

# $D = 3, N = 8$ conformal supergravity and the Dragon window

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ABSTRACT: We give a superspace description of  $D = 3, N = 8$  supergravity. The formulation is off-shell in the sense that the equations of motion are not implied by the superspace constraints (but an action principle is not given). The multiplet structure is unconventional, which we connect to the existence of a “Dragon window”, that is modules occurring in the supercurvature but not in the supertorsion. According to Dragon’s theorem this cannot happen above three dimensions. We clarify the relevance of this window for going on the conformal shell, and discuss some aspects of coupling to conformal matter.

KEYWORDS: Extended Supersymmetry, Supergravity Models, Conformal and W Symmetry

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**Contents**

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>The supergravity multiplet</b>	<b>3</b>
<b>3</b>	<b>Solving the Bianchi identities</b>	<b>5</b>
3.1	Conventional constraints	6
3.2	Implementation of the constraints	7
3.3	Solution of the Bianchi identities	8
<b>4</b>	<b>Equations of motion and matter couplings</b>	<b>15</b>
4.1	Conformal equations of motion	15
4.2	Matter coupling	16
<b>5</b>	<b>Conclusions and comments</b>	<b>17</b>
<b>A</b>	<b>The field content in a scalar superfield</b>	<b>18</b>

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**1 Introduction**

The recent discovery in M-theory of Lagrangian formulations for stacks of M2-branes at the IR fix-point has led to an enormous boost in the interest in three-dimensional superconformal Chern-Simons matter theories. This development was triggered by the construction of a theory with eight conformal supersymmetries, related to a stack with two M2-branes, by Bagger and Lambert [1, 2], and independently by Gustavsson [3] (BLG). This was soon generalised by Aharony et al. [4] to an arbitrary number of M2-branes in terms of a Lagrangian exhibiting, however, only six supersymmetries (ABJM).

One may ask how these theories are related to string theory in ten dimensions, that is to fundamental strings and D2-branes. Based on a new kind of Higgs mechanism, a connection of the BLG theory to D2-branes was found by Mukhi and Papageorgakis in ref. [5], that turns the BLG  $SU(2) \times SU(2)$  Chern-Simons theory into an ordinary  $SU(2)$  Yang-Mills theory appropriate for a 2-stack of D2-branes. This mechanism has subsequently been applied to ABJM and a number of other theories, see ref. [6] and the references therein.

However, in the case of the fundamental string the situation is different. While it is well understood [7, 8] how to relate the Bergshoeff et al. [9, 10] non-conformal theory for one M2-brane to the string, much less is known about how the Polyakov formulation of the string is related to the superconformal BLG theory. One might suspect that to make such a relation explicit also the M2-branes should have a version involving non-dynamical supergravity. Given the Lagrangian for extended conformal supergravity for any number

of supersymmetries [11–13], constructions of this type, referred to as topologically gauged matter theories in the following references, were obtained in the BLG case in ref. [14], where  $N = 8$  conformal supergravity was coupled to the BLG theory and the Lagrangian partly derived, and in the ABJM case in ref. [15], where the coupled Lagrangian was obtained in full detail. A curious result pointed out in the latter work is that if this topologically gauged ABJM theory is higgsed one finds that the theory ends up at a chiral point similar to the one of Li, Song and Strominger [16]. This connection to D2-branes at the chiral point was further discussed in ref. [6] where many of the details were worked out.

The aim of this paper is to find a superspace formulation of three-dimensional supergravity that can accommodate not only Poincaré supergravity but also conformal supergravity together with the currents needed for the matter couplings in the topologically gauged BLG theory [14]. The natural starting point is the standard set of superspace Bianchi identities (BI's) for the supertorsion and supercurvature. Prior to imposing any constraints on the fields, they describe "off-shell" supergravity in the sense that no dynamical equations are hidden in the BI's. The task is thus to find constraints that lead to dynamical equations of the required kind. If these equations can be obtained containing arbitrary currents then we say that the theory is still off-shell. This usage of the concept "off-shell" has been adopted in other circumstances as well. For a more elaborate explanation in the context of eleven-dimensional supergravity, see ref. [17, 18]. In general these methods are well suited for the derivation of Poincaré supergravity theories in eleven dimensions and below, and for any number of supersymmetries.

For conformal supergravities these superspace methods have been much less studied. Although related issues have been discussed both in the past [19–23] and more recently [24], to our knowledge no field equations for conformal supergravity have so far been derived with these methods. In three dimensions one needs to understand how to obtain the Cotton equation and its spin  $\frac{3}{2}$  analog, the Cottino equation, from the BI's. By essentially extending the analysis in ref. [19], we will here demonstrate how this can be done by solving the relevant BI equations to the required level (dimension). This calculation is more complicated than the corresponding one for Poincaré supergravity simply because the conformal field equations have one extra derivative which forces one to carry the calculation one level (dimension) higher. We should remark also that the constraints used to derive the field equations for Poincaré supergravity must be relaxed in the superconformal case since e.g. the Ricci tensor is not itself constrained by the field equations (as in the Poincaré case) but appear in the Cotton equation with an extra derivative acting on it. This will be explained in detail below.

Our final results rely on a phenomenon unique to three dimensions which we call the "Dragon window". It refers to the fact that there are irreducible components in the supercurvature tensor that do not appear in the supertorsion. In dimensions above three a classic theorem by Dragon [25] guarantees that this does not happen, which has the further consequence that the Bianchi identities for the supercurvature are automatically satisfied once the Bianchi identities for the supertorsion are solved. Thus in the case of three dimensions investigated here also independent curvature BI's must be solved. The Dragon window phenomenon was implicitly found in refs. [19, 23]<sup>1</sup> and more recently discussed in ref. [24].

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<sup>1</sup>We thank Paul Howe for pointing this out to us.

This paper is organised as follows. We continue in section 2 with a discussion of the structure of the supermultiplet that we expect to find when solving the BI's. This is partly carried out in the language of pure spinor cohomology which gives independent insight into the multiplet structure. In section 3 we first analyse the available constraints and then go on to solve the BI's both for the torsion and the curvature up to the level needed to conclude that the conformal supergravity equations, like the Cotton equation, are all present and fairly easily derivable. The dynamical equations are further discussed in section 4 along with some comments and observations about possible matter couplings. Some final conclusions are collected in section 5.

## 2 The supergravity multiplet

Let us denote 3-dimensional vector indices  $a, b, \dots$ , and collective spinor indices  $\alpha A, \beta B, \dots$ , where  $\alpha, \beta, \dots = 1, 2$  are  $so(1, 2) \approx sl(2)$  spinor indices and  $A, B, \dots = 1, \dots, 8$  chiral  $so(8)$  spinor indices ( $8_s$ ). We also write the spinor index of opposite chirality ( $8_c$ ) as  $A', B', \dots$  and  $so(8)$  vectors ( $8_v$ ) as  $I, J, \dots$ . Collective superspace flat tangent indices are written  $\mathcal{A}, \mathcal{B}, \dots$ , with  $\mathcal{A} = (a, \alpha A)$ . Dynkin labels have the  $sl(2)$  node in the first position. We use the triality convention  $8_v = (1000)$ ,  $8_s = (0010)$ ,  $8_c = (0001)$ . Conventions for 3-dimensional spinors are that indices are lowered and raised with  $\varepsilon_{\alpha\beta}$  and its inverse. They are raised and lowered by matrix multiplication, so that e.g.  $M^\alpha_\beta = M^{\alpha\gamma}\varepsilon_{\gamma\beta}$  and  $M_{\alpha\beta} = \varepsilon_{\alpha\gamma}M^{\gamma\beta}$ . Then the sign issues associated with symplectic spinor modules are kept to a minimum, although one still has to remember details like  $(\gamma_a)_\alpha^\beta = -(\gamma_a)^\beta_\alpha$ . We use conventions where  $(\gamma_{abc})_\alpha^\beta = +\varepsilon_{abc}\delta_\alpha^\beta$ .

