Published for SISSA by Springer

Received: July 26, 2011
Revised: September 9, 2011
Accepted: September 16, 2011
Published: September 27, 2011

Baxter's T-Q equation, $\mathrm{SU}(N)/\,\mathrm{SU}(2)^{N-3}$ correspondence and Ω-deformed Seiberg-Witten prepotential

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Abstract: We study Baxter's T-Q equation of XXX spin-chain models under the semi-classical limit where an intriguing $\mathrm{SU}(N)/\,\mathrm{SU}(2)^{N-3}$ correspondence is found. That is, two kinds of 4D $\mathcal{N}=2$ superconformal field theories having the above different gauge groups are encoded simultaneously in one Baxter's T-Q equation which captures their spectral curves. For example, while one is $\mathrm{SU}(N_c)$ with $N_f = 2N_c$ flavors the other turns out to be $\mathrm{SU}(2)^{N_c-3}$ with N_c hyper-multiplets ($N_c > 3$). It is seen that the corresponding Seiberg-Witten differential supports our proposal.

Keywords: Integrable Equations in Physics, Duality in Gauge Field Theories, Bethe Ansatz, String Duality

ArXiv ePrint: 1107.3756

doi:10.1007/JHEP09(2011)125

JHEP09(2011)125

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1 Introduction and summary

Recently there have been new insight into the duality between integrable systems and 4D $\mathcal{N} = 2$ gauge theories. In [1–3] Nekrasov and Shatashvili (NS) have found that Yang-Yang functions as well as Bethe Ansatz equations of a family of integrable models are indeed encoded in a variety of Nekrasov's partition functions [4, 5] restricted to the two-dimensional Ω-background.[1] As a matter of fact, this mysterious correspondence can further be extended to the full Ω-deformation in view of the birth of AGT conjecture [10]. Let us briefly refine the latter point.

Recall that AGT claimed that correlators of primary states in Liouville field theory (LFT) can get re-expressed in terms of Nekrasov's partition function Z_{Nek} of 4D $\mathcal{N} = 2$ quiver-type SU(2) superconformal field theories (SCFTs). In particular, every Riemann surface $C_{g,n}$ (whose doubly-sheeted cover is called Gaiotto curve [11]) on which LFT dwells is responsible for one specific SCFT called $\mathcal{T}_{g,n}(A_1)$ such that the following equality

$$\text{Conformal block w.r.t. } C_{g,n} = \text{Instanton part of } Z_{\text{Nek}}\Big(\mathcal{T}_{g,n}(A_1)\Big)$$

holds. Because of $\epsilon_1 : \epsilon_2 = b : b^{-1}$ [10] the one-parameter version of AGT conjecture directly leads to the semiclassical LFT at $b \to 0$. Quote further the geometric Langlands correspondence [12] which associates Gaudin integrable models on the projective line with LFT at $b \to 0$. It is then plausible to put both insights of NS and AGT into one unified scheme.

[1] See also recent [6–9] which investigated XXX spin-chain models along this line.

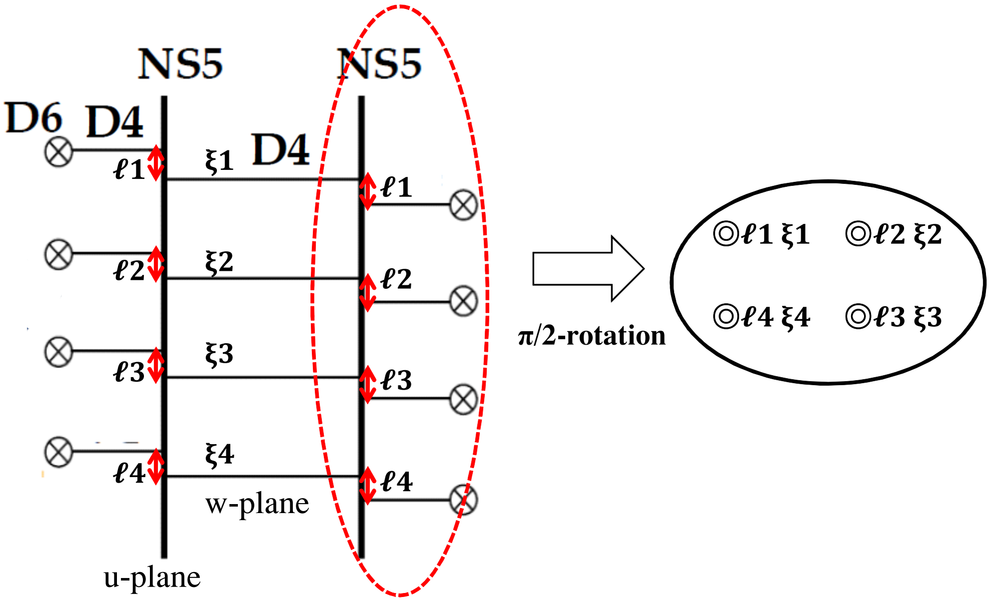

Figure 1. Main idea: $\mathrm{SU}(N)/\,\mathrm{SU}(2)^{N-3}$ correspondence. L.h.s.: M-theory curve of SU(4) $N_f = 8$ Yang-Mills theory embedded in $\mathbf{C} \times \mathbf{C}^*$ parameterized by (u, w) $(w = \exp(-s/R),\ R = \ell_s g_s :$ M-circle radius); r.h.s.: (ξ, ℓ) labeling (coordinate, weight) of each puncture on $\mathbf{CP}^1$ here but indicating each flavor D6-brane location along u-plane before.

In this letter, we add a new element into the above 2D/4D correspondence. Starting from Baxter's T-Q equation of XXX spin-chain models we found a novel interpretation of it. That is, under the semiclassical limit it possesses two aspects simultaneously. It describes

- 4D $\mathcal{N} = 2$ SU(N_c) Yang-Mills with $N_f = 2N_c$ flavors $\mathcal{T}_{0,4}(A_{N_c-1})$ on the one hand and
- SU$(2)^{N_c-3}$ $(N_c > 3)$ quiver-type Yang-Mills with N_c (four fundamental and $N_c - 4$ bi-fundamental) hyper-multiplets $\mathcal{T}_{0,N_c}(A_1)$ on the other hand.

