Singular points in $\mathcal{N} = 2$ SQCD

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ABSTRACT: We revisit the study of singular points in $\mathcal{N} = 2$ SQCD with classical gauge groups. Using a technique proposed recently by Gaiotto, Seiberg and Tachikawa we find that the low-energy physics at the maximally singular point involves two superconformal sectors coupled to an infrared free SU(2) gauge group. When one softly breaks extended supersymmetry to $\mathcal{N} = 1$ adding a mass term for the chiral multiplet in the adjoint representation, a finite number of vacua remain and the theory becomes confining. Our analysis allows us to identify the low-energy physics at these distinguished points in the moduli space. In some cases, which we will describe in detail, two sectors coupled to an infrared free SU(2) gauge group emerge as before. For USp and SO gauge groups one of these sectors is always free, contrary to the SU case.

Contents

1 Introduction

Theories with $\mathcal{N} = 2$ supersymmetry represent an interesting theoretical laboratory to study nonperturbative effects in four dimensions. The great virtue of these models is that the low energy dynamics is explicitly known and can be encoded in a family of algebraic curves, as shown for SU(2) gauge theories by Seiberg and Witten in [1, 2] and then for more general gauge groups in [3, 4] (see also [5] and references therein).

It was soon realized that superconformal fixed points are ubiquitous in these models and their study was initiated in [6] and [7]. In particular, in the latter reference general properties of conformal theories with $\mathcal{N} = 2$ supersymmetry were derived, such as the fact that mass parameters associated to a nonAbelian global symmetry cannot acquire anomalous dimension. A more systematic analysis was initiated in [8] and especially [9], in which the authors classified singular points in SU(N) SQCD with N_f flavors. All these papers are based on the idea that the SW curve in a neighbourhood of the singular point should exhibit scale invariance. Combining this with the requirement that the SW differential has scaling dimension one fixes the scaling dimensions of all the chiral operators. This analysis revealed the existence (for any value of the bare mass m of the flavors) of singular submanifolds in the moduli space such that the SW curve factorizes as $y^2 = (x+m)^{2r}Q(x)$ ($2r \leq N_f$). The low energy dynamics is described by a nonAbelian SU(r) theory with N_f massless matter fields in the fundamental. Only in the case $2r = N_f$ we have an interacting fixed point. Tuning m appropriately one can find points in the moduli space where the curve becomes more singular. In this latter case the approach of [9] leads to anomalous dimensions for the casimirs of the nonAbelian flavor group, contrary to the argument given in [7].

More recently the situation has been reanalyzed in [10], in which the authors show that this problem can be solved if one allows for the existence of two scale invariant sectors, weakly coupled by a gauge field. This is analogous to Argyres-Seiberg duality [11] (and

generalization thereof), apart from the fact that the gauge group appearing in the dual description is infrared free. This proposal also removes a possible counterexample to the a-theorem [12] (see also [13, 14]).

In [9] the authors analyzed singular points in $\mathrm{USp}(2N)$ and $\mathrm{SO}(N)$ gauge theories as well. The result of their study is that, as long as the flavors are massive, the singular points are identical to those of $\mathrm{SU}(N)$ SQCD. However, when the masses are set to zero the flavor symmetry enhances and one finds a different class of fixed points. At that time no tools were available to study them but now, with the techniques of [11] and the methods developed by Gaiotto in [15] and Tachikawa in [16], the problem can be approached. The scope of this note is to make a systematic analysis of these singular points.

Another fundamental aspect of $\mathcal{N}=2$ gauge theories is that breaking softly extended supersymmetry with a mass term for the chiral superfield in the $\mathcal{N}=2$ vector multiplet, the theory becomes confining [1]: the moduli space is lifted and a finite number of vacua remain in which magnetically charged objects condense realizing the 't Hooft-Mandelstam mechanism for confinement. The low energy dynamics of these models have been extensively studied (an incomplete list of references is [17–20]). It turned out that the relevant vacua are generically characterized by a nonAbelian gauge symmetry in the infrared. When the flavors are massive the properties of these vacua are "universal" and do not depend on the gauge group (for classical gauge groups). In the massless limit this picture does not change for $\mathrm{SU}(N)$ theories, whereas a different phenomenon occurs for SO and USp theories [18, 20]: only two vacua remain. One is characterized by the condensation of baryonic-like composite objects and is in a nonAbelian Coulomb phase (this vacuum was identified in [19]). The second one (called Chebyshev point in [18, 20]) arises from the collision of the other vacua and is in general characterized by a strongly interacting low energy theory, which exhibits conformal invariance in the $\mathcal{N}=2$ limit. In these sense the study of confinement in the softly broken theory and the analysis of singular points in the parent $\mathcal{N}=2$ theory are linked. As we will see, the analysis of maximally singular points will allow us to understand the low energy physics at the Chebyshev point as well. The properties of the theory once the $\mathcal{N}=1$ perturbation has been turned on will be studied in a separate publication.

The paper is organized as follows: in section 2 we review the argument given in [10], which will be the key ingredient of our analysis, and explain the properties of vacua relevant for the perturbation to $\mathcal{N}=1$. In section 3 we determine the structure of the maximally singular point and the Chebyshev point in $\mathrm{USp}(2N)$ gauge theory with $2n$ flavors. In section 4 we repeat this analysis for SO gauge theories and we conclude with a discussion in section 5. As a byproduct we will recover many of the infinite coupling dualities proposed recently.

2 SU(N) SQCD with $2n$ flavors and r-vacua

We will now sketch the argument presented in [10]. As is well known, the SW curve and differential for SU(N) SQCD with $N_f = 2n$ flavors are

$$y^2 = P_N(x) - 4\Lambda^{2N-2n} \prod_{i=1}^{2n} (x + m_i), \quad \lambda = xd \log \frac{P_N - y}{P_N + y},$$

with $P_N(x) = x^N - \sum_{k \geq 2} u_k x^{N-k}$. For our purposes, it is convenient to rewrite the curve in the following way (see [10] for the details):

$$y^2 = (x^N - \cdots - u_N)(x^N - \cdots + (4\Lambda^{N-n} - u_{N-n})x^n - \cdots - u_N) - \sum_{k=2}^{2n} c_k x^{2n-k}.$$

Setting all the coulomb branch coordinates u_i and casimirs of the flavor symmetry c_i to zero, we find the maximally singular (EHIY) point, where the SW curve and SW differential become:

$$y^2 = x^{N+n}(x^{N-n} + 4\Lambda^{N-n}), \quad \lambda \approx \frac{y}{x^n}dx.$$

Requiring that the SW differential has dimension one gives the relation $[y] = 1 + (n-1)[x]$. If we further impose the scale invariance of the curve we find the equation $2[y] = (N+n)[x]$. These relations fix the scaling dimensions of x and y and in particular imply an anomalous dimension for the cubic and higher casimirs of the flavor group ($[c_i] = (2N+i)/(N+1)$). So, when n is at least two, the above analysis is inconsistent with the general constraints for theories with $\mathcal{N} = 2$ superconformal symmetry (e.g. the c_i's should have canonical dimension [7]).

