Published for SISSA by Springer

Received: March 4, 2024
Revised: June 19, 2024
Accepted: July 4, 2024
Published: August 20, 2024

Low-energy theorem revisited and OPE in massless QCD

Marco Bochicchio[a] and **Elisabetta Pallante**[b,c]

[a] *Physics Department, INFN Roma1, Piazzale Aldo Moro 2, 00185, Rome, Italy*

[b] *Van Swinderen Institute for Particle Physics and Gravity, University of Groningen, 9747 AG, The Netherlands*

[c] *Nikhef, Science Park, Amsterdam, The Netherlands*

E-mail: marco.bochicchio@roma1.infn.it, e.pallante@rug.nl

Abstract: We revisit a low-energy theorem (LET) of NSVZ type in SU(N) QCD with N_f massless quarks derived in [1] by implementing it in dimensional regularization. The LET relates n-point correlators in the l.h.s. to $n+1$-point correlators with the extra insertion of $\mathrm{Tr}F^2$ at zero momentum in the r.h.s. We demonstrate that, for 2-point correlators of an operator O in the l.h.s., the LET implies that, in general, the integrated 3-point correlator in the r.h.s. needs in perturbation theory an infinite additive renormalization in addition to the multiplicative one. We relate the above counterterm to a corresponding divergent contact term in a certain coefficient of the OPE of $\mathrm{Tr}F^2$ with O in the momentum representation, thus extending to any operator O an independent argument that first appeared for $O = \mathrm{Tr}F^2$ in [2]. Finally, we demonstrate that in the asymptotically free phase of QCD the aforementioned counterterm in the LET is actually finite nonperturbatively after resummation to all perturbative orders. We also briefly recall the implications of the LET in the gauge-invariant framework of dimensional regularization for the perturbative and nonperturbative renormalization in large-N QCD. The implications of the LET inside and above the conformal window of SU(N) QCD with N_f massless quarks will appear in a forthcoming paper.

Keywords: Renormalization and Regularization, Renormalization Group

ArXiv ePrint: 2402.16490

https://doi.org/10.1007/JHEP08(2024)145

JHEP08(2024)145

Contents

1 Introduction and physics motivations

In the present paper we work out in dimensional regularization several equivalent versions of a low-energy theorem (LET) of NSVZ type in SU(N) QCD with N_f massless quarks derived in [1].

In fact, in the original NSVZ papers [4, 5] an identity has been demonstrated that relates nonperturbatively the glueball condensate $\langle \mathrm{Tr}\, F^2\rangle$ to the glueball propagator at zero momentum $\int \langle \mathrm{Tr}\, F^2(x)\, \mathrm{Tr}\, F^2(0)\rangle\, d^4x$ — or, more generally, a condensate to a suitable zero-momentum 2-point correlator — hence, the name LET.

In the present paper — following [1] — we study a vast generalization of the above identity, which we also refer to as LET — that relates n-point correlators in the l.h.s. to $n+1$-point correlators with the extra insertion of $\mathrm{Tr}\, F^2$ integrated over space-time — i.e. at zero momentum — in the r.h.s. Apart from its intrinsic interest, the above study has several applications that we divide in two installments.

The first installment — in the present paper — concerns the LET applied to 2-point correlators $C_0^{(O,O)'}(z) = \langle O(z)O(0)\rangle'$ at distinct points $z \neq 0$ of a multiplicatively renormalizable operator O in the l.h.s.

We demonstrate that the corresponding integrated 3-point correlator in the r.h.s. needs in general an infinite additive renormalization as $\epsilon \to 0$ in addition to the multiplicative one order by order in perturbation theory. The corresponding counterterm consists in the subtraction of an integrated — i.e. at zero momentum — divergent contact term proportional to a $\delta^{(4)}$ multiplying the 2-point correlator of O.

We also work out the relation of the LET — in our gauge-invariant framework provided by dimensional regularization — with a certain OPE coefficient of $\mathrm{Tr}\, F^2$ with an operator O defined by the solution of the Callan-Symanzik (CS) equation in the momentum representation, $C_1^{(F^2,O)}(p)$, its perturbative and nonperturbative renormalization in the asymptotically free (AF) phase of QCD, and the nonperturbative renormalization in large-N QCD briefly recalled below.

The second installment — in a forthcoming paper [6] — concerns the LET in the phases inside and above the conformal window of QCD, and at the Wilson-Fisher conformal fixed point in $4 - 2\epsilon$ dimensions. In fact, it depends crucially on some delicate results in the first one that we report in detail in the conclusions.

We now recall further relations of the LET in the present paper with the existing literature.

The LET has two versions [1], one involving in the l.h.s. the logarithmic derivative with respect to the gauge coupling, and one involving the logarithmic derivative with respect to the RG-invariant scale, Λ_{UV}, in YM theories AF in the UV.

Originally, the LET has been employed [1] to study the nonperturbative renormalization of large-N confining massless QCD-like theories and, in particular, massless QCD [7–10], since the second version of the LET and the nonperturbative renormalization [1] of Λ_{QCD} in the large-N 't Hooft [7] and Veneziano [8] expansions control [1] the structure of the nonperturbative counterterms for the YM action.

In fact, the second version of the LET provides an obstruction [11] to the existence of canonical string models implementing the open/closed string duality that would realize nonperturbatively the large-N 't Hooft expansion of QCD and massive $\mathcal{N} = 1$ SUSY QCD in the confining phase.

Besides, the first version of the LET demonstrates how the open/closed string duality may be implemented in canonical string models [11] perturbatively realizing massless QCD-like theories to order g^2. In this context an essential tool to actually compute the possible divergences in the r.h.s. of the LET has been the OPE of O with $\mathrm{Tr}\, F^2$, both in perturbation theory and in its RG-improved form by a hard-cutoff regularization of the space-time integral in the r.h.s. [11, 12].

The logic above can be inverted and the relevant contribution to order g^2 in the r.h.s. of the LET for the OPE of O with $\mathrm{Tr}\, F^2$ can be recovered [12] from the anomalous dimension of O in the l.h.s. by assuming the LET for the 2-point correlator of O.

We should also mention the approach to the OPE in [15] that is closely related to the LET, but actually independent of the present paper that employs dimensional regularization.

JHEP08(2024)145

2 Plan of the paper

In section 3 we work out several equivalent versions of the LET in dimensional regularization and establish their main implications with respect to the aforementioned additive renormalization.

In section 4 we investigate the relation between the LET and the OPE coefficient $C_1^{(F^2,O)}$ including contact terms, both in the coordinate and momentum representation. Moreover, we explicitly compute the corresponding finite and divergent contact terms for $C_1^{(F^2,F^2)}$ to order g^4.

In section 5 we compare the perturbative version of the LET with its nonperturbative resummation to all perturbative orders.

In section 6 we solve the LET in perturbation theory to order g^2, where the theory is exactly conformal.

In section 7 we investigate — analogously to $C_1^{(F^2,F^2)}$ in section 4 — the occurrence of finite and divergent contact terms for $C_0^{(F^2,F^2)}$ to order g^2, both in the coordinate and momentum representation.

In section 8 we employ one of our versions of the LET in dimensional regularization to verify in a gauge-invariant framework the implications of the aforementioned version in hard-cutoff regularization [11, 12], both perturbatively and nonperturbatively.

In section 9 we summarize the main results of the present paper.

In the appendices A–F we report several ancillary computations.

3 LET

The LET applies to YM theories in $d = 4$ dimensions.

3.1 LET and Wilsonian normalization of the action

Starting from the Euclidean functional integral of SU(N) YM theories in $d = 4$ dimensions, the LET has been derived for bare correlators with the Wilsonian normalization of the action [1]:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 = \frac{\int \mathcal{O}_1 \cdots \mathcal{O}_n \, e^{-\frac{N}{2g_0^2}\int \operatorname{Tr} \mathcal{F}^2 d^4x + \cdots}}{\int e^{-\frac{N}{2g_0^2}\int \operatorname{Tr} \mathcal{F}^2 d^4x + \cdots}} \tag{3.1}$$

with g_0 the bare 't Hooft coupling $g_0^2 = g_{0YM}^2 N$, Tr the trace in the fundamental representation, $\mathcal{O}_i$ local bare operators independent of g_0, $\operatorname{Tr} \mathcal{F}^2 \equiv \operatorname{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu})$, $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, and the sum over repeated indices understood. We explicitly write in eq. (3.1) only the term in the action that depends on g_0 and, therefore, enters the derivation of the LET. We immediately arrive at the LET by deriving eq. (3.1) with respect to $-1/g_0^2$:

$$\frac{\partial}{\partial \log g_0} \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 = \frac{N}{2g_0^2} \int \langle \mathcal{O}_1 \cdots \mathcal{O}_n \mathcal{F}^2(x) \rangle_0 - \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 \langle \mathcal{F}^2(x) \rangle_0 \, d^4x \tag{3.2}$$

with $\mathcal{F}^2 \equiv 2 \operatorname{Tr} \mathcal{F}^2$.

3.2 LET and canonical normalization of the action

We rescale the gauge fields in eq. (3.1) by the factor $\frac{g_0}{\sqrt{N}}$ in order to rewrite the LET for the canonically normalized YM action [12]. Defining $\frac{g_0^2}{N}\,\mathrm{Tr}\, F^2 = \mathrm{Tr}\,\mathcal{F}^2$ and $(\frac{g_0}{\sqrt{N}})^{c_{O_k}} O_k = \mathcal{O}_k$ for some c_{O_k} — for example, $c_{F^2} = 2$ — after the rescaling, with O_k now dependent on g_0 but canonically normalized and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{g_0}{\sqrt{N}}[A_\mu, A_\nu]$, we get the identity [12]:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 = \prod_{k=1}^{k=n} \left(\frac{g_0}{\sqrt{N}}\right)^{c_{O_k}} \langle O_1 \cdots O_n \rangle_0 \tag{3.3}$$

with the expectation values in the l.h.s. and r.h.s. defined by the Wilsonian and canonical normalization, respectively. The LET in eq. (3.2) is rewritten in terms of canonically normalized bare local operators [12] by employing eq. (3.3):

$$\begin{aligned}
&\sum_{k=1}^{k=n} c_{O_k} \langle O_1 \cdots O_n \rangle_0 + \frac{\partial}{\partial \log g_0} \langle O_1 \cdots O_n \rangle_0 \\
&\quad = \frac{1}{2} \int \langle O_1 \cdots O_n F^2(x) \rangle_0 - \langle O_1 \cdots O_n \rangle_0 \langle F^2(x) \rangle_0 \, d^4x
\end{aligned} \tag{3.4}$$

with $F^2 \equiv 2\,\mathrm{Tr}\, F^2$. In fact, we only consider the applications of the LET for $n = 2$ and $O_1 = O_2 = O$:

$$\begin{aligned}
&2c_O \langle O(z)O(0) \rangle_0 + \frac{\partial}{\partial \log g_0} \langle O(z)O(0) \rangle_0 \\
&\quad = \frac{1}{2} \int \langle O(z)O(0)F^2(x) \rangle_0 - \langle O(z)O(0) \rangle_0 \langle F^2(x) \rangle_0 \, d^4x
\end{aligned} \tag{3.5}$$

3.3 Dimensional regularization

In general, the bare correlators are divergent and need regularization and, eventually, renormalization. Therefore, the first issue to apply the LET to YM theories is to find a regularization of the bare correlators and, correspondingly, of the LET. Dimensional regularization preserves gauge invariance to all orders of perturbation theory and it is our natural choice. It may be performed both in the coordinate and momentum representation for the correlators of the fundamental fields and the corresponding composite operators.

3.4 Dimensional regularization of YM theories

In dimensional regularization the space-time dimension is shifted:

$$d \to \tilde{d} = d - 2\epsilon \tag{3.6}$$

with ϵ positive and small and, in the present paper, $d = 4$. The canonical dimension of the bare operators in the action follows by dimensional analysis, for the action to be dimensionless in $\tilde{d}$ dimensions. In particular, the canonically normalized YM action formally reads in $\tilde{d}$ dimensions:

$$S = \frac{1}{2} \int \mathrm{Tr}\, F_0^2 \, d^{\tilde{d}}x \tag{3.7}$$

with $\operatorname{Tr} F_0^2 = \operatorname{Tr}(F_{\mu\nu}F_{\mu\nu})$ the bare operator. Given $[d^{\tilde{d}}x] = -\tilde{d}$, with $[.]$ the mass dimension of its argument, $[S] = 0$ implies:

$$[\operatorname{Tr} F_0^2] \equiv \tilde{\Delta}_{F_0^2} = \tilde{d} \tag{3.8}$$

As $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{g_0}{\sqrt{N}}[A_\mu, A_\nu]$, the canonical dimension of the bare gauge field A_μ follows:

$$\begin{aligned} [A_\mu] &= \frac{\tilde{\Delta}_{F_0^2}}{2} - 1 \\ &= \frac{\tilde{d}}{2} - 1 \\ &\overset{d=4}{=} 1 - \epsilon \end{aligned} \tag{3.9}$$

Besides, since $[\partial_\mu] = 1$, eq. (3.9) implies for the bare gauge coupling:

$$\begin{aligned} [g_0] &= -2[A_\mu] + \frac{\tilde{d}}{2} \\ &= 2 - \frac{\tilde{d}}{2} \\ &\overset{d=4}{=} \epsilon \end{aligned} \tag{3.10}$$

We recall some relations needed to rewrite the LET in terms of the renormalized coupling and correlators in dimensionally regularized YM theories. According to eq. (3.10) we get:

$$g_0 = Z_g(g, \epsilon)\mu^\epsilon g \tag{3.11}$$

where g is the dimensionless renormalized gauge coupling and Z_g the renormalization factor that in $\overline{MS}$-like renormalization schemes is a series of pure poles in ϵ — as it is the renormalization factor Z_O for a multiplicatively renormalizable operator O. Moreover, the beta function reads in $\tilde{d} = 4 - 2\epsilon$ dimensions:

$$\beta(g, \epsilon) = -\epsilon g + \beta(g) \tag{3.12}$$

with:

$$\beta(g, \epsilon) = \frac{dg}{d \log \mu} \tag{3.13}$$

and:

$$\begin{aligned} \beta(g) &= -g \frac{d \log Z_g}{d \log \mu} \\ &= \left. \frac{dg}{d \log \mu} \right|_{\epsilon=0} \end{aligned} \tag{3.14}$$

the beta function in $d = 4$ dimensions. We obtain from eq. (A.6):

$$\begin{aligned} \frac{\partial \log g}{\partial \log g_0} &= \left(1 + \frac{\partial \log Z_g}{\partial \log g}\right)^{-1} \\ &= 1 - \frac{\beta(g)}{\epsilon g} \\ &= Z_{F^2}^{-1}(g, \epsilon) \end{aligned} \tag{3.15}$$

that follows from the logarithmic derivative of eq. (3.11) by equating eq. (3.12) to:

$$\beta(g,\epsilon) = -\epsilon g \left(1 + \frac{\partial \log Z_g}{\partial \log g}\right)^{-1} \tag{3.16}$$

Finally, from the anomalous dimension γ_O of a multiplicatively renormalizable operator $O = Z_O O_0$:

$$\gamma_O(g) = -\frac{d \log Z_O}{d \log \mu} \tag{3.17}$$

we obtain:

$$\begin{aligned}
\frac{\partial \log Z_O^{-2}}{\partial \log g_0} &= -2\frac{\partial \log Z_O}{\partial \log g_0} \\
&= -2\frac{d \log Z_O}{d \log \mu}\frac{d \log \mu}{d \log g}\frac{\partial \log g}{\partial \log g_0} \\
&= 2\gamma_O(g)\left(\frac{1}{g}\frac{dg}{d \log \mu}\right)^{-1}\left(1 + \frac{\partial \log Z_g}{\partial \log g}\right)^{-1} \\
&= 2\gamma_O(g)\left(\frac{\beta(g,\epsilon)}{g}\right)^{-1}\left(1 - \frac{\beta(g)}{\epsilon g}\right) \\
&= -2\frac{\gamma_O(g)}{\epsilon}\left(1 - \frac{\beta(g)}{\epsilon g}\right)^{-1}\left(1 - \frac{\beta(g)}{\epsilon g}\right) \\
&= -2\frac{\gamma_O(g)}{\epsilon}
\end{aligned} \tag{3.18}$$

employing eqs. (3.17) and (3.15).

3.5 LET in dimensional regularization

The LET in $\tilde{d}$ dimensions with the Wilsonian normalization of bare operators may be written in terms of the dimensionless bare coupling $\bar{g}_0 = g_0 \mu^{\tilde{d}/2-2}$ according to eq. (3.10):

$$\frac{\partial}{\partial \log \bar{g}_0}\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 = \frac{N}{2\bar{g}_0^2}\mu^{2-\frac{\tilde{d}}{2}} \int \langle \mathcal{O}_1 \cdots \mathcal{O}_n \mathcal{F}^2(x) \rangle_0 - \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_0 \langle \mathcal{F}^2(x) \rangle_0 \, d^{\tilde{d}}x \tag{3.19}$$

where the μ dependence in the r.h.s. is due to $[\mathcal{F}_0^2] = 4$. Rewriting eq. (3.19) in terms of canonically normalized bare operators we get in $\tilde{d}$ dimensions:

$$\begin{aligned}
&\sum_{k=1}^{k=n} c_{O_k} \langle O_1 \cdots O_n \rangle_0 + \frac{\partial}{\partial \log g_0}\langle O_1 \cdots O_n \rangle_0 \\
&\quad = \frac{1}{2}\int \langle O_1 \cdots O_n F^2(x) \rangle_0 - \langle O_1 \cdots O_n \rangle_0 \langle F^2(x) \rangle_0 \, d^{\tilde{d}}x
\end{aligned} \tag{3.20}$$

where the integral over space-time in the r.h.s. is defined by analytic continuation and, since $[F_0^2] = \tilde{d}$, no dependence on the scale μ appears in the r.h.s. As for the notation, every time that in an equation the integral over space-time is denoted by:

$$\int d^{\tilde{d}}x \tag{3.21}$$

dimensional regularization of the correlators is understood as well. For $n = 2$ with $O_1 = O_2 = O$ we get:

$$2c_O \langle O(z)O(0)\rangle_0 + \frac{\partial}{\partial \log g_0} \langle O(z)O(0)\rangle_0 \\ = \frac{1}{2}\int \langle O(z)O(0)F^2(x)\rangle_0 - \langle O(z)O(0)\rangle_0 \langle F^2(x)\rangle_0 \, d^{\tilde{d}}x \tag{3.22}$$

3.6 Bare LET

We limit ourselves to correlators of gauge-invariant scalar operators $O(z)$ that are multiplicatively renormalizable up to the mixing with operators that are BRST variations and operators that vanish by the equations of motion (EOM) [16–18]. We neglect the mixing with BRST operators, since their insertion in gauge-invariant correlators vanishes. The mixing with EOM operators produces at most contact terms. Therefore, we set $z \neq 0$ unless otherwise stated, so that we may ignore the above mixing both in the l.h.s. and r.h.s. of the LET and express the LET in terms of the multiplicatively renormalized operator $O(z)$. Then, the LET reads in terms of the bare operator $F_0^2(x)$ and the multiplicatively renormalized one $O(z)$:

$$\langle O(z)O(0)\rangle \left(2c_O - \frac{2\gamma_O(g)}{\epsilon} + Z_{F^2}^{-1}\frac{\partial \log \langle O(z)O(0)\rangle}{\partial \log g}\right) \\ = \frac{1}{2}\int \langle O(z)O(0)F_0^2(x)\rangle - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle \, d^{\tilde{d}}x \tag{3.23}$$

with:

$$Z_{F^2}^{-1} = 1 - \frac{\beta(g)}{\epsilon g} \tag{3.24}$$

Eq. (3.23) is obtained from eq. (3.22) as follows. The renormalized operator O is related to the bare one O_0 by a multiplicative renormalization, hence:

$$O_0(z) = Z_O^{-1}(g,\epsilon)O(z) \tag{3.25}$$

The l.h.s. of eq. (3.23) is then obtained by multiplying by the factor Z_O^2 the l.h.s. of eq. (3.22) that reads:

$$2c_O Z_O^{-2}\langle O(z)O(0)\rangle + g_0\frac{\partial}{\partial g_0}Z_O^{-2}\langle O(z)O(0)\rangle + Z_O^{-2}g_0\frac{\partial}{\partial g_0}\langle O(z)O(0)\rangle \\ = Z_O^{-2}\langle O(z)O(0)\rangle \left(2c_O + \frac{\partial \log Z_O^{-2}}{\partial \log g_0} + \frac{\partial \log \langle O(z)O(0)\rangle}{\partial \log g_0}\right) \\ = Z_O^{-2}\langle O(z)O(0)\rangle \left(2c_O - \frac{2\gamma_O(g)}{\epsilon} + \left(1 - \frac{\beta(g)}{\epsilon g}\right)\frac{\partial \log \langle O(z)O(0)\rangle}{\partial \log g}\right) \tag{3.26}$$

where in the last line we have employed:

$$\frac{\partial}{\partial \log g_0} = \frac{\partial \log g}{\partial \log g_0}\frac{\partial}{\partial \log g} \tag{3.27}$$

and eqs. (3.15) and (3.18).

3.7 Renormalized LET

In general, both sides of the LET in eq. (3.23) diverge as $\epsilon \to 0$ order by order in perturbation theory. Hence, we derive from eq. (3.23) two renormalized versions of the LET requiring that the l.h.s. — and therefore also the r.h.s. — is finite as $\epsilon \to 0$.

They differ by a finite term in the renormalized object in the l.h.s. and we find it convenient to present each of them, (I) and (II), in three — though totally equivalent — ways as follows.

Version (IA):

$$\begin{aligned}\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} &= \frac{1}{2}Z_{F^2}\int \langle O(z)O(0)F_0^2(x)\rangle - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle\, d^{\tilde{d}}x \\ &\quad - Z_{F^2}\left(2c_O - \frac{2\gamma_O(g)}{\epsilon}\right)\langle O(z)O(0)\rangle \end{aligned} \tag{3.28}$$

(IA) shows that, since the l.h.s. above is by construction finite as $\epsilon \to 0$ because it is expressed only in terms of a renormalized object, also the r.h.s. must be finite. Moreover, it shows that the integrated correlator in the r.h.s. needs an additive renormalization in addition to the multiplicative one.

Version (IB):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} \\ &= \frac{1}{2}Z_{F^2}\int \langle O(z)O(0)F_0^2(x)\rangle - \left(2c_O - \frac{2\gamma_O(g)}{\epsilon}\right)\left(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x)\right)\langle O(z)O(0)\rangle \\ &\quad - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle\, d^{\tilde{d}}x \end{aligned} \tag{3.29}$$

where $\delta^{(\tilde{d})}(x)$ is defined by analytic continuation by means of the equation [19]:

$$\int \delta^{(\tilde{d})}(x)\, d^{\tilde{d}}x = 1 \tag{3.30}$$

and it is a well-defined distribution in integer dimensions. (IB) shows that the aforementioned additive renormalization is equivalent to the subtraction of finite and divergent contact terms as $\epsilon \to 0$ from the bare 3-point correlator in the r.h.s. By locality and dimensional analysis, in $d = 4$ dimensions this counterterm is proportional to the product of the 2-point correlator $\langle O(z)O(0)\rangle$ and the sum of contact terms $\delta^{(4)}(x) + \delta^{(4)}(x-z)$. Only after the subtraction involving the above mentioned finite and divergent contact terms, the integrated bare 3-point correlator becomes multiplicatively renormalizable.

Version (IC):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} \\ &= \frac{1}{2}Z_{F^2}\int \langle O(z)O(0)F_0^2(x)\rangle - 2c_O\left(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x)\right)\langle O(z)O(0)\rangle \\ &\quad - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle\, d^{\tilde{d}}x + Z_{F^2}\frac{2\gamma_O(g)}{\epsilon}\langle O(z)O(0)\rangle \end{aligned} \tag{3.31}$$

(IC) shows that subtracting only the finite contact terms in (IB) from the bare correlator in the r.h.s. changes the divergent contact terms to be subtracted from the multiplicatively renormalized one, in order to obtain the very same finite fully renormalized object defined in the l.h.s.

In fact, there is a finite ambiguity also in the definition of the fully renormalized object in the l.h.s. that affects the infinite additive renormalization of the integrated bare 3-point correlator in the r.h.s.

Version (IIA):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} + 2c_O \langle O(z)O(0)\rangle \\ &= \frac{1}{2} Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle \, d^{\tilde{d}}x \\ &+ Z_{F^2} \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \langle O(z)O(0)\rangle \end{aligned} \tag{3.32}$$

where we have employed (IA) and the identity:

$$-\frac{2c_O}{1 - \frac{\beta(g)}{\epsilon g}} + 2c_O = -\frac{2c_O \frac{\beta(g)}{g}}{\epsilon \left(1 - \frac{\beta(g)}{\epsilon g}\right)} \tag{3.33}$$

As anticipated, the finite change in (IIA) with respect to (IA) in the definition of the fully renormalized object in the l.h.s. of the LET produces an infinite change in the needed additive renormalization of the bare 3-point correlator in the r.h.s.

Version (IIB):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} + 2c_O \langle O(z)O(0)\rangle \\ &= \frac{1}{2} Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle + \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} (\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x)) \langle O(z)O(0)\rangle \\ &- \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle \, d^{\tilde{d}}x \end{aligned} \tag{3.34}$$

that is the analogue of (IB).

Version (IIC):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} + 2c_O \langle O(z)O(0)\rangle \\ &= \frac{1}{2} Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle - \frac{2c_O \frac{\beta(g)}{g}}{\epsilon} (\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x)) \langle O(z)O(0)\rangle \\ &- \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle \, d^{\tilde{d}}x + Z_{F^2} \frac{2\gamma_O(g)}{\epsilon} \langle O(z)O(0)\rangle \end{aligned} \tag{3.35}$$

that is the analogue of (IC).

3.8 Renormalized LET and operator mixing of F^2 in massless QCD

The various forms of the LET in terms of the bare operator F_0^2 apply to any YM theory. However, it is convenient to express the LET in terms of the renormalized operator F^2. The renormalization of F^2 depends on the specific choice of the YM theory.

In massless QCD F^2 mixes [20] with the quark Lagrangian density $\bar\psi\gamma_\mu D_\mu\psi$ that vanishes by the EOM, whose insertion in the correlators produces at most contact terms. Yet, contact terms cannot be ignored in the r.h.s. of the LET because of the space-time integration.

Nevertheless, even if F_0^2 mixes with $\bar\psi\gamma_\mu D_\mu\psi$, (IIA) shows that the integrated correlator in the r.h.s. is made finite by the multiplicative renormalization Z_{F^2} in addition to the additive one, despite the possible nontrivial mixing of F^2. Hence, as far as the LET is concerned, we may define $F^2 \equiv Z_{F^2}F_0^2$ and ignore the mixing.

3.9 RG-invariant form of the LET

It is worth writing the LET in a form that involves the RG-invariant operator (appendix A):

$$\begin{aligned}
-\epsilon F_0^2 &= -\epsilon Z_{F^2}^{-1}(g,\epsilon)Z_{F^2}(g,\epsilon)F_0^2 \\
&= -\epsilon Z_{F^2}^{-1}(g,\epsilon)F^2 \\
&= -\epsilon\left(1-\frac{\beta(g)}{\epsilon g}\right)F^2 \\
&= \frac{\beta(g,\epsilon)}{g}F^2
\end{aligned} \tag{3.36}$$

Multiplying eq. (3.23) by $-\epsilon$ and employing eq. (3.15) we obtain:

$$\begin{aligned}
&\langle O(z)O(0)\rangle\left(2\gamma_O(g)-2c_O\epsilon+\frac{\beta(g,\epsilon)}{g}\frac{\partial\log\langle O(z)O(0)\rangle}{\partial\log g}\right)\\
&=\frac{1}{2}\frac{\beta(g,\epsilon)}{g}\int\langle O(z)O(0)F^2(x)\rangle-\langle O(z)O(0)\rangle\langle F^2(x)\rangle\, d^{\tilde d}x
\end{aligned} \tag{3.37}$$

Interestingly, in this form of the LET no infinite additive renormalization arises.