A convenient way to investigate linearised supermultiplets is to use pure spinor cohomology [26–30] (see also refs. [31–33] for early work on the rôle of pure spinors for supersymmetric theories). Since the  $N = 8$  supergravity multiplet is half-maximally supersymmetric, we expect to find an off-shell supermultiplet. We take the pure spinor  $\lambda^{\alpha A}$  to transform as (1)(0010) of  $sl(2) \oplus so(8)$ . An arbitrary symmetric spinor bilinear is in (2)(0000)  $\oplus$  (0)(0100)  $\oplus$  (2)(0020). The torsion at dimension 0 is  $T_{\alpha A, \beta B}^c = 2\delta_{AB}\gamma_{\alpha\beta}^c$ . It is straightforward to check (see the following section) that conventional constraints [34–36] may be used to set everything else except a component in (4)(0020) to zero in the dimension 0 torsion. The pure spinor field, whose lowest component is the diffeomorphism ghost, is of course taken to be  $\Phi^a$  in (2)(0000), with an extra gauge invariance

$$\Phi^a \approx \Phi^a + (\lambda\gamma^a\rho) \tag{2.1}$$

(as is standard in the description of gravity multiplets, and more generally, for non-scalar pure spinor superfields). The nilpotency of the BRST operator  $Q = \lambda^{\alpha A}D_{\alpha A}$  implies that the  $D = 3$  vector  $\lambda^{\alpha A}\lambda^{\beta A}$  has to vanish. It is not *a priori* clear whether or not also the bilinear in (0)(0100) will vanish. We know from the superspace formulation of the BLG model [37, 38] that the scalar multiplet relies on pure spinors with non-vanishing bilinear in (0)(0100). Here, it is straightforward to show that the pure spinor constraint in (2)(0000) together with the gauge invariance (2.1) ensures that the power expansion in the pure spinor gives the irreducible module  $(n+2)(00n0)$  at order  $\lambda^n$ . Therefore, at  $\lambda^2$  the correct

torsion in (4)(0020) is reproduced. The supergravity is formulated using the same pure spinors as in the BLG model, and there is no need to constrain the bilinear in (0)(0100).

The zero-mode cohomology looks as follows:

	gh# = 1	0	-1	-2
dim = -1	(2)(0000)			
$-\frac{1}{2}$	(1)(0010)	•		
0	(0)(0100)	(4)(0000)	•	
$\frac{1}{2}$	•	(3)(0010)	•	•
1	•	(0)(2000) $\oplus$ (2)(0100) $\oplus$ (0)(0002)	•	•
$\frac{3}{2}$	•	(1)(1001) $\oplus$ (1)(1001)	•	•
2	•	(0)(0002) $\oplus$ (0)(2000) $\oplus$ (2)(0100)	•	•
$\frac{5}{2}$	•	(3)(0010)	•	•
3	•	(4)(0000)	(0)(0100)	•
$\frac{7}{2}$	•	•	(1)(0010)	•
4	•	•	(2)(0000)	•
$\frac{9}{2}$	•	•	•	•

The structure of the pure spinor cohomology is intriguing, in that it provides an off-shell formulation, but still contains a “current multiplet” in the same field (the component  $E_{\alpha A}^a$  of the super-vielbein) as the physical fields. In the first column we recognise the ghosts (or gauge parameters) for superdiffeomorphisms and local  $so(8)$  transformations. In the second column, the off-shell supergravity multiplet contains the linearised graviton, gravitino and  $so(8)$  gauge potential, as well as auxiliary fields at dimensions 1,  $\frac{3}{2}$  and 2. By implementing an extra constraint setting the field in (0)(0002) at dimension 1 to zero, the fields in grey are eliminated. The constraint implements a kind of selfduality in the multiplet structure — there is nothing in the previously introduced data that distinguishes the two modules (0)(2000) (selfdual in four spinor indices) and (0)(0002) (anti-selfdual).

The current (or antifield) multiplet can be obtained in a similar way. It comes in a field in the module (0)(2000) (there is also a gauge invariance analogous to that of eq. (2.1)). This antifield zero-mode cohomology is:

	gh# = -1	-2	-3	-4
dim = 1	(0)(2000)			
$\frac{3}{2}$	(1)(1001)	•		
2	(0)(00002) $\oplus$ (2)(0100)	•	•	
$\frac{5}{2}$	(3)(0010)	•	•	•
3	(4)(0000)	(0)(0100)	•	•
$\frac{7}{2}$	•	(1)(0010)	•	•
4	•	(2)(0000)	•	•
$\frac{9}{2}$	•	•	•	•
5	•	•	•	•

Note that this multiplet consists of a triality rotated version of the modules in grey in the field cohomology (present before the extra constraint is imposed), and that there is room for the Cotton and Cottino equations at dimensions 3 and  $\frac{5}{2}$ .

The field multiplet has the peculiar feature that supersymmetry connects the component fields in a tree-like structure:



This property is read off from the  $sl(2)$  modules appearing. The field at the “branch” of the tree is the auxiliary field  $C_{IJ}^+$  in (0)(2000). The antifield/current multiplet of course has the same structure, but turned upside down. In the following sections we will demonstrate that its appearance in the superspace geometry is a result of what we choose to call a “Dragon window” — a module appearing in the supercurvature but not in the supertorsion. The proof of Dragon’s theorem [25] assumes the space-time dimension to be greater than 3. This field plays a central rôle for the equations of motion in conformal supergravity.

The conformal multiplet for all  $N$  were discussed in ref. [19] and more recently in refs. [24, 39], based on an investigation of the superspace Bianchi identities, while the  $N = 8$  multiplet with the selfduality constraint on the dimension 1 scalars was given in refs. [19, 23]. There, the authors presented the proper constraints and indicated how the BI’s result in the above structure of the  $N = 8$  supermultiplet. In the following section we continue this analysis to the extent that we can conclude that the conformal field equations are actually derivable in this approach.

### 3 Solving the Bianchi identities

In this section, we will show how the solution of the the superspace Bianchi identities leads to the off-shell supergravity multiplet. Torsion is defined as  $T^A = DE^A = dE^A + E^B \wedge \Omega_B^A$ , where  $E^A$  is the superspace vielbein (frame 1-form) and  $\Omega_A^B$  the superspace connection 1-form, taking values in the structure algebra. The torsion Bianchi identity therefore reads  $DT^A = E^B \wedge R_B^A$ . Since, as will be explained below, Dragon’s theorem does not fully apply in  $D = 3$ , we also need the curvature Bianchi identity  $DR_A^B = 0$ .

The structure algebra would naïvely be taken as the sum of the Lorentz algebra and the R-symmetry algebra, but we will also include Weyl scalings, so that the total structure algebra is  $so(1, 2) \oplus so(8) \oplus \mathbb{R}$ . Although not necessary for conformal symmetry, the inclusion of Weyl scalings will facilitate the treatment of scaling properties and the elimination of unphysical degrees of freedom through conventional constraints. An element  $T$  in the fundamental representation of the structure algebra (i.e., acting on a superspace tangent vector) has the non-vanishing matrix elements

$$\begin{aligned}
 T_a^b &= L_a^b + N\delta_a^b, \\
 T_{\alpha A}^{\beta B} &= \frac{1}{4}L_a^b(\gamma^{a,b})_{\alpha}^{\beta}\delta_A^B + M_A^B\delta_{\alpha}^{\beta} + \frac{1}{2}N\delta_{\alpha}^{\beta}\delta_A^B,
 \end{aligned}
 \tag{3.1}$$

where  $L \in so(1, 2)$ ,  $M \in so(8)$  and  $N \in \mathbb{R}$ .

### 3.1 Conventional constraints

As usual in superspace geometry, the number of fields are reduced using conventional constraints [34–36]. The constraints we will impose are of different types. The property they have in common is that they are effectuated by fixing some components of the torsion. This ensures the gauge covariance of the constraints, and therefore of the resulting physical system. In principle, some of the constraints have the effect of eliminating certain superfluous components of the vielbein, i.e. components that after solving the Bianchi identities occur in combinations such that they can be removed by field redefinitions (as can be seen by not enforcing these constraints). However, imposing them explicitly in terms of vielbeins would not be an optimal procedure, since such constraints could potentially break diffeomorphism invariance. The vielbeins carry one coordinate index and one inertial index, and the coordinate index can not be converted into an inertial index (the result would be the unit matrix). The torsion components, on the other hand, carry an inertial index and in addition two lower indices that can be taken in the inertial as well as in the coordinate basis. All constraints are formulated in terms of the torsion, and in terms of components with inertial indices only. Since such components are scalars under diffeomorphisms, this is the only covariant procedure to impose constraints. As long as they are formulated in a way that respects the local structure symmetry, all symmetries will be preserved.

