

Antenna subtraction for processes with identified particles at hadron colliders

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ABSTRACT: Collider processes with identified hadrons in the final state are widely studied in view of determining details of the proton structure and of understanding hadronization. Their theory description requires the introduction of fragmentation functions, which parametrise the transition of a produced parton into the identified hadron. To compute higher-order perturbative corrections to these processes requires a subtraction method for infrared singular configurations. We extend the antenna subtraction method to hadron fragmentation processes in hadronic collisions up to next-to-next-to-leading order (NNLO) in QCD by computing the required fragmentation antenna functions in initial-final kinematics. The integrated antenna functions retain their dependence on the momentum fractions of the incoming and fragmenting partons.

KEYWORDS: Higher-Order Perturbative Calculations, Jets and Jet Substructure, Specific QCD Phenomenology

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1 Introduction

The transition from a parton into a hadron (fragmentation) is a non-perturbative process, whose probability is described by a fragmentation function (FF, [1, 2]). Like parton distribution functions (PDFs), these FFs fulfil perturbative evolution equations [3] in their resolution scales, whose non-perturbative initial conditions can not be determined from first principles. Instead, FFs are determined from global fits [4–12] to data on the production of a selected hadron species in various collider processes. The FF evolution kernels are known [13, 14] to next-to-next-to-leading order (NNLO) in QCD.

In terms of hadron production observables, one distinguishes semi-inclusive observables, which are differential in the hadron momentum but fully inclusive in all other particles in the event, and exclusive observables, where the hadron is identified in final states that have been selected based on specific properties, such as the identification of a jet or a gauge boson. Hadron production in e^+e^- annihilation or deep-inelastic scattering has typically been studied in terms of semi-inclusive observables. In contrast, hadron production cross sections at hadron colliders are often of exclusive type, such as gauge-boson-plus hadron production or identified hadron spectra inside jets.

The coefficient functions for semi-inclusive hadron production are known to NNLO accuracy in compact analytical form for e^+e^- annihilation [15–17] and for deeply-inelastic lepton-proton scattering [15, 18–22], thereby allowing for the consistent inclusion of the respective data sets into global FF fits at NNLO.

Semi-inclusive hadron production at hadron colliders [23] as well as any type of exclusive identified hadron production cross section can not be computed to higher orders using the methods that yielded the respective semi-inclusive coefficient functions in e^+e^- annihilation or lepton-proton scattering. Instead, these processes require a numerical implementation of all parton-level subprocess contributions to a given perturbative order, using an appropriate method to identify and recombine infrared singular contributions among different subprocesses.

Generic subtraction methods for NLO [24, 25] and NNLO [26–37] calculations are available and have been used widely for jet cross sections. For processes involving hadron fragmentation, any subtraction method requires an extension in order to keep track of parton momentum fractions in unresolved emissions, which are usually integrated over. Such an extension is available at NLO for dipole subtraction [24]. At NNLO, recent work towards fragmentation processes yielded results for heavy hadron production in top quark decays [38] in the residue subtraction method [30] and photon fragmentation [39, 40] in the antenna subtraction method [26, 27, 41].

Subsequently, the antenna subtraction method was extended [42] to accommodate generic hadron fragmentation processes by deriving the required phase space factorizations and mappings and by devising the construction of the antenna subtraction terms. The newly introduced fragmentation antenna functions contain one parton with identified momentum fraction in the final state and another radiator parton in the final state (final-final fragmentation antenna) or in the initial state (initial-final fragmentation antenna). The unintegrated forms of these antenna functions are identical to their inclusive counterparts [26], while their integration must keep proper track of the identified parton’s momentum fraction.

In [42], the full set of integrated final-final fragmentation antenna functions was derived. A subset of integrated initial-final fragmentation antenna functions was computed in [39], focusing on identified photons in the final state. In this paper, we complete the construction of the NNLO antenna subtraction method for processes with identified hadrons. We derive the full set of integrated initial-final fragmentation antenna functions and we describe how the integrated antenna functions can be systematically combined with mass factorization counter-terms for PDFs and FFs to yield so-called dipole operators [27, 43]. Such dipole operators conveniently combine with purely virtual corrections, thereby ensuring the infrared finiteness of the numerical implementation of all parton-level contributions to a given observable.

In section 2, we briefly recapitulate the formulation of the antenna subtraction method for fragmentation processes [39, 42]. The integration of the fragmentation antenna functions in initial-final kinematics is described in detail in section 3, with the relevant master integrals tabulated in appendix A. The resulting integrated antenna functions are combined with mass factorization counter-terms, described in appendix B, to yield integrated fragmentation dipoles in section 4. We conclude with an outlook in section 5.

2 Antenna subtraction for fragmentation processes

The generic form of a cross section with an identified final-state hadron is given as convolution over parton-level cross sections with PDFs and FFs:

$$d\sigma^H = \sum_{i,j,p} d\hat{\sigma}_{ij}^p(\xi_1, \xi_2, \eta, \mu_r, \mu_f, \mu_a) \otimes f_i(\xi_1, \mu_f) \otimes f_j(\xi_2, \mu_f) \otimes D_p^H(\eta, \mu_a), \quad (2.1)$$

where $f_{i,j}$ denote the parton distributions in the two incoming hadrons, defined at the factorization scale μ_f and D_p^H is the fragmentation function describing the transition of parton p into hadron H , defined at factorization scale μ_a . The parton-level cross section $d\hat{\sigma}$ for the process $i + j \rightarrow p + X$ contains the full definition of the selected final state, including specific cuts on X , which can select other particles (such as gauge bosons) accompanying the identified hadron or involve a jet reconstruction. It can be computed in perturbative QCD as an expansion in the renormalized coupling constant α_s , defined at the renormalization scale μ_r :

$$d\hat{\sigma} = d\hat{\sigma}_{\text{LO}} + \left(\frac{\alpha_s}{2\pi}\right) d\hat{\sigma}_{\text{NLO}} + \left(\frac{\alpha_s}{2\pi}\right)^2 d\hat{\sigma}_{\text{NNLO}} + \dots, \quad (2.2)$$

with the leading order (Born-level) cross section

$$d\hat{\sigma}_{\text{LO}} = \mathcal{N}_B \int d\Phi_n(k_1, \dots, k_p, \dots, k_n; p_i, p_j) \times \frac{1}{S_n} M_n^0(k_1, \dots, k_p, \dots, k_n; p_i, p_j) J(k_1, \dots, k_p, \dots, k_n; p_i, p_j; \xi_1, \xi_2, \eta), \quad (2.3)$$

where \mathcal{N}_B is the Born-level normalization factor (potentially containing further powers of α_s), Φ_n is the Born-level phase space, S_n is the symmetry factor appropriate for the final state, M^0 represents the Born-level squared matrix element and J the final state definition, based on the Born-level kinematics of the parton-level process $\{k_1, \dots, k_p, \dots, k_n; p_i, p_j\}$ and the incoming (p_i/ξ_1 , p_j/ξ_2) and outgoing (ηk_p) hadron momenta.

At higher orders in perturbation theory, the parton-level cross sections receive contributions from real and virtual corrections (or combinations thereof), which contain infrared singularities and are defined on phase-spaces of varying multiplicity ($n + i$ for i real emissions relative to the Born-level process). These singularities are made explicit by working in dimensional regularization in $d = 4 - 2\epsilon$ dimensions. The PDFs and FFs are redefined by mass factorization (absorption of initial- and final-state real radiation singularities) at each order. Only the sum of all contributions at a given order is infrared-finite and well-defined. In order to extract the infrared singularities from each subprocess, one typically applies a subtraction scheme which introduces subtraction terms that enable to define finite remainders at the level of each phase space multiplicity.

