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Massless geodesics in $AdS_5 \times Y(p,q)$ as a superintegrable system

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Abstract: A constant of motion of Carter type for a probe particle in the $Y(p,q)$ Einstein-Sasaki backgrounds is presented. This quantity is functionally independent with respect to the five known constants for these geometries. As the metric is five dimensional and the number of independent constants of motion is at least six, the geodesic equations turn out to be superintegrable. This result applies to the configuration of massless geodesic in $AdS_5 \times Y(p,q)$ studied by Benvenuti and Kruczenski [86], which are matched to long BPS operators in the dual N=1 supersymmetric gauge theory.

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1 Introduction

The present work deals with a superintegrable problem. Roughly speaking, a mechanical system is superintegrable if the number of its functionally independent constants of motion is larger than its number of degrees of freedom. A classical example is the Kepler problem. For a particle motion in a generic central field, the energy E and the component of the angular momenta perpendicular to the plane of motion L_z are conserved. Given that the motion takes place in a plane, all these problems are integrable. For the Kepler problem, there is a further conserved quantity namely, a component of the Runge-Lenz vector. The set of these three constant of motion is functionally independent, therefore the Kepler problem is superintegrable. Similar considerations apply to the central harmonic oscillator in three dimensions. For both systems, the closed trajectories are ellipses.

For any mechanical system with n degrees of freedom, the maximal number of functionally independent constants of motion is $2n-1$. Systems possessing this number of constants of motion are known as maximally superintegrable. When $n=2$ a maximal superintegrable system admits three constants of motions. Thus the Kepler problem is maximally superintegrable.

Superintegrable systems are gifted with special properties, some of them are intrinsic and some others depend on the problem under consideration. For the quantum version of the Kepler motion, some interesting features appear. The Runge-Lenz vector becomes, by the correspondence principle, into an operator which commutes with the hamiltonian of the particle. The algebra constituted by the hamiltonian, the angular momentum and the Runge-Lenz vector is not closed, in fact it is an infinite dimensional twisted loop algebra [1]. But when this algebra is restricted to subspaces of constant negative energy, which correspond to bound states, the resulting is isomorphic to SO(4). Thus the expected symmetry group for a central field, which is SO(3), is enhanced for the Newton potential to SO(4). This enhancement explains the accidental degeneration of the energy levels of the hydrogen atom, i.e, the independence of the energy levels with respect to the total angular momenta of the particle. In fact Pauli, Bargmann and Fock [2]–[4] have shown that, due to the presence of the Runge-Lenz operator, it is possible to obtain the bound state spectra of the hydrogen algebraically, without solving the Schrodinger equation explicitly.

The discussion given above suggest that the discovery of superintegrability for a given problem may dramatically simplify the study of its properties. It was Sommerfeld who pointed out that if the Hamilton-Jacobi equation for a given potential is separable in more than one coordinate system, then the problem is superintegrable [5]. This statement was extensive studied by Smorodinskyi, Winternitz and collaborators, who were able to classify all the potentials in two dimensions which are separable in more than one coordinate system [6]–[7]. For the three dimensional flat space, it was Eisenhart who classified the possible coordinate systems for which separation takes place, together with the form of the potentials that permits such separation [9]–[10]. Further investigation was made in [8] where the three dimensional potentials that are separable in spherical coordinates and one more coordinate system were found. The remaining possibilities in three dimensions were classified in [11] later on.

The achievements described above motivated intense research on the subject. In recent years several integrable systems were found, some examples are in [12]–[66] and references therein. These consist in wide variety of physical systems such as the Kepler problem in arbitrary dimensions and its extensions in presence of magnetic monopoles, and generalizations of known systems to spaces of non zero curvature in several dimensions.

In the present work it will be shown that the equations for the geodesic motion over the Einstein-Sasaki metrics defined on the $Y(p,q)$ manifolds [67]–[69] are superintegrable. The main technical tool for obtaining this result are the Killing and Killing-Yano tensors [70]. These tensors play a significant role for the integrability of the geodesic equations in the rotating black hole background [71]–[79]. This topic was reviewed in [80] and we refer the reader to this paper for a list of references. More recent ones are [81]–[85].

The present text is organized as follows. In section 2.1 some important features of Einstein-Sasaki manifolds and Calabi-Yau cones are briefly reviewed, together with a description of the $Y(p,q)$ geometries in different coordinate systems. In section 2.2 it is shown that the defining equations for the configurations of massless geodesics on $AdS_5 \times Y(p,q)$ considered in [86] are integrable. The material in section 2 is, of course, not new. In section 3.1 it is reviewed the role of the Killing and the Killing-Yano tensors as generators of hidden symmetries. These tools are applied in section 3.2 to show that the configurations of massless geodesics mentioned above admit a further constant of motion which is functionally independent with respect to the ones found in [86]. This is checked explicitly, and it is concluded that the configuration of massless geodesics in the geometry is superintegrable. In section 4 some consequences related to the presence of hidden symmetries are derived. It is found that the spinning particle action [87]–[93] with the $Y(p,q)$ geometries as background metric admit some symmetries which were not considered in the literature before. Additionally, it is shown that this symmetry is not anomalous, in the sense that it corresponds to an operator which commutes with the laplacian defined over the $Y(p,q)$ geometry. Properties of the Dirac operator are also briefly commented. In section 5, some open perspectives and future lines of work are discussed. Although the validity of our results are checked in the text, some mathematical statements which were used for obtaining them are collected in the appendix. This is in order to separate the statement of the results from the description of how they were obtained. Hopefully this organization makes the text more readable.

2 Preliminary material

2.1 A brief description of the Einstein-Sasaki metrics on $Y(p,q)$

Since the present work is related to Einstein-Sasaki metrics and there exist a direct connection between these metrics and non compact Calabi-Yau cones, it will be convenient to describe their main properties of the both geometries from the very beginning. A non compact Calabi-Yau metric g_{2n} is by definition a $2n$ dimensional one defined over a space M_{2n} and whose holonomy is SU(n) or a subgroup of SU(n). The reduction of the holonomy from SO($2n$) to SU(n) imply that these metrics are Ricci-flat, and that there exist always a local choice of the basis e^a for which the metric takes the diagonal form

$$g = \delta_{ab} e^a \otimes e^b,$$

and for which the symplectic two form

$$\omega = e^1 \wedge e^2 + .. + e^{2n-1} \wedge e^{2n}, \tag{2.1}$$

and the $(n,0)$ form

$$\Omega = (e^1 + ie^2) \wedge \ldots \wedge (e^{n-1} + ie^n), \tag{2.2}$$

are closed. The converse of these statements are also true. The closure condition $d\omega = 0$ implies that Calabi-Yau metrics are sympletic. The closure of Ω implies that the almost complex structure J defined by $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$ is integrable and thus, the manifold M_{2n} is complex. Complex sympletic manifolds of this type are Kahler and therefore, any Calabi-Yau metric is automatically Kahler. In fact, a Ricci flat Kahler metric is locally Calabi-Yau. Details of these assertions can be found in standard books on the subject [94, 95].