It is helpful to have a rough idea through figure 1. Pictorially, a punctured $\mathbf{CP}^1$ (Gaiotto curve) in r.h.s. results from the encircled part in l.h.s. after a $\pi/2$-rotation. In other words, the conventional Type IIA Seiberg-Witten (SW) curve (see table 1) in fact contains another important piece of information while seen from (u, v)-space $(u = x^4 + ix^5, v = x^7 + ix^8)$.[2] Here, "$\pi/2$-rotation" just means that SW differentials of the above two theories are related by exchanging holomorphic coordinates $(u, s = x^6 + ix^{10})$ as will be found.

[2]This aspect of $\mathcal{N} = 2$ curves is also stressed in [13].

	0, 1, 2, 3	$u = x^4 + ix^5$	6	7, 8, 9
D6	○	-	-	○
NS5	○	○	-	-
D4	○	-	○	-

Table 1. Type IIA D6-NS5-D4 brane configuration

# of UV parameter	l.h.s.	r.h.s.
Coulomb moduli	$N_c - 1$ (ξ)	$\odot N_c - 3$ (a)
bare flavor mass	N_c ($\xi \pm \ell$)	N_c (m)
gauge coupling	$\odot 1 \exp\left(\dfrac{\Delta x^6 + i\Delta x^{10}}{R}\right)$	$N_c - 3$ (q)

Table 2. Comparison of UV parameters of two gauge theories in figure 1.

This quite unexpected phenomenon will be explained later by combining a couple of mathematical topics, say, Bethe Ansatz, Gaudin model and Liouville theory. The relationship between parameters is summarized in table 2. Roughly speaking, the spin-chain parameter ℓ (ξ) is related to m (q). This implies that N_c Coulomb moduli $\xi \in \mathbf{C}$ (one dummy U(1) factor) are mapped to $N_c - 3$ gauge coupling constants $q = \exp(2\pi i\tau) \in \mathbf{C}^*$ where three of them are fixed to $(0, 1, \infty)$ on $\mathbf{C}^*$. Those entries marked by $\odot$ do not have direct comparable counterparts. See section 4 for more complete discussion.

We organize this letter as follows. Section 2 is devoted to a further study of figure 2 on which our main idea figure 1 is based. Then section 3 unifies three elements: Gaudin model, LFT and matrix model as shown in figure 3. Finally, in section 4 we complete our proposal by examining λ_{SW} (SW differential) and shortly discuss XYZ Gaudin models.

2 XXX spin chain

Baxter's T-Q equation [14, 15] plays an underlying role in various spin-chain models. It emerges within the context of quantum inverse scattering method (QISM) or algebraic Bethe Ansatz. On the other hand, it has long been known that the low-energy Coulomb sectors of $\mathcal{N} = 2$ gauge theories are intimately related to a variety of integrable systems [16–20]. Here, by integrable model (or solvable model) we mean that there exists some spectral curve which gives enough integrals of motion (or conserved charges). In the case of $\mathcal{N} = 2$ SU(N_c) Yang-Mills theory with N_f fundamental hyper-multiplets, its SW curve [21, 22] is identified with the spectral curve of an inhomogeneous periodic Heisenberg XXX spin chain on N_c sites:

$$w + \frac{1}{w} = \frac{P_{N_c}(u)}{\sqrt{Q_{N_f}(u)}}. \tag{2.1}$$

Here, two polynomials P_{N_c} and Q_{N_f} encode respectively parameters of $\mathcal{N}=2$ vector- and hyper-multiplets. Meanwhile, the meromorphic SW differential $\lambda_{\rm SW}=ud\log w$ provides a set of "special coordinates" through its period integrals. From table 2 one has

$$\xi_i=\oint_{\alpha_i}\lambda_{\rm SW},\qquad \frac{\partial\mathcal{F}_{\rm SW}}{\partial\xi_i}=\xi_i^D=\oint_{\beta_i}\lambda_{\rm SW},\qquad \xi_i\pm\ell_i=\oint_{\gamma_i}\lambda_{\rm SW} \tag{2.2}$$

where $\mathcal{F}_{\rm SW}$ is the physical prepotential.

2.1 Baxter's T-Q equation

Indeed, (2.1) arises from (up to $w\to\sqrt{Q_{N_f}}w$)

$$\det\big(w-T(u)\big)=0\quad\to\quad w^2-{\rm tr}\,T(u)w+\det T(u)=0,\qquad T(u):\ \text{monodromy matrix},$$
$$\det T(u)=Q_{N_f}(u)=\prod_{i=1}^{N_c}(u-m_i^-)(u-m_i^+),\qquad m_i^\pm=\xi_i\pm\ell_i.$$

${\rm tr}\,T(u)=t(u)=P_{N_c}(u)=\langle\det(u-\Phi)\rangle$, called transfer matrix, encodes the quantum vev of the adjoint scalar field Φ. In fact, (2.1) belongs to the conformal case where $N_f=2N_c$ bare flavor masses are indicated by $m_i^\pm$. It is time to quote Baxter's T-Q equation:

$$t(u)Q(u)=\triangle_+(u)Q(u-2\eta)+\triangle_-(u)Q(u+2\eta). \tag{2.3}$$

Some comments follow:

- η is Planck-like and ultimately gets identified with ϵ_1 (one of two Ω-background parameters) in section 4.
- As a matter of fact, (2.3) boils down to (2.1) (up to $w\to\sqrt{Q_{N_f}}w$) as $\eta\to0$. Curiously, its $\lambda_{\rm SW}$ signals the existence of another $\mathcal{N}=2$ theory. The situation is pictorially shown in figure 1. Note that SW differentials of two theories are related by exchanging two holomorphic coordinates (u,s) but their M-lifted [23] Type IIA NS5-D4 brane systems[3] are not. The $\pi/2$-rotated part is closely related to $\mathcal{N}=2$ Gaiotto curves. A family of quiver-type SU(2) SCFTs $\mathcal{T}_{0,n}(A_1)$ discovered by Gaiotto [11] is hence made contact with.

