

# Higher-spin algebras and cubic interactions for simple mixed-symmetry fields in AdS spacetime

---

Nicolas Boulanger<sup>a,1</sup> and E.D. Skvortsov<sup>b</sup>

<sup>a</sup>*Service de Mécanique et Gravitation, Université de Mons — UMONS,  
20 Place du Parc, 7000 Mons, Belgium*

<sup>b</sup>*P.N.Lebedev Physical Institute,  
Leninsky prospect 53, 119991, Moscow, Russia*

*E-mail:* [nicolas.boulanger@umons.ac.be](mailto:nicolas.boulanger@umons.ac.be), [skvortsov@lpi.ru](mailto:skvortsov@lpi.ru)

**ABSTRACT:** Nonabelian Fradkin-Vasiliev cubic interactions for dual-graviton-like gauge fields with gravity and themselves are constructed in anti-de Sitter spacetime. The Young diagrams of gauge potentials have shapes of “tall-hooks”, i.e. two columns the second of height one.

The underlying nonabelian algebra is a Clifford algebra with the anti-de Sitter signature. We also discuss the universal enveloping realization of higher-spin algebras, showing that there is a one-parameter family of algebras compatible with unitarity, which is reminiscent of  $d = 3$  deformed oscillators.

**KEYWORDS:** Gauge Symmetry, Field Theories in Higher Dimensions

**ARXIV EPRINT:** [1107.5028](https://arxiv.org/abs/1107.5028)

---

<sup>1</sup>Research Associate of the Fund for Scientific Research-FNRS (Belgium).

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Two-column gauge fields</b>	<b>3</b>
2.1	Notation and conventions	3
2.2	Short review on two-column gauge fields in $AdS_d$	3
<b>3</b>	<b>Fradkin-Vasiliev cubic interactions</b>	<b>12</b>
<b>4</b>	<b>Gravitational interactions</b>	<b>16</b>
<b>5</b>	<b>Cubic interactions: AdS-covariant formulation</b>	<b>18</b>
<b>6</b>	<b>Developments and discussion</b>	<b>21</b>
<b>7</b>	<b>Conclusion</b>	<b>27</b>

---

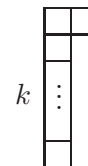
## 1 Introduction

Since the pioneering works by Fradkin and Vasiliev [1, 2] where the gravitational interaction problem (as well as self-interactions) for higher-spin gauge fields was solved at the first nontrivial order by going to a four-dimensional (anti-)de Sitter  $(A)dS_4$  background, there has been a lot of attention devoted to the study of totally-symmetric gauge fields around  $AdS_d$  background, culminating with Vasiliev's fully nonlinear and consistent equations for totally symmetric gauge fields [3–5]. These equations admit constantly-curved spacetimes as exact solutions, where it is crucial that the curvature be nonvanishing. For a review of the key mechanisms of higher-spin extensions of gravity, see [6], while various reviews on Vasiliev's equations can be found in [7–9].

Depending on one's taste, one may view higher-spin gauge theory as a limit of string theory [10], or string theory as a broken phase of higher-spin gauge theory. See however recent works [11–13] where higher-spin fields were realized as certain vertex operators in exotic pictures without taking any limits of superstring theory. It is expected that, for a better understanding of both higher-spin gauge theory and strings together with their interconnections, the role played by the fields at high levels, whose Young symmetries are neither totally symmetric (first Regge trajectory) nor totally antisymmetric ( $p$ -forms), will be crucial. For a recent discussion and results, see [14]. At present, free mixed-symmetry gauge fields are very well understood off-shell, in both flat and  $(A)dS_d$  backgrounds. For a non-exhaustive list of recent works on quadratic action principles and equations for mixed-symmetry gauge fields in constantly-curved backgrounds, see e.g. [15–35] and references therein.

As far as the problem of finding consistent interactions for mixed-symmetry fields is concerned, some analysis have been done in flat background [36–41], but in  $(A)dS_d$  background almost nothing has been achieved apart from the very recent works [42–44] (see also the earlier works [45, 46]) and [47].

It is the goal of the present paper to study the interaction problem for the simplest class of mixed-symmetry gauge fields, i.e. those that are described in the metric-like formalism by potentials of Young shape  $[k, 1]$ , i.e. with two columns, the first of arbitrary height (albeit constrained by the spacetime dimension  $d$ ), the second of height one. We call such fields “tall hooks”. In particular the graviton belongs to the spectrum as  $[1, 1]$  Young shape. The spin-one Maxwell field is also present as degenerate  $[0, 1]$  Young shape. A detailed analysis of the case  $[2, 1]$  corresponding to the dual graviton in five dimensions has been provided very recently in [44] where special emphasis was put on the Stückelberg formulation and the flat limit that will not be considered here.



Higher-spin fields are to be organized in the adjoint multiplet of some higher-spin algebra  $\mathfrak{g}$ . Any reasonable higher-spin algebra must contain  $\mathfrak{so}(d - 1, 2)$  as a subalgebra.<sup>1</sup> Gauging of  $\mathfrak{so}(d - 1, 2)$  itself describes gravity by virtue of the MacDowell-Mansouri-Stelle-West [51, 52] approach. In addition  $\mathfrak{g}$  contains some other generators corresponding to higher-spin fields.

The higher-spin algebra whose gauging leads to the spectrum built of tall hooks is just the Clifford algebra,  $\mathcal{C}\ell_{d-1,2}$ , with the anti-de Sitter signature. The graviton appears when gauging  $\mathcal{C}\ell_{d-1,2}$  since  $\mathfrak{so}(d - 1, 2) \in \mathcal{C}\ell_{d-1,2}$ . The spin-one field corresponds to the unity of  $\mathcal{C}\ell_{d-1,2}$ . Higher products of  $\gamma$ -matrices represents generators for genuine tall hooks. The Clifford algebra  $\mathcal{C}\ell_{d-1,2}$  can be thought of as the simplest higher-spin algebra because it is finite-dimensional.

We show that the structure constants of  $\mathcal{C}\ell_{d-1,2}$  are compatible with switching on cubic interactions of tall hooks and construct the cubic action using the Fradkin-Vasiliev approach, [1, 2]. Yang-Mills groups can be easily activated too, leading to colored tall hooks.

Aimed at giving an invariant definition for general higher-spin algebras we consider the quotients of the universal enveloping algebra of  $\mathfrak{so}(d - 1, 2)$ ,  $U(\mathfrak{so}(d - 1, 2))$ . The restrictions by unitarity lead naturally to a one-parameter family  $hs(\nu)$  of higher-spin algebras with the even Clifford algebra and the Vasiliev higher-spin algebra [5] belonging to it at  $\nu = \frac{d(d+1)}{8}$  and at the “singleton point”  $\nu = -\frac{(d-1)(d-3)}{4}$ , respectively.

The algebra  $hs(\nu)$  can be thought of as a generalization to arbitrary dimension of the  $3d$  higher-spin algebra of [53], which is based on the deformed oscillator algebra [54]. Recently there has been a lot of interest in  $3d$   $hs(\nu)$  in the context of  $AdS_3/CFT^2$ , see e.g. [55–58].

The outline of the paper is as follows. In section 2 we recall some results about two-column gauge fields in  $(A)dS_d$  background. Section 3 reviews the Fradkin-Vasiliev construction for consistent cubic interactions around  $(A)dS$  background. In section 4 we present nonabelian gravitational interactions for tall hooks. The manifestly  $AdS_d$ -covariant formulation for cubic interactions of tall hooks is given in section 5, where an extension of

---

<sup>1</sup>See [48–50] for more detailed exposition and original results.

the previous interactions is proposed, whereby the self-interactions are considered as well. We discuss the realization of higher-spin algebras in terms of  $U(\mathfrak{so}(d-1,2))$  and some other developments in section 6, before we give the conclusions in section 7.

## 2 Two-column gauge fields

In this section we briefly review some features pertaining to the kinematics of irreducible gauge fields in  $AdS_d$  backgrounds.

### 2.1 Notation and conventions

Base manifold indices, or world indices, are denoted by Greek letters  $\mu, \nu, \dots$ , while Lorentz indices are denoted by lower-case Latin letters. The Lorentz algebra  $\mathfrak{so}(d-1,1)$  is associated with the metric  $\eta_{ab} = \text{diag}(-, +, \dots, +)$  where the indices  $a, b, \dots$  run over the values  $0, 1, \dots, d-1$ , while the anti-de Sitter algebra  $\mathfrak{so}(d-1,2)$  is associated with the metric  $\eta_{AB} = \text{diag}(-, +, \dots, +, -)$  where the indices  $A, B, \dots$  run over the values  $0, 1, \dots, d-1, d$ .

Square brackets indicate total antisymmetrization with strength one, i.e.

$$[a_1 a_2 \dots a_p] = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^{|\sigma|} a_{\sigma(1)} \dots a_{\sigma(p)} \quad , \quad (2.1)$$

where  $\mathfrak{S}_p$  is the group of permutations of  $p$  elements and  $|\sigma| \in \{0, 1\}$  is the signature of the permutation  $\sigma \in \mathfrak{S}_p$ . Similarly, curved brackets  $(a_1 a_2 \dots a_p)$  denote total symmetrization with strength one. A group of  $p$  totally antisymmetrized Lorentz indices will be denoted by  $[a_1 a_2 \dots a_p] = a[p]$ , while  $(a_1 a_2 \dots a_p) = a(p)$  for totally symmetrized indices, and similarly for world and  $AdS_d$  indices. To further simplify the notation we will often use conventions whereby like letters imply complete (anti)symmetrization with strength one, so that for example  $e^a \omega^a [3] = e^{[a} \omega^{aaa]}$ . If not explicitly specified, the context will make clear whether we mean complete symmetrization or antisymmetrization.

The components of an irreducible  $\mathfrak{gl}_d$  tensor whose symmetry type consists of a Young diagram with two columns, the first of length  $p$  and the second of length  $q$ , will be denoted  $\varphi^{\mu[p], \nu[q]}$ ,  $p \geq q$ . The Young symmetry described above is abbreviated by  $\varphi \sim [p, q]$ . The components of a Lorentz tensor of type  $[p, q]$  are denoted by  $\varphi^{a[p], b[q]}$ , *idem* for an  $\mathfrak{so}(d-1,2)$ -tensor  $\varphi^{A[p], B[q]}$ . Note that, by abuse of notation, we do not consider (anti) self-duality constraints on what we call Lorentz (or  $AdS_d$ ) tensors in  $d = 2n$ , so that a Lorentz tensor, in our conventions, only obeys over-(anti)symmetrization and trace constraints. The torsion-free, Lorentz-connection on the base manifold is denoted by the symbol  $D$ . In flat background,  $[D_\mu, D_\nu] = 0$  while in  $AdS_d$ , one has  $[D_\mu, D_\nu] = \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab}$  where  $M_{ab}$  are the generators of  $\mathfrak{so}(d-1,1)$  and the components of the background curvature two-form  $R_{\mu\nu}{}^{ab} = -\lambda^2 (h_\mu^a h_\nu^b - h_\mu^b h_\nu^a)$  in terms of the  $AdS_d$  background vielbein components  $h_\mu^a$  and the parameter  $\lambda^2$  related to the cosmological constant  $\Lambda$  by  $-\lambda^2 = \frac{\Lambda}{(d-1)(d-2)}$ . We denote  $D_a = h_a^\mu D_\mu$  featuring the inverse vielbein, or tetrad, components  $h_a^\mu$ .

### 2.2 Short review on two-column gauge fields in $AdS_d$

In this section we are mainly concerned with mixed-symmetry fields of type- $[p, q]$ , but also review some other background material.

**On-mass shell.** Considering a massive type- $[p, q]$  tensor in  $AdS_d$  spacetime, the complete set of on-mass-shell conditions is

$$(\square + m^2)\phi^{a[p],b[q]} = 0 , \tag{2.2}$$

$$D_a\phi^{a[p],b[q]} = 0 = D_b\phi^{a[p],b[q]} , \tag{2.3}$$

$$\eta_{ab}\phi^{a[p],b[q]} = 0 . \tag{2.4}$$

For generic values of  $m^2$ , the equations (2.2)–(2.4) possesses no gauge symmetries and describe a massive type- $[p, q]$  field. Instead of the parameter  $m^2$ , it is convenient to use the minimal eigenvalue  $E_0$  (the lowest energy) of the  $\mathfrak{so}(2)$  generator of the maximal compact subalgebra  $\mathfrak{so}(2) \oplus \mathfrak{so}(d-1) \subset \mathfrak{so}(d-1, 2)$ . The lowest energy  $E_0$  is related to  $m^2$  by [59]

$$m^2 = \lambda^2 [E_0(E_0 - d + 1) - p - q] . \tag{2.5}$$

There are three special values of  $E_0$  at which certain gauge symmetry appears. At each critical point the corresponding field equations modulo gauge transformations define an irreducible  $\mathfrak{so}(d-1, 2)$ -module, the corresponding on-shell field therefore qualifying for being an elementary field.

- (i)  **$E_0 = d - q, p \geq q$** : There is a gauge symmetry with the gauge parameter having the symmetry type  $[p, q - 1]$ . The corresponding gauge transformations read

$$\delta_\epsilon\phi^{a[p],b[q]} = D^b\epsilon^{a[p],b[q-1]} + \frac{p(-1)^{p+1}}{(p-q+1)} D^a\epsilon^{a[p-1]b,b[q-1]} , \tag{2.6}$$

where the last term is needed for the whole expression to have the  $\mathfrak{gl}_d$  symmetry type  $[p, q]$ . The gauge parameter obeys equations analogous to (2.2)–(2.4), but with a different value for the critical mass. The quotient Verma module is unitary. The field is called *massless* [59] because of the gauge transformations with one derivative, as occurring for a massless field in flat spacetime. It is that kind of field, with  $q = 1$ , that we will be considering in the present paper.

- (ii)  **$E_0 = d - p - 1, p > q$** : That critical point corresponds to the second possibility for a gauge parameter, namely the one having the type  $[p - 1, q]$ ,

$$\delta_\epsilon\phi^{a[p],b[q]} = D^a\epsilon^{a[p-1],b[q]} . \tag{2.7}$$

The projector onto the symmetry type  $[p, q]$  trivializes down to a single term. In this case, however, the quotient Verma module is non-unitary, which removes such massless fields in any reasonable field theory in  $AdS_d$ . These types of fields are also called massless [59] due to first order gauge transformation law. Due to the non-unitarity property, we will not be considering them in the present paper.

- (iii)  **$E_0 = d - q - 1, p = q$** : If one allows for higher derivatives in gauge transformations, then there can be a third possibility [60] with the gauge parameter having the symmetry type  $[p - 1, p - 1]$ :

$$\delta_\epsilon\phi^{a[p],b[p]} = D^a D^b\epsilon^{a[p-1],b[p-1]} + \lambda^2\eta^{ab}\epsilon^{a[p-1],b[p-1]} , \tag{2.8}$$

where the second term above represents a lower-derivative  $\Lambda$ -correction to the first one.<sup>2</sup> In  $AdS_d$  these fields give rise to non-unitary modules so that they will not be considered in this paper. They are called “partially-massless” [61] because the gauge parameter has fewer indices in comparison with the massless case, and thus one may think of a gauge symmetry weakening. In general, partially-massless fields constitute a discrete chain interpolating between non-gauge massive fields with the highest number of physical degrees of freedom, and massless ones with the lowest number thereof. For more details and the description of partially-massless fields in the frame-like (or Cartan) formalism, we refer to [62].