Finally, setting $O_0 = F_0^2$ and multiplying eq. (3.22) by $(-\epsilon)^3$ we get:

$$\begin{aligned}
&-2c_O\epsilon\,\langle\frac{\beta(g,\epsilon)}{g}F^2(z)\frac{\beta(g,\epsilon)}{g}F^2(0)\rangle+\frac{\beta(g,\epsilon)}{g}\frac{\partial}{\partial\log g}\langle\frac{\beta(g,\epsilon)}{g}F^2(z)\frac{\beta(g,\epsilon)}{g}F^2(0)\rangle\\
&=\frac{1}{2}\int\langle\frac{\beta(g,\epsilon)}{g}F^2(z)\frac{\beta(g,\epsilon)}{g}F^2(0)\frac{\beta(g,\epsilon)}{g}F^2(x)\rangle\\
&\quad-\langle\frac{\beta(g,\epsilon)}{g}F^2(z)\frac{\beta(g,\epsilon)}{g}F^2(0)\rangle\langle\frac{\beta(g,\epsilon)}{g}F^2(x)\rangle\, d^{\tilde d}x
\end{aligned} \tag{3.38}$$

3.10 LET involving Λ_{UV}

In an AF YM theory the LET can be rewritten in terms of the RG-invariant scale Λ_{UV} [1]. We obtain in dimensional regularization:

$$\begin{aligned}
&\langle O(z)O(0)\rangle\left(2\gamma_O(g)-2c_O\epsilon-\frac{\partial\log\langle O(z)O(0)\rangle}{\partial\log\Lambda_{UV}}\right)\\
&=\frac{1}{2}\frac{\beta(g,\epsilon)}{g}\int\langle O(z)O(0)F^2(x)\rangle-\langle O(z)O(0)\rangle\langle F^2(x)\rangle\, d^{\tilde d}x
\end{aligned} \tag{3.39}$$

employing the chain rule $\frac{\partial}{\partial \log g} = \frac{\partial \Lambda_{UV}}{\partial \log g} \frac{\partial}{\partial \Lambda_{UV}}$, the defining relation of the RG invariance of Λ_{UV} (appendix D.1):

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g,\epsilon) \frac{\partial}{\partial g}\right) \Lambda_{UV} = 0 \tag{3.40}$$

and the identity:

$$\frac{\partial \Lambda_{UV}}{\partial \log \mu} = \Lambda_{UV} \tag{3.41}$$

that follows from $\Lambda_{UV} = \mu f(g,\epsilon) = e^{\log \mu} f(g,\epsilon)$ for a certain function $f(g,\epsilon)$ with $g = g(\mu)$. The l.h.s. of the LET has a finite limit for $\epsilon \to 0$. Therefore, also the r.h.s. must be finite as $\epsilon \to 0$. Hence, in the AF phase of the theory this in turn implies the finiteness of the space-time integral itself in the r.h.s. as $\epsilon \to 0$, so that eq. (3.39) reads in the limit $\epsilon \to 0$:

$$\begin{aligned} &\langle O(z)O(0)\rangle \left(2\gamma_O(g) - \frac{\partial \log \langle O(z)O(0)\rangle}{\partial \log \Lambda_{UV}}\right) \\ &= \frac{1}{2}\frac{\beta(g)}{g} \int \langle O(z)O(0)F^2(x)\rangle - \langle O(z)O(0)\rangle \langle F^2(x)\rangle \, d^4x \end{aligned} \tag{3.42}$$

where the integral in the r.h.s. and all the correlators are in $d = 4$ dimensions.

4 LET and OPE

We show in the following that the infinite additive renormalization of the integrated 3-point correlator in the r.h.s. of the LET is related to the corresponding divergence of the multiplicatively renormalized OPE coefficient $Z_{F^2} C_1^{(F_0^2,O)}(p)$ in the momentum representation:

$$\frac{1}{2}[Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle \, d^{\tilde{d}}x]_{\text{div}} = [Z_{F^2} C_1^{(F_0^2,O)}(p)]_{\text{div}} \langle O(z)O(0)\rangle \tag{4.1}$$

The divergence is represented by a contact term in the momentum representation, i.e. a polynomial in momentum, and it is just a constant in our specific case. The OPE coefficient in the coordinate representation reads:

$$Z_{F^2} F_0^2(x) O(0) = \cdots + Z_{F^2} C_1^{(F_0^2,O)}(x) O(0) + \cdots \tag{4.2}$$

Actually, we define $C_1^{(F_0^2,O)}(x)$ as a certain extension, including a distribution supported at coinciding points $x = 0$, of the OPE coefficient $C_1^{(F_0^2,O)'}(x)$ at distinct points $x \neq 0$. The above extension is naturally defined as the inverse Fourier transform of the OPE coefficient in the momentum representation $Z_{F^2} C_1^{(F_0^2,O)}(p)$ in the sense of distributions, both in dimensional regularization and $d = 4$ by taking the limit $\epsilon \to 0$:

$$Z_{F^2} C_1^{(F_0^2,O)}(x) = \frac{1}{(2\pi)^{4-2\epsilon}} \int Z_{F^2} C_1^{(F_0^2,O)}(p) e^{ip\cdot x} d^{4-2\epsilon}p \tag{4.3}$$

that is inverted as:

$$Z_{F^2} C_1^{(F_0^2,O)}(p) = \int Z_{F^2} C_1^{(F_0^2,O)}(x) e^{-ip\cdot x} d^{4-2\epsilon}x \tag{4.4}$$

This definition involves implicitly an interplay between the solutions of the CS equation in the momentum and coordinate representation and the corresponding contact terms.

JHEP08(2024)145

4.1 OPE and contact terms from the CS equation in the momentum representation

We start by recalling the CS equation in the coordinate representation for 2-point correlators and OPE coefficients of multiplicatively renormalizable operators at distinct points. The renormalized 2-point correlator at distinct points — denoted by a prime — in $d = 4$ dimensions satisfies the CS equation (appendix E):

$$\left(z \cdot \frac{\partial}{\partial z} + \beta(g) \frac{\partial}{\partial g} + 2(\Delta_{O_0} + \gamma_O(g)) \right) \langle O(z) O(0) \rangle \Big|_{d=4} = 0 \tag{4.5}$$

and analogously for the OPE coefficient at distinct points:

$$\left(x \cdot \frac{\partial}{\partial x} + \beta(g) \frac{\partial}{\partial g} + 4 + \gamma_{F^2}(g) \right) C_1^{(F^2,O)'}(x) \Big|_{d=4} = 0 \tag{4.6}$$

Similarly, the dimensionally regularized correlator satisfies (appendix D):

$$\left(z \cdot \frac{\partial}{\partial z} + \beta(g,\epsilon) \frac{\partial}{\partial g} + 2(\tilde{\Delta}_{O_0} + \gamma_O(g)) \right) \langle O(z) O(0) \rangle \Big|_{\tilde{d}=4-2\epsilon} = 0 \tag{4.7}$$

and the OPE coefficient:

$$\left(x \cdot \frac{\partial}{\partial x} + \beta(g,\epsilon) \frac{\partial}{\partial g} + 4 - 2\epsilon + \gamma_{F^2}(g) \right) C_1^{(F^2,O)'}(x) \Big|_{\tilde{d}=4-2\epsilon} = 0 \tag{4.8}$$

where Δ_{O_0}, $\tilde{\Delta}_{O_0} = \Delta_{O_0} - \delta_O \epsilon$ are the canonical dimensions of O in $d = 4$ and $\tilde{d} = 4 - 2\epsilon$ dimensions, $\Delta_O = \Delta_{O_0} + \gamma_O$, $\tilde{\Delta}_O = \tilde{\Delta}_{O_0} + \gamma_O$ the scaling dimensions of O, and γ_O its anomalous dimension, respectively.

We stress that all the above equations only capture the multiplicative renormalization of the correlators and OPE coefficients at distinct points, so that their solutions cannot be extended at coinciding points, even in the dimensionally regularized case. Indeed, to take into account contact terms, they need to be modified to include additive renormalizations, as we demonstrate momentarily.

The general solution for the 2-point correlator at distinct points in $d = 4$ dimensions is (appendix E):

$$\langle O(z) O(0) \rangle_{d=4} = \frac{\mathcal{G}_2^{(O)}(g(z))}{|z|^{2\Delta_{O_0}}} Z^{(O)2}(g(z), g(\mu)) \tag{4.9}$$

with $|z| = \sqrt{z^2}$, and for the OPE coefficient:

$$C_1^{(F^2,O)'}(x)_{d=4} = \frac{\mathcal{G}^{(F^2,O)}(g(x))}{|x|^4} Z^{(F^2)}(g(x), g(\mu)) \tag{4.10}$$

Moreover, in $\tilde{d} = 4 - 2\epsilon$ dimensions we get for the 2-point correlator (appendix D):

$$\langle O(z) O(0) \rangle_{\tilde{d}=4-2\epsilon} = \frac{\mathcal{G}_2^{(O)}(\tilde{g}(z))}{|z|^{2\tilde{\Delta}_{O_0}}} Z^{(O)2}(\tilde{g}(z), g(\mu)) \tag{4.11}$$

and OPE coefficient:

$$C_1^{(F^2,O)'}(x)_{\tilde{d}=4-2\epsilon} = \frac{\mathcal{G}^{(F^2,O)}(\tilde{g}(x))}{|x|^{\tilde{\Delta}_{F_0^2}}} Z^{(F^2)}(\tilde{g}(x), g(\mu)) \tag{4.12}$$

The above solutions may be extended at coinciding points as distributions to include contact terms. In fact, the — possibly divergent — contact terms are most conveniently obtained in the momentum representation as $\epsilon \to 0$, where the operators are not multiplicatively renormalizable in general.

To clarify this issue, we discuss the CS equations for the 2-point correlator and OPE coefficient in the momentum representation that include the aforementioned additive renormalization (appendix F).

We denote the fully renormalized 2-point correlator in the momentum representation in $\tilde{d} = 4 - 2\epsilon$ dimensions as the OPE coefficient of the identity $C_0^{(O,O)}(p)$:

$$C_0^{(O,O)}(p) = Z_O^2 C_0^{(O_0,O_0)}(p) + p^{2\Delta_{O_0}-4}\mu^{2(1-\delta_O)\epsilon} Z_{0\text{c.t.}} \tag{4.13}$$

It decomposes into the sum of the multiplicatively renormalized contribution $Z_O^2\, C_0^{(O_0,O_0)}(p)$ and the — in general divergent as $\epsilon \to 0$ — additive renormalization proportional to $Z_{0\text{c.t.}}$. The corresponding CS equation in dimensional regularization reads (appendix F):

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g,\epsilon)\frac{\partial}{\partial g} + 2\gamma_O(g)\right) C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4}\mu^{2(1-\delta_O)\epsilon}\gamma_{0\text{c.t.}}(g) \tag{4.14}$$

where:

$$\begin{aligned}\gamma_{0\text{c.t.}}(g) &= \mu\frac{dZ_{0\text{c.t.}}}{d\mu} + 2\gamma_O Z_{0\text{c.t.}} + 2(1-\delta_O)\epsilon Z_{0\text{c.t.}} \\ &= \beta(g,\epsilon)\frac{\partial Z_{0\text{c.t.}}}{\partial g} + 2\gamma_O Z_{0\text{c.t.}} + 2(1-\delta_O)\epsilon Z_{0\text{c.t.}}\end{aligned} \tag{4.15}$$

Hence, the CS equation in $d = 4$ dimensions reads:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_O(g)\right) C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4}\gamma_{0\text{c.t.}}(g) \tag{4.16}$$

Perturbatively, in dimensional regularization as $\epsilon \to 0$, the fully renormalized OPE coefficient $C_1^{(F^2,O)}(p)$ similarly decomposes into the sum of a nontrivial function of $\frac{p^2}{\mu^2}$ — $Z_{F^2}C_1^{(F_0^2,O)}(p)$ — that is obtained by multiplicative renormalization of the bare OPE coefficient $C_1^{(F_0^2,O)}(p)$ and a — in general divergent as $\epsilon \to 0$ — constant term $Z_{1\text{c.t.}}$:

$$C_1^{(F^2,O)}(p) = Z_{F^2}C_1^{(F_0^2,O)}(p) + Z_{1\text{c.t.}} \tag{4.17}$$

The corresponding CS equation in dimensional regularization reads (appendix F):

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g,\epsilon)\frac{\partial}{\partial g} + \gamma_{F^2}(g)\right) C_1^{(F^2,O)}(p,\mu,g(\mu)) = \gamma_{1\text{c.t.}}(g) \tag{4.18}$$

with:

$$\gamma_{1\text{c.t.}}(g) = \mu \frac{dZ_{1\text{c.t.}}}{d\mu} + \gamma_{F^2}(g) Z_{1\text{c.t.}}$$
$$= \beta(g,\epsilon) \frac{\partial Z_{1\text{c.t.}}}{\partial g} + \gamma_{F^2}(g) Z_{1\text{c.t.}} \tag{4.19}$$

Hence, the CS equation in $d = 4$ dimensions reads:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma_{F^2}(g)\right) C_1^{(F^2,O)}(p,\mu,g(\mu)) = \gamma_{1\text{c.t.}}(g) \tag{4.20}$$

We demonstrate by explicit computation for $O = F^2$ (section 4.2) that — contrary to eqs. (4.3) and (4.4), which by definition are the inverse of each other in dimensional regularization at least in the limit $\epsilon \to 0$ — in general:

$$Z_{F^2} C_1^{(F_0^2,O)}(p) \neq \int Z_{F^2} C_1^{(F_0^2,O)'}(x) e^{-ip\cdot x} d^{4-2\epsilon}x \tag{4.21}$$

as $\epsilon \to 0$, with the objects in the coordinate and momentum representation defined by means of the corresponding CS equations above. The interpretation of the inequality is that in general not all the divergent contact terms that occur in the momentum representation as $\epsilon \to 0$ arise from the divergence of the Fourier transform of the OPE coefficient at distinct points as $\epsilon \to 0$.

In general, in order for the Fourier transform of $Z_{F^2} C_1^{(F_0^2,O)}(x)$ to reproduce $Z_{F^2} C_1^{(F_0^2,O)}(p)$, $Z_{F^2} C_1^{(F_0^2,O)}(x)$ must contain in the sense of distributions some extra divergent contact terms as $\epsilon \to 0$, $[Z_{F^2} C_1^{(F_0^2,O)}(x) - Z_{F^2} C_1^{(F_0^2,O)'}(x)]_{\text{div}}$, that add to the ones that arise from the Fourier transform of $Z_{F^2} C_1^{(F_0^2,O)'}(x)$ at distinct points. We refer to (minus) the divergent contact terms of the multiplicatively renormalized OPE coefficient in the momentum representation as additive counterterms, while we refer to the divergent contact terms that are extensions of the OPE coefficient in the coordinate representation at coinciding points $[Z_{F^2} C_1^{(F_0^2,O)}(x) - Z_{F^2} C_1^{(F_0^2,O)'}(x)]_{\text{div}}$ as divergent proper contact terms.

To summarize, according to eq. (4.21), in the above language the divergent proper contact terms do not vanish in general, as we demonstrate below.

4.2 A paradigmatic example: the OPE coefficient for $O = F^2$

In massless QCD the occurrence of the additive renormalization in the OPE coefficient $C_1^{(F^2,F^2)}(p)$ of F^2 with itself — i.e. in the special case $O = F^2$ — in the momentum representation has been discovered to order g^4 [3] and g^6 [14] in perturbation theory, and computed in a closed form in terms of the beta function and its first derivative [2] (appendix B), so that the fully renormalized object reads in our notation:

$$C_1^{(F^2,F^2)}(p) = Z_{F^2} C_1^{(F_0^2,F^2)}(p) + Z_{F^2} \frac{2g \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) - 4\frac{\beta(g)}{g}}{\epsilon} \tag{4.22}$$

Interestingly, the above additive counterterm is scheme independent to order g^4 [3]:

$$Z_{F^2} \frac{2g \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) - 4\frac{\beta(g)}{g}}{\epsilon} = -\frac{4\beta_1 g^4}{\epsilon} + \cdots \tag{4.23}$$

JHEP08(2024)145

and must cancel a corresponding scheme-independent divergence in $Z_{F^2} C_1^{(F_0^2,F^2)}(p)$ to order g^4 that we report in our notation in the $\overline{MS}$ scheme [3]:

$$Z_{F^2} C_1^{(F_0^2,F^2)}(p) = 4 - 4B_{1,1}g^2 - 4B_{1,2}g^4 + \frac{4\beta_1 g^4}{\epsilon}$$
$$-4\beta_0 g^2 \log\frac{p^2}{\mu^2} + 4\beta_0^2 g^4 \log^2\frac{p^2}{\mu^2} + 4B_{1,3}g^4 \log\frac{p^2}{\mu^2} + \cdots \tag{4.24}$$

with the beta-function universal coefficients:

$$\beta_0 = \frac{1}{(4\pi)^2}\left(\frac{11}{3} - \frac{2}{3}\frac{N_f}{N}\right)$$
$$\beta_1 = \frac{1}{(4\pi)^4}\left(\frac{34}{3} - \frac{13}{3}\frac{N_f}{N} + \frac{N_f}{N^3}\right) \tag{4.25}$$

and the scheme-dependent coefficients:

$$B_{1,1} = \frac{1}{(4\pi)^2}\left(-\frac{49}{9} + \frac{10}{9}\frac{N_f}{N}\right)$$
$$B_{1,2} = \frac{1}{(4\pi)^4}\left(-\frac{11509}{81} + 66\zeta_3 + (13-12\zeta_3)\frac{N_f}{N^2}\frac{N^2-1}{N} + \left(\frac{3095}{81} + 12\zeta_3\right)\frac{N_f}{N} - \frac{100}{81}\frac{N_f^2}{N^2}\right)$$
$$B_{1,3} = -4\beta_1 + 2\beta_0 B_{1,1} \tag{4.26}$$

where we have conveniently rewritten $B_{1,3}$ [3] in terms of β_0, β_1 and $B_{1,1}$.

Hence, as pointed out in [3], it follows from eq. (4.24) that the bare OPE coefficient $C_1^{(F_0^2,F^2)}(p)$ in the momentum representation is not multiplicatively renormalizable because a divergent contact term $+\frac{4\beta_1 g^4}{\epsilon}$ arises in $Z_{F^2} C_1^{(F_0^2,F^2)}(p)$ to order g^4 that is cancelled by the additive renormalization in eq. (4.22) to order g^4.

Our aim now is to reconstruct the OPE coefficient $Z_{F^2} C_1^{(F_0^2,F^2)}(x)$ in the coordinate representation to order g^4 from the OPE coefficient $Z_{F^2} C_1^{(F_0^2,F^2)}(p)$ in the momentum representation according to eqs. (4.3) and (4.4) as $\epsilon \to 0$. In doing so we verify the inequality in eq. (4.21) up to order g^4, thus demonstrating the occurrence of both additive counterterms and divergent proper contact terms, according to the terminology introduced above.

To find the OPE coefficient in the coordinate representation whose Fourier transform is $Z_{F^2} C_1^{(F_0^2,F^2)}(p)$ in eq. (4.24) it is mandatory to start with an ansatz for the multiplicatively renormalized coefficient in the coordinate representation at distinct points:

$$C_1^{(F^2,F^2)'}(x) = \frac{Z^{(F^2)}(\tilde{g}(x), g(\mu))}{x^{4-2\epsilon}}(a\tilde{g}^2(x) + b\tilde{g}^4(x) + \cdots) \tag{4.27}$$

that is solution of the CS equation in $\tilde{d} = 4 - 2\epsilon$ dimensions in eq. (4.8) for $O = F^2$, where to the relevant order (appendix D):

$$\frac{\tilde{g}^2(x)}{g^2(\mu)} = \frac{|x\mu|^{2\epsilon}}{1 - \beta_0 g^2(\mu)\frac{|x\mu|^{2\epsilon}-1}{\epsilon}} \tag{4.28}$$

and:

$$Z^{(F^2)}(\tilde{g}(x), g(\mu)) = \frac{\tilde{g}^2(x)}{g^2(\mu)}|x\mu|^{-2\epsilon} \tag{4.29}$$

By inserting eqs. (4.28) and (4.29) into eq. (4.27) and setting $g(\mu) = g$, the matching of the logarithmic terms to order g^2 in eq. (4.24) fixes $a = 4\beta_0/\pi^2$ and $b = -4\tilde{B}_{1,3}/\pi^2$ with $\tilde{B}_{1,3}$ to be determined a posteriori. Hence, we get:

$$
\begin{aligned}
C_1^{(F^2,F^2)'}(x) &= \frac{4\beta_0}{\pi^2}\frac{\mu^{2\epsilon}g^2}{x^{4-4\epsilon}}\left(1-\beta_0 g^2\frac{|x\mu|^{2\epsilon}-1}{\epsilon}\right)^{-2} \\
&\quad -\frac{4\tilde{B}_{1,3}}{\pi^2}\frac{\mu^{4\epsilon}g^4}{x^{4-6\epsilon}}\left(1-\beta_0 g^2\frac{|x\mu|^{2\epsilon}-1}{\epsilon}\right)^{-3}+\cdots \\
&= \frac{4\beta_0}{\pi^2}\frac{\mu^{2\epsilon}g^2}{x^{4-4\epsilon}}\left(1+2\beta_0 g^2\frac{|x\mu|^{2\epsilon}-1}{\epsilon}\right)-\frac{4\tilde{B}_{1,3}}{\pi^2}\frac{\mu^{4\epsilon}g^4}{x^{4-6\epsilon}}+\cdots \\
&= \frac{4\beta_0}{\pi^2}\frac{\mu^{2\epsilon}g^2}{x^{4-4\epsilon}}\left(1-\frac{2\beta_0 g^2}{\epsilon}\right)+\frac{8\beta_0^2}{\pi^2\,\epsilon}\frac{\mu^{4\epsilon}g^4}{x^{4-6\epsilon}}-\frac{4\tilde{B}_{1,3}}{\pi^2}\frac{\mu^{4\epsilon}g^4}{x^{4-6\epsilon}}+\cdots
\end{aligned}
\tag{4.30}
$$

where the second equality is the perturbative expansion for small g with ϵ fixed and the dots stand for order g^6 contributions. Its Fourier transform (FT) is:

$$
\begin{aligned}
&\mathrm{FT}\left[C_1^{(F^2,F^2)'}(x)\right] = \frac{4\beta_0 g^2}{\pi^2}\left(1-\frac{2\beta_0 g^2}{\epsilon}\right)\mathrm{FT}\left[\frac{\mu^{2\epsilon}}{x^{4-4\epsilon}}\right]+\frac{4g^4}{\pi^2}\left(\frac{2\beta_0^2}{\epsilon}-\tilde{B}_{1,3}\right)\mathrm{FT}\left[\frac{\mu^{4\epsilon}}{x^{4-6\epsilon}}\right]+\cdots \\
&= 4\beta_0 g^2\left(\frac{1}{\epsilon}-\log\frac{p^2}{\mu^2}+2+3\Gamma'(1)-\log\frac{\pi}{4}\right)+8\beta_0^2 g^4\Big\{-\frac{1}{2\epsilon^2}+\frac{1}{2}\log^2\frac{p^2}{\mu^2} \\
&\quad -\left(\frac{1}{\epsilon}-\log\frac{p^2}{\mu^2}\right)\left(2+3\Gamma'(1)-\log\frac{\pi}{4}\right)+\left(\frac{1}{\epsilon}-2\log\frac{p^2}{\mu^2}\right)\left(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\right) \\
&\quad -\frac{1}{2}\left(2+3\Gamma'(1)-\log\frac{\pi}{4}\right)^2+\left(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\right)^2+\frac{1}{4}-\frac{1}{4}\Gamma'^2(1)+\frac{1}{4}\Gamma''(1)\Big\} \\
&\quad -4\tilde{B}_{1,3}g^4\left(\frac{1}{2\epsilon}-\log\frac{p^2}{\mu^2}+\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\right)+\cdots
\end{aligned}
\tag{4.31}
$$

where in the second equality we have employed the FT in d dimensions:

$$
\mathrm{FT}\left[\frac{\mu^{d-2\Delta}}{x^{2\Delta}}\right] \equiv \int d^d x\,\frac{\mu^{d-2\Delta}}{x^{2\Delta}}e^{-ipx} = \pi^{d/2}\frac{\Gamma(d/2-\Delta)}{\Gamma(\Delta)}\left(\frac{p^2}{4\mu^2}\right)^{\Delta-d/2}
\tag{4.32}
$$

that for $d = 4-2\epsilon$ and perturbatively in ϵ gives us for $\Delta = 2-2\epsilon$:

$$
\begin{aligned}
\mathrm{FT}\left[\frac{\mu^{2\epsilon}}{x^{4-4\epsilon}}\right] &= \pi^2\left(\frac{\pi p^2}{4\mu^2}\right)^{-\epsilon}\frac{\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} \\
&= \pi^2\left(\frac{\pi p^2}{4\mu^2}\right)^{-\epsilon}\left(\frac{1}{\epsilon}+2+3\Gamma'(1)+\epsilon\Big(4+6\Gamma'(1)(1+\Gamma'(1))-\frac{3}{2}\Gamma''(1)\Big)+\cdots\right) \\
t &= \pi^2\bigg(\frac{1}{\epsilon}-\log\frac{\pi p^2}{4\mu^2}+2+3\Gamma'(1)+\epsilon\Big(\frac{1}{2}\Big(\log\frac{\pi p^2}{4\mu^2}-2-3\Gamma'(1)\Big)^2 \\
&\quad +2+\frac{3}{2}\Gamma'^2(1)-\frac{3}{2}\Gamma''(1)\Big)+\cdots\bigg)
\end{aligned}
\tag{4.33}
$$

and for $\Delta = 2 - 3\epsilon$:

$$\begin{aligned}
\mathrm{FT}\left[\frac{\mu^{4\epsilon}}{x^{4-6\epsilon}}\right] &= \pi^2\left(\frac{\sqrt{\pi}p^2}{4\mu^2}\right)^{-2\epsilon}\frac{\Gamma(2\epsilon)}{\Gamma(2-3\epsilon)} \\
&= \pi^2\left(\frac{\sqrt{\pi}p^2}{4\mu^2}\right)^{-2\epsilon}\left(\frac{1}{2\epsilon}+\frac{3}{2}+\frac{5}{2}\Gamma'(1)+\frac{\epsilon}{2}\Big(9+15\Gamma'(1)(1+\Gamma'(1))-\frac{5}{2}\Gamma''(1)\Big)+\cdots\right) \\
&= \pi^2\bigg(\frac{1}{2\epsilon}-\log\frac{\sqrt{\pi}p^2}{4\mu^2}+\frac{3}{2}+\frac{5}{2}\Gamma'(1)+\epsilon\bigg(\Big(\log\frac{\sqrt{\pi}p^2}{4\mu^2}-\frac{3}{2}-\frac{5}{2}\Gamma'(1)\Big)^2 \\
&\quad +\frac{9}{4}+\frac{5}{4}\Gamma'^2(1)-\frac{5}{4}\Gamma''(1)\bigg)+\cdots\bigg) \qquad (4.34)
\end{aligned}$$

Setting in eq. (4.31):

$$\tilde{B}_{1,3} = B_{1,3} - 2\beta_0^2\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big) + 4\beta_0^2\Big(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big) \qquad (4.35)$$

we obtain:

$$\begin{aligned}
\mathrm{FT}\left[C_1^{(F^2,F^2)'}(x)\right] &= 4\beta_0 g^2\left(\frac{1}{\epsilon}-\log\frac{p^2}{\mu^2}+2+3\Gamma'(1)-\log\frac{\pi}{4}\right) \\
&+ 8\beta_0^2 g^4\bigg\{-\frac{1}{2\epsilon^2}+\frac{1}{2}\log^2\frac{p^2}{\mu^2}-\frac{1}{2\epsilon}\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big) \\
&-\frac{1}{2}\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big)^2-\Big(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big)^2 \\
&+\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big)+\frac{1}{4}-\frac{1}{4}\Gamma'^2(1)+\frac{1}{4}\Gamma''(1)\bigg\} \\
&-4B_{1,3}g^4\Big(\frac{1}{2\epsilon}-\log\frac{p^2}{\mu^2}+\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big)+\cdots \\
&= 4\beta_0 g^2\left(\frac{1}{\epsilon}-\log\frac{p^2}{\mu^2}\right)+8\beta_0^2 g^4\Big(-\frac{1}{2\epsilon^2}+\frac{1}{2}\log^2\frac{p^2}{\mu^2}\Big)+4B_{1,3}g^4\log\frac{p^2}{\mu^2}+\frac{8\beta_1 g^4}{\epsilon} \\
&-\frac{4\beta_0 g^4}{\epsilon}\Big(B_{1,1}+\beta_0\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big)+\text{finite contact terms}+\cdots \qquad (4.36)
\end{aligned}$$

where in the last equality we have employed $B_{1,3} = -4\beta_1 + 2\beta_0 B_{1,1}$ in eq. (4.26) to rewrite the order g^4/ϵ terms, and:

$$\begin{aligned}
\text{finite contact terms} &= 4\beta_0 g^2\left(2+3\Gamma'(1)-\log\frac{\pi}{4}\right) \\
&+8\beta_0^2 g^4\bigg\{-\frac{1}{2}\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big)^2-\Big(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big)^2 \\
&+\Big(2+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big)+\frac{1}{4}-\frac{1}{4}\Gamma'^2(1)+\frac{1}{4}\Gamma''(1)\bigg\} \\
&-4B_{1,3}g^4\Big(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\Big) \qquad (4.37)
\end{aligned}$$

From eq. (4.36) it is clear that the Fourier transform of the OPE coefficient at distinct points does not reproduce the object in eq. (4.24), thus proving the inequality in eq. (4.21) up to order g^4. Specifically, it differs from it by finite and divergent proper contact terms.