2.2 More detail

Let us refine the above argument. Consider a quantum spin-chain built over an N-fold tensor product $\mathcal{H}=\otimes_{n=1}^N V_n$. Namely, at each site labeled by n we assign an irreducible (ℓ_n+1)-dimensional representation V_n of $\mathfrak{sl}_2$ (ℓ_n: highest weight). Within the context of QISM, monodromy and transfer matrices are defined respectively by

$$T(u)=\begin{pmatrix}A_N(u)&B_N(u)\\C_N(u)&D_N(u)\end{pmatrix}=L_N(u-\xi_N)\cdots L_1(u-\xi_1), \tag{2.4}$$
$$\widehat{t}(u)=A_N(u)+D_N(u).$$

[3] In [24] this symmetry has been notified in the context of Toda-chain models because two kinds of Lax matrices exist there.

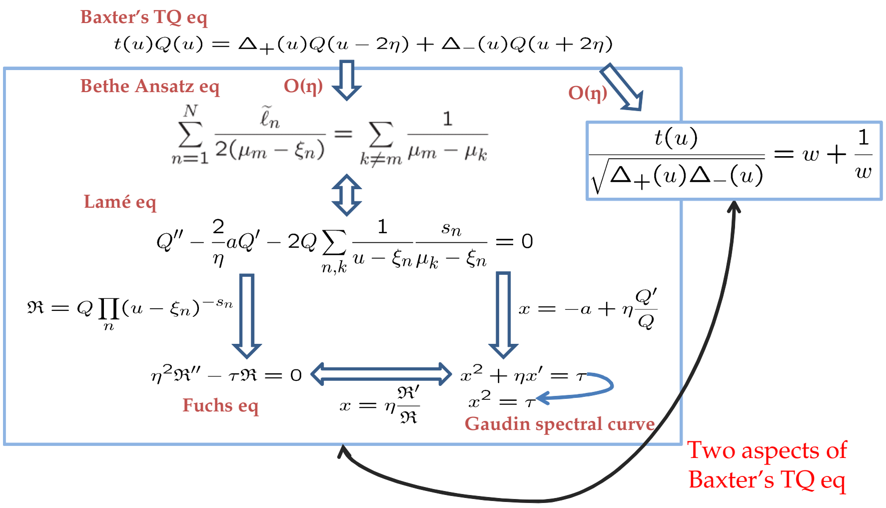

Figure 2. Mathematical description of figure 1.

V_n is acted on by the n-th Lax operator L_n. By *inhomogeneous* one means that the spectral parameter u has been shifted by ξ. Conventionally, $\widehat{t}(u)$ or its eigenvalue $t(u)$ is called generating function because a series of conserved charges can be extracted from its coefficients owing to $[\widehat{t}(u), \widehat{t}(v)] = 0$. The commutativity arises just from the celebrated Yang-Baxter equation.

As far as the inhomogeneous periodic XXX spin chain is concerned, its T-Q equation reads

$$\begin{aligned} t(u)Q(u) &= \triangle_+(u)Q(u-2\eta) + \triangle_-(u)Q(u+2\eta), \\ Q(u) &= \prod_{k=1}^{K}(u-\mu_k), \qquad \triangle_\pm = \prod_{n=1}^{N}(u-\xi_n \pm \widetilde{\ell}_n\eta), \end{aligned} \tag{2.5}$$

where each Bethe root μ_k satisfies a set of Bethe Ansatz equations ($\eta\widetilde{\ell} = \ell$):

$$\frac{\triangle_+(\mu_k)}{\triangle_-(\mu_k)} = \prod_{n=1}^{N} \frac{(\mu_k - \xi_n + \widetilde{\ell}_n\eta)}{(\mu_k - \xi_n - \widetilde{\ell}_n\eta)} = \prod_{l(\neq k)}^{K} \frac{\mu_k - \mu_l + 2\eta}{\mu_k - \mu_l - 2\eta}. \tag{2.6}$$

Here, the η dependence enables us to carry out a semiclassical limit later on. Through

$$\frac{t(u)}{\sqrt{\triangle_+\triangle_-}} = \frac{Q(u-2\eta)}{Q(u)}\sqrt{\frac{\triangle_+}{\triangle_-}} + \frac{Q(u+2\eta)}{Q(u)}\sqrt{\frac{\triangle_-}{\triangle_+}} \tag{2.7}$$

and

$$w \equiv \sqrt{\frac{\triangle_+}{\triangle_-}}(1 - 2\eta\frac{Q'}{Q}) \tag{2.8}$$

as $\eta \to 0$ and $\ell =$ fixed we arrive at

$$\frac{t(u)}{\sqrt{\triangle_+(u)\triangle_-(u)}} = w + \frac{1}{w} \tag{2.9}$$

which is nothing but (2.1). From now on we call $\lambda_{\rm SW} \equiv \lambda^{\eta}_{\rm SW} = ud \log w$ of (2.9) "η-deformed" SW differential as in [25, 26] because

$$\lambda^{\eta}_{\rm SW} = 2\eta ud \left(\frac{\Psi'}{\Psi}\right) + \mathcal{O}(\eta^2), \qquad \Psi = \frac{1}{Q(u)} \prod_n (u - \xi_n)^{\widetilde{\ell}_n/2}. \tag{2.10}$$

It is this form that signals the existence of r.h.s. in figure 1. As $\eta \to 0$ the classical version of (2.6) reads

$$\sum_{n=1}^{N} \frac{\widetilde{\ell}_n}{2(\mu_k - \xi_n)} = \sum_{l(\neq k)}^{K} \frac{1}{(\mu_k - \mu_l)}. \tag{2.11}$$

Let us emphasize that the standard classical limit is taken by $\eta \to 0$ with $\widetilde{\ell}_n$ fixed.