The parton-level cross section at NLO reads as follows:

$$d\hat{\sigma}_{\text{NLO}} = \int_{n+1} \left(d\hat{\sigma}_{\text{NLO}}^R - d\hat{\sigma}_{\text{NLO}}^S \right) + \int_n \left(d\hat{\sigma}_{\text{NLO}}^V - d\hat{\sigma}_{\text{NLO}}^T \right), \quad (2.4)$$

where $d\hat{\sigma}_{\text{NLO}}^{R,V}$ denote the real and virtual NLO corrections to the Born-level process and $d\hat{\sigma}_{\text{NLO}}^{S,T}$ are the respective subtraction terms. These fulfil

$$d\hat{\sigma}_{\text{NLO}}^T = -d\hat{\sigma}_{\text{NLO}}^{MF} - \int_1 d\hat{\sigma}_{\text{NLO}}^S \quad (2.5)$$

with the NLO mass factorization counter-term contribution $d\hat{\sigma}_{\text{NLO}}^{MF}$ and \int_1 denoting the analytical integration over the phase space relevant to the emission of one extra parton. Likewise, at NNLO we have:

$$\begin{aligned}
 d\hat{\sigma}_{\text{NNLO}} = & \int_{n+2} \left(d\hat{\sigma}_{\text{NNLO}}^{RR} - d\hat{\sigma}_{\text{NNLO}}^S \right) \\
 & + \int_{n+1} \left(d\hat{\sigma}_{\text{NNLO}}^{RV} - d\hat{\sigma}_{\text{NNLO}}^T \right) \\
 & + \int_n \left(d\hat{\sigma}_{\text{NNLO}}^{VV} - d\hat{\sigma}_{\text{NNLO}}^U \right), \tag{2.6}
 \end{aligned}$$

with $d\hat{\sigma}_{\text{NNLO}}^{RR,RV,VV}$ being the double-real, real-virtual and double-virtual corrections to the Born-level process and $d\hat{\sigma}_{\text{NNLO}}^{S,T,U}$ the respective subtraction terms. These are related as follows:

$$\begin{aligned}
 d\hat{\sigma}_{\text{NNLO}}^S &= d\hat{\sigma}_{\text{NNLO}}^{S,1} + d\hat{\sigma}_{\text{NNLO}}^{S,2} \\
 d\hat{\sigma}_{\text{NNLO}}^T &= d\hat{\sigma}_{\text{NNLO}}^{VS,1} - d\hat{\sigma}_{\text{NNLO}}^{MF,1} - \int_1 d\hat{\sigma}_{\text{NNLO}}^{S,1} \\
 d\hat{\sigma}_{\text{NNLO}}^U &= -d\hat{\sigma}_{\text{NNLO}}^{MF,2} - \int_1 d\hat{\sigma}_{\text{NNLO}}^{VS,1} - \int_2 d\hat{\sigma}_{\text{NNLO}}^{S,2}. \tag{2.7}
 \end{aligned}$$

In here, $d\hat{\sigma}_{\text{NNLO}}^{MF,1}$ and $d\hat{\sigma}_{\text{NNLO}}^{MF,2}$ are the mass factorization terms for one and two unresolved emissions and $\int_{1,2}$ are analytical integrations over the single- and double-real emission components of the phase space. Each integration in (2.4) and (2.6) is numerically well-defined and finite. It can be implemented in a numerical parton-level event generator code that analyses each parton-level momentum configuration according to the observable definition J .

The antenna subtraction method [26, 27, 41] describes a systematic procedure to construct the subtraction terms up to NNLO in QCD. The method is based on exact factorizations of the $(n + 1)$ and $(n + 2)$ particle phase spaces into a reduced n -particle phase space and an antenna phase space, which allows to build real radiation subtraction terms from products of antenna functions with lower-multiplicity matrix elements, evaluated over the reduced phase space. Owing to the phase space factorization, each antenna function can be integrated analytically over its corresponding antenna phase space, thus entering the construction of the virtual subtraction terms. An antenna function encapsulates all unresolved radiation between two colour-ordered hard radiator partons. For subtraction at NLO, only three-parton (two hard radiators, one unresolved parton) tree level antenna functions X_3^0 are required. At NNLO, these are supplemented by four-parton tree-level antenna functions X_4^0 and three-parton one-loop antenna functions X_3^1 as well as by large-angle single-soft antennae S [27]. The antenna functions depend on the parton species of the two hard radiators (quark-antiquark [44], quark-gluon [45] or gluon-gluon [46]) and are derived from matrix elements for the decay of a colour-neutral current to a three- or four-parton system. Alternatively, they can also be constructed in an iterative manner from the unresolved behaviour in all limits [47–49]. An extension of antenna subtraction to N3LO, involving up to five-parton antenna functions, has been put forward most recently [50–52]. Each of the two radiators can be in the initial or in the final state, yielding different integrated antenna functions for final-final [26], initial-final [53], and initial-initial [54, 55] cases. These integrated antenna functions are fully inclusive in the final-state radiation, but differential in the momentum fractions

of the initial-state radiation. They can be combined with the mass factorization terms of the PDFs to yield integrated antenna dipoles [27, 43] J_2^1 and J_2^2 that make the infrared pole structure of the integrated subtraction terms explicit and allow for an analytical cancellation of ϵ -poles between subtraction terms and virtual corrections to the matrix elements.

The extension of the antenna subtraction method to incorporate final-state hadron fragmentation requires to keep track of the momentum fraction of the fragmenting final-state parton in the phase space factorization and mappings, as well as in the resulting integrated antenna subtraction terms. It requires the introduction of fragmentation antenna functions [39, 42], which are differential in the momentum of one final-state parton, which is associated with a final-state radiator. The other radiator can be either in the final state (final-final fragmentation antenna functions) or in the initial state (initial-final fragmentation antenna functions). These fragmentation antenna functions appear in those contributions to the antenna subtraction terms that involve the fragmenting parton p , indicated by a superscript ‘id. p ’. In the following we briefly illustrate the structure of the antenna subtraction terms with identified final-state particles. We aim at a contained description of the key features of each subtraction component, rather than a detailed discussion of the whole antenna subtraction method, which is thoroughly documented in [26, 27, 43]. In particular, we highlight how fragmentation antenna functions are employed up to NNLO to subtract infrared divergences associated to the emission of unresolved radiation from the fragmenting parton. Clearly, for a generic process, standard antenna functions are also required to fully remove singularities originated from soft or collinear emission from other external legs.

2.1 Final-final configurations

At NLO, with fragmenting parton k_p and non-identified partons k_j, k_k all in the final state, we use three-parton tree-level fragmentation antenna function $X_3^{0,\text{id}.p}$ to construct the real subtraction term:

$$\begin{aligned} d\hat{\sigma}_{\text{NLO}}^{\text{S,id}.p} \supseteq & 8\pi^2 \mathcal{N}_{BD} d\Phi_{n+1}(k_1, \dots, k_p, k_j, k_k, \dots, k_{n+1}; p_i, p_j) \frac{1}{S_{n+1}} X_3^{0,\text{id}.p}(k_p, k_j, k_k) \\ & \times M_n^0(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j) \\ & \times J(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j; \xi_1, \xi_2, z\eta). \end{aligned} \quad (2.8)$$

The final-final phase space mapping [42] is defined by the momentum fraction z , used to define $\tilde{k}_p = k_p/z$, and by a recoil momentum \tilde{K} :

$$z = \frac{s_{pj} + s_{pk}}{s_{pj} + s_{pk} + s_{jk}}, \quad \tilde{K} = k_j + k_k - (1-z)\frac{k_p}{z}. \quad (2.9)$$

At NNLO for final-final kinematics, the removal of double-unresolved divergences at the real-real level is done using final-final four-parton tree-level fragmentation antenna functions $X_4^{0,\text{id}.p}$:

$$\begin{aligned} d\hat{\sigma}_{\text{NNLO}}^{\text{S,id}.p} \supseteq & (8\pi^2)^2 \mathcal{N}_{BD} d\Phi_{n+2}(k_1, \dots, k_p, k_j, k_k, k_l, \dots, k_{n+1}; p_i, p_j) \frac{1}{S_{n+1}} X_4^{0,\text{id}.p}(k_p, k_j, k_k, k_l) \\ & \times M_n^0(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j) \\ & \times J(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j; \xi_1, \xi_2, z\eta), \end{aligned} \quad (2.10)$$

now with

$$z = \frac{s_{pj} + s_{pk} + s_{pl}}{s_{pj} + s_{pk} + s_{jk} + s_{pl} + s_{jl} + s_{kl}}, \quad \tilde{K} = k_j + k_k + k_l - (1 - z) \frac{k_p}{z}. \quad (2.11)$$

