr}(451) - \text{Tr}(154)], & T_{19} &= \text{Tr}(24) [\text{Tr}(351) - \text{Tr}(153)], \\
 T_{15} &= \text{Tr}(34) [\text{Tr}(512) - \text{Tr}(215)], & T_{20} &= \text{Tr}(35) [\text{Tr}(412) - \text{Tr}(214)], \\
 T_{16} &= \text{Tr}(45) [\text{Tr}(123) - \text{Tr}(321)], & T_{21} &= \text{Tr}(41) [\text{Tr}(523) - \text{Tr}(325)], \\
 T_{17} &= \text{Tr}(51) [\text{Tr}(234) - \text{Tr}(432)], & T_{22} &= \text{Tr}(52) [\text{Tr}(134) - \text{Tr}(431)]. \quad (\text{B.5})
 \end{aligned}$$

The vectors (5.9) that span the irreducible representations of the double-trace representation  $R_{2;3}^{(\text{ind})}$  can be expanded in this basis as

$$v_i(2;3 | \tau, j) = \sum_{\lambda=13}^{22} T_{\lambda} v_{\lambda i}(2;3 | \tau) \quad (\text{B.6})$$

where

$$\begin{aligned}
 v_{\lambda i}(2;3 | \overline{\square}^{\square}) &= \frac{1}{5} \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 3 & 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 3 & 1 & 1 \\ 0 & -1 & -1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & 3 & 1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 & 3 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ -1 & 1 & 3 & 0 & 1 & -1 \end{pmatrix}, \\
 v_{\lambda i}(2;3 | \overline{\square}^{\square}) &= \frac{2}{5} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}. \quad (\text{B.7})
 \end{aligned}$$

By using these explicit decompositions, it is straightforward to show that the vectors spanning odd- and even-loop five-point null spaces (5.14) and (5.16) are linear combinations of the odd- and even-loop null vectors written down in ref. [14].

## References

- [1] M.L. Mangano and S.J. Parke, *Multiparton amplitudes in gauge theories*, *Phys. Rept.* **200** (1991) 301 [[hep-th/0509223](#)] [[INSPIRE](#)].
- [2] Z. Bern and D.A. Kosower, *Color decomposition of one loop amplitudes in gauge theories*, *Nucl. Phys. B* **362** (1991) 389 [[INSPIRE](#)].
- [3] Z. Bern, J. Carrasco and H. Johansson, *New relations for gauge-theory amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [[arXiv:0805.3993](#)] [[INSPIRE](#)].
- [4] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative quantum gravity as a double copy of gauge Theory*, *Phys. Rev. Lett.* **105** (2010) 061602 [[arXiv:1004.0476](#)] [[INSPIRE](#)].
- [5] J.J.M. Carrasco and H. Johansson, *Generic multiloop methods and application to  $N = 4$  super-Yang-Mills*, *J. Phys. A* **44** (2011) 454004 [[arXiv:1103.3298](#)] [[INSPIRE](#)].
- [6] N. Bjerrum-Bohr, P.H. Damgaard and P. Vanhove, *Minimal basis for gauge theory amplitudes*, *Phys. Rev. Lett.* **103** (2009) 161602 [[arXiv:0907.1425](#)] [[INSPIRE](#)].