To further proceed we define an extension of the OPE coefficient including the finite and divergent proper contact terms according to the ansatz:

$$\begin{aligned} C_1^{(F^2,F^2)}(x) &= C_1^{(F^2,F^2)'}(x) \\ &\quad + \delta^{(\tilde{d})}(x)\left[Z_{F^2}\left(4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 + \cdots\right)\right]_{\text{up to order } g^4} \end{aligned} \tag{4.38}$$

The corresponding Fourier transform reads:

$$\begin{aligned} \mathrm{FT}\left[C_1^{(F^2,F^2)}(x)\right] &= \mathrm{FT}\left[C_1^{(F^2,F^2)'}(x)\right] \\ &\quad + \left[Z_{F^2}\left(4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 + \cdots\right)\right]_{\text{up to order } g^4} \end{aligned} \tag{4.39}$$

where:

$$\tilde{B}_{1,1} = B_{1,1} + \beta_0\left(2 + 3\Gamma'(1) - \log\frac{\pi}{4}\right) \tag{4.40}$$

$\tilde{B}_{1,2}$ is determined a posteriori to match the finite contributions to order g^4 in eq. (4.24), and (appendix A):

$$Z_{F^2}(g,\epsilon) = 1 - \frac{\beta_0 g^2}{\epsilon} - \frac{\beta_1 g^4}{\epsilon} + \frac{\beta_0^2 g^4}{\epsilon^2} + \cdots \tag{4.41}$$

The ansatz above is obtained multiplying by Z_{F^2} the finite proper contact terms up to order g^4, thus obtaining:

$$\begin{aligned} &\left[Z_{F^2}\left(4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 + \cdots\right)\right]_{\text{up to order } g^4} = \\ &\left[\left(1 - \frac{\beta_0 g^2}{\epsilon} - \frac{\beta_1 g^4}{\epsilon} + \frac{\beta_0^2 g^4}{\epsilon^2} + \cdots\right)\left(4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 + \cdots\right)\right]_{\text{up to order } g^4} \\ &= 4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 - 4\frac{\beta_0 g^2}{\epsilon} + 4\frac{\beta_0^2 g^4}{\epsilon^2} - 4\frac{\beta_1 g^4}{\epsilon} + 4\tilde{B}_{1,1}\frac{\beta_0 g^4}{\epsilon} + \cdots \end{aligned} \tag{4.42}$$

Hence, by means of eqs. (4.36), (4.40) and (4.42), eq. (4.39) reads:

$$\begin{aligned} \mathrm{FT}\left[C_1^{(F^2,F^2)}(x)\right] &= 4\beta_0 g^2\left(\frac{1}{\epsilon} - \log\frac{p^2}{\mu^2}\right) + 8\beta_0^2 g^4\left(-\frac{1}{2\epsilon^2} + \frac{1}{2}\log^2\frac{p^2}{\mu^2}\right) \\ &\quad + 4B_{1,3}g^4\log\frac{p^2}{\mu^2} + \frac{8\beta_1 g^4}{\epsilon} - 4\tilde{B}_{1,1}\frac{\beta_0 g^4}{\epsilon} + \text{finite contact terms} \\ &\quad + 4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 - 4\frac{\beta_0 g^2}{\epsilon} + 4\frac{\beta_0^2 g^4}{\epsilon^2} - 4\frac{\beta_1 g^4}{\epsilon} + 4\tilde{B}_{1,1}\frac{\beta_0 g^4}{\epsilon} + \cdots \\ &= 4 - 4\tilde{B}_{1,1}g^2 - 4\tilde{B}_{1,2}g^4 + \text{finite contact terms} \\ &\quad - 4\beta_0 g^2\log\frac{p^2}{\mu^2} + 4\beta_0^2 g^4\log^2\frac{p^2}{\mu^2} + 4B_{1,3}g^4\log\frac{p^2}{\mu^2} + \frac{4\beta_1 g^4}{\epsilon} + \cdots \\ &= 4 - 4B_{1,1}g^2 - 4B_{1,2}g^4 + \frac{4\beta_1 g^4}{\epsilon} \\ &\quad - 4\beta_0 g^2\log\frac{p^2}{\mu^2} + 4\beta_0^2 g^4\log^2\frac{p^2}{\mu^2} + 4B_{1,3}g^4\log\frac{p^2}{\mu^2} + \cdots \end{aligned} \tag{4.43}$$

where in the last equality we have employed eq. (4.40) and:

$$\tilde{B}_{1,2} = B_{1,2} - B_{1,3}\left(\frac{3}{2}+\frac{5}{2}\Gamma'(1) - \log\frac{\sqrt{\pi}}{4}\right) + 2\beta_0^2\left\{-\frac{1}{2}\left(2+3\Gamma'(1)-\log\frac{\pi}{4}\right)^2 \right.$$
$$-\left(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\right)^2 + \left(2+3\Gamma'(1)-\log\frac{\pi}{4}\right)\left(\frac{3}{2}+\frac{5}{2}\Gamma'(1)-\log\frac{\sqrt{\pi}}{4}\right)$$
$$\left. +\frac{1}{4}-\frac{1}{4}\Gamma'^2(1)+\frac{1}{4}\Gamma''(1)\right\} \tag{4.44}$$

The result in eq. (4.43) reproduces eq. (4.24) to order g^4. We have thus verified the inequality in eq. (4.21) for $O = F^2$ up to order g^4 in perturbation theory. Indeed, the object in the coordinate representation whose Fourier transform reproduces the multiplicatively renormalized coefficient $Z_{F^2}C_1^{(F_0^2,F^2)}(p)$ in eq. (4.24) is provided by eq. (4.38) as $\epsilon \to 0$, with $C_1^{(F^2,F^2)'}(x)$ in eq. (4.30). It differs from the multiplicatively renormalized $C_1^{(F^2,F^2)'}(x)$ at distinct points by divergent and finite proper contact terms. Specifically, to order g^2 the multiplicatively renormalized $C_1^{(F^2,F^2)}(p)$ is finite, so that no additive counterterm occurs, while the Fourier transform of the multiplicatively renormalized $C_1^{(F^2,F^2)'}(x)$ at distinct points is divergent, thus implying the occurrence of divergent proper contact terms in $C_1^{(F^2,F^2)}(x)$ to cancel the latter divergence. To order g^4 the multiplicatively renormalized $C_1^{(F^2,F^2)}(p)$ is divergent and the additive renormalization occurs, but the divergence — as to order g^2 — does not entirely arise from the Fourier transform of $C_1^{(F^2,F^2)'}(x)$.

4.3 Contact terms in the OPE from the LET

Having clarified the nature of contact terms in the Fourier transform of the OPE coefficient of $Z_{F^2}F_0^2(x)$ with $O(0)$, we demonstrate the relation between its divergences and the ones of the integrated 3-point correlator in the r.h.s. of the LET according to eq. (4.1). Obviously, it suffices to consider the OPE of $F^2(x)$ with $O(0)$, the OPE with $O(z)$ being entirely analogous:

$$Z_{F^2}F_0^2(x)O(0) = \cdots + Z_{F^2}C_1^{(F_0^2,O)}(x)O(0) + \cdots \tag{4.45}$$

In general, for dimensional reasons the above OPE coefficient is the only one that may contribute in the integrated 3-point correlator the product of a dimension 4 — possibly divergent — contact term times the 2-point correlator at distinct points $\langle O(z)O(0)\rangle$ that has to be cancelled by the counterterm with the same structure in the r.h.s. of the LET, both in its perturbative and RG-improved version. The argument is as follows.

It is convenient to consider at the same time the theory in $d = 4$ dimensions and its dimensionally regularized version in $\tilde{d} = 4 - 2\epsilon$ dimensions. The theory in $d = 4$ dimensions is exactly conformal perturbatively to order g^2 in a conformal scheme [21], which may differ by a finite renormalization from a generic scheme, because the beta function $\beta(g) = -\beta_0 g^3 + \cdots$ affects the solution of the CS equation only starting from order g^4. The theory in $\tilde{d} = 4 - 2\epsilon$ dimensions is exactly conformal perturbatively to order g^0, because the beta function $\beta(g,\epsilon) = -\epsilon g + \cdots$ affects the solution of the CS equation only starting from order g^2.

It follows that up to the perturbative order where the theory is conformal, under the standard assumption that O belongs to a basis of mutually orthogonal conformal primary operators, along with the term in the OPE in eq. (4.45) only the terms in the OPE of $Z_{F^2}F_0^2(x)$ with the descendants of $O(0)$ may a priori also contribute to the r.h.s. of the LET, once inserted in the v.e.v. with $O(z)$. Indeed, all the 2-point correlators of $O(z)$ with different primary operators and their descendants vanish identically, given the orthogonality of the basis. Such a basis of Hermitian primary operators certainly exists, since the 2-point correlators of primary operators with different conformal dimensions vanish by the conformal symmetry, and the 2-point correlators of operators with the same conformal dimension are a real symmetric matrix that can always be diagonalized — times a universal conformal structure.

In fact, the OPE of $Z_{F^2}F_0^2(x)$ with the descendants of $O(0)$ cannot contribute to the divergence of the integrated 3-point correlator: the corresponding OPE coefficients have conformal dimensions lower than 4 — their integrals being therefore UV finite — and the insertion of the descendants gives rise to derivatives of the 2-point correlator, instead of the correlator itself.

A similar argument holds to higher orders in perturbation theory and nonperturbatively, both in $d = 4$ and $\tilde{d} = 4 - 2\epsilon$ dimensions, in AF massless QCD. In both cases, for dimensional reasons, only operators O' with the same canonical dimension of O may a priori contribute to the relevant OPE. Yet, their insertion gives rise to 2-point correlators $\langle O(z)O'(0)\rangle$ that — by the above conformal argument, because of the orthogonality of O and O' in the conformal limit — are necessarily suppressed by powers of the coupling $g^{2n}(\mu)$ perturbatively, with n a positive integer. To actually rule out their contribution to the divergences in the r.h.s. of the LET in the AF case, we show that $\langle O(z)O(0)\rangle$ is in fact linearly independent of $\langle O(z)O'(0)\rangle$ with coefficients valued in the renormalized coupling $g(\mu)$, so that no linear combination of $\langle O(z)O'(0)\rangle$ may occur in the divergent part of the integrated 3-point correlator in the r.h.s. of the LET. To this aim, it suffices to observe that in $d = 4$ dimensions in general the asymptotics of $\langle O(z)O'(0)\rangle$ as $z \to 0$:

$$\begin{aligned}\langle O(z)O'(0)\rangle_{d=4} &= \frac{\mathcal{G}_2^{(O,O')}(g(z))}{|z|^{2\Delta_{O_0}}} Z^{(O)}(g(z),g(\mu))Z^{(O')}(g(z),g(\mu)) \\ &\sim \frac{g^{2n}(z)}{|z|^{2\Delta_{O_0}}} g^{\frac{\gamma_0^{(O)}+\gamma_0^{(O')}}{\beta_0}}(z)\end{aligned} \tag{4.46}$$

is linearly independent of the one of $\langle O(z)O(0)\rangle_{d=4}$:

$$\begin{aligned}\langle O(z)O(0)\rangle_{d=4} &= \frac{\mathcal{G}_2^{(O)}(g(z))}{|z|^{2\Delta_{O_0}}} Z^{(O)2}(g(z),g(\mu)) \\ &\sim \frac{1}{|z|^{2\Delta_{O_0}}} g^{\frac{2\gamma_0^{(O)}}{\beta_0}}(z)\end{aligned} \tag{4.47}$$

and analogously in $\tilde{d} = 4 - 2\epsilon$ dimensions (appendix D). The only exception may occur for $\gamma_0^{(O')} + 2n\beta_0 = \gamma_0^{(O)}$ that, in the terminology of [22, 23], is the resonant condition for the operator mixing of O with O', necessary for the nonexistence of a renormalization scheme where O, O' are multiplicatively renormalizable.

We conclude that, if the nonresonant condition $\gamma_0^{(O')} + 2n\beta_0 \neq \gamma_0^{(O)}$ is satisfied, the OPE coefficient of $Z_{F^2}F_0^2$ with O', the latter having the same canonical dimension of O, does not contribute to the divergent part of the integrated 3-point correlator in the r.h.s. of the LET for a given 2-point correlator of O in the l.h.s.

However, we should stress that the nonresonant condition is sufficient but by no means necessary. First, even if the UV asymptotics of $\langle O(z)O(0)\rangle$ and $\langle O(z)O'(0)\rangle$ coincide, the actual 2-point correlators may still be functional independent to higher orders in the running coupling. Second, even if the resonant condition is realized, the resulting correlators may still be functional independent for different reasons. For example, for $O = F^2$ in massless QCD, despite the resonant condition with $n = 1$ is satisfied for $O' = \bar{\psi}\gamma_\mu D_\mu \psi$, the correlator $\langle F^2(z)\bar{\psi}\gamma_\mu D_\mu \psi(0)\rangle$ at distinct points vanishes because of the EOM both in $d = 4$ and $\tilde{d} = 4 - 2\epsilon$ dimensions.

Hence, under the above assumptions, we can relate the divergent contact terms that arise in the integrated 3-point correlator in the r.h.s. of the LET as $\epsilon \to 0$ to those of the corresponding multiplicatively renormalized OPE coefficient in the momentum representation:

$$\frac{1}{2}[Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle \, d^{\tilde{d}}x]_{\text{div}} = [Z_{F^2} C_1^{(F_0^2,O)}(p)]_{\text{div}} \, \langle O(z)O(0)\rangle \tag{4.48}$$

where the Fourier transform in the r.h.s. may be conveniently computed at nonvanishing momentum p to avoid infrared divergences in perturbation theory, without affecting the divergent contact terms:

$$[Z_{F^2} C_1^{(F_0^2,O)}(p)]_{\text{div}} = [Z_{F^2} C_1^{(F_0^2,O)}(0)]_{\text{div contact}} \tag{4.49}$$

because they are the Fourier transform of a $\delta^{(4)}$ — or a $\delta^{(\tilde{d})}$ in $\tilde{d} = 4-2\epsilon$ dimensions (section 3.7) — thus a momentum-independent constant. We should notice that in perturbation theory there is a finite ambiguity in eq. (4.48), due to the fact that the l.h.s. is computed in the coordinate representation and the r.h.s. in the momentum representation.

Finally, once the lowest order finite contact term $2c_O\delta^{(4)}$ is determined consistently with perturbation theory according to (IIA), under the above assumptions, the LET unambiguously predicts the fully renormalized OPE coefficient in the momentum representation:

$$C_1^{(F^2,O)}(p) = Z_{F^2} C_1^{(F_0^2,O)}(p) + Z_{F^2} \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \tag{4.50}$$

in terms of the multiplicatively renormalized one and the divergent contact term occurring in the r.h.s. of the LET. Indeed, given that $\gamma_{F^2}(g) = g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)$ (appendix A), the contact term above coincides for $O = F^2$ with the one computed in eq. (4.22) [2].

5 Perturbative versus nonperturbative LET

5.1 Perturbative LET

(IIA) shows that in general the 3-point correlator with the insertion of $Z_{F^2}F_0^2(x)$ at zero momentum needs — order by order in perturbation theory — an infinite additive renormalization as $\epsilon \to 0$ to yield the finite fully renormalized object defined by the l.h.s.:

$$
\begin{aligned}
&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} + 2c_O \langle O(z)O(0)\rangle \\
&= \frac{1}{2} Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle \, d^{\tilde{d}}x \\
&+ Z_{F^2} \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \langle O(z)O(0)\rangle
\end{aligned} \tag{5.1}
$$

Indeed, we have already compared the additive counterterm in (IIA) to order g^4 in perturbation theory for $O = F^2$ with the corresponding counterterm in the OPE coefficient $C_1^{(F^2,F^2)}(p)$ finding perfect agreement (section 4). Besides, in doing so we have discovered the existence of divergent proper contact terms in the coordinate representation.

5.2 Nonperturbative LET

We take the limit $\epsilon \to 0$ of the LET in eq. (5.1) in the AF phase:

$$
\begin{aligned}
&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} + 2c_O \langle O(z)O(0)\rangle \\
&= \frac{1}{2} \int \langle O(z)O(0)F^2(x)\rangle - \langle O(z)O(0)\rangle \langle F^2(x)\rangle \, d^4x \\
&- \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\frac{\beta(g)}{g}} \langle O(z)O(0)\rangle
\end{aligned} \tag{5.2}
$$

where we have employed:

$$
Z_{F^2} \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} = \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon(1 - \frac{\beta(g)}{\epsilon g})} \tag{5.3}
$$

Hence, eq. (5.2) reads identically:

$$
\begin{aligned}
&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} \\
&= \frac{1}{2} \int \langle O(z)O(0)F^2(x)\rangle - \langle O(z)O(0)\rangle \langle F^2(x)\rangle \, d^4x \\
&- \frac{2\gamma_O(g)}{\frac{\beta(g)}{g}} \langle O(z)O(0)\rangle
\end{aligned} \tag{5.4}
$$

Somehow unexpectedly, no infinite additive renormalization occurs nonperturbatively in $d = 4$ dimensions in the AF phase, since in the last term no zero of the beta function arises in the denominator but the one at $g = 0$ that is cancelled by the zero of the same order due to the anomalous dimension $\gamma_O(g)$ in the numerator.

Another important difference with the perturbative case is that in the nonperturbative AF case an infinite additive renormalization cannot either arise from the integrated OPE coefficient at distinct points, since the latter turns out to be UV finite because of the asymptotic freedom for small x^2 [11]:

$$
\int C_1^{(F^2,O)'}(x) e^{-ip\cdot x} d^4x \sim \int \frac{g^2(x)}{g^2(\mu)} \frac{g^2(x)}{x^4} e^{-ip\cdot x} d^4x < +\infty \tag{5.5}
$$

where the first factor in the last integrand arises from the anomalous dimension of F^2 and the second one from the fact that the first contribution to the OPE coefficient at distinct points occurs to order g^2 in perturbation theory. As a consequence, no divergent proper contact term occurs nonperturbatively in the AF phase.

6 Solving for the perturbative correlators to order g^2 by the LET

We find an a-priori solution of the LET for the correlators up to order g^2 in perturbation theory, where a massless QCD-like theory is conformal invariant in $d = 4$ dimensions. Though LET (IC) and (IIB) are strictly equivalent, for technical reasons it is interesting to compute to order g^2 both versions: (IC) is further employed to verify (section 8.1) the corresponding version that involves a hard-cutoff regularization [11, 12], while (IIB) is suitable to match the finite and divergent proper contact terms in the OPE coefficient $C_1^{(F^2,O)}(x)$ up to order g^2 in perturbation theory.

6.1 LET (IC) and (IIB) to order g^2

We find the common solution of (IC):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g}\\&= \frac{1}{2} Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle - 2c_O(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x)) \langle O(z)O(0)\rangle \, d^{\tilde{d}}x\\&\quad + Z_{F^2} \frac{2\gamma_O(g)}{\epsilon} \langle O(z)O(0)\rangle\end{aligned} \tag{6.1}$$

and (IIB):

$$\begin{aligned}&\frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} + 2c_O \langle O(z)O(0)\rangle\\&= \frac{1}{2} Z_{F^2} \int \langle O(z)O(0)F_0^2(x)\rangle\\&\quad + Z_{F^2} \frac{2\gamma_O(g) - 2c_O\frac{\beta(g)}{g}}{\epsilon} (\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x)) \langle O(z)O(0)\rangle \, d^{\tilde{d}}x\end{aligned} \tag{6.2}$$

up to order g^2, where we have set $z \neq 0$ once for all, and $\langle F_0^2(x)\rangle = 0$ perturbatively in dimensional regularization.

The l.h.s. of the LET is computed from the exact solution of the CS equation at distinct points in dimensional regularization by means of (appendix D.5):

$$\frac{\partial \langle O(z)O(0)\rangle_{\tilde{d}=4-2\epsilon}}{\partial \log g} = \langle O(z)O(0)\rangle_{\tilde{d}=4-2\epsilon} \frac{\partial \log \langle O(z)O(0)\rangle_{\tilde{d}=4-2\epsilon}}{\partial \log g} \tag{6.3}$$

expanded up to order g^2. We obtain:

$$\frac{\partial \log \langle O(z)O(0)\rangle_{\tilde{d}=4-2\epsilon}}{\partial \log g}\Big|_{\text{up to order } g^2} = \frac{\partial \log N_2(g)}{\partial \log g} - 2g\frac{\partial \gamma_O}{\partial g} \log |z\mu| + \cdots \Big|_{\text{up to order } g^2} \tag{6.4}$$

JHEP08(2024)145

that coincides, up to terms that vanish as $\epsilon \to 0$, with the corresponding object computed by means of the conformal renormalized 2-point correlator in $d = 4$ dimensions to order g^2:

$$\langle O(z)O(0)\rangle_{d=4} = \frac{N_2(g)\mu^{2\Delta_{O_0}-2\Delta_O}}{|z|^{2\Delta_O}} \tag{6.5}$$

Hence, (IC) reads:

$$\begin{aligned}
&\langle O(z)O(0)\rangle \left(\frac{\partial \log N_2(g)}{\partial \log g} - 2g\frac{\partial \gamma_O}{\partial g}\log|z\mu| + \cdots\right)\Big|_{\text{up to order } g^2} \\
&= \frac{1}{2}Z_{F^2}\int \langle O(z)O(0)F_0^2(x)\rangle - 2c_O(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x))\,\langle O(z)O(0)\rangle\, d^{\tilde{d}}x \\
&\quad + Z_{F^2}\frac{2\gamma_O(g)}{\epsilon}\langle O(z)O(0)\rangle\Big|_{\text{up to order } g^2}
\end{aligned} \tag{6.6}$$

and (IIB) reads:

$$\begin{aligned}
&\langle O(z)O(0)\rangle \left(2c_O + \frac{\partial \log N_2(g)}{\partial \log g} - 2g\frac{\partial \gamma_O}{\partial g}\log|z\mu| + \cdots\right)\Big|_{\text{up to order } g^2} \\
&= \frac{1}{2}Z_{F^2}\int \langle O(z)O(0)F_0^2(x)\rangle \\
&\quad + Z_{F^2}\frac{2\gamma_O(g) - 2c_O\frac{\beta(g)}{g}}{\epsilon}(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x))\,\langle O(z)O(0)\rangle\, d^{\tilde{d}}x\Big|_{\text{up to order } g^2}
\end{aligned} \tag{6.7}$$

Because of the structure of (IC) the following ansatz is natural for the bare 3-point correlator extended at coinciding points:

$$\begin{aligned}
\langle O(z)O(0)F_0^2(x)\rangle_{\tilde{d}=4-2\epsilon}\Big|_{\text{up to order } g^2} &= (2c_O + \tilde{B}g^2)(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x))\,\langle O(z)O(0)\rangle \\
&\quad + Z_{F^2}^{-1}\langle O(z)O(0)F^2(x)\rangle'\Big|_{\text{up to order } g^2} \\
&= (2c_O + \tilde{B}g^2)(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x))\,\langle O(z)O(0)\rangle \\
&\quad + \langle O(z)O(0)F^2(x)\rangle'\Big|_{\text{up to order } g^2}
\end{aligned} \tag{6.8}$$

where the second equality follows from the fact that $\langle O(z)O(0)F^2(x)\rangle'$ is necessarily of order g^2 by the LET. Therefore, the bare correlator in eq. (6.8) differs from the multiplicatively renormalized one at distinct points by finite proper contact terms and it is finite up to order g^2. It follows from eq. (6.8) the multiplicatively renormalized correlator extended at coinciding points:

$$\begin{aligned}
Z_{F^2}\langle O(z)O(0)F_0^2(x)\rangle\Big|_{\text{up to order } g^2} &= Z_{F^2}(2c_O + \tilde{B}g^2)(\delta^{(\tilde{d})}(x-z) + \delta^{(\tilde{d})}(x))\,\langle O(z)O(0)\rangle \\
&\quad + \langle O(z)O(0)F^2(x)\rangle'\Big|_{\text{up to order } g^2}
\end{aligned} \tag{6.9}$$

Hence, inserting eq. (6.8) into the r.h.s. of (IC), we conclude that to order g^2 the additive counterterm in the last line of the r.h.s. of (IC) may only be compensated for by the divergence of the space-time integral of the bare 3-point correlator at distinct points:

$$\frac{1}{2}\int \langle O(z)O(0)F^2(x)\rangle' \, d^{\tilde{d}}x\Big|_{\text{to order } g^2} \tag{6.10}$$

that to order g^2 coincides with the nultiplicatively renormalized one.

Instead, the multiplicatively renormalized 3-point correlator extended at coinciding points in eq. (6.9) that occurs in (IIB) differs from the correlator at distinct points by finite and divergent proper contact terms to order g^2, consistently with the explicit computation of the OPE coefficient for $O = F^2$. Therefore, the finiteness of the r.h.s. of (IIB) arises from a more involved cancellation between the integral of the correlator at distinct points, the divergent proper contact terms in eq. (6.9) and the additive counterterm in the r.h.s. of (IIB) to order g^2.

Yet, the two versions of the LET are equivalent, and we employ (IC) in the following. We need an ansatz for the 3-point correlator at distinct points to order g^2 in $\tilde{d} = 4-2\epsilon$ dimensions:

$$\langle O(z)O(0)F^2(x)\rangle'_{\tilde{d}=4-2\epsilon} = \frac{N_3(g)\mu^{2\tilde{\Delta}_O+\Delta^*_{F^2}-2\tilde{\Delta}_{O_0}-\tilde{\Delta}_{F_0^2}}}{|z|^{2\tilde{\Delta}_O-\Delta^*_{F^2}}|x|^{\Delta^*_{F^2}}|x-z|^{\Delta^*_{F^2}}}\Big|_{\text{to order } g^2} \tag{6.11}$$

where $\Delta^*_{F^2}$ is uniquely determined by the two conditions:

(I) The correlator should be conformal in the limit $\epsilon \to 0$.

(II) The correlator should imply the correct OPE coefficient to order g^2 in $\tilde{d} = 4-2\epsilon$ dimensions.

The ansatz in eq. (6.11) fulfills property (I) provided that $\Delta^*_{F^2} \to 4$ as $\epsilon \to 0$, since $N_3(g)$ is of order g^2.

We point out that (II) is in general incompatible with an exact conformal symmetry in $\tilde{d} = 4-2\epsilon$ dimensions because of the soft breaking of the latter in the CS equation to order g^2, due to the nonvanishing beta function $\beta(g,\epsilon) = -\epsilon g + \cdots$ induced by dimensional regularization. We determine $\Delta^*_{F^2}$ from (II) by observing that up to order g^2:

$$\langle O(z)O(0)F^2(x)\rangle' \sim C_1^{(F^2,O)'}(x)\, \langle O(z)O(0)\rangle\Big|_{\text{to order } g^2} \tag{6.12}$$

should hold as x^2 approaches zero, with $C_1^{(F^2,O)'}(x)$ the OPE coefficient at distinct points. $C_1^{(F^2,O)'}(x)$ satisfies eq. (4.8), and the perturbative expansion to order g^2 of its solution in eq. (4.12) reads:

$$\begin{aligned} C_1^{(F^2,O)'}(x) &= \frac{\mathcal{G}^{(F^2,O)}(\tilde{g}(x))}{|x|^{4-2\epsilon}} Z^{(F^2)}(\tilde{g}(x),g) \\ &= \frac{(a\tilde{g}^2(x)+\cdots)}{|x|^{4-2\epsilon}}\frac{\tilde{g}^2(x)}{g^2}|x\mu|^{-2\epsilon} \\ &= \frac{ag^2}{|x|^{4-2\epsilon}}\left(\frac{|x\mu|^{2\epsilon}}{1-\beta_0 g^2\frac{|x\mu|^{2\epsilon}-1}{\epsilon}}\right)^2|x\mu|^{-2\epsilon}+\cdots \\ &= \frac{a\mu^{2\epsilon}g^2}{|x|^{4-4\epsilon}}+\cdots\Big|_{\text{to order } g^2} \end{aligned} \tag{6.13}$$

where we have employed $\tilde{\Delta}_{F_0^2} = 4 - 2\epsilon$, eq. (4.28) and eq. (4.29), with $g(\mu) = g$. On the other hand, the short-distance singularity as $x^2 \to 0$ of the 3-point correlator in eq. (6.11) reads to order g^2:

$$\begin{aligned}\langle O(z)O(0)F^2(x)\rangle'_{\tilde{d}=4-2\epsilon} &\sim \frac{N_3(g)\mu^{\Delta^*_{F^2}-\tilde{\Delta}_{F_0^2}}}{|x|^{\Delta^*_{F^2}}} \frac{\mu^{2\tilde{\Delta}_O-2\tilde{\Delta}_{O_0}}}{|z|^{2\tilde{\Delta}_O}}\Big|_{\text{to order } g^2} \\ &\sim \frac{N_3(g)}{N_2(g)} \frac{\mu^{\Delta^*_{F^2}-\tilde{\Delta}_{F_0^2}}}{|x|^{\Delta^*_{F^2}}} \langle O(z)O(0)\rangle\Big|_{\text{to order } g^2}\end{aligned} \tag{6.14}$$

The comparison of eq. (6.13) with eq. (6.14) and the asymptotics in eq. (6.12) then imply that the corrected dimension due to the nonvanishing beta function in dimensional regularization is $\Delta^*_{F_0^2} = \tilde{\Delta}_{F_0^2} - 2\epsilon = 4 - 4\epsilon$ in the denominator of eq. (6.11), while the powers of μ in the numerator compensate for the change of dimension. From eq. (6.12) we also deduce the normalization of the OPE coefficient:

$$ag^2 = \frac{N_3(g)}{N_2(g)}\Big|_{O(g^2)} \equiv \frac{N_3}{N_2} g^2 \tag{6.15}$$

in terms of the normalization of the 3- and 2-point correlators. Therefore, to order g^2 we obtain:

$$\begin{aligned}&\frac{1}{2}\int \langle O(z)O(0)F^2(x)\rangle' \, d^{\tilde{d}}x\Big|_{\text{up to order } g^2} \\ &= \frac{1}{2}\int \frac{\mu^{2\epsilon}N_3(g)}{|z|^{2\tilde{\Delta}_{O_0}-\tilde{\Delta}_{F_0^2}+2\epsilon}|x|^{\tilde{\Delta}_{F_0^2}-2\epsilon}|x-z|^{\tilde{\Delta}_{F_0^2}-2\epsilon}} \, d^{\tilde{d}}x\Big|_{\text{up to order } g^2} \\ &= \langle O(z)O(0)\rangle_{\tilde{d}=4-2\epsilon} \frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}\left(\frac{2}{\epsilon} + 4\log|z\mu| - 2\log\pi + 2 + 2\Gamma'(1)\right)\Big|_{\text{up to order } g^2}\end{aligned} \tag{6.16}$$

where the above integral has been computed in appendix C. Hence, the finiteness of the r.h.s. of (IC) in eq. (6.6) implies:

$$-\frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}\Big|_{\text{to order } g^2} = \gamma_O(g)\Big|_{\text{to order } g^2} \tag{6.17}$$

with $\gamma_O(g) = -\gamma_0^{(O)} g^2 + \cdots$, that in turn implies for the OPE coefficient of a generic operator O with F^2 at distinct points to order g^2:

$$C_1^{(F^2,O)'}(x) = \frac{2\gamma_0^{(O)}}{\pi^2}\frac{\mu^{2\epsilon}g^2}{|x|^{4-4\epsilon}} + \cdots \tag{6.18}$$

by means of eqs. (6.13) and (6.15). Besides, the matching to order g^2 of the logarithmic terms in the l.h.s. and r.h.s. of eq. (6.6), which is independent of the regularization, leads to the constraint:

$$-2g\frac{\partial\gamma_O}{\partial g}\log|z\mu|\Big|_{\text{order } g^2} = 4\frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}\log|z\mu|\Big|_{\text{order } g^2} \tag{6.19}$$

that implies:

$$g\frac{\partial\gamma_O}{\partial g}\bigg|_{\text{order } g^2} = -\pi^2\frac{N_3(g)}{N_2(g)}\bigg|_{\text{order } g^2} \tag{6.20}$$

Putting the above results together, we get:

$$g\frac{\partial\gamma_O}{\partial g}\bigg|_{\text{order } g^2} = 2\gamma_O(g)\bigg|_{\text{order } g^2} \tag{6.21}$$

whose only solution is that the anomalous dimension $\gamma_O(g) = -\gamma_0^{(O)}g^2$ is one-loop exact, which is obviously consistent with our perturbative assumption to order g^2.