A natural resolution of this inconsistency is to identify subsectors with different scalings of x; clearly the $N + n$ colliding branch points will distribute among the subsectors. The proposal of [10] is precisely along this line: the authors introduce a particular scaling limit in which two subsectors emerge: one is a D_{N-n+2} Argyres-Douglas theory [9, 21] (or maximally singular point of SU($N-n+1$) gauge theory with 2 flavors [10]; see also [22, 23]) and the other can be described as a three punctured sphere in the Gaiotto framework [15], as we will now see.

The first step is to rewrite the curve in a "6d form": one defines $t = y/x^{n-1}$ (so the SW differential becomes $\lambda \approx t\,dx/x$) and writes the curve as

$$
\begin{aligned}
t^2 = &\left(x^{N-n+2} - u_1 x^{N-n+1} - \cdots - u_{N-n+2} - \cdots - \frac{u_N}{x^{n-2}} \right) \\
&\times \left(x^{N-n} - \cdots + (4\Lambda^{N-n} - u_{N-n}) - \cdots - \frac{u_N}{x^n} \right) - \sum_{k=2}^{2n} c_k x^{2-k}.
\end{aligned}
\tag{2.1}
$$

Notice the presence of u_1, which is a parameter proportional to $\sum_i m_i$ and not a coordinate on the Coulomb branch. This will be important in later sections. To account for the presence of two sectors, we now introduce two scales $\epsilon_A, \epsilon_B \ll 1$. Notice that with the above rescaling the condition $[\lambda] = 1$ implies $[t] = 1$ in both sectors. In the A sector ($|x| \sim \epsilon_A$) we will impose $[x] = 1$, in order to satisfy the constraint $[c_k] = k$. This leads to the relation $c_k \sim O(\epsilon_A^k)$. Consider now the term $4\Lambda^{N-n}x^{N-n+2}$. It is clearly negligible for $|x| \sim \epsilon_A$ (with respect to, e.g. $\sum_{k=2}^{2n} c_k x^{2-k}$), and consequently has to appear in the B sector ($|x| \sim \epsilon_B$). This implies $t^2 \sim \epsilon_B^{N-n+2}$, and since t has scaling dimension one in both sectors, we deduce the relation

$$\epsilon_A^2 = \epsilon_B^{N-n+2}. \tag{2.2}$$

Interestingly, the above considerations and the requirement that all the coulomb branch coordinates appear in at least one sector necessarily imply

$$u_k \sim O(\epsilon_B^k), \quad k = 1, \ldots, N-n+2; \quad u_k \sim O(\epsilon_A^{k+n-N}) \quad k = N-n+2, \ldots, N.$$

Now it is possible to read from (2.1) the curves for the two subsectors just collecting the leading order terms:

1. For $|x| \sim \epsilon_A$ we are left with

$$t^2 = -\left(u_{N-n+2} + \cdots + \frac{u_N}{x^{n-2}}\right)\left(4\Lambda^{N-n} - \frac{u_{N-n+2}}{x^2} - \cdots - \frac{u_N}{x^n}\right) - \sum_{k=2}^{2n} c_k x^{2-k}. \tag{2.3}$$

As discussed in [10], this is the SW curve (when $n > 2$) for the Gaiotto theory obtained compactifying n M5 branes on a sphere with three regular punctures (two are maximal and one is described by a Young tableau with columns of height $\{n-2, 1, 1\}$). Its global symmetry group is $SU(2) \times SU(2n)$ and the corresponding casimirs are c_i and $u_{N-n+2} + c_2/4\Lambda^{N-n}$. This is precisely the interacting theory that enters in the S-dual description of $SU(n)$ theory with 2n flavors and its properties have been studied in detail in [24]. For $n = 3$ this S-duality coincides with Argyres-Seiberg duality and the A sector describes the E_6 theory of Minahan and Nemeschansky [25]. For $n = 2$ the theory becomes free and describes three doublets of $SU(2)$ (the global symmetry is $SU(2) \times SO(6) \simeq SU(4)$).

2. For $|x| \sim \epsilon_B$ we find instead

$$t^2 = 4\Lambda^{N-n}(x^{N-n+2} - u_1 x^{N-n+1} - \cdots - u_{N-n+2}) - c_2. \tag{2.4}$$

This is the SW curve for the D_{N-n+2} theory. For $N = n$ this sector is free and describes a doublet of hypermultiplets.

Both theory 1 and theory 2 have an $SU(2)$ flavor symmetry; in our context the diagonal combination has been gauged and the SW curve for $\epsilon_A < |x| < \epsilon_B$ describes the tubular region associated to this gauge group ($t^2 = -4\Lambda^{N-n}u_{N-n+2} - c_2$).

As explained in [10], one can now evaluate the beta function for this $SU(2)$ gauge group from the above curve: a closed BPS string located in the tubular region at constant $|x|$ describes a W-boson with central charge a, whereas a geodesic connecting a branch point at $|x| \sim \epsilon_B$ and another at $|x| \sim \epsilon_A$ describes a monopole with central charge a_D (see [26] for a detailed discussion on this point).

$$a = \int_{|x|=\text{const.}} \lambda = 2\pi i \alpha; \quad \alpha^2 = -4\Lambda^{N-n} u_{N-n+2} - c_2,$$

$$a_D = \int_{|x|\sim\epsilon_A}^{|x|\sim\epsilon_B} \lambda = \alpha\left(\frac{N-n}{N-n+2}\log\epsilon_A + \text{const.}\right).$$

Using then the relation $\tau = \partial a_D/\partial a$ and identifying ϵ_A with the renormalization group scale we obtain

$$\frac{d\tau}{d(\log\epsilon_A)} = \frac{b_1}{2\pi i} = \frac{1}{2\pi i}\frac{N-n}{N-n+2},$$

where b_1 is the one-loop coefficient of the beta function. We thus learn that this SU(2) group is infrared free. Since the contribution to the beta function from the three punctured sphere is 3, we can read out the contribution given by the D_{N-n+2} theory:

$$b_{D_{N-n+2}} = 2\left(1 - \frac{1}{N-n+2}\right).$$

Indeed, this matches the result of [21] (the calculation can also be performed using the techniques presented in [27]).