The relation between Einstein-Sasaki manifolds and Calabi-Yau cones we mentioned above can be described as follows. Consider the family of $2n$ dimensional cones given by

$$g_{2n} = dr^2 + r^2 g_{2n-1}, \tag{2.3}$$

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with a metric g_{2n-1} which does not depends on the coordinate r and is defined over a $2n-1$ manifold M_{2n-1}. The distance element (2.3) is singular at the tip of the cone $r=0$ unless g_{2n-1} is the canonical metric on the sphere S^{2n-1}. If in addition, the cone (2.3) is Calabi-Yau, then it is Ricci-flat, thus the Sasaki metric g_{2n-1} is Einstein. Such metrics are known as Einstein-Sasaki, for obvious reasons. In fact, the last relation can be taken as the definition of an Einstein-Sasaki metric, in a local sense. The converse of these statements are also true, that is, any Einstein-Sasaki metric defines a Calabi-Yau cone by (2.3). Several properties for these metrics are collected in the appendix but were extensively reviewed in [96]–[98].

The Einstein-Sasaki metrics which we will be concerned with are defined over the $Y(p,q)$ manifolds [67]–[69]. There exists a local coordinate system for which these metrics takes the following form

$$\begin{aligned} g_{p,q} = \frac{1-y}{6}(d\theta^2+\sin^2\theta d\phi^2) + \frac{dy^2}{6p(y)} + \frac{q(y)}{9}(d\psi-\cos\theta d\phi)^2 \\ + w(y)[d\alpha + f(y)(d\psi - \cos\theta d\phi)]^2 \end{aligned} \tag{2.4}$$

with

$$\begin{aligned} w(y) &= 2\frac{a-y^2}{1-y}, \\ q(y) &= \frac{a-3y^2+2y^3}{a-y^2}, \\ f(y) &= \frac{a-2y+y^2}{6(a-y^2)}, \end{aligned} \tag{2.5}$$

and

$$p(y) = \frac{w(y)q(y)}{6} = \frac{a-3y^2+2y^3}{3(1-y)}. \tag{2.6}$$

The range of the coordinates $(\theta, \phi, y, \alpha, \psi)$ is given by

$$\begin{aligned} &0 \le \theta \le \pi, && 0 \le \phi \le 2\pi, && y_1 \le y \le y_2, \\ &0 \le \alpha \le 2\pi l, && 0 \le \psi \le 2\pi. \end{aligned} \tag{2.7}$$

The constant a appearing in the expression for the metric and the constants $y_{1,2}$ and l determining the range of the coordinates can be expressed in terms of two integers p y q defining the manifold

$$\begin{aligned} a &= 3y_1^2 - 2y_1^3, \\ l &= \frac{q}{3q^2-2p^2+p\sqrt{4p^2-3q^2}}, \\ y_{1,2} &= \frac{1}{4p}\left(2p \mp 3q - \sqrt{4p^2-3q^2}\right). \end{aligned}$$

The constants $y_{1,2}$ are the zeros of the function $p(y)$ defining the metric. In addition, there is a third zero for $p(y)$ given by $y_3 = \frac{3}{2} - y_1 - y_2$. A global analysis of these metrics can be found in [69].

There exist other standard expressions for these metrics, which describe them as U(1) fibrations over Kahler-Einstein metrics. The coordinate change $\alpha = -\frac{\beta}{6} - \frac{\psi'}{6}$, $\psi = \psi'$ takes the distance element (2.4) to the following form

$$g_{p,q} = \frac{1-y}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dy^2}{6p(y)} + \frac{1}{36}q(y)w(y)(d\beta + \cos\theta d\phi)^2 \qquad (2.8)$$
$$+\frac{1}{9}[d\psi' - \cos\theta d\phi + y(d\beta + \cos\theta d\phi)]^2.$$

This can be written more concisely as

$$g_{p,q} = \frac{1}{9}(d\psi' + A)^2 + g_4, \qquad (2.9)$$

with

$$A = -\cos\theta d\phi + y(d\beta + \cos\theta d\phi), \qquad (2.10)$$
$$g_4 = \frac{1-y}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{dy^2}{6p(y)} + \frac{1}{36}q(y)w(y)(d\beta + \cos\theta d\phi)^2. \qquad (2.11)$$

It follows from (2.9)–(2.11) that the vector field $V = \partial_{\psi'}$ is Killing. The form (2.9) is quite general in the theory of Einstein-Sasaki manifolds [96]–[98]. The four dimensional metric g_4 is in general Kahler-Einstein, with Kahler form $\omega = dA$, and the Killing vector for which the metric takes the form (2.9) is known as the Reeb vector. This statement is true even if the Einstein condition is relaxed, in these cases, the resulting metric g_4 is only Kahler.

2.2 Massless strings in $AdS_5 \times Y(p,q)$ as an integrable system

The $Y(p,q)$ geometries described above are relevant in the context of the AdS/CFT correspondence [99]. For instance, the study of semiclassical strings in backgrounds of the form $AdS_5 \times Y(p,q)$ gives information about the anomalous dimensions of the dual $N = 1$ supersymmetric gauge theory, by the gauge/gravity duality [86]. The local distance element for these backgrounds is

$$g_{10} = -dt^2 \cosh^2\rho + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + g_{p,q}, \qquad (2.12)$$