Vice versa, since $\gamma_O(g) = -\gamma_0^{(O)}g^2$ to order g^2, no divergent contact term may arise in the bare correlator in eq. (6.8), otherwise the LET would not be satisfied. The LET further implies the matching of the finite terms to order g^2:

$$\frac{\partial\log N_2(g)}{\partial\log g}\bigg|_{\text{to order } g^2} = \tilde{B}g^2 - \gamma_O(g)\Big(2+2\Gamma'(1)-2\log\pi\Big)\bigg|_{\text{to order } g^2} \tag{6.22}$$

that agrees with eq. (D.57) for $O = F^2$.

6.2 Verifying the LET for $O = F^2$

It is interesting to further verify the predictions of the LET for $O = F^2$ by comparing with the perturbative computation [2, 3, 12–14]. First, to order g^0:

$$F^2(x)F^2(0) = 2c_{F^2}\,\delta^{(4)}(x)F^2(0) + \cdots \tag{6.23}$$

with $c_{F^2} = 2$ [2, 3, 12–14] according to the LET.

Second, eq. (6.20):

$$-\frac{\pi^2}{2}\frac{4\beta_0}{\pi^2}g^2 = -2\beta_0 g^2 = \gamma_{F^2}(g) \tag{6.24}$$

is satisfied to order g^2, where $\gamma_{F^2}(g) = -2\beta_0 g^2 + \cdots$ (appendix A) and we read from the OPE of F^2 with itself [2, 3, 12–14] that $N_2 = \frac{48(N^2-1)}{\pi^4}$ and $N_3 = N_2\frac{4\beta_0}{\pi^2}g^2$ to their leading order, respectively.

6.3 Conformal resummation to order g^2

We find an a-priori solution of the resummed version of the LET in eq. (5.4) to order g^2 by employing the resummed conformal 3-point correlator in $d = 4$ dimensions expanded to order g^2 after the space-time integration:

$$\begin{aligned}
&\langle O(z)O(0)\rangle\left(\frac{\partial\log N_2(g)}{\partial\log g} - 2g\frac{\partial\gamma_O}{\partial g}\log|z\mu|\right)\bigg|_{\text{up to order } g^2}\\
&= \frac{1}{2}\int Z_{F^2}\,\langle O(z)O(0)F_0^2(x)\rangle\\
&\quad - \frac{2\gamma_O(g)}{\frac{\beta(g)}{g}}\left(\delta^{(4)}(x-z)+\delta^{(4)}(x)\right)\langle O(z)O(0)\rangle\,d^4x\bigg|_{\text{up to order } g^2}
\end{aligned} \tag{6.25}$$

In the resummed conformal theory the multiplicatively renormalized 3-point correlator in the coordinate representation must be finite, since no contact term may arise for $\gamma_{F^2} \neq 0$ for dimensional reasons:

$$\langle O(z)O(0)F_0^2(x)\rangle\Big|_{\text{up to order } g^2} = Z_{F^2}^{-1}\langle O(z)O(0)F^2(x)\rangle'\Big|_{\text{up to order } g^2} \tag{6.26}$$

Moreover, in $d=4$ dimensions the space-time integral of the resummed conformal 3-point correlator at distinct points:

$$\begin{aligned}\int \langle O(z)O(0)F^2(x)\rangle' d^4x\Big|_{\text{up to order } g^2} &= \int \frac{N_3(g)\mu^{-2\gamma_O-\gamma_{F^2}}}{|z|^{2\Delta_O-\Delta_{F^2}}|x|^{\Delta_{F^2}}|x-z|^{\Delta_{F^2}}}d^4x\Big|_{\text{up to order } g^2}\\ = \langle O(z)O(0)\rangle\,\pi^2\frac{N_3(g)}{N_2(g)}|z\mu|^{-\gamma_{F^2}}&\frac{\Gamma(2+\gamma_{F^2})\Gamma(-\frac{\gamma_{F^2}}{2})^2}{\Gamma(2+\frac{\gamma_{F^2}}{2})^2\Gamma(-\gamma_{F^2})}\Big|_{\text{up to order } g^2}\\ = \langle O(z)O(0)\rangle\,\pi^2\frac{N_3(g)}{N_2(g)}&\left(\frac{2}{\beta_0 g^2}+4\log|z\mu|+\cdots\right)\Big|_{\text{up to order } g^2}\end{aligned} \tag{6.27}$$

is finite because $\Delta_{F^2} = 4+\gamma_{F^2}$, with nonzero $\gamma_{F^2} = -2\beta_0 g^2$ to order g^2 (appendix C). Hence, the LET implies the following matching of the logarithmic terms in the l.h.s. and r.h.s. up to order g^2:

$$-2g\frac{\partial\gamma_O}{\partial g}\log|z\mu|\Big|_{\text{up to order } g^2} = \frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}4\log|z\mu|\Big|_{\text{up to order } g^2} \tag{6.28}$$

that is regularization independent, and leads to the constraint:

$$g\frac{\partial\gamma_O}{\partial g}\Big|_{\text{up to order } g^2} = -\pi^2\frac{N_3(g)}{N_2(g)}\Big|_{\text{up to order } g^2} \tag{6.29}$$

The latter implies that $\frac{N_3(g)}{N_2(g)}$ is of order g^2 and it reproduces the same constraint implied by (IC) and (IIB) in perturbation theory. The finite contribution in the r.h.s. of eq. (6.25) reads to order g^0 by means of eq. (6.27):

$$\begin{aligned}0 &= \frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}\frac{2}{\beta_0 g^2} - \frac{2\gamma_O(g)}{\frac{\beta(g)}{g}}\Big|_{\text{order } g^0}\\ &= -\frac{1}{\beta_0 g^2}g\frac{\partial\gamma_O}{\partial g} - \frac{2\gamma_O(g)}{\frac{\beta(g)}{g}}\Big|_{\text{order } g^0}\end{aligned} \tag{6.30}$$

that vanishes identically for $\gamma_O(g) = -\gamma_0^{(O)}g^2$ and $\frac{\beta(g)}{g} = -\beta_0 g^2$. The LET is then fulfilled because $\frac{\partial\log N_2(g)}{\partial\log g}$ in the l.h.s. vanishes identically to order g^0. Yet, the finite terms cannot be determined to order g^2 consistently with the conformal symmetry, since:

$$\begin{aligned}&\frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}\frac{2}{\beta_0 g^2} - \frac{2\gamma_O(g)}{\frac{\beta(g)}{g}}\Big|_{\text{order } g^2}\\ &= \frac{\pi^2}{2}\frac{N_3(g)}{N_2(g)}\Big|_{\text{order } g^4}\frac{2}{\beta_0 g^2} - \frac{2\gamma_O(g)}{\frac{\beta(g)}{g}}\Big|_{\text{order } g^2}\end{aligned} \tag{6.31}$$

involve the evaluation of $\frac{N_3(g)}{N_2(g)}\Big|_{\text{order } g^4}$ and $\frac{2\gamma_O(g)}{\frac{\beta(g)}{g}}\Big|_{\text{order } g^2}$ that is not consistent with the conformal symmetry.

JHEP08(2024)145

7 Contact terms revisited

The occurrence of divergent proper contact terms in $C_1^{(F^2,O)}(x)$ may seem surprising. In the following we establish whether they are an isolated phenomenon or a more general one, by investigating whether they exist in $C_0^{(F^2,F^2)}(z)$ as well, where in this section we allow $z=0$.

Indeed, we demonstrate that to order g^0 the multiplicatively renormalized $Z_{F^2}\, C_0^{(F_0^2,F_0^2)}(p)$ needs an infinite additive renormalization that fully originates from the Fourier transform of $C_0^{(F^2,F^2)'}(z)$ at distinct points, so that no divergent proper contact term occurs to this order. Yet, we show that divergent proper contact terms do occur to order g^2 in $C_0^{(F^2,F^2)}(z)$ as in $C_1^{(F^2,F^2)}(x)$.

7.1 $\langle F^2(z)F^2(0)\rangle$ to order g^2 in perturbation theory

We expand the solution of the CS equation in $\tilde{d}=4-2\epsilon$ dimensions at distinct points (appendix D):

$$\langle F^2(z)F^2(0)\rangle' = \frac{1}{z^{8-4\epsilon}}\,\mathcal{G}_2^{(F^2)}(\tilde{g}(z))\,Z^{(F^2)2}(\tilde{g}(z),g(\mu)) \tag{7.1}$$

perturbatively up to order g^2. The RG-invariant function $\mathcal{G}_2^{(F^2)}$ reads:

$$\mathcal{G}_2^{(F^2)}(\tilde{g}(z)) = \mathcal{G}_2^{(F^2)}(0)(1+\delta_2^{(F^2)}\tilde{g}^2(z)+\cdots) \tag{7.2}$$

and, by means of eq. (D.35), we obtain:

$$\langle F^2(z)F^2(0)\rangle' = \frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\,(1+\delta_2^{(F^2)}\tilde{g}^2(z)+\cdots)\,\frac{\tilde{g}^4(z)}{g^4(\mu)}|z\mu|^{-4\epsilon}+\cdots \tag{7.3}$$

By employing the perturbative expansion of the running coupling $\tilde{g}(z)$ in eq. (D.23):

$$\begin{aligned}\frac{\tilde{g}^2(z)}{g^2(\mu)} &= \frac{|z\mu|^{2\epsilon}}{1-\beta_0 g^2(\mu)\frac{|z\mu|^{2\epsilon}-1}{\epsilon}+\cdots}\\ &= |z\mu|^{2\epsilon}\left(1+\beta_0 g^2(\mu)\frac{|z\mu|^{2\epsilon}-1}{\epsilon}+\cdots\right)\end{aligned} \tag{7.4}$$

and setting $g(\mu)=g$, eq. (7.3) yields:

$$\begin{aligned}\langle F^2(z)F^2(0)\rangle' &= \frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\left(1+\delta_2^{(F^2)}|z\mu|^{2\epsilon}g^2+\cdots\right)\left(1+2\beta_0 g^2\frac{|z\mu|^{2\epsilon}-1}{\epsilon}+\cdots\right)+\cdots\\ &= \frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\left(1+g^2\left(\delta_2^{(F^2)}|z\mu|^{2\epsilon}+2\beta_0\frac{|z\mu|^{2\epsilon}-1}{\epsilon}\right)+\cdots\right)\\ &= \mathcal{G}_2^{(F^2)}(0)\left\{\frac{1}{z^{8-4\epsilon}}\left(1-\frac{2\beta_0 g^2}{\epsilon}\right)+\frac{\mu^{2\epsilon}}{z^{8-6\epsilon}}g^2\left(\delta_2^{(F^2)}+\frac{2\beta_0}{\epsilon}\right)+\cdots\right\}\end{aligned} \tag{7.5}$$

Its Fourier transform is:

$$\begin{aligned}\mathrm{FT}[\,\langle F^2(z)F^2(0)\rangle'\,] &= \int d^{\tilde{d}}z\, e^{ipz}\,\langle F^2(z)F^2(0)\rangle'\\ &= \mathcal{G}_2^{(F^2)}(0)\left\{\mathrm{FT}\left[\frac{1}{z^{8-4\epsilon}}\right]\left(1-\frac{2\beta_0 g^2}{\epsilon}\right)\right.\\ &\qquad\left.+\mathrm{FT}\left[\frac{\mu^{2\epsilon}}{z^{8-6\epsilon}}\right]g^2\left(\delta_2^{(F^2)}+\frac{2\beta_0}{\epsilon}\right)+\cdots\right\}\end{aligned} \tag{7.6}$$

with $\tilde{d} = 4 - 2\epsilon$. It is convenient to first Fourier transform the correlator in eq. (7.5) and then to expand in ϵ. To this aim, we employ the general FT:

$$\mathrm{FT}\left[\frac{1}{x^{2\Delta}}\right] = \pi^{\frac{d}{2}} \frac{\Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta)} \left(\frac{p^2}{4}\right)^{\Delta - \frac{d}{2}} \tag{7.7}$$

that for $d = 4 - 2\epsilon$ and $\Delta = 4 - 2\epsilon$ yields to the relevant order in ϵ:

$$\begin{aligned} \mathrm{FT}\left[\frac{1}{x^{8-4\epsilon}}\right] &= \left(\frac{\pi p^2}{4}\right)^2 \left(\frac{\pi p^2}{4}\right)^{-\epsilon} \frac{\Gamma(-2+\epsilon)}{\Gamma(4-2\epsilon)} \\ &= \frac{\pi^2 p^4}{16} \left(\frac{\pi p^2}{4}\right)^{-\epsilon} \frac{1}{12\epsilon} \Bigg(1 + \epsilon\left(\frac{31}{6} + 3\Gamma'(1)\right) \\ &\quad + \epsilon^2 \left(\frac{601}{36} + \frac{31}{2}\Gamma'(1) - \frac{3}{2}\Gamma''(1) + 6\Gamma'^2(1)\right) + \cdots \Bigg) \end{aligned} \tag{7.8}$$

and, for $\Delta = 4 - 3\epsilon$:

$$\begin{aligned} \mathrm{FT}\left[\frac{\mu^{2\epsilon}}{x^{8-6\epsilon}}\right] &= \left(\frac{\pi p^2}{4}\right)^2 \mu^{2\epsilon} \left(\frac{\sqrt{\pi} p^2}{4}\right)^{-2\epsilon} \frac{\Gamma(-2+2\epsilon)}{\Gamma(4-3\epsilon)} \\ &= \frac{\pi^2 p^4}{16} \mu^{2\epsilon} \left(\frac{\sqrt{\pi} p^2}{4}\right)^{-2\epsilon} \frac{1}{24\epsilon} \Bigg(1 + \epsilon\left(\frac{17}{2} + 5\Gamma'(1)\right) \\ &\quad + \epsilon^2 \left(\frac{179}{4} + \frac{85}{2}\Gamma'(1) - \frac{5}{2}\Gamma''(1) + 15\Gamma'^2(1)\right) + \cdots \Bigg) \end{aligned} \tag{7.9}$$

By means of eqs. (7.8) and (7.9), eq. (7.6) reads as $\epsilon \to 0$:

$$\begin{aligned} \mathrm{FT}[\langle F^2(z) F^2(0)\rangle'] = \frac{\pi^2 \mathcal{G}_2^{(F^2)}(0)}{192} p^4 \Bigg\{ &\frac{1}{\epsilon} - \log p^2 - \log\frac{\pi}{4} + \frac{31}{6} + 3\Gamma'(1) \\ &- \frac{g^2\beta_0}{\epsilon^2} + \frac{g^2}{2\epsilon}\left(\delta_2^{(F^2)} - 2\beta_0\left(\frac{11}{6} + \Gamma'(1) - \log\pi\right)\right) + \frac{g^2\beta_0}{\epsilon}\log\mu^2 \\ &+ g^2\beta_0 \log^2\frac{p^2}{\mu^2} - \frac{g^2\beta_0}{2}\log^2\mu^2 \\ &- g^2\left(\delta_2^{(F^2)} + 2\beta_0\left(\frac{10}{3} + 2\Gamma'(1) + \log 4\right)\right)\log\frac{p^2}{\mu^2} \\ &- \frac{g^2}{2}\left(\delta_2^{(F^2)} - 2\beta_0\left(\frac{11}{6} + \Gamma'(1) - \log\pi\right)\right)\log\mu^2 \\ &- g^2\delta_2^{(F^2)}\left(\log\frac{\sqrt{\pi}}{4} - \frac{17}{4} - \frac{5}{2}\Gamma'(1)\right) + g^2\beta_0\Bigg(\left(\frac{11}{6} + \Gamma'(1)\right)\log\pi \\ &+ 4\left(\frac{5}{3} + \Gamma'(1)\right)\log 4 + \frac{409}{36} + \frac{23}{2}\Gamma'(1) + \frac{1}{2}\Gamma''(1) + 3\Gamma'^2(1)\Bigg) + \cdots \Bigg\} \end{aligned} \tag{7.10}$$

We now compare the above result with the perturbative computation of $C_0^{(F^2,F^2)}(p)$ in the $\overline{MS}$ scheme [2, 3]:

$$C_0^{(F^2,F^2)}(p) = Z_{F^2}^2 C_0^{(F_0^2,F_0^2)}(p) + p^4 Z_{0\mathrm{c.t.}} \tag{7.11}$$

where in our notation the multiplicatively renormalized contribution to order g^2 reads [2, 3]:

$$Z^2_{F^2} C_0^{(F_0^2,F_0^2)}(p) = \frac{N^2-1}{4\pi^2} p^4 \Big\{ 1 + \frac{1}{\epsilon} - \log\frac{p^2}{\mu^2} + g^2\beta_0 \log^2\frac{p^2}{\mu^2} - \frac{g^2\beta_0}{\epsilon^2} + 2g^2(c_1-\beta_0)\log\frac{p^2}{\mu^2} - \frac{g^2 c_1}{\epsilon} + \frac{g^2}{16\pi^2}\Big(-12\zeta_3 + \frac{485}{12} - \frac{17}{2}\frac{N_f}{N}\Big) + \cdots \Big\} \tag{7.12}$$

and the additive counterterm up to order g^2 reads [2, 3]:

$$p^4 Z_{0\text{c.t.}} = \frac{N^2-1}{4\pi^2} p^4 \Big\{ -\frac{1}{\epsilon} + \frac{g^2\beta_0}{\epsilon^2} + \frac{g^2 c_1}{\epsilon} + \cdots \Big\} \tag{7.13}$$

where, for later convenience, we have defined:

$$c_1 = -\frac{1}{16\pi^2}\Big(\frac{17}{2} - \frac{5}{3}\frac{N_f}{N}\Big) \tag{7.14}$$

so that we obtain the coefficient of the subleading $\log\frac{p^2}{\mu^2}$ to order g^2 in eq. (7.12):

$$2(c_1-\beta_0) = -\frac{1}{16\pi^2}\Big(\frac{73}{3} - \frac{14}{3}\frac{N_f}{N}\Big) \tag{7.15}$$

Eq. (7.10) reproduces the leading divergence as well as the leading $\log\frac{p^2}{\mu^2}$ to order g^0 of the multiplicatively renormalized coefficient in eq. (7.12), once the overall normalization has been matched:

$$\mathcal{G}_2^{(F^2)}(0) = \frac{48(N^2-1)}{\pi^4} \tag{7.16}$$

We conclude that no divergent proper contact term occurs to order g^0.

The matching of the subleading $\log\frac{p^2}{\mu^2}$ to order g^2 in eqs. (7.10) and (7.12) implies:

$$\delta_2^{(F^2)} = -2(c_1-\beta_0) - 2\beta_0\Big(\frac{10}{3} + 2\Gamma'(1) + \log 4\Big) \tag{7.17}$$

that inserted in the subleading divergence to order g^2 in eq. (7.10), by neglecting temporarily the pure $\log\mu^2$ terms, yields:

$$\begin{aligned}
&\frac{g^2}{2\epsilon}\Big(\delta_2^{(F^2)} - 2\beta_0\Big(\frac{11}{6} + \Gamma'(1) - \log\pi\Big)\Big)\\
&= \frac{g^2}{2\epsilon}\Big(-2c_1 + 2\beta_0 - 2\beta_0\Big(\frac{10}{3} + 2\Gamma'(1) + \log 4\Big) - 2\beta_0\Big(\frac{11}{6} + \Gamma'(1) - \log\pi\Big)\Big)\\
&= -\frac{g^2 c_1}{\epsilon} + \frac{g^2}{2\epsilon}\Big(-2\beta_0\Big(\frac{25}{6} + 3\Gamma'(1) - \log\frac{\pi}{4}\Big)\Big)
\end{aligned} \tag{7.18}$$

The last line differs from the corresponding divergent term in eq. (7.12) because of the last contribution proportional to β_0. Hence, the above calculation implies that both a finite and divergent proper contact term $\mu^{-2\epsilon}\tilde{C}\,\Delta^2\delta^{(\tilde{d})}(z)$ occur to order g^2, with:

$$\tilde{C} = \frac{N^2-1}{4\pi^2}\frac{g^2}{2\epsilon} 2\beta_0\Big(\frac{25}{6} + 3\Gamma'(1) - \log\frac{\pi}{4}\Big) + \frac{N^2-1}{4\pi^2}\Big(\tilde{A}_0 + \tilde{A}_1 g^2 + \cdots\Big) \tag{7.19}$$

so that the FT of:

$$\langle F^2(z)F^2(0)\rangle' + \mu^{-2\epsilon}\tilde{C}\,\Delta^2\delta^{(\tilde{d})}(z) \tag{7.20}$$

reproduces the multiplicatively renormalized OPE coefficient in eq. (7.12) up to pure $\log\mu^2$ terms, with Δ the Laplacian in $\tilde{d}$ Euclidean space-time dimensions, $\langle F^2(z)F^2(0)\rangle'$ given by eqs. (7.5) and (7.16), and:

$$\begin{aligned}
\tilde{A}_0 &= 1 - \frac{31}{6} - 3\Gamma'(1) + \log\frac{\pi}{4} \\
\tilde{A}_1 &= \delta_2^{(F^2)}\Big(\log\frac{\sqrt{\pi}}{4} - \frac{17}{4} - \frac{5}{2}\Gamma'(1)\Big) - \beta_0\Big(\Big(\frac{11}{6}+\Gamma'(1)\Big)\log\pi + 4\Big(\frac{5}{3}+\Gamma'(1)\Big)\log 4 \\
&\quad + \frac{409}{36} + \frac{23}{2}\Gamma'(1) + \frac{1}{2}\Gamma''(1) + 3\Gamma'^2(1)\Big) + \frac{1}{16\pi^2}\Big(-12\zeta_3 + \frac{485}{12} - \frac{17}{2}\frac{N_f}{N}\Big)
\end{aligned} \tag{7.21}$$

with $\delta_2^{(F^2)}$ in eq. (7.17) and c_1 in eq. (7.14).

Finally, as a further check, the fully renormalized OPE coefficient $C_0^{(F^2,F^2)}(p)$ in the momentum representation is obtained by adding to the FT of eq. (7.20) the counterterm proportional to $Z_{0\text{c.t.}}$ in eq. (7.13):

$$\begin{aligned}
C_0^{(F^2,F^2)}(p) = \;& \mathrm{FT}\big[\langle F^2(z)F^2(0)\rangle'\big] + \mathrm{FT}\big[\mu^{-2\epsilon}\tilde{C}\,\Delta^2\delta^{(\tilde{d})}(z)\big] \\
& + \mathrm{FT}\big[\mu^{-2\epsilon}Z_{0\text{c.t.}}\,\Delta^2\delta^{(\tilde{d})}(z)\big]
\end{aligned} \tag{7.22}$$

where:

$$\begin{aligned}
\mu^{-2\epsilon}Z_{0\text{c.t.}} &= \frac{N^2-1}{4\pi^2}\Big(1 - \epsilon\log\mu^2 + \frac{1}{2}\epsilon^2\log^2\mu^2 + \cdots\Big)\Big(-\frac{1}{\epsilon} + \frac{g^2\beta_0}{\epsilon^2} + \frac{g^2 c_1}{\epsilon} + \cdots\Big) \\
&= \frac{N^2-1}{4\pi^2}\Big(-\frac{1}{\epsilon} + \frac{g^2\beta_0}{\epsilon^2} + \frac{g^2 c_1}{\epsilon} + \log\mu^2 - \frac{g^2\beta_0}{\epsilon}\log\mu^2 \\
&\quad + \frac{g^2\beta_0}{2}\log^2\mu^2 - g^2 c_1\log\mu^2 + \cdots\Big)
\end{aligned} \tag{7.23}$$

Indeed, the identity in eq. (7.18) provides the complete cancellation of the divergences and pure $\log\mu^2$ terms in eq. (7.22), thus reproducing the fully renormalized $C_0^{(F^2,F^2)}(p)$ in eq. (7.11).

8 LET versus perturbative and nonperturbative renormalization

8.1 Perturbative renormalization

A perturbative computation of the bare LET to order g^2 has been worked out for $O = F^2$ with a hard-cutoff regularization [11]. The corresponding bare LET (IC) reads in dimensional regularization (section 3.5):

$$\begin{aligned}
&\frac{\partial}{\partial\log g_0}\langle O(z)O(0)\rangle_0\Big|_{\text{dim. reg.}} \\
&= \frac{1}{2}\int \langle O(z)O(0)F^2(x)\rangle_0 - \langle O(z)O(0)\rangle_0\langle F^2(x)\rangle_0\, d^4x - 2c_O\langle O(z)O(0)\rangle_0\Big|_{\text{dim. reg.}}
\end{aligned} \tag{8.1}$$

Eq. (8.1) is the most convenient to verify to order g^2 in perturbation theory the compatibility of the LET in [11] regularized by a hard-cutoff in $d = 4$ with our gauge-invariant formulation in dimensional regularization. Indeed, in [11] the occurrence of the finite contact terms in the r.h.s. has not been taken into account because of the hard-cutoff regularization in $d = 4$, which is consistent with their automatic subtraction in the r.h.s. of eq. (8.1).

The bare LET expressed in terms of renormalized objects follows by employing eq. (3.26):

$$\begin{aligned}&Z_{F^2}^{-1} Z_O^{-2} \frac{\partial \langle O(z)O(0)\rangle}{\partial \log g} - \frac{2\gamma_O(g)}{\epsilon} Z_O^{-2} \langle O(z)O(0)\rangle \Big|_{\text{dim. reg.}} \\ &= \frac{1}{2} Z_O^{-2} \int \langle O(z)O(0)F_0^2(x)\rangle - 2c_O(\delta^{(4)}(x-z) + \delta^{(4)}(x)) \langle O(z)O(0)\rangle \\ &\quad - \langle O(z)O(0)\rangle \langle F_0^2(x)\rangle \, d^4x \Big|_{\text{dim. reg.}} \end{aligned} \tag{8.2}$$

Then, the above LET allows us to verify to order g^2 in perturbation theory for the special case $O = F^2$ the corresponding version in [11], with the identification $\frac{1}{\epsilon} = \log(\frac{\Lambda^2}{\mu^2})$. Indeed, to this order we can set $Z_{F^2} = Z_O = 1$, since both sides of the LET are already of order g^2. It follows that all the perturbative arguments in [11] — strictly limited to order g^2 — hold unmodified.

8.2 Nonperturbative large-N renormalization

In the AF nonperturbative case, once the l.h.s. of the LET is expressed in terms of the derivative with respect to Λ_{UV}, the r.h.s. necessarily contains the insertion of the RG-invariant object according to eq. (3.39):

$$\begin{aligned}&\langle O(z)O(0)\rangle \left(2\gamma_O(g) - 2c_O\epsilon - \frac{\partial \log \langle O(z)O(0)\rangle}{\partial \log \Lambda_{UV}}\right) \\ &= \frac{1}{2}\frac{\beta(g,\epsilon)}{g} \int \langle O(z)O(0)F^2(x)\rangle - \langle O(z)O(0)\rangle \langle F^2(x)\rangle \, d^{\tilde{d}}x \end{aligned} \tag{8.3}$$

It follows that no divergent contact term arises in the limit $\epsilon \to 0$ both in the l.h.s. and r.h.s. according to eq. (3.42), so that all the arguments involving the nonperturbative large-N renormalization in $d = 4$ dimensions in [11] hold unmodified.

9 Conclusions

We summarize our main results.