The removal of single-unresolved divergences at the real-virtual level is done analogously to (2.8) using final-final one-loop fragmentation antenna functions $X_3^{1,\text{id},p}$:

$$\begin{aligned} d\hat{\sigma}_{\text{NNLO}}^{\text{T},\text{id},p} \supseteq & 8\pi^2 \mathcal{N}_B d\Phi_{n+1}(k_1, \dots, k_p, k_j, k_k, \dots, k_{n+1}; p_i, p_j) \frac{1}{S_{n+1}} X_3^{1,\text{id},p}(k_p, k_j, k_k) \\ & \times M_n^0(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j) \\ & \times J(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j; \xi_1, \xi_2, z\eta). \end{aligned} \quad (2.12)$$

The integration of the final-final fragmentation antenna functions is based on the phase space factorizations (with $q = k_p + k_j + k_k(+k_l)$ time-like):

$$\begin{aligned} d\Phi_{n+1}(k_1, \dots, k_p, k_j, k_k, \dots, k_{n+1}; p_i, p_j) &= d\Phi_n(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+1}; p_i, p_j) \\ & \times \frac{q^2}{2\pi} d\Phi_2(k_j, k_k; q - k_p) z^{1-2\epsilon} dz, \\ d\Phi_{n+2}(k_1, \dots, k_p, k_j, k_k, k_l, \dots, k_{n+2}; p_i, p_j) &= d\Phi_n(k_1, \dots, \tilde{k}_p, \tilde{K}, \dots, k_{n+2}; p_i, p_j) \\ & \times \frac{q^2}{2\pi} d\Phi_3(k_j, k_k, k_l; q - k_p) z^{1-2\epsilon} dz. \end{aligned} \quad (2.13)$$

These lead to the integrated antenna functions

$$\mathcal{X}_3^{0,\text{id},p}(z) = \frac{1}{C(\epsilon)} \int d\Phi_2(k_j, k_k; q - k_p) \frac{q^2}{2\pi} z^{1-2\epsilon} X_3^{0,\text{id},p}(k_p, k_j, k_k), \quad (2.14)$$

$$\mathcal{X}_3^{1,\text{id},p}(z) = \frac{1}{C(\epsilon)} \int d\Phi_2(k_j, k_k; q - k_p) \frac{q^2}{2\pi} z^{1-2\epsilon} X_3^{1,\text{id},p}(k_p, k_j, k_k), \quad (2.15)$$

$$\mathcal{X}_4^{0,\text{id},p}(z) = \frac{1}{C(\epsilon)^2} \int d\Phi_3(k_j, k_k, k_l; q - k_p) \frac{q^2}{2\pi} z^{1-2\epsilon} X_4^{0,\text{id},p}(k_p, k_j, k_k, k_l), \quad (2.16)$$

with

$$C(\epsilon) = \frac{(4\pi e^{-\gamma_E})^\epsilon}{8\pi^2}, \quad (2.17)$$

which were all computed in [42].

Integrated fragmentation antenna functions are employed to subtract explicit infrared singularities from virtual corrections. It is possible to organize them within the so-called integrated dipoles [27, 43, 56, 57]. We discuss in detail the construction and the properties of integrated dipoles in the context of antenna subtraction with fragmentation in section 4. Here we simply summarize their generic structure at NLO and NNLO to present the typical expressions for virtual subtraction terms.

One-loop integrated dipoles $J_2^{(1)}$ contain integrated fragmentation three-parton tree-level antenna functions and NLO mass factorization kernels $\Gamma^{(1)}$:

$$J_2^{(1)}(p, a) = c_{\mathcal{X}_3^0} \mathcal{X}_3^{0,\text{id},p} + c_{\Gamma^{(1)}} \Gamma^{(1)}, \quad (2.18)$$

where $c_{\mathcal{X}_3^0}$ and $c_{\Gamma^{(1)}}$ are constants which depend on the specific partonic and kinematical configuration. Parton a is a non-identified hard radiator. Since both integrated fragmentation

antenna functions and splitting kernels depend on z , the integrated dipoles will also depend on it, even if we keep this dependence implicit for ease of notation. One-loop integrated dipoles can be used to assemble the NLO virtual subtraction term, aimed at removing the explicit infrared singularities in one-loop matrix elements:

$$\begin{aligned}
 d\hat{\sigma}_{\text{NLO}}^{\text{T,id},p} &\supseteq \mathcal{N}_B d\Phi_n(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j) \frac{1}{S_n} J_2^{(1)}(p, a) \\
 &\quad \times M_n^0(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j) \\
 &\quad \times J(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j; \xi_1, \xi_2, z\eta).
 \end{aligned} \tag{2.19}$$

The specific form of the integrated dipole $J_2^{(1)}$ depends on the partonic species of p and a as encoded by its arguments in the expression above. An analogous contribution is used at partonic multiplicity $(n+1)$ to subtract the explicit poles of the real-virtual matrix element.

Two-loop integrated dipoles $J_2^{(2)}$ have a more complicated structure:

$$\begin{aligned}
 J_2^{(2)}(p, a) &= c_{\mathcal{X}_4^0} \mathcal{X}_4^{0,\text{id},p} + c_{\mathcal{X}_3^1} \mathcal{X}_3^{1,\text{id},p} + c_{\mathcal{X}_3^0 \mathcal{X}_3^0} \mathcal{X}_3^{0,\text{id},p} \otimes \mathcal{X}_3^{0,\text{id},p} + c_{\beta_0} \frac{\beta_0}{\epsilon} \mathcal{X}_3^{0,\text{id},p} \\
 &\quad + c_{\bar{\Gamma}^{(2)}} \bar{\Gamma}^{(2)} + c_{\Gamma^{(1)}\Gamma^{(1)}} \Gamma^{(1)} \otimes \Gamma^{(1)} + c_{\Gamma^{(1)}\mathcal{X}_3^0} \Gamma^{(1)} \otimes \mathcal{X}_3^0,
 \end{aligned} \tag{2.20}$$

where, as at one-loop, all constants depend on the partonic content and the kinematical configuration of the dipole. For a complete list of two-loop mass factorization kernels $\bar{\Gamma}^{(2)}$ see e.g. [27] for the space-like expressions and appendix B for the time-like ones. Two-loop integrated dipoles are needed to construct the double-virtual subtraction terms. However, to properly remove the infrared singularities of two-loop matrix elements, one also has to consider the combination of one-loop integrated dipoles with one-loop matrix-elements and the convolution of two one-loop integrated dipoles [27, 43]. The structures present in the double-virtual subtraction term are then:

$$\begin{aligned}
 d\hat{\sigma}_{\text{NNLO}}^{\text{U,id},p} &\supseteq \mathcal{N}_B d\Phi_n(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j) \frac{1}{S_n} J_2^{(2)}(p, a) \\
 &\quad \times M_n^0(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j) \\
 &\quad \times J(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j; \xi_1, \xi_2, z\eta)
 \end{aligned} \tag{2.21}$$

$$\begin{aligned}
 d\hat{\sigma}_{\text{NNLO}}^{\text{U,id},p} &\supseteq \mathcal{N}_B d\Phi_n(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j) \frac{1}{S_n} J_2^{(1)}(p, a) \\
 &\quad \times M_n^1(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j) \\
 &\quad \times J(k_1, \dots, k_p, k_a, \dots, k_n; p_i, p_j; \xi_1, \xi_2, z\eta),
 \end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
 d\hat{\sigma}_{\text{NNLO}}^{\text{U,id},p} &\supseteq \mathcal{N}_B d\Phi_n(k_1, \dots, k_a, k_b, \dots, k_b, k_c, \dots, k_n; p_i, p_j) \frac{1}{S_n} J_2^{(1)}(p, a) \otimes J_2^{(1)}(b, c) \\
 &\quad \times M_n^0(k_1, \dots, k_p, k_a, \dots, k_b, k_c, \dots, k_n; p_i, p_j) \\
 &\quad \times J(k_1, \dots, k_p, k_a, \dots, k_b, k_c, \dots, k_n; p_i, p_j; \xi_1, \xi_2, z\eta),
 \end{aligned} \tag{2.23}$$

where b and c can also coincide with either p or a .