Comments on (2.9) follow:

- In M-theory D6-branes correspond to singular loci of $xy = \triangle_+(u)\triangle_-(u)$. This simply means that one incorporates flavors via replacing a flat $\mathbf{R}^4$ over (u, s) by a resolved $\mathbf{A}_{2N_c-1}$-type singularity.

- Without flavors one sees that $\oint \lambda^{\eta}_{\rm SW}$ reduces to a logarithm of the usual Vandermonde evaluated over Bethe roots. This sounds like the familiar Dijkgraaf-Vafa story [27–29] without any tree-level potential which brings $\mathcal{N} = 2$ pure Yang-Mills to $\mathcal{N} = 1$ descendants.

- Surely, this intuition is noteworthy because (2.11) manifests itself as the saddle-point condition within the context of matrix models. To pursue this interpretation, one should regard μ's as diagonal elements of a Hermitian matrix $\mathcal{M}$ (of size $K \times K$). Besides, the tree-level potential now obeys

$$\mathcal{W}'(x) = \sum_{n=1}^{N} \frac{\ell_n}{(x - \xi_n)}.$$

  In other words, we are equivalently dealing with "$\mathcal{N} = 2$" Penner-type matrix models which have been heavily investigated recently in connection with AGT conjecture due to [30]. In what follows, our goal is to show that $\lambda^{\eta}_{\rm SW}$ does reproduce the ϵ_1-deformed SW prepotential w.r.t. $\mathcal{T}_{0,N}(A_1)$. This is accomplished by means of the chart drawn in figure 3. This phenomenon is referred to as the advertised SU(N)/SU$(2)^{N-3}$ $(N > 3)$ correspondence.

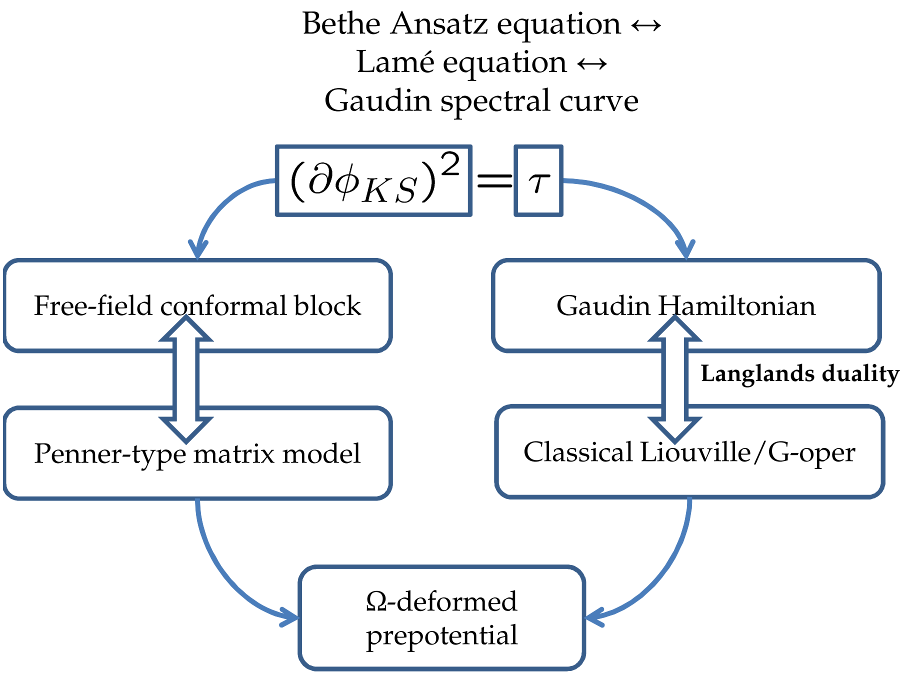

Figure 3. Flow chart of section 3.

3 XXX Gaudin model

Momentarily, we turn to another well-studied integrable model: XXX Gaudin model. It turns out that $\lambda^{\eta}_{\rm SW}$ naturally emerges as the holomorphic one-form (2.10) of Gaudin's spectral curve which captures Gaiotto's curve of $\mathcal{T}_{0,N}(A_1)$.

The essential difference between Heisenberg and Gaudin models amounts to the definition of their generating functions. Following figure 3 we want to explain two important aspects of Gaudin's spectral curve.

3.1 R.h.s. of figure 3

Expanding around small η, we yield

$$L_n(u) = 1 + 2\eta\mathcal{L}_n + O(\eta^2), \tag{3.1}$$
$$T(u) = 1 + 2\eta\mathcal{T} + \eta^2\mathcal{T}^{(2)} + O(\eta^3), \tag{3.2}$$
$$t(u) = 1 + \eta^2{\rm tr}\mathcal{T}^{(2)} + O(\eta^3), \tag{3.3}$$
$$\tau(u) \equiv \frac{1}{2}{\rm tr}\mathcal{T}^2, \qquad \mathcal{T} = \sum_n \mathcal{L}_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(u) \end{pmatrix} \tag{3.4}$$

where

$$A(u) = \sum_{n=1}^{N} \frac{J^z_n}{u-\xi_n}, \qquad B(u) = \sum_{n=1}^{N} \frac{J^-_n}{u-\xi_n}, \qquad C(u) = \sum_{n=1}^{N} \frac{J^+_n}{u-\xi_n}. \tag{3.5}$$

