Seiberg-Witten geometries revisited

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ABSTRACT: We provide a uniform solution to 4d $\mathcal{N} = 2$ gauge theory with a single gauge group $G = A, D, E$ when the one-loop contribution to the beta function from any irreducible component R of the hypermultiplets is less than or equal to half of that of the adjoint representation. The solution is given by a non-compact Calabi-Yau geometry, whose defining equation is built from explicitly known polynomials W_G and X_R, associated respectively to the gauge group G and each irreducible component R. We provide many pieces of supporting evidence, for example by analyzing the system from the point of view of the 6d $\mathcal{N} = (2,0)$ theory compactified on a sphere.

Dedicated to the memory of Professor Sung-Kil Yang

Contents

1 Introduction

It is by now well-known that the non-perturbative quantum effect of 4d $\mathcal{N} = 2$ gauge theory can be captured by classical complex geometry [1, 2]. This geometry is usually called the Seiberg-Witten curve when it is a curve, or just the Seiberg-Witten geometry when it is not. Many methods to obtain the solutions were devised over the years:

- One is to consider the geometry as the spectral curve of the integrable system [3–6]. It is, however, not straightforward to pin down the integrable system given the gauge theory. For reviews on this approach, see e.g. [7–9]

- Another is to use the Calabi-Yau compactification of the string theory. Originally, the solutions were extracted from the decoupling limit of compact Calabi-Yaus [10–12], but eventually the method was distilled into the geometric engineering [13, 14], which directly gives a non-compact Calabi-Yau which is fibered by ALE spaces, or equivalently 6d $\mathcal{N} = (2, 0)$ theory put on a Riemann surface. This requires the identification of the heterotic dual to the Type IIB on non-compact Calabi-Yau manifolds, and it is not always easy. For a review, see e.g. [15].

- Yet another is to use branes in the type IIA theory which is then lifted to M-theory [16]. This method suits very well classical gauge groups with usual representations, but it is not very helpful for exceptional cases. For a review on this approach, see e.g. [17].

- Lately, it was realized that 6d $\mathcal{N} = (2,0)$ compactified on a Riemann surface can be understood from the properties of codimension-2 defects of the 6d theory [18, 19]. Then, finding a Seiberg-Witten solution reduces to the identification of the combination of the codimension-2 defects.

These techniques are all inter-related, and each is complementary to another. The state of the art is that, for almost all of the choice of the gauge group and the matter content, at least one of the method is applicable, and the Seiberg-Witten solution can be found, and indeed the solution to most of the choices has been written down.

Glancing through the solutions available in the literature [4, 20–32], it was noted in [30] that the solution to a theory with single simply-laced gauge group $G = A, D, E$ almost always is given uniformly by the Seiberg-Witten geometry

$$z + \frac{\Lambda^{2h^\vee}}{z} \prod_R \Lambda^{-b_R} X_R(x_1, x_2, x_3; w_i; m) = W_G(x_1, x_2, x_3; w_i). \tag{1.1}$$

where

- $h^\vee$ is the dual Coxeter number of G,

- W_G is the equation of the ALE space of type G deformed by the Casimirs w_i given in table 1, 2.

- b_R is the contribution to the one-loop beta function from the hypermultiplet in the irreducible representation R, given in table 3,

- and X_R is the polynomial representing the hypermultiplet in the irreducible representation R, tabulated in the appendix A.

Note that we can consider half-hypermultiplets when R is a pseudo-real representation. We will denote them by $\frac{1}{2}R$, and $b_{\frac{1}{2}R} = \frac{1}{2}b_R$.

X_R is known for all R for which $b_R \leq h^\vee$, and there is a reason to suspect that a matter representation R with $h^\vee < b_R \leq 2h^\vee$, although still asymptotically free or conformal, is not realizable in this form. We call the former 'nice' representations ($b_R \leq h^\vee$), and the latter 'reasonable' representations. ($h^\vee < b_R \leq 2h^\vee$). The main objective of this paper is to give a rationale behind this regularity of the solution to the $\mathcal{N} = 2$ gauge theory with single simply-laced gauge group and nice hypermultiplets. Here we note that most of the possible matter representations are in the "nice" representations. Indeed, as seen from table 3, the only non-nice reasonable representations are only the 2-index symmetric tensor for SU(n), the 3-index antisymmetric tensor for SU(7) and SU(8), the spinor for SO(14), and the adjoint representations.[1]

G	$h^\vee$	x_1	x_2	x_3	w_i
$\mathrm{SU}(n) = A_{n-1}$	n	1	$n/2$	$n/2$	$2, 3, \ldots, n$
$\mathrm{SU}(n) = A'_{n-1}$	n	1	$n-2$	2	$2, 3, \ldots, n$
$\mathrm{SO}(2n) = D_n$	$2n-2$	2	$n-2$	$n-1$	$2, 4, \ldots, 2n-2; n$
E_6	12	3	4	6	$2, 5, 6, 8, 9, 12$
E_7	18	4	6	9	$2, 6, 8, 10, 12, 14, 18$
E_8	30	6	10	15	$2, 8, 12, 14, 18, 20, 24, 30$

Table 1. The dual Coxeter number for the gauge group A_{n-1}, D_n and $E_{6,7,8}$, together with the mass dimension of x_i in the corresponding ALE spaces and of the Casimirs w_i.

$$
\begin{aligned}
W_{A_{n-1}} &= x_1^n + x_2^2 + x_3^2 + w_2 x_1^{n-2} + w_3 x_1^{n-3} + \cdots + w_n, \\
W_{A'_{n-1}} &= x_1^n + x_2 x_3 + w_2 x_1^{n-2} + w_3 x_1^{n-3} + \cdots + w_n, \\
W_{D_n} &= x_1^{n-1} + x_1 x_2^2 - x_3^2 + w_2 x_1^{n-2} + w_4 x_1^{n-3} + \cdots + w_{2n-2} + \tilde{w}_n x_2, \\
W_{E_6} &= x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12}, \\
W_{E_7} &= x_1^3 x_2 + x_2^3 + x_3^2 \\
&\quad + w_2 x_2^2 x_1 + w_6 x_2^2 + w_8 x_2 x_1 + w_{10} x_1^2 + w_{12} x_2 + w_{14} x_1 + w_{18} \\
W_{E_8} &= x_1^5 + x_2^3 + x_3^2 + \\
&\quad + w_2 x_2 x_1^3 + w_8 x_2 x_1^2 + w_{12} x_1^3 + w_{14} x_2 x_1 + w_{18} x_1^2 + w_{20} x_2 + w_{24} x_1 + w_{30}
\end{aligned}
$$

Table 2. The equation defining the ALE spaces. The explicit formula for w_i via the Cartan of G was given in [35–37], and is reproduced in appendix A.

Although X_R for all nice R can be found and will be tabulated in this paper, the mathematics behind them is not yet as clear as that for W_G. One way to obtain a handle to X_R is to view it from the perspective of punctures of 6d $\mathcal{N} = (2,0)$ theory, which is most straightforward when $\sum_{R_i} b_{R_i} = h^\vee$ [18, 38, 39]. In this case we expect that $\mathcal{N} = (2,0)$ theory compactified on a sphere with three regular punctures realize free hypermultiplets in the representation $\oplus R_i$. We give a detailed analysis of two cases, one involving the three-index antisymmetric tensor of SU(6) and another involving **56** of E_7.

The rest of the paper is organized as follows: in section 2, we interpret the polynomial X_R as giving a Calabi-Yau which represents a free hypermultiplet in the representation R, and give an explanation of the uniformity of the solution (1.1). We also discuss how X_R for different G and R is related to each other. In section 3, we perform three detailed case studies: the first is the 2-index antisymmetric tensor of SU(N), the second is **20** of SU(6), and the third is **56** of E_7. Among others, we will find that the same X_R can arise from a completely different combination of punctures from the point of view of the 6d $\mathcal{N} = (2,0)$ theory. We conclude with a short discussion in section 4. The appendix A contains the list of all X_R. In appendix B B we describe the details of obtaining the curve from the geometry.

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[1]The Seiberg-Witten curves which include the 2-index symmetric tensor and the adjoint representations are given in [33] and [34], respectively. Thus, the exact solutions for any matter content for the gauge group are known except for the ones with the spinor representation of SO(14) and the 3-index antisymmetric tensor of SU(7) and SU(8). We comment on this point again in section 4.

2 Generalities

2.1 Motivating examples

2.1.1 $\mathcal{N} = 2$ gauge theory without matters

Let us start by the analysis of the solution to the pure $\mathcal{N} = 2$ gauge theory with gauge group $G = A, D, E$ [4, 13, 20, 21, 24, 40]:

$$\Lambda^{h^\vee} z + \frac{\Lambda^{h^\vee}}{z} = W_G(x_1, x_2, x_3; w_k) \tag{2.1}$$

where the Seiberg-Witten three-form is given by

$$\omega = \frac{1}{2\pi i} \oint \frac{dx_1 \wedge dx_2 \wedge dx_3}{W_G(x_1, x_2, x_3; w_k) - \Lambda^{h^\vee}(z + 1/z)} \wedge \frac{dz}{z} \tag{2.2}$$

where the contour integral is taken around the poles of the denominator.

It is helpful to recall the property of the deformed ALE space $W_G = 0$. It is quasihomogeneous in the variables x_i and the coefficients w_k whose degrees are given in table 1; we use the convention that w_k has degree k. When w_i are generic, there are $r = \text{rank}\, G$ two-cycles $C_1, \dots C_r$ intersecting according to the Dynkin diagram of G. Then, we can define an element ϕ in the Cartan of the Lie algebra of G such that

$$\alpha_i \cdot \phi = \int_{C_i} \oint \frac{dx_1 \wedge dx_2 \wedge dx_3}{W_G} \tag{2.3}$$

where α_i is the i-th simple root. Then w_k is given by the degree-k Casimir constructed from ϕ [35, 36]. The geometry $W_G = 0$ becomes singular when $\alpha \cdot \phi = 0$ for a root α. It is well known that the low-energy limit of the Type IIB string on $W_G = 0$ gives 6d $\mathcal{N} = (2,0)$ theory of type G.