Let us start by considering the conventional constraints. There are two kinds of conventional constraints that can be associated with transformations of the spin connection and the vielbein respectively, while the other is held constant. These two transformations have the property that they leave the torsion Bianchi identities invariant and therefore take a solution of the Bianchi identities into a new solution. This is the reason why we can use these kinds of transformations in order to find an as simple solution to the Bianchi identities as possible. The two kinds of transformations clearly commute with each other.<sup>2</sup>

The first kind shifts the spin connection by an arbitrary 1-form (with values in the structure algebra) and leaves the vielbein invariant:

$$\left. \begin{aligned} E^A &\rightarrow E^A \\ \Omega_{\mathcal{A}}^B &\rightarrow \Omega_{\mathcal{A}}^B + \Delta_{\mathcal{A}}^B \end{aligned} \right\} \implies T^A \rightarrow T^A + E^B \wedge \Delta_B^A. \quad (3.2)$$

This kind of redefinition serves to remove the independent degrees of freedom in  $\Omega$ , which can be achieved by constraints on  $T$  as long as there are no irreducible modules of the structure group residing in  $\Omega$  that do not occur in  $T$  (all structure groups under consideration fulfill this requirement, as will be seen later). This shift is often expressed as the torsion being absorbed in the spin connection. The canonical example is ordinary bosonic geometry, where one gets  $T_{ab}{}^c \rightarrow T_{ab}{}^c + 2\Delta_{[ab]}{}^c$ , where  $\Delta$  is antisymmetric in the last two indices, meaning that the transformation can be used to set the torsion identically to zero, leaving the vielbeins as the only independent variables. In supergravity the analysis is more subtle. Only certain modules in the torsion can be brought to zero.

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<sup>2</sup>Note, however, that the commutativity of the constraints does not imply that they do not interfere with each other.

The second kind of transformation consists of a change of tangent bundle, while the connection is left invariant:

$$\left. \begin{array}{l} E^A \rightarrow E^B M_B^A \\ \Omega_A^B \rightarrow \Omega_A^B \end{array} \right\} \implies T^A \rightarrow T^B M_B^A + E^B \wedge D M_B^A. \quad (3.3)$$

Again, it is essential that one implements the constraints on the torsion. This will mean that not all components in  $M$  can be used. In fact, the remaining degrees of freedom will all reside in the component  $E_\mu^a$  of negative dimension. The form of the transformation of  $T$  will in practice mean that the transformations have to be implemented sequentially in increasing dimension, in order for the second term not to interfere with constraints obtained by using the first term. This second kind of transformation has no relevance in purely bosonic geometry — there  $M$  has dimension 0, and can not be used to algebraically eliminate torsion components of dimension 1 (which are taken care of by the first kind of transformation, anyway). It should also be noted that not all matrices  $M$  are relevant. If  $M$  is an element in the structure group, the transformations in eq. (3.3) can be supplemented by a transformation of the first kind from eq. (3.2) with suitable parameter ( $\Delta = M^{-1}dM + M^{-1}\Omega M - \Omega$ ) so that the total transformation is a gauge transformation.

### 3.2 Implementation of the constraints

An examination of the transformations of the previous subsection at dimension 0 shows that conventional constraints corresponding to the parameters  $M_a^b$  and  $M_{\alpha A}^{\beta B}$  can be used to eliminate all degrees of freedom in the torsion at dimension 0, except an element in (4)(0020), so that

$$T_{\alpha A, \beta B}^c = 2\delta_{AB}\gamma_{\alpha\beta}^c + X_{AB}^{cd}(\gamma_d)_{\alpha\beta}, \quad (3.4)$$

where  $X$  is symmetric and traceless in both pairs of indices. This is what motivates the pure spinor constraints in section 2. Dimension zero torsion with field dependent terms like  $X$  has been used in other contexts to take the theory off-shell, in particular for ten-dimensional supergravity as discussed in refs. [40, 41].<sup>3</sup>

In addition to the conventional constraints, one needs to make one additional choice in order to obtain the physical multiplet. We call this constraint,  $X = 0$ , a physical, or non-conventional, constraint. Setting  $X = 0$  is the constraint implied by the  $Q$ -closedness of section 2. This constraint was contained in the definition of a superconformal structure given in ref. [19]. Note that the requirement of super-Weyl invariance provides an independent argument for setting  $X = 0$  to zero (see e.g. ref. [21]). Super-Weyl transformation are treated in detail in refs. [23, 24].

At dimension  $\frac{1}{2}$ , one has the constraints corresponding to the parameters  $M_a^{\beta B}$ , as well as  $(\Delta_{\alpha A})_a^b$ ,  $(\Delta_{\alpha A})_A^B$  and  $\Delta_{\alpha A}$ . It turns out that there is a slight excess of possible transformations, when Weyl scalings are included in the structure algebra. All four parameters contain one spinor (1)(0010) each, but it turns out that a certain combination

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<sup>3</sup>The situation in ten-dimensional supergravity is special in the sense that the pure spinor cohomology (i.e., the linearised fields obtained without this extra tensor) is not entirely off-shell although the theory is only half-maximally supersymmetric [22]. No such phenomenon occurs in the present three-dimensional model.



of these transformations leaves the dimension- $\frac{1}{2}$  torsion invariant. There are a priori five independent spinors in the torsion at dimension  $\frac{1}{2}$ , which transforms as

$$\begin{aligned} \delta T_{\alpha A, b}{}^c &= (\gamma_b{}^c(\chi_1 - 2\chi_4))_{\alpha A} + (\chi_3 + 2\chi_4)_{\alpha A} \delta_b{}^c, \\ \delta T_{\alpha A, \beta B}{}^{\gamma C} &= \varepsilon_{\alpha\beta} \left( \frac{3}{2}\chi_1 + \frac{1}{2}\chi_2 + \frac{1}{2}\chi_3 \right)_{[A}^{\gamma} \delta_{B]}^C + \gamma_{\alpha\beta}^i \delta_{AB} \left( \gamma_i \left( \frac{1}{2}\chi_2 + 2\chi_4 \right) \right)^{C\gamma} \\ &\quad + \gamma_{\alpha\beta}^i \left( \gamma_i \left( -\frac{1}{2}\chi_1 - \frac{1}{2}\chi_2 + \frac{1}{2}\chi_3 \right) \right)^{\gamma} ({}^A\delta_B^C), \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} (\Delta_{\alpha A})_b{}^c &= (\gamma_b{}^c \chi_1)_{\alpha A}, \\ (\Delta_{\alpha A})_B{}^C &= \frac{1}{2}(\delta_{AB} \chi_{2\alpha}{}^C - \delta_A^C \chi_{2\alpha B}), \\ \Delta_{\alpha A} &= \chi_{3\alpha A}, \\ M_a{}^{\beta B} &= (\gamma_a \chi_4)^{\beta B}. \end{aligned} \quad (3.6)$$

A shift  $\chi_3$  of the Weyl connection can be compensated by conventional transformations with  $\chi_1 = -\chi_3$ ,  $\chi_2 = 2\chi_3$  and  $\chi_4 = -\frac{1}{2}\chi_3$ . Such a compensating transformation will affect torsion at higher dimension, due to derivatives in eq. (3.3). We will be especially interested in flat Weyl connections, with gauge transformations  $\delta\omega_{\alpha A} = D_{\alpha A}\phi$ . Such Weyl rescalings clearly lead to shifts in the dimension 1 torsion with second spinorial derivatives of  $\phi$ , i.e., in the modules  $\wedge^2(1)(0010) = (0)(0000) \oplus (0)(0020) \oplus (2)(0100)$ . In the following subsection, this argument is used to gauge away such unwanted (component) degrees of freedom. In appendix A, the complete component expansion of an unconstrained scalar superfield is given. We should make clear that, unlike in  $D = 11$  supergravity [21], inclusion of scaling in the structure group is not necessary for the elimination of dimension  $\frac{1}{2}$  torsion, but used here only to systematise the form of the scale transformations.