2.2 Initial-final configurations

For configurations with momentum p_i in the initial state, fragmenting parton momentum k_p and unresolved momentum k_k in the final state, the NLO real subtraction term is assembled using initial-final three-parton tree-level fragmentation antenna functions $X_{3,i}^{0,\text{id},p}$:

$$\begin{aligned} d\hat{\sigma}_{\text{NLO}}^{\text{S,id},p} \supseteq & 8\pi^2 \mathcal{N}_B d\Phi_{n+1}(k_1, \dots, k_p, k_k, \dots, k_{n+1}; p_i, p_j) \frac{1}{S_{n+1}} X_{3,i}^{0,\text{id},p}(k_p, k_k; p_i) \\ & \times M_n^0(k_1, \dots, \tilde{k}_p, \dots, k_{n+1}; xp_i, p_j) \\ & \times J(k_1, \dots, \tilde{k}_p, \dots, k_{n+1}; xp_i, p_j; \xi_1/x, \xi_2, z\eta). \end{aligned} \quad (2.24)$$

In here, the phase space mapping is the initial-final one [41], with final-state momentum fraction z and initial-state momentum fraction x defined as

$$z = \frac{s_{ip}}{s_{ip} + s_{ik}} \equiv z_3, \quad x = \frac{s_{ip} + s_{ik} - s_{pk}}{s_{ip} + s_{ik}}. \quad (2.25)$$

At NNLO, we use initial-final four-parton fragmentation antenna functions $X_{4,i}^{0,\text{id},p}$ to remove double-unresolved singularities:

$$\begin{aligned} d\hat{\sigma}_{\text{NNLO}}^{\text{S,id},p} \supseteq & (8\pi^2)^2 \mathcal{N}_B d\Phi_{n+2}(k_1, \dots, k_p, k_k, k_l, \dots, k_{n+2}; p_i, p_j) \frac{1}{S_{n+2}} X_{4,i}^{0,\text{id},p}(k_p, k_k, k_l; p_i) \\ & \times M_n^0(k_1, \dots, \tilde{k}_p, \dots, k_{n+2}; xp_i, p_j) \\ & \times J(k_1, \dots, \tilde{k}_p, \dots, k_{n+2}; xp_i, p_j; \xi_1/x, \xi_2, z\eta), \end{aligned} \quad (2.26)$$

with the initial-final phase space mapping at NNLO, and x and z defined as:

$$z = \frac{s_{ip}}{s_{ip} + s_{ik} + s_{il}} \equiv z_4, \quad x = \frac{s_{ip} + s_{ik} + s_{il} - s_{pk} - s_{pl} - s_{kl}}{s_{ip} + s_{ik} + s_{il}}. \quad (2.27)$$

The removal of single-unresolved divergences from the real-virtual corrections is done analogously to (2.24) using initial-final one-loop fragmentation antenna functions $X_{3,i}^{1,\text{id},p}$:

$$\begin{aligned} d\hat{\sigma}_{\text{NNLO}}^{\text{T,id},p} \supseteq & 8\pi^2 \mathcal{N}_B d\Phi_{n+1}(k_1, \dots, k_p, k_k, \dots, k_{n+1}; p_i, p_j) \frac{1}{S_{n+1}} X_{3,i}^{1,\text{id},p}(k_p, k_k; p_i) \\ & \times M_n^0(k_1, \dots, \tilde{k}_p, \dots, k_{n+1}; xp_i, p_j) \\ & \times J(k_1, \dots, \tilde{k}_p, \dots, k_{n+1}; xp_i, p_j; \xi_1/x, \xi_2, z\eta). \end{aligned} \quad (2.28)$$

To integrate the initial-final fragmentation antenna functions, we employ the phase space mappings (with $q = -p_i + k_p + k_k (+k_l)$ space-like and $Q^2 = -q^2$):

$$\begin{aligned} d\Phi_{n+1}(k_1, \dots, k_p, k_k, \dots, k_{n+1}; p_i, p_j) &= d\Phi_n(k_1, \dots, \tilde{k}_p, \dots, k_{n+1}; xp_i, p_j) \\ & \quad \times \frac{dx}{x} \frac{Q^2}{2\pi} d\Phi_2(k_p, k_k; q, p_i) \delta(z - z_3) dz, \\ d\Phi_{n+2}(k_1, \dots, k_p, k_k, k_l, \dots, k_{n+2}; p_i, p_j) &= d\Phi_n(k_1, \dots, \tilde{k}_p, \dots, k_{n+2}; xp_i, p_j) \\ & \quad \times \frac{dx}{x} \frac{Q^2}{2\pi} d\Phi_3(k_p, k_k, k_l; q, p_i) \delta(z - z_4) dz, \end{aligned} \quad (2.29)$$

resulting in the integrated antenna functions

$$\mathcal{X}_{3,i}^{0,\text{id}.p}(x, z) = \frac{1}{C(\epsilon)} \int d\Phi_2(k_p, k_k; q, p_i) \frac{Q^2}{2\pi} X_{3,i}^{0,\text{id}.p}(k_p, k_k; p_i) \delta(z - z_3), \quad (2.30)$$

$$\mathcal{X}_{3,i}^{1,\text{id}.p}(x, z) = \frac{1}{C(\epsilon)} \int d\Phi_2(k_p, k_k; q, p_i) \frac{Q^2}{2\pi} X_{3,i}^{1,\text{id}.p}(k_p, k_k; p_i) \delta(z - z_3), \quad (2.31)$$

$$\mathcal{X}_{4,i}^{0,\text{id}.p}(x, z) = \frac{1}{C(\epsilon)^2} \int d\Phi_3(k_p, k_k, k_l; q, p_i) \frac{Q^2}{2\pi} X_{4,i}^{0,\text{id}.p}(k_p, k_k, k_l; p_i) \delta(z - z_4). \quad (2.32)$$

The method for the analytic computation of these integrated initial-final fragmentation antenna functions has been developed in the context of extending the antenna subtraction method to photon fragmentation processes in [39]. In the case of the $\mathcal{X}_{3,i}^{0,\text{id}.p}(x, z)$ and $\mathcal{X}_{3,i}^{1,\text{id}.p}(x, z)$, the $\delta(z - z_3)$ trivializes the $d\Phi_2$ phase space integration, which subsequently amounts only to an expansion in distributions in $(1 - z)$ and $(1 - x)$. This has been performed for all three-parton initial-final fragmentation antenna functions in [39]. The integration of the four-parton initial-final fragmentation antenna functions is described in detail in section 3 below, where the full set of $\mathcal{X}_{4,i}^{0,\text{id}.p}(x, z)$ is computed.

As for the final-final case, the subtraction of explicit singularities at the virtual level occurs by means of one- and two-loop integrated dipoles. In this case, initial-final integrated fragmentation antenna functions are combined with initial- or final-state splitting kernels to construct the integrated dipoles, but their general expressions still follow (2.18) and (2.20). Clearly, integrated dipoles in the initial-final configuration depend on both z and x . We refer the reader to section 4 for the explicit representation of final-final and initial-final integrated dipoles. The structure of the virtual subtraction terms is identical to the one in equations (2.19) and (2.21)–(2.23).