From this viewpoint, the geometry for the pure gauge theory (2.1) can also be written as

$$W_G(x_1, x_2, x_3; w_k(z)) = 0 \qquad \text{where} \begin{cases} w_k(z) = w_k & (k < h^\vee), \\ w_{h^\vee}(z) = -\Lambda^{h^\vee} z + w_{h^\vee} - \Lambda^{h^\vee}/z. \end{cases} \tag{2.4}$$

This describes how the ALE space is deformed as one changes z. It can be regarded as the compactification of the 6d $\mathcal{N} = (2,0)$ theory of type G on the cylinder parameterized by z. Then $w_k(z)$ is the worldvolume fields of the 6d theory which depends on the position on the cylinder.

Let us choose the vev so that $w_k \sim a^k$. When the system is weakly coupled, i.e. $\Lambda \ll a$, the geometry can be visualized as in figure 1a). When $|z| \sim O(1)$, $w_k(z)$ is almost constant. Let us take an element ϕ in the Cartan of the Lie algebra of G so that $w_k = w_k(\phi)$. Take a three-cycle of the form $A_i = A \times C_i$ where A is a cycle wrapping around the cylinder once. Then

$$\int_{A_i} \omega \sim \alpha_i \cdot \phi. \tag{2.5}$$

This means that a D3-brane wrapped around A_i gives rise to the W-boson of the gauge group G corresponding to the root α_i [13, 19, 41].
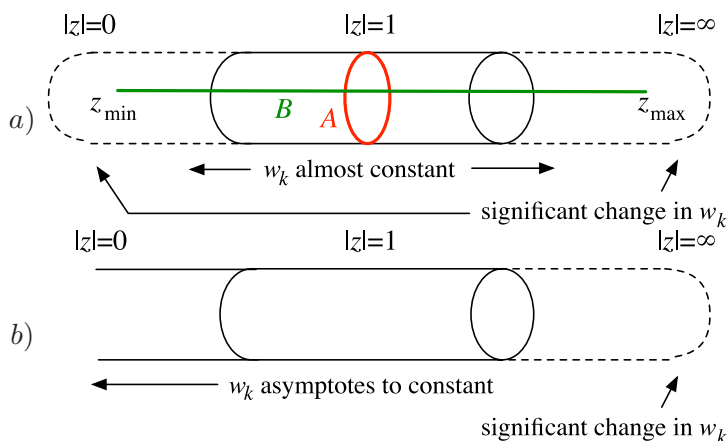
Figure 1. a) The base of the Seiberg-Witten geometry in the weakly-coupled region. b) a half of the Seiberg-Witten geometry. This can be though of as the geometry representing free hypermultiplets.

The geometry is terminated by the divergence of $w_{h^\vee}(z)$. Take the path B connecting $z = z_{\max}$ and $z = z_{\min}$ where the two-cycle C_i shrinks. The geometry is significantly modified when the right hand side on (2.1) is of the same order with the left hand side. So we can estimate $|z_{\max}| \sim |1/z_{\min}| \sim (a/\Lambda)^{h^\vee}$.

Consider the three-cycle of the form $B_i = B \times C_i$ see figure 1a). Then we find

$$\int_{B_i} \omega \sim (\alpha_i \cdot \phi) \frac{1}{2\pi i} \int_{z_{\min}}^{z_{\max}} \frac{dz}{z} \sim \frac{2h^\vee}{2\pi i} (\alpha_i \cdot \phi) \log \frac{\Lambda}{a} \tag{2.6}$$

which is the mass of the BPS 't Hooft-Polyakov monopole. The ratio of the masses (2.5) and (2.6),

$$\tau(a) = \frac{2h^\vee}{2\pi i} \log \frac{\Lambda}{a} \tag{2.7}$$

correctly gives the one-loop running of the low-energy gauge coupling on the Coulomb branch.

The geometry around $|z| \sim \infty$ can be studied by dropping the term inversely linear in z in (2.1):

$$\Lambda^{h^\vee} z = W_G(x_1, x_2, x_3; w_i), \tag{2.8}$$

see figure 1b). This should represent the presence of no vector multiplet at all. Indeed this geometry is smooth for all values of w_i, because of the linear term in z.

2.1.2 SU(n) with fundamental hypermultiplets

Let us next consider the geometry for the SU(n) gauge theory with $n_f < 2n$ fundamental flavors with masses m_i [22, 23]:

$$z + \frac{\Lambda^{2n-n_f}}{z} \prod_{i=1}^{n_f} (x_1 - m_i) = x_2^2 + x_3^2 + P(x_1) \tag{2.9}$$

where $P(x) = \prod(x - a_i) = x_1^n + w_2 x_1^{n-2} + \cdots + w_n$. We can split $n_f = n'_f + n''_f$ so that $n'_f, n''_f \leq n$. Then we can write the same geometry as

$$\Lambda^{n-n''_f} z \prod_{i=1}^{n''_f} (x_1 - m_i) + \frac{\Lambda^{n-n'_f}}{z} \prod_{i=1}^{n'_f} (x_1 - \tilde{m}_i) = x_2^2 + x_3^2 + P(x_1). \qquad (2.10)$$

where we redefined z. Now the degree of the left hand side is less or equal to that of the right hand side, and thus the system can be thought of as the compactification of 6d $\mathcal{N} = (2,0)$ theory of type A_{n-1}. When Λ is very small, $w_k(z)$ is again almost constant around $|z| \sim O(1)$, which again fits the situation shown in figure 1, a). We can find the three-cycles representing the W-bosons and the monopoles as before; the only difference is that now $z_{\max} \sim (a/\Lambda)^{n-n''_f}$ and $z_{\min} \sim (\Lambda/a)^{n-n'_f}$. Then the ratio of the masses is now

$$\tau(a) = \frac{2n - n'_f - n''_f}{2\pi i} \log \frac{\Lambda}{a}, \qquad (2.11)$$

correctly reproducing the one-loop running.

The divergences of $w_k(z)$ at $|z| \sim 0$ and $|z| \sim \infty$ should then represent the presence of n'_f and n''_f hypermultiplets in respective regions. To isolate the behavior at $|z| \sim \infty$, we drop the term inversely proportional to z in (2.10), and consider the geometry

$$\Lambda^{n-n''_f} z \prod_{i=1}^{n''_f} (x_1 - m_i) = x_2^2 + x_3^2 + \prod_{i=1}^{n} (x_1 - a_i). \qquad (2.12)$$

The geometry can be visualized as in figure 1b). As can be easily checked, this geometry becomes singular when $a_i = m_j$ for some i and j, signifying the presence of the fundamental hypermultiplets with masses m_i. Note also that the prefactor $\Lambda^{n-n''_f}$ was responsible for reproducing the one-loop running. This geometry will be obtained by the weak coupling limit, i.e. $\Lambda \to 0$ although only the n''_f hypermultiplets can be represented.

2.2 Hypermultiplet geometries

The discussion so far suggests the following general picture: for a hypermultiplet in representation R of G with mass m, we should find a Calabi-Yau geometry[2] representing it of the form

$$\Lambda^{h^\vee - b_R} z X_R(x_1, x_2, x_3; w_k; m) = W_G(x_1, x_2, x_3; w_k) \qquad (2.13)$$

such that there is a three-cycle C_v for each weight v of R such that

$$\int_{C_v} \omega = v \cdot \phi - m \qquad (2.14)$$

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[2]Calabi-Yau geometries which produce massless hypermultiplets in various representations when used in Type IIA compactifications were identified in [42, 43]. In our case, the Calabi-Yau geometries should be used in Type IIB compactifications. These should be mirror to each other, but since they are non-compact and typically non-toric, it is hard to show that it is indeed the case.

G		R		b	X_R
SU(n) = A_{n-1}		fund.	$\mathbf{n}$	1	(A.4)
		2-index antisym.	$\frac{\mathbf{n(n-1)}}{\mathbf{2}}$	$n-2$	(A.5)
SU(6) = A_5	half 3-index antisym.		$\mathbf{20}$	3	(A.8)
SO(n) = D_n		vector	$\mathbf{2n}$	2	(A.13)
SO(8) = D_4		spinor	$\mathbf{8s}$	2	(A.14)
SO(8) = D_4		conj. spinor	$\mathbf{8c}$	2	(A.15)
SO(10) = D_5		spinor	$\mathbf{16}$	4	(A.16)
SO(12) = D_6	half	spinor	$\mathbf{32s}$	4	(A.17)
SO(12) = D_6	half	conj. spinor	$\mathbf{32c}$	4	(A.19)
E_6			$\mathbf{27}$	6	(A.24)
E_7	half		$\mathbf{56}$	6	(A.25)

a) 'Nice' representations: $b \leq h^\vee$.

G	R	b
SU(n) = A_{n-1}	2-index sym.	$n+2$
	adj.	$2n$
SU(7) = A_6	3-index antisym.	10
SU(8) = A_6	3-index antisym.	15
SO($2n$) = D_n	adj.	$4n-4$
SO(14) = D_7	spinor	16
E_6	adj.	24
E_7	adj.	36
E_8	adj.	60

b) 'Reasonable' representations: $h^\vee < b \leq 2h^\vee$.

Table 3. List of 'nice' and 'reasonable' representations of $G = A_{n-1}$, D_n and $E_{6,7,8}$.

where ϕ is such that $w_k = w_k(\phi)$. Finding three-cycles is a big problem, so in practice we impose a weaker condition, that the geometry becomes singular whenever

$$v \cdot \phi = m \tag{2.15}$$

for a weight v of R. It is not, a priori, obvious that we can find such a geometry for all R, or why it should be linear in z.