In the massless  $\Lambda \rightarrow 0$  limit, all the critical values for  $E_0$  simultaneously go to zero and one has a degeneracy of gauge symmetry types: the partially-massless cases do not exist any more and one observes that the equations (2.2)–(2.4) become simultaneously invariant under both types of gauge transformations with parameters of type  $[p, q - 1]$  and  $[p - 1, q]$ . Therefore, due to this gauge symmetry enhancement, the Minkowski spin- $[p, q]$  field has fewer degrees of freedom in comparison with any of the two massless fields in anti-de Sitter background. See [63] and [29, 30] for extended discussions.

So far the description was on-shell. In order to give an off-shell description, the constraints (2.3)–(2.4) have to be relaxed both for the field and its gauge parameter. The field content has to be enlarged with new fields associated with certain nonvanishing traces of the  $\mathfrak{gl}_d$  tensor  $\phi^{\mu[p],\nu[q]}$ . In the frame-like, or Cartan, approach to gauge fields in  $AdS_d$  background, powerful tools exist in order to tackle the interaction problem, so we now review the Cartan approach to gauge fields in  $AdS_d$ .

**Generalized connections.** It has been known since [51, 52] that anti-de Sitter gravity can be understood as a Yang-Mills-like theory with the gauge algebra being  $\mathfrak{so}(d - 1, 2)$

$$\begin{array}{|c|} \hline \square \\ \hline \end{array}, \quad W_{\mu}^{A,B} dx^{\mu} \quad \longrightarrow \quad \delta_{\epsilon} \phi_{\mu\nu} = D_{\mu} \epsilon_{\nu} + D_{\nu} \epsilon_{\mu} \quad (2.9)$$

where the gauge transformations above have been written for the linearized theory.

Vasiliev showed in [64] that a free spin- $s$  gauge field can be described by a one-form that, as an irreducible tensor of  $\mathfrak{so}(d - 1, 2)$ , has the symmetry of a rectangular two-row Young diagram of length- $(s - 1)$ :

$$\begin{array}{|cc|} \hline s - 1 \\ \hline s - 1 \\ \hline \end{array}, \quad W_{\mu}^{A(s-1),B(s-1)} dx^{\mu} \quad \longrightarrow \quad \delta_{\epsilon} \phi_{\mu(s)} = D_{\mu} \epsilon_{\mu(s-1)} . \quad (2.10)$$

This lead [18, 29, 30, 62, 65] to the study of the generalized connections (or Yang-Mills-like fields) of the anti-de Sitter algebra that are defined by differential forms of arbitrary degrees taking their values in arbitrary irreducible representation of  $\mathfrak{so}(d - 1, 2)$ . In the most general cases, the relation between any given gauge theory (not necessarily unitary) and the corresponding generalized  $\mathfrak{so}(d - 1, 2)$ -connections was given in [60], with the details left in [33].

---

<sup>2</sup>In the case of a  $[p, p, \dots, p]$ -type field, one has a tail of lower-derivative terms with increasing powers of  $\lambda^2$ . In the simplest  $[p, p]$ -type case where the leading terms in the gauge transformation bring in two covariant derivatives, one has only  $\mathcal{O}(\lambda^2)$  corrections.

**AdS background for generalized connections.** Pick a generalized  $\mathfrak{so}(d-1,2)$ -connection and denote it by  $W_{\mathbf{q}}^{\mathbf{A}}$ . It is defined by the form degree  $q$  and the finite-dimensional tensor module  $\mathbf{A}$  of  $\mathfrak{so}(d-1,2)$ . The anti-de Sitter background can be described by a connection  $\Omega^{A,B}$  that is a one-form taking its values in the adjoint representation of  $\mathfrak{so}(d-1,2)$ . In order for it to describe  $AdS_d$ , it must be a nondegenerate solution of

$$d\Omega^{A,B} + \Omega^A{}_C \wedge \Omega^{C,B} = 0 \quad \longleftrightarrow \quad D_0^2 = 0, \quad D_0 = d + \Omega. \quad (2.11)$$

The linearized gauge theory with connection  $W_{\mathbf{q}}^{\mathbf{A}}$  is defined off-shell by specifying gauge transformations and a gauge invariant curvature

$$\delta_\epsilon W_{\mathbf{q}}^{\mathbf{A}} = D_0 \epsilon_{\mathbf{q}-1}^{\mathbf{A}}, \quad (2.12)$$

$$R_{\mathbf{q}+1}^{\mathbf{A}} = D_0 W_{\mathbf{q}}^{\mathbf{A}}, \quad \delta_\epsilon R_{\mathbf{q}+1}^{\mathbf{A}} = 0, \quad D_0 R_{\mathbf{q}+1}^{\mathbf{A}} = 0, \quad (2.13)$$

where both  $\epsilon_{\mathbf{q}-1}^{\mathbf{A}}$  and  $R_{\mathbf{q}+1}^{\mathbf{A}}$  take their values in the same  $\mathfrak{so}(d-1,2)$ -module  $\mathbf{A}$ . At some point one may need to break the manifest  $\mathfrak{so}(d-1,2)$ -symmetry of the construction to extract the background vielbein  $h_\mu^a$  and spin-connection  $\varpi_\mu^{a,b}$  out of  $\Omega^{A,B}$ . This can be achieved via introducing [52, 64] a normalized vector  $V^A V_A = -1$ , called compensator, which carries no propagating degrees of freedom. The stability algebra of  $V^A$  is identified with the local Lorentz subalgebra  $\mathfrak{so}(d-1,1) \subset \mathfrak{so}(d-1,2)$ . The appropriate  $\mathfrak{so}(d-1,2)$ -covariant definitions of the background vielbein  $H^A$  and Lorentz connection  $\Omega_L^{A,B}$  have the form [64]

$$\lambda H^A = D_0 V^A, \quad \Omega_L^{A,B} = \Omega^{A,B} - \lambda(V^A H^B - H^A V^B), \quad (2.14)$$

where  $H_\mu^A$  is required to have the maximal rank  $d$  in order to give rise to a nondegenerate vielbein, a necessary condition entering the definition of the  $AdS_d$  background. As a consequence of the above definitions, one has the following natural constraints on  $H^A$ ,  $V^A$ :

$$H^B V_B = 0, \quad D V^A = 0, \quad D H^A = 0. \quad (2.15)$$

in terms of the Lorentz covariant derivative  $D$  defined by

$$D = d + \Omega_L. \quad (2.16)$$

Let the  $(d+1)$ th value of an  $\mathfrak{so}(d-1,2)$  index be denoted by the symbol  $\bullet$ , so that  $A = (a, \bullet)$ . Taking  $V^A = \delta_\bullet^A$ , the definitions (2.14) yield

$$\lambda H_\mu^a = \Omega_\mu^{a, \bullet}, \quad H_\mu^\bullet = 0, \quad \Omega_L^{a,b} = \Omega^{a,b}. \quad (2.17)$$

Therefore (2.14) is an  $\mathfrak{so}(d-1,2)$ -covariant way to decompose an antisymmetric matrix  $\Omega^{A,B}$  into a vector to be identified with  $h^a$  and an antisymmetric matrix of a lower rank to be identified with  $\varpi^{a,b}$ . For a somewhat more geometric presentation of the above material, see e.g. [30].

**Lorentz view on generalized connections.** Upon decomposing  $\mathbf{A}$  of  $W_{\mathbf{q}}^{\mathbf{A}}$  into irreducible Lorentz modules with the help of the embedding of  $\mathfrak{so}(d-1, 1)$  into  $\mathfrak{so}(d-1, 2)$  defined by  $V^A$ , the  $\mathfrak{so}(d-1, 2)$ -connection  $W_{\mathbf{q}}^{\mathbf{A}}$  gives rise to a set of  $\mathfrak{so}(d-1, 1)$ -fields which are certain generalizations of the dynamical vielbein and Lorentz connection.

To put the theory on-shell one may impose equations by setting certain components of  $R_{\mathbf{q}+1}^{\mathbf{A}}$  to zero. Similarly to the case of gravity theories with vanishing torsion, the generalized spin-connections are expressed in terms of derivatives of the vielbein field. For a spin- $s$  field the appropriate on-mass-shell condition reads [64]

$$R_2^{A(s-1), B(s-1)} = H_M H_N C_0^{A(s-1)M, B(s-1)N}, \quad C_0^{A(s), B(s-1)C} V_C = 0, \quad (2.18)$$

where the zero-form  $C_0^{A(s), B(s)}$  is called the spin- $s$  Weyl tensor. It is an irreducible tensor of  $\mathfrak{so}(d-1, 2)$  type  $(s, s)$ . The condition that the Weyl tensor be  $V^A$ -transversal means that it is effectively an irreducible  $\mathfrak{so}(d-1, 1)$ -tensor of the same type  $(s, s)$ . The curvature two-form  $R_2^{A(s-1), B(s-1)}$  decomposes under  $\mathfrak{so}(d-1, 1) \subset \mathfrak{so}(d-1, 2)$  into the following set of two-forms

$$R_2^{A(s-1), B(s-1)} \rightsquigarrow \left\{ R_2^{a(s-1)}, R_2^{a(s-1), b}, \dots, R_2^{a(s-1), b(s-2)}, R_2^{a(s-1), b(s-1)} \right\}. \quad (2.19)$$

Therefore, the equations (2.18) set all but one Lorentz components of the curvature two-form to zero. For a spin-two dynamical field described by  $e^a$  and  $\omega^{a,b}$ , this gives the linearized Einstein equations

$$D e^a + \omega^a_c h^c = 0, \quad (2.20)$$

$$D \omega^{a,b} + \lambda^2 h^a e^b + \lambda^2 e^a h^b = h_m h_n C^{am, bn}, \quad (2.21)$$

in a form that tells us that all components of the Riemann tensor but the traceless one are zero, i.e. the equations set the linearized Ricci tensor to zero. It should be noted that the spin-two and the generalized spin- $s$  Weyl tensors are differentially constrained, in the sense that they satisfy differential Bianchi identities following from  $D_0 R_{\mathbf{q}+1}^{\mathbf{A}} \equiv 0$ .

The problem of identifying the on-shell physical fields, the Weyl tensors and equations of motion can be reduced to a cohomological problem [18, 29, 30, 62, 66, 67] with the answer known in full generality [33].

**Generalized connections for  $[k, q]$ -type gauge fields.** As was shown in [60] in full generality — the unitary cases were previously discussed in [65] — a family of spin- $[k, q]$  fields in  $AdS_d$  is described by generalized gauge connections of the form

$$W_{\mathbf{q}}^{A[k+1]} \equiv W_{\mu_1 \dots \mu_q}^{A_1 \dots A_{k+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_q}, \quad \text{antisymmetric in } A_1 \dots A_{k+1}, \quad (2.22)$$

where the two parameters  $k$  and  $q$  run over all admissible values ( $q \geq 1$  since we are working with gauge fields). The precise correspondence [60] is given below.

In table 1 we summarize the main features of the theories for the various  $[k, q]$ -type gauge fields. From the discussion at the beginning of the present section it is obvious that to specify a gauge theory in  $AdS_d$  it is enough to specify the spin of the field and



$k$ and $q$	unitarity	massless(m.)/ p.-massless(p.m.)	field	gauge parameter	Weyl tensor
$k + 1 = 0$	yes	m.	$\begin{array}{ c } \hline \\ \hline \end{array}_q$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q-1}$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q+1}$
$k > q$	yes	m.	$\begin{array}{ c } \hline \\ \hline \end{array}_k \begin{array}{ c } \hline \\ \hline \end{array}_q$	$\begin{array}{ c } \hline \\ \hline \end{array}_k \begin{array}{ c } \hline \\ \hline \end{array}_{q-1}$	$\begin{array}{ c } \hline \\ \hline \end{array}_k \begin{array}{ c } \hline \\ \hline \end{array}_{q+1}$
$k = q$	yes	m.	$\begin{array}{ c } \hline \\ \hline \end{array}_q \begin{array}{ c } \hline \\ \hline \end{array}_q$	$\begin{array}{ c } \hline \\ \hline \end{array}_q \begin{array}{ c } \hline \\ \hline \end{array}_{q-1}$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q+1} \begin{array}{ c } \hline \\ \hline \end{array}_{q+1}$
$k = q - 1$	no	p.m.	$\begin{array}{ c } \hline \\ \hline \end{array}_q \begin{array}{ c } \hline \\ \hline \end{array}_q$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q-1} \begin{array}{ c } \hline \\ \hline \end{array}_{q-1}$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q+1} \begin{array}{ c } \hline \\ \hline \end{array}_q$
$k < q - 1$	no	m.	$\begin{array}{ c } \hline \\ \hline \end{array}_q \begin{array}{ c } \hline \\ \hline \end{array}_k$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q-1} \begin{array}{ c } \hline \\ \hline \end{array}_k$	$\begin{array}{ c } \hline \\ \hline \end{array}_{q+1} \begin{array}{ c } \hline \\ \hline \end{array}_k$

**Table 1.** Fields, gauge parameters and primary Weyl tensors for the various possible  $[k, q]$ -type gauge fields in  $AdS_d$ .

the symmetry type of its gauge parameter, or equivalently the spin of the field and the symmetry of its primary Weyl tensor. The various case depicted in table 1 are described as follows:

$k + 1 = 0$  : Gauge  $q$ -forms, i.e. massless fields with spin given by  $[q]$ .

$k = q$  : Unitary massless fields of spin  $[q, q]$ . These fields are very close to the Minkowski massless fields. The graviton, whose spin is  $[1, 1]$ , belongs to this class.

$k > q$  : Unitary massless fields with spin- $[k, q]$ . The gauge parameter has spin  $[k, q - 1]$ .

$k = q - 1$  : Nonunitary partially massless field of spin  $[q, q]$  and depth two, i.e. the gauge parameter has spin  $[q - 1, q - 1]$  and the gauge transformations contain two derivatives.

$k < q - 1$  : Nonunitary massless fields of spin  $[q, k]$ . The gauge parameter has the symmetry type  $[q - 1, k]$ .