3 Integration of initial-final fragmentation antenna functions

The initial-final phase space for three-parton fragmentation antenna functions $\mathcal{X}_{3,i}^{0,\text{id}.p}$ is fully constrained, such that no integration is required:

$$\begin{aligned} \mathcal{X}_{3,i}^{0,\text{id}.p}(x, z) &= \frac{1}{C(\epsilon)} \int d\Phi_2(k_p, k_k; q, p_i) X_{3,i}^{0,\text{id}.p} \frac{Q^2}{2\pi} \delta\left(z - \frac{s_{ip}}{s_{ip} + s_{ik}}\right) \\ &= \frac{Q^2}{2} \frac{e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} (Q^2)^{-\epsilon} \mathcal{J}(x, z) X_{3,i}^{0,\text{id}.p}(x, z), \end{aligned} \quad (3.1)$$

with $q^2 = (p_i - k_j - k_k)^2 = -Q^2 < 0$ and the Jacobian factor is given by

$$\mathcal{J}(x, z) = (1 - x)^{-\epsilon} x^\epsilon z^{-\epsilon} (1 - z)^{-\epsilon}. \quad (3.2)$$

After expressing the invariants in the antenna function in terms of x and z , all terms of the form $(1 - x)^{-1-\epsilon}$ and $(1 - z)^{-1-\epsilon}$ are expanded in distributions, where we use the notation

$$\mathcal{D}_n(u) = \left[\frac{\ln^n(1 - u)}{1 - u} \right]_+, \quad n \in \mathbb{N}_0. \quad (3.3)$$

The same feature also holds true for the one-loop three-parton antenna functions $\mathcal{X}_{3,i}^{1,\text{id}.p}$, where some additional care has to be taken to isolate the kinematical endpoint contributions

in $x = 1$ and $z = 1$ from the one-loop box integrals. The integration of the initial-final fragmentation antenna functions $\mathcal{X}_{3,i}^{0,\text{id}.p}$ and $\mathcal{X}_{3,i}^{1,\text{id}.p}$ is described in detail in [39]. Although the main focus of [39] was on photon fragmentation processes, the full set of $\mathcal{X}_{3,i}^{0,\text{id}.p}$ and $\mathcal{X}_{3,i}^{1,\text{id}.p}$ required for generic hadron fragmentation was already computed there.

The integration of four-parton initial-final fragmentation antenna functions $\mathcal{X}_{4,i}^{0,\text{id}.p}$ is also described in [39], where only a limited set of antenna functions relevant to photon fragmentation has been computed. To complete the computation for all antenna functions relevant to hadron fragmentation processes $\mathcal{X}_{4,i}^{0,\text{id}.p}$, technical extensions of the methods outlined in [39] are required. The antenna functions relevant to photon fragmentation processes were all integrable at $z = 1$, since the photon must always be accompanied by a quark which can not become soft. In contrast, general hadron fragmentation processes can be accompanied by soft gluons only, thereby requiring an expansion of the relevant fragmentation antenna functions around $z = 1$ (in addition to the expansion around $x = 1$). Another technical aspect concerns the integration of the initial-final fragmentation antenna functions over z , which recovers the inclusive initial-final antenna functions [53]. These calculations help to determine several integration constants in the master integrals relevant for $\mathcal{X}_{4,i}^{0,\text{id}.p}$ and they must now account for potentially singular behaviour in one or both endpoints $z = 0$ and $z = 1$.

The kinematics of the X_4^0 initial-final antenna functions is given by

$$q + p_i \rightarrow k_p + k_l + k_k, \tag{3.4}$$

with $p_i^2 = k_p^2 = k_l^2 = k_k^2 = 0$ and $q^2 = -Q^2 < 0$. Fully inclusive integration over the four-particle phase space yields the integrated \mathcal{X}_4^0 antenna functions [53]:

$$\mathcal{X}_{4,i}^0(x) = \frac{1}{C(\epsilon)^2} \int d\Phi_3(k_p, k_k, k_l; p_i, q) \frac{Q^2}{2\pi} X_{4,i}^0, \tag{3.5}$$

with $x = Q^2/(2p \cdot q)$.

For fragmentation antenna functions in initial-final kinematics, the integration remains differential in the final-state momentum fraction z of the identified parton p

$$\mathcal{X}_{4,i}^{0,\text{id}.p}(x, z) = \frac{1}{C(\epsilon)^2} \int d\Phi_3(k_p, k_k, k_l; p_i, q) \delta\left(z - x \frac{(p_i + k_p)^2}{Q^2}\right) \frac{Q^2}{2\pi} X_{4,i}^{0,\text{id}.p}, \tag{3.6}$$

where as in (3.1), the initial-state momentum p is used as a reference momentum to define z :

$$z = x \frac{(k_p + p_i)^2}{Q^2} = \frac{s_{ip}}{s_{ip} + s_{ik} + s_{il}}. \tag{3.7}$$

Using the reverse unitarity relation

$$2\pi i \delta(k^2) = \frac{1}{k^2 + i\epsilon} - \frac{1}{k^2 - i\epsilon}, \tag{3.8}$$

the phase space integrals (3.6) are rewritten as $2 \rightarrow 2$ three-loop-integrals in forward scattering kinematics with four cut propagators (three on-shell conditions and the definition of z). These are amenable to standard integral reduction techniques based on integration-by-parts (IBP) relations [58] in the Laporta algorithm [59]. The resulting integrals all contain four cut

propagators and up to three linearly independent ordinary propagators. After applying momentum conservation $k_k = q + p_i - k_p - k_l$, the following set of denominator factors appears in the antenna functions:

$$\begin{aligned}
 D_1 &= (q - k_p)^2, \\
 D_2 &= (p_i + q - k_p)^2, \\
 D_3 &= (p_i - k_l)^2, \\
 D_4 &= (q - k_l)^2, \\
 D_5 &= (p_i + q - k_l)^2, \\
 D_6 &= (q - k_p - k_l)^2, \\
 D_7 &= (p_i - k_p - k_l)^2, \\
 D_8 &= (k_p + k_l)^2, \\
 D_9 &= k_p^2, \\
 D_{10} &= k_l^2, \\
 D_{11} &= (q + p_i - k_p - k_l)^2, \\
 D_{12} &= (p_i - k_p)^2 + Q^2 \frac{z}{x},
 \end{aligned} \tag{3.9}$$

where the cut propagators are D_9 to D_{12} . Combining the cut propagators with any subset of three linearly independent ordinary propagators yields an integral family, for which an IBP reduction to master integrals can be performed. We use the `Reduze2` [60] code for this task.

The master integrals are labelled by their propagators factors (omitting the cut propagators, which we require in each integral), for example:

$$I[-3, 7] = \frac{Q^2 (2\pi)^{-2d+3}}{x} \int d^d k_p d^d k_l \delta(D_9) \delta(D_{10}) \delta(D_{11}) \delta(D_{12}) \frac{D_3}{D_7}, \tag{3.10}$$

where a negative sign on the propagator label indicates its occurrence in the numerator. We find 12 integral families and in total 21 master integrals which are summarised in table 1. The integral family F derives from the $I[1, 3, 7]$ top-level integral which is reducible to known integrals from other families. The master integrals are calculated using differential equations [61] in the two kinematic variables x and z . The boundary conditions are fixed by integrating the solution of the differential equations over z and comparing the result with the inclusive master integrals calculated in [53]. This procedure is described in detail in [39], where the master integrals of families A and B were computed already.

The master integrals and the antenna functions potentially contain end-point singularities in $x = 1$, $z = 1$ and $z = 0$. These are regulated by factors $(1 - x)^{-2\epsilon}$, $z^{-n\epsilon}$ and $(1 - z)^{-n\epsilon}$ with $n = 1, 2$ that need to be retained in exact form in solving the differential equations. Their ϵ -expansion subsequently yields distributions around the endpoints. The singular behaviour at $z = 0$, which is relevant to match the boundary conditions onto the inclusive master integrals, requires particular attention.

The phase space integral $I[0]$ can be derived by direct integration:

$$I[0](Q^2, x, z) = N_\Gamma (Q^2)^{1-2\epsilon} (1 - x)^{1-2\epsilon} x^{-1+2\epsilon} z^{-\epsilon} (1 - z)^{1-2\epsilon}, \tag{3.11}$$

family	master	deepest pole	at $x = 1$	at $z = 1$
	$I[0]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
A	$I[5]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[2, 3, 5]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
B	$I[7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[-2, 7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[-3, 7]$	ϵ^0	$(1-x)^{1-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[2, 3, 7]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
C	$I[5, 7]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{1-2\epsilon}$
	$I[3, 5, 7]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
D	$I[1]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[1, 4]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[1, 3, 4]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
E	$I[1, 3, 5]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
G	$I[1, 3, 8]$	ϵ^{-2}	$(1-x)^{-2\epsilon}$	$(1-z)^{-1-2\epsilon}$
H	$I[1, 4, 5]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
I	$I[2, 4, 5]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
J	$I[4, 7]$	ϵ^0	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
	$I[3, 4, 7]$	ϵ^{-1}	$(1-x)^{-2\epsilon}$	$(1-z)^{-2\epsilon}$
K	$I[3, 5, 8]$	ϵ^{-2}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
L	$I[4, 5, 7]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$
M	$I[4, 5, 8]$	ϵ^{-1}	$(1-x)^{-1-2\epsilon}$	$(1-z)^{-2\epsilon}$