A particular configuration of interest is given by the non massive geodesics in the reduced metric

$$g = -dt^2 + ds^2_{p,q} = -dt^2 + g_{ab}dx^a dx^b, \qquad (2.13)$$

together with its conserved quantities. In (2.13), g_{ab} denotes the $Y(p,q)$ metric described in (2.4), t is the global time coordinate in AdS_5, the non massive point like string is located in $\rho = 0$ and the movement takes place in the internal space $Y(p,q)$. The action for such configuration is

$$S = \frac{\sqrt{\lambda}}{2}\int d\tau \left(-\dot{t}^2 + g_{ab}\dot{x}^a\dot{x}^b\right), \qquad (2.14)$$

where $\sqrt{\lambda} = (R/l_s)^2$ is the effective string tension. The equations of motion should be supplemented with the null geodesic constraint

$$-\dot{t}^2 + g_{ab}\dot{x}^a\dot{x}^b = 0. \qquad (2.15)$$

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The Euler-Lagrange equation for t implies that $t = P_t \tau$ with P_t the conjugate momenta of t. Here P_t is a constant and represent the energy of the string configuration. By taking this into account it follows that the action can be reduced to

$$\begin{aligned} S &= \int d\tau L = \frac{\sqrt{\lambda}}{2} \int d\tau (g_{ab}\dot{x}^a \dot{x}^b) \\ &= \frac{\sqrt{\lambda}}{2} \int d\tau \bigg\{ \frac{1-y}{6}(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) + \frac{1}{w(y)q(y)}\dot{y}^2 + \frac{q(y)}{9}(\dot{\psi} - \cos\theta\dot{\phi})^2 \\ &\qquad + w(y)[\dot{\alpha} + f(y)(\dot{\psi} - \cos\theta\dot{\phi})]^2 \bigg\}. \end{aligned} \tag{2.16}$$

This is clearly the action of a free particle moving along geodesics in the Einstein-Sasaki geometry. The conjugate momenta are

$$P_a = \frac{\partial L}{\partial \dot{x}^a}, \tag{2.17}$$

and, in terms of these momenta, the hamiltonian for the particle is

$$H = \frac{1}{2} g^{ab} P_a P_b. \tag{2.18}$$

It can be deduced from the isometries of (2.4) that the quantities P_ϕ, P_ψ y P_α are conserved. Additionally, the square of the SU(2) angular momenta

$$J^2 = P_\theta^2 + \frac{1}{\sin^2\theta}\left(P_\phi + \cos\theta P_\psi\right)^2 + P_\psi^2, \tag{2.19}$$

is also conserved. The full set of momenta can be expressed in terms of the velocities as follows

$$\frac{1}{\sqrt{\lambda}} P_y = \frac{1}{6p(y)}\dot{y}, \tag{2.20}$$

$$\frac{1}{\sqrt{\lambda}} P_\theta = \frac{1-y}{6}\dot{\theta}, \tag{2.21}$$

$$\frac{1}{\sqrt{\lambda}}\left(P_\phi + \cos\theta P_\psi\right) = \frac{1-y}{6}\sin^2\theta\dot{\phi}, \tag{2.22}$$

$$\frac{1}{\sqrt{\lambda}}\left(P_\psi - f(y)P_\alpha\right) = \frac{q(y)}{9}\left(\dot{\psi} - \cos\theta\dot{\phi}\right), \tag{2.23}$$

$$\frac{1}{\sqrt{\lambda}} P_\alpha = w(y)\left(\dot{\alpha} + f(y)\left(\dot{\psi} - \cos\theta\dot{\phi}\right)\right) \tag{2.24}$$

and the Hamiltonian may be expressed in these terms as

$$\begin{aligned} 2\lambda H = \lambda\kappa^2 &= \frac{1}{2} 6p(y)P_y^2 + \frac{6}{1-y}\left(J^2 - P_\psi^2\right) + \frac{1-y}{2(a-y^2)}P_\alpha^2 \\ &+ \frac{9(a-y^2)}{a-3y^2+2y^3}\left(P_\psi - \frac{a-2y+y^2}{6(a-y^2)}P_\alpha\right)^2. \end{aligned} \tag{2.25}$$

The relation between κ and H in the last equation follows from formula (2.15).

The discussion given above shows that there are five functionally independent conserved quantities for the problem namely P_ϕ, P_ψ, P_α, J^2 y H and therefore, the equations defining the problem constitute an integrable system [86]. The purpose of the following sections is to present a further conserved quantity which is functionally independent with respect to these, which will imply that the problem is superintegrable.

3 Superintegrability of the massless strings in $AdS_5 \times Y(p,q)$

The following discussion about Killing and Killing-yano tensors is central for our construction. We review below their role for finding constants of motion for particle actions such as (2.14). After this brief exposition, we present explicit Killing and Killing-Yano tensors for the $Y(p,q)$ geometries, together with the corresponding conserved quantities. This result will imply that the geodesic equations for the $Y(p,q)$ geometries are a superintegrable system.

3.1 Killing and Killing-Yano tensors

The motion of a free particle on an arbitrary space-time $(M, g_{\mu\nu})$ takes place along a geodesic, described by the following action

$$S = \int_{\tau_0}^{\tau_1} L\, d\tau = \int_{\tau_0}^{\tau_1} \frac{1}{2}\, g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu d\tau. \tag{3.1}$$

In particular (2.14) is of this form. Here τ is the proper time. The variation of (3.1) with respect to infinitesimal transformations of the trajectory δx and $\delta \dot{x}$ is

$$\begin{aligned} \delta S &= \int_{\tau_0}^{\tau_1} \left[\frac{\delta L}{\delta x^\mu} - \frac{d}{d\tau}\left(\frac{\delta L}{\delta \dot{x}^\mu}\right)\right]\delta x^\mu d\tau + \int_{\tau_0}^{\tau_1} \frac{d}{d\tau}\left(\frac{\delta L}{\delta \dot{x}^\mu}\delta x^\mu\right) d\tau \\ &= \int_{\tau_0}^{\tau_1} \left[-\delta x^\mu g_{\mu\nu}\frac{D\dot{x}^\nu}{D\tau} + \frac{d}{d\tau}\Big(\delta x^\mu p_\mu\Big)\right] d\tau\,, \end{aligned} \tag{3.2}$$

with

$$p_\mu = \frac{\delta L}{\delta \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu, \tag{3.3}$$

the conjugated momenta of the particle. For variations with fixed endpoints the total derivative in (3.2) may be discarded. The variation will then vanish if and only if the equations of motion

$$\frac{D\dot{x}^\mu}{D\tau} = \ddot{x}^\mu + \Gamma^\mu_{\nu\alpha}\dot{x}^\nu\dot{x}^\alpha = 0, \tag{3.4}$$

are satisfied. Here $\Gamma^\mu_{\nu\alpha}$ denotes the usual Christoffel symbols

$$\Gamma^k_{ij} = \frac{g^{kl}}{2}(g_{il,j} + g_{jl,i} - g_{ij,l}), \tag{3.5}$$

and the first two terms of (3.4) are simply the definition of the derivative $\frac{D\dot{x}^\nu}{D\tau}$. The system of equations (3.4) states that the free particle in the geometry moves along a geodesic.