Instead of $\mathrm{tr}\,\mathcal{T}^{(2)}$ ($\mathrm{tr}\,\mathcal{T}=0$) the generating function one would like to adopt is ($s=\widetilde{\ell}/2=\ell/2\eta$)

$$\tau(u)=\sum_{n=1}^{N}\left\{\frac{\eta^2 s_n(s_n+1)}{(u-\xi_n)^2}+\frac{c_n}{u-\xi_n}\right\},\qquad c_n=\sum_{i\neq n}^{N}\frac{2\vec{J}_n\cdot\vec{J}_i}{\xi_n-\xi_i},\qquad \vec{J}_n\cdot\vec{J}_n=\eta^2 s_n(s_n+1).\quad(3.6)$$

$\vec{J}=(J^z,J^{\pm})$ represents generators of $\mathfrak{sl}_2$ Lie algebra while c_n's are called Gaudin Hamiltonians which commute with one another as a result of the classical Yang-Baxter equation. In fact, the N-site Gaudin spectral curve is expressed by $\Sigma:\ x^2=\tau\subset T^*C$, a doubly-sheeted cover of $C=\mathbf{CP}^1\backslash\{\xi_1,\cdots,\xi_N\}$. According to the geometric Langlands correspondence,[4] c_n's give exactly *accessory* parameters of a G-oper:

$$\mathcal{D}=-\partial_z^2+\sum_{n=1}^{N}\frac{\delta_n}{(z-\xi_n)^2}+\sum_{n=1}^{N}\frac{\widetilde{c}_n}{z-\xi_n},\qquad \delta=s(s+1),\qquad c=\eta^2\widetilde{c}$$

defined over $C=\mathbf{CP}^1\backslash\{\xi_1,\cdots,\xi_N\}$. The non-singular behavior of $\mathcal{D}$ is ensured by imposing

$$\sum_{n=1}^{N}\widetilde{c}_n=0,\qquad \sum_{n=1}^{N}(\xi_n\widetilde{c}_n+\delta_n)=0,\qquad \sum_{n=1}^{N}(\xi_n^2\widetilde{c}_n+2\xi_n\delta_n)=0.$$

Certainly, one soon recalls that $\tau(u)$ here is nothing but the holomorphic LFT stress-tensor as the central charge $1+6Q^2$ goes to infinity (or $b\to 0$). Namely,

$$\tau=\frac{1}{2}\partial_z^2\varphi_{cl}-\frac{1}{4}(\partial_z\varphi_{cl})^2=\sum_{n=1}^{N}\frac{\delta_n}{(z-\xi_n)^2}+\sum_{n=1}^{N}\frac{\widetilde{c}_n}{z-\xi_n}.$$

Here, φ_{cl} which satisfies Liouville's equation stands for the unique saddle-point w.r.t. $\mathcal{S}_{\mathrm{tot}}[\varphi]$ being specified below. Meanwhile, Polyakov conjectured that one obtains $\widetilde{c}_n$ by computing LFT correlation functions of primary fields $\langle\prod_n V_{\alpha_n}\rangle$ subject to $b\to 0$ with $V_\alpha=\exp(2\alpha\phi)$ ($\Delta_\alpha=\alpha(Q-\alpha),\ Q=b+b^{-1}$). More precisely,

$$\widetilde{c}_n=-\frac{\partial\mathcal{S}_{\mathrm{tot}}[\varphi_{cl}]}{\partial\xi_n},\qquad \widetilde{\alpha}_n=b\alpha_n=s_n+1\qquad(3.7)$$

where on a large disk Γ

$$\mathbf{S}_{\mathrm{tot}}=\int_\Gamma d^2z\left(\frac{1}{4\pi}|\partial_z\phi|^2+\mu e^{2b\phi}\right)+\text{boundary terms},\qquad \mathbf{S}_{\mathrm{tot}}[\phi]=\frac{1}{b^2}\mathcal{S}_{\mathrm{tot}}[\varphi].$$

3.2 L.h.s. of figure 3

As shown in [34], $\tau(u)$ has another form in terms of $a(u)$,[5] i.e. eigenvalue of $A(u)$:

$$\tau(u)=a^2-\eta a'-2\eta\sum_k\frac{a(u)-a(\mu_k)}{u-\mu_k},\qquad a(u)=\sum_{n=1}^{N}\frac{\eta s_n}{(u-\xi_n)}\qquad(3.8)$$

[4] See [31–33] for more detail.

[5] We hope that readers will not confuse $a(u)$ here with Coulomb moduli a.

with μ_k's being Bethe roots. This expression is extremely illuminating in connection with Penner-type matrix models. Borrowing $Q(u)$ from (2.5) and defining

$$\Re(u) \equiv Q(u) \exp\Big(-\frac{1}{\eta} \int^u a(y) dy \Big) = \prod_k (u - \mu_k) \prod_n (u - \xi_n)^{-s_n}, \tag{3.9}$$

we can verify that there holds

$$\eta x' + x^2 = \tau, \qquad x(u) = \eta \frac{\Re'(u)}{\Re(u)} = -a + \sum_k \frac{\eta}{u - \mu_k}. \tag{3.10}$$

This is the so-called Lamé equation in disguise. Equivalently, $\Re(u)$ satisfies a Fuchs-type equation with N regular singularities on the projective line, i.e. $\big(\eta^2 \partial_u^2 - \tau(u)\big)\Re(u) = 0$ (see figure 2).