At dimension 1, the shift in the Lorentz spin connection is used to set  $T_{ab}{}^c = 0$ , as usual.

### 3.3 Solution of the Bianchi identities

We will summarise the results of solving the Bianchi identities up to dimension 2. In reference [24], they were solved up to dimension  $\frac{3}{2}$  for arbitrary number of supersymmetries.

After the implementation of the conventional and physical constraints at dimension 0 and  $\frac{1}{2}$ , we have

$$\begin{aligned} T_{\alpha A, \beta B}{}^c &= 2\gamma_{\alpha\beta}^c \delta_{AB}, \\ T_{\alpha A, b}{}^c &= 0, \\ T_{\alpha A, \beta B}{}^{\gamma C} &= 0. \end{aligned} \quad (3.7)$$

At dimension 1, it is straightforward to show that the Bianchi identities allow the three modules  $(2)(0100) \oplus (0)(0020) \oplus (0)(0000)$  in the torsion. They are introduced as

$$T_{a, \beta B}{}^{\gamma C} = \varepsilon_a{}^{ij} (\gamma_i)_{\beta}{}^{\gamma} Y_{jB}{}^C + (\gamma_a)_{\beta}{}^{\gamma} (\tilde{Z}_B{}^C + \delta_B{}^C Z) \quad (3.8)$$

(in agreement with ref. [19]). The curvature at dimension 1 also contains these superfields. In addition, fields in  $(0)(2000) \oplus (0)(0002)$  are allowed in the dimension 1 curvature, without appearing in the torsion. This was observed already in ref. [19] based on the above constraint. This phenomenon is unique to  $D = 3$ , and forbidden by Dragon’s theorem in higher dimensions. We call it the “Dragon window”. The appearance of a Dragon window is connected to the peculiar “branched” structure of the supermultiplet demonstrated in section 2. As described in that section, we can take the second of these modules,  $(0)(0002)$ , to vanish (while staying off-shell), and denote the first one  $C_{ABCD}^+$ , which is completely antisymmetric and selfdual. The complete curvature at dimension 1 is

$$\begin{aligned} (R_{\alpha A, \beta B})_{cd} &= 4\varepsilon_{\alpha\beta}\varepsilon_{cd}{}^i Y_{iAB} + 4\varepsilon_{cd}{}^i (\gamma_i)_{\alpha\beta} (\tilde{Z}_{AB} + \delta_{AB}Z), \\ (F_{\alpha A, \beta B})_{CD} &= -2(\gamma_i)_{\alpha\beta} \delta_{AB} Y^i{}_{CD} + 8(\gamma_i)_{\alpha\beta} \delta_{[C(A} Y^i{}_{B)D]} \\ &\quad - 8\varepsilon_{\alpha\beta} \delta_{[A[C} (\tilde{Z}_{D]B]} + \delta_{D]B]} Z) + \varepsilon_{\alpha\beta} C_{ABCD}^+. \end{aligned} \tag{3.9}$$

The Weyl component of the curvature vanishes. The argument concerning Weyl rescalings in the previous subsection shows that the fields  $Y$ ,  $\tilde{Z}$  and  $Z$ , although needed in the off-shell supergeometry, have lowest components that are pure gauge [19] (see ref. [42] for a similar phenomenon in two dimensions and the appendix for the  $\theta$  expansion of a  $N = 8$  scalar superfield in three dimensions). It can of course be checked explicitly that they transform inhomogeneously under Weyl rescalings, with the compensating transformations included to keep the dimension  $\frac{1}{2}$  torsion vanishing.

As can be seen from section 2, the field  $C^+$  seems to be where conformal couplings are to be inserted.<sup>4</sup> This will also be verified later by explicitly checking that setting it to zero yields the equations of motion of pure conformal supergravity.

Solving the Bianchi identities at dimension  $\frac{3}{2}$  shows that the modules  $(3)(0110)$  and  $(3)(1001)$  in  $DY$ ,  $(1)(0030)$  in  $D\tilde{Z}$  and  $(1)(2010)$  in  $DC^+$  are set to 0 (the latter due to the curvature Bianchi identity). In addition, the  $(1)(1001)$ ’s in  $DY$  and  $DC^+$  are related as well as the  $(1)(0110)$ ’s in  $DY$  and  $D\tilde{Z}$ . There is also one linear relation between the  $(1)(0010)$ ’s in  $DY$ ,  $D\tilde{Z}$  and  $DZ$ . The torsion component  $(3)(0010)$  is identified with the one in  $DY$  and the one in  $(1)(0010)$  with a certain linear combination of the two surviving  $(1)(0010)$ ’s. We have not yet bothered to write down explicit expressions for the curvatures at dimension  $\frac{3}{2}$ , but they are completely and straightforwardly solved for using the torsion Bianchi identities. With the concrete Ansatz where every term represents an irreducible component (the vanishing modules above are left out)

$$\begin{aligned} D_{\alpha A} Y_{bCD} &= \delta_{A[C} y_{D]b\alpha} + \delta_{A[C} (\gamma_{b|} y_{D]})_{\alpha} + (\gamma_b y_{CD, A})_{\alpha} + (\gamma_b y_{ACD})_{\alpha}, \\ D_{\alpha A} \tilde{Z}_{BC} &= \delta_{A(B} \tilde{z}_{C)\alpha} + \tilde{z}_{A(B, C)\alpha}, \\ D_{\alpha A} Z &= z_{\alpha A}, \\ D_{\alpha A} C_{BCDE}^+ &= 4\delta_{A[B} c_{CDE]\alpha} + \frac{1}{6} \varepsilon_{ABCDE}{}^{FGH} c_{FGH\alpha}, \\ T_{ab}{}^{\gamma C} &= \varepsilon_{ab}{}^i (\tilde{t}_i + \gamma_i t)^{\gamma C} \end{aligned} \tag{3.10}$$

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<sup>4</sup>Such couplings are derived in ref. [14] but one should note that there are still unresolved questions concerning the exact structure of such conformal couplings in the  $N = 8$  case. See section 4.2 for further comments on this issue.

(primed symmetrisation includes subtraction of the trace), the relations become

$$\begin{aligned}
 (1)(1001) : 0 &= y_{ABC\alpha} + \frac{1}{6}c_{ABC\alpha}, \\
 (1)(0110) : 0 &= 2y_{AB,C\alpha} + \tilde{z}_{AB,C\alpha}, \\
 (1)(0010) : 0 &= \frac{3}{2}y_{A\alpha} - \frac{1}{4}\tilde{z}_{A\alpha} - 2z_{A\alpha},
 \end{aligned} \tag{3.11}$$

and the torsion is

$$\begin{aligned}
 \tilde{t}_{a\alpha A} &= \frac{1}{2}y_{a\alpha A}, \\
 t_{\alpha A} &= \frac{1}{2}y_{\alpha A} + \frac{1}{2}\tilde{z}_{\alpha A}.
 \end{aligned} \tag{3.12}$$

Also some of the remaining component fields at dimension  $\frac{3}{2}$  can be removed by a Weyl rescaling, since they come at the  $\theta^3$  level of a scalar superfield. This applies to one of the spinors (1)(0010) and the field in (1)(0110). This leaves at dimension  $\frac{3}{2}$  precisely the torsion modules together with the field in (1)(1001) in the current multiplet.

Now to dimension 2. The Bianchi identity with fermionic indices is

$$(R_{ab})_{\gamma C}{}^{\delta D} = 2D_{[a}T_{b]\gamma C}{}^{\delta D} + D_{\gamma C}T_{ab}{}^{\delta D} + 2T_{[a|\gamma C}{}^{\varepsilon E}T_{b]\varepsilon E}{}^{\delta D}. \tag{3.13}$$

It is *a priori* not clear to what extent the torsion Bianchi identities should give new information here, in addition to the one obtained by acting with fermionic derivatives on the relations on dimension  $\frac{3}{2}$ .