**Lorentz view on two-column fields.** We restrict ourselves to the unitary cases only, investigated in great details in [65]. The generalized gauge connection we will consider

$$W_{\mathbf{q}}^{A[k+1]}, \tag{2.23}$$

decomposes into two Lorentz connections

$$e_{\mathbf{q}}^{a[k]} = W_{\mathbf{q}}^{a[k]\bullet} \quad \text{and} \quad \omega_{\mathbf{q}}^{a[k+1]} = W_{\mathbf{q}}^{a[k+1]}. \tag{2.24}$$

After some  $\lambda$ -rescalings, the curvature  $R_{\mathbf{q}+1}^{A[k+1]}$  in Lorentz components reads

$$R_{\mathbf{q}+1}^{a[k]} = D\epsilon_{\mathbf{q}}^{a[k]} - h_m \omega_{\mathbf{q}}^{ma[k]}, \quad (2.25)$$

$$R_{\mathbf{q}+1}^{a[k+1]} = D\omega_{\mathbf{q}}^{a[k+1]} - \lambda^2 h^a \epsilon_{\mathbf{q}}^{a[k]}, \quad (2.26)$$

and is invariant under gauge transformations of the form<sup>3</sup>

$$\delta_\epsilon \epsilon_{\mathbf{q}}^{a[k]} = D\epsilon_{\mathbf{q}-1}^{a[k]} - h_m \epsilon_{\mathbf{q}-1}^{ma[k]}, \quad (2.27)$$

$$\delta_\epsilon \omega_{\mathbf{q}}^{a[k+1]} = D\epsilon_{\mathbf{q}-1}^{a[k+1]} - \lambda^2 h^u \epsilon_{\mathbf{q}-1}^{a[k]}. \quad (2.28)$$

The potential  $\varphi^{a[k],b[q]}$  is to be identified with the corresponding component of the generalized vielbein  $e_{\mathbf{q}}^{a[k]}$ . This can be easily done with a choice of the symmetric notation for the field

$$\varphi^{a[k],b[q]} \longrightarrow \varphi^{\overbrace{aa,bb,\dots,cc}^{q \text{ pairs}}, \overbrace{u,v,\dots,w}^{k-q}} \quad (2.29)$$

then the potential is a maximally symmetric part of the vielbein

$$\varphi^{aa,bb,\dots,cc,u,\dots,w} = e^{ab\dots cu\dots w|ab\dots c}, \quad e^{a[k]|u\dots w} = e_{\mu_1\dots\mu_q}^{a[k]} h^{\mu_1 u} \dots h^{\mu_q w}, \quad (2.30)$$

where  $h^{\mu c}$  is the inverse of the background vielbein  $h_\mu^a$ :  $h^{\mu c} h_\mu^a = \eta^{ac}$ . One has to use  $h^{\mu c}$  or  $h_\mu^a$  in order to interpret the gauge connections in terms of the potentials. The fiber version of (2.27) reads

$$\delta_\epsilon e^{a[k]|v[q]} = D^v \epsilon^{a[k]|v[q-1]} - \epsilon^{va[k]|v[q-1]}. \quad (2.31)$$

The second term is a shift (Stückelberg-like) symmetry, a kind of local Lorentz transformations for the vielbein, whose purpose is to remove the unwanted components of the vielbein in order for it to match with the content of the field potential. Indeed, one observes that the second term does not shift  $\varphi$ . The same kind of shift symmetry acts on the gauge parameters too, as a result of reducibility of gauge transformations:

$$\delta_\epsilon W_{\mathbf{q}}^{\mathbf{A}} \equiv 0 \quad \text{for} \quad \epsilon_{\mathbf{q}-1}^{\mathbf{A}} = D_0 \chi_{\mathbf{q}-2}^{\mathbf{A}}. \quad (2.32)$$

This is exactly the cohomological origin<sup>4</sup> of the problem of finding physically relevant components in  $\epsilon_{\mathbf{q}-1}^{\mathbf{A}}$ ,  $W_{\mathbf{q}}^{\mathbf{A}}$  and  $R_{\mathbf{q}+1}^{\mathbf{A}}$ . Let us denote by  $\sigma_-$  the operator taking some Lorentz  $[k+1]$ -type  $q$ -form  $B_{\mathbf{q}}^{a[k+1]}$  to the Lorentz  $[k]$ -type  $(q+1)$ -form  $B_{\mathbf{q}+1}^{a[k]} = h_m B_{\mathbf{q}}^{ma[k]}$

$$0 \longrightarrow B_{\mathbf{q}}^{a[k+1]} \xrightarrow{\sigma_-} B_{\mathbf{q}+1}^{a[k]} \longrightarrow 0. \quad (2.33)$$

Then the cohomology groups  $H^{q-1}(\sigma_-)$  and  $H^q(\sigma_-)$  correspond to the differential (as opposed to Stückelberg) gauge parameters and the potential  $\varphi^{a[k],b[q]}$ , including all the

<sup>3</sup>To see this one has to use  $D^2 \xi^a = -\lambda^2 h^a \wedge h_m \xi^m$  for any vector  $\xi^a$ .

<sup>4</sup>This is so-called  $\sigma_-$ -cohomology [66], which have been one of the main technical tools in many works, e.g. [30, 33, 67, 68].

traces that are necessary to formulate the theory off-shell. By solving the cohomology problem, which is very simple in this case, one concludes that the symmetries of the potential and gauge parameter match the required ones.

The problem of identifying the content of the curvature is more subtle and requires the use of (2.25)–(2.26) together with the Bianchi identities  $D_0 R_{\mathbf{q}+1}^{\mathbf{A}} \equiv 0$ :

$$DR_{\mathbf{q}+1}^{a[k]} - h_m R_{\mathbf{q}+1}^{ma[k]} \equiv 0, \tag{2.34}$$

$$DR_{\mathbf{q}+1}^{a[k+1]} + \lambda^2 h^a R_{\mathbf{q}+1}^{a[k]} \equiv 0. \tag{2.35}$$

Actually,  $H^{q+1}(\sigma_-)$  gives all linearly independent, gauge-invariant by construction, components of the curvatures. The  $\sigma_-$ -exact pieces correspond to those components that can be shift to zero by redefining  $\omega_{\mathbf{q}}^{a[k+1]}$ . The equations that do not belong to  $H^{q+1}(\sigma_-)$  either express certain components of  $\omega_{\mathbf{q}}^{a[k+1]}$  in terms of derivatives of  $\varphi$ , or are the differential consequences of more fundamental equations that lie in  $H^{q+1}(\sigma_-)$ .<sup>5</sup>

Which representatives of  $H^{q+1}(\sigma_-)$  should be set to zero depends on the dynamics one wishes to describe. For the example of gravity, if we set the Weyl tensor to zero keeping the Ricci tensor unconstrained, then the equations will describe various conformally flat backgrounds [68] (in  $d > 3$  the manifold is locally conformally flat if and only if the Weyl tensor vanishes). Instead, if one only keeps nonzero the traceless part of the curvature tensor (i.e. the Weyl tensor), then one effectively imposes the Einstein equations.

Generally, for  $W_{\mathbf{q}}^{\mathbf{A}}$  to describe an irreducible propagating field one should set all the components of the curvature to zero but the generalized Weyl tensor. To give a concrete example, we will consider a spin- $\square$  field  $\varphi \sim [2, 1]$ , i.e.  $k = 2, q = 1$ , represented in the one-form  $W_{\mathbf{1}}^{A[k+1]}$ , but otherwise all the statements hold true for any spin- $[k, q]$  field.

The space of gauge-invariant candidate equations, i.e. the representatives of  $H^{q+1}(\sigma_-)$  for  $\mathbf{A}$  having type  $[k + 1]$ , is parameterized as follows [30, 33]. The (primary) Weyl tensor is to be extracted from the components of  $R_{\mathbf{q}+1}^{a[k]}$ . It is a traceless tensor of type  $[k, q + 1]$  (see also the last column of table 1):

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \hline \end{array} \longleftrightarrow \text{traceless part of } \left( 2D^{(a}e^{a)(b|b)} + 2D^{(b}e^{b)(a|a)} = D^a\varphi^{bb,a} + D^b\varphi^{aa,b} \right). \tag{2.36}$$

Indeed, one can see that the  $\sigma_-$ -exact part  $\sigma_-(\omega)$  of the curvature in (2.25) drops out for such a permutation of indices. The above tensor, of first order in derivatives of the potential field, is a representative of  $H^{q+1}(\sigma_-)$ . There are more representatives of  $H^{q+1}(\sigma_-)$  in  $R_{\mathbf{q}+1}^{a[k]}$  that are collectively given by a traceful tensor with the symmetry  $[k - 1, q]$  called the torsion  $T^{a[k-1],b[q]}$ :

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \hline \end{array} \oplus \bullet \longleftrightarrow D^{(a}e^{a)m|}_m + D_m e^{m(a|a)} = D^{(a}\varphi^{a)m}_m - D_m \varphi^{aa,m}. \tag{2.37}$$

The reason to call it *torsion* is that it is a part of the  $(De + \omega)$ -like curvature that coincides with the torsion for spin-two. There is no nontrivial representative of the  $\sigma_-$ -cohomology

---

<sup>5</sup>Evidently, taking a derivative of a gauge-invariant equation produces one more gauge-invariant equation of higher order.

in the spin-two torsion tensor, of course, as  $T_{\mu\nu}{}^a = 0$  does not restrict the spin-two vielbeins  $e^a$  but instead expresses the spin connection in terms of them.

The intersection of  $H^{q+1}(\sigma_-)$  with  $R_{\mathbf{q}+1}^{a[k+1]}$  is empty. However, as we keep the primary Weyl tensor nonzero, we have to keep all its descendants (secondary Weyl tensor, etc.) that are produced by taking various appropriately projected derivatives of it. The first such descendant, the secondary Weyl tensor, is a traceless type- $[k+1, q+1]$  component of  $R_{\mathbf{q}+1}^{a[k+1]}$ . All other types of tensors appearing as first derivative of the primary Weyl tensor or the torsion must be set to zero by imposing equations of motion. These are given by a tensor  $G^{a[k],b[q]}$  that has the same  $\mathfrak{gl}_d$ -symmetry and trace properties as  $\varphi^{a[k],b[q]}$ . As a differential expression,  $G^{a[k],b[q]}[\varphi]$  defines a second-order operator starting like  $\square\varphi^{a[k],b[q]} + \dots$ . It represents the equations of motion. Actually, one apparently still needs to impose  $T = 0$  to get an irreducible module.<sup>6</sup> Fortunately, for a two-column spin- $[k, q]$  field the problem is cured by noting that  $D_m G^{a[k-1]m,b[q]} \equiv \Lambda T^{a[k-1],b[q]}$ . Hence, once the second-order differential equations  $G[\varphi] = 0$  are imposed,  $T = 0$  becomes a consequence of it.<sup>7</sup>

To summarize, the necessary equations of motion that leave the primary Weyl tensor and its descendant free read [29, 30, 60, 65]

$$R_{\mathbf{q}+1}^{a[k]} = h_m \dots h_m C^{a[k],m[q+1]}, \quad R_{\mathbf{q}+1}^{a[k+1]} = h_m \dots h_m C^{a[k+1],m[q+1]} \quad (2.38)$$

where the zero-form  $C^{a[k],m[q+1]}$  is the primary Weyl tensor, while its descendant, the secondary Weyl tensor  $C^{u[k+1],m[q+1]}$ , is given by a derivative of the primary Weyl tensor as can be seen from (2.34)–(2.35). Both zero forms are irreducible Lorentz tensors. Equivalently, one can use manifestly  $\mathfrak{so}(d-1, 2)$ -covariant form of equations

$$R_{\mathbf{q}+1}^{U[k+1]} = H_M \dots H_M C^{U[k+1],M[q+1]}, \quad C^{U[k+1],M[q]B} V_B \equiv 0 \quad (2.39)$$

where the  $\mathfrak{so}(d-1, 2)$ -irreducible zero-form  $C^{U[k+1],M[q+1]}$ , subject to *non*-complete  $V$ -transversality condition, contains two Lorentz components corresponding to the primary and secondary Weyl tensors  $C^{a[k],m[q+1]}$  and  $C^{a[k+1],m[q+1]}$ .

**Action.** The next problem is to find an action yielding  $G^{a[k],b[q]}[\varphi] = 0$ . We will see that the use of the generalized gauge connections together with the rules of exterior differential algebra leave almost no freedom here as compared to the metric-like tensor  $\varphi^{a[k],b[q]}$  for which one can write many terms with two derivatives as an Ansatz for the Lagrangian.

Let us define the following two volume forms in  $AdS_d$ - and Lorentz-covariant formulations:

$$\text{vol}_{U[k]} = \epsilon_{U_1 \dots U_k B_{k+1} \dots B_d W} H^{B_{k+1}} \wedge \dots \wedge H^{B_d} V^W, \quad \text{vol}_{u[k]} = \epsilon_{u_1 \dots u_k b_{k+1} \dots b_d} h^{b_{k+1}} \wedge \dots \wedge h^{b_d}.$$

---

<sup>6</sup>The constraint  $T = 0$  is the price one has to pay in AdS because one of the two gauge symmetries of massless spin- $[k, q]$  field in Minkowski space gets broken in  $AdS_d$ .

<sup>7</sup>This is similar to  $\partial^\mu A_\mu = 0$  that appears as a consequence of the Proca field equation  $\partial^\mu (\square A_\mu - \partial_\mu \partial^\nu A_\nu + m^2 A_\mu) = 0$ . Let us note that the case of fields with type  $[k, q, q, \dots, q]$ ,  $k \geq q$ , is degenerate. In general, one cannot achieve  $T = 0$  by acting on  $G = 0$  with derivatives, so that  $T = 0$  is an independent equation that must be imposed.

The volume form  $\text{vol}_{u^{[k]}}$  obeys the identity

$$h^c \wedge \text{vol}_{u_1 \dots u_k} = \frac{(-1)^k}{(d-k+1)} \sum_{i=1}^{i=k} (-1)^i \delta_{u_i}^c \text{vol}_{u_1 \dots \hat{u}_i \dots u_k}, \quad (2.40)$$

and a similar identity for  $\text{vol}_{U^{[k]}}$ .

The most general parity-even, manifestly gauge-invariant Ansatz for the action reads [65]

$$S = \frac{1}{\lambda^2} \int \left( a_1 R^{u[q+1]a[k-q-1]} \wedge R^{u[q+1]}_{a[k-q-1]} + a_2 R^{u[q+1]a[k-q]} \wedge R^{u[q+1]}_{a[k-q]} \right) \wedge \text{vol}_{u[2q+2]} + \frac{\alpha}{\lambda^2} \int D \left( R^{u[q]a[k-q]} \wedge R^{u[q+1]}_{a[k-q]} \wedge \text{vol}_{u[2q+1]} \right), \quad (2.41)$$

where the second line includes a boundary term. In manifestly  $AdS_d$ -covariant terms:

$$S = \frac{1}{\lambda^2} \int \left( \alpha_1 R^{U[q+1]A[k-q-1]C} \wedge R^{U[q+1]}_{A[k-q-1]}{}^C V_C V_C + \alpha_2 R^{U[q+1]A[k-q]} \wedge R^{U[q+1]}_{A[k-q]} \right) \wedge \text{vol}_{U[2q+2]} + \frac{\alpha}{\lambda^2} \int D_0 \left( R^{U[q]A[k-q]C} V_C \wedge R^{U[q+1]}_{A[k-q]} \wedge \text{vol}_{U[2q+1]} \right). \quad (2.42)$$

It consists of two terms and one boundary term [65] that can be used to adjust the ratio  $\frac{\alpha_1}{\alpha_2}$  at will. As we will see, in order to switch on the gravitational interactions, the ratio  $\frac{\alpha_1}{\alpha_2}$  has to be fixed in the right way.

It appears that  $d \geq 2q + 2$  must be true in order for the action to make sense. For  $d$  odd it automatically follows that there is no action for a spin- $[k, k]$  field with  $k = \frac{d-1}{2}$ . The rectangular spin diagrams with height equal to the rank of  $\mathfrak{so}(d-1)$ , the Wigner little algebra in  $AdS_d$ , correspond to the higher-spin singletons (or doubletons for  $d = 5$ ), see e.g. [69, 70] for some recent works. A singleton is an irreducible representation of  $\mathfrak{so}(d-1, 2)$  that is too short to possess any bulk degrees of freedom in  $AdS_d$ . It is not possible to write the bulk action for singletons, which is manifested in  $d \geq 2q + 2$ .

The Lagrangian equations of motion

$$R^{u[q]a[k-q]} \wedge \delta \omega^{u[q+1]}_{a[k-q]} \wedge \text{vol}_{u[2q+1]} = 0 \quad (2.43)$$

$$R^{u[q+1]a[k-q]} \wedge \delta e^{u[q]}_{a[k-q]} \wedge \text{vol}_{u[2q+1]} = 0 \quad (2.44)$$

express  $\omega$  as  $D\varphi$  and then impose  $G^{a[k], b[q]}[\varphi] = 0$ , as required.

### 3 Fradkin-Vasiliev cubic interactions

In this section we first review the Fradkin-Vasiliev procedure [2] (see also [42, 64, 71]), putting emphasis on the situation where a higher-spin algebra  $\mathfrak{g}$  containing  $\mathfrak{so}(d-1, 2)$  and decomposable under  $\mathfrak{so}(d-1, 2)$  is given from the outset.