In view of the breaking to $\mathcal{N}=1$, the relevant points in the moduli space are those such that the SW curve factorizes as

$$y^2 = (x-\alpha)(x-\beta)Q^2(x). \tag{2.5}$$

As shown in [17, 18], in the case of equal masses m_i, these vacua are labelled by an integer r ($0 \leq r \leq N_f/2$), corresponding to the fact that $Q(x)$ factorizes as $Q(x) = (x+m)^r \tilde{Q}(x)$. When the $\mathcal{N}=1$ perturbation is turned on, for $m \gg \Lambda$ the theory is in the Higgs phase, whereas in the limit $m \ll \Lambda$ it becomes confining. For each value of r there are $2N - N_f$ vacua and the low energy effective theory is characterized by an abelian sector and a nonAbelian one with U(r) gauge group and N_f massless matter fields in the fundamental representation. For $r < N_f/2$ the low energy theory is infrared free and admits a lagrangian description. More interesting is the situation for $r = N_f/2$: in this case the nonAbelian sector of the low energy theory is superconformal and the scaling dimensions of chiral operators can be determined as in [9] (in this case the casimirs have canonical dimension and there is no need to introduce two sectors). For generic values of the mass parameter m, this is the whole story. However, if one sets m equal to

$$m = \omega_{2N-N_f}^k \frac{2N-N_f}{N}\Lambda,$$

one can show [28][1] that some of the r vacua (one for each value of r) collide and the curve becomes more singular. This signals the transition from Higgs to confinement phase [28]. In this limit α or β in equation (2.5) become equal to $-m$ and the SW curve and differential can be approximated as

$$y^2 \approx (x+m)^{N_f+1}, \quad \lambda \approx \frac{y}{x^n}dx.$$

Here we recognize the EHIY point when $N = n + 1$. Indeed, as was argued in [9], the physics of this singular point is that for the EHIY point of SU($n+1$) gauge theory with $2n$ flavors. In this case the B sector is given by the D_3 theory.

We thus propose that the low-energy physics at this point is described by:

- An abelian U(1)$^{N-n-1}$ sector with massless hypermultiplets charged under each U(1) factor.

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[1]In [28] the formula for m is different. The discrepancy is simply due to a different convention (which is the one adopted in [29]): indicating with $\tilde{m}$ and $\tilde{\Lambda}$ the parameters used in those papers, we have $m = \tilde{m}/\sqrt{2}$ and $\Lambda^{2N-N_f} = \sqrt{2}^{N_f}\tilde{\Lambda}^{2N-N_f}$. This point was overlooked in [30].

- The D_3 theory (B sector).

- The scale invariant theory entering in the S-dual description of SU(n) SQCD with $2n$ flavors (A sector).

- An infrared free SU(2) gauge multiplet coupled to sectors A and B.

This is identical to the proposal made in [10], apart from the fact that the abelian sector includes hypermultiplets charged under the various U(1) factors (one for each U(1)). This comes from the requirement that the point we are discussing is not lifted by the $\mathcal{N} = 1$ perturbation [17, 18]. This is not the case for the EHIY point discussed in [10] (apart from the case $N = n + 1$).

3 USp($2N$) SQCD with $2n$ flavors

Let us turn to $\mathcal{N} = 2$ gauge theories with USp gauge group and N_f hypermultiplets in the fundamental representation (we consider only the equal mass case as before). If the bare mass m for the matter fields is different from zero, the flavour symmetry is U(N_f) and, as we said in the introduction, one recovers the results found in the previous section; in particular the vacua surviving the $\mathcal{N} = 1$ perturbation have exactly the same structure as the r-vacua of SU(N) SQCD (see [4, 9, 18, 19]) and all the superconformal points are analogous to those described in the previous section.

More interesting is the case of massless matter fields: the first main difference is the flavor symmetry, which is enhanced to SO($2N_f$). Moreover, all the r vacua merge into a single superconformal point in this limit [18] (we will refer to it as the Chebyshev point from now on, because its location in the Coulomb branch is determined by Chebyshev polynomials [18]). As the symmetry enhancement suggests, this fixed point is different from those we have seen so far. The purpose of this section is to study the superconformal points of massless USp($2N$) SQCD with $N_f = 2n$.

The SW curve and SW differential for this model are [4]:

$$xy^2 = [xP_N(x) + 2\Lambda^{2N-2n+2}\prod_i m_i]^2 - 4\Lambda^{4N-4n+4}\prod_i(x - m_i^2), \tag{3.1}$$

$$\lambda = \frac{\sqrt{x}}{2\pi i}d\log\left(\frac{xP_N(x) + 2\Lambda^{2N-2n+2}\prod_i m_i - \sqrt{x}y}{xP_N(x) + 2\Lambda^{2N-2n+2}\prod_i m_i + \sqrt{x}y}\right). \tag{3.2}$$

where $P_N(x) = x^N - u_1 x^{N-1} - \cdots - u_N$ and m_i are the masses for the flavors. Note that in the SU(N) case u_1 was a parameter while in this case is a coordinate on the Coulomb branch. We can now rewrite the curve as

$$xy^2 = (x^{N+1} - u_1 x^N - \cdots - u_N x + \tilde{c}_{2n})^2 - 4\Lambda^{4N-4n+4}x^{2n} - \sum_i c_{2i}x^{2n-i}, \tag{3.3}$$

where $c_{2i}, \tilde{c}_{2n}$ are the SO($4N$) casimirs. We can further rewrite it as

$$xy^2 = (x^{N+1}\cdots - u_{N-n+1}x^n\cdots + \tilde{c}_{2n})(x^{N+1}\cdots$$
$$- (u_{N-n+1} - 4\Lambda^{2N-2n+2})x^n\cdots + \tilde{c}_{2n}) - \sum_i c_{2i}x^{2n-i}, \tag{3.4}$$

where we just redefined u_{N-n+1}. If we set to zero all c_i and u_k, we find the maximally singular point, where the curve and differential become

$$y^2 = x^{N+n}(x^{N-n+1} + 4\Lambda^{2N-2n+2}), \quad \lambda = \frac{y}{x^n}dx. \tag{3.5}$$