First, we have worked out in dimensional regularization the renormalized LET for 2-point correlators $C_0^{(O,O)'}(z) = \langle O(z)O(0)\rangle'$ at distinct points $z \neq 0$ of a multiplicatively renormalizable operator O in the l.h.s. Our key result is that the corresponding integrated 3-point correlator in the r.h.s. needs in general an infinite additive renormalization as $\epsilon \to 0$ — completely controlled by the LET — in addition to the multiplicative one order by order in perturbation theory. The corresponding counterterm consists in the subtraction of an integrated — i.e. at zero momentum — divergent contact term proportional to a $\delta^{(4)}$ multiplying the 2-point correlator of O. In fact, we have provided two equivalent versions of the above LET — (I) and (II) — that differ by a finite contribution in the l.h.s. Besides, each version has three equivalent formulations (IA), (IB), (IC) in eqs. (3.28), (3.29), (3.31) and (IIA), (IIB), (IIC) in eqs. (3.32), (3.34), (3.35) respectively, corresponding to different ways of writing the aforementioned counterterm in the r.h.s.

Second, we have worked out two RG-invariant forms of the LET involving the insertion of $\frac{\beta(g,\epsilon)}{g}F^2$ in the r.h.s., in terms of the derivative with respect to the gauge coupling in the l.h.s. of eq. (3.37) and, in AF gauge theories, in terms of the derivative with respect to RG-invariant scale in the l.h.s. of eq. (3.39). Perhaps not surprisingly, no infinite additive renormalization occurs in this case.

Third, we have demonstrated in eq. (4.48) that the aforementioned divergent contact term in the r.h.s. of the LET coincides with the corresponding countertem for $Z_{F^2}C_1^{(F_0^2,O)}(p)$, so that the fully renormalized OPE coefficient $C_1^{(F^2,O)}(p)$ in the momentum representation is unambigoulsy fixed by the LET in eq. (4.50). We have also verified that, for the special case of $O = F^2$, the fully renormalized $C_1^{(F^2,F^2)}(p)$ in eq. (4.22) agrees with a previous result [2] obtained by means of an independent argument.

Fourth, we have argued according to eq. (4.21) that, in general, the aforementioned counterterm for $C_1^{(F^2,O)}(p)$ satisfying eq. (4.18) does not totally arise in dimensional regularization from the Fourier transform of $C_1^{(F^2,O)'}(x)$ restricted at distinct points satisfying eq. (4.8). As a consequence, an intrinsically divergent contact term — that we have referred to as a divergent proper contact term — must be added to $C_1^{(F^2,O)'}(x)$ in order to reproduce the correct infinite additive renormalization of $C_1^{(F^2,O)}(p)$. Indeed, we have verified the above statements by direct computation for $O = F^2$ in eqs. (4.38), (4.39) and (4.43).

These observations will play a key role in our second installment, where we apply the LET to QCD inside and above the conformal window.

Fifth, taking advantage that YM theory is conformal invariant to order g^2, we have employed the LET to reconstruct to order g^2 the relation between the normalization of the involved 2- and 3-points correlators and the anomalous dimension in eqs. (6.17), (6.20), (6.21) and (6.22). The above technique will play an important role in our second installment as well.

Sixth, we have demonstrated in eq. (5.4) that in the AF phase of the theory the additive renormalization in the r.h.s. of the LET turns out to be finite after the nonperturbative resummation to all perturbative orders.

Finally, we have employed the LET in dimensional regularization to verify in a manifestly gauge-invariant framework in eqs. (8.2) and (8.3) some perturbative and nonperturbative computations [11] involving a hard-cutoff regularization of the bare LET, respectively.

A Anomalous dimension of F^2

We compute in two different ways the anomalous dimension of $\mathrm{Tr}\, F^2 = \frac{1}{2}F^2$ in massless QCD and $\mathcal{N} = 1$ SUSY YM theory: first, from the multiplicative renormalization of $\mathrm{Tr}\, F^2$ in $\overline{MS}$-like schemes. Second, from the RG invariance of the trace anomaly that reads in $d = 4$ dimensions [20]:

$$T_{\alpha\alpha} = \frac{\beta(g)}{g}\,\mathrm{Tr}\, F^2 \tag{A.1}$$

The two computations are not actually independent. Indeed:

$$T_{\alpha\alpha} = -\epsilon\,\mathrm{Tr}\, F_0^2 \tag{A.2}$$

in $\tilde{d} = 4 - 2\epsilon$ dimensions, with $\operatorname{Tr} F_0^2$ the bare operator. Hence, we get:

$$\begin{aligned} T_{\alpha\alpha} &= -\epsilon Z_{F^2}^{-1}(g,\epsilon) Z_{F^2}(g,\epsilon) \operatorname{Tr} F_0^2 \\ &= -\epsilon Z_{F^2}^{-1}(g,\epsilon) \operatorname{Tr} F^2 \\ &= -\epsilon \left(1 - \frac{\beta(g)}{\epsilon g}\right) \operatorname{Tr} F^2 \\ &= \frac{\beta(g,\epsilon)}{g} \operatorname{Tr} F^2 \end{aligned} \tag{A.3}$$

where:

$$\frac{dg}{d\log\mu} = \beta(g,\epsilon) = -\epsilon g + \beta(g) \tag{A.4}$$

is the beta function in $\tilde{d} = 4 - 2\epsilon$ dimensions. It follows that the trace anomaly is a consequence of the multiplicative renormalization of $\operatorname{Tr} F^2$ that also implies the anomalous dimension of $\operatorname{Tr} F^2$ (appendix A.1). Vice versa, the anomalous dimension of $\operatorname{Tr} F^2$ follows from the trace anomaly as well (appendix A.2).

A.1 Multiplicative renormalization of F^2

In gauge-invariant correlators of massless QCD and $\mathcal{N} = 1$ SUSY YM theory $\operatorname{Tr} F^2$ is multiplicatively renormalizable in $\overline{MS}$-like schemes up to operators proportional to the EOM:

$$\operatorname{Tr} F^2 = Z_{F^2}(g,\epsilon) \operatorname{Tr} F_0^2 \tag{A.5}$$

where $Z_{F^2}(g,\epsilon)$ [3, 20]:

$$Z_{F^2}(g,\epsilon) = 1 + g\frac{\partial}{\partial g} \log Z_g(g,\epsilon) = \left(1 - \frac{\beta(g)}{\epsilon g}\right)^{-1} \tag{A.6}$$

with:

$$g_0 = Z_g(g,\epsilon)\mu^\epsilon g(\mu) \tag{A.7}$$

and $\beta(g) = -g\,\frac{d\log Z_g}{d\log\mu}$. Thus, by means of $\frac{d}{d\log\mu} = \beta(g,\epsilon)\frac{\partial}{\partial g} + \frac{\partial}{\partial\log\mu}$ and eq. (A.6), we get the anomalous dimension of $\operatorname{Tr} F^2$:

$$\gamma_{F^2}(g) = -\frac{d\log Z_{F^2}}{d\log\mu} = -\beta(g,\epsilon)\frac{\partial \log Z_{F^2}}{\partial g} = g\frac{\partial}{\partial g}\left(\frac{\beta(g,\epsilon)}{g}\right) = g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) \tag{A.8}$$

Besides, from eq. (A.6) it follows:

$$Z_{F^2}(g,\epsilon) = 1 - \frac{\beta_0 g^2}{\epsilon} - \frac{\beta_1 g^4}{\epsilon} + \frac{\beta_0^2 g^4}{\epsilon^2} + O(g^6) \tag{A.9}$$

and:

$$Z_g(g,\epsilon) = 1 - \frac{\beta_0 g^2}{2\epsilon} - \frac{\beta_1 g^4}{4\epsilon} + \frac{3\beta_0^2 g^4}{8\epsilon^2} + O(g^6) \tag{A.10}$$

up to two loops. As a consequence:

$$\begin{aligned} Z_g^2(g,\epsilon) &= 1 - 2\frac{\beta_0 g^2}{2\epsilon} - 2\frac{\beta_1 g^4}{4\epsilon} + 2\frac{3\beta_0^2 g^4}{8\epsilon^2} + \left(\frac{\beta_0 g^2}{2\epsilon}\right)^2 + O(g^6) \\ &= 1 - \frac{\beta_0 g^2}{\epsilon} - \frac{\beta_1 g^4}{2\epsilon} + \frac{\beta_0^2 g^4}{\epsilon^2} + O(g^6) \end{aligned} \tag{A.11}$$

Hence, only to one loop Z_{F^2} coincides with Z_g^2 [3].

A.2 $\gamma_{F^2}(g)$ from the trace anomaly

Eq. (A.8) follows from the trace anomaly as well [24, 25]. Because of the RG invariance of the trace anomaly the CS equation reads in $d = 4$ dimensions:

$$\left(x \cdot \frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g} + 4\right)\left(\frac{\beta(g)}{g}\,\mathrm{Tr}\,F^2(x)\right) = 0 \tag{A.12}$$

Hence:

$$\left(x \cdot \frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g} + 4 + \beta(g)\frac{\partial}{\partial g}\log\left(\frac{\beta(g)}{g}\right)\right)\mathrm{Tr}\,F^2(x) = 0 \tag{A.13}$$

that implies:

$$\left(x \cdot \frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g} + 4 + g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\right)\mathrm{Tr}\,F^2(x) = 0 \tag{A.14}$$

Comparing the above equation with the CS equation for $\mathrm{Tr}\,F^2$:

$$\left(x \cdot \frac{\partial}{\partial x} + \beta(g)\frac{\partial}{\partial g} + 4 + \gamma_{F^2}(g)\right)\mathrm{Tr}\,F^2(x) = 0 \tag{A.15}$$

we obtain:

$$\gamma_{F^2}(g) = g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) \tag{A.16}$$

that coincides with eq. (A.8).[1] The anomalous dimension up to two loops in $\overline{MS}$-like schemes follows:

$$\gamma_{F^2}(g) = -2\beta_0 g^2 - 4\beta_1 g^4 + \cdots \tag{A.17}$$

with:

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \cdots \tag{A.18}$$

The first coefficient $\gamma_0^{(F^2)} = -2\beta_0$ is universal, i.e. scheme independent, as for the one-loop coefficient of the anomalous dimension of any canonically normalized operator. Instead, $\gamma_1^{(F^2)} = -4\beta_1$ is scheme dependent, and it may be changed [22] by a finite multiplicative renormalization of $\mathrm{Tr}\,F^2$ reducing to the identity for $g = 0$, contrary to a reparametrization of g that would not affect β_1 — the second universal coefficient of the beta function.

B Contact terms in the OPE for F^2 [2, 3]

The Euclidean version of the Minkowskian operator O_1 [2, 3] reads in our notation:

$$\begin{aligned}
O_1 &= -\frac{1}{2g_{YM}^2}\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} \\
&= -\frac{N}{2g^2}\,\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}_{\mu\nu} \\
&= -\frac{N}{2g^2}\frac{g^2}{N}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu} \\
&= -\frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}F_{\mu\nu} \\
&= -\frac{1}{4}F^2
\end{aligned} \tag{B.1}$$

[1] The analogous reasoning in $\tilde{d} = 4 - 2\epsilon$ dimensions also leads to eq. (A.8).

where $g^2 = N g_{YM}^2$ and we have expressed $\mathcal{F}_{\mu\nu}$ in the Wilsonian normalization in terms of $F_{\mu\nu}$ in the canonical one (section 3). It follows the fully renormalized OPE coefficient of O_1 with itself [2] in the momentum representation:

$$\begin{aligned}&\int O_1(x)O_1(0)e^{-ip\cdot x}d^{4-2\epsilon}x \\ &= \int \cdots + \left(C^{(S)}_{1CZ}(x) - \delta^{(4-2\epsilon)}(x)\left(\frac{Z^L_{11}}{Z_{11}} + \cdots\right)\right) O_1(0) + \cdots e^{-ip\cdot x}d^{4-2\epsilon}x \end{aligned} \tag{B.2}$$

with:

$$Z_{11} = \left(1 - \frac{\beta(\alpha_s)}{\epsilon}\right)^{-1} \tag{B.3}$$

and [2]:

$$\begin{aligned}\frac{Z^L_{11}}{Z_{11}} &= \frac{1}{\epsilon}\left(1 - \frac{\beta(\alpha_s)}{\epsilon}\right)^{-1} \alpha_s^2 \frac{\partial}{\partial \alpha_s}\left(\frac{\beta(\alpha_s)}{\alpha_s}\right) \\ &= \frac{1}{\epsilon}\left(1 - \frac{\beta(\alpha_s)}{\epsilon}\right)^{-1} (\alpha_s\beta'(\alpha_s) - \beta(\alpha_s))\end{aligned} \tag{B.4}$$

where $\beta(\alpha_s)$ reads in our notation:

$$\begin{aligned}\beta(\alpha_s) &= \frac{d\log\alpha_s}{d\log\mu^2} = \frac{1}{2}\frac{1}{\alpha_s}\frac{\partial\alpha_s}{\partial g}\frac{dg}{d\log\mu} \\ &= \frac{1}{2}\frac{4\pi N}{g^2}\frac{2g}{4\pi N}\frac{dg}{d\log\mu} = \frac{1}{g}\frac{dg}{d\log\mu} \\ &= \frac{\beta(g)}{g}\end{aligned} \tag{B.5}$$

with $\alpha_s = g_{YM}^2/4\pi = g^2/4\pi N$. Hence, employing:

$$\begin{aligned}\alpha_s^2\frac{\partial}{\partial\alpha_s}\left(\frac{\beta(\alpha_s)}{\alpha_s}\right) &= \frac{1}{2}\frac{1}{4\pi N}g^3\frac{\partial}{\partial g}\left(4\pi N\frac{\beta(g)}{g^3}\right) = \frac{1}{2}g^3\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g^3}\right) \\ &= \frac{1}{2}g^3\frac{\partial}{\partial g}\left(\frac{1}{g^2}\frac{\beta(g)}{g}\right) = \frac{1}{2}g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) - \frac{\beta(g)}{g} \\ &= \frac{1}{2}\gamma_{F^2}(g) - \frac{\beta(g)}{g}\end{aligned} \tag{B.6}$$

we obtain from eq. (B.4):

$$\frac{Z^L_{11}}{Z_{11}} = \frac{1}{\epsilon}\left(1 - \frac{\beta(g)}{\epsilon g}\right)^{-1}\left(\frac{1}{2}g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) - \frac{\beta(g)}{g}\right) \tag{B.7}$$

According to $F^2 = -4O_1$ in eq. (B.1), we get the OPE coefficients in both notations:

$$\int C_1^{(F^2,F^2)}(x)e^{-ip\cdot x}d^{4-2\epsilon}x = -4\int C^{(S)}_{1CZ}(x)e^{-ip\cdot x}d^{4-2\epsilon}x \tag{B.8}$$

Then, the infinite additive renormalization of $C_1^{(F^2,F^2)}(p)$ is given by:

$$4\frac{Z^L_{11}}{Z_{11}} \tag{B.9}$$

that coincides with eq. (4.50) for $O = F^2$.

C Integral $I_{\tilde{d},\Delta_{F^2},\Delta_{F^2}}$

Eqs. (6.16) and (6.27) involve special cases of the integral [26]:

$$\begin{aligned} I_{d,\Delta_1,\Delta_2} &= \int \frac{1}{|x|^{\Delta_1}|x-z|^{\Delta_2}} d^d x \\ &= (2\pi)^d \, C_{d,\frac{\Delta_1}{2},\frac{\Delta_2}{2}} \, |z|^{d-\Delta_1-\Delta_2} \end{aligned} \tag{C.1}$$

with:

$$C_{d,\frac{\Delta_1}{2},\frac{\Delta_2}{2}} = \frac{\Gamma(\frac{\Delta_1+\Delta_2-d}{2})\Gamma(\frac{d-\Delta_1}{2})\Gamma(\frac{d-\Delta_2}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{\Delta_1}{2})\Gamma(\frac{\Delta_2}{2})\Gamma(d-\frac{\Delta_1}{2}-\frac{\Delta_2}{2})} \tag{C.2}$$

For $d \to \tilde{d}$ and $\Delta_1 = \Delta_2 \to \Delta_{F^2}$ in eqs. (C.1) and (C.2) we get:

$$\begin{aligned} I_{\tilde{d},\Delta_{F^2},\Delta_{F^2}} &= \int \frac{1}{|x|^{\Delta_{F^2}}|x-z|^{\Delta_{F^2}}} d^{\tilde{d}} x \\ &= (2\pi)^{\tilde{d}} \, C_{\tilde{d},\frac{\Delta_{F^2}}{2},\frac{\Delta_{F^2}}{2}} \, |z|^{\tilde{d}-2\Delta_{F^2}} \end{aligned} \tag{C.3}$$

with:

$$C_{\tilde{d},\frac{\Delta_{F^2}}{2},\frac{\Delta_{F^2}}{2}} = \frac{\Gamma(\frac{2\Delta_{F^2}-\tilde{d}}{2})\Gamma(\frac{\tilde{d}-\Delta_{F^2}}{2})^2}{(4\pi)^{\frac{\tilde{d}}{2}}\Gamma(\frac{\Delta_{F^2}}{2})^2\Gamma(\tilde{d}-\Delta_{F^2})} \tag{C.4}$$

Eq. (6.16) is obtained for $\tilde{d} = 4-2\epsilon$ and $\Delta_{F^2} = 4-4\epsilon$ in eqs. (C.3) and (C.4):

$$\begin{aligned} I_{4-2\epsilon,4-4\epsilon,4-4\epsilon} &= (2\pi)^{4-2\epsilon} \, C_{4-2\epsilon,2-2\epsilon,2-2\epsilon} \, |z|^{-4+6\epsilon} \\ &= (2\pi)^{4-2\epsilon} \frac{\Gamma(2-3\epsilon)\Gamma(\epsilon)^2}{(4\pi)^{2-\epsilon}\Gamma(2-2\epsilon)^2\Gamma(2\epsilon)} \, |z|^{-4+6\epsilon} \\ &= \pi^2\pi^{-\epsilon}\Big(\frac{2}{\epsilon}+2+2\Gamma'(1)+\cdots\Big)|z|^{-4+6\epsilon} \\ &= \pi^2\Big(\frac{2}{\epsilon}+2+2\Gamma'(1)-2\log\pi+\cdots\Big)|z|^{-4+6\epsilon} \end{aligned} \tag{C.5}$$

where in the ϵ expansion of the ratio of Γ functions we have repeatedly employed $z\Gamma(z) = \Gamma(z+1)$ and the Taylor expansion $\Gamma(1+\epsilon) = 1+\epsilon\Gamma'(1)+\cdots$. Eq. (C.5) multiplied by $\frac{\mu^{2\epsilon}}{|z|^{-4+4\epsilon}}$ yields as $\epsilon \to 0$:

$$\begin{aligned} \frac{\mu^{2\epsilon}}{|z|^{-4+4\epsilon}} I_{4-2\epsilon,4-4\epsilon,4-4\epsilon} &= \pi^2|z\mu|^{2\epsilon}\Big(\frac{2}{\epsilon}+2+2\Gamma'(1)-2\log\pi+\cdots\Big) \\ &= \pi^2\Big(\frac{2}{\epsilon}+4\log|z\mu|+2+2\Gamma'(1)-2\log\pi\Big) \end{aligned} \tag{C.6}$$

reported in eq. (6.16). Eq. (6.27) is obtained for $\tilde{d} \to d = 4$ and $\Delta_{F^2} = 4+\gamma_{F^2}$ in eqs. (C.3) and (C.4):

$$\begin{aligned} I_{4,4+\gamma_{F^2},4+\gamma_{F^2}} &= (2\pi)^4 \, C_{4,2+\frac{\gamma_{F^2}}{2},2+\frac{\gamma_{F^2}}{2}} \, |z|^{-4-2\gamma_{F^2}} \\ &= (2\pi)^4 \frac{\Gamma(2+\gamma_{F^2})\Gamma(\frac{-\gamma_{F^2}}{2})^2}{(4\pi)^2\Gamma(2+\frac{\gamma_{F^2}}{2})^2\Gamma(-\gamma_{F^2})} \, |z|^{-4-2\gamma_{F^2}} \\ &= \pi^2\Big(-\frac{4}{\gamma_{F^2}}+O(\gamma_{F^2})\Big)|z|^{-4-2\gamma_{F^2}} \end{aligned} \tag{C.7}$$

expanded perturbatively in γ_{F^2}. Eq. (C.7) multiplied by $\frac{\mu^{-\gamma_{F^2}}}{|z|^{-4-\gamma_{F^2}}}$ yields perturbatively in γ_{F^2}:

$$\begin{aligned}\frac{\mu^{-\gamma_{F^2}}}{|z|^{-4-\gamma_{F^2}}} I_{4,4+\gamma_{F^2},4+\gamma_{F^2}} &= \pi^2\Big(-\frac{4}{\gamma_{F^2}}+O(\gamma_{F^2})\Big)|z\mu|^{-\gamma_{F^2}} \\ &= \pi^2\Big(-\frac{4}{\gamma_{F^2}}+4\log|z\mu|+O(\gamma_{F^2})\Big)\end{aligned} \tag{C.8}$$

reported in eq. (6.27) for $\gamma_{F^2} = -2\beta_0 g^2 + \cdots$.

D Callan-Symanzik equation in $\tilde{d} = 4 - 2\epsilon$ dimensions in the coordinate representation

The CS equation [27, 28] in a massless QCD-like theory for connected 2-point correlators $G^{(2)} \equiv \langle O(z)O(0)\rangle'$ at distinct points $z \neq 0$ of a multiplicatively renormalizable gauge-invariant scalar operator O with canonical dimension $\tilde{D}$ is a consequence of the independence of the bare correlator $G_0^{(2)} \equiv \langle O(z)O(0)\rangle_0'$:

$$G_0^{(2)}(z,\epsilon,g_0) = Z_O^{-2}(\epsilon,g(\mu))G^{(2)}(z,\mu,g(\mu)) \tag{D.1}$$

from the renormalization scale μ:

$$\mu\frac{d}{d\mu}G_0^{(2)}\Big|_{\epsilon,g_0} = 0 \tag{D.2}$$

with fixed bare parameters g_0 and ϵ of the dimensionally regularized theory. Substituting eq. (D.1) into eq. (D.2) we get the CS equation [27–30]:

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta(g,\epsilon)\frac{\partial}{\partial g}+2\gamma_O(g)\Big)\,G^{(2)}(z,\mu,g(\mu)) = 0 \tag{D.3}$$

with:

$$\beta(g,\epsilon) = \frac{dg}{d\log\mu}\Big|_{\epsilon,g_0} = -\epsilon g + \beta(g) \tag{D.4}$$

and:

$$\gamma_O(g) = -\frac{d\log Z_O}{d\log\mu}\Big|_{\epsilon,g_0} \tag{D.5}$$

the anomalous dimension of O. As the theory is massless to all orders of perturbation theory we define:

$$G^{(2)}(z,\mu,g(\mu)) = \frac{1}{z^{2\tilde{D}}}\bar{G}^{(2)}(z\mu,g(\mu)) \tag{D.6}$$

with the dimensionless correlator $\bar{G}^{(2)}$ that satisfies eq. (D.3) as well. Besides:

$$\Big(z\cdot\frac{\partial}{\partial z}+\beta(g,\epsilon)\frac{\partial}{\partial g}+2\gamma_O(g)\Big)\,\bar{G}^{(2)}(z\mu,g(\mu)) = 0 \tag{D.7}$$

as $\bar{G}^{(2)}$ depends on z via the product $z\mu$ only. Therefore:

$$\Big(z\cdot\frac{\partial}{\partial z}+\beta(g,\epsilon)\frac{\partial}{\partial g}+2\tilde{D}+2\gamma_O(g)\Big)\,G^{(2)}(z,\mu,g(\mu)) = 0 \tag{D.8}$$

The solution of eqs. (D.8) and (D.3) may then be written as:

$$G^{(2)}(z,\mu,g(\mu)) = \frac{1}{z^{2\tilde{D}}}\,\bar{G}^{(2)}(z\mu,g(\mu))$$
$$= \frac{1}{z^{2\tilde{D}}}\,\mathcal{G}_2^{(O)}(\tilde{g}(z))\,Z^{(O)2}(\tilde{g}(z),g(\mu)) \tag{D.9}$$

where $\mathcal{G}_2^{(O)}$ is an RG-invariant function of the running coupling $\tilde{g}(z) \equiv \tilde{g}(z\mu, g(\mu))$ in $\tilde{d}$ dimensions that solves:

$$-\frac{d\tilde{g}(z)}{d\log|z|} = \beta(\tilde{g}(z),\epsilon) \tag{D.10}$$

with the initial condition $\tilde{g}(1,g(\mu)) = g(\mu)$, and the renormalized multiplicative factor $Z^{(O)}(\tilde{g}(z),g(\mu))$:

$$Z^{(O)}(\tilde{g}(z),g(\mu)) = \exp\int_{g(\mu)}^{\tilde{g}(z)} \frac{\gamma_O(g)}{\beta(g,\epsilon)}dg \tag{D.11}$$

solves:

$$\gamma_O(g(\mu)) = -\frac{d\log Z^{(O)}}{d\log\mu} \tag{D.12}$$

Moreover, it follows from the RG invariance of $\tilde{g}(z)$:

$$0 = \frac{d\tilde{g}(z)}{d\log\mu}$$
$$= \frac{\partial\tilde{g}(z)}{\partial\log\mu} + \frac{\partial\tilde{g}(z)}{\partial g(\mu)}\frac{dg(\mu)}{d\log\mu}$$
$$= \frac{d\tilde{g}(z)}{d\log|z|} + \frac{\partial\tilde{g}(z)}{\partial g(\mu)}\frac{dg(\mu)}{d\log\mu}$$
$$= -\beta(\tilde{g}(z),\epsilon) + \frac{\partial\tilde{g}(z)}{\partial g(\mu)}\beta(g(\mu),\epsilon) \tag{D.13}$$

according to eq. (D.10) and the fact that $\tilde{g}(z)$ depends on z via the product $z\mu$ only. As a consequence:

$$\frac{\partial\tilde{g}(z)}{\partial g(\mu)} = \frac{\beta(\tilde{g}(z),\epsilon)}{\beta(g(\mu),\epsilon)} \tag{D.14}$$

D.1 Asymptotics of the running coupling

In $\tilde{d} = 4 - 2\epsilon$ dimensions the beta function:

$$\beta(g,\epsilon) = -\epsilon g + \beta(g)$$
$$= -\epsilon g - \beta_0 g^3 - \beta_1 g^5 + \cdots \tag{D.15}$$

has a UV zero at $g = 0$. The running coupling satisfies:

$$\int_{g(\mu)}^{\tilde{g}(z)} \frac{dg}{\beta(g,\epsilon)} = -\int_{\mu^{-1}}^{|z|} d\log|z| \tag{D.16}$$

We evaluate asymptotically eq. (D.16) including the leading order of the perturbative expansion of $\beta(g) = -\beta_0 g^3 + \cdots$, with $\beta_0 > 0$:

$$\int_{g(\mu)}^{\tilde{g}(z)} \frac{dg}{-\epsilon g - \beta_0 g^3 + \cdots} = -\int_{\mu^{-1}}^{|z|} d\log|z| \tag{D.17}$$

where both length scales, $|z| = \sqrt{z^2}$ and μ^{-1}, are assumed to be close to zero in order for $\tilde{g}(z)$ and $g(\mu)$ to stay in a neighborhood of $g = 0$. By employing:

$$\int_{g(\mu)}^{\tilde{g}(z)} \frac{dg}{g(\epsilon + \beta_0 g^2)} = \frac{1}{2\epsilon} \log \frac{g^2}{\epsilon + \beta_0 g^2}\Bigg|_{g(\mu)}^{\tilde{g}(z)} \tag{D.18}$$

eq. (D.17) yields:

$$\log\left(\frac{\tilde{g}^2(z)}{g^2(\mu)} \frac{\epsilon + \beta_0 g^2(\mu) + \cdots}{\epsilon + \beta_0 \tilde{g}^2(z) + \cdots}\right) = 2\epsilon \log|z\mu| \tag{D.19}$$

After exponentiating both sides:

$$\frac{\tilde{g}^2(z)}{g^2(\mu)} = |z\mu|^{2\epsilon} \frac{\epsilon + \beta_0 \tilde{g}^2(z) + \cdots}{\epsilon + \beta_0 g^2(\mu) + \cdots} \tag{D.20}$$

we obtain:

$$\frac{\epsilon}{\tilde{g}^2(z)} + \beta_0 + \cdots = |z\mu|^{-2\epsilon}\left(\frac{\epsilon}{g^2(\mu)} + \beta_0 + \cdots\right) \tag{D.21}$$

that implies:

$$\frac{g^2(\mu)}{\tilde{g}^2(z)} = |z\mu|^{-2\epsilon}\left(1 - \beta_0 g^2(\mu)\frac{|z\mu|^{2\epsilon} - 1}{\epsilon} + \cdots\right) \tag{D.22}$$

whose inverse is:

$$\frac{\tilde{g}^2(z)}{g^2(\mu)} = \frac{|z\mu|^{2\epsilon}}{1 - \beta_0 g^2(\mu)\frac{|z\mu|^{2\epsilon}-1}{\epsilon} + \cdots} \tag{D.23}$$

For $\epsilon \to 0$ eq. (D.23) provides the solution in $d = 4$ dimensions (appendix E.1):

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{\tilde{g}^2(z)}{g^2(\mu)} &= \lim_{\epsilon\to 0} \frac{1 + 2\epsilon \log|z\mu| + \cdots}{1 - 2\beta_0 g^2(\mu) \log|z\mu| + \cdots} \\ &= \frac{1}{1 - 2\beta_0 g^2(\mu)\log|z\mu| + \cdots}\end{aligned} \tag{D.24}$$

Instead, for fixed $\epsilon > 0$ and around the $\frac{1}{\tilde{g}(z)}$ singularity as $\tilde{g}(z) \to 0$, i.e. for large $\log|z\mu|$, the asymptotics of the solution in eq. (D.23) reads:

$$\tilde{g}^2(z) \sim \frac{\epsilon}{\beta_0}|z\mu|^{2\epsilon} \tag{D.25}$$

Moreover, eq. (D.19):

$$-\frac{1}{2\epsilon}\log\beta_0\left(1 + \frac{\epsilon}{\beta_0\tilde{g}^2(z)} + \cdots\right) - \log|z| = \frac{1}{2\epsilon}\log\beta_0\left(1 + \frac{\epsilon}{\beta_0 g^2(\mu)} + \cdots\right) + \log\mu \tag{D.26}$$

implies that both sides are independent of $|z|$ and μ^{-1}, i.e. RG-invariant, though scheme dependent. After subtracting the infinite constant $-\frac{1}{2\epsilon}\log\beta_0$ from both sides and introducing a scheme-dependent integration constant C, eq. (D.26) thus defines the RG-invariant mass scale Λ_{UV} in $\tilde{d}=4-2\epsilon$ dimensions:

$$\log\Lambda_{UV} = -\frac{1}{2\epsilon}\log\left(1+\frac{\epsilon}{\beta_0\tilde{g}^2(z)}+\cdots\right)-\log|z|-\frac{C}{2\beta_0} \tag{D.27}$$

whose exponential reads:

$$\Lambda_{UV} = |z|^{-1}\left(1+\frac{\epsilon}{\beta_0\tilde{g}^2(z)}+\cdots\right)^{-\frac{1}{2\epsilon}}\exp\left(-\frac{C}{2\beta_0}\right) \tag{D.28}$$

Solving eq. (D.28) in terms of $\tilde{g}(z)$ as $\epsilon\to 0$ and asymptotically as $|z|\to 0^+$ we get:

$$\begin{aligned}\tilde{g}^2(z) &\sim \frac{1}{\beta_0}\frac{1}{\frac{\left(-1+|z\Lambda_{UV}|^{-2\epsilon}+\cdots\right)e^{-\frac{\epsilon C}{\beta_0}}}{\epsilon}}\\ &\sim \frac{1}{\beta_0}\frac{1}{-2\log|z|\Lambda_{UV}-\frac{C}{\beta_0}+\cdots}\\ &\sim \frac{1}{-2\beta_0\log|z|\Lambda_{UV}}\left(1-\frac{C}{2\beta_0\log|z|\Lambda_{UV}}+\cdots\right)\end{aligned} \tag{D.29}$$

Eqs. (D.28) and (D.29) yield as $\epsilon\to 0$ the corresponding eqs. (E.12) and (E.13) in $d=4$ dimensions.