Imposing the semiclassical limit $\eta \to 0$ with $\eta s_n = \mathcal{O}(\eta^0)$ resembles that in LFT when $\widetilde{\alpha} = b\alpha$ is kept fixed during $b \to 0$. Note that $\epsilon_1 = \eta \approx b$ has been claimed by AGT.[6] By omitting the term $\eta x'$, we consequently arrive at Gaudin's spectral curve

$$x^2 = \tau. \tag{3.11}$$

In view of (3.10), it is tempting to introduce ϕ_{KS}, i.e. Kodaira-Spencer field of Z^M in (3.14). That is,

$$2x \equiv \partial \phi_{\text{KS}} = -\mathcal{W}' + 2\eta \, \text{tr} \big\langle \frac{1}{u - \mathcal{M}} \big\rangle. \tag{3.12}$$

Subsequently, (3.11) becomes precisely the spectral curve of Z^M. Because of (3.12) using

$$\oint \partial \phi_{\text{KS}} du = - \oint \lambda^{\eta}_{\text{SW}} \tag{3.13}$$

we can obtain the tree-level free energy $\mathcal{F}_0$ of Z^M:

$$Z^M = \int \mathcal{D}\mathcal{M} \exp\left[-\frac{1}{\eta} \mathcal{W}(\mathcal{M}) \right], \qquad \mathcal{W}' = \sum_{n=1}^{N} \frac{\ell_n}{(u - \xi_n)}. \tag{3.14}$$

Of course, the saddle-point of Z^M is dictated by (2.11). We want to dispaly in section 4 that $\mathcal{F}_0$ thus yielded does characterize $\mathcal{T}_{0,N}(A_1)$.

Remark that $\partial \phi_{\text{KS}} du = -\lambda^{\eta}_{\text{SW}}$ up to a total derivative term. Since $(x, u) \in \mathbf{C} \times \mathbf{C}^*$ so we introduce $v = xu$ such that xdu here and the former $ud \log w$ look more symmetrical. Besides, the proposed $\text{SU}(N)/\,\text{SU}(2)^{N-3}$ correspondence does not rigorously mean just a $\pi/2$-rotation noted in figure 1 which naively leads to $\text{SU}(N)/\,\text{SU}(2)^{N-1}$ correspondence instead.[7]

Rewriting Z^M in terms of a multi-integral over diagonal elements of $\mathcal{M}$ is another crucial step:

$$Z^M \;\Rightarrow\; \oint dz_1 \cdots \oint dz_K \prod_{i<j} (z_i - z_j)^2 \prod_{i,n} (z_i - \xi_n)^{-\widetilde{\ell}_n} \prod_{n<m} (\xi_n - \xi_m)^{\widetilde{\ell}_n \widetilde{\ell}_m/2} \tag{3.15}$$

[6] Bear in mind $\eta = \hbar/b$ such that when $b \to 0$ η is kept fixed and small.

[7] We thank Yuji Tachikawa for his comment on this point.

which stands for the Dotsenko-Fateev_type free-field realization of an N-point conformal block of primary states in LFT subject to $b \to 0$.[8] In addition, K screening operators $\oint dz \exp 2b^{-1}\phi(z)$ and the free propagator $\langle\phi(z_1)\phi(z_2)\rangle_{\text{free}} = -\log(z_1 - z_2)^{1/2}$ were taken into account. We then have ($\ell_n = \eta\widetilde{\ell}_n,\ \eta = \hbar/b$)

$$\lim_{b\to 0} \log \left\langle V_{\ell_1/2}(\xi_1)\cdots V_{\ell_N/2}(\xi_N)\right\rangle_{\text{conformal block}} = -\eta^2 \widetilde{F},$$
$$\widetilde{F} = -\log Z^M, \qquad Z^M = \exp(-\eta^{-2}\mathcal{F}_0 + \cdots).$$

$\widetilde{F}$ is named *classical* conformal block in the pioneering work of Zamolodchikov and Zamolodchikov [35]. Obviously, the tree-level free energy $\mathcal{F}_0$ gets equal to $\eta^2\widetilde{F}$ at small enough η. Next, $\widetilde{F}$ will be identified with the Ω-deformed SW prepotential of $\mathcal{T}_{0,N}(A_1)$ such that our proposal becomes completed.

4 Application and discussion

By examining a concrete example, we display that $\widetilde{F}$ and $\lambda^{\eta}_{\text{SW}} = -\partial\phi_{\text{KS}} du$ are indeed the very ϵ_1-deformed SW prepotential and differential of $\mathcal{T}_{0,N}(A_1)$, respectively.

Let us focus only on $N = 4$ and quote the known $\tau(u)$ from [35] with $\delta = \widetilde{\alpha}(1-\widetilde{\alpha})$:

$$\frac{1}{\eta^2}\tau(u) = \frac{\delta_1}{u^2} + \frac{\delta_2}{(u-q)^2} + \frac{\delta_3}{(1-u)^2} + \frac{\delta_1+\delta_2+\delta_3-\delta_4}{u(1-u)} + \frac{q(1-q)\widetilde{c}(q)}{u(u-q)(1-u)} \tag{4.1}$$

where via projective invariance q represents the cross-ratio of four marked points $(\xi_1,\xi_2,\xi_3,\xi_4) \equiv (0,1,q,\infty)$ on $\mathbf{CP}^1$. According to Polyakov's conjecture (3.7) as well as (3.11) and the residue of τ around $u = 1$ we write down ($v = xu$)

$$q\widetilde{c}(q) = -\eta^{-2}\oint ux^2 du = -\frac{1}{2}\eta^{-2}\oint v\lambda^{\eta}_{\text{SW}} = -q\frac{\partial}{\partial q}\widetilde{F}_{\delta,\delta_n}(q), \qquad n = 1,\cdots,4, \tag{4.2}$$
$$\hbar^2\widetilde{F}_{\delta,\delta_n}(q) = (\delta-\delta_1-\delta_2)\log q + \frac{(\delta+\delta_1-\delta_2)(\delta+\delta_3-\delta_4)}{2\delta}q + \mathcal{O}(q^2).$$

Notice that only the holomorphic classical block $\widetilde{F}$ survives $\partial/\partial q$. Even without quoting Polyakov's conjecture the equality for accessory parameters, $c = -\partial_q\mathcal{F}_0$, arises directly once the stress-tensor nature of the spectral curve $(\partial\phi_{\text{KS}})^2 = 4\tau$ in Hermitian matrix models is recalled.