Given a quadratic (free) action  $S_0 = S_0[\varphi]$  invariant under abelian gauge transformations  $\delta\varphi = \delta_0\varphi$ , the general perturbative procedure for finding cubic and higher interaction vertices leads to considering a formal expansion in powers of some coupling constant  $g$

$$S = S_0 + g S_1 + \mathcal{O}(g^2), \quad \delta\varphi = \delta_0\varphi + g \delta_1\varphi + \mathcal{O}(g^2), \quad (3.1)$$

and look order by order in powers of  $g$  for solutions of

$$0 = \delta S = \delta_0 S_0 + g (\delta_0 S_1 + \delta_1 S_0) + \mathcal{O}(g^2) , \quad (3.2)$$

where the order- $g^0$  terms vanish by gauge invariance of the quadratic action  $S_0$ . The cubic interaction vertices are governed by terms of order  $g$ ,

$$0 = \delta_0 S_1 + \delta_1 S_0 = \delta_1 \varphi \frac{\delta S_0}{\delta \varphi} + \delta_0 \varphi \frac{\delta S_1}{\delta \varphi} . \quad (3.3)$$

Taking into account that  $\frac{\delta S_0}{\delta \varphi}$  are the linear equations of motion, the problem of cubic interactions is reduced to finding  $S_1$  such that its gauge variation under abelian transformation  $\delta_0 \varphi$  vanishes on-mass-shell,

$$\left. \frac{\delta S_1}{\delta \varphi} \delta_0 \varphi \right|_{\frac{\delta S_0}{\delta \varphi} = 0} = 0 , \quad (3.4)$$

i.e. is proportional to  $\frac{\delta S_0}{\delta \varphi}$ . Having achieved this,  $\delta_1 \varphi$  can be extracted by inspection of the terms  $\delta_1 \varphi \frac{\delta S_0}{\delta \varphi}$  in (3.3). Now this general consideration will be specified to higher-spin theories following the pioneering work [2].

Suppose there is a set of generalized one-form connections  $\omega = \{W_1^k\}$ ,  $k = 1, \dots$ , collectively denoted by  $\omega$ , with the  $\mathfrak{so}(d-1, 2)$ -connection  $W_1^{A,B}$  belonging to the set. Suppose also that there is an associative algebra structure  $\mathfrak{g}$  (with product denoted by  $\star$  in the sequel) on the set  $\{W^k\}$  such that  $\mathfrak{so}(d-1, 2) \subset \mathfrak{g}$  acts on  $\mathfrak{g}$  via the adjoint action. Then  $\mathfrak{g}$  may be called a higher-spin algebra [49, 50]. There is a well-defined linear gauge theory (2.12)–(2.13) on individual components  $W_1^k$  of  $\omega$ , i.e. the spectrum of  $W_1^k$ 's is given by decomposing  $\omega$  of  $\mathfrak{g}$  into  $\mathfrak{so}(d-1, 2)$ -submodules  $W^k$ . This linear gauge theory can be understood as a linearization of  $R = d\omega + \omega \star \omega$ ,  $\delta\omega = d\epsilon + [\omega, \epsilon]_\star$  over the anti-de Sitter background defined by  $\Omega^{A,B}$ , (2.11):

$$R = R_0 + gR_1 = D_0\omega + g\omega \star \omega , \quad (3.5)$$

$$\delta\omega = \delta_0\omega + g\delta_1\omega = D_0\epsilon + g[\omega, \epsilon]_\star , \quad (3.6)$$

$$\delta R = \delta_0 R_0 + g(\delta_1 R_0 + \delta_1 R_1) + g^2 \delta_1 R_1 = 0 + g[R_0, \epsilon]_\star + g^2 [R_1, \epsilon]_\star . \quad (3.7)$$

In terms of individual components one has

$$R^k = D_0 W^k + g \sum_{m,n} W^m \star W^n , \quad (3.8)$$

$$\delta W^k = D_0 \epsilon^k + g \sum_{m,n} [W^m, \epsilon^n]_\star , \quad (3.9)$$

$$\delta R^k = g \sum_{m,n} [R_0^m, \epsilon^n]_\star + \mathcal{O}(g^2) . \quad (3.10)$$

One then tries to find a quadratic action  $S_0$  for individual fields  $W^k \in \omega$ , bilinear in the linearized curvatures  $R_0^k$  ( $H$  is the background tetrad  $H^A$ ):

$$S_0 = \sum_k \alpha_k S_0^k , \quad S_0^k[\omega, V] = \frac{1}{2} \int_{M_d} \left( R_0^k \wedge R_0^k \right)^{ABCD} \wedge \text{vol}_{ABCD} , \quad (3.11)$$

where  $(R_0^k \wedge R_0^k)^{[ABCD]}$  means that all but four indices carried by curvatures are contracted among themselves or with a number of compensators  $V^A$ . In general there may be a lot of such nontrivial contractions that can contribute, e.g. two in (2.42). It is the problem of constructing quadratic action to determine all free coefficients inside  $S_0^k$  up to an overall factor  $\alpha_k$ . The indices  $ABCD$  appear implicitly antisymmetrized because they are contracted with  $\text{vol}_{ABCD}$ . The integrand is a  $d$ -form as is required, this is why  $\text{vol}_{ABCD}$  has to carry four free indices. Note that according to table 1 none of the nonunitary fields can be given a Lagrangian as they are described by gauge connections that do not have enough antisymmetric indices as compared to the form degree. Hence the FV procedure cannot be applied directly to nonunitary type-[p,q] fields.

The action  $S_0^k$  is manifestly gauge invariant. It is a generalization of the Stelle-West action [52]. The coefficients  $\alpha_k$  account for possible different choices of normalization for each individual quadratic action entering the sum. The action  $S_0^k$  for a spin- $s$  field was found in [64]. For two-column gauge fields, quadratic actions were elaborated along those lines in [20, 65].

The idea of Fradkin and Vasiliev [2] was to use the same Ansatz for a cubic action, i.e. to replace  $R_0$  with the full Yang-Mills-like curvature  $R$  and consider the action modulo terms of order higher than cubic<sup>8</sup>

$$S[\omega, V] = \frac{1}{2} \sum_k \alpha_k \int_{M_d} (R^k \wedge R^k)^{ABCD} \wedge \text{vol}_{ABCD} + \mathcal{O}(\omega^4). \quad (3.12)$$

Firstly, in order to have a nonabelian gauge algebra, we make appear in the transformation laws  $\delta_1 \varphi$  a part denoted  $\delta_1^{hs} \varphi$  associated with the algebra  $\mathfrak{g}$ , plus some extra contribution  $\delta_1^{ex}$  such that  $\delta_1 \varphi = \delta_1^{hs} \varphi + \delta_1^{ex} \varphi$ . As a result (3.4) reads

$$\left( \delta_1^{hs} \varphi \frac{\delta S_0}{\delta \varphi} + \delta_0 \varphi \frac{\delta S_1}{\delta \varphi} \right) \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = 0 \quad (3.13)$$

and it is  $\delta_1^{ex} \varphi$  that has to be extracted once the solution to (3.13) is found. The most complicated work is to extract  $\delta_1^{ex} \varphi$ . Fortunately, whatever  $\delta_1^{ex} \varphi$  is, the vertex is constructed once (3.13) is solved for  $S_1$ , so in practice one does not need to struggle with finding  $\delta_1^{ex} \varphi$  if one is only interested in the vertex and not in the complete expression for the gauge transformations. For example, in the case of pure gravity the diffeomorphism  $\delta_\xi$  along the vector field  $\xi = \xi^\mu \partial_\mu$  is equivalent to a combination of gauge transformations with  $\epsilon^a = \xi^\nu e_\nu^a$ ,  $\epsilon^{a,b} = \xi^\nu \omega_\nu^{a,b}$  and a curvature-dependent term, e.g.

$$\delta_\xi e_\mu^a = \delta_\epsilon e_\mu^a - \xi^\nu (de^a + \omega^a{}_b \wedge e^b)_{\mu\nu} \quad \delta_\epsilon e^a = de^a - \epsilon^a{}_b e^b + \omega^a{}_b \epsilon^b. \quad (3.14)$$

The first term of the first equation represents  $\delta_1^{hs}$  while the second one corresponds to  $\delta_1^{ex}$  and does not have a nice form in general. For higher-spin fields  $\delta_1^{ex}$  is much more complicated [2].

---

<sup>8</sup>It might seem that background tetrads  $H^A$  that are hidden in  $\text{vol}_{ABCD}$  has to be replaced with dynamical ones. Fortunately [2], the gauge variation of such terms can always be compensated. Therefore, it is everywhere implied that  $\text{vol}_{ABCD}$  is a volume element with respect to the background.

Having eq. (3.10) in mind, the gauge variation of (3.12) under  $\mathfrak{g}$  is easy to evaluate:

$$\delta^{\mathfrak{g}} S = \delta_0 S_1 + \delta_1^{hs} S_0 = g \sum_k \alpha_k \sum_{m,n} \int \left( R_0^k \wedge [R_0^m, \epsilon^n]_{\star} \right)^{ABCD} \wedge \text{vol}_{ABCD} + \mathcal{O}(\omega^3 \epsilon). \quad (3.15)$$

To find cubic interactions among elementary fields sitting in  $\omega$  one needs to adjust  $\alpha_k$  such that (3.15) vanishes on free shell, up to terms  $\mathcal{O}(\omega^3 \epsilon)$ . A simplification results from the central on-mass-shell theorem, originally formulated for  $4d$  higher-spin fields in [72]. It turns out that almost all components of  $R_0$  are zero on free mass-shell except for some of them parameterized by the primary Weyl tensors (and certain of its descendants for mixed-symmetry fields), i.e.

$$R_0^k \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = H \wedge \dots \wedge H C^k, \quad (3.16)$$

where the generalized Weyl tensors collectively denoted by  $C$  carry some indices that are contracted with a number of background vielbeins to match the symmetry type and form degree on both sides. The explicit examples are presented in (2.18), (2.20)–(2.21), (2.38) and (2.39).

Having replaced  $R_0^k$  with Weyl tensors on account of the central on-mass-shell theorem (3.16), one can use the identity (2.40) that basically gives

$$H^U H^U H^U H^U \text{vol}_{AAAA} \sim \delta_A^{[U} \delta_A^U \delta_A^U \delta_A^U] \text{vol}. \quad (3.17)$$

Together with the algebraic properties of the Weyl tensors this results in

$$\delta S \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = g \sum_k \frac{\alpha_k}{\beta_k} \sum_{m,n} \int_{M_d} \text{tr} (C^k \star [C^m, \epsilon^n]_{\star}) \wedge \text{vol} \quad (3.18)$$

where  $\text{tr}(\dots)$  means a projection to a singlet component, i.e. just a total contraction of all indices (up to some factor), which is unique. The extra coefficient  $\beta_k^{-1}$  originates from using Young symmetry properties to rearrange indices carried by  $C^k$  and from the fact that there can be a nontrivial normalization for  $\text{tr}(C^k \star C^k)$  that depends on  $k$ ;  $\text{vol}$  is a  $d$ -volume form.

Later, we will argue that any higher-spin algebra admits a natural trace operation. Then by making the choice  $\alpha_k = \beta_k$ , one can make appear traces of commutators using

$$\text{tr} \left( C^k \star [C^m, \epsilon^n]_{\star} + C^m \star [C^k, \epsilon^n]_{\star} \right) \equiv -\text{tr} \left( [C^m \star \epsilon^n, C^k]_{\star} + [C^k \star \epsilon^n, C^m]_{\star} \right), \quad (3.19)$$

so that this expression identically vanishes by the definition of the trace. Hence

$$\delta S \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = -g \sum_{k,n,m} \int \text{tr}([C^k \star \epsilon^n, C^m]_{\star}) \wedge \text{vol} + \mathcal{O}(\omega^3 \epsilon). \quad (3.20)$$

Note that the action itself cannot be written in a trace-like form  $\text{tr}(R \wedge R)$ , the latter action being topological. Fortunately, when taken on free mass-shell, the gauge variation of the action *can* be written in terms of the trace on the algebra  $\mathfrak{g}$ . Also note that one



has to consider the individual components  $S^k$  in order to compute the factors  $\beta_k$ , and it is important that  $\beta_k$  depends on  $k$  only and not on  $n$  and  $m$  appearing in  $\delta R^k$  via (3.10). The latter property is expected to hold true for any action of type (3.11).

Let us also note that an action for a spin-one field  $S_{YM} = -\frac{1}{4} \int \text{tr}(F_{\mu\nu})^2$  cannot be written in the form (3.11) even if a spin-one field belongs to the spectrum of the higher-spin algebra. Nevertheless, as was pointed out in the original paper [2], one can add  $S_{YM}$  to (3.11), where  $F_{\mu\nu}$  is a projection of the full curvature (3.5) to the spin-one sector, to get a cubic action that includes vertices with spin-one. We will not emphasize this subtlety in the sequel.

To conclude this section, once a higher-spin algebra is found it leads to an action consistent up to the cubic order, and one still has to find the free quadratic action (3.11) and compute  $\beta_k$ .

#### 4 Gravitational interactions

In this section we would like to test gravitational interactions for the simplest case of spin- $[k, 1]$  gauge fields, i.e. we are interested in  $[k, 1] - [k, 1] - [1, 1]$  cubic vertices. This is a direct generalization of the case  $k = 2$  presented in [44].

We introduce the following set of one-form gauge fields  $\{e^a, \omega^{ab}, e^{a[k]}, \omega^{a[k+1]}\}$  where  $\{e^a, \omega^{ab}\}$  are the dynamical one-form gauge fields in the spin-2 sector. As recalled in section 2, the two fields  $\{e^{a[k]}, \omega^{a[k+1]}\}$  correspond to the one-forms needed to describe an irreducible and unitary  $[k, 1]$ -type gauge field in  $AdS_d$ .