We thus come across the same problem found in the previous section: imposing $[\lambda] = 1$ and $[y] = \frac{N+n}{2}[x]$ leads to anomalous dimensions for the nonAbelian casimirs ($c_{2i} = 2 + (i-1)[x] = 2\frac{N-n+1+i}{N-n+2}$). In order to determine the structure of this infrared fixed point, we can adopt the technique seen before and introduce two different sectors.

It is now convenient to define $t = y/x^{n-1}$ and rewrite the curve as

$$t^2 = \left(x^{N+2-n} \cdots + \frac{\tilde{c}_{2n}}{x^{n-1}}\right)\left(x^{N+1-n} \cdots + \frac{\tilde{c}_{2n}}{x^n}\right) - \sum_i c_{2i}x^{2-i}. \tag{3.6}$$

The constraint $[c_{2i}] = 2i$ can be satisfied introducing the scale ϵ_A and setting $c_{2i} \sim O(\epsilon_A^{2i})$, $|x| \sim \epsilon_A^2$. This gives $t \sim \epsilon_A$. A second sector emerges as we introduce the scale ϵ_B and set $|x| \sim \epsilon_B^2$. The same reasoning adopted in the previous section leads to the relation $t^2 \sim x^{N+2-n}$, from which we deduce

$$\epsilon_B^{2N+4-2n} = \epsilon_A^2.$$

The Coulomb branch coordinates are then scaled to zero as

$$u_i \sim O(\epsilon_B^{2i}) \quad i = 1, \ldots, N-n+2; \quad u_{N-n+2+i} \sim O(\epsilon_A^{2+2i}) \quad i = 0, \ldots, n-2.$$

Collecting the leading terms as before we can now determine the SW curves for the two sectors.

For $|x| \sim \epsilon_A^2$ the curve becomes

$$t^2 = \left(u_{N+2-n} + \cdots + \frac{\tilde{c}_{2n}}{x^{n-1}}\right)\left(4\Lambda^{2N+2-2n} - \frac{u_{N-n+2}}{x} + \cdots + \frac{\tilde{c}_{2n}}{x^n}\right) - \sum_i c_{2i}x^{2-i}. \tag{3.7}$$

It has $2n - 2$ branch points.

The remaining $N - n + 2$ branch points appear in the second sector, for $|x| \sim \epsilon_B^2$. The curve becomes in this case

$$t^2 = 4\Lambda^{2N+2-2n}(x^{N+2-n} - \cdots - u_{N-n+2}) - c_2. \tag{3.8}$$

Let us analyze these two regions.

For $|x| \sim \epsilon_B^2$ we recognize the curve we have seen before: this is the SW curve for the D_{N-n+2} theory. The only difference with respect to the SU(N) case is that, as we noticed before, u_1 is a coordinate on the Coulomb branch in the present context. The flavor symmetry of this theory is thus just SU(2). Two special cases are $N = n$, when the theory becomes free and describes a doublet of SU(2), and $N = n - 1$, when the curve becomes trivial and describes an "empty" theory [11].

The curve for the region $|x| \sim \epsilon_A^2$ is new; it has SU(2) $\times$ SO($4n$) flavor symmetry and can be described as the compactification on a three punctured sphere of the 6d $(2,0)$ D_n theory [16], as we will now see, for $n > 2$. For $n = 2$ it becomes a free theory.

The SW curve for this class of theories can be written in the form

$$\lambda^{2N} = \sum_k \phi_{2k}(z)\lambda^{2N-2k}, \quad \lambda = v\frac{dz}{z}. \tag{3.9}$$

The theory is specified by the singularities on the Riemann surface, which are labelled by Young tableaux (in the case of regular punctures) as in the A_N case. From the Young tableaux one can read out the pole structure of the various k-differentials and then determine the Coulomb branch coordinates using Riemann-Roch theorem.[2] The pole structure at each puncture can be determined as follows [16, 31, 32]:

- Take the longest even row in the Young tableau which occurs with odd multiplicity (in our case the row of length four) and remove the last box. Place it at the end of the next available row (such that the result is a Young tableau). Repeat this operation until it stops (the resulting Young tableau does not contain even rows with odd multiplicity).

- Number the boxes of the "corrected" Young tableau as follows: start with zero in the first box and number the boxes in the first row with successive integers. When you reach the end of the row, repeat that number in the first box of the following row and continue.

The numbers inserted in boxes number $2, 4, \ldots, 2N$ are the orders of the pole of $\phi_2, \phi_4, \ldots, \phi_{2N} = (\tilde{\phi}_N)^2$ at the given puncture. The algorithm for A_N punctures is obtained just neglecting the first step (odd degree differentials ϕ_{2k+1} do not vanish in the A_N case, and the corresponding degree of the pole is the integer contained in boxes number $2k+1$).