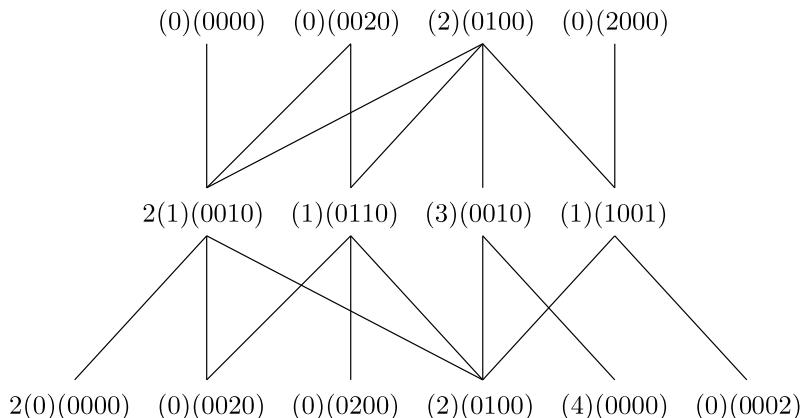
We will go through the modules appearing at dim. 2 systematically, and show that all essential information off-shell is provided by the equations at dimension  $\frac{3}{2}$ . The analysis will also provide information concerning the absence of equations of motion at dimension 2, the appearance of the physical curvature and some implications of going on a conformal shell by specifying  $C^+$ . The propagation of constraints appearing up to dimension  $3/2$  to dimension 2 is depicted in figure 1.

**(0)(0020):** the module (0)(0020) at dim. 2 comes from (1)(0010), (1)(0110) and (1)(0030) at dim.  $\frac{3}{2}$ . One must therefore use the identities  $D\tilde{Z}|_{(1)(0030)} = 0$ , i.e.,  $D_{\alpha(A}\tilde{Z}_{BC)} - \frac{1}{5}\delta_{(AB}D|_{\alpha}{}^D\tilde{Z}_{C)D} = 0$ , together with  $2y_{AB,C\alpha} + \tilde{z}_{AB,C\alpha} = 0$ , and of course the Bianchi identity. We act on the former equation with one more fermionic derivative and contract the spinor index and one of the SO(8) indices to get (0)(0020). In the process, we have to divide the two fermionic derivatives in symmetric and antisymmetric parts, of which the symmetric part gets contribution from the curvature at dim. 1 (and in general from torsion, but not in this case). This gives a relation between the two (0)(0020)'s in  $D^{\wedge 2}\tilde{Z}$ :

$$0 = \left\{ (D_{(A}D_{C)})\tilde{Z}_B{}^C + \frac{5}{8}(D^C D_C)\tilde{Z}_{AB} - 24(\tilde{Z} + Z\mathbb{1})_{AB}^2 \right\}_{(AB)'}, \tag{3.14}$$

where the overall traceless symmetrisation in  $(AB)'$  is taken after the other ones. The derivative of the relation  $2y_{AB,C\alpha} + \tilde{z}_{AB,C\alpha} = 0$  is treated in the same way. Taking one derivative of the identity (the inversion of how  $y_{AB,C\alpha}$  appears in the Ansatz (3.10))

$$y_{AB,C\alpha} = \frac{1}{3}(\gamma_i D)_{\alpha C} Y^i{}_{AB} - \frac{1}{3}(\gamma_i D)_{\alpha[C} Y^i{}_{AB]} + \frac{2}{21}\delta_{C[A}(\gamma^i D)_{|\alpha}^E Y_{i|B]E} \tag{3.15}$$



**Figure 1.** The above figure is obtained by subtracting all “descendants” from the dimension  $\frac{3}{2}$  constraint. Continued subtraction at higher dimensions indicates that no fields arise at dimension  $\frac{5}{2}$  or higher.

gives

$$(D^A y_{A(B,C)}) = \left\{ \frac{8}{21} (D_{[B} \gamma^i D_{E]}) Y_{iC}{}^E - \frac{32}{7} (Y^i Y_i)_{BC} \right\}_{(BC)'}, \quad (3.16)$$

and similarly on  $\tilde{Z}$ , where also eq. (3.14) has been used,

$$(D^A \tilde{z}_{A(B,C)}) = \frac{8}{7} (D^E D_E) \tilde{Z}_{BC} - \frac{64}{7} (\tilde{Z} + Z \mathbb{1})^2_{(BC)'}. \quad (3.17)$$

Combining this information, we have, already before using the Bianchi identity at dim. 2,

$$0 = \left\{ 2(D_{[A} \gamma^i D_{E]}) Y_{iB}{}^E - 24(Y^i Y_i)_{AB} + 3(D^E D_E) \tilde{Z}_{AB} - 24(\tilde{Z} + Z \mathbb{1})^2_{AB} \right\}_{(AB)'}. \quad (3.18)$$

To obtain similar information from the Bianchi identity, we need to perform a similar calculation obtaining (0)(0020) as a fermionic derivative on the fields in (1)(0010), since they sit in the torsion. Using the “inversions”  $y_{\alpha A} = -\frac{2}{21} (\gamma^i D)_{\alpha}^E Y_{iAE}$  and  $\tilde{z}_{\alpha A} = \frac{8}{35} D_{\alpha}^E \tilde{Z}_{AE}$ , one obtains

$$\begin{aligned} (D_{(A} y_{B)'}) &= \left\{ -\frac{2}{21} (D_{[A} \gamma^i D_{E]}) Y_{iB}{}^E - \frac{32}{21} (Y^i Y_i)_{AB} \right\}_{(AB)'}, \\ (D_{(A} \tilde{z}_{B)'}) &= -\frac{1}{7} (D^E D_E) \tilde{Z}_{AB} + \frac{64}{7} (\tilde{Z} + Z \mathbb{1})^2_{(AB)'}. \end{aligned} \quad (3.19)$$

These equations are then used in the (0)(0020) component of the Bianchi identity, using the form of the lower-dimensional torsion from above. The result turns out to be exactly eq. (3.18) again (after an intricate combination of a number of numerical coefficients), so no new information is obtained. We consider this quite striking result a check that the calculation is correct. This means that there is one unrestricted (0)(0020) field left at dim. 2.

The (0)(0020) in  $Z$  can of course also be solved for by using the relation  $\frac{3}{2}y_{A\alpha} - \frac{1}{4}\tilde{z}_{A\alpha} - 2z_{A\alpha} = 0$ . It becomes

$$\begin{aligned} & (D_{(A}D_{B)})Z \\ &= -\frac{1}{56} \left\{ 4(D_{[A}\gamma^i D_{E]})Y_{iB}{}^E + 64(Y^i Y_i)_{AB} - (D^E D_E)\tilde{Z}_{AB} + 64(\tilde{Z} + Z\mathbb{1})_{AB}^2 \right\}_{(AB)'} \quad (3.20) \\ &= -\left\{ \frac{1}{12}(D_{[A}\gamma^i D_{E]})Y_{iB}{}^E + (Y^i Y_i)_{AB} + (\tilde{Z} + Z\mathbb{1})_{AB}^2 \right\}_{(AB)'} \end{aligned}$$

**(2)(0100):** the module (2)(0100) is interesting, since this is where the SO(8) field strength occurs. *A priori* there are 3 (2)(0100)'s in  $D^{\wedge 2}Y$  and one each in  $D^{\wedge 2}\tilde{Z}$ ,  $D^{\wedge 2}Z$  and  $D^{\wedge 2}C^+$ . We will use the shorthand

$$\begin{aligned} Y_1^{(2)} &= (D^E D_E)Y_{aAB}, \\ Y_2^{(2)} &= \{(D_{(A}D_{E)})Y_{aB}{}^E\}_{[AB]}, \\ Y_3^{(2)} &= \{\varepsilon_{aij}(D_{[A}\gamma^i D_{E]})Y^j{}_{B}{}^E\}_{[AB]}, \\ \tilde{Z}^{(2)} &= \{(D_{[A}\gamma_a D_{E]})\tilde{Z}_B{}^E\}_{[AB]}, \\ Z^{(2)} &= (D_{[A}\gamma_{|a} D_{B]})Z, \\ C^{+(2)} &= (D^C \gamma_a D^D)C^+_{ABCD}, \\ \varepsilon F &= \varepsilon_a{}^{ij}(F_{ij})_{AB}. \end{aligned} \quad (3.21)$$