Quadratic corrections  $R_1$  to curvatures (2.25)–(2.26) are made by replacing background tetrad  $h^a$  and Lorentz spin-connection  $\varpi^{a,b}$  with  $h^a + e^a$  and  $\varpi^{a,b} + \omega^{a,b}$ , respectively. Quadratic contributions to the torsion and Riemann curvature are determined from the most general Ansatz by requiring curvatures to be gauge invariant up to order  $g$ . Denoting the total vielbein  $\mathbf{e}^a = h^a + e^a$ , the result is

$$T^a = D e^a + \mathbf{e}_b \omega^{ba} - g \omega^{ab[k]} e_{b[k]}, \quad (4.1)$$

$$R^{ab} = D \omega^{ab} + \omega^a_c \omega^{cb} - \Lambda \mathbf{e}^a \mathbf{e}^b - k \Lambda g e^{ac[k-1]} e^b_{c[k-1]} - g \omega^{ac[k]} \omega^b_{c[k]}, \quad (4.2)$$

$$T^{a[k]} = D e^{a[k]} - \mathbf{e}_b \omega^{ba[k]} + \omega^a_b e^{ba[k-1]}, \quad (4.3)$$

$$R^{a[k+1]} = D \omega^{a[k+1]} + \Lambda \mathbf{e}^a e^{a[k]} + \omega^a_b \omega^{ba[k]}. \quad (4.4)$$

The Yang-Mills-like gauge transformation are

$$\delta e^a = d\xi^a + \xi_b \omega^{ab} - e_b \xi^{ab} + g \eta^{ab[k]} e_{b[k]} - g \omega^{ab[k]} \eta_{b[k]}, \quad (4.5)$$

$$\begin{aligned} \delta \omega^{ab} = d\xi^{ab} - \xi^a_c \omega^{cb} + \omega^{ac} \xi^b_c + \Lambda (\xi^a e^b - e^a \xi^b) + k \Lambda g \eta^{ac[k-1]} e^b_{c[k-1]} \\ - k \Lambda g e^{ac[k-1]} \eta^b_{c[k-1]} + g \eta^{ac[k]} \omega^b_{c[k]} - g \omega^{ac[k]} \eta^b_{c[k]}, \end{aligned} \quad (4.6)$$

$$\delta e^{a[k]} = d\eta^{a[k]} + \xi_b \omega^{ba[k]} - e_b \eta^{ba[k]} - \xi^a_b e^{ba[k-1]} + \omega^a_b \eta^{ba[k-1]}, \quad (4.7)$$

$$\delta \omega^{a[k+1]} = d\eta^{a[k+1]} - \Lambda \xi^a e^{a[k]} + \Lambda T^a \eta^{a[k]} - \xi^a_b \omega^{ba[k]} + \omega^a_b \eta^{ba[k]}, \quad (4.8)$$

and accordingly, for the curvatures:

$$\delta T^a = \xi_b R^{ab} - T_b \xi^{ab} + g \eta^{ab[k]} R_{b[k]} - g R^{ab[k]} \eta_{b[k]} , \quad (4.9)$$

$$\begin{aligned} \delta R^{ab} = & -\xi^a_c R^{cb} + R^{ac} \xi_c^b + \Lambda (\xi^a T^b - T^a \xi^b) + k \Lambda g \eta^{ac[k-1]} R^b_{c[k-1]} \\ & - k \Lambda g R^{ac[k-1]} \eta^b_{c[k-1]} + g \eta^{ac[k]} R^b_{c[k]} - g R^{ac[k]} \eta^b_{c[k]} , \end{aligned} \quad (4.10)$$

$$\delta T^{a[k]} = \xi_b R^{ba[k]} - T_b \eta^{ba[k]} - \xi^a_b T^{ba[k-1]} + R^a_b \eta^{ba[k-1]} , \quad (4.11)$$

$$\delta R^{a[k+1]} = -\Lambda \xi^a T^{a[k]} + \Lambda T^a \eta^{a[k]} - \xi^a_b R^{ba[k]} + R^a_b R^{ba[k]} . \quad (4.12)$$

The on-mass-shell linearized conditions for a free  $[k, 1]$ -type fields read (2.38)

$$\mathcal{T}^{a[k]} = h_b h_b \mathcal{C}^{a[k], b[2]} , \quad \mathcal{R}^{a[k+1]} = h_b h_b \mathcal{C}^{a[k+1], b[2]} , \quad (4.13)$$

while the spin-2 sector ( $k = 1$ ) gives the constraints

$$\mathcal{T}^a = 0 , \quad \mathcal{R}^{a[2]} = h_b h_b \mathcal{W}^{aa, bb} , \quad (4.14)$$

where the linearized quantities are indicated by calligraphic symbols.

The  $\mathfrak{so}(d-1, 1)$ -tensors  $\{\mathcal{C}^{a[k], b[2]}, \mathcal{C}^{a[k+1], b[2]}, \mathcal{W}^{a[2], b[2]}\}$  are irreducible tensors of symmetry type  $[k, 2]$ ,  $[k+1, 2]$  and  $[2, 2]$ , respectively.

We take the following Ansatz (2.41) for the action, dropping the boundary term

$$S = \frac{1}{2} \int (R^{uu} R^{vv} + a_1 T^{uu a[k-2]} T^{vv}_{a[k-2]} + a_2 \int R^{uu a[k-1]} R^{vv}_{a[k-1]}) \wedge \text{vol}_{uuuv} \quad (4.15)$$

where it is understood that the quartic terms are neglected at this order in perturbation. The variation of the above action can be evaluated using (4.9)–(4.12), keeping only terms bilinear in the fields and linear in the gauge parameter. In other words, after taking the gauge variation inside the action, the curvatures are replaced by their linearized expressions that are then constrained according to (4.13) and (4.14).

Denoting

$$\mathbf{A} = \int \text{vol} \mathcal{C}_{uu, vv} \eta^v_{a[k-1]} \mathcal{W}^{va[k-1], uu} , \quad (4.16)$$

$$\mathbf{B} = \int \text{vol} \mathcal{C}_{uu, vv} \eta^v_{a[k]} \mathcal{W}^{va[k], uu} , \quad (4.17)$$

$$\mathbf{C} = \int \text{vol} \mathcal{W}_{a[k], mn} \xi_c \mathcal{W}^{ca[k], mn} , \quad (4.18)$$

the Fradkin-Vasiliev consistency condition gives the following constraint on the free parameters entering the action  $S[e^a, \omega^{ab}, e^{a[k]}, \omega^{a[k+1]}]$ :

$$\left[ k \Lambda g - \frac{a_1}{(k-1)} \right] \mathbf{A} + \left[ g - \frac{a_2}{k} \right] \mathbf{B} + \left[ \frac{a_1}{k(k-1)} - \frac{a_2 \Lambda}{k} \right] \mathbf{C} = 0 . \quad (4.19)$$

This admits the solution

$$a_1 = g k(k-1) \Lambda , \quad a_2 = g k . \quad (4.20)$$

Since the ratio  $\frac{a_1}{a_2}$  is completely fixed by the consistency of the action (4.15), one must set  $\alpha = 0$  in the action (2.41).

Thus, a natural requirement to include cubic interactions with gravity, gives us a less general Ansatz for action, in fact only one term is possible in  $\mathfrak{so}(d-1, 2)$ -covariant language:

$$S_0^\ell = \frac{\alpha\ell}{2} \int R^{UU A[\ell-2]} \wedge R^{VV}{}_{A[\ell-2]} \wedge \text{vol}_{UVV} \quad (4.21)$$

and the choice for compensator  $V^A = |\Lambda|^{1/2} \delta_{d+1}^A$  gives exactly the same value  $(k-1)\Lambda$  for the ratio  $\frac{a_1}{a_2}$ , (4.20), that is required by consistency of cubic interactions.

The second term possible, in which  $R$  is contracted with  $V$ , turns out to be forbidden. For the case of general mixed-symmetry fields, it is still easy to see that all terms in the action that (i) are contracted with a number of  $V$ ; (ii) to which generalized Weyl tensors contribute; must vanish. The reason is in that such terms must cancel with analogous terms coming from the spin-two action, i.e. from projections of  $\omega \star \omega$  to  $R^{A,B}$ . In the latter Weyl tensors contribute via  $\star$ -product that does not depend on  $V$ . Hence the cancelation takes place only for the terms with no  $V$ -contractions. Therefore, switching on gravitational interactions reduces the freedom to add boundary terms.

## 5 Cubic interactions: AdS-covariant formulation

One of the simplest candidate higher-spin algebras one can consider is the Clifford algebra  $\mathcal{C}\ell_{d-1,2}$  for  $\mathfrak{so}(d-1, 2)$ , which can be realized as an algebra of anticommuting symbol variables  $\phi^A$  with the Clifford  $\star$ -product on functions of  $\phi^A$  instead of usual Grassmann multiplication. The star product between two functions  $F$  and  $G$  of  $\phi$  can be realized by

$$F(\phi) \star G(\phi) = F(\phi) \exp\left(\frac{\overleftarrow{\partial}_r}{\partial\phi^C} \eta^{CD} \frac{\overrightarrow{\partial}_l}{\partial\phi^D}\right) G(\phi) = F(\phi) \exp[\overleftarrow{\partial}_r \cdot \overrightarrow{\partial}_l] G(\phi). \quad (5.1)$$

It features left ( $\partial_l$ ) and right ( $\partial_r$ ) derivatives with respect to the Grassmann-odd variables  $\{\phi^A\}$  and the arrows on the derivatives indicate on which function they act. This  $\star$ -product corresponds to Weyl ordering of the symbols  $F(\phi)$  and  $G(\phi)$ . In particular, it leads to

$$\{\phi^A, \phi^B\}_\star = 2\eta^{AB}, \quad \phi^A \star \phi^B = \phi^A \phi^B + \eta^{AB}, \quad (5.2)$$

$$\phi^A \star F(\phi) = \left(\phi^A + \frac{\overrightarrow{\partial}_l}{\partial\phi_A}\right) F(\phi), \quad F(\phi) \star \phi^A = F(\phi) \left(\phi^A + \frac{\overleftarrow{\partial}_r}{\partial\phi_A}\right). \quad (5.3)$$

The trace is a projection to the singlet,  $\phi$ -independent, component, i.e.  $\text{tr}(F(\phi)) = F(0)$ .

We would like to consider one-forms  $W_1(\phi)$ , whose expansion coefficients  $W^{A[h]}$  in

$$W_1(\phi) = \sum_h \frac{1}{h!} W_1^{A[h]} \phi_{A_1} \dots \phi_{A_h} \quad (5.4)$$

are one-forms that take their values in totally antisymmetric tensor  $\mathfrak{so}(d-1, 2)$ -modules and hence unify all spin- $[k, 1]$  fields according to section 2. However, one has to truncate the algebra to even polynomials in  $\phi$  since the gauge field  $W_1^A \phi_A$  is known to describe

a partially-massless graviton [62], which is nonunitary in anti-de Sitter, and we see no other way to truncate away only the linear term in  $\phi$  from  $W_1(\phi)$  while maintaining the associative algebra structure. We thus restrict ourselves<sup>9</sup> to the even subalgebra  $\mathcal{C}\ell_{d-1,2}^0$  of  $\mathcal{C}\ell_{d-1,2}$ :

$$W(\phi) = W(-\phi) . \quad (5.5)$$

Therefore, the gauging of  $\mathcal{C}\ell_{d-1,2}^0$  describes fields with spins<sup>10</sup>  $[2h - 1, 1]$ ,  $h = 1, \dots, [(d + 1)/2]$  and a spin-1. The important point, in order to have the spin-2 field in the spectrum, is that  $\mathfrak{so}(d - 1, 2) \subset \mathcal{C}\ell_{d-1,2}^0$ , the generators of  $\mathfrak{so}(d - 1, 2)$  being

$$M_{AB} = \frac{1}{4}[\phi_A, \phi_B]_{\star} . \quad (5.6)$$

**Component form.** Expanding the curvature  $R(\phi) = D_0 W(\phi) + g W(\phi) \wedge \star W(\phi)$  in terms of its Taylor components  $\frac{1}{(2h)!} R^{A[2h]} \phi_{A_1} \dots \phi_{A_{2h}}$ , one arrives at

$$R^{A[2h_1]} = D_0 W^{A[2h_1]} + g \sum_{h_2, h_3} a_{h_2, h_3}^{h_1} W^{A[\mathbf{h}-2h_3]C[\mathbf{h}-2h_1]} \wedge W^{A[\mathbf{h}-2h_2]}_{C[\mathbf{h}-2h_1]} , \quad (5.7)$$

where

$$\mathbf{h} = h_1 + h_2 + h_3 . \quad (5.8)$$

The Clifford  $\star$ -product gives the following expression for the structure coefficients  $a_{h_2, h_3}^{h_1}$ :

$$a_{h_2, h_3}^{h_1} = (-1)^{\mathbf{h}(\mathbf{h}+1)/2+h_1} \frac{(2h_1)!}{\prod_{i=1,2,3} (\mathbf{h} - 2h_i)!} . \quad (5.9)$$

Note that if  $W^{A[2h_i]}$  were taken to be a form of degree  $q_i$  then one would have

$$\begin{aligned} & W_{\mathbf{q}_2}^{A[\mathbf{h}-2h_3]C[\mathbf{h}-2h_1]} \wedge W_{\mathbf{q}_3}^{A[\mathbf{h}-2h_2]}_{C[\mathbf{h}-2h_1]} \\ &= (-1)^{\mathbf{h}^2+q_2q_3} W_{\mathbf{q}_3}^{A[\mathbf{h}-2h_2]C[\mathbf{h}-2h_1]} \wedge W_{\mathbf{q}_2}^{A[\mathbf{h}-2h_3]}_{C[\mathbf{h}-2h_1]} \end{aligned}$$

from which it follows that in certain cases there are accidental zeros in the couplings. If all fields are one-forms then it is easy to see that there is always a nonvanishing contribution of  $\{W^{A[2k]}, \wedge \star W^{A[2k]}\}$  to the graviton curvature  $R^{A[2]}$  and  $\{W^{A[2k]}, \wedge \star W^{A[2]}\}$  to  $R^{A[2k]}$  itself, i.e. all the fields interact with gravity and contribute to the gravitational energy-momentum tensor as they should. For one-forms in general all mutual “two-to-one”  $2h_1 - 2h_2 \rightarrow 2h_3$  couplings are nonzero if and only if  $\mathbf{h}$  is odd and all couplings vanish otherwise, when  $\mathbf{h}$  is even.

<sup>9</sup>This is not enough, however, because of dual descriptions as  $W^{A[k+1]}$  and  $W^{A[d-k]}$  describe the same spin- $[k, 1]$  field, [60]. Thus, both  $(k + 1)$  and  $(d - k)$  must be even, i.e.  $d$  is odd.

<sup>10</sup>There is a natural limitation on the spin due to the spacetime dimension. We remark that a free spin- $[2, 1]$  field in  $AdS_4$  is equivalent to a Fierz-Pauli (or Fronsdal) spin-2 field in  $AdS_4$ .

**Cubic action.** Following the Fradkin-Vasiliev procedure, we replace  $R_0$  in (4.21) with the nonabelian Yang-Mills-like  $R$ 's. The variation of

$$S = \sum_{h=1}^{[(d+1)/2]} \frac{\alpha_h}{2} \int R^{UU A[2h-2]} \wedge R^{VV}_{A[2h-2]} \wedge \text{vol}_{UVV} \quad (5.10)$$

reads

$$\delta S = \sum_{h_1, h_2, h_3} \alpha_{h_1} \delta S_{h_2, h_3}^{h_1} = \sum_{h_1} \alpha_{h_1} \sum_{h_2, h_3} \int R^{UU}_{A[2h_1-2]} [R_0^{2h_2}, \xi^{2h_3}]_{\star}^{VV A[2h_1-2]} \text{vol}_{UVV}. \quad (5.11)$$

Consider then the term  $\delta S_{h_2, h_3}^{h_1}$  of the variation and use the on-mass-shell theorem (2.39):

$$\delta S_{h_2, h_3}^{h_1} = \int H_M H_N C^{UU}_{A[2h_1-2]}, MN [H_P H_Q C^{2h_2, PQ}, \xi^{2h_3}]_{\star}^{VV A[2h_1-2]} \text{vol}_{UVV}, \quad (5.12)$$

where the notation  $C^{2h_2, PQ}$  means that the indices  $A[2h_2]$  of  $C^{A[2h_2], PQ}$  are involved in the  $\star$ -product. The result of  $C^{2h_2} \star \xi^{2h_3}$  has indices  $VVA[2h_1-2]$  and carries indices  $PQ$  that are not affected by the  $\star$ -product. Using (3.17) and the fact that the Weyl tensors are traceless, the indices  $MN$  must be contracted with  $VV$  only, whence  $PQ$  have to be contracted with  $UU$

$$\delta S_{h_2, h_3}^{h_1} \sim \frac{2}{2h_1(2h_1-1)} \int C_{A[2h_1], UU} [C^{2h_2, UU}, \xi^{2h_3}]_{\star}^{A[2h_1]} \text{vol} \quad (5.13)$$

where the prefactor comes from the simple identity

$$C_{A[M-N]U[N], A[N]} \phi^A \dots \phi^A = \frac{(M-N)!N!}{M!} C_{A[M], U[N]} \phi^A \dots \phi^A \quad (5.14)$$

used in the case  $M = 2h_1$ ,  $N = 2$ . In order to pass from the total contraction of indices  $C_{A[2h_1]} B^{A[2h_1]}$  to the trace  $\text{tr}(C \star B) = a_{h_1, h_1}^0 C_{A[2h_1]} B^{A[2h_1]}$  one needs to compensate for  $a_{h_1, h_1}^0 = \frac{(-)^{h_1}}{(2h_1)!}$ , which results in

$$\delta S_{h_2, h_3}^{h_1} \sim 2(-)^{h_1} (2h_1-2)! \int \text{tr} \left( C^{2h_1, UU} \star [C^{2h_3, UU}, \xi^{2h_2}]_{\star} \right) \text{vol}. \quad (5.15)$$

Therefore, taking into account the analysis of section 3, the choice

$$\alpha_h = \frac{(-1)^h}{2(2h-2)!} \quad (5.16)$$

is such that the variation of the cubic action, evaluated on free shell and taken at order  $g$ , can be presented as a trace of commutators, thereby solving the problem:

$$\delta S \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = \sum_{h_1, h_2, h_3} \alpha_{h_1} \delta S_{h_2, h_3}^{h_1} \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = 0 + \mathcal{O}(g^2).$$

Introducing additional Clifford oscillators  $\psi^U$ , which are to be contracted with the second group of indices of Weyl tensors,  $\frac{1}{2(2h_1)!} C^{A[2h_1], UU} \phi_A \dots \phi_A \psi_U \psi_U$ , the variation of the action

becomes a trace on the  $\mathcal{C}\ell_{d-1,2} \times \mathcal{C}\ell_{d-1,2}$ . Actually, to cancel the variation of the action, only the indices carried by gauge potentials must be hidden into the trace, so one can keep the pair  $UU$  in plain view.