Let us apply the above algorithm to a sphere (depicted in figure 1) with two maximal punctures (labelled by a Young tableau with a single row of length $2n$) and a third one labelled by a Young tableau (always with $2n$ boxes) with a row of length 4 and the others of length one (these are all grey punctures in the notation of [16]). The pole structure at the maximal puncture is $\{1, 3, \ldots, 2n-3; n-1\}$, whereas the other puncture assigns pole orders $\{1, 2, \ldots, 2; 1\}$. The last entry represents the order of the pole of $\tilde{\phi}_n$. The k differentials can thus be written as

$$\phi_{2k} = 2\frac{u_{2k}z}{(z-1)^2}\left(\frac{dz}{z}\right)^{2k} \quad 2k = 4, \ldots, 2n-2; \quad \phi_2 = \tilde{\phi}_n = 0.$$

The SW curve can then be derived just by plugging this result in (3.9). If we now multiply both sides by $(z-1)^2/v^{2n}$ and define $y = z-1$, $x = v^2$ we find

$$y^2 = 2\sum_{k=2}^{n-1}\frac{u_{2k}}{x^k}(y+1) \Longrightarrow \left(y - \sum_{k=2}^{n-1}\frac{u_{2k}}{x^k}\right)^2 = \left(\sum_{k=2}^{n-1}\frac{u_{2k}}{x^k}\right)\left(2 + \sum_{k=2}^{n-1}\frac{u_{2k}}{x^k}\right).$$

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[2]Contrary to the A_N theory, in which this is the general recipe, the D_N theory has a further complication: the coefficients one extracts using Riemann-Roch theorem obey in general non-trivial polynomial relations and one must take this into account in order to extract the true coordinates on the Coulomb branch (see [32] for a detailed analysis of this issue). However, this will not be important in the present case.
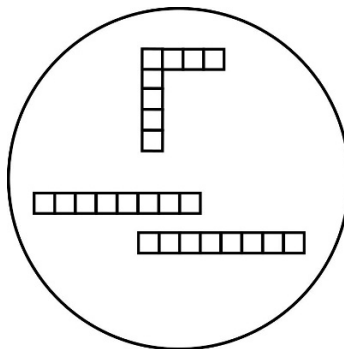
Figure 1. The three punctured sphere that represents the theory entering in the S dual description of $\mathrm{USp}(2N)$ with $2N+2$ flavors (in this case $N=3$).

Defining now $(y - \sum_{k=2}^{n-1} \frac{u_{2k}}{x^k})^2 = t^2/x^2$ and multiplying both terms by x^2 we immediately recognize (3.7), with c_i and u_{N-n+2} set to zero. These are the mass parameters associated with the $\mathrm{SO}(4n = 2N_f) \times \mathrm{SU}(2)$ flavor symmetry of the theory.

Following [10], our interpretation is that the infrared physics at the maximally singular point can be described by the two sectors A ($|x| \sim \epsilon_A^2$) and B ($|x| \sim \epsilon_B^2$); both sectors have $\mathrm{SU}(2)$ global symmetry and the diagonal combination is promoted to a gauge symmetry.

The A sector we have just described (see figure 1) already appeared in [16] (and for $N=3$ was studied in [31]), where it was recognized that it enters in the S dual description of the scale invariant $\mathrm{USp}(2N)$ theory with $2N+2$ flavors: in the infinite coupling limit the two simple punctures collide and this three punctured sphere emerges from the collision. Indeed, we can understand this duality using the analysis of the maximally singular point given above: if we apply the same strategy to the scale invariant case, so that the maximally singular point is precisely the origin of the Coulomb branch, we find a B sector which is trivial and thus the S dual description is given by the A sector, with an $\mathrm{SU}(2)$ subgroup of the global symmetry group gauged. The commutant $\mathrm{SO}(2N_f)$ is the flavor group of the original theory. For $N=1$ the theory is $\mathrm{USp}(2) \simeq \mathrm{SU}(2)$ with 4 flavors, which has $\mathrm{SO}(8)$ flavor symmetry. In this case the A sector becomes free and describes four doublets of $\mathrm{SU}(2)$. For $N=2$ the theory is $\mathrm{USp}(4)$ with six fields in the fundamental. This case has been studied by Seiberg and Argyres in [11], where it was recognized that the A sector coincides with the E_7 SCFT of Minahan and Nemeschansky [33] (so in this case there is an enhancement from the naive $\mathrm{SU}(2) \times \mathrm{SO}(12)$ to E_7). Indeed, the scale invariance of the curve requires (for $N = n-1$) that the $\mathrm{SU}(2)$ beta function is zero, so the A sector should give the same contribution as 4 flavors. This obviously works in the $N=1$ case and tells us that the $\mathrm{SU}(2)$ central charge is 8 in the other cases. This is precisely the value found in [31] for the puncture with partition $\{2n-3, 1, 1, 1\}$.

We can now determine the beta function of the $\mathrm{SU}(2)$ gauge group emerging at the maximally singular point with the same technique adopted in the previous section: the curve for $\epsilon_A^2 < |x| < \epsilon_B^2$ represents a tubular region associated with the $\mathrm{SU}(2)$ gauge group. We can thus compute a and a_D and then determine the generalized coupling constant

$$\tau = \partial a_D / \partial a.$$

$$a = \int_{|x|=\text{const.}} \lambda = 2\pi i \alpha; \quad \alpha^2 = -4\Lambda^{2N+2-2n} u_{N-n+2} - c_2,$$

$$a_D = \int_{|x|\sim\epsilon_A^2}^{|x|\sim\epsilon_B^2} \lambda = \alpha \left(\frac{2N-2n+2}{N-n+2} \log \epsilon_A + \text{const.} \right).$$

Identifying as before ϵ_A with the energy scale we find

$$b_1 = 2 \left(1 - \frac{1}{N-n+2} \right),$$

which is the contribution to the beta function of the D_{N-n+2} theory. Indeed, this is the expected result, since the contribution from the A sector, as we have just seen, saturates the SU(2) beta function.

The case $n = 1$ deserves some comments: the A sector becomes trivial and we are left with the D_{N+1} theory, which has SU(2) flavor symmetry and not SO(4)! Let us analyze the curve carefully in this case:

$$xy^2 = (x^{N+1} - \cdots - u_N x + \tilde{c}_2)(x^{N+1} - \cdots - (u_N \pm 4\Lambda^{2N})x + \tilde{c}_2) - c_2 x - \tilde{c}_2^2.$$

Scaling towards the small x region we find

$$y^2 = \pm 4\Lambda^{2N}(x^{N+1} - \cdots - u_N x) - c_2 \pm 4\Lambda^{2N}\tilde{c}_2,$$

where the casimir associated to the SU(2) global symmetry is $c_2 \pm 4\Lambda^{2N}\tilde{c}_2$. The $\pm$ term reflects the fact that there are two maximally singular vacua, corresponding to $u_N = \pm 2\Lambda^{2N}$ (clearly, this is true also for $n > 1$). The $n = 1$ case is special because the two quadratic casimirs of SO(4) enter symmetrically in the scaled curve and in each one of the two singular points only an SU(2) subgroup acts. Of course, it is well known that this occurs in the $N = 1$ case (i.e. the USp(2) $\simeq$ SU(2) gauge theory with two massless flavors): in this case the two singular points describe two hypermultiplets which are neutral under an SU(2) subgroup of the SO(4) flavor symmetry group.