First, in order for the left hand side to be a deformation of the singularity, b_R can be at most $h^\vee$, or in our terminology, R needs to be 'nice'. Therefore, the adjoint representation for which $b = 2h^\vee$ is out of the question in this approach. Fortunately, one finds that most of the representations which can be used to construct asymptotically free or conformal theory with simply-laced gauge group $G = A, D, E$ are nice. The exceptions are the adjoint representation of each G, the symmetric 2-index tensor representation of SU(N), the antisymmetric 3-index representations of SU(7) and SU(8), and the spinor representation of SO(14). Therefore, representations with $b_R \leq h^\vee$ are not that a big restriction, see table 3.
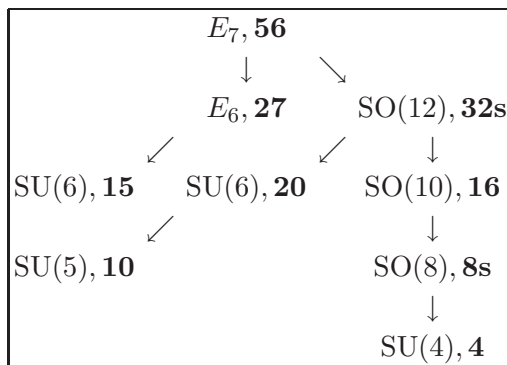
$$\begin{array}{ccc}
E_7, \mathbf{56} & & \\
\downarrow & \searrow & \\
E_6, \mathbf{27} & & SO(12), \mathbf{32s} \\
\swarrow & \swarrow & \downarrow \\
SU(6), \mathbf{15} \quad SU(6), \mathbf{20} & & SO(10), \mathbf{16} \\
\swarrow & & \downarrow \\
SU(5), \mathbf{10} & & SO(8), \mathbf{8s} \\
& & \downarrow \\
& & SU(4), \mathbf{4}
\end{array}$$

Table 4. Relations among 'nice' exceptional representations under decoupling.

Second, the table shows that there are two types of irreducible representations with $b_R \leq h^\vee$: one which comes in infinite series, fundamental and two-index antisymmetric tensor of $SU(n)$, and vector of $SO(2n)$. There are only finite number of exceptions, which we call *exceptional representations*. The curves for the infinite series are standard [22, 23, 25, 26, 32], and can be translated into corresponding polynomials X_R. As for the exceptional representations, the biggest is $\mathbf{56}$ of E_7, and it can be seen that any other exceptional representations can be obtained by a repeated application of the decoupling procedure, see table 4. Namely, given a geometry for the representation R of G, one can give a vev $\langle \phi \rangle$ breaking G to G'. At the same time, we can tune the mass M of the hypermultiplet in R of the order $M \sim \langle \phi \rangle$ so that the mass of the hypermultiplet in a subrepresentation R' of G' remains finite. This decoupling method produces $X_{R'}$ for G' given X_R for G, which we illustrate in an example in section 3.2. Now, $X_{\mathbf{56}}$ of E_7 was determined in [31]. Therefore, every other X_R for exceptional representations follow.

Third, suppose we find such polynomials X_{R_1} and X_{R_2} for two representations R_1 and R_2 of G. Then the geometry for the combined representation can be easily found:

$$\Lambda^{h^\vee - b_{R_1} - b_{R_2}} z X_{R_1} X_{R_2} = W_G \tag{2.16}$$

Indeed, defining $\tilde{z} = z\Lambda^{-b_{R_2}} X_{R_2}$, we get just $\Lambda^{h^\vee - b_{R_1}} \tilde{z} X_{R_1} = W_G$. Therefore, assuming a three-cycle representing the hypermultiplets in R_1 is known in the latter geometry, the same three-cycle can be found in the geometry (2.16). Therefore the geometry for the reducible representations can be found by multiplying X_R for the irreducible representations.

2.3 General recipe

Now we come to a general method to write down the Seiberg-Witten geometry for $\mathcal{N} = 2$ theory with simply-laced gauge group G with matter content $\oplus_i R_i$, such that $b_{R_i} \leq h^\vee$ to have X_R and $\sum_i b_{R_i} \leq 2h^\vee$ to be asymptotically free or conformal. We split the irreducible components into two groups, $\oplus_a R_a$ and $\oplus_{\tilde{a}} \tilde{R}_{\tilde{a}}$ so that $\sum_a b_{R_a} \leq h^\vee$ and $\sum_{\tilde{a}} b_{\tilde{R}_{\tilde{a}}} \leq h^\vee$. Then we consider the geometry

$$\Lambda^{h^\vee} z \prod_a \Lambda^{-b_{R_a}} X_{R_a} + \frac{\Lambda^{h^\vee}}{z} \prod_{\tilde{a}} \Lambda^{-b_{\tilde{R}_{\tilde{a}}}} X_{\tilde{R}_{\tilde{a}}} = W_G(x_1, x_2, x_3; w_k). \tag{2.17}$$

When Λ is very small, the geometry is of the form shown in figure 1 a), and thus we find three-cycles representing W-bosons and monopoles, together with three-cycles representing hypermultiplets in $\oplus_i R_i$ in the regions $|z| \sim 0$ and $|z| \sim \infty$. At least the one-loop running is reproduced by construction. We can hope that the holomorphy guarantees that the geometry is correct even in the strongly-coupled regime.

We needed to divide $\oplus_i R_i$ into two subsets, but the resulting geometry in fact does not depend on the choice, because it can be easily rewritten to

$$z + \frac{\Lambda^{2h^\vee}}{z} \prod_{\tilde{a}} \Lambda^{-b_{\tilde{R}_{\tilde{a}}}} X_{\tilde{R}_{\tilde{a}}} = W_G. \tag{2.18}$$

Admittedly, there are many gaps in the rough argument presented above. In the next section, we provide detailed checks of the construction in three examples.

We note that the Seiberg-Witten geometry (2.17) can be always rewritten in the form

$$W_G(x'_1, x'_2, x'_3; \tilde{w}) = 0, \tag{2.19}$$

by a redefinition $x'_i = x'_i(x_k, w_k, m, \Lambda, z)$ and $\tilde{w}_k = \tilde{w}_k(w_k, m, \Lambda, z)$.[3] Then, we can construct a corresponding Seiberg-Witten curve:

$$P_R(x, \tilde{w}_i) = 0 \tag{2.20}$$

from the deformed Casimirs $\tilde{w}_i$ defined by (2.19) and the characteristic polynomial of a representation R of G, $P_R(x, w_i) = \prod_\alpha (x - a \cdot \lambda_\alpha)$, where λ_α is a weight of R. We expect that the low energy effective actions computed from this curve and the Seiberg-Witten geometry are the same. The equivalence was explictly verified by [40] for pure E_6 theory, but it holds universally.

In particular, the physics does not depend on the choice of the representation R as discussed in [41], in which the cycles to be integrated on were explicitly specified. Thus, we can use the curve instead of the geometry if we want to do so. The Seiberg-Witten differential will be given by[4]

$$\lambda = x \frac{dz}{z}. \tag{2.21}$$

These two descriptions are related in string theory by the duality between the IIB string on non-compact Calabi-Yau (2.17) and the 6d (2,0) theory (or the M5-branes in the non-trivial background) on the curve.

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[3]Below, we will simply denote $\tilde{w}_k(w_k, m, \Lambda, z)$ as $w_k(z)$.

[4]Precisely speaking, this is not completely fixed by the considerations so far partially because there are ambiguities for the definition of $\tilde{w}_i$. We will explain how to fix it in the appendix B.

3 Case studies

3.1 2-index antisymmetric tensor of $SU(n)$

Derivation. The curve of $SU(n)$ with two hypermultiplets in the 2-index antisymmetric tensor representation with equal mass m was determined in [32]:[5]

$$\Lambda^2 \left(z + \frac{1}{z} \right) = x^2 - u, \tag{3.1}$$

$$0 = R(u) + xS(u) \tag{3.2}$$

where polynomials $R(u)$ and $S(u)$ are determined by the condition

$$R(x^2) + xS(x^2) = \prod_i \left(x - a_i + \frac{m}{2} \right), \tag{3.3}$$

and the SW differential is given by

$$\lambda = x \frac{dz}{z}. \tag{3.4}$$

This curve is equivalent to the following non-compact Calabi-Yau, as we will show momentarily:

$$\Lambda^2 \left(z + \frac{1}{z} \right) X_{\text{asym}}(x_1, x_2, x_3; w_i; m) = x_2 x_3 + P(x_1; w_i) \tag{3.5}$$

where

$$P(x_1; w_i) = \prod_i (x_1 - a_i), \tag{3.6}$$

$$X_{\text{asym}}(x_1, x_2, x_3; w_i; m) = -x_3 + T(x_1, x_2; w_i; m). \tag{3.7}$$

Here, the polynomial $T(x_1, x_2; w_i; m)$ is defined via the relation

$$x_2 T(x_1, x_2) = R\left(\left(x_1 - \frac{m}{2} \right)^2 + x_2 \right) + \left(x_1 - \frac{m}{2} \right) S\left(\left(x_1 - \frac{m}{2} \right)^2 + x_2 \right) - P(x_1). \tag{3.8}$$

Note that the right hand side has a factor of x_2 due to (3.3) and (3.6), guaranteeing that $T(x_1, x_2)$ is a polynomial.

Let us show the equivalence of the Calabi-Yau and the curve. Note that x_3 only appears linearly in (3.5) and can be 'integrated out'. The derivative with respect to x_3 of (3.5) sets

$$-x_2 = \Lambda^2 \left(z + \frac{1}{z} \right). \tag{3.9}$$

Plugging it back to (3.5) and using (3.8), we find

$$R\left(\left(x_1 - \frac{m}{2} \right)^2 + x_2 \right) + \left(x_1 - \frac{m}{2} \right) S\left(\left(x_1 - \frac{m}{2} \right)^2 + x_2 \right) = 0. \tag{3.10}$$

Introducing $x = x_1 - \frac{m}{2}$ and $u = (x_1 - \frac{m}{2})^2 + x_2$, we find the curve (3.1) and (3.2).