We will for the moment not deal with  $Z^{(2)}$  and  $C^{+(2)}$ . The dimension  $\frac{3}{2}$  torsion does not contain  $DC^+$  and can be written without  $DZ$ . We also use shorthand for the other structures appearing:

$$\begin{aligned} DY &= \varepsilon_a{}^{ij}D_i Y_{jAB}, \\ YY &= \varepsilon_a{}^{ij}(Y_i Y_j)_{AB}, \\ \tilde{Z}Y &= (\tilde{Z}Y_a)_{[AB]}, \\ ZY &= ZY_{aAB}, \\ C^+Y &= C^+_{AB}{}^{CD}Y_{aCD}. \end{aligned} \quad (3.22)$$

The three structures in  $Y$  have two relation among themselves (and lower-dimensional fields) thanks to the vanishing of the modules (3)(0110) and (3)(1001) in  $DY$ . The equations at dim.  $\frac{3}{2}$  read (the (3)(0110) equation as stated also contains (3)(1001))

$$\begin{aligned} (3)(1001) : 0 &= D_{\alpha[A}Y_{|a|BC]} - \frac{1}{3}(\gamma_a \gamma^i D)_{\alpha[A}Y_{i|BC]}, \\ (3)(0110) : 0 &= D_{\alpha C}Y_{aAB} + \frac{2}{7}\delta_{C[A}D_{|\alpha}^E Y_{a|B]E} \\ &\quad - \frac{1}{3}(\gamma_a \gamma^i D)_{\alpha C}Y_{iAB} - \frac{2}{21}\delta_{C[A}(\gamma_{|a} \gamma^i D)_{\alpha}^E Y_{i|B]E}. \end{aligned} \quad (3.23)$$

Acting on these equation with one further fermionic derivative, forming (2)(0100), separating into antisymmetric and symmetric products of derivatives (the former leading to the

$Y^{(2)}$ 's, the latter to torsion and curvature) leads to the equations

$$\begin{aligned} 0 &= Y_1^{(2)} + 2Y_2^{(2)} - Y_3^{(2)} - 6DY + 8YY + 8\tilde{Z}Y + 72ZY - 2C^+Y, \\ 0 &= 14Y_1^{(2)} + 4Y_2^{(2)} + 2Y_3^{(2)} - 108DY + 144YY - 48\tilde{Z}Y + 336ZY + 4C^+Y. \end{aligned} \quad (3.24)$$

As for the module (0)(0020), a relation at dim. 2 is obtained by applying a fermionic derivative on the (1)(0110) relation  $2y_{AB,C\alpha} + \tilde{z}_{AB,C\alpha} = 0$ , and forming (2)(0100). This results in

$$0 = 7Y_1^{(2)} - 4Y_2^{(2)} - 10Y_3^{(2)} - 24\tilde{Z}^{(2)} + 120DY - 40YY - 400\tilde{Z}Y - 720ZY + 10C^+Y. \quad (3.25)$$

At this point, there is one independent field left. The (2)(0100)'s in  $Z$  and  $C^+$  are related to the ones in  $Y$  and  $\tilde{Z}$  via (1)(0010) and (1)(1001) respectively, but they are not needed in the torsion Bianchi identity.

The torsion Bianchi identity contains two (2)(0100)'s. One of these is obtained as  $\varepsilon_a^{ij}\varepsilon^{\alpha\beta}(R_{ij})_{\alpha[A,|\beta|B]} = \dots$ , and contains  $F$ ; the other one as  $(\gamma^i)^{\alpha\beta}(R_{ai})_{\alpha[A,|\beta|B]} = \dots$ , and does not contain curvature. The latter one can potentially give one more relation among the fields. Some calculation gives the equation

$$0 = 5Y_3^{(2)} + 8\tilde{Z}^{(2)} - 60DY + 40YY + 112\tilde{Z}Y + 240ZY. \quad (3.26)$$

This equation is a linear combination of the three earlier ones. The other equation in the Bianchi identity expresses  $F$  in terms of the remaining field. The result is

$$\varepsilon F = \frac{1}{7}Y_2^{(2)} - \frac{4}{35}\tilde{Z}^{(2)} - \frac{10}{7}YY + \frac{24}{35}\tilde{Z}Y - \frac{24}{7}ZY + \frac{1}{7}C^+Y. \quad (3.27)$$

Setting  $C^+ = 0$  implies going on shell in pure conformal supergravity. This constraint leads to additional relations at dim. 2, since it then follows that  $DY|_{(1)(1001)} = 0$ . Taking another fermionic derivative on this equation and forming (2)(0100) adds one more relation between the fields in this module at dim. 2, which means that there is no free components left. The extra equation is

$$0 = Y_1^{(2)} + 2Y_2^{(2)} + 2Y_3^{(2)} + 12DY - 40YY + 32\tilde{Z}Y. \quad (3.28)$$

Solving for the (2)(0100) components of the superfields and inserting in eq. (3.27) gives the very nice equation

$$(F_{ab})_{CD} = -2D_{[a}Y_{b]CD} + 2(Y_{[a}Y_{b]})_{CD} \quad (3.29)$$

(the terms with  $\tilde{Z}Y$  and  $ZY$  cancel out), or equivalently as forms,  $F = -DY - Y \wedge Y$ . It should be noted that this expression, for this specific combination of numerical coefficients, automatically satisfies the Bianchi identity for  $F$ . It can be written in the more suggestive way

$$F(A + Y) = 0, \quad (3.30)$$

where  $A$  is the  $SO(8)$  gauge potential, stating that  $A + Y$  is a flat connection. We remind that the lowest component of  $Y$  is gauged away by a Weyl transformation. The only other

effect of setting  $C^+ = 0$  (or equal to some current superfield) is that the auxiliary field in (0)(0002) at dim. 2, which is unconstrained by the Bianchi identities, will vanish.

More generally,  $Z_{aAB}^{(2)}$  and  $C_{aAB}^{+(2)}$  may be expressed as

$$\begin{aligned} Z^{(2)} &= \frac{1}{70}(-5Y_2^{(2)} + 5Y_3^{(2)} - 2\tilde{Z}^{(2)} + 10DY + 20YY + 32\tilde{Z}Y + 80ZY - 5C^+Y), \\ C^{+(2)} &= \frac{10}{3}(Y_1^{(2)} + 2Y_2^{(2)} + 2Y_3^{(2)} + 12DY - 40YY + 32\tilde{Z}Y - 2C^+Y). \end{aligned} \quad (3.31)$$

This allows for reexpressing the field strength as

$$\varepsilon_a{}^{bc}F(A+Y)_{bcAB} = \frac{1}{60}C_{aAB}^{+(2)} + \frac{1}{3}C_{AB}^+{}^{CD}Y_{aCD}. \quad (3.32)$$

**(0)(0100):** there are from the beginning two (0)(0100)'s, one each in  $D^2Y$  and  $D^2\tilde{Z}$ . The only other term that can enter is  $D^iY_{iAB}$ . The constraints in (1)(0010), (1)(1001) and (1)(0110) propagate to this module. There is also a Bianchi identity in this module. We define  $Y_{AB}^{(2)} = \{(D_{[A}\gamma^i D_{C]})Y_{iB}{}^C\}_{[AB]}$  and  $\tilde{Z}_{AB}^{(2)} = \{(D_{(A}D_{C)})\tilde{Z}_{B}{}^C\}_{[AB]}$ . The (1)(1001) constraint gives  $Y_{AB}^{(2)} + 6D^iY_{iAB} = 0$ . The (1)(0010) and (1)(0110) constraints give respectively  $Y_{AB}^{(2)} + 2D^iY_{iAB} + \frac{2}{5}\tilde{Z}_{AB}^{(2)} = 0$  and  $Y_{AB}^{(2)} - 12D^iY_{iAB} + \frac{9}{5}\tilde{Z}_{AB}^{(2)} = 0$ . There is a linear dependency, and the solutions are  $Y_{AB}^{(2)} = -6D^iY_{iAB}$ ,  $\tilde{Z}_{AB}^{(2)} = 10D^iY_{iAB}$ . The Bianchi identity, finally, gives  $Y_{AB}^{(2)} + 30D^iY_{iAB} - \frac{12}{5}\tilde{Z}_{AB}^{(2)} = 0$ , which provides no further information.