We are now in a position to determine the infrared physics at the Chebyshev point. The SW curve and differential are [18]

$$y^2 \approx x^{2n}, \quad \lambda \approx \frac{y}{x^n} dx.$$

This are precisely the curve and differential at the maximally singular point of USp($2n$) theory with $2n$ flavors, in which the B sector describes a doublet of SU(2). We thus propose that the low-energy description at the Chebyshev point of USp($2N$) theory with $2n$ massless flavors includes:

- An abelian $U(1)^{N-n}$ sector, with massless particles charged under each U(1) subgroup.

- The A sector described above, with global symmetry SU(2) $\times$ SO($4n$).

- A third sector consisting of two hypermultiplets, whose symmetry is SU(2). The gauging of the diagonal SU(2) couples the last two sectors.

Notice that for $n = 2$ the A sector becomes free and describes four doublets of SU(2).

4 SO(N) SQCD

We can extend this analysis to theories with gauge group SO(N), that exhibit the same phenomena described in the previous section (coalescence of r vacua and flavor symmetry enhancement in the massless limit). We analyze first theories with N even and then those with N odd.

4.1 SO(2N) SQCD with 2n flavors

Let us consider SO($2N$) gauge theory with $N_f = 2n$ flavors. The theory becomes superconformal for $n = N - 1$ and in the massless limit has USp($4n$) flavor symmetry. The SW curve and differential are

$$y^2 = xP_N^2(x) - 4\Lambda^{4N-4n-4}x^3\prod_i(x - m_i^2), \tag{4.1}$$

$$\lambda = \frac{\sqrt{x}}{2\pi i}d\log\left(\frac{xP_N(x) - \sqrt{x}y}{xP_N(x) + \sqrt{x}y}\right), \tag{4.2}$$

where $P_N(x) = x^N - \sum_{k=1}^{N-1}u_k x^{N-k} - (u_N)^2$ (u_1 is a Coulomb branch coordinate in this case as well). With usual manipulations we can rewrite the curve as

$$y^2 = x(x^N - \cdots - u_N^2)(x^N - \cdots + (4\Lambda^{2N-2n-2} - u_{N-n-1})x^{n+1} - \cdots - u_N^2)$$
$$- \sum_{k=1}^{2n}c_{2k}x^{2n+3-k}. \tag{4.3}$$

Here we have simply redefined $u_{N-n-1} + 2\Lambda^{2N-2n-2} \to u_{N-n-1}$. Turning off all the parameters we then get the maximally singular point:

$$y^2 = x^{N+n+2}(x^{N-n-1} + 4\Lambda^{2N-2n-2}), \quad \lambda = \frac{y}{x^{n+2}}dx. \tag{4.4}$$

Rescaling the curve as before we obtain ($t = y/x^{n+1}$)

$$t^2 = -\sum_{k=1}^{2n}c_{2k}x^{1-k} + \left(x^{N-n} - \cdots - u_{N-n} - \cdots - \frac{u_N^2}{x^n}\right)$$
$$\times \left(x^{N-n-1} - \cdots + (4\Lambda^{2N-2n-2} - u_{N-n-1}) - \cdots - \frac{u_N^2}{x^{n+1}}\right). \tag{4.5}$$

We can now introduce the two sectors imposing $|x| \sim \epsilon_A^2$ and $|x| \sim \epsilon_B^2$. Setting $c_{2k} \sim O(\epsilon_A^{2k})$ leads to $t \sim \epsilon_A$ in the A sector and $t \sim \epsilon_B^{N-n}$ in the second one, so we deduce

$$\epsilon_A = \epsilon_B^{N-n}.$$

The same argument we gave in sections 2 and 3 then assigns

$$u_i \sim O(\epsilon_B^{2i})\ i = 1,\ldots,N-n; \quad u_{N-n+i} \sim O(\epsilon_A^{2+2i})\ i < n; \quad u_N \sim O(\epsilon_A^{n+1}).$$

The SW curve in the B sector is the by now familiar curve for the D_{N-n} theory:

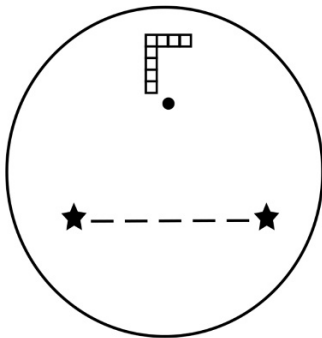$$t^2 = 4\Lambda^{2N-2n-2}(x^{N-n} - \cdots - u_{N-n}) - c_2.$$

Figure 2. The three punctured sphere associated to the SCFT entering in the S dual description of SO(2N) with $2N - 2$ flavors (in this case $N = 4$). The black dot indicates the D-partition and we have drawn the corresponding Young tableau. The two maximal C-partitions are indicated with ⋆. To visualize the presence of the twist we draw a dashed line.

In the conformal case $N = n + 1$ this sector is trivial and describes a doublet of hypermultiplets when $N = n + 2$. The A sector is described by the curve

$$t^2 = \left(u_{N-n} - \cdots - \frac{u_N^2}{x^n} \right) \left(4\Lambda^{2N-2n-2} - \frac{u_{N-n}}{x} - \cdots - \frac{u_N^2}{x^{n+1}} \right) - \sum_{k=1}^{2n} c_{2k} x^{1-k}.$$

This curve has $2n + 2$ branch points and contrary to the USp(2N) case the A sector is never free. The global symmetry group is $SU(2) \times USp(4n)$ and, as usual, the SU(2) gauge group is gauged. In the scale invariant case we recover a S-dual description similar to the one for USp(2N): the B sector is trivial and we are left with the A sector with an SU(2) subgroup of the flavor symmetry group gauged. Now one can repeat the calculation of the SU(2) beta function with the same technique adopted in sections 2 and 3; the result is

$$b_1 = 2 \left(1 - \frac{1}{N - n} \right).$$

This coincides again with the contribution of the D_{N-n} theory, so the contribution from the A sector must saturate the SU(2) beta function as before.