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[5]Their coordinate system and ours are related as $z_{\text{theirs}} = u_{\text{ours}}$, $y_{\text{theirs}}/\hat{x}_{\text{theirs}} = x_{\text{ours}}$, $\hat{x}_{\text{theirs}} = z_{\text{ours}}/\Lambda^2$, and $m_{A,\text{theirs}} = -2m_{\text{ours}}$. We set $\hat{w}_{\text{theirs}} = 1$. Then our (3.1) is their (5.52), our (3.2) is their (5.53), and our (3.3) is their (5.40). Their (5.53) has $x - 2m_A$ instead of x in (3.2), but that was a typo. The authors thank P. Argyres for correspondences on this point.

Confirming the discriminant. From the discussion above, we conclude that the hypermultiplet geometry representing a 2-index antisymmetric representation of SU(n) is given by

$$\Lambda^2 z X_{\text{asym}}(x_1, x_2, x_3; w_i; m) = x_2 x_3 + P(x_1; w_i). \tag{3.11}$$

Let us check that this geometry becomes singular when a component of the hypermultiplet becomes massless. The geometry (3.11) becomes singular when its derivatives with respect to x_1, x_2, x_3 all becomes zero. One easily sees that this is equivalent to the fact that the equations $R(u) = 0$ and $S(u) = 0$ defined in (3.3) have a common zero $u = u_0$, i.e. $R(u_0) = S(u_0) = 0$. This happens when their resultant is zero; and in general the resultant is given by

$$\text{Resultant of } R(u) \text{ and } S(u) = \text{const.} \prod_{i<j} (a_i + a_j - m). \tag{3.12}$$

Therefore, the geometry becomes singular when a component of the antisymmetric tensor becomes massless. To confirm the resultant, assume $a_i + a_j = m$ for a pair $i < j$. Let $x_0 = a_i - m/2 = -a_j + m/2$. Using (3.3), we find $R(x_0^2) + x_0 S(x_0^2) = R(x_0^2) - x_0 S(x_0^2) = 0$. x_0 is generically nonzero, so we conclude $R(x_0^2) = S(x_0^2) = 0$. This means that the resultant has a factor of $a_i + a_j - m$ for each pair $i < j$. One can independently calculate the degree of the resultant because we know the degrees of $R(u)$ and $S(u)$, and we conclude that there is no other factor.

3.2 3-index antisymmetric tensor 20 of SU(6)

Decoupling from 32 of SO(12). One representation for which X_R has not been written down is the three-index antisymmetric tensor **20** of SU(6).[6] This can be derived from the known case, **32** of SO(12) via decoupling. We start from $X_{\mathbf{32}}$, obtained in [31]:

$$X_{\mathbf{32}}(x_i; W_i; m) = \frac{1}{256}(8ix_2 + 8x_1^2 + 4W_2 x_1 + 4W_4 - W_2^2)^2$$
$$+ m^2 \left(i\tilde{W}_6 + W_6 - \frac{1}{4} W_2 W_4 + \frac{1}{16} W_2^3 + 2W_2 x_1^2 - \frac{1}{8} W_2^2 x_1 + \frac{3}{2} W_4 x_1 + 3x_1^3 + 3ix_1 x_2 + \frac{i}{2} W_2 x_2 \right)$$
$$+ 4im^3 x_3 + m^4 \left(3ix_2 + \frac{1}{2} W_2 x_1 + \frac{3}{8} W_2^2 - \frac{1}{2} W_4 \right) + (2x_1 + W_2)m^6 + m^8. \tag{3.13}$$

With this, we consider the geometry

$$\Lambda^{10-8} z X_{\mathbf{32}} = x_1^5 + x_1 x_2^2 - x_3^2 + W_2 x_1^4 + W_4 x_1^3 + W_6 x_1^2 + W_8 x_1 + W_{10} + \tilde{W}_6. \tag{3.14}$$

Now we give a vev to W_i so that SO(12) is broken to SU(6) × U(1). The spin representation decomposes as

$$\mathbf{32} \to \mathbf{20}_0 \oplus \mathbf{6}_1 \oplus \bar{\mathbf{6}}_{-1}. \tag{3.15}$$

The vev gives the mass to the hypermultiplets proportional to the U(1) charges through the $\sqrt{2}\tilde{Q}\Phi Q$ term. Therefore, if we take the vev infinite with a shift of the mass m to cancel

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[6]There was a paper [44] in which it was attempted to determine the Seiberg-Witten curve for gauge theories with three-index antisymmetric tensors.

this additional contribution, we will have SU(6) gauge theory with **20** or **6**, depending on the shift. Here, we should rescale the dynamical scale using the scale matching condition as usual and the U(1) sector decouples by the limit. In order to obtain **20** we do not need to shift the mass and just give the vev and take the limit. The geometry in the limit will be obtained with a non-trivial redefinition of the coordinates $x_i' = x_i'(x_1, x_2, x_3, m, W_k)$ such that the geometry is written as

$$W_{\mathrm{SO}(12)}(x_1, x_2, x_3, W_k) = M^4 W_{\mathrm{SU}(6)}(x_1', x_2', x_3', w_k) + \mathcal{O}(M^3), \tag{3.16}$$

where M is the scale of the vev. This procedure was explained in detail in [30]. Explicitly doing this with the help of a computer, we finally obtain the geometry for **20** of SU(6):

$$\Lambda^{6-6} z X_{\mathbf{20}} = x_2 x_3 + x_1^6 + w_2 x_1^4 + w_3 x_1^3 + w_4 x_1^2 + w_5 x_1 + w_6 \tag{3.17}$$

where

$$\begin{aligned}
X_{\mathbf{20}}(x_i; w_i; m) = &-(w_2 x_1 + w_3 + 2x_1^3 + 2ix_2)^2 \\
&+ m^2(-12x_1^4 - 12ix_1 x_2 - 6w_2 x_1^2 - 6w_3 x_1 - 4w_4 + w_2^2) \\
&+ 8m^3 x_3 + m^4(3x_1^2 + 2w_2) + m^6. \quad (3.18)
\end{aligned}$$

It is relieving to find that it becomes a square when m is set to zero, because **20** is pseudo-real and one can consider a half-hypermultiplet in **20**:

$$X_{\frac{1}{2}\mathbf{20}}(x_i; w_i) = i(w_2 x_1 + w_3 + 2x_1^3 + 2ix_2). \tag{3.19}$$

Comparison with 6d $\mathcal{N} = (2,0)$ theory: introductory remarks. Let us study this geometry from the point of view of the 6d $\mathcal{N} = (2,0)$ theory of type A_5. It is done by rewriting the geometry in the form

$$x_2 x_3 + x_1^6 + w_2(z) x_1^4 + w_3(z) x_1^3 + w_4(z) x_1^2 + w_5(z) x_1 + w_6(z) = 0 \tag{3.20}$$

and considering the multi-differentials

$$\varphi_k(z) = w_k(z) \frac{dz^k}{z^k} \tag{3.21}$$

as the worldvolume fields of the 6d theory. A further compactification on S^1 makes the system into 5d maximally supersymmetric Yang-Mills on a sphere parameterized by z. Then there is a complex adjoint scalar field $\Phi(z)$ which is a differential such that

$$\lambda^6 + \varphi_2(z)\lambda^4 + \cdots + \varphi_6(z) = \det(\lambda - \Phi(z)) \tag{3.22}$$

where λ is the Seiberg-Witten differential.

The main point of the approach using the adjoint field $\Phi(z)$ [18, 19] is to identify its singularity. This method is most developed when the total b equals $h^\vee$. In this case, it is expected that there are three singularities, say at $z = 0, 1, \infty$. Let t be a local coordinate at a singularity so that the singularity is at $t = 0$. Then Φ has the form

$$\Phi \sim \Phi_{-1} \frac{dt}{t} + \text{regular} \tag{3.23}$$

with a residue Φ_{-1}. The residue Φ_{-1} is diagonal when the Seiberg-Witten differential λ has a single pole there, representing the hypermultiplet mass term. Even when the hypermultiplet mass term is zero, Φ_{-1} can be a nonzero nilpotent matrix. The Jordan decomposition of this nilpotent matrix captures the important data of a singularity. There is now a method of reproducing the number of hypermultiplets and its flavor symmetry given the type of three singularities [18, 38].

Half 20 and three fundamentals. To apply this technology, first take a half-hypermultiplet of **20** and add three copies of fundamentals so that the total b is 6:

$$z(x_1 - m_1)(x_1 - m_2)(x_1 - m_3)i(w_2 x_1 + w_3 + 2x_1^3 + 2ix_2) =$$
$$x_2 x_3 + x_1^6 + w_2 x_1^4 + w_3 x_1^3 + w_4 x_1^2 + w_5 x_1 + w_6, \quad (3.24)$$

which can be readily made into the form (3.20). One finds the following:

- at $z = 0$, Φ_{-1} is a generic diagonal matrix. This puncture is called the full puncture.

- at $z = 1$, $\Phi_{-1} \propto \mathrm{diag}(1, 1, 1, 1, -2, -2)$ when $m_1 + m_2 + m_3 \neq 0$. When $m_1 + m_2 + m_3 = 0$, the order p_k of the pole of φ_k is given by $(p_2, p_3, p_4, p_5, p_6) = (1, 1, 2, 2, 2)$, which corresponds to a nilpotent $\Phi_{-1} = N_2 \oplus N_2 \oplus N_1 \oplus N_1$ where N_c is a $c \times c$ Jordan block. This corresponds to the puncture labeled by a partition $[4, 2]$.