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Consider now variations $\delta x^\mu = K^\mu$ of (3.1) without fixed endpoints. In this situation the time derivative can not be discarded. By taking into account the equations of motion (3.4) it follows that

$$\delta S = \int_{\tau_0}^{\tau_1} \delta L \, d\tau = \int_{\tau_0}^{\tau_1} \frac{d}{d\tau}(K^\mu p_\mu) d\tau. \tag{3.6}$$

If there exists a variation $\delta x^\mu = K^\mu$ such that $\delta S = 0$, then this transformation is referred as a local symmetry of L. From (3.6) it follows that this is possible only if the quantity

$$E_K = K_\mu \dot{x}^\mu, \tag{3.7}$$

is a constant of motion associated to the symmetry. Thus, there exist a constant of motion for every symmetry the lagrangian (3.1) admits.

A well known example of a symmetry are the usual isometries, which corresponds to local variations of the form $\delta x^\mu = K^\mu(x)$ which leave the action invariant. The vanishing of (3.6) in this case is

$$\frac{d}{d\tau}(K_\mu \dot{x}^\mu) = \dot{x}^\nu \nabla_\nu K_\mu \dot{x}^\mu + K_\mu \frac{D\dot{x}^\mu}{D\tau} = 0. \tag{3.8}$$

But since the first term is zero due to (3.4) it follows that

$$\nabla_{(\nu} K_{\mu)} = 0, \tag{3.9}$$

is satisfied for the generators of the isometry. Here the parenthesis denote the totally symmetric component of the tensor. The vectors satisfying (3.9) are known as Killing vectors. Nevertheless, the isometries are not the most general symmetries. One may consider more general transformations of the form $\delta x^\mu = K(x, \dot{x})$, which are local with respect to the phase space coordinates $(x^\mu,\ \dot{x}^\mu)$. This form is quite general, since any dependence in higher order time derivatives such that $\ddot{x}$ will be reduced to a combination of $(x,\ \dot{x})$ by the equations of motion (3.4) and thus it is redundant. Given such a symmetry one may impose a Taylor like expansion of the form

$$\delta x^\mu = K^\mu + K^\mu_\alpha \dot{x}^\alpha + K^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + \ldots, \tag{3.10}$$

with tensors $K^\mu_{\mu_1..\mu_n}(x)$ which do not depend on the velocities $\dot{x}_i$. In these terms, it may be shown that (3.10) is a symmetry of (3.1) when

$$\nabla_{(\mu} K_{\mu_1..\mu_n)} = 0, \tag{3.11}$$

is satisfied. The method for reaching this conclusion is analogous to the one giving (3.9) and in fact (3.11) is a generalization of the Killing condition for symmetric tensors of higher order. The tensors satisfying that condition are known as *Killing tensors* and the quantities

$$C_n = K_{\mu_1..\mu_n} \dot{x}^{\mu_1}..\dot{x}^{\mu_n}, \tag{3.12}$$

are all constants of motion. The constants (3.12) are homogeneous polynomial expressions in the velocities whose degree is equal to the order of the associated Killing tensor. For

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tensors of order larger than one, the corresponding symmetries are not as visual or intuitive as usual isometries. For this reason these are known as *hidden symmetries*.

There exists another important generalization of the Killing vector equation, known as the Killing-Yano equation [70]. Although there are several well written reviews about the subject such as [100], let us explain some of their main properties. The constants of motion (3.12) are all scalars. If one consider instead tensor quantities, then in order to compare its components in different points on the manifolds, a notion of parallel transport is required. The statement that a tensor quantity $C_{\mu_1 \dots \mu_{n-1}}$ is "conserved" means that it is parallel transported along a geodesic. This is satisfied when

$$\dot{x}^{\alpha} \nabla_{\alpha} C_{\mu_1 \dots \mu_{n-1}} = 0. \tag{3.13}$$

A Killing-Yano tensor $f_{\mu_1 \dots \mu_n}$ is an *antisymmetric* one and which generate tensor "constants" of motion, which are linear in the velocities

$$C_{\mu_1 \dots \mu_{n-1}} = f_{\mu_1 \dots \mu_{n-1} \mu} \dot{x}^{\mu}. \tag{3.14}$$

The parallel transport condition (3.13) implies that

$$f_{\mu_1 \dots \mu_{n-1}(\alpha;\beta)} = 0. \tag{3.15}$$

This is a generalization of the Killing vector equation for antisymmetric tensors of higher order [70]–[76]. It is usual in the literature to present (3.15) in the following equivalent form

$$\nabla_X f = \frac{1}{p+1} i_X df, \tag{3.16}$$

for a Killing-Yano $p-$form $f_{\mu_1 \dots \mu_p}$, with X an arbitrary vector field and i_X the usual contraction operation. The equivalence between (3.15) and (3.16) follows directly by writing the last in components, and it is straightforward.

Given an arbitrary Killing-Yano tensor $f_{\mu_1 \dots \mu_{n-1} \mu}$ one may construct the following scalar quantity

$$C^2 = C_{\mu_1 \dots \mu_{n-1}} C^{\mu_1 \dots \mu_{n-1}} = f_{\mu_1 \dots \mu_{n-1} \mu} f_{\nu}^{\mu_1 \dots \mu_{n-1}} \dot{x}^{\mu} \dot{x}^{\nu}, \tag{3.17}$$

which is obviously a constant of motion quadratic in the velocities. This means that

$$K_{\mu\nu} = f_{\mu_1 \dots \mu_{n-1} \mu} f_{\nu}^{\mu_1 \dots \mu_{n-1}}, \tag{3.18}$$

is a Killing tensor of order two. This result is usually paraphrased by saying that the square (3.18) of a Killing-Yano tensor $f_{\mu_1 \dots \mu_{n-1} \mu}$ of arbitrary order is a Killing tensor $K_{\mu\nu}$ of order two. This statement is also true when the Killing tensor is constructed out of two different Killing-Yano tensors as follows

$$K_{\mu\nu} = f^{(1)}_{\mu_1 \dots \mu_{n-1}(\mu} f^{(2)\mu_1 \dots \mu_{n-1}}_{\nu)}.$$