Also in this case a description in terms of 6d $(2,0)$ D_{n+1} theory compactified on a three punctured sphere is available (for $n > 1$). The only new ingredient is the presence of black puntures (in the notation of [16]), or C-partitions in the language of [31, 32]. To determine the theory, the simplest way is to notice that the A sector emerges in the dual description (of the strong coupling limit) of the scale invariant SO(2N) SQCD. The collision of the simple punctures produces as before the D-partition (or grey puncture) described by Young tableau with $2n + 2$ boxes, organized in a row of length four and the others of unit length (see figure 2). This puncture gives rise to an SU(2) global symmetry group with central charge $k = 8$, which is precisely the value needed to saturate the beta function. The remaining two punctures are described by a Young tableau with $2n$ boxes and a single row. The pole structure for the k-differentials encoded in this puncture has

been determined in [16] and is $\{1, \ldots, 2n - 1; n + 1/2\}$.[3] The fractional degree of the pole for $\tilde{\phi}_{n+1}$ is not a problem, since we have two such punctures. Turning around one of them we find $\tilde{\phi}_{n+1} \rightarrow -\tilde{\phi}_{n+1}$, which is precisely the action of the $\mathbb{Z}_2$ outer automorphism of the D_{n+1} Lie algebra [34] (see figure 2). The k-differentials can thus be written as

$$\phi_{2k} = \frac{u_{2k}z}{(z-1)^2} \left(\frac{dz}{z}\right)^{2k} k = 2, \ldots, n; \quad \tilde{\phi}_{n+1} = \frac{u_{n+1}\sqrt{z}}{z-1} \left(\frac{dz}{z}\right)^{n+1}.$$

Using (3.9) we find the following SW curve:

$$v^{2n+2} = \frac{z}{(z-1)^2} \left(\sum_{k=2}^{n} u_{2k} v^{2n+2-2k} + u_{n+1}^2\right).$$

With the same manipulations described in section 3 we get precisely the curve for the A sector.

The case $n = 1$ deserves some comments: the A sector has (at least) $SU(2) \times USp(4)$ global symmetry and turning off all the mass deformations its curve becomes

$$t^2 = -\frac{u_N^2}{x} \left(4\Lambda^{2N-4} - \frac{u_N^2}{x^2}\right).$$

It describes a rank one, scale invariant theory with a Coulomb branch coordinate of dimension 2. Notice also that the discriminant is proportional to u_N^6. The possible scale invariant singularities of rank one curves were classified in [25] (see also [35]) and just from these data we can associate the above curve with the D_4 singularity.[4] In [36] it was pointed out that this is not enough to identify the A sector: there are different theories associated with the same singularity and one should turn on the mass deformations in order to distinguish them. In the case of the D_4 singularity there are two possibilities: one theory has $SO(8)$ global symmetry and the second one $SU(2)$ (corresponding to $SU(2)$ gauge theory with 4 hypermultiplets in the doublet or one in the adjoint). The second possibility can be ruled out since our theory has a larger global symmetry group. We can thus identify the A sector with the origin of the Coulomb branch of $SU(2)$ theory with 4 massless flavors. This theory has central charges $a = 23/24$, $c = 7/6$ and $SO(8)$ global symmetry with central charge $k_{SO(8)} = 4$ (see e.g. [37]). Here we see once again the phenomenon first described by Argyres and Seiberg in [11]: $SO(8)$ has a maximal $SU(2) \times USp(4)$ subgroup and by gauging the $SU(2)$ factor we recover the $USp(4)$ symmetry of the parent gauge theory.

We can confirm our claim looking at the conformal case, namely $SO(4)$ gauge theory with two flavors: this theory is equivalent to the $SU(2)$ gauging of the A sector. We expect the $SU(2)$ beta function to vanish and the value of the central charges a and c should match

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[3]The algorithm for determining the pole structure for general C-partitions is different with respect to the one described in section 3 and in this paper we will not need it. The interested reader can find an exhaustive discussion on this point in [32].

[4]In the present context the A-D-E nomenclature comes from a correspondence between the A-D-E affine Lie algebra extended Dynkin diagrams and the pattern of blowups resolving those singularities. This should not be confused with the D_4 Argyres-Douglas theory we have discussed so far (see e.g. [35]).

in both descriptions. The SU(2) central charge can be computed using the formula given in [11]

$$k_{\mathrm{SU(2)}} = \mathcal{I}_{\mathrm{SU(2)} \hookrightarrow \mathrm{SO(8)}} k_{\mathrm{SO(8)}},$$

where $\mathcal{I}$ is the embedding index. Using for example that the $\mathbf{8_V}$ of SO(8) decomposes as $\mathbf{8_V} = (\mathbf{3,1}) \oplus (\mathbf{1,5})$ under SU(2) × USp(4), (indeed the result does not depend on the representation chosen) we obtain [11]

$$\mathcal{I}_{\mathrm{SU(2)} \hookrightarrow \mathrm{SO(8)}} = \frac{T(\mathbf{3}) + 5 \cdot T(\mathbf{1})}{T(\mathbf{8_V})} = 2.$$

We thus find that the SU(2) central charge is 8, which is precisely the value needed to saturate the beta function. Finally, summing the contribution to a and c coming from the SU(2) gauge group and from the A sector we get precisely the central charges for the SO(4) theory with two flavors, which is nothing but the SU(2) × SU(2) gauge theory with two hypermultiplets in the $(\mathbf{2,2})$.

4.2 SO(2N + 1) SQCD with 2n + 1 flavors

The above analysis can be repeated for SO(2N + 1) gauge theories with odd number of flavors $N_f = 2n + 1$. The SW curve and differential are

$$y^2 = x P_N^2(x) - 4\Lambda^{4N-4n-4} x^2 \prod_i (x - m_i^2), \tag{4.6}$$

$$\lambda = \frac{\sqrt{x}}{2\pi i} d \log \left(\frac{x P_N(x) - \sqrt{x}y}{x P_N(x) + \sqrt{x}y} \right), \tag{4.7}$$

Redefining $u_{N-n-1} + 2\Lambda^{2N-2n-2} \to u_{N-n-1}$ we can rewrite the curve as

$$y^2 = -\sum_{i=1}^{2n+1} c_{2i} x^{2n+3-i}$$

$$+ x(x^N - \cdots - u_N)(x^N - \cdots + (4\Lambda^{2N-2n-2} - u_{N-n-1})x^{n+1} - \cdots - u_N). \tag{4.8}$$

The most singular point can be found setting all C_{2i} and u_i to zero:

$$y^2 = x^{N+n+2}(x^{N-n-1} + 4\Lambda^{2N-2n-2}); \quad \lambda \approx \frac{y}{x^{n+2}} dx.$$

As before, when $N = n + 1$ the theory is conformal and this point coincides with the origin of the Coulomb branch. When $N = n + 2$ we recover the Chebyshev point, where the curve degenerates as $y^2 \approx x^{N_f + 3}$.