Note that there are some terms in the variation that vanish by themselves. These originate from cubic self-interactions. The statement whose applicability spreads far beyond the case of type  $[k, 1]$  fields<sup>11</sup> is that the Fradkin-Vasiliev condition for cubic self-interactions is identically satisfied. The reason is that there is no representation carried by gauge parameter  $\xi$  in the symmetric tensor product of two  $\mathfrak{so}(d-1, 2)$ -modules corresponding to Weyl tensor, e.g. for the case of gravity

$$\text{no singlet in } \left( \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right)_{\text{Sym}} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} \iff C^{AA,BC} C_{AA,B}{}^D \xi_{C,D} \equiv 0. \quad (5.17)$$

It is worth noting that  $\alpha_h$  can be determined from a more simple requirement that the quadratic action can be represented as a trace on free shell. The same  $\alpha_h$  (5.16) ensures that

$$S_0 \Big|_{\frac{\delta S_0}{\delta \varphi} = 0} = \int \text{tr} (C_{UU} \star C^{UU}) \text{ vol} . \quad (5.18)$$

Notice that the quantity  $\alpha_h$  in (5.16) is equal to  $a_{h-1, h-1}^0$  up to a constant factor independent of  $h$ . That suggests writing the cubic action in the form

$$S = \frac{1}{2} \int \text{tr} (R^{UU} \wedge \star R^{VV}) \wedge \text{vol}_{UVV} \quad (5.19)$$

where one still has to take off a pair of oscillators on each curvature to undress two pairs of indices, these are not involved in taking  $\star$ -product.

**Unitarity.** When expressed in terms of metric-like field  $\phi$ , any frame-like action  $\int R \wedge R$  is proportional to  $(-)^{\text{spin}} (\partial\phi)^2$ , where the sign factor depends linearly on spin. This factor is irrelevant for type- $[2h-1, 1]$  fields considered here as they differ by an even number of indices. Thus, in order to make action (5.19) unitary one has to insert imaginary unit to the definition of the star product, which compensates for unwanted  $(-)^h$  in  $\alpha_h$ . Equivalently, one can inherit the reality conditions from the  $\mathfrak{osp}(1|2)$ -algebra [50].

## 6 Developments and discussion

**Extension to higher degree forms.** The previous analysis has been carried over in the case of one-form gauge potentials taking their values in the even Clifford algebra  $\mathcal{C}\ell_{d-1,2}^0$ . At least two problems appear when trying to include forms of higher degree. Firstly, the number of forbidden values for  $k$  in  $W_{\mathfrak{q}}^{A[k]}$  grows with  $q$ . For example, among the one-forms only  $W_1^A$  corresponding to a partially-massless graviton in  $AdS_d$  was forbidden by unitarity. For two-forms, there are two such fields  $W_2^A$  and  $W_2^{AA}$ , etc. It is not evident how to truncate the algebra without breaking the associativity in a way that the star-product of two unitary

---

<sup>11</sup>One can check that cubic self-interactions are always consistent for all gauge fields described by one-form gauge potentials.

fields gives no contribution to the sector of nonunitary fields. Secondly, since higher degree forms cannot contribute to lower degree forms no pairwise cancelation of terms is possible now. Generically, it is difficult to imagine a theory for nonabelian  $p$ -forms with  $p > 1$  since the closure of the gauge algebra would require gauge parameters with form degrees unbounded from above, which is hardly compatible with a finite spacetime dimensionality.

**Extension to higher orders.** As is known [49], the candidate higher-spin algebra  $\mathfrak{g}$  must satisfy the *admissibility condition* in order for interaction of higher than cubic order to exist. The *admissibility condition* demands that gauging  $\mathfrak{g}$ , which describes certain field content, must match with the field content of some unitary representation of  $\mathfrak{g}$ . For example, a little bit tautological though, gauging of  $\mathfrak{so}(d-1, 2)$  itself leads to  $e^a, \omega^{a,b} \in W^{A,B}$  that describes a free spin-two field at the linearized level, the same time there exists a unitary irreducible representation of  $\mathfrak{so}(d-1, 2)$  that is a spin-two field.

The *admissibility condition* becomes highly nontrivial and restrictive for genuine higher-spin algebras. In particular it was found in [72], that certain higher-spin algebras do not give rise to consistent theories beyond the cubic approximation, these defective higher-spin algebras were shown [49, 73] not to meet the *admissibility condition*, i.e. not to have unitary representations with spectrum giving by gauging thereof.

Therefore, the cubic approximation is insensitive to the *admissibility condition*. Nevertheless, we may argue that  $\mathcal{C}\ell_{d-1,2}^0$  does not satisfy the *admissibility condition* because its gauging leads to a too small spectrum, which is much smaller than the one resulting from tensoring the minimal representations  $|Di\rangle, |Rac\rangle$  corresponding to conformal scalar and spinor [50]. It is still interesting to see if the action closes at the quartic level as it happens for pure gravity.

**Universal enveloping realization.** <sup>12</sup> In the spirit of the approach used in [69], let us put some remarks on the universal enveloping algebra  $U(\mathfrak{h})$ ,  $\mathfrak{h} \cong \mathfrak{so}(d-1, 2)$ , and on the explicit realization of  $\mathfrak{g} = \mathcal{C}\ell_{d-1,2}^0$  as a quotient of  $U(\mathfrak{h})$ . Consider  $U(\mathfrak{h})$ , generated by  $M_{AB}$  modulo relations ( $\star$  denotes the product in  $U(\mathfrak{h})$ )

$$[M_{AB}, M_{CD}]_{\star} = M_{AD}\eta_{BC} - M_{BD}\eta_{AC} - M_{AC}\eta_{BD} + M_{BC}\eta_{AD} . \tag{6.1}$$

It is useful to write down the decomposition of the first several levels of  $U(\mathfrak{h})$  in terms of the standard adjoint action of  $\mathfrak{h}$ , which by the Poincare-Birkhoff-Witt theorem is equivalent to computing symmetric products of  $M_{AB} \sim \square$ ,

$$U(\mathfrak{h})|_{\mathfrak{h}} \cong \underbrace{\bullet}_0 \oplus \underbrace{\left(\square\right)}_1 \oplus \underbrace{\left(\begin{array}{cc} \square & \square \\ \square & \square \end{array} \oplus \begin{array}{c} \square \\ \square \\ \square \end{array} \oplus \square \oplus \bullet\right)}_2 \oplus \underbrace{\left(\begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \end{array} \oplus \dots\right)}_3 \oplus \dots \tag{6.2}$$

where the singlet  $\bullet$  at the level zero is the identity of  $U(\mathfrak{h})$  and another one at the level two is the quadratic Casimir operator  $C_2 = -\frac{1}{2}M_{AB} \star M^{AB}$ . All singlets in  $U(\mathfrak{h})$  are by

---

<sup>12</sup>E.S. is grateful to Per Sundell for many valuable discussions on [69] and to M.A.Vasiliev for sharing his draft.

definition certain functions of the Casimir operators  $C_{2i}$ ,  $i = 1, \dots, N$ ,  $N = \lfloor \frac{d+1}{2} \rfloor$ . The center of  $U(\mathfrak{h})$  is a free field  $K[C_2, \dots, C_{2N}]$  in  $N$  variables.

According to [69] a higher-spin algebra  $\mathfrak{g}$  can be constructed as a quotient algebra of  $U(\mathfrak{h})$  over a given two-sided ideal. The ideal corresponding to the Vasiliev  $sp(2)$  higher-spin algebra [5], whose gauging describes all totally-symmetric massless fields, is generated by two  $\mathfrak{h}$ -covariant elements,

$$I \cong U(\mathfrak{h}) \star \left( \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) \star U(\mathfrak{h}) , \tag{6.3}$$

where

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} = M_{[AB} \star M_{CD]}, \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = M_A{}^C \star M_{AC} - \frac{2}{(d+1)} \eta_{AA} C_2 . \tag{6.4}$$

Roughly speaking, to quotient by  $I$  means that all diagrams with more than two rows as well as all the elements  $M \star \dots \star M$  where at least two  $\mathfrak{h}$ -indices are contracted must be set to zero. The resulting  $\mathfrak{h}$ -adjoint spectrum of  $U(\mathfrak{h})/I$  is given by all rectangular two-row diagrams,

$$\mathfrak{g} = U(\mathfrak{h})/I|_{\mathfrak{h}} \cong \bullet \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \oplus \dots \tag{6.5}$$

The salient feature of the ideal  $I$  is that besides removing ‘unwanted’ diagrams it also restricts all Casimirs,  $C_{2i}$ , to particular values  $\mu_{2i}$ . When inside  $U(\mathfrak{h})$ , to determine the values of  $C_{2i}$  one [69] has to verify the consistency of the ideal by multiplying its elements and inspecting if the result belongs to the ideal too. This procedure leads for example to relations of the type  $(C_{2i} - \mu_{2i}) \star M_{AB} \sim 0$ , of which the nontrivial solution is  $C_{2i} - \mu_{2i} \sim 0$ . Note that if  $C_{2i}$  were free it would lead to a degeneracy of the spectrum due to the center  $K[C_2, \dots]$  of  $U(\mathfrak{h})/I$ .

As was noticed in [69, 74], the ideal  $I$  is in fact the annihilator,  $\text{Ann}(|\text{Rac}\rangle)$ , of the remarkable Dirac scalar singleton representation  $|\text{Rac}\rangle$  of  $\mathfrak{h}$ , which fixes all  $C_{2i}$  accordingly. Therefore, quite generally one may think of any higher-spin algebra  $\mathfrak{g}$  as the universal enveloping algebra  $U(\mathfrak{h})$  of  $\mathfrak{h}$  evaluated in some  $\mathfrak{h}$ -module, say  $V$ ,

$$\mathfrak{g} = U(\mathfrak{h})|_V \sim \text{End}(V) \sim V^* \otimes V . \tag{6.6}$$

However, the scalar  $|\text{Rac}\rangle$  and spinor  $|\text{Di}\rangle$  singletons are very distinguished representations. We see no natural way to generalize  $|\text{Rac}\rangle$ ,  $|\text{Di}\rangle$  to some other representation, say  $V$ , such that  $\mathfrak{g} = U(\mathfrak{h})/\text{Ann}(V)$  would contain  $\mathfrak{h}$ -modules  $A_i$ ,  $\mathfrak{g} = \oplus_i A_i$  in its  $\mathfrak{h}$ -adjoint decomposition that by means of  $A_i$ -valued generalized connection  $\omega_{\mathfrak{q}}^{A_i}$  would describe unitary mixed-symmetry fields for some  $q$ . As noted in [30, 70] higher-spin singletons should give examples of higher-spin algebras, whose gauging leads to certain multiplets of mixed-symmetry fields. However, these exist only in odd dimensions and we do not expect that consistent theories with mixed-symmetry fields are confined to odd dimensions. Moreover, the tensor product of two higher-spin singletons with high enough spins does not



contain a graviton. As the most simple example, using [45] one can evaluate the product  $|j_1, 0\rangle \otimes |0, j_2\rangle$  of two  $\mathfrak{so}(4, 2)$  higher-spin singletons with spins  $j_1$  and  $j_2$ , which are also called doubletons, [75],

$$|j_1, 0\rangle \otimes |0, j_2\rangle = \bigoplus_k \left[ \begin{array}{l} \text{massless unitary} \\ \text{mixed-symmetry} \\ \text{fields in } AdS_5 \end{array} \right] \text{ of type } \begin{array}{l} j_1 + j_2 + k \\ j_1 - j_2 \end{array} . \quad (6.7)$$

The sign of  $(j_1 - j_2)$  distinguishes between selfdual and anti-selfdual fields. This result reduces at  $j_1 = j_2 = 0$  to the Flato-Fronsdal-type theorem [76] of [45] that the product of two scalar singletons decomposes into a sum over all totally symmetric bosonic higher-spin fields, see also [50, 77].

Thus the ability of higher-spin singletons to describe a world with gravity and mixed-symmetry fields is very restricted. Therefore, as we have no candidates for the annihilator, we would like to define  $I$  directly by specifying which diagrams are ‘unwanted’.

The adjoint spectrum of  $\mathfrak{g} = \mathcal{C}\ell_{d-1,2}^0$  consists of all  $[2k]$  types with multiplicity one,

$$\mathfrak{g} = \underbrace{\bullet}_0 \oplus \underbrace{\left( \begin{array}{|c|} \hline \square \\ \hline \end{array} \right)}_1 \oplus \underbrace{\left( \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right)}_2 \oplus \dots \quad (6.8)$$

This suggests the ideal be generated by

$$\left( \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \right) \oplus \left( \begin{array}{|c|} \hline \square \\ \hline \end{array} \right) \sim I_{AA, BB} = M_{AB} \star M_{AB} + \frac{2C_2}{d(d+1)} (\eta_{AA}\eta_{BB} - \eta_{AB}\eta_{AB}) \sim 0. \quad (6.9)$$

Indeed, in verifying the compatibility condition  $M^{AB} \star I_{AA, BB} \sim 0$ , which can be done by using the following relation, which holds true modulo terms proportional to  $I_{AA, B}^B$ ,

$$M_A^C \star M_{BC} \sim \frac{-2C_2}{d+1} \eta_{AB} - \frac{d-1}{2} M_{AB}, \quad (6.10)$$

one finds that the Casimir must be a fixed number  $C_2 = \frac{d(d+1)}{8}$ ,

$$0 \sim M^{AB} \star I_{AA, BB} \sim \left( C_2 - \frac{d(d+1)}{8} \right) \star M_{AB} \sim 0, \quad (6.11)$$

which is exactly  $C_2$  computed in the representation (5.6) and is, as expected, equal to the Casimir of the spinor module. Therefore,<sup>13</sup>

$$\mathfrak{g} = \mathbf{U}(\mathfrak{h}) / (\mathbf{U}(\mathfrak{h}) \star I_{AA, BB} \star \mathbf{U}(\mathfrak{h})) . \quad (6.12)$$

The analog of  $|Rac\rangle$  representation for  $\mathcal{C}\ell_{d-1,2}^0$  is the spinor representation, which is finite dimensional in accordance with finiteness of  $\mathcal{C}\ell_{d-1,2}^0$  (the symmetry algebra of a field equation, e.g. conformal scalar, must be an infinite dimensional algebra as it contains arbitrary powers of translation generators).