The Killing and the Killing-Yano conditions admit conformal generalizations [111, 112]. A symmetric tensor $K_{\mu_1 .. \mu_p}$ is known as a conformal Killing tensor if it satisfies the following equation

$$\nabla_{(\nu} K_{\mu_1 \dots \mu_p)} = g_{\mu(\nu_1} \widetilde{K}_{\mu_2 \dots \mu_p)}.$$

Here $\widetilde{K}_{\mu_2\ldots\mu_p}$ is obtained taking the trace on both sides, for instance, for a tensor $K_{\mu\nu}$ of order two in n dimensions it is obtained that $\widetilde{K}_\mu = \frac{1}{n+2}K^\nu_{\nu,\mu} + \frac{2}{n+1}K^\nu_{\mu;\nu}$. Obviously the last condition does not change it form under the transformation $g \to \Omega^2 g$. The conformal generalization of the Killing-Yano condition is a bit more complicated to deduce, the result is

$$\nabla_X f = \frac{1}{p+1} i_X df - \frac{1}{n-p+1} X^* \wedge d^* f. \tag{3.19}$$

Here n is the dimension of the metric, X^* is the dual 1-form to the vector field X and the operation $d^* f = (-1)^p *^{-1} d * f$ has been introduced, in which

$$*^{-1} = \epsilon_p *, \qquad \epsilon_p = (-1)^{p(n-p)} \frac{\det g}{|\det g|}.$$

In the particular case defined by $d * f = 0$ the definition (3.19) reduces to the usual one for a Killing-Yano tensor (3.16).

3.2 A new quadratic constant for the $Y(p,q)$ geometries

The next task is to construct a quadratic constant of motion for the $Y(p,q)$ manifolds, which is functionally independent with respect to the known ones. This construction is based in a series of mathematical results which apply to any Sasaki geometry. Since these statements are general, it will be convenient to enunciate the main proposition here and leave the details of the deduction for the appendix.

As it was pointed out below (2.9), any Einstein-Sasaki metric may be expressed in the form

$$g_5 = \eta^2 + g_4,$$

with $3\eta = d\psi + A$ and $V = \partial_\psi$ the Reeb vector. The distance element g_4 is Kahler-Einstein. This statement is true even if the Einstein condition is relaxed. Moreover, it can be generalized to higher dimensional Sasaki metrics, which arise as fibrations of the form

$$g_{2n+1} = \eta^2 + g_{2n},$$

with g_{2n} a Kahler metric. For all these metrics the following $(2k+1)$-forms

$$\omega_k = \eta \wedge (d\eta)^k, \tag{3.20}$$

are Killing-Yano tensors [113]. Note that this affirmation is not restricted to the $Y(p,q)$ case.[1]

The formula (3.20) can be interpreted as a recipe for constructing hidden symmetries. When this construction is applied to the $Y(p,q)$ geometries, it follows from (2.9) and (2.10) that

$$\eta = d\psi' + A = d\psi' - \cos\theta d\phi + y(d\beta + \cos\theta d\phi), \tag{3.21}$$

[1] Nevertheless for other Sasaki geometries, the geodesic equations may not be superintegrable as for the $Y(p,q)$ case.

and for $k = 1$ the Killing-Yano 3-form that follows from (3.20) and (3.21) is

$$\begin{aligned}\omega_3 = \frac{1}{9}\,&[(1-y)\sin\theta\; d\theta\wedge d\phi\wedge d\psi' - \cos\theta\; d\theta\wedge dy\wedge d\psi' + dy\wedge d\beta\wedge d\psi' \\ &+ \sin\theta\; y(1-y)\; d\theta\wedge d\phi\wedge d\beta - d\phi\wedge dy\wedge d\beta].\end{aligned} \tag{3.22}$$

From the discussion below (3.18) it follows that the "square" $K_{\mu\nu} = (\omega_3)_{\mu\alpha\beta}(\omega_3)_\nu^{\alpha\beta}$ is a Killing tensor for the $Y(p,q)$ geometries. In components it reads

$$K_{\mu\nu} = \begin{pmatrix} K_{\theta\theta} & 0 & 0 & 0 & 0 \\ 0 & K_{\phi\phi} & 0 & K_{\phi\beta} & K_{\phi\psi'} \\ 0 & 0 & K_{yy} & 0 & 0 \\ 0 & K_{\beta\phi} & 0 & K_{\beta\beta} & K_{\beta\psi'} \\ 0 & K_{\psi'\phi} & 0 & K_{\psi'\beta} & K_{\psi'\psi'} \end{pmatrix}, \tag{3.23}$$

with

$$\begin{aligned} K_{\theta\theta} &= \frac{4}{3}(1-y), \\ K_{\phi\phi} &= \frac{1}{9}\left\{[1+\cos(2\theta)][q(y)w(y)+8y^2] + \cos(2\theta)[2-10y] + 14 - 22y\right\}, \\ K_{yy} &= \frac{8}{q(y)w(y)}, \\ K_{\beta\beta} &= \frac{2}{9}[q(y)w(y)+8y^2], \\ K_{\psi'\psi'} &= \frac{16}{9}, \\ K_{\phi\beta} = K_{\beta\phi} &= \frac{2}{9}[q(y)w(y)+8y^2-8y]\cos(\theta), \\ K_{\phi\psi'} = K_{\psi\phi} &= \frac{16}{9}(y-1)\cos(\theta), \\ K_{\beta\psi'} = K_{\psi'\beta} &= \frac{16}{9}y. \end{aligned} \tag{3.24}$$

The Killing condition means that the infinitesimal transformation $\delta\dot{x}^\mu = K^\mu_\alpha \dot{x}^\alpha$ represents a hidden symmetry for the geodesic motion in the $Y(p,q)$ Einstein-Sasaki metric (2.8). The associated constant of motion is

$$\begin{aligned} C = K_{\mu\nu}\dot{x}^\nu\dot{x}^\mu \;=\; & K_{\theta\theta}\dot{\theta}^2 + K_{\phi\phi}\dot{\phi}^2 + K_{yy}\dot{y}^2 + K_{\beta\beta}\dot{\beta}^2 + K_{\psi'\psi'}\dot{\psi'}^2 \\ & + 2K_{\phi\beta}\dot{\phi}\dot{\beta} + 2K_{\phi\psi'}\dot{\phi}\dot{\psi}' + 2K_{\beta\psi'}\dot{\beta}\dot{\psi}'. \end{aligned} \tag{3.25}$$