We can now set $t = y/x^{n+1}$ and introduce the A and B sectors, in which $|x| \simeq \epsilon_A^2$ and $|x| \simeq \epsilon_B^2$ respectively. The same argument given in the previous sections leads us to the relation $\epsilon_A = \epsilon_B^{N-n}$ and to the assignment

$$u_i \sim O(\epsilon_B^{2i}) \quad i = 1, \ldots, N - n; \quad u_{N-n+i} \sim O(\epsilon_A^{2+2i}).$$

The curves describing the theories in the two sectors can now be readily identified: the B sector is the D_{N-n} theory (when $N = n + 2$ it describes a doublet of SU(2) and when $N = n + 1$ becomes trivial) and the curve for the A sector is

$$t^2 = -\left(u_{N-n} + \cdots + \frac{u_N}{x^n}\right)\left(4\Lambda^{2N-2-2n} - \frac{u_{N-n}}{x} - \cdots - \frac{u_N}{x^{n+1}}\right) - \sum_{i=1}^{2n+1} c_{2i}x^{1-i}. \quad (4.9)$$

One can easily see from the above curve that this sector has $SU(2) \times USp(4n + 2)$ global symmetry. To identify the theory, let us start from the $n = 1$ case. Turning off the mass parameters we are left with the curve (setting $4\Lambda^{2N-4} = 2$)

$$t^2 = -\frac{u}{x}\left(2 - \frac{u}{x^2}\right).$$

To bring it to a more familiar form, it is now convenient to define $y = tx^2$. We then find

$$y^2 = -ux(2x^2 - u); \quad \frac{\partial\lambda}{\partial u} = \frac{dx}{y}.$$

Making then the change of variables

$$y = -\frac{\tilde{y}}{2u}; \quad x = -\frac{\tilde{x}}{2u},$$

we recognize the curve for the E_7 SCFT of Minahan and Nemeschansky [33]:

$$\tilde{y}^2 = \tilde{x}^3 - 2u^3\tilde{x}; \quad \frac{\partial\lambda}{\partial u} = \frac{d\tilde{x}}{\tilde{y}}.$$

The fact that only an $SU(2) \times USp(6)$ subgroup of E_7 appears in (4.9) can be understood in terms of the phenomenon mentioned in the previous section [36, 38]: there is more than one theory described by the above curve and in order to distinguish them one has to turn on the mass deformations. They are characterized by different flavor symmetries and the Minahan-Nemeschansky theory is just the one with the largest symmetry group. In [38] the authors recognized that the $SU(2) \times USp(6)$ theory enters in the S-dual description of $SO(5)$ gauge theory with three flavors at the infinite coupling point and called it submaximal mass deformation of the E_7 theory.

We already encountered the E_7 theory in section 3: it is the A sector for $USp(2N)$ SQCD with six flavors. Comparing equations (3.7) and (4.9), we can see that the analogy between the A sectors of these two theories is not limited to this case! The A sectors of $SO(2N+1)$ SQCD with $N_f = 2n+1$ flavors and $USp(2N)$ SQCD with $N_f = 2n+4$ flavors are described by the same curve (once we have set to zero the mass deformations), with Coulomb branch coordinates of the same scaling dimension. However, the flavor symmetry groups are different: $USp(4n + 2)$ and $SO(4n + 8)$ respectively. Here we see some higher rank examples of the phenomenon studied in [36, 38] (in the language of Argyres and Wittig the first theory represents a submaximal mass deformation of the second one). This is not surprising since the SW curves and differentials for $SO(2N + 1)$ SQCD with $N_f = 2N - 1$ and $USp(2N)$ SQCD with $N_f = 2N + 2$ coincide in the massless case.

5 Conclusions

We have made a systematic analysis of singular points in $\mathcal{N} = 2$ SQCD with classical gauge groups, focusing on the maximally singular points in the moduli space. We have seen that, in order to satisfy the constraint on the scaling dimensions of mass parameters, we are forced to introduce different scale invariant sectors. The introduction of two sectors, which is the simplest possibility, leads to a unique answer which is consistent with all the strong coupling dualities found recently and allows to satisfy the constraints imposed by superconformal invariance.

We found a common structure for the low-energy description at these points, which is schematically given by:

- An abelian sector.

- The B sector, which is always given either by a D_N theory (with $N > 2$) or by a doublet of hypermultiplets. In both cases the flavor symmetry is SU(2).

- The A sector, with (at least) $SU(2) \times G$ flavor symmetry, where G is the flavor symmetry of the parent gauge theory. This is the only sector that changes as we vary the gauge group and in most cases admits a six-dimensional description.

- An infrared free SU(2) gauge multiplet coupled to sectors A and B.

At the maximally singular point the abelian sector just describes a number of decoupled vector multiplets, as pointed out in [10], whereas at points which are not lifted by the $\mathcal{N} = 1$ perturbation it includes massless hypermultiplets charged under each U(1) factor. Chebyshev points in USp and SO gauge theories fall in this second class and are characterized by a free B sector, which describes two massless hypermultiplets. This is not the case for SU(N) gauge theories. A particularly simple case is given by USp theory with 4 flavors: in this case the A sector becomes free and it turns out that the Chebyshev point admits a lagrangian description.

One possible future direction is to analyze more general theories and see whether different structures emerge in the infrared at the maximally singular points. The Argyres-Douglas theories studied in [39] certainly play an important role. In particular, it would be interesting to see whether the constraints we imposed in this note always lead to a unique answer. Another interesting question is how to use our results to explore the properties of the $\mathcal{N} = 1$ theories obtained adding a superpotential for the chiral field in the adjoint.

Acknowledgments

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