Table 1. Summary of the double-real radiation master integrals.

with

$$N_\Gamma = \frac{2^{-5+4\epsilon} \pi^{-3+2\epsilon} \Gamma^2(2-\epsilon)}{\Gamma^2(3-2\epsilon)}. \tag{3.12}$$

In the limit $z \rightarrow 0$, the identified particle momentum k_p becomes soft, such that propagators $D_5 = (k_p + k_k)^2$ and $D_8 = (k_p + k_l)^2$ become singular. The computation of the master integrals that contain both these propagators, $I[3, 5, 8]$ (family K) and $I[4, 5, 8]$ (family M), must account for this behaviour and retain the exact dependence on the dimensional regulator at least in the $z \rightarrow 0$ limit. For both integrals, a naive approach of solving the differential equations in x and z as a Laurent expansion in ϵ with symbolic boundary conditions at a regular point in z yields integrals that contain at most ϵ^{-1} , while the corresponding z -integrated inclusive initial-final master integrals with this combination of propagators diverge as ϵ^{-3} .

In the following, we provide a detailed description of the computation of $I[3, 5, 8] = I[358]$ in a closed form in ϵ . The differential equations for this master integral contain only $I[0]$ and

$$I[5](Q^2, x, z) = N_\Gamma \left(\frac{1-2\epsilon}{\epsilon} \right)^2 (Q^2)^{-2\epsilon} (1-x)^{-2\epsilon} x^{2\epsilon} \times \left(z^{-\epsilon} {}_2F_1(\epsilon, 2\epsilon, 1+\epsilon; z) - z^{-2\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} \right). \quad (3.13)$$

as subtopologies, and the differential equations in z and x are fully separable. Moreover, its differential equation in x is homogeneous, implying that the x -dependence of $I[358]$ factorises fully. The differential equations read:

$$\begin{aligned} \frac{\partial I[358](Q^2, x, z)}{\partial Q^2} &= -\frac{2(1+\epsilon)}{Q^2} I[358](Q^2, x, z), \\ \frac{\partial I[358](Q^2, x, z)}{\partial x} &= \left(\frac{1+2\epsilon}{1-x} + \frac{2+2\epsilon}{x} \right) I[358](Q^2, x, z), \\ \frac{\partial I[358](Q^2, x, z)}{\partial z} &= -\frac{1+2\epsilon}{z} I[358](Q^2, x, z) - \frac{2x^3(1-2\epsilon)^2(1+z)}{(Q^2)^3(1-x)^2\epsilon z^2(1-z)^2} I[0](Q^2, x, z) \\ &\quad + \frac{2x^2\epsilon}{(Q^2)^2(1-x)z^2} I[5](Q^2, x, z). \end{aligned} \quad (3.14)$$

From the above equations, the functional dependence of $I[358]$ on Q^2 and on x can be read off. Moreover, the z -differential equation can be solved by means of an integrating factor $z^{1+2\epsilon}$. Introducing

$$I[358](Q^2, x, z) = N_\Gamma \left(\frac{1-2\epsilon}{\epsilon} \right)^2 (Q^2)^{-2-2\epsilon} (1-x)^{-1-2\epsilon} x^{2+2\epsilon} z^{-1-2\epsilon} I'[358](z), \quad (3.15)$$

and inserting (3.11), (3.13) yields

$$\begin{aligned} \frac{\partial I'[358](z)}{\partial z} &= -\frac{4(1-2\epsilon)^2}{\epsilon} z^\epsilon (1-z)^{-1-2\epsilon} - \frac{2(1-2\epsilon)^2}{\epsilon} z^{-1+\epsilon} (1-z)^{-2\epsilon} \\ &\quad + \frac{2(1-2\epsilon)^2}{\epsilon} z^{-1+\epsilon} {}_2F_1(\epsilon, 2\epsilon; 1+\epsilon; z) \\ &\quad - \frac{2(1-2\epsilon)^2}{\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} z^{-1}. \end{aligned} \quad (3.16)$$

This equation can be integrated in a straightforward manner in the form of a primitive:

$$\begin{aligned} I'[358](z) &= -\frac{4(1-2\epsilon)^2}{\epsilon(1+\epsilon)} z^{1+\epsilon} {}_2F_1(1+\epsilon, 1+2\epsilon; 2+\epsilon; z) \\ &\quad - \frac{2(1-2\epsilon)^2}{\epsilon^2} z^\epsilon {}_2F_1(\epsilon, 2\epsilon; 1+\epsilon; z) \\ &\quad + \frac{2(1-2\epsilon)^2}{\epsilon^2} z^\epsilon {}_3F_2(\epsilon, \epsilon, 2\epsilon; 1+\epsilon, 1+\epsilon; z) \\ &\quad - \frac{2(1-2\epsilon)^2}{\epsilon} \frac{\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} \ln(z) + C'. \end{aligned} \quad (3.17)$$

where C' is the constant of integration. In (3.17), which is exact in ϵ , we note the simultaneous appearance of z^ϵ and $\ln z$. This observation implies that the singular behaviour of the master

integral $I[358](Q^2, x, z)$ at $z \rightarrow 0$ can not be expressed by an ansatz containing a finite number i of terms of the form $z^{-1-n\epsilon}$ with integer $n \leq i$. Moreover, a naive ϵ -expansion of (3.17) shows that its most singular piece is only $1/\epsilon$. The a priori unknown boundary constant C' can be determined by computing the z -integral of $I[358](Q^2, x, z)$ and matching it onto the known inclusive result. The inclusive $I_{\text{inc}}[358](Q^2, x)$ is itself not a master integral but can be expressed in terms of the inclusive phase space $I_{\text{inc}}[0](Q^2, x)$:

$$I_{\text{inc}}[358](Q^2, x) = \frac{3(1-2\epsilon)(4-6\epsilon)(2-6\epsilon)}{\epsilon^3} \frac{x^3}{(Q^2)^3(1-x)^2} I[0](Q^2, x), \quad (3.18)$$

with

$$I_{\text{inc}}[0](Q^2, x) = N_{\Gamma}(Q^2)^{1-2\epsilon} (1-x)^{1-2\epsilon} x^{-1+2\epsilon} \frac{\Gamma(2-2\epsilon)\Gamma(1-\epsilon)}{\Gamma(3-3\epsilon)}. \quad (3.19)$$

We note that (3.18) diverges as $1/\epsilon^3$. Dividing (3.17) by the integrating factor $z^{1+2\epsilon}$ and integrating over z , we obtain standard integrals yielding hypergeometric functions at unit argument as well as from the last two terms:

$$\int_0^1 dz z^{-1-2\epsilon} = -\frac{1}{2\epsilon}, \quad \int_0^1 dz \ln(z) z^{-1-2\epsilon} = -\frac{1}{4\epsilon^2}, \quad (3.20)$$

where in particular the occurrence of a double pole in the second integral is noteworthy. By matching onto the inclusive integral, we then obtain a closed form expression:

$$\begin{aligned} I[358](Q^2, x, z) = N_{\Gamma} \left(\frac{1-2\epsilon}{\epsilon} \right)^2 (Q^2)^{-2-2\epsilon} (1-x)^{-1-2\epsilon} x^{2+2\epsilon} z^{-1-2\epsilon} \\ \times \left(-2(1-z)^{-2\epsilon} z^{\epsilon} + 2z^{\epsilon} {}_3F_2(\epsilon, \epsilon, 2\epsilon, 1+\epsilon, 1+\epsilon, z) \right. \\ \left. - \frac{2\epsilon\Gamma(1-2\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-\epsilon)} (\pi \cot(\pi\epsilon) + \ln(z)) \right). \end{aligned} \quad (3.21)$$

The master integral $I[458]$ has considerably more subtologies than $I[358]$ and its differential equations in z and x do not separate. After extracting its dominant behaviour at $z = 0$, the computation for $I[458]$ follows in principle the same steps as for $I[358]$. It is however much more cumbersome and consequently less intuitive to describe.