By going to the coordinates $(\theta, \phi, y, \alpha, \psi)$ which takes the metric to the form (2.4) the constant may be expressed as follows

$$\begin{aligned} C =& \frac{4}{3}(1-y)\dot{\theta}^2 + \frac{8}{q(y)w(y)}\dot{y}^2 + 8\left[q(y)w(y)+8y^2\right]\dot{\alpha}^2 + \left\{16\left[q(y)w(y)+8y^2\right] - \frac{16}{3}\right\}\dot{\alpha}\dot{\psi} \\ &+ \frac{1}{9}\left\{[1+\cos(2\theta)]\left[q(y)w(y)+8y^2\right] + \cos(2\theta)\,[2-10y] + 14 - 22y\right\}\dot{\phi}^2 - \frac{24}{9}\left[q(y)w(y)\right. \\ &\left. + 8y^2 - 8y\right]\cos(\theta)\dot{\phi}\dot{\alpha} + \left\{\frac{32}{9}(y-1)\cos\theta - \frac{24}{9}\left[q(y)w(y)+8y^2-8y\right]\cos(\theta)\right\}\dot{\phi}\dot{\psi} \\ &+ \left\{\frac{16}{9} - \frac{64}{3}y + 8\left[q(y)w(y)+8y^2\right]\right\}\dot{\psi}^2. \end{aligned} \tag{3.26}$$

The results presented above are not enough to state that the geodesic equations on $Y(p,q)$ are superintegrable. This will be the case if the set $\{P_\phi, P_\psi, P_\alpha, J^2, H, C\}$ constitute a functionally independent set of constants of motion for the massless geodesics on $AdS_5 \times Y(p,q)$ geometry considered in previous sections. To prove the functional independence one should construct the $(n+1)\times 2n$ Jacobian

$$J = \frac{\partial(P_\phi, P_\psi, P_\alpha, J^2, H, C)}{\partial(\theta, \phi, y, \alpha, \psi, \dot{\theta}, \dot{\phi}, \dot{y}, \dot{\alpha}, \dot{\psi})}, \tag{3.27}$$

with $n = 5$ and to calculate its rank. The result is

$$Rank(J) = 6, \tag{3.28}$$

and was obtained by use of the Wolfram Mathematica program. Therefore it is safe to say that the configuration of massless geodesics on $AdS_5 \times Y(p,q)$ defined in previous sections are superintegrable, since the number of degrees of freedom is five and the number of functionally independent constants of motion is at least six.

4 Comparison with the literature

Having shown the superintegrability of the geodesic equations for the $Y(p,q)$ geometries, we turn our attention in deriving consequences of this fact. Our analysis will rely in some standard results in the literature, which we will cite below.

4.1 Separability of the Laplace and the Dirac operators

The first consequence is that the hidden symmetry that the Killing tensor (3.26) generates is not anomalous. This statement may be explained as follows. The quantum mechanical analogue of the hamiltonian for the free particle in a curved geometry is the laplacian

$$\Delta = \nabla_\mu(g^{\mu\nu}\nabla_\nu).$$

Any Killing vector $V = V^\mu\partial_\mu$ for $g_{\mu\nu}$ is in correspondence with a quantum mechanical operator $\hat{V} = V^\mu\nabla_\mu$ which commutes with the laplacian Δ defined above. But this is not the case for Killing tensors [75], unless some extra conditions are satisfied. In fact, the commutator between the laplacian Δ and the operator $\hat{K} = \nabla_\mu(K^{\mu\nu}\nabla_\nu)$ associated to the Killing tensor $K_{\mu\nu}$ is given by [75]

$$[\widehat{K}, \widehat{H}] = -\frac{4}{3}\nabla_\nu(R_\mu^{[\nu}K^{\sigma]\mu})\nabla_\sigma,$$

and it follows that it is not zero in general. This means that the symmetry a Killing tensor generates is anomalous unless the integrability condition

$$\nabla_\nu(R_\mu^{[\nu}K^{\sigma]\mu}) = 0, \tag{4.1}$$

is satisfied. This condition holds when the space is Einstein $R_{\mu\nu} = \Lambda g_{\mu\nu}$. This is the case for the $Y(p,q)$ geometries, and thus the operator $\hat{K} = \nabla_\mu K^{\mu\nu}\nabla_\nu$ corresponding to (3.23) represents a genuine symmetry at the quantum level.

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The presence of the Killing-Yano tensor (3.22) is also relevant for studying the separability of the Dirac operator for the $Y(p,q)$ geometry [101, 102]. In general, for geometries admiting spinors one can consider an irreducible representation of the Clifford algebra, which is composed by elements e^a for which

$$e^a e^b + e^b e^a = g^{ab}. \tag{4.2}$$

The Dirac operator for these geometries is

$$D = e^a \nabla_{X_a}. \tag{4.3}$$

Given an arbitrary p-form $f_{\mu_1 \ldots \mu_p}$ one may construct an operator D_f with special properties. This operator is given explicitly as

$$D_f = L_f - (-1)^p f D, \tag{4.4}$$

with

$$L_f = e^a f \nabla_{X^a} + \frac{p}{p+1} df + \frac{n-p}{n-p+1} d^* f. \tag{4.5}$$

By defining the graded commutator

$$\{D, D_f\} = D D_f + (-1)^p D_f D, \tag{4.6}$$

it follows from (4.4) and (4.3) that

$$\{D, D_f\} = RD, \qquad R = \frac{2(-1)^p}{n-p+1} d^* f D. \tag{4.7}$$

If the p-form $f_{\mu_1 \ldots \mu_p}$ is Killing-Yano, then from (3.19) and (3.16), it follows that $d^* f = 0$ and therefore $R = 0$. By comparing this with (4.7) it is seen immediately that for any Killing-Yano tensor $f_{\mu_1 \ldots \mu_p}$ of arbitrary order there exist an operator D_f acting on spinors and whose graded commutator with the Dirac operator D is zero [101]. We remark that for p odd the graded commutator becomes the usual commutator. Therefore the presence of the Killing-Yano 3-form (3.22) for the $Y(p,q)$ geometries implies that there is an operator commuting with the Dirac operator defined on the background.