Another ingredient we need is the ϵ_1-deformed version of Matone's relation [36–38] proposed in [25, 26, 39]:

$$\langle \text{tr}\,\Phi^2\rangle_{\epsilon_1} = 2\bar{q}\partial_{\bar{q}}W, \qquad \bar{q} = \exp(2\pi i\bar{\tau}_{UV}) \tag{4.3}$$

for, say, $\mathcal{N} = 2$ $\mathcal{T}_{0,4}(A_1)$ theory where

$$\frac{1}{\epsilon_1\epsilon_2}W(\epsilon_1) \equiv \lim_{\epsilon_2\to 0}\log Z_{\text{Nek}}\big(a,\vec{m},\bar{q},\epsilon_1,\epsilon_2\big),$$
$$a:\ \text{UV vev of }\Phi, \qquad \vec{m}:\ \text{linearly-combined bare flavor mass.}$$

[8] One should be aware of the validity of the free-field approximation for LFT correlators. It is valid only when the "momentum conservation" $bK = Q - \sum\alpha_n$ (for generic b) is respected.

Theory of r.h.s. in figure 1 ($N=4$)
$q=\dfrac{(\xi_1-\xi_3)(\xi_2-\xi_4)}{(\xi_2-\xi_3)(\xi_1-\xi_4)}$
$\epsilon_1 m\equiv\widetilde{\ell}$
$(\epsilon_1,\epsilon_2)=(\eta,0)$
$a=\oint_\alpha\lambda^\eta_{\rm SW}$
$\epsilon_1^2\dfrac{\partial\widetilde{F}}{\partial a}=\oint_\beta\lambda^\eta_{\rm SW}$

Table 3. Quantities of r.h.s. in figure 1 in terms of spin-chain variables.

Now, (4.2) and (4.3) together manifest $\lambda^\eta_{\rm SW}$ as the ϵ_1-deformed SW differential for $\mathcal{T}_{0,4}(A_1)$ if there holds in (4.2)

$$\frac{1}{b^2}\widetilde{F}_{\delta,\delta_n}(q)=\lim_{\epsilon_2\to0}\frac{1}{\epsilon_1\epsilon_2}W(\epsilon_1) \tag{4.4}$$

under $q=\bar{q}$, $\epsilon_1=\eta$ and $\epsilon_1\epsilon_2=\hbar^2$. In fact, (4.4) has alreay been verified in [40]. To conclude, we have found that Baxter's T-Q equation characterizes simultaneously two kinds of $\mathcal{N}=2$ theories, $\mathcal{T}_{0,N}(A_1)$ and $\mathcal{T}_{0,4}(A_{N-1})$, by examining $\lambda^\eta_{\rm SW}$. We call this remarkable property $\mathrm{SU}(N)/\mathrm{SU}(2)^{N-3}$ correspondence.

4.1 Discussion

- At the level of $\lambda^\eta_{\rm SW}$, based on (2.9) and (2.10) we have

$$\log w=2\eta\frac{\Psi'}{\Psi},\qquad w=\frac{A+\sqrt{A^2-4}}{2},\qquad A=\frac{P_{N_c}}{\sqrt{Q_{N_f}}}. \tag{4.5}$$

  Namely, all quantum $\mathrm{SU}(N_c)$ Coulomb moduli inside $P_{N_c}(u)\equiv\langle\det(u-\Phi)\rangle$ are determined by spin-chain variabes (η,ξ,ℓ). This fact is consistent with (2.2).

- Besides, from table 3 we find that the transformation between $\mathcal{F}_{\rm SW}$ in (2.2) and $\widetilde{F}$ is quite complicated. Although sharing the same SW differential (up to a total derivative term), two theories have diverse IR dynamics because both of their gauge group and matter content differ. To pursue a concrete interpolation between them is under investigation.

4.2 Other Gaudin models

There are still two other Gaudin models, say, hyperbolic and elliptic ones. Let us briefly discuss the elliptic type because it sheds light on $\mathcal{N}=2^*$ $\mathcal{T}_{1,1}(A_1)$. Now Bethe roots satisfy the following classical Bethe Ansatz equation:

$$\sum_{n=1}^{N}\frac{s_n\theta'_{11}(\mu_k-\xi_n)}{\theta_{11}(\mu_k-\xi_n)}=-\pi i\nu+\sum_{l(\neq k)}\frac{\theta'_{11}(\mu_k-\mu_l)}{\theta_{11}(\mu_k-\mu_l)},\qquad \nu\in\text{integer}. \tag{4.6}$$

Regarding it as a saddle-point condition, we are led to the spectral curve analogous to (3.11)

$$x^2 = \left[\sum_{n=1}^{N} \frac{s_n \theta'_{11}(u-\xi_n)}{\theta_{11}(u-\xi_n)} - \sum_{k=1}^{K} \frac{\theta'_{11}(u-\mu_k)}{\theta_{11}(u-\mu_k)}\right]^2$$
$$= \sum_{n=1}^{N} \wp(u-\xi_n)\eta^2 s_n(s_n+1) + \sum_{n=1}^{N} H_n \zeta(u-\xi_n) + H_0$$

where

$$H_n = \sum_{i\neq n}^{N} \sum_{a=1}^{3} w_a(\xi_n-\xi_i) J_n^a J_i^a,$$
$$H_0 = \sum_{n=1}^{N} \sum_{a=1}^{3} \Big\{ -\wp\left(\frac{\omega_{5-a}}{2}\right) J_n^a J_n^a + \sum_{i\neq n} w_a(\xi_i-\xi_n) \left[\zeta\left(\xi_n-\xi_i+\frac{\omega_{5-a}}{2}\right) - \zeta\left(\frac{\omega_{5-a}}{2}\right)\right] J_n^a J_i^a \Big\}. \tag{4.7}$$

$\wp(u)$ and $\zeta(u)$ respectively denote Weierstrass $\wp$- and ζ-function. Periods of $\wp(u)$ are (see appendix A for w_a)

$$\omega_1 = \omega_4 = 1, \qquad \omega_2 = \tau, \qquad \omega_3 = \tau + 1. \tag{4.8}$$

Notice that H_n's ($\sum H_n = 0$) are known as elliptic Gaudin Hamiltonians [41, 42]. Since all these are elliptic counterparts of those in the rational XXX model, according to the logic of figure 3 it will be interesting to see whether the one-form xdu is $\lambda_{\rm SW}$ of the ϵ_1-deformed $\mathcal{N} = 2^*$ SW theory when $n = 1$.[9]

Acknowledgments

TST thanks Kazuhiro Sakai and Hirotaka Irie for encouragement and helpful comments. RY is supported in part by Grant-in-Aid for Scientific Research No. 23540316 from Japan Ministry of Education. NY and RY are also supported in part by JSPS Bilateral Joint Projects (JSPS-RFBR collaboration).