---

<sup>13</sup>Unfortunately, it is very complicated to inspect all relations that come from  $\mathbf{U}(\mathfrak{h}) \star I \star \mathbf{U}(\mathfrak{h})$  for some  $I$ , in particular, to find out if there are some additional relations at higher levels, e.g. at the level  $[(d+1)/2]$ , which restricts  $d$  to be odd.

In general, we see from (6.2) that any reasonable ideal  $I$  must take away  $\square\square$  at the least,

$$I_1 = (\mathbf{U}(\mathfrak{h}) \star \square\square \star \mathbf{U}(\mathfrak{h})) \quad (6.13)$$

as the generalized connection  $W_{\mathfrak{q}}^{AB} = W_{\mathfrak{q}}^{BA}$  describes a nonunitary theory for any  $q$ , [18, 60]. This requirement already forces higher Casimirs,  $C_{2i}$ ,  $i = 2, \dots$ ,

$$C_{2k} = \frac{1}{2} M_A^B \star M_B^C \star \dots \star M_U^A \quad (6.14)$$

to be certain functions  $F_{2i}(C_2)$  of the quadratic one

$$F_{2i}(X) = \frac{X((\lambda_+)^{2i-1} - (\lambda_-)^{2i-1})}{\lambda_+ - \lambda_-}, \quad \lambda_{\pm} = \frac{d-1}{4} \left( 1 \pm \sqrt{1 - \frac{32}{(d+1)(d-1)^2} X} \right),$$

$$F_2(X) = X, \quad F_4(X) = \frac{X}{4} \left( \frac{8X}{d+1} + (d-1)^2 \right).$$

Therefore, the only ‘degree of freedom’ left if is due to  $C_2$ . In the spirit of Feigin’s  $\mathfrak{gl}(\lambda)$ , [78], which is equivalent to the deformed oscillators of [53, 54], we can quotient further  $(C_2 - \nu)$  and define a one parameter family of higher-spin algebras

$$hs(\nu) = \mathbf{U}(\mathfrak{h}) / (\mathbf{U}(\mathfrak{h}) \star \{\square\square \oplus (C_2 - \nu)\} \star \mathbf{U}(\mathfrak{h})) . \quad (6.15)$$

At certain values of  $\nu$  the algebra  $hs(\nu)$  acquires an ideal that can be factored out, giving a smaller algebra. The value of  $C_2$  would be fixed by choosing one more ‘unwanted’ diagram in the  $\mathfrak{h}$ -adjoint spectrum of  $\mathbf{U}(\mathfrak{h})$ . The  $\mathfrak{h}$ -adjoint decomposition of  $hs(\nu)$  is easy to describe as

$$hs(\nu)|_{\mathfrak{h}} = \{\text{diagrams made out of } \boxtimes\} , \quad (6.16)$$

where the new elementary cell  $\boxtimes$  denotes  $\square$ . Note that by inspecting generalized connections of [18] (that are allowed by unitarity) we conclude that gauging of  $hs(\nu)$  can describe unitary fields. The additional ideal  $I_2$  leading to the Vasiliev  $sp(2)$  higher-spin algebra, which removes the degeneracy due to  $C_2$ , is generated by  $\begin{smallmatrix} \boxtimes \\ \boxtimes \end{smallmatrix}$ .<sup>14</sup> The ideal  $I_2$  leading to  $\mathcal{C}\ell_{d-1,2}^0$  is  $\begin{smallmatrix} \boxtimes & \boxtimes \\ \boxtimes & \boxtimes \end{smallmatrix}$ . The ideal  $I_2$  leading to the  $\phi$ -even Vasiliev  $osp(1|2)$  higher-spin algebra [50] is  $\begin{smallmatrix} \boxtimes & \boxtimes \\ \boxtimes & \boxtimes \end{smallmatrix}$ . Let us note that there is no room here for the hypothetical algebra whose spectrum was suggested in [79] in the context of reducible multiplets of totally-symmetric fields, to read

$$\underbrace{\bullet}_0 \oplus \underbrace{\boxtimes}_1 \oplus \underbrace{(\boxtimes\boxtimes \oplus \bullet)}_2 \oplus \underbrace{(\boxtimes\boxtimes\boxtimes \oplus \boxtimes)}_3 \oplus \underbrace{(\boxtimes\boxtimes\boxtimes\boxtimes \oplus \boxtimes\boxtimes \oplus \bullet)}_4 \oplus \dots . \quad (6.17)$$

The above spectrum may result simply from extending the field of scalars.

<sup>14</sup>Note that by  $\begin{smallmatrix} \boxtimes \\ \boxtimes \end{smallmatrix}$  we do not mean the antisymmetric tensor product of  $\square$  with itself, but simply the Young diagram with four cells in the same column.

Note that factoring out any nontrivial component of  $U(\mathfrak{h})$  at the level- $(k+1)$  expresses higher Casimirs  $C_{2k+2}, \dots$  in terms of lower ones  $C_2, \dots, C_{2k}$ . For example, which is relevant below,  $\begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array} \sim 0$  leads to

$$M_{[UU] \star M_{UU}} \sim 0 \implies M_{[UU] \star \dots \star M_{UU}} \star M_{[VV] \star \dots \star M_{VV}} \eta^{UV} \dots \eta^{UV} \sim 0 \quad (6.18)$$

that allows to express any  $C_{2k}$ ,  $k > 1$  as a function  $G_{2k}(C_2)$  of  $C_2$ , e.g.  $G_4$  is

$$C_4 = G_4(C_2) = (C_2)^2 + \frac{(d-2)(d-1)}{2} C_2. \quad (6.19)$$

It is interesting to find out if there exist ideals generated by more than two diagrams, the natural restrictions coming from the condition that any  $k$ -generated ideal lies in the intersection of  $k$  algebraic functions of Casimirs. For example, the scalar singleton point corresponding to the Vasiliev higher-spin algebra is the unique intersection of  $F_{2i}(X) = G_{2i}(X)$ .

If one factors out only  $\begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array}$  without factoring out  $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$ , the resulting spectrum is rich enough to describe all massive totally-symmetric fields in the spirit of [80]. The degeneracy due to  $C_2$  enlarges the spectrum with Stückelberg companions. Flow with respect to  $C_2$  should pass all critical points where massive fields decompose into partially-massless fields plus massive fields of lower spin.

It seems natural that in addition to  $I_1$  one may pick any  $\begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array}$ -diagram, say  $\mathbf{Y}$ , from the spectrum of  $A_1$  and build the quotient algebra whose spectrum does not contain the  $\begin{array}{|c|} \hline \times \\ \hline \times \\ \hline \end{array}$ -diagrams for which  $\mathbf{Y}$  is a subdiagram. The value of  $\nu$  is to be determined by consistency of  $I_{\mathbf{Y}} = U(\mathfrak{h}) \star \mathbf{Y} \star U(\mathfrak{h})$  with  $I_1$ .

As was mentioned in the introduction,  $\mathcal{C}\ell_{d-1,2}^0$  is just the simplest higher-spin algebra in the hierarchy, which now can be depicted as

$$\mathfrak{so}(d-1, 2) \xrightarrow{\text{finite-dimensional}} \mathcal{C}\ell_{d-1,2}^0 \rightarrow \mathfrak{hs}(\nu_i) \xrightarrow{\text{infinite-dimensional}} \mathfrak{osp}(1|2)_e \rightarrow \mathfrak{hs}(\nu)$$

$\swarrow$   
 $\mathfrak{sp}(2)$

where  $\rightarrow$  means succession of gauging, i.e. the spectrum of fields resulting from gauging one algebra belongs to the gauging of the next one. Together with the Clifford algebra one can construct other finite-dimensional algebras that correspond to  $\mathfrak{hs}(\nu)$  at certain  $\nu_i$  of finite-dimensional modules as well as infinite-dimensional algebras that correspond to  $\mathfrak{hs}(\nu)$  at generic  $\nu$  or specific  $\nu$ 's of infinite-dimensional modules.

That the generators of a higher-spin algebra  $\mathfrak{g}$  obtained from  $U(\mathfrak{h})$  have only even ranks (number of indices of  $\mathfrak{h}$ ) seems to be a drawback of  $U(\mathfrak{h})$  as of any two mixed-symmetry fields whose ranks differ by one index one can be realized as a part of such a  $\mathfrak{g}$  while another one cannot. The possible way out is shown by the  $\mathfrak{osp}(1|2)$ -algebra [50], that is to extend  $U(\mathfrak{h})$  with Clifford algebra. However, in this way it is still not possible to have an algebra whose gauging leads, for example, to two-row mixed-symmetry fields only, as Clifford algebra brings a one-column Young diagram of height up to  $d+1$  that is attached at the bottom.

The above consideration within the universal enveloping algebra is not the most general one. The  $\star$ -product of  $U(\mathfrak{h})$  prescribes a particular way of contracting indices when expanding expressions like  $\omega \star \omega$  in components.

For example, for the Vasiliev  $\mathfrak{sp}(2)$  algebra  $\star$ -product is induced by  $\exp(s)$ , where  $s = t^{\alpha\beta} C_{\alpha\beta}$ ,  $t^{\alpha\beta} = \frac{\overleftarrow{\partial}}{\partial Y^A} \frac{\overrightarrow{\partial}}{\partial Y^B} \eta^{AB}$ . This star product is in fact  $\mathfrak{sp}(2d+2)$  invariant as it makes use of  $\eta^{AB} C_{\alpha\beta}$ . Various other  $\mathfrak{sp}(2) \oplus \mathfrak{so}(d-1, 2) \subset \mathfrak{sp}(2d+2)$  invariants built with  $t^{\alpha\beta}$  can be used to contract indices. The field of  $\mathfrak{sp}(2)$  invariants for  $t^{\alpha\beta}$  is generated by  $s$  and  $p = \det t^{\alpha\beta}$ . An arbitrary nontrivial polynomial  $s^k p^n$  gives rise to consistent cubic interactions at least at the cubic level [81], the  $\star$ -product being different from the one dictated by  $U(\mathfrak{h})$ . This is to be compared with the general formalism of cubic interactions developed in the upcoming paper by Vasiliev [82].

In general one is led to study the full tensor algebra of  $\mathfrak{h}$ . The Grothendieck ring of tensor category of  $\mathfrak{h}$ -modules is an associative ring whose basis  $e_\mu$  is enumerated by all finite-dimensional  $\mathfrak{h}$ -modules  $\mu$  and the structure constants are defined by the decomposition of  $\mu \otimes \nu$  into irreducibles. Inevitably any higher-spin algebra corresponds to a subring of the Grothendieck ring, the additional requirement being that  $\mathfrak{h}$  itself as its adjoint module must belong to the higher-spin algebra. Finding an appropriate truncation of the Grothendieck ring does not solve the problem yet, as one has to choose a particular way to contract indices, which we believe can be done by classifying invariants, like  $s$  and  $p$ , corresponding to the truncation.

Then, the trace operation  $tr$  can be defined as a projection to the singlet component since it is unique.  $tr(AB - BA) = 0$  holds automatically because if  $A$  is isomorphic to  $B$  as  $\mathfrak{h}$  modules then there is no singlet component in  $A \wedge B$ , otherwise  $A \otimes B$  does not contain a singlet component either. Studying subrings of the Grothendieck ring might be useful for finding higher-spin algebras whose gauging leads to a desired spectrum of fields. We hope to come back to this issue in a future work [81].

## 7 Conclusion

We have proven that the Clifford algebra correctly produces not only the structure but all the coefficients that are required for type  $[2k-1, 1]$  fields, to have consistent interactions with gravity and themselves at the cubic order. It is instructive to investigate if the same action can be made consistent up to the quartic order as it does for the sector of pure gravity and if not where the obstructions come from.

We have seen that the Fradkin-Vasiliev approach is a powerful machinery, which allows one to construct cubic vertices for a multiplet of higher-spin fields once the candidate higher-spin algebra  $\mathfrak{g}$  is known. The spectrum of fields is given by the adjoint decomposition of  $\mathfrak{g}$  with respect to the anti-de Sitter algebra, leading to a number of generalized connections of  $\mathfrak{so}(d-1, 2)$ . One requires quadratic actions of type  $\int R_0 \wedge R_0$  as an input. The Fradkin-Vasiliev recipe is to replace the linearized curvatures  $R_0$  with the nonlinear ones  $R$  that are dictated by the algebra and adjust coefficients in front of individual actions to push the gauge invariance to the next nontrivial order. The procedure determines all coupling constants in front of different cubic vertices in terms of just one constant.

Once the quadratic actions built with curvatures  $R_0$  for generalized connections are known one needs to look at those parts of the action to which the generalized Weyl tensors  $C$  contribute via the on-mass-shell theorem. The coefficients are then determined by requiring  $R_0 \wedge R_0$  terms to have the form  $tr(C_{UU} \star C^{UU})$  on-mass-shell, which is reminiscent of Yang-Mills'  $tr(F_{\mu\nu} \star F^{\mu\nu})$ . This gives a combinatoric factor originating from using Young symmetry properties and the normalization of the  $\mathfrak{g}$ -trace; the computation can be done for a free action and is quite simple.

Following [69] we believe that the universal enveloping algebra of  $\mathfrak{so}(d-1, 2)$  is a natural framework for description of higher-spin fields and we have treated  $\mathcal{C}\ell_{d-1,2}^0$  from this point of view. We have discussed some general features of embedding a higher-spin algebra into  $U(\mathfrak{so}(d-1, 2))$ , which results in that any reasonable higher-spin algebra built from  $U(\mathfrak{so}(d-1, 2))$  should correspond to  $hs(\nu)$  at a particular  $\nu$ .

## Acknowledgments

E.S. would like to thank E. Feigin, R. Metsaev, Yu. Zinoviev, M.A. Grigoriev, O.V. Shaynkman, P. Sundell, K.B. Alkalaev, A. Campoleoni and M. A. Vasiliev for many valuable discussions. N.B. thanks F. Buisseret, P. P. Cook, A. Campoleoni, P. Sundell and Yu. Zinoviev for discussions. E.S. acknowledges communications with M. A. Vasiliev and thanks the service de Mécanique et Gravitation at UMONS for hospitality. The work of E.S. was supported in parts by RFBR grant No.11-02-00814 and President grant No.5638. The work of N.B. was supported in parts by an ARC contract No. AUWB-2010-10/15-UMONS-1.

## References

- [1] E.S. Fradkin and M.A. Vasiliev, *On the Gravitational Interaction of Massless Higher Spin Fields*, *Phys. Lett.* **B 189** (1987) 89 [SPIRES].
- [2] E.S. Fradkin and M.A. Vasiliev, *Cubic Interaction in Extended Theories of Massless Higher Spin Fields*, *Nucl. Phys.* **B 291** (1987) 141 [SPIRES].
- [3] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B 243** (1990) 378 [SPIRES].
- [4] M.A. Vasiliev, *More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions*, *Phys. Lett.* **B 285** (1992) 225 [SPIRES].
- [5] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in (A)dS(d)*, *Phys. Lett.* **B 567** (2003) 139 [hep-th/0304049] [SPIRES].
- [6] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, [arXiv:1007.0435](https://arxiv.org/abs/1007.0435) [SPIRES].
- [7] M.A. Vasiliev, *Higher spin symmetries, star-product and relativistic equations in AdS space*, [hep-th/0002183](https://arxiv.org/abs/hep-th/0002183) [SPIRES].
- [8] M.A. Vasiliev, *Higher spin gauge theories in various dimensions*, *Fortsch. Phys.* **52** (2004) 702 [hep-th/0401177] [SPIRES].