4.2 New symmetries for the spinning particle action

Killing-Yano tensors are also generators for symmetries of the spinning particle, which is a supersymmetric extension of the bosonic particle (3.1) [87]–[93]. This action can be written in terms of a superfield X which maps a supermanifold parameterized by two coordinates (t, θ) into a space time manifold M with metric $g_{\mu\nu}$. The variable θ is a Grassman variable, which means that $\theta^2 = 0$. In these terms the spinning particle action reads

$$I = -\frac{i}{2} \int dt d\theta g_{\mu\nu} D X^\mu \partial_t X^\nu. \tag{4.8}$$

Here D is the worldline superspace derivate, for which $D^2 = i\partial_t$. An old example of symmetries for (4.8) are exotic supersymmetries, which correspond to Killing-Yano tensors of order two. These are symmetries imitate the supersymmetry property of mixing the bosonic and fermionic components of the superfield X [103]–[106]. Also the parameter of the symmetry is anti-commuting. These of course, are not genuine supersymmetries, in fact they are not described by the usual super Poincare algebra. In a more general context, when the space-time metric $g_{\mu\nu}$ admits a set of Killing-Yano tensors $f^i_{\mu_1\ldots\mu_p}$ of arbitrary order, then there appear new symmetries for (4.8) of the form [107]

$$\delta X^\mu = a_l f^l_{\mu_1\ldots\mu_p} DX^{\mu_1}..DX^{\mu_p}, \tag{4.9}$$

with a_l infinitesimal parameters. Therefore, when the background geometries are the $Y(p,q)$ ones, there exist a symmetry of the form (4.9) for (4.8) with the Killing-Yano tensors (3.22) playing the role of any of the p-forms $f^i_{\mu_1\ldots\mu_p}$.

5 Discussion and open perspectives

In the present work a constant of motion of Carter type for the geodesic motion of a probe particle in any of the $Y(p,q)$ Einstein-Sasaki geometries was constructed. The constant constructed is functionally independent with respect to the five known constant of motion for the geometry. The complete set of functionally independent constants for the problem is at least six and since there are five degrees of freedom, the geodesic equations turns out to be superintegrable.

We have derived several consequences of these results. We have shown that the additional symmetry is preserved at the quantum level, that is, it corresponds to an operator which commutes with the laplacian defined over the $Y(p,q)$ geometries. It also corresponds to an operator which commutes with the Dirac operator defined on the background. In addition to these results, we have shown that the spinning particle action [87]–[93] with any $Y(p,q)$ geometry as target metric possesses a symmetry of the form (4.9) with our Killing-Yano tensor (3.22) playing the role of the symmetry generator.

We ignore if the Killing-Yano tensor (3.22) and the Killing tensor (3.23) were presented in the literature before. By looking the references we have in hand, we suggest that the present work may overlap with [109, 110]. That reference studies the eigenfunctions of the Laplacian defined over a family of Einstein-Sasaki spaces. This family is obtained by reduction of certain black hole space-times which possess hidden symmetries, and contains the $Y(p,q)$ geometries as a particular case. The authors of [109, 110] are able to separate the eigenvalue equations completely and they find an accidental quadratic constant of motion C during the process. Their interpretation is that C is related to the hidden symmetries coming from the higher dimensional black hole geometry. What we do not know is that, when the $Y(p,q)$ limit is taken, the constant C turns out to be equivalent to (3.23). Note that equivalent does not mean equal, but that the two constants are the same up to a quadratic combination of the other symmetry generators. If the resulting constants are not equivalent, then the $Y(p,q)$ geometries admit two hidden symmetries instead of one, which is a possibility that deserves to be investigated further. But even if they happen to coincide, the advantage of the construction we presented in (3.22) and (3.26) is that it is intrinsic, that is, it does not takes into account reductions coming from higher dimensional

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black holes or other geometries. Additionally, the arguments we presented can be applied to more general backgrounds, such as the ones considered in [114]. For this reason, even if we happen to rediscover some results of [109, 110], we are using different arguments which can be applied to more general situations.

To conclude, we remark that it may be an interesting task to realize whether or not the fake supersymmetries considered in [108] are related to hidden symmetries. This is a possibility that deserves further attention.

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A Sasaki and Einstein-Sasaki metrics

A.1 Defining equations for Sasaki estructures

Let us consider a generic cone in six dimensions

$$g_6 = dr^2 + r^2 g_5. \tag{A.1}$$

Here the metric g_5 does not depend on the radial coordinate r and is defined over a five dimensional variety which we denote as M_5. The metric g_6 is defined over $R_{>0} \times M_5$ and is singular at the tip of the cone $r = 0$, unless g_5 is the canonical metric of S^5. When the cone g_6 is Kahler the metric g_5 is known as Sasaki and the converse is also true. This fact can be taken as the definition of a Sasaki metric, in a local sense. Let us recall that a metric g_6 is Kahler if there exist a basis e^a such that

$$g_6 = \delta_{ab} e^a \otimes e^b,$$

and for which the almost complex structure

$$J = e_1 \otimes e^2 - e_2 \otimes e^1 + e_3 \otimes e^4 - e_4 \otimes e^3 + e_5 \otimes e^6 - e_6 \otimes e^5, \tag{A.2}$$

is covariantly constant, i.e, $\nabla_X J = 0$ for any vector field X in TM_6. This condition implies that the manifold is complex and sympletic with respect to the two form $\omega = g(\cdot, J\cdot)$. In the basis e^i this sympletic form is expressed as follows

$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6.$$

In addition, if g_6 is Ricci flat then g_5 is Einstein. Since a Ricci flat Kahler metric is Calabi-Yau,[2] it follows that there is a one to one correspondence between Calabi-Yau cones (A.1) and Einstein-Sasaki manifolds.

[2]For the non compact case, Calabi-Yau metrics may be defined as $2n$ dimensional ones with holonomy group included or equal to SU(n).