D.2 Asymptotics of $\langle O(z)O(0)\rangle'$

In perturbation theory:[2]

$$\gamma_O(g) = -\gamma_0^{(O)}g^2-\gamma_1^{(O)}g^4+\cdots \tag{D.30}$$

with the first coefficient $\gamma_0^{(O)}$ universal, i.e. scheme independent. We employ $\gamma_O(g)$ to leading order to evaluate asymptotically the renormalized multiplicative factor $Z^{(O)}(\tilde{g}(z),g(\mu))$ for small coupling, i.e. $\tilde{g}(z),g(\mu)\to 0$:

$$\begin{aligned}Z^{(O)}(\tilde{g}(z),g(\mu)) &\sim \exp\int_{g(\mu)}^{\tilde{g}(z)}\frac{-\gamma_0^{(O)}g^2+\cdots}{-\epsilon g-\beta_0 g^3+\cdots}dg\\ &\sim \exp\left(\frac{\gamma_0^{(O)}}{2}\int_{g^2(\mu)}^{\tilde{g}^2(z)}\frac{1+\cdots}{\epsilon+\beta_0 g^2+\cdots}dg^2\right)\\ &\sim \exp\left(\frac{\gamma_0^{(O)}}{2\beta_0}\log\left(\frac{\epsilon+\beta_0\tilde{g}^2(z)+\cdots}{\epsilon+\beta_0 g^2(\mu)+\cdots}\right)\right)\\ &\sim \exp\left(\frac{\gamma_0^{(O)}}{2\beta_0}\log\left(\frac{\tilde{g}^2(z)}{g^2(\mu)}|z\mu|^{-2\epsilon}\right)\right)\\ &\sim \left(\frac{\tilde{g}^2(z)}{g^2(\mu)}|z\mu|^{-2\epsilon}\right)^{\frac{\gamma_0^{(O)}}{2\beta_0}}\end{aligned} \tag{D.31}$$

[2] In the present paper the convention about the sign of the coefficients $\gamma_i^{(O)}$ agrees with [12, 24, 31], but it is opposite to the standard one [22].

where we have employed eq. (D.20). It follows the short-distance asymptotics of the 2-point correlator in eq. (D.9) as $\tilde{g}(z), g(\mu) \to 0$:

$$\langle O(z)O(0)\rangle' \sim \frac{\mathcal{G}_2^{(O)}(0)}{z^{2\tilde{D}}} \left(\frac{\tilde{g}^2(z)}{g^2(\mu)} |z\mu|^{-2\epsilon} \right)^{\frac{\gamma_0^{(O)}}{\beta_0}} \tag{D.32}$$

D.3 Asymptotics of $\langle F^2(z)F^2(0)\rangle'$

$Z^{(F^2)}$ admits a closed form in terms of the beta function according to eqs. (A.16) and (D.11):

$$\begin{aligned} Z^{(F^2)}(\tilde{g}(z), g(\mu)) &= \exp \int_{g(\mu)}^{\tilde{g}(z)} \frac{\gamma_{F^2}(g)}{\beta(g,\epsilon)} dg \\ &= \exp \int_{g(\mu)}^{\tilde{g}(z)} \frac{\frac{\partial}{\partial g}\left(\frac{\beta(g,\epsilon)}{g}\right)}{\frac{\beta(g,\epsilon)}{g}} dg \\ &= \frac{\beta(\tilde{g}(z),\epsilon)}{\tilde{g}(z)} \frac{g(\mu)}{\beta(g(\mu),\epsilon)} \end{aligned} \tag{D.33}$$

Correspondingly, we get for the solution of the CS equation:

$$\langle F^2(z)F^2(0)\rangle' = \frac{\mathcal{G}_2^{(F^2)}(\tilde{g}(z))}{z^{8-4\epsilon}} \left(\frac{\beta(\tilde{g}(z),\epsilon)}{\tilde{g}(z)} \right)^2 \left(\frac{g(\mu)}{\beta(g(\mu),\epsilon)} \right)^2 \tag{D.34}$$

with $\tilde{D} = \tilde{d} = 4 - 2\epsilon$ the canonical dimension of F^2. For small coupling we evaluate asymptotically eq. (D.33):

$$\begin{aligned} Z^{(F^2)}(\tilde{g}(z), g(\mu)) &= \frac{\beta(\tilde{g}(z),\epsilon)}{\tilde{g}(z)} \frac{g(\mu)}{\beta(g(\mu),\epsilon)} \\ &\sim \frac{\epsilon + \beta_0 \tilde{g}^2(z) + \cdots}{\epsilon + \beta_0 g^2(\mu) + \cdots} \\ &\sim \frac{\tilde{g}^2(z)}{g^2(\mu)} |z\mu|^{-2\epsilon} \end{aligned} \tag{D.35}$$

that, indeed, for $\gamma_0^{(F^2)} = 2\beta_0$ coincides with eq. (D.31). The short-distance asymptotics of eq. (D.34) as $\tilde{g}(z), g(\mu) \to 0$ follows:

$$\langle F^2(z)F^2(0)\rangle' \sim \frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}} \frac{\tilde{g}^4(z)}{g^4(\mu)} |z\mu|^{-4\epsilon} \tag{D.36}$$

D.4 Change of renormalization scheme

Replacing $z \to \tilde{\mu}^{-1}$ in eq. (D.23) and expanding for small $g(\mu)$, we obtain:

$$g^2(\tilde{\mu}) = \left(\frac{\mu}{\tilde{\mu}}\right)^{2\epsilon} g^2(\mu) \left(1 + 2\beta_0 g^2(\mu) \log\frac{\mu}{\tilde{\mu}} + \cdots\right) \tag{D.37}$$

Hence, the change of scheme $\tilde{\mu} = \alpha\mu$ yields for the coupling:

$$\begin{aligned} g^2(\tilde{\mu} = \alpha\mu) &= \alpha^{-2\epsilon} g^2(\mu)(1 - 2\beta_0 g^2(\mu) \log\alpha + \cdots) \\ &= \alpha^{-2\epsilon} g^2(\mu)(1 + \cdots) \end{aligned} \tag{D.38}$$

JHEP08(2024)145

where the dots contain $O(g^2)$ contributions. We determine the scheme dependence of the complete 2-point correlator entering the FT in eq. (7.22) and the 2-point correlator at distinct points in eq. (7.5) by replacing $\mu \to \tilde{\mu} = \alpha\mu$ and rewriting $g^2 \to g^2(\tilde{\mu} = \alpha\mu)$ in terms of $g(\mu) = g$ by means of eq. (D.38). For the correlator at distinct points in eq. (7.5), we obtain to order g^2:

$$
\begin{aligned}
&\langle F^2(z)F^2(0)\rangle'\big|_{\tilde{\mu}}\\
&= \mathcal{G}_2^{(F^2)}(0)\Big\{\frac{1}{z^{8-4\epsilon}}\left(1-\frac{2g^2(\tilde{\mu})\beta_0}{\epsilon}\right)+\frac{\tilde{\mu}^{2\epsilon}}{z^{8-6\epsilon}}g^2(\tilde{\mu})\left(\delta_2^{(F^2)}+\frac{2\beta_0}{\epsilon}\right)+\cdots\Big\}\\
&= \mathcal{G}_2^{(F^2)}(0)\Big\{\frac{1}{z^{8-4\epsilon}}\left(1-\frac{2g^2\beta_0\alpha^{-2\epsilon}}{\epsilon}\right)+\frac{\mu^{2\epsilon}\alpha^{2\epsilon}}{z^{8-6\epsilon}}g^2\alpha^{-2\epsilon}\left(\delta_2^{(F^2)}+\frac{2\beta_0}{\epsilon}\right)+\cdots\Big\}\\
&= \mathcal{G}_2^{(F^2)}(0)\Big\{\frac{1}{z^{8-4\epsilon}}\left(1-\frac{2g^2\beta_0}{\epsilon}\right)+\frac{\mu^{2\epsilon}}{z^{8-6\epsilon}}g^2\left(\delta_2^{(F^2)}+\frac{2\beta_0}{\epsilon}\right)\Big\}\\
&\quad+\frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\frac{2g^2\beta_0}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots\\
&= \langle F^2(z)F^2(0)\rangle'\big|_{\mu}+\frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\frac{2g^2\beta_0}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots\\
&= \langle F^2(z)F^2(0)\rangle'\big|_{\mu}+\frac{48(N^2-1)}{\pi^4 z^{8-4\epsilon}}\frac{2g^2\beta_0}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots
\end{aligned}
\tag{D.39}
$$

where we have employed eq. (7.16) in the last equality. For the complete correlator entering the FT in eq. (7.22) we obtain to order g^2:

$$
\begin{aligned}
&\langle F^2(z)F^2(0)\rangle\big|_{\tilde{\mu}}\\
&= \langle F^2(z)F^2(0)\rangle'\big|_{\tilde{\mu}}+\frac{N^2-1}{4\pi^2}\Delta^2\delta^{(\tilde{d})}(z)\tilde{\mu}^{-2\epsilon}\Big\{-\frac{1}{\epsilon}+\frac{g^2(\tilde{\mu})\beta_0}{\epsilon^2}\\
&\quad-\frac{g^2(\tilde{\mu})}{\epsilon}\Big(\frac{1}{16\pi^2}\Big(\frac{17}{2}-\frac{5}{3}\frac{N_f}{N}\Big)-\beta_0\Big(\frac{25}{6}+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big)+\tilde{A}_0+\tilde{A}_1 g^2(\tilde{\mu})\Big\}+\cdots\\
&= \langle F^2(z)F^2(0)\rangle'\big|_{\mu}+\frac{48(N^2-1)}{\pi^4 z^{8-4\epsilon}}\frac{2g^2\beta_0}{\epsilon}(1-\alpha^{-2\epsilon})\\
&\quad+\frac{N^2-1}{4\pi^2}\Delta^2\delta^{(\tilde{d})}(z)\mu^{-2\epsilon}\alpha^{-2\epsilon}\Big\{-\frac{1}{\epsilon}+\frac{g^2\alpha^{-2\epsilon}\beta_0}{\epsilon^2}\\
&\quad-\frac{g^2\alpha^{-2\epsilon}}{\epsilon}\Big(\frac{1}{16\pi^2}\Big(\frac{17}{2}-\frac{5}{3}\frac{N_f}{N}\Big)-\beta_0\Big(\frac{25}{6}+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big)\\
&\quad+\tilde{A}_0+\tilde{A}_1 g^2\alpha^{-2\epsilon}\Big\}+\cdots\\
&= \langle F^2(z)F^2(0)\rangle\big|_{\mu}+\frac{48(N^2-1)}{\pi^4 z^{8-4\epsilon}}\frac{2g^2\beta_0}{\epsilon}(1-\alpha^{-2\epsilon})\\
&\quad+\frac{N^2-1}{4\pi^2}\Delta^2\delta^{(\tilde{d})}(z)\mu^{-2\epsilon}\Big\{\frac{1-\alpha^{-2\epsilon}}{\epsilon}+(1-\alpha^{-4\epsilon})\Big[-\frac{g^2\beta_0}{\epsilon^2}\\
&\quad+\frac{g^2}{\epsilon}\Big(\frac{1}{16\pi^2}\Big(\frac{17}{2}-\frac{5}{3}\frac{N_f}{N}\Big)-\beta_0\Big(\frac{25}{6}+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big)-\tilde{A}_1 g^2\Big]\Big\}+\cdots
\end{aligned}
\tag{D.40}
$$

Hence, the scheme dependence of its FT in eq. (7.22) to order g^2 reads as $\epsilon \to 0$:

$$\begin{aligned}
&\mathrm{FT}[\langle F^2(z)F^2(0)\rangle|_{\tilde\mu}] = \mathrm{FT}[\langle F^2(z)F^2(0)\rangle|_{\mu}] \\
&+\frac{N^2-1}{4\pi^2}p^4\Big\{\frac{2g^2\beta_0}{\epsilon^2}\Big(\frac{\pi p^2}{4}\Big)^{-\epsilon}\Big(1+\epsilon\Big(\frac{31}{6}+3\Gamma'(1)\Big)+\cdots\Big)(1-\alpha^{-2\epsilon})\Big\} \\
&+\frac{N^2-1}{4\pi^2}p^4\mu^{-2\epsilon}\Big\{\frac{1-\alpha^{-2\epsilon}}{\epsilon}+(1-\alpha^{-4\epsilon})\Big[-\frac{g^2\beta_0}{\epsilon^2} \\
&+\frac{g^2}{\epsilon}\Big(\frac{1}{16\pi^2}\Big(\frac{17}{2}-\frac{5}{3}\frac{N_f}{N}\Big)-\beta_0\Big(\frac{25}{6}+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big)-\tilde A_1 g^2\Big]\Big\}+\cdots \\
&=\mathrm{FT}[\langle F^2(z)F^2(0)\rangle|_{\mu}]+\frac{N^2-1}{4\pi^2}p^4\Big\{ \\
&\frac{2g^2\beta_0}{\epsilon^2}\Big(2\epsilon\log\alpha-2\epsilon^2\log^2\alpha-2\epsilon^2\log\alpha\Big(\log p^2+\log\frac{\pi}{4}-\frac{31}{6}-3\Gamma'(1)\Big)\Big) \\
&+2\log\alpha-\frac{g^2\beta_0}{\epsilon^2}\Big(4\epsilon\log\alpha-8\epsilon^2\log^2\alpha-4\epsilon^2\log\alpha\log\mu^2\Big) \\
&+4g^2\log\alpha\Big(\frac{1}{16\pi^2}\Big(\frac{17}{2}-\frac{5}{3}\frac{N_f}{N}\Big)-\beta_0\Big(\frac{25}{6}+3\Gamma'(1)-\log\frac{\pi}{4}\Big)\Big)\Big\}+\cdots \\
&=\mathrm{FT}[\langle F^2(z)F^2(0)\rangle|_{\mu}]+\frac{N^2-1}{4\pi^2}p^4\Big\{2\log\alpha-4g^2\beta_0\log\alpha\log\frac{p^2}{\mu^2} \\
&+4g^2\beta_0\log^2\alpha+2g^2\log\alpha\frac{1}{16\pi^2}\Big(\frac{73}{3}-\frac{14}{3}\frac{N_f}{N}\Big)\Big\}+\cdots
\end{aligned} \tag{D.41}$$

where we have employed eqs. (7.8) and (7.18). Eq. (D.41) manifestly agrees with the scheme dependence obtained by replacing $\mu \to \tilde\mu = \alpha\mu$ in $C_0^{(F^2,F^2)}(p)$ in eq. (7.11).

D.5 Derivative of the 2-point correlator $\frac{\partial \log\langle O(z)O(0)\rangle'}{\partial \log g}$ in the l.h.s. of the LET

The l.h.s. of the LET (IC) in $\tilde d = 4-2\epsilon$ dimensions for $z \neq 0$ reads:

$$\text{l.h.s.} = \langle O(z)O(0)\rangle' \frac{\partial \log\langle O(z)O(0)\rangle'}{\partial \log g} \tag{D.42}$$

where $g = g(\mu)$ and the 2-point correlator is the solution in eq. (D.9) of the CS equation in $4-2\epsilon$ dimensions. By explicit computation we obtain:

$$\begin{aligned}
\frac{\partial \log\langle O(z)O(0)\rangle'}{\partial \log g} &= \frac{\partial \log \mathcal{G}_2^{(O)}(\tilde g(z))}{\partial \log g}+2\frac{\partial \log Z^{(O)}(\tilde g(z), g(\mu))}{\partial \log g} \\
&= g\frac{\partial \log \mathcal{G}_2^{(O)}(\tilde g(z))}{\partial g}+2g\frac{\partial}{\partial g}\int_g^{\tilde g(z)}\frac{\gamma_O(g')}{\beta(g',\epsilon)}dg' \\
&= g\frac{\partial \log \mathcal{G}_2^{(O)}(\tilde g(z))}{\partial g}+2g\Big(\frac{\partial \tilde g(z)}{\partial g}\frac{\gamma_O(\tilde g(z))}{\beta(\tilde g(z),\epsilon)}-\frac{\gamma_O(g)}{\beta(g,\epsilon)}\Big) \\
&= g\frac{\partial \log \mathcal{G}_2^{(O)}(\tilde g(z))}{\partial g}+2g\Big(\frac{\beta(\tilde g(z),\epsilon)}{\beta(g,\epsilon)}\frac{\gamma_O(\tilde g(z))}{\beta(\tilde g(z),\epsilon)}-\frac{\gamma_O(g)}{\beta(g,\epsilon)}\Big) \\
&= g\frac{\partial \log \mathcal{G}_2^{(O)}(\tilde g(z))}{\partial g}+\frac{2g}{\beta(g,\epsilon)}(\gamma_O(\tilde g(z))-\gamma_O(g)) \\
&= g\frac{\partial \log \mathcal{G}_2^{(O)}(\tilde g(z))}{\partial g}-\frac{2Z_{F^2}}{\epsilon}(\gamma_O(\tilde g(z))-\gamma_O(g))
\end{aligned} \tag{D.43}$$

where — in the order — we have employed eq. (D.11) for $Z^{(O)}$, eq. (D.14) for the ratio of the running couplings, eq. (A.6) and $\beta(g,\epsilon) = -\epsilon g + \beta(g)$ that imply $Z_{F^2}^{-1} = -\frac{\beta(g,\epsilon)}{\epsilon g}$. To order g^2 in perturbation theory:

$$\begin{aligned}\mathcal{G}_2^{(O)}(\tilde{g}(z)) &= \mathcal{G}_2^{(O)}(0)(1+\delta_2^{(O)}\tilde{g}^2(z)+\cdots) \\ &= \mathcal{G}_2^{(O)}(0)(1+\delta_2^{(O)}|z\mu|^{2\epsilon}g^2+\cdots)\end{aligned} \tag{D.44}$$

and:

$$\begin{aligned}\gamma_O(\tilde{g}(z)) - \gamma_O(g) &= -\gamma_0^{(O)}(\tilde{g}^2(z)-g^2)+\cdots \\ &= -\gamma_0^{(O)}g^2(|z\mu|^{2\epsilon}-1)+\cdots\end{aligned} \tag{D.45}$$

so that eq. (D.43) reads to order g^2 as $\epsilon \to 0$:

$$\begin{aligned}\frac{\partial \log \langle O(z)O(0)\rangle'}{\partial \log g}\bigg|_{\text{to order } g^2} &= 2\delta_2^{(O)}|z\mu|^{2\epsilon}g^2 + \frac{2}{\epsilon}\gamma_0^{(O)}g^2(|z\mu|^{2\epsilon}-1) \\ &= 2\delta_2^{(O)}g^2 + 4\gamma_0^{(O)}g^2\log|z\mu| + \cdots\end{aligned} \tag{D.46}$$

where the dots stand for $O(\epsilon)$ contributions. For $O = F^2$, eq. (D.46) yields:

$$\frac{\partial \log \langle F^2(z)F^2(0)\rangle'}{\partial \log g}\bigg|_{\text{to order } g^2} = 2\delta_2^{(F^2)}g^2 + 8\beta_0 g^2\log|z\mu| + \cdots \tag{D.47}$$

where we have employed $\gamma_0^{(F^2)} = 2\beta_0$.

D.6 Scheme dependence of the l.h.s. of the LET for $O = F^2$ to order g^2

The scheme dependence of the derivative in eq. (D.47) is implied by the one of the correlator at distinct points in eq. (D.39). For $\tilde{\mu} = \alpha\mu$ and setting $g(\mu) = g$, the derivative of eq. (D.39) yields:

$$\begin{aligned}\frac{\partial \log \langle F^2(z)F^2(0)\rangle'|_{\tilde{\mu}}}{\partial \log g(\tilde{\mu})} &= \frac{\partial}{\partial \log g}\log\left(\langle F^2(z)F^2(0)\rangle'|_{\mu} + \frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\frac{2\beta_0 g^2}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots\right) \\ &= \frac{\partial}{\partial \log g}\Bigg\{\log\langle F^2(z)F^2(0)\rangle'|_{\mu} \\ &\quad +\log\left(1+\frac{1}{\langle F^2(z)F^2(0)\rangle'|_{\mu}}\frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\frac{2\beta_0 g^2}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots\right)\Bigg\} \\ &= \frac{\partial \log\langle F^2(z)F^2(0)\rangle'|_{\mu}}{\partial \log g} \\ &\quad +\frac{\partial}{\partial \log g}\log\left(1+\frac{1}{\frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}(1+O(g^2))}\frac{\mathcal{G}_2^{(F^2)}(0)}{z^{8-4\epsilon}}\frac{2\beta_0 g^2}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots\right) \\ &= \frac{\partial \log\langle F^2(z)F^2(0)\rangle'|_{\mu}}{\partial \log g} + \frac{\partial}{\partial \log g}\log\left(1+\frac{2\beta_0 g^2}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots\right) \\ &= \frac{\partial \log\langle F^2(z)F^2(0)\rangle'|_{\mu}}{\partial \log g} + \frac{4\beta_0 g^2}{\epsilon}(1-\alpha^{-2\epsilon})+\cdots \\ &= \frac{\partial \log\langle F^2(z)F^2(0)\rangle'|_{\mu}}{\partial \log g} + 8\beta_0 g^2\log\alpha+\cdots\end{aligned} \tag{D.48}$$

where we have employed to the relevant order as $\epsilon \to 0$:

$$\frac{\partial}{\partial \log g(\tilde{\mu})} = \frac{\partial}{\partial \log g}\frac{\partial \log g}{\partial \log g(\tilde{\mu})} = \frac{\partial}{\partial \log g}(1+\cdots) \tag{D.49}$$

The result in eq. (D.48) can also be straightforwardly obtained by replacing $\mu \to \tilde{\mu} = \alpha\mu$ in eq. (D.47). Hence, after factoring out the 2-point correlator $\langle F^2(z)F^2(0)\rangle'$ in eq. (D.42) for $O = F^2$, the remaining scheme dependence of the l.h.s. of the LET to order g^2 amounts to:

$$\frac{\partial \log \langle F^2(z)F^2(0)\rangle' |_{\tilde{\mu}}}{\partial \log g(\tilde{\mu})}\Bigg|_{\text{to order } g^2} = \frac{\partial \log \langle F^2(z)F^2(0)\rangle' |_{\mu}}{\partial \log g}\Bigg|_{\text{to order } g^2} + 8\beta_0 g^2 \log\alpha \tag{D.50}$$

D.7 Scheme dependence of the r.h.s. of the LET for $O = F^2$ to order g^2

For $O = F^2$, the r.h.s. of the LET (IC) to order g^2 in $\tilde{d} = 4-2\epsilon$ dimensions reads for $z \neq 0$:

$$\begin{aligned}
\text{r.h.s.} &= \frac{1}{2}Z_{F^2}\int \langle F^2(z)F^2(0)F_0^2(x)\rangle\, d^{\tilde{d}}x \\
&\quad -\frac{1}{2}Z_{F^2}\int 4(\delta^{(\tilde{d})}(x-z)+\delta^{(\tilde{d})}(x))\,\langle F^2(z)F^2(0)\rangle'\, d^{\tilde{d}}x \\
&\quad +Z_{F^2}\frac{2\gamma_{F^2}(g)}{\epsilon}\,\langle F^2(z)F^2(0)\rangle'\Bigg|_{\text{up to order } g^2} \\
&= \frac{1}{2}\int \langle F^2(z)F^2(0)F^2(x)\rangle'\, d^{\tilde{d}}x \\
&\quad +\frac{1}{2}\int \tilde{B}_0^{(F^2)}g^2\Big(\delta^{(\tilde{d})}(x)+\delta^{(\tilde{d})}(x-z)\Big)\,\langle F^2(z)F^2(0)\rangle'\, d^{\tilde{d}}x \\
&\quad +\frac{2\gamma_{F^2}(g)}{\epsilon}\,\langle F^2(z)F^2(0)\rangle'\Bigg|_{\text{up to order } g^2} \\
&= 2\beta_0 g^2\Big(\frac{2}{\epsilon}+4\log|z\mu|-2\log\pi+2+2\Gamma'(1)\Big)\,\langle F^2(z)F^2(0)\rangle' \\
&\quad +\tilde{B}_0^{(F^2)}g^2\,\langle F^2(z)F^2(0)\rangle' - \frac{4\beta_0 g^2}{\epsilon}\,\langle F^2(z)F^2(0)\rangle'\Bigg|_{\text{to order } g^2}
\end{aligned} \tag{D.51}$$

where we have employed eq. (6.8) for $O = F^2$:

$$\begin{aligned}
\langle F^2(z)F^2(0)F_0^2(x)\rangle\Bigg|_{\text{up to order } g^2} &= (4+\tilde{B}_0^{(F^2)}g^2)(\delta^{(\tilde{d})}(x-z)+\delta^{(\tilde{d})}(x))\,\langle F^2(z)F^2(0)\rangle' \\
&\quad +Z_{F^2}^{-1}\,\langle F^2(z)F^2(0)F^2(x)\rangle'\Bigg|_{\text{up to order } g^2}
\end{aligned} \tag{D.52}$$

and eqs. (6.16) and (6.17), with $\gamma_{F^2}(g) = -2\beta_0 g^2+\cdots$. After factoring out $\langle F^2(z)F^2(0)\rangle'$ in eq. (D.51), the scheme dependence of the r.h.s. of the LET to order g^2 amounts to:

$$\begin{aligned}
&2\beta_0 g^2\Big(\frac{2}{\epsilon}+4\log|z\mu|-2\log\pi+2+2\Gamma'(1)\Big)+\tilde{B}_0^{(F^2)}g^2-\frac{4\beta_0 g^2}{\epsilon} \\
&= 2\beta_0 g^2\Big(4\log|z\mu|-2\log\pi+2+2\Gamma'(1)\Big)+\tilde{B}_0^{(F^2)}g^2
\end{aligned} \tag{D.53}$$

where the divergent contact term has cancelled the divergence of the integrated contribution at distinct points. Correspondingly, the result in the $\tilde{\mu}$-scheme is related to the one in the μ-scheme, with $\tilde{\mu} = \alpha\mu$ and $g(\mu) = g$, as $\epsilon \to 0$ as follows:

$$\begin{aligned} &2\beta_0 g^2(\tilde{\mu})\Big(4\log|z\tilde{\mu}| - 2\log\pi + 2 + 2\Gamma'(1)\Big) + \tilde{B}_0^{(F^2)} g^2(\tilde{\mu}) \\ &= 2\beta_0 g^2\Big(4\log|z\mu| + 4\log\alpha - 2\log\pi + 2 + 2\Gamma'(1)\Big) + \tilde{B}_0^{(F^2)} g^2 \\ &= 2\beta_0 g^2\Big(4\log|z\mu| - 2\log\pi + 2 + 2\Gamma'(1)\Big) + \tilde{B}_0^{(F^2)} g^2 + 8\beta_0 g^2\log\alpha \end{aligned} \tag{D.54}$$

where we have employed eq. (D.38) for $\epsilon \to 0$.