- at $z = \infty$, $\Phi_{-1} \propto \mathrm{diag}(a, a, b, b, c, c)$ where (a, b, c) is the traceless part of (m_1, m_2, m_3). When it is zero, the orders of the poles of φ_k are given by $(p_2, p_3, p_4, p_5, p_6) = (1, 2, 2, 3, 4)$, which corresponds to a nilpotent $\Phi_{-1} = N_3 \oplus N_3$. This corresponds to the puncture labeled by a partition $[2, 2, 2]$.

Using the method explained in [38], it is straightforward to find that this three-punctured sphere has the requisite properties to be identified with $\frac{1}{2}\mathbf{20}$ plus three copies of $\mathbf{6}$. For example, let us check that the $\mathcal{N} = 2$ system defined by this three-punctured sphere does not have any Coulomb branch, as is required for any theory consisting only of hypermultiplets. In general, the dimension of the Coulomb branch of a sphere with many punctures is obtained by the formula [18, 38, 45]

$$\dim(\text{Coulomb branch}) = \sum_i \dim_{\mathbb{C}} O_G(\Phi_{-1,i}) - 2 \dim G \quad (3.25)$$

where $\Phi_{-1,i}$ is the residue at the i-th singularity and $O_G(\Phi)$ is the space of elements conjugate to Φ under $G_{\mathbb{C}}$. In our current situation, we have

$$\dim O(\text{generic}) = 30, \quad \dim O(N_2 \oplus N_2 \oplus N_1 \oplus N_1) = 16, \quad \dim O(N_3 \oplus N_3) = 24. \quad (3.26)$$

Then the dimension of the Coulomb branch is

$$\dim(\text{Coulomb branch}) = 30 + 16 + 24 - 2 \cdot 35 = 0, \quad (3.27)$$

which is zero as it should be.

Full 20. Next, consider the full half-hypermultiplet **20**, which in itself satisfy $b_{20} = 6$. We rewrite the geometry (3.17) into (3.20). We find three singularities of φ_k:

- at $z = 0$, one has again the full puncture, with the flavor symmetry SU(6).

- at $z = 1$, one finds that φ_3 and φ_5 have the branch cut of the form $\sim (z-1)^{1/2}$. We find that the orders of poles are given by $(p_2, p_3, p_4, p_5, p_6) = (1, \frac{3}{2}, 3, \frac{5}{2}, 3)$, but by redefining the Casimirs as

$$\varphi_2' = \varphi_2, \ \varphi_3' = \varphi_3, \ \varphi_4' = \varphi_4 - \frac{1}{4}\varphi_2{}^2, \ \varphi_5' = \varphi_5 - \frac{1}{2}\varphi_2\varphi_3, \ \varphi_6' = \varphi_6 - \frac{1}{4}\varphi_3{}^2, \quad (3.28)$$

we find the poles are given by $(p_2', p_3', p_4', p_5', p_6') = (1, \frac{3}{2}, 1, \frac{3}{2}, 2)$.

- at $z = \infty$, one similarly finds that φ_3 and φ_5 have branch cuts. The orders of poles in the massless case are given by $(p_2, p_3, p_4, p_5, p_6) = (1, \frac{3}{2}, 3, \frac{7}{2}, 4)$, which can not be further reduced by redefinitions of Casimirs. When the hypermultiplet is massive, we have

$$\varphi_2(z) \sim -3m^2 \frac{(dz)^2}{z^2}, \qquad \varphi_3(z) \sim \frac{(dz)^3}{z^{3/2}}, \qquad \varphi_4(z) \sim +3m^4 \frac{(dz)^4}{z^4}, \qquad (3.29)$$

$$\varphi_5(z) \sim \frac{(dz)^5}{z^{7/2}}, \qquad \varphi_6(z) \sim -m^6 \frac{(dz)^6}{z^6}. \qquad (3.30)$$

The presence of branch cuts means that the 6d construction involves the $\mathbb{Z}_2$ outer automorphism, as in [46]. This means that the Hitchin field is transposed, $\Phi(z) \to \sigma(\Phi(z))$, when we go around the puncture. Here $\sigma(X)$ can be

$$\sigma(X) = -X^t. \qquad (3.31)$$

But this choice of σ does not leave any diagonal matrix invariant, and thus inconvenient. Instead let us use σ defined by

$$\sigma(E_{i,j}) = -(-1)^{i+j} E_{7-j,7-i} \qquad (3.32)$$

where $E_{i,j}$ is the matrix with 1 at (i,j)-th entry and zero otherwise. Then the diagonal matrix of the form

$$\text{diag}(a, b, c, -c, -b, -a) \qquad (3.33)$$

is preserved; we can check that the twist σ preserves USp(6) subgroup of SU(6). So, when we have branch cuts, we expect Φ to behave as

$$\Phi(t) \sim \Phi_{-1} \frac{dt}{t} + \Phi_{-1/2} \frac{dt}{t^{1/2}} + \Phi_0 \frac{dt}{t^0} + \text{lower order terms} \qquad (3.34)$$

so that $\sigma(\Phi_{-1}) = \Phi_{-1}$, $\sigma(\Phi_{-1/2}) = -\Phi_{-1/2}$ and $\sigma(\Phi_0) = \Phi_0$. The leading term Φ_{-1} determines the property of the puncture, and $\Phi_{-1/2}$ and Φ_0 will be generic. We find that

- at $z = 1$, the choice $\Phi_{-1} = 0$ reproduces the behavior of φ_k.

- at $z = \infty$, we can reproduce the behavior of φ_k by choosing $\Phi_{-1} = N_2 \oplus N_2 \oplus N_2$ when $m = 0$, and $\Phi_{-1} \propto \text{diag}(1,1,1,-1,-1,-1)$ when $m \neq 0$. These massive and massless Φ_{-1} are the one associated to the partition $[3,3]$ if there is no branch cut.

- at $z = 0$, we do not have any branch cut, and Φ_{-1} is just a generic diagonal matrix.

Thus, at least we have checked that our polynomial X_R can come from a singularity of the Hitchin field Φ. Let us check that the $\mathcal{N} = 2$ theory defined by these punctures does not have the Coulomb branch, as it should be for a theory of free hypermultiplets. In our situation, the formula (3.25) cannot be directly used, because some of the punctures have branch cuts. In this case we need to use the formula [45]

$$\dim(\text{Coulomb branch}) = \sum_i \dim_{\mathbb{C}} O_{G_i}(\Phi_{-1,i}) - 2 \dim G_{\text{common}} \tag{3.35}$$

where G_i is the gauge group preserved by the twist at the i-th puncture, and G_{common} is the common subgroup of all G_i. To apply this formula to our current case, we use

$$O_{\text{SU}(6)}(\text{generic}) = 30, \quad O_{\text{USp}(6)}(0) = 0, \quad O_{\text{USp}(6)}(N_2 \oplus N_2 \oplus N_2) = 12 \tag{3.36}$$

and $G_{\text{common}} = \text{USp}(6)$. Then

$$\dim(\text{Coulomb branch}) = 30 + 0 + 12 - 2 \cdot 21 = 0, \tag{3.37}$$

as it should be.

It is not yet developed how the properties of the hypermultiplet can be recovered the choice of the singularities of the Hitchin field in the presence of the branch cuts of 6d A_{n-1} theory, so we cannot perform further checks in this case.

Six fundamentals. As a comparison, consider six fundamentals, which also have total $b = 6$. The geometry is

$$z \prod_{i=1}^{6} (x - m_i) = x_2^2 + x_3^2 + x_1^6 + w_2 x_1^4 + w_3 x_1^3 + w_4 x_1^2 + w_5 x_1 + w_6, \tag{3.38}$$

which can be made into the form (3.20) very easily. One finds [18]

- At $z = 0$ and $z = \infty$, one finds the full puncture, each carrying SU(6) flavor symmetry.

- At $z = 1$, the poles behave as $(p_2, p_3, p_4, p_5, p_6) = (1,1,1,1,1)$. The residue of the Hitchin field is given by $\Phi_{-1} \propto M(1,1,1,1,1,-5)$ where $M = \sum m_i$ in general; it becomes $\Phi_1 = N_2 \oplus N_1 \oplus N_1 \oplus N_1 \oplus N_1$ when $M = 0$. This singularity is usually called the simple puncture.

Summary. Thus, we saw the three cases, i) half **20** and three fundamentals, ii) full **20**, iii) six fundamentals, all have realizations in terms of 6d $\mathcal{N} = (2,0)$ theory of type A_5 on three-punctured spheres. We have found that the choice of punctures in each case was completely different from each other; the second case involved even a $\mathbb{Z}_2$ outer automorphism. These diverse examples, however, all came from the same ingredient $X_{\mathbf{20}}$ and $X_{\mathbf{6}}$ within the approach of this paper.

As another intriguing consequence of this analysis, let us consider superconformal $SU(6)$ gauge theory with one **20** and six fundamentals. The Seiberg geometry is given by

$$z + \frac{q}{z} X_{\mathbf{20}} X_{\mathbf{6}}{}^6 = W_{A_5}. \tag{3.39}$$

One can redefine z to rewrite the geometry in the form

$$z' X_{\mathbf{6}}{}^6 + \frac{q}{z'} X_{\mathbf{20}} = W_{A_5}. \tag{3.40}$$

As a six-dimensional theory living on the sphere parameterized by z', we have a simple puncture and a full puncture on the left giving six fundamentals, and two punctures with branch cuts on the right giving a full **20**. One can also redefine z in such a way to have the geometry of the form

$$z'' X_{\mathbf{6}}{}^3 X_{\frac{1}{2}\mathbf{20}} + \frac{q}{z''} X_{\mathbf{6}}^3 X_{\mathbf{20}} = W_{A_5}. \tag{3.41}$$

Then, the 6d theory living on the sphere parameterized on z'' has, on both sides, one puncture of type $[4, 2]$ and another of $[2, 2, 2]$. This illustrates the fact that the same Seiberg-Witten geometry (3.39) can lead to two completely different combinations of codimension-two defects on the 6d theory, when the choice of the base was changed from z' to z'', see figure 2.