It is convenient for the following discussion to select a basis for g_6 of the form

$$e_i = r\widetilde{e}_i, \qquad e_6 = dr, \tag{A.3}$$

with $i = 1, .., 5$ and $\widetilde{e}_i$ a basis for g_5. In this basis (A.3) the almost complex structure (A.2) is expressed as

$$J = \widetilde{e}_1 \otimes \widetilde{e}^2 - \widetilde{e}_2 \otimes \widetilde{e}^1 + \widetilde{e}_3 \otimes \widetilde{e}^4 - \widetilde{e}_4 \otimes \widetilde{e}^3 + r\widetilde{e}_5 \otimes \partial_r - \frac{dr}{r} \otimes \widetilde{e}^5. \tag{A.4}$$

Alternatively (A.4) may be written as

$$J = \phi + r\eta \otimes \partial_r - \frac{dr}{r} \otimes \xi, \tag{A.5}$$

with $\eta = e_5$, $\xi = e^5 = \eta^*$ and

$$\phi = \widetilde{e}_1 \otimes \widetilde{e}^2 - \widetilde{e}_2 \otimes \widetilde{e}^1 + \widetilde{e}_3 \otimes \widetilde{e}^4 - \widetilde{e}_4 \otimes \widetilde{e}^3. \tag{A.6}$$

Although the expression (A.6) for ϕ involves four elements of the basis, it should not be concluded that ϕ is a quantity defined in a subvariety four dimensional M_4 of M_5. In general, the elements $\widetilde{e}^a$ with $a = 1, 2, 3, 4$ are 1-forms generically defined over M_5, and not over a submanifold. Nevertheless, when the metric is Kahler these elements are defined on a subvariety M_4, and the restriction of ϕ to M_4 becomes a complex structure for M_4. This statement can be shown as follows. A vector field $\widetilde{X} \in R_{>0} \times M_5$ may be decomposed in a radial part a and an angular part X, we write it as $\widetilde{X} = (a, X)$. From (A.5) it may be deduced that the action of J over $\widetilde{X}$ is

$$J(a, X) = (r\eta(X), \phi X - \frac{a}{r}\xi). \tag{A.7}$$

Furthermore, the Levi-Civita connection $\widetilde{\nabla}$ for the cone may be decomposed in the following way

$$\begin{aligned} \widetilde{\nabla}_{\partial_r}\partial_r &= 0, \qquad \widetilde{\nabla}_X\partial_r = \widetilde{\nabla}_{\partial_r}X = \frac{X}{r} \\ \widetilde{\nabla}_X Y &= \nabla_X Y - r g(X, Y)\partial_r. \end{aligned} \tag{A.8}$$

Here ∇ is the connection for g_5. Formula (A.8) is elementary and arise directly by comparing the Christofell symbols of g_5 with the ones of g_6. From (A.8) and (A.5) it is obtained that

$$\begin{aligned} (\widetilde{\nabla}_{\partial_r}J)\partial_r &= (0, 0), \qquad (\widetilde{\nabla}_{\partial_r})X = (0, 0), \\ (\nabla_X J)\partial_r &= \left(0, \frac{1}{r}(-\nabla_X\xi + \phi X)\right) \\ (\widetilde{\nabla}_X J)Y &= (r\nabla_X\eta(Y) - r g_5(X, \phi Y), (\nabla_X\phi)Y - g_5(X, Y)\xi + \eta(Y)X). \end{aligned} \tag{A.9}$$

As it was remarked above, if $\nabla_X J = 0$ the cone g_6 is Kahler. This means that all the covariant derivatives (A.9) are zero, which is satisfied if and only if

$$\nabla_X\xi = \phi X, \tag{A.10}$$

$$\nabla_X\eta(Y) = g_5(X, \phi Y), \tag{A.11}$$

$$(\nabla_X\phi)Y = g_5(X, Y)\xi - \eta(Y)X. \tag{A.12}$$

The radial coordinate r does not appear in these formulas, which are then conditions over the metric g_5. The metrics which fullfill (A.10), (A.11) and (A.12) are Sasaki by definition.

A.2 Derivation of the main formulas (3.22) and (3.23) of the text

The relations (A.10)–(A.12) written above may not be illuminating, but they may be clarified by examining their consequences. They are in fact crucial for showing that Sasakian metrics are locally U(1) fibrations over a Kahler metric. The relation (A.11) implies that

$$\nabla_X \eta(Y) + \nabla_Y \eta(X) = g_5(X, \phi Y) + g_5(Y, \phi X), \tag{A.13}$$

but it may be directly deduced from the definition (A.6) of ϕ that $g_5(X, \phi Y) = -g_5(Y, \phi X)$. Thus our last equation is

$$\eta_{(i;j)} = 0, \tag{A.14}$$

which implies that $\xi = \eta^*$ is a Killing vector and we have the local decomposition $M_5 = U_\xi(1) \times M_4$. The metric takes the form

$$g_5 = \eta^2 + g_4,$$

with

$$g_4 = \tilde{e}^1 \otimes \tilde{e}^1 + \tilde{e}^2 \otimes \tilde{e}^2 + \tilde{e}^3 \otimes \tilde{e}^3 + \tilde{e}^4 \otimes \tilde{e}^4.$$

This is the local form (2.9) presented in the text. The vector field $\xi = \eta^*$ is the Reeb vector. Additionally if we define I the restriction of ϕ to M_4 then it follows from (A.11) and (A.14) that

$$g_4(u, Iv) = d\eta(u, v),$$

for arbitrary vectors u and v in TM_4. Denoting

$$f(u, v) = d\eta(u, v) \tag{A.15}$$

it follows that $d_4 f = 0$, thus M_4 is sympletic with respect to f. In addition the antisymmetric part of (A.12) can be written by taking into account (A.10) and uppering indices with the metric g_5 as follows [113]

$$\nabla_X(d\eta) = -2X^* \wedge \eta. \tag{A.16}$$

Note that for vectors u in TM_4 the right hand side of (A.16) is zero, since vector fields in TM_4 are orthogonal to η. This means that

$$\nabla_4 d\eta = \nabla_4 f = 0,$$

where we took into account the definition (A.15). The last condition says that the metric g_4 defined on M_4 is not only sympletic, but Kahler. This is one of the features discussed below formula (2.11) in the text.

Finally, let us explain how the Killing-Yano tensor (3.22) and the Killing tensor (3.23) are obtained. Consider again (A.16). A direct consequence of this condition is that $d^* d\eta = 2(n-1)\eta$, with $n = 5$. This means that (A.16) can be expressed alternatively as

$$\nabla_X(d\eta) = -\frac{1}{n-1} X^* \wedge d^* d\eta. \tag{A.17}$$

By taking into account that $d\eta$ is closed and by denoting $f = d\eta$, it follows by comparison by (A.17) with (3.19) that $d\eta$ is a conformal Killing tensor. This, together with the fact that η is a Killing 1-form implies that the combinations

$$\omega_k = \eta \wedge (d\eta)^k, \tag{A.18}$$

are all Killing-Yano tensors of order $2k + 1$. This calculation is straightforward and we learned it from proposition 3.4 of [113]. Note that (A.18) is a *generic* statement for all the Sasakian structures and that the Einstein condition do not play any role in obtaining these results.

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