**(2)(0000):** similarly, there is one (2)(0000) in  $D^2Y$ . It becomes constrained by the (1)(0010) constraint and by two (2)(0000)'s in the Bianchi identity (neither of which contain curvature, due to  $(R_{[ab]c})^d = 0$ ). The only other expression in this module is  $D_aZ$ . The components of the Bianchi identity are obtained as  $(R_{ab})_{\gamma C}{}^{\gamma C} = \dots$  and  $(R_{ab})_{\gamma C}{}^{\delta C}(\gamma^b)_{\delta}{}^{\gamma} = \dots$ . The former gets contribution only from the second term on the right hand side of eq. (3.13), and is trivially fulfilled due to the easily derived equation  $(D^E y_{aE}) + (D^E \gamma_a y_E) = 0$ . The second one gives  $\varepsilon_a{}^{ij}(D^E \gamma_i D^F)Y_{jEF} = -224D_aZ$ . Finally, one gets an equation in (2)(0000) by differentiation of the (1)(0010) equation  $\frac{3}{2}y_{A\alpha} - \frac{1}{4}\tilde{z}_{A\alpha} - 2z_{A\alpha}$ , with contribution from the first and third terms. It turns out to yield the same information again. Thus, there is no restriction on  $D_aZ$ , and no remaining degrees of freedom in (2)(0000) at dim. 2.

**(2)(0020):** in (2)(0020), there are two fields in  $D^2Y$  and one in  $D^2\tilde{Z}$ . The constraints in (1)(0010), (1)(0030), (1)(0110) and (3)(0110) propagate to this module, and there are two (2)(0020)'s in the Bianchi identity. This looks dangerous; there is naïvely 6 equations for 3 fields. The equations may also contain  $D_a\tilde{Z}_{AB}$  and  $(Y_a\tilde{Z})_{(AB)}$ . Let

$$\begin{aligned} Y_{1aAB}^{(2)} &= \{(D_{(A}D_{E)})Y_{aB}{}^E\}_{(AB)}, \\ Y_{2aAB}^{(2)} &= \{\varepsilon_a{}^{ij}(D_{[A}\gamma_i D_{E]})Y_{jB}{}^E\}_{(AB)}, \\ \tilde{Z}_{aAB}^{(2)} &= \{(D_{[A}\gamma_a D_{E]})\tilde{Z}_{B}{}^E\}_{(AB)}. \end{aligned} \quad (3.33)$$

Beginning with the constraint  $DY|_{(3)(0110)} = 0$ , it gives  $3Y_{1aAB}^{(2)} - 2Y_{2aAB}^{(2)} - 36(Y_a\tilde{Z})_{(AB)} = 0$ . The constraint  $D\tilde{Z}|_{(1)(0030)} = 0$  propagates to  $\tilde{Z}_{aAB}^{(2)} + 8D_a\tilde{Z}_{AB} + 4(Y_a\tilde{Z})_{(AB)} = 0$ . The

constraints in (1)(0110) and (1)(0010) give the two equations  $Y_{1aAB}^{(2)} + \frac{4}{3}Y_{2aAB}^{(2)} - \tilde{Z}_{aAB}^{(2)} + 16D_a\tilde{Z}_{AB} - 64(Y_a\tilde{Z})_{(AB)} = 0$  and  $-5Y_{1aAB}^{(2)} + 5Y_{2aAB}^{(2)} - 2\tilde{Z}_{aAB}^{(2)} + 4D_a\tilde{Z}_{AB} + 12(Y_a\tilde{Z})_{(AB)} = 0$ . Finally, the two equations from the Bianchi identity are  $5Y_{1aAB}^{(2)} - 4\tilde{Z}_{aAB}^{(2)} + 8D_a\tilde{Z}_{AB} - 156(Y_a\tilde{Z})_{(AB)} = 0$  and  $5Y_{2aAB}^{(2)} + 8\tilde{Z}_{aAB}^{(2)} + 124D_a\tilde{Z}_{AB} - 88(Y_a\tilde{Z})_{(AB)} = 0$ . Only three of the six equations are linearly independent, and the solutions are

$$\begin{aligned} Y_{1aAB}^{(2)} &= -8D_a\tilde{Z}_{AB} + 28(Y_a\tilde{Z})_{(AB)}, \\ Y_{2aAB}^{(2)} &= -12D_a\tilde{Z}_{AB} + 24(Y_a\tilde{Z})_{(AB)}, \\ \tilde{Z}_{aAB}^{(2)} &= -8D_a\tilde{Z}_{AB} - 4(Y_a\tilde{Z})_{(AB)}. \end{aligned} \quad (3.34)$$

**(0)(0000):** concerning the singlet at dim. 2, we have initially one field each in  $Y$ ,  $\tilde{Z}$  and  $Z$ . One relation is obtained from the (1)(0010) identity at dim.  $\frac{3}{2}$ , and reads

$$0 = \frac{1}{7}Y^{(2)} + \frac{2}{35}\tilde{Z}^{(2)} + 2Z^{(2)}, \quad (3.35)$$

where  $Y^{(2)} = (D^A\gamma^i D^B)Y_{iAB}$ ,  $\tilde{Z}^{(2)} = (D^A D^B)\tilde{Z}_{AB}$ ,  $Z^{(2)} = (D^E D_E)Z$ . The (0)(0000) part of the Bianchi identity determines the scalar curvature as

$$R = -\frac{1}{14}Y^{(2)} + \frac{6}{35}\tilde{Z}^{(2)} + 2\text{tr}(Y^i Y_i) - 6\text{tr}(\tilde{Z} + Z\mathbb{1})^2. \quad (3.36)$$

**(4)(0000):** in (4)(0000), the Bianchi identity relates the traceless part of the Ricci tensor to the field  $Y_{ab}^{(2)} = (D^E\gamma_{(a} D^F)Y_{b)EF}$  as

$$R_{(ab)'} = -\frac{1}{56}Y_{ab}^{(2)} + \frac{1}{2}\text{tr}(Y_{(a} Y_{b)'}). \quad (3.37)$$

The last components of the Bianchi identity lie in (4)(0100) and (4)(0020). The fields in  $D^2Y$  in these modules (one each) must become determined in terms of  $D_{(a}Y_{b)'}_{AB}$  and  $(Y_{(a}Y_{b)'}_{AB})$ , respectively, by the vanishing of the (3)(1001) and (3)(0110) parts of  $DY$ . The Bianchi identities must then constitute consistency checks on those relations. We have not yet checked this.

Of the surviving components at dimension 2, some can be removed by a Weyl transformation. This applies to one of the scalars (0)(0000) and the modules (0)(0020) and (0)(0200).

## 4 Equations of motion and matter couplings

### 4.1 Conformal equations of motion

In  $D = 3$ , the Riemann tensor is completely determined by the Ricci tensor:

$$R_{ab,cd} = \varepsilon_{ab}{}^i \varepsilon_{cd}{}^j \left( R'_{ij} - \frac{1}{6}\eta_{ij}R \right), \quad (4.1)$$

where  $R'_{ab} = R_{(ab)'}$  is the traceless part of  $R_{ab}$ . The curvature Bianchi identity is here a vector equation (after dualisation). It reads

$$D^b R'_{ab} - \frac{1}{6}D_a R = 0. \quad (4.2)$$



This equations eliminates the curvature scalar as local degree of freedom. It also coincides with the vector part of the conformal equation of motion  $D_{[a}(R_{b]c} - \frac{1}{4}\eta_{b]c}R) = 0$ , whose only other content is given by the Cotton tensor in (4)(0000). The linearised equations of motion for pure conformal supergravity read.

$$\varepsilon_{(a}{}^{ij}D_{|i}R'_{j|b)'} = 0. \tag{4.3}$$

The Bianchi identity for the gravitino at dim.  $\frac{5}{2}$  is  $D^i\tilde{t}_i + \not{D}t = 0$ . The conformal equation of motion in (3)(0010) for the gravitino is

$$\varepsilon_a{}^{ij}D_i\tilde{t}_{j\alpha A} - (\gamma\text{-trace}) = 0. \tag{4.4}$$

We will now show that going on the conformal shell by specifying  $C^+$  gives these last two equations. In doing this we will not deal with any specific type of matter couplings, only pure conformal supergravity, obtained from  $C^+ = 0$ . We will also restrict ourselves to a linearised treatment.