3.3   56 of E_7

Let us next consider **56** of E_7, whose polynomial is given by

$$\begin{aligned}
X_{\mathbf{56}}(x_i; w_i; m) = {}& x_2^2 + m^2(-6x_1 x_2 - 4w_{10}) + 8im^3 x_3 \\
& + m^4(-3x_1^2 - 6w_2 x_2 - 4w_8) + m^6(2w_2 x_1 - 10x_2 - 4w_6) \\
& + m^8(6x_1 + w_2^2) + 2w_2 m^{10} + m^{12}. \quad (3.42)
\end{aligned}$$

For a half-hypermultiplet, the polynomial is very simple, $X_{\frac{1}{2}\mathbf{56}} = x_2$.

Behavior of Casimirs. The total b becomes $h^\vee = 18$ if we use three half-hypermultiplets. The geometry is then

$$\begin{aligned}
z x_2 X_{\mathbf{56}}(x_i; w_i; m) = {}& x_1^3 x_2 + x_2^3 + x_3^2 \\
& + w_2 x_2^2 x_1 + w_6 x_2^2 + w_8 x_2 x_1 + w_{10} x_1^2 + w_{12} x_2 + w_{14} x_1 + w_{18}. \quad (3.43)
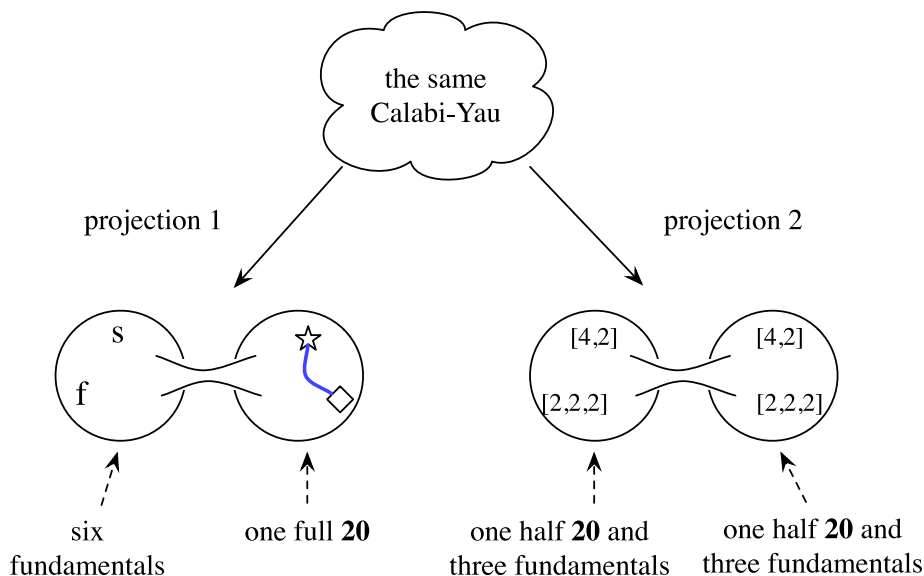\end{aligned}$$

Figure 2. The same non-compact Calabi-Yau geometry (3.39) leads to two completely different sets of codimension-two "punctures" as 6d $\mathcal{N} = (2,0)$ theory on a sphere. On the left, we chose the projection to the sphere parameterized by z', (3.40). The pair of a simple puncture and a full puncture gives us six fundamentals, and the pair of two punctures with $\mathbb{Z}_2$ branch cuts produces one full hypermultiplet in **20**. On the right, the projection was to the sphere parameterized by z'', (3.41). Two pairs of a puncture of type $[4,2]$ and another of type $[2,2,2]$ each give one half-hypermultiplet in **20** and three fundamentals.

Let us study this geometry from the viewpoint of $\mathcal{N} = (2,0)$ theory of type E_7. We first rewrite it in the form

$$0 = x_1^3 x_2 + x_2^3 + x_3^2 + w_2(z)x_2{}^2 x_1 + w_6(z)x_2{}^2$$
$$+ w_8(z)x_2 x_1 + w_{10}(z)x_1{}^2 + w_{12}(z)x_2 + w_{14}(z)x_1 + w_{18}(z), \quad (3.44)$$

and consider $\varphi_k(z) = w_k(z)(dz/z)^k$ as the worldvolume fields. We find the following singularities:

- at $z = 0$, we have a singularity where $\varphi_k(z) \sim w_k(dz/z)^k$.

- at $z = 1$, the order of the poles of $\varphi_k(z)$ is given by

$$(p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}) = (1, 2, 2, 2, 3, 3, 4). \quad (3.45)$$

  This cannot be further reduced by redefinition of the Casimirs. This does not depend on m, either.

- at $z = \infty$ with $m \neq 0$, the leading order of $\varphi_k(z)$ behaves as

$$w_2(t) \sim -6m^2, \qquad w_6(t) \sim +10m^6, \qquad w_8(t) \sim -3m^8, \qquad w_{10}(t) \sim 0,$$
$$w_{12}(t) \sim -2m^{12}, \qquad w_{14}(t) \sim 0, \qquad w_{18}(t) \sim 0 \qquad (3.46)$$

where $t = 1/z$. $\varphi_k(z)$ is obtained by multiplying $w_k(t)$ by $(dt/t)^k$. When $m = 0$, the order of the poles is given by

$$(p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}) = (1, 4, 6, 8, 9, 11, 14). \tag{3.47}$$

Behavior of the Hitchin field. Any one of these behaviors should come from a singularity of the Hitchin field, $\Phi(t) \sim \Phi_{-1} dt/t + \text{regular}$. Indeed, by trial and error, one finds that the following choices do the job:

- at $z = 0$, we just have to take Φ_{-1} to be a general Cartan element h:

$$\Phi(z) = h \frac{dz}{z} + \text{regular}. \tag{3.48}$$

  This is the full puncture.

- at $z = 1$, we can take Φ_{-1} to be a raising operator N_α corresponding to any root α:

$$\Phi(t) = N_\alpha \frac{dt}{t} + \text{regular} \tag{3.49}$$

  where $t = z - 1$. The pole behavior (3.45) was checked by choosing a random regular element, taking $\text{tr}\Phi(t)^k$ for $k = 2, \ldots, 18$, and finally extracting the Casimirs using the formulas in [35, 36].[7] This orbit N_α is known to be rigid, i.e. cannot be deformed to include mass terms, see e.g. appendix A of [47].

- at $z = \infty$, we can take Φ_{-1} to be the fundamental weight vector v_{center} corresponding to the central node of E_7 Dynkin diagram where three legs are joined:

$$\Phi(t) = m v_{\text{center}} \frac{dt}{t} + \text{regular}. \tag{3.50}$$

  This can be readily translated to the behavior of $w_k(t)$ using the formula given in [35], reproducing (3.46). When the mass parameter m is turned off, the adjoint element conjugate to $m v_{\text{center}}$ is known (see e.g. [48]) to degenerate to a nilpotent element $N_{A_4+A_2}$ of the Bala-Carter label $A_4 + A_2$:

$$\Phi(t) = N_{A_4+A_2} \frac{dt}{t} + \text{regular}. \tag{3.51}$$

  The Bala-Carter label means the following: one takes a subgroup of the corresponding type,

$$\text{SU}(5) \times \text{SU}(3) \subset E_7, \tag{3.52}$$

  which is apparent from the Dynkin diagram. Then, we take the biggest Jordan blocks in the subgroup:

$$N_5 \oplus N_3 \in \text{su}(5) \oplus \text{su}(3) \subset E_7. \tag{3.53}$$

  Then $N_{A_4+A_2} = N_5 \oplus N_3$ is the same element regarded as an element of E_7. Again, the behavior of the poles can be computed and it reproduces (3.47).

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[7]The structure constants of any simple Lie algebra are available in electronically readable form in the Lie algebra package of GAP. The authors thank David Vegh for the help.

Thus we identified the type of the three punctures on the sphere, on which 6d $\mathcal{N} = (2,0)$ theory is compactified. Let us perform one final check by confirming that the Coulomb branch of this theory is free is zero dimensional. Again by referring to e.g. [48], we find

$$\dim_{\mathbb{C}} O(h) = 126, \quad \dim_{\mathbb{C}} O(N_\alpha) = 34, \quad \dim_{\mathbb{C}} O(N_{A_4+A_2}) = 106. \tag{3.54}$$

Recalling $\dim E_7 = 133$ and using (3.25), we find

$$\dim(\text{Coulomb branch}) = 126 + 34 + 106 - 2 \cdot 133 = 0, \tag{3.55}$$

as it should be.

4 Conclusions

In this paper we argued that the Seiberg-Witten geometry of any single simply-laced gauge group with matter content such that any irreducible representation R in it satisfies $b_R \leq h^\vee$ has a universal form given in (1.1). The basic point was that Type IIB string on the non-compact Calabi-Yau given by

$$zX_R(x_1, x_2, x_3; w_i; m) = W_G(x_1, x_2, x_3; w_i) \tag{4.1}$$

represents the hypermultiplets in the representation R of the group G. We saw how the general geometry (1.1) can be built from the basic ingredients of the form (4.1).

We then analyzed these Seiberg-Witten solutions from the point of view of the 6d $\mathcal{N} = (2,0)$ theory in detail for three cases: the first was the 2-index antisymmetric tensor of SU(n), the second was **20** of SU(6), and the third was **56** of E_7. We studied the types of the punctures of 6d $\mathcal{N} = (2,0)$ theory, starting from the equations of the non-compact Calabi-Yau, and identified the residues of the adjoint one-form Φ on the sphere. We found that the Calabi-Yau geometries previously found via various methods can all consistently be interpreted as arising from the 6d $\mathcal{N} = (2,0)$ theory on a sphere.