D.8 Matching the finite terms in the LET for $O = F^2$ to order g^2

Importantly, we verify that the matching of the l.h.s. and r.h.s. of the LET is scheme independent. Indeed, after factoring out $\langle F^2(z)F^2(0)\rangle'$ on both sides of the LET, the scheme dependence of the l.h.s. is given by eq. (D.50), i.e. it consists of the shift by the term $8\beta_0 g^2\log\alpha$ that equates the one of the r.h.s. of the LET in eq. (D.54).

Hence, for $O = F^2$ the LET implies the following scheme-independent matching to order g^2 as $\epsilon \to 0$:

$$\frac{\partial\log\langle F^2(z)F^2(0)\rangle'}{\partial\log g}\bigg|_{\text{to order } g^2} = 2\beta_0 g^2\Big(4\log|z\mu| - 2\log\pi + 2 + 2\Gamma'(1)\Big) + \tilde{B}_0^{(F^2)} g^2 \tag{D.55}$$

that, by means of eq. (D.47), reads:

$$2\delta_2^{(F^2)} g^2 + 8\beta_0 g^2\log|z\mu| = 2\beta_0 g^2\Big(4\log|z\mu| - 2\log\pi + 2 + 2\Gamma'(1)\Big) + \tilde{B}_0^{(F^2)} g^2 \tag{D.56}$$

The terms proportional to $\log|z\mu|$ indeed match. The matching of the remaining finite terms implies the relation:

$$\delta_2^{(F^2)} = \frac{1}{2}\tilde{B}_0^{(F^2)} + \beta_0\Big(2 + 2\Gamma'(1) - 2\log\pi\Big) \tag{D.57}$$

between the coefficients $\delta_2^{(F^2)}$ and $\tilde{B}_0^{(F^2)}$ computed in the same RG scheme.

The LET thus relates in a scheme-independent way two scheme-dependent quantities that enter different OPE coefficients computed in perturbation theory. Specifically, $\delta_2^{(F^2)}$ — hence the coefficient $2(c_1 - \beta_0)$ of the subleading $\log\frac{p^2}{\mu^2}$ to order g^2 — that enters $C_0^{(F^2,F^2)}(p)$ computed in [2, 3, 13] and $\tilde{B}_0^{(F^2)}$ — hence $B_{1,1}$ — that enters $C_1^{(F^2,F^2)}(p)$ computed in [3, 14].

The value of $\tilde{B}_0^{(F^2)}$ determined in [3, 14] in the $\overline{MS}$ scheme in our notation reads:

$$\begin{aligned} \tilde{B}_{0\,CZ}^{(F^2)} &= -4\tilde{B}_{1,1} \\ &= -4B_{1,1} - 4\beta_0\Big(2 + 3\Gamma'(1) - \log\frac{\pi}{4}\Big) \\ &= -4\frac{1}{(4\pi)^2}\Big(-\frac{49}{9} + \frac{10}{9}\frac{N_f}{N}\Big) - 4\beta_0\Big(2 + 3\Gamma'(1) - \log\frac{\pi}{4}\Big) \end{aligned} \tag{D.58}$$

where in the last line we have employed the value of $B_{1,1}$ reported in eq. (4.26). The value of $\delta_2^{(F^2)}$ determined in [2, 3] in the $\overline{MS}$ scheme in our notation reads:

$$\begin{aligned}
\delta^{(F^2)}_{2\,CZ} &= -2(c_1-\beta_0) - 2\beta_0\Big(\frac{10}{3} + 2\Gamma'(1) + \log 4\Big) \\
&= \frac{1}{(4\pi)^2}\Big(\frac{73}{3} - \frac{14}{3}\frac{N_f}{N}\Big) - 2\beta_0\Big(\frac{10}{3} + 2\Gamma'(1) + \log 4\Big) \\
&= -2\frac{1}{(4\pi)^2}\Big(-\frac{49}{9} + \frac{10}{9}\frac{N_f}{N}\Big) - 2\beta_0\Big(\frac{3}{2} + 2\Gamma'(1) + \log 4\Big) \\
&= -2B_{1,1} - 2\beta_0\Big(\frac{3}{2} + 2\Gamma'(1) + \log 4\Big) \\
&= \frac{1}{2}\tilde{B}^{(F^2)}_{0\,CZ} + 2\beta_0\Big(2 + 3\Gamma'(1) - \log\frac{\pi}{4}\Big) - 2\beta_0\Big(\frac{3}{2} + 2\Gamma'(1) + \log 4\Big) \\
&= \frac{1}{2}\tilde{B}^{(F^2)}_{0\,CZ} + \beta_0\Big(1 + 2\Gamma'(1) - 2\log\pi\Big)
\end{aligned} \tag{D.59}$$

where we have employed eqs. (7.17), (7.15), (4.26) and (D.58). Eq. (D.59) differs from the LET constraint in eq. (D.57) for a term β_0:

$$\Big(\delta^{(F^2)}_{2\,CZ} - \frac{1}{2}\tilde{B}^{(F^2)}_{0\,CZ}\Big) = \Big(\delta^{(F^2)}_{2} - \frac{1}{2}\tilde{B}^{(F^2)}_{0}\Big) - \beta_0 \tag{D.60}$$

whereas all terms with $\Gamma'(1)$, $\log\pi$ and $\log 4$ do agree. Tentatively, we suggest two alternative explanations.

Either, eq. (D.59) for the coefficient in $C_0^{(F^2,F^2)}(p)$ in [2, 3] or eq. (D.58) for the coefficient in $C_1^{(F^2,F^2)}(p)$ in [3, 14] need a correction. Or, our ansatz conformal in the limit $\epsilon \to 0$ for the solution of the LET to order g^2 — despite reproducing exactly the OPE to the relevant order — introduces an error of order ϵ that somehow generates a finite discrepancy once multiplied by some divergence in the r.h.s. of the LET.

E Callan-Symanzik equation in $d = 4$ dimensions in the coordinate representation

The CS equation in $d = 4$ dimensions for $G^{(2)} \equiv \langle O(z)O(0)\rangle'$ at distinct points $z \neq 0$ and a multiplicatively renormalizable gauge-invariant scalar operator O with canonical dimension D follows from eq. (D.3) as $\epsilon \to 0$:

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_O(g)\Big)\, G^{(2)}(z,\mu,g(\mu)) = 0 \tag{E.1}$$

whose solution reads:

$$\begin{aligned}
G^{(2)}(z,\mu,g(\mu)) &= \frac{1}{z^{2D}}\,\bar{G}^{(2)}(z\mu,g(\mu)) \\
&= \frac{1}{z^{2D}}\,\mathcal{G}_2^{(O)}(g(z))\, Z^{(O)2}(g(z),g(\mu))
\end{aligned} \tag{E.2}$$

in terms of the RG-invariant function $\mathcal{G}_2^{(O)}$ of the running coupling $g(z) \equiv g(z\mu, g(\mu))$ that solves:

$$-\frac{dg(z)}{d\log|z|} = \beta(g(z)) \tag{E.3}$$

with the initial condition $g(1, g(\mu)) = g(\mu)$ and the renormalized multiplicative factor $Z^{(O)}(g(z), g(\mu))$:

$$Z^{(O)}(g(z), g(\mu)) = \exp \int_{g(\mu)}^{g(z)} \frac{\gamma_O(g)}{\beta(g)} dg \tag{E.4}$$

that solves:

$$\gamma_O(g(\mu)) = -\frac{d \log Z^{(O)}}{d \log \mu} \tag{E.5}$$

Moreover, the RG invariance of $g(z)$ implies:

$$\begin{aligned} 0 &= \frac{dg(z)}{d \log \mu} \\ &= \frac{\partial g(z)}{\partial \log \mu} + \frac{\partial g(z)}{\partial g(\mu)} \frac{dg(\mu)}{d \log \mu} \\ &= \frac{dg(z)}{d \log |z|} + \frac{\partial g(z)}{\partial g(\mu)} \frac{dg(\mu)}{d \log \mu} \\ &= -\beta(g(z)) + \frac{\partial g(z)}{\partial g(\mu)} \beta(g(\mu)) \end{aligned} \tag{E.6}$$

according to eq. (E.3) and the fact that $g(z)$ depends on z via the product $z\mu$ only. As a consequence:

$$\frac{\partial g(z)}{\partial g(\mu)} = \frac{\beta(g(z))}{\beta(g(\mu))} \tag{E.7}$$

E.1 Asymptotics of the running coupling

The beta function has a zero at $g = 0$:

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 + \cdots \tag{E.8}$$

and for $\beta_0 > 0$ the theory is AF at short distances. From the integral of the inverse of the beta function:

$$\int_{g(\mu)}^{g(z)} \frac{dg}{-\beta_0 g^3 - \beta_1 g^5 + \cdots} = -\int_{\mu^{-1}}^{|z|} d \log |z| \tag{E.9}$$

where the length scales, $|z| = \sqrt{z^2}$ and μ^{-1}, are assumed to be close to zero in order for $g(z)$ and $g(\mu)$ to stay in a neighborhood of $g = 0$, we obtain:

$$\frac{1}{g^2(z)} - \frac{1}{g^2(\mu)} = -2\beta_0 \log |z\mu| - 2\frac{\beta_1}{\beta_0} \log \left(\frac{g(z)}{g(\mu)} \right) + \cdots \tag{E.10}$$

or, equivalently:

$$\frac{1}{g^2(z)} + 2\beta_0 \log |z| + 2\frac{\beta_1}{\beta_0} \log g(z) + C + \cdots = \frac{1}{g^2(\mu)} - 2\beta_0 \log \mu + 2\frac{\beta_1}{\beta_0} \log g(\mu) + C + \cdots \tag{E.11}$$

where the scheme-dependent integration constant C arises from the indefinite version of the integral in eq. (E.9). The above equation implies the existence of an RG-invariant mass scale Λ_{UV}:

$$\begin{aligned}\Lambda_{UV} &= \mu \exp\left(-\frac{1}{2\beta_0 g^2(\mu)}\right) g^{-\frac{\beta_1}{\beta_0^2}}(\mu) \exp\left(-\frac{C}{2\beta_0}+\cdots\right) \\ &= |z|^{-1} \exp\left(-\frac{1}{2\beta_0 g^2(z)}\right) g^{-\frac{\beta_1}{\beta_0^2}}(z) \exp\left(-\frac{C}{2\beta_0}+\cdots\right)\end{aligned} \tag{E.12}$$

with the dots a scheme-dependent series in even powers of $g(\mu)$ and $g(z)$ respectively. Solving eq. (E.12) asymptotically as $|z| \to 0^+$ we get:

$$g^2(z) \sim \frac{1}{-2\beta_0 \log(|z|\Lambda_{UV})}\left(1+\frac{\beta_1}{2\beta_0^2}\frac{\log(-2\beta_0\log(|z|\Lambda_{UV}))}{\log(|z|\Lambda_{UV})} - \frac{C}{2\beta_0\log(|z|\Lambda_{UV})}+\cdots\right) \tag{E.13}$$

Incidentally, the $\overline{MS}$ scheme is defined by choosing $C = (\beta_1/\beta_0)\log\beta_0$ that cancels the last term in eq. (E.13) against the term $\frac{\beta_1}{2\beta_0^2}\frac{\log 2\beta_0}{\log(|z|\Lambda_{UV})}$ in the first line. The universal asymptotics of the running coupling as $|z| \to 0^+$ follows:

$$g^2(z) \sim \frac{1}{-2\beta_0 \log(|z|\Lambda_{UV})}\left(1+\frac{\beta_1}{2\beta_0^2}\frac{\log(-2\beta_0\log(|z|\Lambda_{UV}))}{\log(|z|\Lambda_{UV})}\right) \tag{E.14}$$

Perturbatively to order $g^4(\mu)$ we obtain from eq. (E.10):

$$\begin{aligned}\frac{g^2(\mu)}{g^2(z)} &= 1 - g^2(\mu)2\beta_0\log|z\mu| + g^2(\mu)\frac{\beta_1}{\beta_0}\log\left(\frac{g^2(\mu)}{g^2(z)}\right)+\cdots \\ &= 1 - g^2(\mu)2\beta_0\log|z\mu| - g^4(\mu)2\beta_1\log|z\mu| + \cdots\end{aligned} \tag{E.15}$$

for $\log|z\mu|$ of order one, where we have employed in the second equality:

$$\log\left(\frac{g^2(\mu)}{g^2(z)}\right) = \log(1-g^2(\mu)2\beta_0\log|z\mu|+\cdots) = -g^2(\mu)2\beta_0\log|z\mu|+\cdots \tag{E.16}$$

Eq. (E.15) yields:

$$g^2(z) = g^2(\mu)\left(1+g^2(\mu)2\beta_0\log|z\mu| + g^4(\mu)(2\beta_1\log|z\mu| + 4\beta_0^2\log^2|z\mu|)+\cdots\right) \tag{E.17}$$

E.2 Asymptotics of $\langle O(z)O(0)\rangle'$

From eq. (E.4) it follows asymptotically at short distances by means of eq. (D.30):

$$\begin{aligned}Z^{(O)}(g(z),g(\mu)) &\sim \left(\frac{g(z)}{g(\mu)}\right)^{\frac{\gamma_0^{(O)}}{\beta_0}} \exp\left(\frac{\gamma_1^{(O)}\beta_0-\gamma_0^{(O)}\beta_1}{2\beta_0^2}(g^2(z)-g^2(\mu))+\cdots\right) \\ &\sim \left(\frac{1}{-g^2(\mu)2\beta_0\log(|z|\Lambda_{UV})}\left(1+\frac{\beta_1}{2\beta_0^2}\frac{\log(-\beta_0\log(|z|\Lambda_{UV}))}{\log(|z|\Lambda_{UV})}\right)\right)^{\frac{\gamma_0^{(O)}}{2\beta_0}} Z^{(O)\prime}(g(\mu))\end{aligned} \tag{E.18}$$

where:

$$Z^{(O)\prime}(g(\mu)) = \exp\left(-\frac{\gamma_1^{(O)}\beta_0-\gamma_0^{(O)}\beta_1}{2\beta_0^2}g^2(\mu)+\cdots\right) \tag{E.19}$$

and we have employed eq. (E.14) for $g(z)$. It follows from eq. (E.18) the asymptotics at short distances of the 2-point correlator [12, 24, 31]:

$$\begin{aligned}\langle O(z)O(0)\rangle' &\sim \frac{\mathcal{G}_2^{(O)}(0)}{z^{2D}}\left(\frac{g(z)}{g(\mu)}\right)^{\frac{2\gamma_0^{(O)}}{\beta_0}}\exp\left(\frac{\gamma_1^{(O)}\beta_0-\gamma_0^{(O)}\beta_1}{\beta_0^2}(g^2(z)-g^2(\mu))+\cdots\right)\\ &\sim \frac{\mathcal{G}_2^{(O)}(0)}{z^{2D}}\Bigg(\frac{1}{-g^2(\mu)2\beta_0\log(|z|\Lambda_{UV})}\\ &\quad\left(1+\frac{\beta_1}{2\beta_0^2}\frac{\log(-\beta_0\log(|z|\Lambda_{UV}))}{\log(|z|\Lambda_{UV})}\right)\Bigg)^{\frac{\gamma_0^{(O)}}{\beta_0}}Z^{(O)\prime 2}(g(\mu))\end{aligned}\tag{E.20}$$

Perturbatively, from eqs. (E.18) and (E.17) we get:

$$Z^{(O)}(g(z),g(\mu)) = 1+g^2(\mu)\gamma_0^{(O)}\log|z\mu|+g^4(\mu)\left(\gamma_1^{(O)}\log|z\mu|+(\gamma_0^{(O)}\beta_0+\frac{\gamma_0^{(O)2}}{2})\log^2|z\mu|\right)+\cdots\tag{E.21}$$

E.3 Asymptotics of $\langle F^2(z)F^2(0)\rangle'$

$Z^{(F^2)}$ admits the closed form:

$$\begin{aligned}Z^{(F^2)}(g(z),g(\mu)) &= \exp\int_{g(\mu)}^{g(z)}\frac{\gamma_{F^2}(g)}{\beta(g)}dg\\ &= \exp\int_{g(\mu)}^{g(z)}\frac{\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)}{\frac{\beta(g)}{g}}dg\\ &= \frac{\beta(g(z))}{g(z)}\frac{g(\mu)}{\beta(g(\mu))}\end{aligned}\tag{E.22}$$

that follows from eq. (D.33) as $\epsilon\to 0$. Correspondingly:

$$\langle F^2(z)F^2(0)\rangle' = \frac{\mathcal{G}_2^{(F^2)}(g(z))}{z^8}\left(\frac{\beta(g(z))}{g(z)}\right)^2\left(\frac{g(\mu)}{\beta(g(\mu))}\right)^2\tag{E.23}$$

whose asymptotics at short distances reads:

$$\begin{aligned}\langle F^2(z)F^2(0)\rangle' &\sim \frac{\mathcal{G}_2^{(F^2)}(0)}{z^8}g^4(z)\beta_0^2\left(\frac{g(\mu)}{\beta(g(\mu))}\right)^2\\ &\sim \frac{\mathcal{G}_2^{(F^2)}(0)}{z^8}\left(\frac{1}{-2\beta_0\log(|z|\Lambda_{UV})}\left(1+\frac{\beta_1}{2\beta_0^2}\frac{\log(-\beta_0\log(|z|\Lambda_{UV}))}{\log(|z|\Lambda_{UV})}\right)\right)^2\left(\frac{\beta_0 g(\mu)}{\beta(g(\mu))}\right)^2\end{aligned}\tag{E.24}$$

by means of eq. (E.14) and:

$$\frac{\beta(g(z))}{g(z)} = -\beta_0 g^2(z)\left(1+\frac{\beta_1}{\beta_0}g^2(z)+\cdots\right)\tag{E.25}$$

It agrees with eq. (E.20) for $O=F^2$, $D=4$, $\gamma_0^{(F^2)}=2\beta_0$ due to:

$$\frac{g(\mu)}{\beta(g(\mu))} = \frac{1}{-\beta_0 g^2(\mu)}\left(1-\frac{\beta_1}{\beta_0}g^2(\mu)+\cdots\right)\tag{E.26}$$

that implies:

$$\left(\frac{\beta_0 g(\mu)}{\beta(g(\mu))}\right)^2 = \frac{1}{g^4(\mu)}\left(1 - 2\frac{\beta_1}{\beta_0}g^2(\mu) + \cdots\right)$$
$$= \frac{1}{g^4(\mu)} Z^{(F^2)\prime 2}(g(\mu)) \tag{E.27}$$

with $Z^{(F^2)\prime}(g(\mu))$ in eq. (E.19) for $O = F^2$ and $\gamma_1^{(F^2)} = 4\beta_1$.

Perturbatively, by means of eq. (E.17) we obtain:

$$Z^{(F^2)}(g(z), g(\mu)) = \left(\frac{\beta(g(z))}{g(z)}\right)\left(\frac{g(\mu)}{\beta(g(\mu))}\right)$$
$$= \frac{g^2(z)}{g^2(\mu)}\left(1 + \frac{\beta_1}{\beta_0}(g^2(z) - g^2(\mu)) + \cdots\right)$$
$$= 1 + g^2(\mu)2\beta_0 \log|z\mu| + g^4(\mu)(4\beta_1 \log|z\mu| + 4\beta_0^2 \log^2|z\mu|) + \cdots \tag{E.28}$$

that agrees with eq. (E.21) for $O = F^2$, $\gamma_0^{(F^2)} = 2\beta_0$ and $\gamma_1^{(F^2)} = 4\beta_1$.

F Callan-Symanzik equation in $\tilde{d} = 4 - 2\epsilon$ dimensions in the momentum representation

F.1 Callan-Symanzik equation for $C_1^{(F^2,O)}(p)$

The CS equation in dimensional regularization for the fully renormalized OPE coefficient in the momentum representation in eq. (4.17):

$$C_1^{(F^2,O)}(p) = Z_{F^2} C_1^{(F_0^2,O)}(p) + Z_{1\text{c.t.}} \tag{F.1}$$

follows from the renormalization-scale independence of the bare coefficient:

$$\mu \frac{dC_1^{(F_0^2,O)}(p)}{d\mu}\Big|_{\epsilon,g_0} = 0 \tag{F.2}$$

for fixed bare parameters ϵ and g_0. In the massless case, employing $\mu\frac{d}{d\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g,\epsilon)\frac{\partial}{\partial g}$ with $\beta(g,\epsilon) = -\epsilon g + \beta(g)$, we obtain:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g,\epsilon)\frac{\partial}{\partial g} + \gamma_{F^2}(g)\right) C_1^{(F^2,O)}(p,\mu,g(\mu)) = \gamma_{1\text{c.t.}}(g) \tag{F.3}$$

where:

$$\gamma_{1\text{c.t.}}(g) = \mu\frac{dZ_{1\text{c.t.}}}{d\mu} + \gamma_{F^2} Z_{1\text{c.t.}}$$
$$= \mu\frac{dg}{d\mu}\frac{\partial Z_{1\text{c.t.}}}{\partial g} + \gamma_{F^2} Z_{1\text{c.t.}}$$
$$= \beta(g,\epsilon)\frac{\partial Z_{1\text{c.t.}}}{\partial g} + \gamma_{F^2} Z_{1\text{c.t.}} \tag{F.4}$$

Given that $Z_{1\text{c.t.}}$ is a series of pure poles in ϵ in $\overline{MS}$-like schemes, $\gamma_{1\text{c.t.}}$ in eq. (F.4) may contain poles in ϵ in addition to $O(\epsilon^0)$ terms, but no $O(\epsilon)$ terms, since in the r.h.s. of eq. (F.4) all the terms that contain factors of ϵ are multiplied by poles. Hence, $\gamma_{1\text{c.t.}}$ is ϵ independent if and only if it is not divergent as $\epsilon \to 0$. Yet, the r.h.s. of eq. (F.3) must be finite as $\epsilon \to 0$ consistently with the finiteness as $\epsilon \to 0$ of the fully renormalized $C_1^{(F^2,O)}$ in the l.h.s. It follows that $\gamma_{1\text{c.t.}}$ is finite as $\epsilon \to 0$ and, therefore, ϵ independent. Since the solution of eq. (F.3) solely depends on the ratio $\frac{p}{\mu}$, we write:

$$\left(p \cdot \frac{\partial}{\partial p} - \beta(g,\epsilon)\frac{\partial}{\partial g} - \gamma_{F^2}(g)\right)C_1^{(F^2,O)}(p,\mu,g(\mu)) = -\gamma_{1\text{c.t.}}(g) \tag{F.5}$$

The CS equation in $d = 4$ dimensions is obtained from eqs. (F.3) and (F.5) by replacing $\beta(g,\epsilon)$ with $\beta(g)$. Hence, for $d = 4$:

$$\left(p \cdot \frac{\partial}{\partial p} - \beta(g)\frac{\partial}{\partial g} - \gamma_{F^2}(g)\right)C_1^{(F^2,O)}(p,\mu,g(\mu)) = -\gamma_{1\text{c.t.}}(g) \tag{F.6}$$

F.2 General solution for $C_1^{(F^2,O)}(p)$

The general solution of eq. (F.3) or (F.5) is:

$$C_1^{(F^2,O)}(p,\mu,g(\mu)) = Z^{(F^2)}(\tilde{g}(p),g(\mu))\Big(\mathcal{G}^{(O)}(\tilde{g}(p)) + \Delta^{(O)}(\tilde{g}(p),g(\mu))\Big) \tag{F.7}$$

where:

$$\begin{aligned} Z^{(F^2)}(\tilde{g}(p),g(\mu)) &= \exp\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{F^2}(g)}{\beta(g,\epsilon)}dg \\ &= \frac{\beta(\tilde{g}(p),\epsilon)}{\tilde{g}(p)}\frac{g(\mu)}{\beta(g(\mu),\epsilon)} \end{aligned} \tag{F.8}$$

with the second equality implied by eq. (A.8), and:

$$\begin{aligned} \Delta^{(O)}(\tilde{g}(p),g(\mu)) &= -\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{1\text{c.t.}}(g)}{\beta(g,\epsilon)} \exp\left(-\int_{g}^{\tilde{g}(p)} \frac{\gamma_{F^2}(g')}{\beta(g',\epsilon)}dg'\right) dg \\ &= -\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{1\text{c.t.}}(g)}{\beta(g,\epsilon)} Z^{(F^2)^{-1}}(\tilde{g}(p),g)\, dg \end{aligned} \tag{F.9}$$

$\mathcal{G}^{(O)}(\tilde{g}(p))$ in eq. (F.7) is a function of the running coupling $\tilde{g}(p)$ in the momentum representation in $\tilde{d} = 4 - 2\epsilon$ dimensions. The latter can be obtained from $\tilde{g}(z)$ in appendix D by the substitution $z \to 1/p$. By means of eqs. (F.8) and (F.9), eq. (F.7) reads:

$$C_1^{(F^2,O)}(p,\mu,g(\mu)) = \frac{g(\mu)}{\beta(g(\mu),\epsilon)}\left(\mathcal{G}^{(O)}(\tilde{g}(p))\frac{\beta(\tilde{g}(p),\epsilon)}{\tilde{g}(p)} - \int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{1\text{c.t.}}(g)}{g}\, dg\right) \tag{F.10}$$

The solution of the CS eq. (F.6) in $d = 4$ dimensions is obtained by replacing $\beta(g,\epsilon)$ with $\beta(g)$ and $\tilde{g}(p)$ with $g(p)$ in eq. (F.10), where $g(p)$ can be obtained from $g(z)$ in appendix E by the substitution $z \to 1/p$. Hence, for $d = 4$:

$$C_1^{(F^2,O)}(p,\mu,g(\mu)) = \frac{g(\mu)}{\beta(g(\mu))}\left(\mathcal{G}^{(O)}(g(p))\frac{\beta(g(p))}{g(p)} - \int_{g(\mu)}^{g(p)} \frac{\gamma_{1\text{c.t.}}(g)}{g}\, dg\right) \tag{F.11}$$

JHEP08(2024)145

Employing eq. (4.50):

$$C_1^{(F^2,O)}(p) = Z_{F^2} C_1^{(F_0^2,O)}(p) + Z_{F^2} \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \tag{F.12}$$

we obtain:

$$Z_{1\text{c.t.}} = Z_{F^2} \frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \tag{F.13}$$

and from eq. (F.4):

$$\begin{aligned}
\gamma_{1\text{c.t.}}(g) &= \beta(g,\epsilon)\frac{\partial Z_{1\text{c.t.}}}{\partial g} + \gamma_{F^2} Z_{1\text{c.t.}} \\
&= \beta(g,\epsilon)\left\{\frac{\partial Z_{F^2}}{\partial g}\frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} + \frac{2Z_{F^2}}{\epsilon}\left(\frac{\partial \gamma_O}{\partial g} - c_O \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\right)\right\} \\
&\quad + Z_{F^2} g \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \\
&= -Z_{F^2} g \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} + \frac{2\beta(g,\epsilon)}{\epsilon} Z_{F^2}\left(\frac{\partial \gamma_O}{\partial g} - c_O \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\right) \\
&\quad + Z_{F^2} g \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\frac{2\gamma_O(g) - 2c_O \frac{\beta(g)}{g}}{\epsilon} \\
&= \frac{2\beta(g,\epsilon)}{\epsilon} Z_{F^2}\left(\frac{\partial \gamma_O}{\partial g} - c_O \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\right) \\
&= -2g\left(\frac{\partial \gamma_O}{\partial g} - c_O \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right)\right)
\end{aligned} \tag{F.14}$$

that is, indeed, manifestly ϵ independent by employing $\beta(g,\epsilon) = -\epsilon g + \beta(g)$, eqs. (A.8), (A.6) and:

$$\beta(g,\epsilon)\frac{\partial Z_{F^2}}{\partial g} = -Z_{F^2} g \frac{\partial}{\partial g}\left(\frac{\beta(g,\epsilon)}{g}\right) = -Z_{F^2} g \frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) \tag{F.15}$$

Then, the solution in eq. (F.10) reads:

$$\begin{aligned}
C_1^{(F^2,O)}(p,\mu,g(\mu)) &= 2c_O + (\mathcal{G}^{(O)}(\tilde{g}(p)) - 2c_O)\frac{\beta(\tilde{g}(p),\epsilon)}{\tilde{g}(p)}\frac{g(\mu)}{\beta(g(\mu),\epsilon)} \\
&\quad + 2g(\mu)\frac{\gamma_O(g(\mu))}{\beta(g(\mu),\epsilon)}\left(\frac{\gamma_O(\tilde{g}(p))}{\gamma_O(g(\mu))} - 1\right)
\end{aligned} \tag{F.16}$$

where the identity:

$$\frac{\beta(\tilde{g}(p),\epsilon)}{\tilde{g}(p)} - \frac{\beta(g(\mu),\epsilon)}{g(\mu)} = \frac{\beta(\tilde{g}(p))}{\tilde{g}(p)} - \frac{\beta(g(\mu))}{g(\mu)} \tag{F.17}$$

has been conveniently employed, and the solution in $d = 4$ dimensions in eq. (F.11) reads:

$$\begin{aligned}
C_1^{(F^2,O)}(p,\mu,g(\mu)) &= 2c_O + (\mathcal{G}^{(O)}(g(p)) - 2c_O)\frac{\beta(g(p))}{g(p)}\frac{g(\mu)}{\beta(g(\mu))} \\
&\quad + 2g(\mu)\frac{\gamma_O(g(\mu))}{\beta(g(\mu))}\left(\frac{\gamma_O(g(p))}{\gamma_O(g(\mu))} - 1\right)
\end{aligned} \tag{F.18}$$

F.3 Specializing to $C_1^{(F^2,F^2)}(p)$

For $O = F^2$ the fully renormalized OPE coefficient in eq. (F.1) has been explicitly computed in perturbation theory up to order g^4 [3, 14]. Moreover, in this case the additive renormalization, i.e. $Z_{1\text{c.t.}}$ in eq. (F.1), has also been determined to all orders in perturbation theory [2]:

$$C_1^{(F^2,F^2)}(p) = Z_{F^2} C_1^{(F_0^2,F^2)}(p) + Z_{F^2} \frac{2g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) - 4\frac{\beta(g)}{g}}{\epsilon} \tag{F.19}$$

where [3, 14]:

$$\begin{aligned} Z_{F^2} C_1^{(F_0^2,F^2)}(p) &= 4 - 4B_{1,1}g^2 - 4B_{1,2}g^4 + \frac{4\beta_1 g^4}{\epsilon} \\ &\quad -4\beta_0 g^2 \log\frac{p^2}{\mu^2} + 4\beta_0^2 g^4 \log^2\frac{p^2}{\mu^2} + 4B_{1,3}g^4 \log\frac{p^2}{\mu^2} + \cdots \end{aligned} \tag{F.20}$$

and [2]:

$$Z_{F^2} \frac{2g\frac{\partial}{\partial g}\left(\frac{\beta(g)}{g}\right) - 4\frac{\beta(g)}{g}}{\epsilon} = -\frac{4\beta_1 g^4}{\epsilon} + \cdots \tag{F.21}$$

so that to order g^4:

$$\begin{aligned} C_1^{(F^2,F^2)}(p) &= 4 - 4B_{1,1}g^2 - 4B_{1,2}g^4 \\ &\quad -4\beta_0 g^2 \log\frac{p^2}{\mu^2} + 4\beta_0^2 g^4 \log^2\frac{p^2}{\mu^2} + 4B_{1,3}g^4 \log\frac{p^2}{\mu^2} + \cdots \end{aligned} \tag{F.22}$$

The fully renormalized coefficient $C_1^{(F^2,F^2)}(p)$ in eq. (F.22) must be a solution of eq. (F.6) and, equivalently, eq. (F.3) for $\epsilon \to 0$. Indeed, we straightforwardly verify it inserting eq. (F.22) into eq. (F.6) and employing for $O = F^2$:

$$\begin{aligned} \gamma_{1\text{c.t.}}(g) &= (-\epsilon g + \beta(g))\frac{\partial Z_{1\text{c.t.}}}{\partial g} + \gamma_{F^2} Z_{1\text{c.t.}} \\ &= -\epsilon g \frac{\partial}{\partial g}\left(-\frac{4\beta_1 g^4}{\epsilon}\right) + \cdots \\ &= 16\beta_1 g^4 + \cdots \end{aligned} \tag{F.23}$$

that follows from eq. (F.4) with $Z_{1\text{c.t.}}$ given by eq. (F.21).