- [9] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, [hep-th/0503128](#) [SPIRES].
- [10] D.J. Gross, *High-Energy Symmetries of String Theory*, *Phys. Rev. Lett.* **60** (1988) 1229 [SPIRES].
- [11] D. Polyakov, *Interactions of Massless Higher Spin Fields From String Theory*, *Phys. Rev. D* **82** (2010) 066005 [[arXiv:0910.5338](#)] [SPIRES].
- [12] D. Polyakov, *Gravitational Couplings of Higher Spins from String Theory*, *Int. J. Mod. Phys. A* **25** (2010) 4623 [[arXiv:1005.5512](#)] [SPIRES].
- [13] D. Polyakov, *A String Model for AdS Gravity and Higher Spins*, [arXiv:1106.1558](#) [SPIRES].
- [14] A. Sagnotti and M. Taronna, *String Lessons for Higher-Spin Interactions*, *Nucl. Phys. B* **842** (2011) 299 [[arXiv:1006.5242](#)] [SPIRES].
- [15] X. Bekaert and N. Boulanger, *On geometric equations and duality for free higher spins*, *Phys. Lett. B* **561** (2003) 183 [[hep-th/0301243](#)] [SPIRES].
- [16] P. de Medeiros and C. Hull, *Geometric second order field equations for general tensor gauge fields*, *JHEP* **05** (2003) 019 [[hep-th/0303036](#)] [SPIRES].
- [17] P. de Medeiros, *Massive gauge-invariant field theories on spaces of constant curvature*, *Class. Quant. Grav.* **21** (2004) 2571 [[hep-th/0311254](#)] [SPIRES].
- [18] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *On the frame-like formulation of mixed-symmetry massless fields in (A)dS(d)*, *Nucl. Phys. B* **692** (2004) 363 [[hep-th/0311164](#)] [SPIRES].
- [19] A. Sagnotti and M. Tsulaia, *On higher spins and the tensionless limit of string theory*, *Nucl. Phys. B* **682** (2004) 83 [[hep-th/0311257](#)] [SPIRES].
- [20] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *Lagrangian formulation for free mixed-symmetry bosonic gauge fields in (A)dS(d)*, *JHEP* **08** (2005) 069 [[hep-th/0501108](#)] [SPIRES].
- [21] K.B. Alkalaev, O.V. Shaynkman and M.A. Vasiliev, *Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in AdS(d)*, [hep-th/0601225](#) [SPIRES].
- [22] X. Bekaert and N. Boulanger, *Tensor gauge fields in arbitrary representations of  $GL(D,R)$ . II: Quadratic actions*, *Commun. Math. Phys.* **271** (2007) 723 [[hep-th/0606198](#)] [SPIRES].
- [23] A. Fotopoulos and M. Tsulaia, *Interacting Higher Spins and the High Energy Limit of the Bosonic String*, *Phys. Rev. D* **76** (2007) 025014 [[arXiv:0705.2939](#)] [SPIRES].
- [24] I.L. Buchbinder, V.A. Krykhtin and H. Takata, *Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields*, *Phys. Lett. B* **656** (2007) 253 [[arXiv:0707.2181](#)] [SPIRES].
- [25] A.A. Reshetnyak, *On Lagrangian formulations for mixed-symmetry HS fields on AdS spaces within BFV-BRST approach*, [arXiv:0809.4815](#) [SPIRES].
- [26] E.D. Skvortsov, *Frame-like Actions for Massless Mixed-Symmetry Fields in Minkowski space*, *Nucl. Phys. B* **808** (2009) 569 [[arXiv:0807.0903](#)] [SPIRES].
- [27] Y.M. Zinoviev, *Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields*, *Nucl. Phys. B* **812** (2009) 46 [[arXiv:0809.3287](#)] [SPIRES].

- [28] A. Campoleoni, D. Francia, J. Mourad and A. Sagnotti, *Unconstrained Higher Spins of Mixed Symmetry. I. Bose Fields*, *Nucl. Phys. B* **815** (2009) 289 [[arXiv:0810.4350](#)] [[SPIRES](#)].
- [29] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding Mixed-Symmetry Fields in AdS and the BMV Conjecture: I. General Formalism*, *JHEP* **07** (2009) 013 [[arXiv:0812.3615](#)] [[SPIRES](#)].
- [30] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding Mixed-Symmetry Fields in AdS and the BMV Conjecture: II. Oscillator Realization*, *JHEP* **07** (2009) 014 [[arXiv:0812.4438](#)] [[SPIRES](#)].
- [31] A. Campoleoni, D. Francia, J. Mourad and A. Sagnotti, *Unconstrained Higher Spins of Mixed Symmetry. II. Fermi Fields*, *Nucl. Phys. B* **828** (2010) 405 [[arXiv:0904.4447](#)] [[SPIRES](#)].
- [32] Y.M. Zinoviev, *Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields. II. General Young tableau with two rows*, *Nucl. Phys. B* **826** (2010) 490 [[arXiv:0907.2140](#)] [[SPIRES](#)].
- [33] E.D. Skvortsov, *Gauge fields in (A)dS within the unfolded approach: algebraic aspects*, *JHEP* **01** (2010) 106 [[arXiv:0910.3334](#)] [[SPIRES](#)].
- [34] K.B. Alkalaev and M. Grigoriev, *Unified BRST description of AdS gauge fields*, *Nucl. Phys. B* **835** (2010) 197 [[arXiv:0910.2690](#)] [[SPIRES](#)].
- [35] E.D. Skvortsov and M.A. Vasiliev, *Reducible multiplets of bosonic massless mixed-symmetry fields*, to appear (2010).
- [36] E.S. Fradkin and R.R. Metsaev, *A Cubic interaction of totally symmetric massless representations of the Lorentz group in arbitrary dimensions*, *Class. Quant. Grav.* **8** (1991) L89 [[SPIRES](#)].
- [37] X. Bekaert, N. Boulanger and M. Henneaux, *Consistent deformations of dual formulations of linearized gravity: A no-go result*, *Phys. Rev. D* **67** (2003) 044010 [[hep-th/0210278](#)] [[SPIRES](#)].
- [38] N. Boulanger and S. Cnockaert, *Consistent deformations of (p,p)-type gauge field theories*, *JHEP* **03** (2004) 031 [[hep-th/0402180](#)] [[SPIRES](#)].
- [39] X. Bekaert, N. Boulanger and S. Cnockaert, *No self-interaction for two-column massless fields*, *J. Math. Phys.* **46** (2005) 012303 [[hep-th/0407102](#)] [[SPIRES](#)].
- [40] R.R. Metsaev, *Cubic interaction vertices for massive and massless higher spin fields*, *Nucl. Phys. B* **759** (2006) 147 [[hep-th/0512342](#)] [[SPIRES](#)].
- [41] R.R. Metsaev, *Cubic interaction vertices for fermionic and bosonic arbitrary spin fields*, [arXiv:0712.3526](#) [[SPIRES](#)].
- [42] K. Alkalaev, *FV-type action for AdS<sub>5</sub> mixed-symmetry fields*, *JHEP* **03** (2011) 031 [[arXiv:1011.6109](#)] [[SPIRES](#)].
- [43] Y.M. Zinoviev, *Gravitational cubic interactions for a massive mixed symmetry gauge field*, [arXiv:1107.3222](#) [[SPIRES](#)].
- [44] N. Boulanger, E.D. Skvortsov and Y.M. Zinoviev, *Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds*, [arXiv:1107.1872](#) [[SPIRES](#)].
- [45] E. Sezgin and P. Sundell, *Doubletons and 5D higher spin gauge theory*, *JHEP* **09** (2001) 036 [[hep-th/0105001](#)] [[SPIRES](#)].
- [46] E. Sezgin and P. Sundell, *Towards massless higher spin extension of D = 5, N = 8 gauged supergravity*, *JHEP* **09** (2001) 025 [[hep-th/0107186](#)] [[SPIRES](#)].

- [47] Y.M. Zinoviev, *On electromagnetic interactions for massive mixed symmetry field*, *JHEP* **03** (2011) 082 [[arXiv:1012.2706](#)] [[SPIRES](#)].
- [48] E.S. Fradkin and M.A. Vasiliev, *Candidate to the Role of Higher Spin Symmetry*, *Ann. Phys.* **177** (1987) 63 [[SPIRES](#)].
- [49] S.E. Konstein and M.A. Vasiliev, *Massless representations and admissibility condition for higher spin superalgebras*, *Nucl. Phys. B* **312** (1989) 402 [[SPIRES](#)].
- [50] M.A. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **12** (2004) 046 [[hep-th/0404124](#)] [[SPIRES](#)].
- [51] S.W. MacDowell and F. Mansouri, *Unified Geometric Theory of Gravity and Supergravity*, *Phys. Rev. Lett.* **38** (1977) 739 [[SPIRES](#)].
- [52] K.S. Stelle and P.C. West, *Spontaneously broken de sitter symmetry and the gravitational holonomy group*, *Phys. Rev. D* **21** (1980) 1466 [[SPIRES](#)].
- [53] S.F. Prokushkin and M.A. Vasiliev, *Higher-spin gauge interactions for massive matter fields in 3D AdS space-time*, *Nucl. Phys. B* **545** (1999) 385 [[hep-th/9806236](#)] [[SPIRES](#)].
- [54] M.A. Vasiliev, *Higher spin algebras and quantization on the sphere and hyperboloid*, *Int. J. Mod. Phys. A* **6** (1991) 1115 [[SPIRES](#)].
- [55] M. Henneaux and S.-J. Rey, *Nonlinear  $W(\infty)$  Algebra as Asymptotic Symmetry of Three-Dimensional Higher Spin Anti-de Sitter Gravity*, *JHEP* **12** (2010) 007 [[arXiv:1008.4579](#)] [[SPIRES](#)].
- [56] A. Campoleoni, S. Fredenhagen, S. Pfenninger and S. Theisen, *Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields*, *JHEP* **11** (2010) 007 [[arXiv:1008.4744](#)] [[SPIRES](#)].
- [57] A. Campoleoni, S. Fredenhagen and S. Pfenninger, *Asymptotic  $W$ -symmetries in three-dimensional higher-spin gauge theories*, [arXiv:1107.0290](#) [[SPIRES](#)].
- [58] M.R. Gaberdiel and T. Hartman, *Symmetries of Holographic Minimal Models*, *JHEP* **05** (2011) 031 [[arXiv:1101.2910](#)] [[SPIRES](#)].
- [59] R.R. Metsaev, *Massless mixed symmetry bosonic free fields in  $d$ -dimensional anti-de Sitter space-time*, *Phys. Lett. B* **354** (1995) 78 [[SPIRES](#)].
- [60] E.D. Skvortsov, *Gauge fields in (anti)-de Sitter space and Connections of its symmetry algebra*, *J. Phys. A* **42** (2009) 385401 [[arXiv:0904.2919](#)] [[SPIRES](#)].
- [61] S. Deser and A. Waldron, *Partial masslessness of higher spins in  $(A)dS$* , *Nucl. Phys. B* **607** (2001) 577 [[hep-th/0103198](#)] [[SPIRES](#)].
- [62] E.D. Skvortsov and M.A. Vasiliev, *Geometric formulation for partially massless fields*, *Nucl. Phys. B* **756** (2006) 117 [[hep-th/0601095](#)] [[SPIRES](#)].
- [63] L. Brink, R.R. Metsaev and M.A. Vasiliev, *How massless are massless fields in  $AdS(d)$* , *Nucl. Phys. B* **586** (2000) 183 [[hep-th/0005136](#)] [[SPIRES](#)].
- [64] M.A. Vasiliev, *Cubic interactions of bosonic higher spin gauge fields in  $AdS_5$* , *Nucl. Phys. B* **616** (2001) 106 [[hep-th/0106200](#)] [[SPIRES](#)].
- [65] K.B. Alkalaev, *Two-column higher spin massless fields in  $AdS(d)$* , *Theor. Math. Phys.* **140** (2004) 1253 [[hep-th/0311212](#)] [[SPIRES](#)].



- [66] V.E. Lopatin and M.A. Vasiliev, *Free massless bosonic fields of arbitrary spin in  $d$ -dimensional de sitter space*, *Mod. Phys. Lett. A* **3** (1988) 257 [SPIRES].
- [67] O.V. Shaynkman and M.A. Vasiliev, *Scalar field in any dimension from the higher spin gauge theory perspective*, *Theor. Math. Phys.* **123** (2000) 683 [hep-th/0003123] [SPIRES].
- [68] M.A. Vasiliev, *On Conformal,  $SL(4, R)$  and  $Sp(8, R)$  Symmetries of 4d Massless Fields*, *Nucl. Phys. B* **793** (2008) 469 [arXiv:0707.1085] [SPIRES].
- [69] C. Iazeolla and P. Sundell, *A Fiber Approach to Harmonic Analysis of Unfolded Higher-Spin Field Equations*, *JHEP* **10** (2008) 022 [arXiv:0806.1942] [SPIRES].
- [70] X. Bekaert and M. Grigoriev, *Manifestly Conformal Descriptions and Higher Symmetries of Bosonic Singletons*, *SIGMA* **6** (2010) 038 [arXiv:0907.3195] [SPIRES].
- [71] K.B. Alkalaev and M.A. Vasiliev,  *$N = 1$  supersymmetric theory of higher spin gauge fields in  $AdS_5$  at the cubic level*, *Nucl. Phys. B* **655** (2003) 57 [hep-th/0206068] [SPIRES].
- [72] M.A. Vasiliev, *Consistent equations for interacting massless fields of all spins in the first order in curvatures*, *Annals Phys.* **190** (1989) 59 [SPIRES].
- [73] S.E. Konstein and M.A. Vasiliev, *Extended higher spin superalgebras and their massless representations*, *Nucl. Phys. B* **331** (1990) 475 [SPIRES].
- [74] M.A. Vasiliev, *Higher spin gauge theories: Star-product and AdS space*, hep-th/9910096 [SPIRES].
- [75] M. Günaydin and D. Minic, *Singletons, doubletons and M-theory*, *Nucl. Phys. B* **523** (1998) 145 [hep-th/9802047] [SPIRES].
- [76] M. Flato and C. Fronsdal, *One Massless Particle Equals Two Dirac Singletons: Elementary Particles in a Curved Space. 6*, *Lett. Math. Phys.* **2** (1978) 421 [SPIRES].
- [77] F.A. Dolan, *Character formulae and partition functions in higher dimensional conformal field theory*, *J. Math. Phys.* **47** (2006) 062303 [hep-th/0508031] [SPIRES].
- [78] B. Feigin, *The Lie algebras  $\mathfrak{gl}(\lambda)$  and cohomologies of Lie algebras of differential operators*, *Russ. Math. Surv.* **43** (1988) 169.
- [79] D.P. Sorokin and M.A. Vasiliev, *Reducible higher-spin multiplets in flat and AdS spaces and their geometric frame-like formulation*, *Nucl. Phys. B* **809** (2009) 110 [arXiv:0807.0206] [SPIRES].
- [80] D.S. Ponomarev and M.A. Vasiliev, *Frame-Like Action and Unfolded Formulation for Massive Higher-Spin Fields*, *Nucl. Phys. B* **839** (2010) 466 [arXiv:1001.0062] [SPIRES].
- [81] N. Boulanger and E. Skvortsov, in preparation.
- [82] M. Vasiliev, to appear.