There are a few obvious directions of further research. One is to extend our analysis to non-simply-laced gauge groups. This will involve $\mathbb{Z}_2$ or $\mathbb{Z}_3$ outer-automorphism, as was first seen in [4]. Another is to find the Seiberg-Witten solution to the 'reasonable' cases, i.e. when the matter content involves a representation R in the range $h^\vee < b_R \leq 2h^\vee$, see table 3. The curves for the theories with massive adjoints or massive 2-index symmetric tensor of SU(n) are known [33, 34]. So, the problematic ones are the 3-index antisymmetric tensor of SU(7) and SU(8), and the spinor of SO(14). If the matter content is just one copy of one of these representations, then they can be obtained by starting from the $\mathcal{N} = 2^*$ theory with E_7 or E_8 gauge group and giving an appropriate vev to the adjoint scalar. Therefore, the curves for these cases are implicitly known. The main problem would be to add additional matter fields in the fundamental representation to these theories.

Finally, we need to note that although we have determined X_R for all nice representations through various means, we do not yet have a direct understanding of the relation between the representation R and the polynomial X_R. It would be desirable to have a more uniform, logical way which allows us to write down the polynomial X_R given the weights of R.

A List of hypermultiplet geometries

A.1 X_R for $SU(n) = A_{n-1}$

The ALE space is given either by

$$W_{A_{n-1}} = x_1^n + x_2^2 + x_3^2 + w_2 x_1^{n-2} + w_3 x_1^{n-3} + \cdots + w_n \tag{A.1}$$

or

$$W_{A'_{n-1}} = x_1^n + x_2 x_3 + w_2 x_1^{n-2} + w_3 x_1^{n-3} + \cdots + w_n. \tag{A.2}$$

The relation between w_k and the Cartan element $\phi = \mathrm{diag}(a_1, \ldots, a_n)$ is standard:

$$x^n + w_2 x^{n-2} + \cdots + w_n = \prod_{i=1}^{n} (x - a_i). \tag{A.3}$$

For the fundamental, we have

$$X_{\mathrm{fund}}(x_i; w_i; m) = x_1 - m. \tag{A.4}$$

This formula goes back to [22, 23].

For the 2-index antisymmetric representation, we use $W_{A'_{n-1}}$ with

$$X_{\text{2-index antisym.}}(x_i; w_i; m) = -x_3 + T(x_1, x_2; w_i; m) \tag{A.5}$$

where $T(x_1, x_2; w_i; m)$ is defined as follows. Let $P(x; w_i) = \prod_{i=1}^{n}(x - a_i)$, and define two polynomials $R(u; w_i; m)$ and $S(u; w_i; m)$ via

$$P\left(x + \frac{m}{2}; w_i\right) = R(x^2; w_i; m) + x S(x; w_i; m). \tag{A.6}$$

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Then $T(x_1, x_2; w_i; m)$ is given by

$$P(x_1; w_i) + x_2 T(x_1, x_2; w_i; m) =$$
$$R\left(\left(x_1 - \frac{m}{2}\right)^2 + x_2; w_i; m\right) + \left(x_1 - \frac{m}{2}\right)^2 S\left(\left(x_1 - \frac{m}{2}\right)^2 + x_2; w_i; m\right) \quad \text{(A.7)}$$

These polynomials can be straightforwardly extracted from the curve given in [32]. The curve was interpreted from the point of view of 6d $\mathcal{N} = (2,0)$ theory in [49] and the punctures were identified there. We can check that these polynomials for $n < 7$ indeed coincide with the those obtained in [30].

For the 3-index antisymmetric representation $\mathbf{20}$ of SU(6), we use W_{A_5} and

$$X_{\mathbf{20}}(x_i; w_i; m) = -(w_2 x_1 + w_3 + 2x_1^3 + 2ix_2)^2$$
$$+ m^2(-12x_1^4 - 12ix_1 x_2 - 6w_2 x_1^2 - 6w_3 x_1 - 4w_4 + w_2^2)$$
$$+ 8m^3 x_3 + m^4(3x_1^2 + 2w_2) + m^6. \quad \text{(A.8)}$$

For a half-hypermultiplet in this representation, we just set $m = 0$ and take the square root:

$$X_{\frac{1}{2} \cdot \mathbf{20}}(x_i; w_i) = i(w_2 x_1 + w_3 + 2x_1^3 + 2ix_2). \quad \text{(A.9)}$$

The detailed analysis of this polynomial was given in section 3.2.

A.2 X_R for SO$(2n) = D_n$

The ALE space is given by

$$W_{D_n} = x_1^{n-1} + x_1 x_2^2 - x_3^2 + w_2 x_1^{n-2} + w_4 x_1^{n-3} + \cdots + w_{2n-2} + \tilde{w}_n x_2. \quad \text{(A.10)}$$

The relation between w_k, $\tilde{w}_n$ and the Casimirs $\phi = \text{diag}(\pm a_i)$ is standard:

$$x^n + w_2 x^{n-1} + w_4 x^{n-2} + \cdots + w_{2n-2} x + w_{2n} = \prod_{i=1}^{n}(x - a_i^2) \quad \text{(A.11)}$$

and

$$\tilde{w}_n = 2i^{n+1} a_1 a_2 \cdots a_n. \quad \text{(A.12)}$$

For the vector representation, we have

$$X_{\text{vector}}(x_i; w_i; m) = x_1 - m^2. \quad \text{(A.13)}$$

This goes back to [25, 26].

For the SO(8) spinor $\mathbf{8s}$, we have

$$X_{\mathbf{8s}}(x_i; w_i; m) = \frac{1}{2}x_1 + \frac{i}{2}x_2 + \frac{1}{4}w_2 + m^2. \quad \text{(A.14)}$$

The polynomial for the conjugate spinor $\mathbf{8c}$ can be obtained by applying the outer automorphism $x_2 \to -x_2$:

$$X_{\mathbf{8c}}(x_i; w_i; m) = \frac{1}{2}x_1 - \frac{i}{2}x_2 + \frac{1}{4}w_2 + m^2. \quad \text{(A.15)}$$

This was determined in [30]. The three-punctured spheres on which 6d $\mathcal{N} = (2,0)$ theory of type D_4 is to be compactified to produce these matter contents were analyzed in [39, 46]; it would be interesting to confirm the agreement.

For the SO(10) spinor **16**, we have

$$X_{\mathbf{16}}(x_i; w_i; m) = -\frac{1}{2}x_1^2 - \frac{1}{4}w_2 x_1 - \frac{1}{4}\left(w_4 - \frac{1}{4}w_2^2\right) + \frac{1}{2}x_3$$
$$- m x_2 + \left(x_1 + \frac{1}{2}w_2\right)m^2 + m^4 \quad \text{(A.16)}$$

The massless limit was determined in [28] and the massive case was found in [30].

For the SO(12) spinor **32s**, we have

$$X_{\mathbf{32s}}(x_i; w_i; m) = \frac{1}{256}(8ix_2 + 8x_1^2 + 4w_2 x_1 + 4w_4 - w_2^2)^2$$
$$+ m^2 \left(i\tilde{w}_6 + w_6 - \frac{1}{4}w_2 w_4 + \frac{1}{16}w_2^3 + 2w_2 x_1^2 - \frac{1}{8}w_2^2 x_1 + \frac{3}{2}w_4 x_1 + 3x_1^3 + 3ix_1 x_2 + \frac{i}{2}w_2 x_2\right)$$
$$+ 4im^3 x_3 + m^4 \left(3ix_2 + \frac{1}{2}w_2 x_1 + \frac{3}{8}w_2^2 - \frac{1}{2}w_4\right) + (2x_1 + w_2)m^6 + m^8. \quad \text{(A.17)}$$

The polynomial for the half-hypermultiplet is then

$$X_{\frac{1}{2}\mathbf{32s}}(x_i; w_i) = \frac{1}{16}(8ix_2 + 8x_1^2 + 4w_2 x_1 + 4w_4 - w_2^2). \quad \text{(A.18)}$$

The polynomials for the conjugate spinor **32c** can be obtained by applying the outer automorphism $x_2 \to -x_2$ and $\tilde{w}_6 \to -\tilde{w}_6$:

$$X_{\mathbf{32c}}(x_i; w_i; m) = \frac{1}{256}(-8ix_2 + 8x_1^2 + 4w_2 x_1 + 4w_4 - w_2^2)^2$$
$$+ m^2 \left(-i\tilde{w}_6 + w_6 - \frac{1}{4}w_2 w_4 + \frac{1}{16}w_2^3 + 2w_2 x_1^2 - \frac{1}{8}w_2^2 x_1 + \frac{3}{2}w_4 x_1 + 3x_1^3 - 3ix_1 x_2 - \frac{i}{2}w_2 x_2\right)$$
$$+ 4im^3 x_3 + m^4 \left(-3ix_2 + \frac{1}{2}w_2 x_1 + \frac{3}{8}w_2^2 - \frac{1}{2}w_4\right) + (2x_1 + w_2)m^6 + m^8. \quad \text{(A.19)}$$

The massless limit was first obtained in [28] and the massive case was found in [31]. For both SO(12) and SO(10) spinors, it should be possible to extend the analysis of [39, 46] to find the realization via 6d D_6 and D_5 theory and compare the resulting curves.