We also verify that $C_1^{(F^2,F^2)}(p)$ in eq. (F.22) is of the form in eq. (F.11) with the RG-invariant coefficient for $O = F^2$:

$$\mathcal{G}^{(F^2)}(g(p)) = 4 - 4B_{1,1}g^2(p) - 4B_{1,2}g^4(p) + \cdots \tag{F.24}$$

Indeed, for $O = F^2$ the perturbative expansion of eq. (F.11) reads:

$$\begin{aligned}
C_1^{(F^2,F^2)}(p,\mu,g(\mu)) &= \frac{g(\mu)}{\beta(g(\mu))}\Big\{\frac{\beta(g(p))}{g(p)}\mathcal{G}^{(F^2)}(g(p))-\int_{g(\mu)}^{g(p)}\frac{\gamma_{1\text{c.t.}}(g)}{g}\,dg\Big\}\\
&= \frac{-1}{\beta_0 g^2}\Big(1-\frac{\beta_1}{\beta_0}g^2\Big)\Big\{-\beta_0 g^2(p)\Big(1+\frac{\beta_1}{\beta_0}g^2(p)\Big)\Big(4-4B_{1,1}g^2(p)-4B_{1,2}g^4(p)\Big)\\
&\quad -4\beta_1 g^4\Big(\frac{g^4(p)}{g^4}-1\Big)\Big\}+\cdots\\
&= \frac{g^2(p)}{g^2}\Big(1+\frac{\beta_1}{\beta_0}g^2\Big(\frac{g^2(p)}{g^2}-1\Big)\Big)\Big(4-4B_{1,1}g^2(p)-4B_{1,2}g^4(p)\Big)\\
&\quad +4\frac{\beta_1}{\beta_0}g^2\Big(\frac{g^4(p)}{g^4}-1\Big)+\cdots\\
&= \left(1-g^2\beta_0\log\frac{p^2}{\mu^2}+g^4\left(-2\beta_1\log\frac{p^2}{\mu^2}+\beta_0^2\log^2\frac{p^2}{\mu^2}\right)\right)\\
&\quad \left(4-4B_{1,1}g^2\left(1-g^2\beta_0\log\frac{p^2}{\mu^2}\right)-4B_{1,2}g^4\right)-8\beta_1 g^4\log\frac{p^2}{\mu^2}+\cdots\\
&= 4-4B_{1,1}g^2-4B_{1,2}g^4\\
&\quad -4\beta_0 g^2\log\frac{p^2}{\mu^2}+4\beta_0^2 g^4\log^2\frac{p^2}{\mu^2}+(-8\beta_1+8\beta_0 B_{1,1})g^4\log\frac{p^2}{\mu^2}\\
&\quad -8\beta_1 g^4\log\frac{p^2}{\mu^2}+\cdots\\
&= 4-4B_{1,1}g^2-4B_{1,2}g^4\\
&\quad -4\beta_0 g^2\log\frac{p^2}{\mu^2}+4\beta_0^2 g^4\log^2\frac{p^2}{\mu^2}+4B_{1,3}g^4\log\frac{p^2}{\mu^2}+\cdots
\end{aligned} \tag{F.25}$$

where we have employed in the second equality $\beta(g) = -\beta_0 g^3 - \beta_1 g^4 + \cdots$, eqs. (F.23) and (F.24), the perturbative expansions for the running coupling $g(p)$ in terms of g:

$$\begin{aligned}
g^2(p) &= g^2\Big(1-g^2\beta_0\log\frac{p^2}{\mu^2}+g^4\big(-\beta_1\log\frac{p^2}{\mu^2}+\beta_0^2\log^2\frac{p^2}{\mu^2}\big)+\cdots\Big)\\
g^4(p) &= g^4\Big(1-g^2 2\beta_0\log\frac{p^2}{\mu^2}+g^4\big(-2\beta_1\log\frac{p^2}{\mu^2}+3\beta_0^2\log^2\frac{p^2}{\mu^2}\big)+\cdots\Big)
\end{aligned} \tag{F.26}$$

in the fourth equality and in the last equality $B_{1,3} = -4\beta_1 + 2\beta_0 B_{1,1}$, with $B_{1,3}$ and $B_{1,1}$ computed in [3] and reported in eq. (4.26). The last equality in eq. (F.25) reproduces the perturbative result in eq. (F.22) to order g^4, as it should be.

F.4 Callan-Symanzik equation for $C_0^{(O,O)}(p)$

The CS equation in dimensional regularization for the fully renormalized OPE coefficient $C_0^{(O,O)}(p)$ in the momentum representation:[3]

$$C_0^{(O,O)}(p) = Z_O^2 C_0^{(O_0,O_0)}(p) + p^{2\Delta_{O_0}-4}\mu^{2(1-\delta_O)\epsilon} Z_{0\text{c.t.}} \tag{F.27}$$

[3] See [32] for an analogous example.

with $\tilde{\Delta}_{O_0} = \Delta_{O_0} - \delta_O \epsilon$ the canonical dimension of O in $\tilde{d} = 4 - 2\epsilon$ dimensions, follows from the renormalization-scale independence of the bare coefficient:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g,\epsilon)\frac{\partial}{\partial g} + 2\gamma_O(g)\right) C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4}\mu^{2(1-\delta_O)\epsilon}\gamma_{0\text{c.t.}}(g) \tag{F.28}$$

with:

$$\mu \frac{d}{d\mu} = \mu \frac{\partial}{\partial \mu} + \beta(g,\epsilon)\frac{\partial}{\partial g} \tag{F.29}$$

and:

$$\begin{aligned} \gamma_{0\text{c.t.}}(g) &= \mu \frac{dZ_{0\text{c.t.}}}{d\mu} + 2\gamma_O Z_{0\text{c.t.}} + 2(1-\delta_O)\epsilon Z_{0\text{c.t.}} \\ &= \beta(g,\epsilon)\frac{\partial Z_{0\text{c.t.}}}{\partial g} + 2\gamma_O Z_{0\text{c.t.}} + 2(1-\delta_O)\epsilon Z_{0\text{c.t.}} \end{aligned} \tag{F.30}$$

where $\gamma_{0\text{c.t.}}$ is finite as $\epsilon \to 0$ and ϵ independent analogously to $\gamma_{1\text{c.t.}}$ in eq. (F.4). For the dimensionless object $C_{0,DL}^{(O,O)}$:

$$C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4}\mu^{2(1-\delta_O)\epsilon} C_{0,DL}^{(O,O)}(\frac{p}{\mu}, g(\mu)) \tag{F.31}$$

that solely depends on the ratio $\frac{p}{\mu}$, we also write:

$$\left(p \cdot \frac{\partial}{\partial p} - \beta(g,\epsilon)\frac{\partial}{\partial g} - 2\gamma_O(g)\right) C_{0,DL}^{(O,O)}(\frac{p}{\mu}, g(\mu)) = -\gamma_{0\text{c.t.}}(g) \tag{F.32}$$

The CS equation in $d = 4$ dimensions is obtained from eq. (F.28) by removing the explicit ϵ dependence. Hence, for $d = 4$:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g)\frac{\partial}{\partial g} + 2\gamma_O(g)\right) C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4}\gamma_{0\text{c.t.}}(g) \tag{F.33}$$

F.5 General solution for $C_0^{(O,O)}(p)$

The general solution of eq. (F.28) is:

$$C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4}\mu^{2(1-\delta_O)\epsilon} Z^{(O)^2}(\tilde{g}(p),g(\mu))\Big(\mathcal{G}_2^{(O)}(\tilde{g}(p)) + \Delta_2^{(O)}(\tilde{g}(p),g(\mu))\Big) \tag{F.34}$$

where:

$$Z^{(O)}(\tilde{g}(p),g(\mu)) = \exp \int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_O(g)}{\beta(g,\epsilon)} dg \tag{F.35}$$

and:

$$\begin{aligned} \Delta_2^{(O)}(\tilde{g}(p),g(\mu)) &= -\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{0\text{c.t.}}(g)}{\beta(g,\epsilon)} \exp\left(-2\int_g^{\tilde{g}(p)} \frac{\gamma_O(g')}{\beta(g',\epsilon)}dg'\right) dg \\ &= -\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{0\text{c.t.}}(g)}{\beta(g,\epsilon)} Z^{(O)^{-2}}(\tilde{g}(p),g)\, dg \end{aligned} \tag{F.36}$$

JHEP08(2024)145

In $d=4$ dimensions, the solution of eq. (F.33) reads:

$$C_0^{(O,O)}(p,\mu,g(\mu)) = p^{2\Delta_{O_0}-4} Z^{(O)^2}(g(p),g(\mu))\Big(\mathcal{G}_2^{(O)}(g(p)) + \Delta_2^{(O)}(g(p),g(\mu))\Big) \tag{F.37}$$

where:

$$Z^{(O)}(g(p),g(\mu)) = \exp \int_{g(\mu)}^{g(p)} \frac{\gamma_O(g)}{\beta(g)} dg \tag{F.38}$$

and:

$$\begin{aligned}\Delta_2^{(O)}(g(p),g(\mu)) &= -\int_{g(\mu)}^{g(p)} \frac{\gamma_{0\mathrm{c.t.}}(g)}{\beta(g)} \exp\Big(-2\int_g^{g(p)} \frac{\gamma_O(g')}{\beta(g')}dg'\Big)\,dg \\ &= -\int_{g(\mu)}^{g(p)} \frac{\gamma_{0\mathrm{c.t.}}(g)}{\beta(g)} Z^{(O)^{-2}}(g(p),g)\,dg\end{aligned} \tag{F.39}$$

Equivalently, eq. (F.37) is obtained by taking $\epsilon \to 0$ in eq. (F.34).

F.6 Specializing to $C_0^{(F^2,F^2)}(p)$

Eq. (F.27) reads for $O = F^2$:

$$C_0^{(F^2,F^2)}(p) = Z_{F^2}^2 C_0^{(F_0^2,F_0^2)}(p) + p^4\mu^{-2\epsilon} Z_{0\mathrm{c.t.}} \tag{F.40}$$

where $\tilde{\Delta}_{F_0^2} = \Delta_{F_0^2} - \delta_{F^2}\epsilon = \tilde{d} = 4-2\epsilon$ is the canonical dimension of F^2 in $\tilde{d} = 4-2\epsilon$ dimensions. The general solution of eq. (F.28) for $O = F^2$ then reads:

$$C_0^{(F^2,F^2)}(p,\mu,g(\mu)) = p^4\mu^{-2\epsilon} Z^{(F^2)^2}(\tilde{g}(p),g(\mu))\Big(\mathcal{G}_2^{(F^2)}(\tilde{g}(p)) + \Delta_2^{(F^2)}(\tilde{g}(p),g(\mu))\Big) \tag{F.41}$$

with $Z^{(F^2)}$ in eq. (F.8):

$$\begin{aligned}Z^{(F^2)}(\tilde{g}(p),g(\mu)) &= \frac{\beta(\tilde{g}(p),\epsilon)}{\tilde{g}(p)} \frac{g(\mu)}{\beta(g(\mu),\epsilon)} \\ &\sim \frac{\tilde{g}^2(p)}{g^2(\mu)}\Big(\frac{\mu}{p}\Big)^{-2\epsilon} \\ &= \frac{1}{1-\beta_0 g^2(\mu)\frac{(\mu/p)^{2\epsilon}-1}{\epsilon}+\cdots} \\ &= 1-\beta_0 g^2(\mu)\log\frac{p^2}{\mu^2}+\cdots\end{aligned} \tag{F.42}$$

where we have employed eq. (D.23) with the replacement $z \to \frac{1}{p}$ and its expansion in $g(\mu)$ as $\epsilon \to 0$, and:

$$\begin{aligned}\Delta_2^{(F^2)}(\tilde{g}(p),g(\mu)) &= -\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{0\mathrm{c.t.}}(g)}{\beta(g,\epsilon)} \exp\Big(-2\int_g^{\tilde{g}(p)} \frac{\gamma_{F^2}(g')}{\beta(g',\epsilon)}dg'\Big)\,dg \\ &= -\int_{g(\mu)}^{\tilde{g}(p)} \frac{\gamma_{0\mathrm{c.t.}}(g)}{\beta(g,\epsilon)} Z^{(F^2)^{-2}}(\tilde{g}(p),g)\,dg\end{aligned} \tag{F.43}$$

Given $Z_{0\text{c.t.}}$ to order g^2 according to eq. (7.13):

$$Z_{0\text{c.t.}} = \frac{N^2-1}{4\pi^2}\left\{-\frac{1}{\epsilon}+\frac{g^2\beta_0}{\epsilon^2}+\frac{g^2 c_1}{\epsilon}\right\}+\cdots \tag{F.44}$$

we obtain for $\gamma_{0\text{c.t.}}$ in eq. (F.30) for $O = F^2$ to order g^2:

$$\begin{aligned}
\gamma_{0\text{c.t.}}(g) &= \beta(g,\epsilon)\frac{\partial Z_{0\text{c.t.}}}{\partial g}+2\gamma_{F^2}Z_{0\text{c.t.}}-2\epsilon Z_{0\text{c.t.}} \\
&= \frac{N^2-1}{4\pi^2}\left\{-\epsilon g\, 2g\left(\frac{c_1}{\epsilon}+\frac{\beta_0}{\epsilon^2}\right)+\frac{4\beta_0 g^2}{\epsilon}-2\epsilon\left(-\frac{1}{\epsilon}+\frac{g^2 c_1}{\epsilon}+\frac{g^2\beta_0}{\epsilon^2}\right)+\cdots\right\} \\
&= \frac{N^2-1}{4\pi^2}\left\{2-4c_1 g^2+\cdots\right\}
\end{aligned} \tag{F.45}$$

that is, indeed, ϵ independent to order g^2. The solution in $d = 4$ dimensions is obtained from eq. (F.37) for $O = F^2$:

$$C_0^{(F^2,F^2)}(p,\mu,g(\mu)) = p^4 Z^{(F^2)^2}(g(p),g(\mu))\Big(\mathcal{G}_2^{(F^2)}(g(p))+\Delta_2^{(F^2)}(g(p),g(\mu))\Big) \tag{F.46}$$

where:

$$\begin{aligned}
Z^{(F^2)}(g(p),g(\mu)) &= \exp\int_{g(\mu)}^{g(p)}\frac{\gamma_{F^2}(g)}{\beta(g)}dg \\
&= \frac{\beta(g(p))}{g(p)}\frac{g(\mu)}{\beta(g(\mu))} \\
&= \frac{-\beta_0 g^2(p)+\cdots}{-\beta_0 g^2(\mu)+\cdots} \\
&= 1-\beta_0 g^2(\mu)\log\frac{p^2}{\mu^2}+\cdots
\end{aligned} \tag{F.47}$$

and the RG-invariant coefficient:

$$\begin{aligned}
\mathcal{G}_2^{(F^2)}(g(p)) &= \bar{\mathcal{G}}_2^{(F^2)}(0)(1+\eta_2^{(F^2)}g^2(p)+\cdots) \\
&= \bar{\mathcal{G}}_2^{(F^2)}(0)(1+\eta_2^{(F^2)}g^2(\mu)+\cdots)
\end{aligned} \tag{F.48}$$

have been perturbatively expanded to order g^2, and:

$$\Delta_2^{(F^2)}(g(p),g(\mu)) = -\int_{g(\mu)}^{g(p)}\frac{\gamma_{0\text{c.t.}}(g)}{\beta(g)}Z^{(F^2)^{-2}}(g(p),g)\,dg \tag{F.49}$$

so that:

$$Z^{(F^2)^2}(g(p),g(\mu))\Delta_2^{(F^2)}(g(p),g(\mu)) = -\left(\frac{g(\mu)}{\beta(g(\mu))}\right)^2\int_{g(\mu)}^{g(p)}\gamma_{0\text{c.t.}}(g)\frac{\beta(g)}{g^2}\,dg \tag{F.50}$$

Inserting eq. (F.45) into eq. (F.50) we further obtain to order g^2:

$$
\begin{aligned}
&Z^{(F^2)^2}(g(p),g(\mu))\Delta_2^{(F^2)}(g(p),g(\mu)) \\
&= -\frac{1}{\beta_0^2 g^4(\mu)}\Big(1-2\frac{\beta_1}{\beta_0}g^2(\mu)+\cdots\Big)\frac{N^2-1}{4\pi^2}\int_{g(\mu)}^{g(p)}(2-4c_1g^2+\cdots)(-\beta_0 g-\beta_1 g^3+\cdots)\,dg \\
&= -\frac{1}{\beta_0^2 g^4(\mu)}\Big(1-2\frac{\beta_1}{\beta_0}g^2(\mu)+\cdots\Big)\frac{N^2-1}{4\pi^2} \\
&\quad\Big\{-\beta_0\Big(g^2(p)-g^2(\mu)\Big)+\Big(c_1\beta_0-\frac{1}{2}\beta_1\Big)\Big(g^4(p)-g^4(\mu)\Big)+\cdots\Big\} \\
&= -\frac{1}{\beta_0^2 g^4(\mu)}\Big(1-2\frac{\beta_1}{\beta_0}g^2(\mu)+\cdots\Big)\frac{N^2-1}{4\pi^2} \\
&\quad\Big\{-\beta_0\Big(-g^4(\mu)\beta_0\log\frac{p^2}{\mu^2}+g^6(\mu)\Big(-\beta_1\log\frac{p^2}{\mu^2}+\beta_0^2\log^2\frac{p^2}{\mu^2}\Big)+\cdots\Big) \\
&\quad+\Big(c_1\beta_0-\frac{1}{2}\beta_1\Big)\Big(-2g^6(\mu)\beta_0\log\frac{p^2}{\mu^2}+\cdots\Big)+\cdots\Big\} \\
&= \frac{N^2-1}{4\pi^2}\frac{1}{\beta_0^2 g^4(\mu)}\Big\{-\beta_0^2 g^4(\mu)\log\frac{p^2}{\mu^2}+g^6(\mu)\Big(\beta_0^3\log^2\frac{p^2}{\mu^2}+2c_1\beta_0^2\log\frac{p^2}{\mu^2}\Big)+\cdots\Big\} \\
&= \frac{N^2-1}{4\pi^2}\Big\{-\log\frac{p^2}{\mu^2}+\beta_0 g^2(\mu)\log^2\frac{p^2}{\mu^2}+2c_1g^2(\mu)\log\frac{p^2}{\mu^2}+\cdots\Big\}
\end{aligned}
\tag{F.51}
$$

where we have employed eq. (F.26). Hence, setting $g(\mu)=g$, the complete solution in eq. (F.46) reads to order g^2:

$$
\begin{aligned}
C_0^{(F^2,F^2)}(p,\mu,g) = p^4\Big\{&\bar{\mathcal{G}}_2^{(F^2)}(0)\Big(1-2\beta_0 g^2\log\frac{p^2}{\mu^2}+\cdots\Big)\Big(1+\eta_2^{(F^2)}g^2+\cdots\Big) \\
&+\frac{N^2-1}{4\pi^2}\Big(-\log\frac{p^2}{\mu^2}+\beta_0 g^2\log^2\frac{p^2}{\mu^2}+2c_1g^2\log\frac{p^2}{\mu^2}\Big)+\cdots\Big\}
\end{aligned}
\tag{F.52}
$$

For:

$$
\bar{\mathcal{G}}_2^{(F^2)}(0) = \frac{N^2-1}{4\pi^2} \tag{F.53}
$$

we get:

$$
\begin{aligned}
C_0^{(F^2,F^2)}(p,\mu,g) = p^4\frac{N^2-1}{4\pi^2}\Big\{&1-\log\frac{p^2}{\mu^2}+g^2\beta_0\log^2\frac{p^2}{\mu^2} \\
&+2g^2(c_1-\beta_0)\log\frac{p^2}{\mu^2}+g^2\eta_2^{(F^2)}+\cdots\Big\}
\end{aligned}
\tag{F.54}
$$

Hence, for a given value of the scheme-dependent coefficient c_1 entering the additive renormalization in eq. (F.44), the scheme-dependent coefficient of the subleading $\log\frac{p^2}{\mu^2}$ to order g^2 that solves the CS equation is $2(c_1-\beta_0)$. Eq. (7.11), given eqs. (7.12) and (7.13), is of the form in eq. (F.54) and it is a solution of the CS equation for any value of c_1.

References

[1] M. Bochicchio, *The large-N Yang-Mills S-matrix is ultraviolet finite, but the large-N QCD S-matrix is only renormalizable*, *Phys. Rev. D* **95** (2017) 054010 [arXiv:1701.07833] [INSPIRE].

[2] M.F. Zoller, *On the renormalization of operator products: the scalar gluonic case*, *JHEP* **04** (2016) 165 [arXiv:1601.08094] [INSPIRE].

[3] M.F. Zoller and K.G. Chetyrkin, *OPE of the energy-momentum tensor correlator in massless QCD*, *JHEP* **12** (2012) 119 [arXiv:1209.1516] [INSPIRE].

[4] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *In Search of Scalar Gluonium*, *Nucl. Phys. B* **165** (1980) 67 [INSPIRE].

[5] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Are All Hadrons Alike?* , *Nucl. Phys. B* **191** (1981) 301 [INSPIRE].

[6] M. Bochicchio and E. Pallante, *Low-energy theorem and OPE in the conformal window of massless QCD*, arXiv:2201.10970, v4 to appear on arXiv [INSPIRE].

[7] G. 't Hooft, *A Planar Diagram Theory for Strong Interactions*, *Nucl. Phys. B* **72** (1974) 461 [INSPIRE].

[8] G. Veneziano, *Some Aspects of a Unified Approach to Gauge, Dual and Gribov Theories*, *Nucl. Phys. B* **117** (1976) 519 [INSPIRE].

[9] A.A. Migdal, *Multicolor QCD as Dual Resonance Theory*, *Annals Phys.* **109** (1977) 365 [INSPIRE].

[10] E. Witten, *Baryons in the 1/n Expansion*, *Nucl. Phys. B* **160** (1979) 57 [INSPIRE].

[11] M. Bochicchio, *Renormalization in large-N QCD is incompatible with open/closed string duality*, *Phys. Lett. B* **783** (2018) 341 [arXiv:1703.10176] [INSPIRE].

[12] M. Becchetti and M. Bochicchio, *OPE and a low-energy theorem in QCD-like theories*, *JHEP* **03** (2019) 088 [arXiv:1810.08527] [INSPIRE].

[13] A.L. Kataev, N.V. Krasnikov and A.A. Pivovarov, *Two Loop Calculations for the Propagators of Gluonic Currents*, *Nucl. Phys. B* **198** (1982) 508 [hep-ph/9612326] [INSPIRE].

[14] M.F. Zoller, *OPE of the energy-momentum tensor correlator and the gluon condensate operator in massless QCD to three-loop order*, *JHEP* **10** (2014) 169 [arXiv:1407.6921] [INSPIRE].

[15] M.B. Fröb, *Recursive construction of the operator product expansion in curved space*, *JHEP* **02** (2021) 195 [arXiv:2007.15668] [INSPIRE].

[16] S.D. Joglekar and B.W. Lee, *General Theory of Renormalization of Gauge Invariant Operators*, *Annals Phys.* **97** (1976) 160 [INSPIRE].

[17] M. Henneaux, *Remarks on the renormalization of gauge invariant operators in Yang-Mills theory*, *Phys. Lett. B* **313** (1993) 35 [*Erratum ibid.* **316** (1993) 633] [hep-th/9306101] [INSPIRE].

[18] J.C. Collins and R.J. Scalise, *The renormalization of composite operators in Yang-Mills theories using a general covariant gauge*, *Phys. Rev. D* **50** (1994) 4117 [hep-ph/9403231] [INSPIRE].

[19] J.C. Collins, *Renormalization*, Cambridge monographs on mathematical physics, Cambridge University Press (1984).

[20] V.P. Spiridonov, *Anomalous Dimension of $G^2_{\mu\nu}$ and β-function*, IYaI-P-0378 (1984), https://lib-extopc.kek.jp/preprints/PDF/1986/8601/8601315.pdf.

[21] V.M. Braun, G.P. Korchemsky and D. Müller, *The uses of conformal symmetry in QCD*, *Prog. Part. Nucl. Phys.* **51** (2003) 311 [hep-ph/0306057] [INSPIRE].

[22] M. Bochicchio, *On the geometry of operator mixing in massless QCD-like theories*, *Eur. Phys. J. C* **81** (2021) 749 [arXiv:2103.15527] [INSPIRE].

[23] M. Becchetti and M. Bochicchio, *Operator mixing in massless QCD-like theories and Poincarè-Dulac theorem*, *Eur. Phys. J. C* **82** (2022) 866 [arXiv:2103.16220] [INSPIRE].

[24] M. Bochicchio and S.P. Muscinelli, *Ultraviolet asymptotics of glueball propagators*, *JHEP* **08** (2013) 064 [arXiv:1304.6409] [INSPIRE].

[25] T. Nunes da Silva, E. Pallante and L. Robroek, *The scalar glueball operator, the a-theorem, and the onset of conformality*, *Phys. Lett. B* **778** (2018) 316 [arXiv:1609.06298] [INSPIRE].

[26] A. Bzowski, P. McFadden and K. Skenderis, *Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies*, *JHEP* **03** (2016) 066 [arXiv:1510.08442] [INSPIRE].

[27] C.G. Callan Jr., *Broken scale invariance in scalar field theory*, *Phys. Rev. D* **2** (1970) 1541 [INSPIRE].

[28] K. Symanzik, *Small distance behavior in field theory and power counting*, *Commun. Math. Phys.* **18** (1970) 227 [INSPIRE].

[29] C. Itzykson nad J.B. Zuber, *Quantum Field Theory*, McGraw-Hill (1985).

[30] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press (1995).

[31] M. Bochicchio, *Glueball and meson propagators of any spin in large-N QCD*, *Nucl. Phys. B* **875** (2013) 621 [arXiv:1305.0273] [INSPIRE].

[32] K.G. Chetyrkin and A. Maier, *Massless correlators of vector, scalar and tensor currents in position space at orders α_s^3 and α_s^4: explicit analytical results*, *Nucl. Phys. B* **844** (2011) 266 [arXiv:1010.1145] [INSPIRE].

JHEP08(2024)145