- [7] S. Stieberger, *Open & closed vs. pure open string disk amplitudes*, [arXiv:0907.2211](#) [[INSPIRE](#)].
- [8] B. Feng, R. Huang and Y. Jia, *Gauge amplitude identities by on-shell recursion relation in s-matrix program*, *Phys. Lett. B* **695** (2011) 350 [[arXiv:1004.3417](#)] [[INSPIRE](#)].
- [9] Y.-X. Chen, Y.-J. Du and B. Feng, *A proof of the explicit minimal-basis expansion of tree amplitudes in gauge field theory*, *JHEP* **02** (2011) 112 [[arXiv:1101.0009](#)] [[INSPIRE](#)].
- [10] R. Kleiss and H. Kuijf, *Multi-gluon cross-sections and five jet production at hadron colliders*, *Nucl. Phys. B* **312** (1989) 616 [[INSPIRE](#)].
- [11] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys. B* **425** (1994) 217 [[hep-ph/9403226](#)] [[INSPIRE](#)].
- [12] Z. Bern, A. De Freitas and L.J. Dixon, *Two loop helicity amplitudes for gluon-gluon scattering in QCD and supersymmetric Yang-Mills theory*, *JHEP* **03** (2002) 018 [[hep-ph/0201161](#)] [[INSPIRE](#)].
- [13] S.G. Naculich, *All-loop group-theory constraints for color-ordered SU(N) gauge-theory amplitudes*, *Phys. Lett. B* **707** (2012) 191 [[arXiv:1110.1859](#)] [[INSPIRE](#)].
- [14] A.C. Edison and S.G. Naculich, *SU(N) group-theory constraints on color-ordered five-point amplitudes at all loop orders*, *Nucl. Phys. B* **858** (2012) 488 [[arXiv:1111.3821](#)] [[INSPIRE](#)].
- [15] N.E.J. Bjerrum-Bohr, P. Damgaard, H. Johansson and T. Sondergaard, *Monodromy-like relations for finite loop amplitudes*, *JHEP* **05** (2011) 039 [[arXiv:1103.6190](#)] [[INSPIRE](#)].
- [16] B. Feng, Y. Jia and R. Huang, *Relations of loop partial amplitudes in gauge theory by unitarity cut method*, *Nucl. Phys. B* **854** (2012) 243 [[arXiv:1105.0334](#)] [[INSPIRE](#)].
- [17] R.H. Boels and R.S. Isermann, *New relations for scattering amplitudes in Yang-Mills theory at loop level*, *Phys. Rev. D* **85** (2012) 021701 [[arXiv:1109.5888](#)] [[INSPIRE](#)].
- [18] R.H. Boels and R.S. Isermann, *Yang-Mills amplitude relations at loop level from non-adjacent BCFW shifts*, *JHEP* **03** (2012) 051 [[arXiv:1110.4462](#)] [[INSPIRE](#)].
- [19] Y.-J. Du, B. Feng and C.-H. Fu, *Note on permutation sum of color-ordered gluon amplitudes*, *Phys. Lett. B* **706** (2012) 490 [[arXiv:1110.4683](#)] [[INSPIRE](#)].
- [20] Y.-J. Du and H. Lüo, *On general BCJ relation at one-loop level in Yang-Mills theory*, [arXiv:1207.4549](#) [[INSPIRE](#)].
- [21] D. Vaman and Y.-P. Yao, *Constraints and generalized gauge transformations on tree-level gluon and graviton amplitudes*, *JHEP* **11** (2010) 028 [[arXiv:1007.3475](#)] [[INSPIRE](#)].
- [22] V. Del Duca, A. Frizzo and F. Maltoni, *Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit*, *Nucl. Phys. B* **568** (2000) 211 [[hep-ph/9909464](#)] [[INSPIRE](#)].
- [23] V. Del Duca, L.J. Dixon and F. Maltoni, *New color decompositions for gauge amplitudes at tree and loop level*, *Nucl. Phys. B* **571** (2000) 51 [[hep-ph/9910563](#)] [[INSPIRE](#)].
- [24] J.J. Carrasco and H. Johansson, *Five-point amplitudes in N = 4 super-Yang-Mills theory and N = 8 supergravity*, *Phys. Rev. D* **85** (2012) 025006 [[arXiv:1106.4711](#)] [[INSPIRE](#)].
- [25] F.A. Berends and W. Giele, *The six gluon process as an example of Weyl-Van Der Waerden spinor calculus*, *Nucl. Phys. B* **294** (1987) 700 [[INSPIRE](#)].

- [26] M.L. Mangano, S.J. Parke and Z. Xu, *Duality and multi-gluon scattering*, *Nucl. Phys. B* **298** (1988) 653 [INSPIRE].
- [27] F.A. Berends and W. Giele, *Recursive calculations for processes with  $n$  gluons*, *Nucl. Phys. B* **306** (1988) 759 [INSPIRE].
- [28] I.V. Schensted, *A course on the application of group theory to quantum mechanics*, NEO Press, Peaks Island U.S.A. (1976).
- [29] D.E. Littlewood, *The theory of group characters and matrix representations of groups*, Oxford University Press, Oxford U.K. (1940).