For their insertion into the integrated antenna functions $\mathcal{X}_{4,i}^{0,\text{id},p}(x, z)$, master integrals are calculated in terms of a Laurent expansion in ϵ , after factoring the relevant regulating factors in $(1-x)$ and $(1-z)$ from each integral. The results for the integrals are collected in appendix A and given in computer-readable format in the supplementary material. The regulating factors combine with potential endpoint singularities in the reduction coefficients of the integrated antenna functions to master integrals, and are subsequently expanded in distributions. The resulting integrated initial-final fragmentation antenna functions are included as an ancillary file in the supplementary material.

3.1 List of integrated antenna functions

In this section we list all integrated antenna functions in the initial-final kinematics with an identified parton in the final state, following the notation of (2.31) and (2.32). These relate to the parent unintegrated antenna, denoting in the argument list the parton crossed to the initial

state with \bar{c} and the identified parton with $^{\text{id.}}$. The functional form of these unintegrated antenna functions is identical for all kinematical crossings and does not change if a final-state parton is identified. To obtain the minimal set of antennae we exploit the symmetries of the unintegrated antennae. For example, if the unintegrated antenna $X_4^0(i, j, k, l)$ is symmetric under the exchange of identical partons i and j , then the two integrated antennae $\mathcal{X}_{4,j}^{0,\text{id.}i}$ and $\mathcal{X}_{4,i}^{0,\text{id.}j}$ are also identical. We avoid listing double-real antennae which only give a finite contribution since they will not be employed in the subtraction.

The \mathcal{X}_3^0 antennae are listed in table 2 along with their symmetries. Due to the presence of convolutions in the integrated dipoles — namely in the subtraction terms — involving \mathcal{X}_3^0 antennae and NLO splitting kernels, the latter must be known up to $\mathcal{O}(\epsilon^2)$. In [39], most of the antennae in table 2 were already computed, with the exception of higher orders in ϵ for $\mathcal{A}_{3,g}^{0,\text{id.}q}$. Furthermore, only partial results were presented for the flavour-changing part of the gluon-initiated \mathcal{D}_3^0 antennae. For identity-changing dipoles in particular, both flavour-preserving and flavour-changing contributions of the gluon-initiated \mathcal{D}_3^0 antenna must be included. $\mathcal{D}_{3,g}^{0,\text{id.}q}$ and $\mathcal{D}_{3,g}^{0,\text{id.}g}$ therefore contain both flavour structures. For $\mathcal{D}_{3,g}^{0,\text{id.}q}$ the flavour-changing contribution is finite. Again, due to the presence of convolutions in the subtraction terms, also the finite $\mathcal{G}_{3,q}^{0,\text{id.}q}$ antenna is needed.

The symmetries of the \mathcal{X}_3^1 antennae are the same as the ones of the corresponding \mathcal{X}_3^0 due to the identical external momentum configuration. \mathcal{A} , \mathcal{E} and \mathcal{G} types have leading (\mathcal{X}_3^1) and sub-leading ($\tilde{\mathcal{X}}_3^1$) colour structures, as well as closed quark loop contributions ($\hat{\mathcal{X}}_3^1$). \mathcal{D} and \mathcal{F} types only have leading-colour and closed quark loop contributions. The \mathcal{X}_3^1 antennae are listed in tables 3–5.

In tables 6–8 we list the leading-colour \mathcal{X}_4^0 and subleading-colour $\tilde{\mathcal{X}}_4^0$ antennae with respective symmetries. The number of antennae within the same family is determined by the number of symmetries of the integrand. For example \mathcal{C} , \mathcal{D} and $\tilde{\mathcal{E}}$ have only one symmetry and therefore we find 7 antennae of \mathcal{C} , \mathcal{D} and $\tilde{\mathcal{E}}$ types respectively. Of the \mathcal{C} -type, only four of these are divergent and thus retained in table 6.

We notice that some identity-changing antennae, despite presenting ϵ -poles, are not employed in the subtraction. This is the case for all four parton tree-level and three-parton one-loop gluon-initiated \mathcal{D} -type antennae with an identified gluon. Indeed, such configuration of identity-changing limits $g \rightarrow q$ can be rendered by quark-antiquark \mathcal{A} -type antennae, due to the freedom in the choice of the spectator parton. Moreover, \mathcal{G} - and \mathcal{E} -type antennae which have the secondary quark (antiquark) in the initial state and the secondary antiquark (quark) identified do not correspond to any physical unresolved configuration and are therefore not needed for subtraction at NNLO. Finally, the $H_4^0(1_q^{\text{id.}}, 2_{\bar{q}}, 3_{q'}, 4_{\bar{q}'})$ antenna function only encodes a double-collinear iterated configuration, which can be targeted with the combination of two NLO antennae. Analogous observations were made in the context of antenna subtraction without identified final-state particles if the identified particle is crossed into the initial state. The full set of final-final and initial-final fragmentation antenna functions is provided as ancillary files in the supplementary material.

Notation	Integral of	Integrand symm.
Hard radiators: quark-quark		
$\mathcal{A}_{3,g}^{0,\text{id}.q}$	$A_3^0(1_q^{\text{id.}}, \hat{3}_g, 2_{\bar{q}})$	
$\mathcal{A}_{3,q}^{0,\text{id}.q}$	$A_3^0(1_q^{\text{id.}}, 3_g, \hat{2}_{\bar{q}})$	$1 \leftrightarrow 2$
$\mathcal{A}_{3,q}^{0,\text{id}.g}$	$A_3^0(1_q, 3_g^{\text{id.}}, \hat{2}_{\bar{q}})$	
Hard radiators: quark-gluon		
$\mathcal{D}_{3,g}^{0,\text{id}.q}$	$D_3^0(1_q^{\text{id.}}, 3_g, \hat{2}_g)$	
$\mathcal{D}_{3,q}^{0,\text{id}.g}$	$D_3^0(\hat{1}_{\bar{q}}, 3_g, 2_g^{\text{id.}})$	$2 \leftrightarrow 3$
$\mathcal{D}_{3,g}^{0,\text{id}.g}$	$D_3^0(1_q, \hat{3}_g, 2_g^{\text{id.}})$	
$\mathcal{E}_{3,q'}^{0,\text{id}.q}$	$E_3^0(1_q^{\text{id.}}, 3_{q'}, \hat{2}_{\bar{q}'})$	
$\mathcal{E}_{3,q}^{0,\text{id}.q'}$	$E_3^0(\hat{1}_{\bar{q}}, 3_{q'}^{\text{id.}}, 2_{\bar{q}'})$	$2 \leftrightarrow 3$
Hard radiators: gluon-gluon		
$\mathcal{F}_{3,g}^{0,\text{id}.g}$	$F_3^0(1_g^{\text{id.}}, 3_g, \hat{2}_g)$	$1 \leftrightarrow 2, 1 \leftrightarrow 3, 2 \leftrightarrow 3$
$\mathcal{G}_{3,q}^{0,\text{id}.g}$	$G_3^0(1_g^{\text{id.}}, 3_q, \hat{2}_{\bar{q}})$	
$\mathcal{G}_{3,g}^{0,\text{id}.q}$	$G_3^0(\hat{1}_g, 3_q^{\text{id.}}, 2_{\bar{q}})$	$2 \leftrightarrow 3$
$\mathcal{G}_{3,q}^{0,\text{id}.q}$	$G_3^0(1_g, 3_q^{\text{id.}}, \hat{2}_{\bar{q}})$	

Table 2. Integrated tree-level three-parton antenna functions.