The lowest-dimensional conformal equation of motion is the one for the  $so(8)$  field strength in (2)(0100), obtained in the previous section. We therefore take a fermionic derivative on the equations in (2)(0100) listed in section 3, and project on (3)(0010). A priori,  $Y$  contains 3 components in (3)(0010) at  $\theta^3$ , while  $\tilde{Z}$  and  $Z$  contain none. This means, that these three components are determined in terms of lower-dimensional fields by taking one spinorial derivative on the three equations in (2)(0100) at dimension 2. If in addition  $C^+ = 0$ , this gives one more equation, so that  $Z^{(2)}$  and  $\tilde{Z}^{(2)}$  in (2)(0100) can be solved for. Linearised, they become proportional to  $\varepsilon_a{}^{ij}D_i Y_{jAB}$ . Taking another spinorial derivative and projecting on (3)(0010) directly gives the gravitino equation of motion (4.4).

Acting with one more spinorial derivative and projecting on the module (4)(0000), straightforwardly gives the Cotton equation (4.3).

We have focused on deriving the equations of motion of conformal supergravity from the off-shell multiplet, and the connection between this question and the occurrence of the Dragon window. Obviously, it is also possible to go on a Poincaré shell. The corresponding physical torsion constraints are clearly present: the tensor  $Y$  should then be identified with the  $SO(8)$  current, which will introduce a dimensionally correct coupling. We do not work out the details of this construction here but the reader may consult ref. [24] for more information on this issue.

## 4.2 Matter coupling

A specific candidate for coupling to the conformal supergravity is the  $N = 8$  scalar multiplet, i.e., the fields of the BLG model. This coupling has been (partly) constructed in a component formalism [14], where, however, there are some issues with the closure of the supersymmetry algebra. It is clear from above that the scalar superfield of the BLG model [37, 38] can be used as a source for the conformal supergravity. The superfield has dimension  $\frac{1}{2}$  and transforms as (0)(1000) (and carries an additional internal 3-algebra index). A natural choice is

$$C^{+IJ} = \text{tr}(\Phi^I\Phi^J), \tag{4.5}$$

the consistency of which is shown as follows. The constraint on the scalar superfield is

$$D_{\alpha A}\Phi^I = (\sigma^I\Psi)_{\alpha A}, \tag{4.6}$$

where  $\Psi$  is the fermion superfield, i.e., the module (1)(1010) in  $D\Phi$  vanishes. This implies that the current superfield  $C^+$  formed as eq. (4.5) fulfills the Bianchi identity  $DC^+|_{(1)(2010)} = 0$ , which is the only constraint on  $C^+$ . A related discussion can be found in ref. [23].

At the same time as matter sources the supergravity, one should formulate the matter dynamics in a conformal supergravity background. This may be achieved if one shows that a nilpotent BRST operator  $Q = \lambda^{\alpha A}D_{\alpha A}$  can be constructed, where  $D_{\alpha A}$  carries the geometric information. Since  $\Phi^I$  transforms under  $so(8)$ , the nilpotency of  $Q$  may potentially be ruined by  $so(8)$  curvature at dimension 1. One also has to take into consideration that  $\Phi^I$  has an extra gauge invariance in addition to the pure spinor constraint,  $\Phi^I \equiv \Phi^I + (\lambda\sigma^I\varrho)$ , for arbitrary  $\varrho$  in (1)(0001).  $Q^2\Phi^I$  needs to vanish only modulo such a term. This gauge invariance turns out to save the nilpotency for the contributions to the  $so(8)$  curvature from  $Y$ ,  $\tilde{Z}$  and  $Z$ , but not from  $C^+$ . The latter curvature component gives

$$Q^2\Phi^I \sim \varepsilon_{\alpha\beta}\lambda^{\alpha A}\lambda^{\beta B}C^+_{ABCD}(\sigma^{IJ})^{CD}\Phi^J, \tag{4.7}$$

which does not take the form of a gauge transformation. Expressing  $C^+$  as  $\Phi^2$  does not change this fact. This looks like an obstruction to a consistent coupling between the BLG model and conformal supergravity, which seems to indicate problems to close the supersymmetry algebra perhaps related to similar unresolved problems in ref. [14]. We feel, however, that a proper supersymmetric treatment of the interacting gravity-matter system calls for a full Batalin-Vilkovisky formulation, and that only BRST is not enough beyond the linearised level. Such a formulation so far exists only for the BLG sector [37], but experience from eleven dimensions [43, 44] gives reason to hope that it is achievable.

## 5 Conclusions and comments

Starting from the  $N = 8$  superspace BI's in three dimensions we have in this paper extended the analysis of ref. [19] and shown that we can generate either Poincaré or conformal supergravity. The field equations are “off-shell” in the sense that they appear with currents that can be chosen arbitrarily opening up for constructions containing supergravity theories, either Poincaré or conformal, coupled to matter multiplets like the BLG theory. One interesting result is the so called “Dragon window”, which refers to the fact that some modules in the supercurvature tensor manage to avoid Dragon’s theorem. This means that in the analysis of the BI’s it is no longer sufficient to solve only the torsion BI’s, since these do not imply all of the curvature BI’s. According to Dragon’s theorem this will not happen in four and higher dimensions. This phenomenon was also used in refs. [19, 23, 24].

One of the motivations for this work was in fact to get a better understanding of the topologically gauged BLG theory presented in ref. [14]. The construction in this work exactly corresponds to the coupling discussed in section 4.2, namely between  $N = 8$  superconformal gravity in three dimensions and the  $N = 8$  BLG theory. However, in ref. [14]

$n$	$R_n$
0	$(0)(0000)$
1	$(1)(0010)$
2	$(0)(0000) \oplus (0)(0020) \oplus (2)(0100)$
3	$(1)(0010) \oplus (1)(0110) \oplus (3)(1001)$
4	$R_2 \oplus (0)(0200) \oplus (2)(1011) \oplus (4)(0002) \oplus (4)(2000)$
5	$R_3 \oplus (1)(1101) \oplus (3)(0012) \oplus (3)(2010) \oplus (5)(1001)$
6	$R_4 \oplus (0)(2002) \oplus (2)(0102) \oplus (2)(2100) \oplus (4)(1011) \oplus (6)(0100)$
7	$R_5 \oplus (1)(1003) \oplus (1)(3001) \oplus (3)(1101) \oplus (5)(0110) \oplus (7)(0010)$
8	$R_6 \oplus (0)(0004) \oplus (0)(4000) \oplus (2)(2002) \oplus (4)(0200) \oplus (6)(0020) \oplus (8)(0000)$

**Table 1.** The modules in a scalar superfield.

the attempt to derive the complete Lagrangian met with some difficulties half way through the construction for reasons that were not completely understood. It would therefore be of some value to have an alternative method by which this theory could be derived. Hopefully the results obtained here will eventually prove to provide such a method.

It should be mentioned also that the corresponding construction in the  $N = 6$  case, then involving the ABJM theory for  $N$ -stacks of M2-branes, was successfully carried out in ref. [15] following the same steps and ideas as applied in ref. [14] in the  $N = 8$  topologically gauged BLG case. For the topologically gauged ABJM theory some complications did arise but of a much milder kind than for  $N = 8$  and could be solved.

During the completion of this paper, related work appeared [39] which analyses the case with maximal supersymmetry in three dimensions, i.e.,  $N = 16$ , and hence does not address the same M2 related questions as in this paper. Nevertheless, this work solves parts of the three dimensional BI's and does contain calculations that partly overlap with ours.

## A The field content in a scalar superfield

Weyl rescalings are performed with a scalar superfield of dimension 0. In order to understand which component fields may be removed by such a rescaling, its component expansion must be examined. The Dynkin labels of the modules  $R_n$  appearing at order  $\theta^n$  are given below. In table 1, we only list the fields up to  $\theta^8$ , since the modules appearing at  $\theta^n$  and  $\theta^{16-n}$  are the same.

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