A Definition of w_n

In appendix A, w_n that appears in (4.7) will be defined according to [41, 42]. We choose periods of $\wp(u)$ as in (4.8). Weierstrass σ-function is as follows:

$$\sigma(u) = \sigma(u; \omega_1, \omega_2)$$
$$= u \prod_{\substack{(n,m)\neq(0,0) \\ n,m\in\mathbf{Z}}} \left(1 - \frac{u}{n\omega_1 + m\omega_2}\right) \exp\left[\frac{u}{n\omega_1+m\omega_2} + \frac{1}{2}\left(\frac{u}{n\omega_1+m\omega_2}\right)^2\right]. \tag{A.1}$$

[9] See [43, 44] for related discussion about Ω-deformed $\mathcal{N} = 2^*$ SW theory.

Note that $\sigma(u)$ satisfies

$$\zeta(u) = \frac{\sigma'(u)}{\sigma(u)}, \qquad \wp(u) = -\zeta'(u). \tag{A.2}$$

We introduce constants $(e_a, \eta_a, \varsigma_a)$ which are related to $\omega_a/2$ such that

$$e_a = \wp(\omega_a/2), \qquad \eta_a = \zeta(\omega_a/2), \qquad \varsigma_a = \sigma(\omega_a/2), \qquad a = 1, 2, 3. \tag{A.3}$$

Using them we further have

$$\begin{aligned}
\sigma_{00}(u) &= \frac{\exp\left[-\left(\eta_1 + \eta_2\right) u\right]}{\varsigma_3} \sigma\left(u + \frac{\omega_3}{2}\right), \\
\sigma_{10}(u) &= \frac{\exp\left(-\eta_1 u\right)}{\varsigma_1} \sigma\left(u + \frac{\omega_1}{2}\right), \\
\sigma_{01}(u) &= \frac{\exp\left(-\eta_2 u\right)}{\varsigma_2} \sigma\left(u + \frac{\omega_2}{2}\right).
\end{aligned} \tag{A.4}$$

Jacobi's ϑ-functions are defined by

$$\begin{aligned}
\vartheta_{00}(u) &= \vartheta(u;\tau) = \vartheta(u) = \sum_{n=-\infty}^{\infty} \exp\left(\pi i n^2 \tau + 2\pi i n u\right), \\
\vartheta_{01}(u) &= \vartheta\left(u + \frac{1}{2}\right), \\
\vartheta_{10}(u) &= \exp\left(\frac{1}{4}\pi i \tau + \pi i u\right) \vartheta\left(u + \frac{1}{2}\tau\right), \\
\vartheta_{11}(u) &= \exp\left(\frac{1}{4}\pi i \tau + \pi i (u + \frac{1}{2})\right) \vartheta\left(u + \frac{1}{2} + \frac{1}{2}\tau\right),
\end{aligned} \tag{A.5}$$

from which Weierstrass σ-functions are listed as below:

$$\omega_1 \exp\left(\frac{\eta_1}{\omega_1} u^2\right) \frac{\vartheta_{11}\left(\frac{u}{\omega_1}\right)}{\vartheta_{11}'(0)} = \sigma(u), \qquad \exp\left(\frac{\eta_1}{\omega_1} u^2\right) \frac{\vartheta_{ab}\left(\frac{u}{\omega_1}\right)}{\vartheta_{ab}'(0)} = \sigma_{ab}(u) \qquad (ab = 0), \tag{A.6}$$

with $\vartheta_{ab}'(t) = d\vartheta_{ab}(t)/dt$. Finally, $w_a(u)$ is as follows:

$$\begin{aligned}
w_1(u) &= \frac{\mathrm{cn}(u\sqrt{e_1 - e_3}; \sqrt{\frac{e_2 - e_3}{e_1 - e_3}})}{\mathrm{sn}(u\sqrt{e_1 - e_3}; \sqrt{\frac{e_2 - e_3}{e_1 - e_3}})} = \frac{\sigma_{10}(u)}{\sigma(u)} = \frac{\vartheta_{11}'(0)}{\vartheta_{10}(0)} \frac{\vartheta_{10}(u)}{\vartheta_{11}(u)}, \\
w_2(u) &= \frac{\mathrm{dn}(u\sqrt{e_1 - e_3}; \sqrt{\frac{e_2 - e_3}{e_1 - e_3}})}{\mathrm{sn}(u\sqrt{e_1 - e_3}; \sqrt{\frac{e_2 - e_3}{e_1 - e_3}})} = \frac{\sigma_{00}(u)}{\sigma(u)} = \frac{\vartheta_{11}'(0)}{\vartheta_{00}(0)} \frac{\vartheta_{00}(u)}{\vartheta_{11}(u)}, \\
w_3(u) &= \frac{1}{\mathrm{sn}(u\sqrt{e_1 - e_3}; \sqrt{\frac{e_2 - e_3}{e_1 - e_3}})} = \frac{\sigma_{01}(u)}{\sigma(u)} = \frac{\vartheta_{11}'(0)}{\vartheta_{01}(0)} \frac{\vartheta_{01}(u)}{\vartheta_{11}(u)}.
\end{aligned} \tag{A.7}$$

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