A.3 X_R for E_n

The ALE space for E_6 is given by

$$W_{E_6} = x_1^4 + x_2^3 + x_3^2 + w_2 x_1^2 x_2 + w_5 x_1 x_2 + w_6 x_1^2 + w_8 x_2 + w_9 x_1 + w_{12}. \quad \text{(A.20)}$$

and the one for E_7 is

$$W_{E_7} = x_1^3 x_2 + x_2^3 + x_3^2 + w_2 x_2^2 x_1 + w_6 x_2^2 + w_8 x_2 x_1 + w_{10} x_1^2 + w_{12} x_2 + w_{14} x_1 + w_{18}. \quad \text{(A.21)}$$

The relation between w_k and the Cartan ϕ is given e.g. in appendices of [35]. Our normalization is slightly different from theirs: for E_6, our w_k and their e_k are related as

$$w_2 = e_2, \;\; w_5 = -\frac{e_5}{4}, \;\; w_6 = \frac{e_6}{4}, \;\; w_8 = \frac{e_8}{16}, \;\; w_9 = -\frac{e_9}{16}, \;\; w_{12} = \frac{e_{12}}{64}. \tag{A.22}$$

For E_7, the relations are

$$w_2 = -\frac{e_2}{4}, \;\; w_6 = \frac{e_6}{4}, \;\; w_8 = \frac{e_8}{16}, \;\; w_{10} = \frac{e_{10}}{64}, \;\; w_{12} = -\frac{e_{12}}{16}, \;\; w_{14} = -\frac{e_{14}}{64}, \;\; w_{18} = \frac{e_{18}}{64}. \tag{A.23}$$

For $\mathbf{27}$ of E_6, the polynomial is given by

$$\begin{aligned}
X_{\mathbf{27}}(x_i; w_i; m) = {}&{-}8\left(x_1^2 - ix_3 + \frac{1}{2}w_6\right) - 4w_2 x_2 \\
&+ 4mw_5 + m^2(w_2^2 - 12x_2) - 8m^3 x_1 + 2m^4 w_2 + m^6.
\end{aligned} \tag{A.24}$$

The massive case was determined in [29]. For $\mathbf{56}$ of E_7, we have

$$\begin{aligned}
X_{\mathbf{56}}(x_i; w_i; m) = {}& x_2^2 + m^2(-6x_1 x_2 - 4w_{10}) + 8im^3 x_3 \\
&+ m^4(-3x_1^2 - 6w_2 x_2 - 4w_8) + m^6(2w_2 x_1 - 10x_2 - 4w_6) \\
&+ m^8(6x_1 + w_2^2) + 2w_2 m^{10} + m^{12}
\end{aligned} \tag{A.25}$$

The polynomial for the half-hypermultiplet is then

$$X_{\frac{1}{2}\mathbf{56}}(x_i; w_i) = x_2. \tag{A.26}$$

The massless case was determined in [27]. The massive case was determined in [31].

B Seiberg-Witten curve from ALE geometry

As explained, the Seiberg-Witten curve can be constructed from the z dependent Casimirs $\tilde{w}$. If we found that the geometry can be written as (2.19), then by the redefinition $x_1' = k(z)^{d_1}\tilde{x}_1', x_2' = k(z)^{d_2}\tilde{y}_2', x_3' = k(z)^{d_3}\tilde{y}_3'$, where $k(z)$ is an arbitrary function z and d_i is the dimension of x_i', the geometry has the form of (2.19) again. This can be considered as the ambiguity of the Seiberg-Witten form λ, which we fixed, because this is just the coordinate change. Thus, in order to read off the $\tilde{w}$, we should fix this ambiguity. If we know the map between the type IIb description and 6d (2,0) theory description, we can fix it. However, for the cases we considered, we do not know it and instead of it we will assume that the Seiberg-Witten form satisfied the usual properties which are related to the positivity of the effective coupling constant.

In order to do that, we now study the behavior of the λ near the singularities. If the solution $x(z)$ of $P_R(x, \tilde{w}_i) = 0$ near $z = 0$ scales as

$$\left(\frac{x}{M}\right)^a \sim z^b, \tag{B.1}$$

where M is a dimension one constant and a, b are positive co-prime numbers, we introduce $\zeta^a = z$ which means $x/M \sim \zeta^b$. Then, we find

$$\lambda \sim aM\zeta^{b-1}d\zeta. \tag{B.2}$$

As usual, it will be required that allowed singularity is single pole and the residue is proportional to the mass of the theory. This implies $b = 0$ and M is a linear combination of the mass parameters (or $b \geq 1$). This means that at $z = 0$ and $z = \infty$, $\tilde{w}_i$ should take finite value which does not depend on w_i.

At another singularity $z = z_0$, we can assume that the solution $x(z)$ behaves

$$\left(\frac{x}{M}\right)^a \sim \frac{1}{h^b}, \tag{B.3}$$

where M is a dimension one constant and a, b are positive co-prime numbers. Here h is the deviation from the singularity: $z/z_0 - 1 = -h$ where c is a constant. We introduce $\zeta^a = h$ which means $x/M \sim \zeta^{-b}$. Then, we find

$$\lambda \sim -M \zeta^{a-b-1} d\zeta. \tag{B.4}$$

Thus, we require that $a - b \geq 1$ or $a = b (= 1)$. For the latter case, M should be independent of w_i. Another thing usually required is the condition

$$\frac{\partial \lambda}{\partial w_i} = \text{a holomorphic form}, \tag{B.5}$$

which indeed follows from the above requirements. If this is satisfied, then, the positive definiteness of the effective coupling constant is assured.

We have explicitly studied the several Seiberg-Witten geometries and found that the ambiguity is indeed fixed uniquely by the requirements for the geometries we studied. Note that it is highly non-trivial to find the $\tilde{w}_i$ which satisfies the requirements. Thus, the fact that we indeed found those $\tilde{w}_i$ is the evidences for the validity of the Seiberg-Witten geometry we proposed.

Finally, we will write down the explicit coordinate change for finding $\tilde{w}$ for some examples. For the geometry (3.17) of the full **20** of SU(6), the correct coordinate change is

$$\begin{aligned}
x_1' &= \frac{1}{\sqrt{h}} x_1 \\
x_2' &= \frac{1}{h} \left(x_2 + 4 i m^3 z \right) \\
x_3' &= \frac{1}{\sqrt{h}} \left(x_3 - 2 i \frac{1}{h} \left(3 m^2 x_1 + w_2 x_1 + 2 x_1{}^3 + w_3 \right) z \right),
\end{aligned} \tag{B.6}$$

where

$$h = 1 + 4z, \tag{B.7}$$

the Seiberg-Witten geometry becomes

$$h^2 W_{A_5}(x_1', x_2', x_3'; \tilde{w}) = 0, \tag{B.8}$$

where $\tilde{w}_i = \tilde{w}_i(z)$ satisfies the above conditions and the correct classical limit $\tilde{w}_i \to w_i$ with $z \to 0$. Of course, we can change $z \to -z/4$ in order to set the singularity at $z = 1$. There are artificial cuts at the singular points $h = 0$ and $h = \infty$, but, we find that the $\tilde{w}_i$ is uniquely defined if we think w_i transforms to $(-1)^i w_i$ when we go around the singular

points. Thus, despite the introduction of the artificial cut at the singular points $h = 0$ and $h = \infty$, we believe that that these $\tilde{w}_i$ give the correct curve.

For the geometry of the **56** of E_7, the correct coordinate change is

$$x_1' = \frac{x_1 - m^4 z}{h}, \quad x_2' = \frac{x_2}{h}, \quad x_3' = x_3 + 4im^3 x_2 z, \tag{B.9}$$

where $h = z + 1$, the geometry becomes

$$h^4 W_{E_7}(x_1', x_2', x_3'; \tilde{w}) = 0, \tag{B.10}$$

where $\tilde{w}_i$ satisfies the conditions.

The last example is the the geometry of the **27** of E_6:

$$z X_{\mathbf{27}}(x_i; w_i; m)^2 + W_{E_6}(x_1, x_2, x_3; w) = 0, \tag{B.11}$$

where we have set the all mass parameter are same. Using the coordinates:

$$\begin{aligned}
x_1' &= \frac{x_1 + 32m^3 z}{h} \\
x_2' &= \frac{h x_2 + \frac{16}{3} z (w_2 + 3m^3)^2}{h^2}, \\
x_3' &= \frac{h x_3 + 8iz(X_{E_6}(x_1, x_2, x_3 = 0))}{h^2},
\end{aligned} \tag{B.12}$$

where

$$h = 1 - 64z, \tag{B.13}$$

the SW geometry becomes

$$h^3 \, W_{E_6}(x_i'; \tilde{w}) = 0, \tag{B.14}$$

where $\tilde{w}_i = \tilde{w}_i(z)$ satisfy following conditions:

- correct classical limit

$$\tilde{w}_i \to w_i, \ as \ z \to 0, \tag{B.15}$$

- finite at $z = \infty$. Indeed, at $z = \infty$ we find[8]

$$\begin{aligned}
W_{E_6}(x_1', x_2', x_3'; \tilde{w}) &= x_1'^4 + x_2'^3 + x_3'^2 \\
&\quad - 3m^2 x_1'^2 x_2' + 3m^5 x_1' x_2' - \frac{3}{2} m^6 x_1'^2 - \frac{3}{4} m^8 x_2' + m^9 x_1' - \frac{3}{16} m^{12}. \tag{B.16}
\end{aligned}$$

- at the singular point $h = 0$, the leading behavior is

$$\begin{aligned}
W_{E_6}(x_1', x_2', x_3'; \tilde{w}) &\sim x_1'^4 + x_2'^3 + x_3'^2 - \frac{3m^2 + w_2}{h} x_1'^2 x_2' - \frac{1}{12} \frac{(3m^2 + w_2)^3}{h^3} x_1'^2 \\
&\quad - \frac{1}{48} \frac{(3m^2 + w_2)^4}{h^4} x_2' + \frac{1}{864} \frac{(3m^2 + w_2)^6}{h^6}. \tag{B.17}
\end{aligned}$$

The characteristic polynomial $P_{\mathbf{27}}$ with the $\tilde{w}$ becomes $x^{15}(x^2 - (6m^2 + 2w_2)/h)^2$ near $h = 0$.

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[8]If two mass parameters are different, say m_1, m_2, it is still finite, but complicated. For following two examples, we found some simple results. For $m_1 = -m_2 = m$, we find $W_{E_6}(x_1', x_2', x_3'; \tilde{w}) = x_1'^4 + x_2'^3 + x_3'^2 - 3m^2 x_1'^2 x_2' + m^6 x_1'^2$. For $m_2 = 0$, the characteristic polynomial is $x^9(x^2 - 4m^2)^3(x^2 - m^2)^6$.

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