Notation	Integral of	Integrand symm.
Hard radiators: quark-quark		
$\mathcal{A}_{3,g}^{1,\text{id}.q}$	$A_3^1(1_q^{\text{id.}}, \hat{3}_g, 2_{\bar{q}})$	
$\mathcal{A}_{3,q}^{1,\text{id}.q}$	$A_3^1(1_q^{\text{id.}}, 3_g, \hat{2}_{\bar{q}})$	
$\mathcal{A}_{3,q}^{1,\text{id}.g}$	$A_3^1(1_q, 3_g^{\text{id.}}, \hat{2}_{\bar{q}})$	
$\tilde{\mathcal{A}}_{3,g}^{1,\text{id}.q}$	$\tilde{A}_3^1(1_q^{\text{id.}}, \hat{3}_g, 2_{\bar{q}})$	
$\tilde{\mathcal{A}}_{3,q}^{1,\text{id}.q}$	$\tilde{A}_3^1(1_q^{\text{id.}}, 3_g, \hat{2}_{\bar{q}})$	$1 \leftrightarrow 2$
$\tilde{\mathcal{A}}_{3,q}^{1,\text{id}.g}$	$\tilde{A}_3^1(1_q, 3_g^{\text{id.}}, \hat{2}_{\bar{q}})$	
$\hat{\mathcal{A}}_{3,g}^{1,\text{id}.q}$	$\hat{A}_3^1(1_q^{\text{id.}}, \hat{3}_g, 2_{\bar{q}})$	
$\hat{\mathcal{A}}_{3,q}^{1,\text{id}.q}$	$\hat{A}_3^1(1_q^{\text{id.}}, 3_g, \hat{2}_{\bar{q}})$	
$\hat{\mathcal{A}}_{3,q}^{1,\text{id}.g}$	$\hat{A}_3^1(1_q, 3_g^{\text{id.}}, \hat{2}_{\bar{q}})$	

Table 3. Integrated quark-quark one-loop three-parton antenna functions.

Notation	Integral of	Integrand symm.
Hard radiators: quark-gluon		
$\mathcal{D}_{3,g}^{1,\text{id}.q}$	$D_3^1(1_q^{\text{id.}}, 3_g, \hat{2}_g)$	$2 \leftrightarrow 3$
$\mathcal{D}_{3,q}^{1,\text{id}.g}$	$D_3^1(\hat{1}_{\bar{q}}, 3_g, 2_g^{\text{id.}})$	
$\mathcal{D}_{3,g}^{1,\text{id}.g}$	$D_3^1(1_q, \hat{3}_g, 2_g^{\text{id.}})$	
$\widehat{\mathcal{D}}_{3,g}^{1,\text{id}.q}$	$\widehat{D}_3^1(1_q^{\text{id.}}, 3_g, \hat{2}_g)$	
$\widehat{\mathcal{D}}_{3,q}^{1,\text{id}.g}$	$\widehat{D}_3^1(\hat{1}_{\bar{q}}, 3_g, 2_g^{\text{id.}})$	
$\widehat{\mathcal{D}}_{3,g}^{1,\text{id}.g}$	$\widehat{D}_3^1(1_q, \hat{3}_g, 2_g^{\text{id.}})$	
$\mathcal{E}_{3,q'}^{1,\text{id}.q}$	$E_3^1(1_{q'}^{\text{id.}}, 3_{q'}, \hat{2}_{\bar{q}'})$	$2 \leftrightarrow 3$
$\mathcal{E}_{3,q}^{1,\text{id}.q'}$	$E_3^1(\hat{1}_{\bar{q}}, 3_{q'}^{\text{id.}}, 2_{\bar{q}'})$	
$\mathcal{E}_{3,q'}^{1,\text{id}.q'}$	$E_3^1(1_{\bar{q}}, 3_{q'}^{\text{id.}}, \hat{2}_{\bar{q}'})$	
$\widetilde{\mathcal{E}}_{3,q'}^{1,\text{id}.q}$	$\widetilde{E}_3^1(1_{q'}^{\text{id.}}, 3_{q'}, \hat{2}_{\bar{q}'})$	
$\widetilde{\mathcal{E}}_{3,q}^{1,\text{id}.q'}$	$\widetilde{E}_3^1(\hat{1}_{\bar{q}}, 3_{q'}^{\text{id.}}, 2_{\bar{q}'})$	
$\widetilde{\mathcal{E}}_{3,q'}^{1,\text{id}.q'}$	$\widetilde{E}_3^1(1_{\bar{q}}, 3_{q'}^{\text{id.}}, \hat{2}_{\bar{q}'})$	
$\widehat{\mathcal{E}}_{3,q'}^{1,\text{id}.q}$	$\widehat{E}_3^1(1_{q'}^{\text{id.}}, 3_{q'}, \hat{2}_{\bar{q}'})$	
$\widehat{\mathcal{E}}_{3,q}^{1,\text{id}.q'}$	$\widehat{E}_3^1(\hat{1}_{\bar{q}}, 3_{q'}^{\text{id.}}, 2_{\bar{q}'})$	
$\widehat{\mathcal{E}}_{3,q'}^{1,\text{id}.q'}$	$\widehat{E}_3^1(1_{\bar{q}}, 3_{q'}^{\text{id.}}, \hat{2}_{\bar{q}'})$	

Table 4. Integrated quark-gluon one-loop three-parton antenna functions.

Notation	Integral of	Integrand symm.
Hard radiators: gluon-gluon		
$\mathcal{F}_{3,g}^{1,\text{id}.g}$	$F_3^1(1_g^{\text{id.}}, 3_g, \hat{2}_g)$	$1 \leftrightarrow 2, 1 \leftrightarrow 3, 2 \leftrightarrow 3$
$\widehat{\mathcal{F}}_{3,g}^{1,\text{id}.g}$	$\widehat{F}_3^1(1_g^{\text{id.}}, 3_g, \hat{2}_g)$	
$\mathcal{G}_{3,q}^{1,\text{id}.g}$	$G_3^1(1_g^{\text{id.}}, 3_q, \hat{2}_{\bar{q}})$	$2 \leftrightarrow 3$
$\mathcal{G}_{3,g}^{1,\text{id}.q}$	$G_3^1(\hat{1}_g, 3_q^{\text{id.}}, 2_{\bar{q}})$	
$\mathcal{G}_{3,q}^{1,\text{id}.q}$	$G_3^1(1_g, 3_q^{\text{id.}}, \hat{2}_{\bar{q}})$	
$\widetilde{\mathcal{G}}_{3,q}^{1,\text{id}.g}$	$\widetilde{G}_3^1(1_g^{\text{id.}}, 3_q, \hat{2}_{\bar{q}})$	
$\widetilde{\mathcal{G}}_{3,g}^{1,\text{id}.q}$	$\widetilde{G}_3^1(\hat{1}_g, 3_q^{\text{id.}}, 2_{\bar{q}})$	
$\widetilde{\mathcal{G}}_{3,q}^{1,\text{id}.q}$	$\widetilde{G}_3^1(1_g, 3_q^{\text{id.}}, \hat{2}_{\bar{q}})$	
$\widehat{\mathcal{G}}_{3,q}^{1,\text{id}.g}$	$\widehat{G}_3^1(1_g^{\text{id.}}, 3_q, \hat{2}_{\bar{q}})$	
$\widehat{\mathcal{G}}_{3,g}^{1,\text{id}.q}$	$\widehat{G}_3^1(\hat{1}_g, 3_q^{\text{id.}}, 2_{\bar{q}})$	
$\widehat{\mathcal{G}}_{3,q}^{1,\text{id}.q}$	$\widehat{G}_3^1(1_g, 3_q^{\text{id.}}, \hat{2}_{\bar{q}})$	

Table 5. Integrated gluon-gluon one-loop three-parton antenna functions.

Notation	Integral of	Integrand symm.
Hard radiators: quark-quark		
$\mathcal{A}_{4,q}^{0,\text{id}.q}$	$A_4^0(1_q^{\text{id.}}, 3_g, 4_g, \hat{2}_{\bar{q}})$	
$\mathcal{A}_{4,g_3}^{0,\text{id}.q}$	$A_4^0(1_q^{\text{id.}}, \hat{3}_g, 4_g, 2_{\bar{q}})$	
$\mathcal{A}_{4,g_4}^{0,\text{id}.q}$	$A_4^0(1_q^{\text{id.}}, 3_g, \hat{4}_g, 2_{\bar{q}})$	$1 \leftrightarrow 2 + 3 \leftrightarrow 4$
$\mathcal{A}_{4,q}^{0,\text{id}.g_3}$	$A_4^0(1_q, 3_g^{\text{id.}}, 4_g, \hat{2}_{\bar{q}})$	
$\mathcal{A}_{4,q}^{0,\text{id}.g_4}$	$A_4^0(1_q, 3_g, 4_g^{\text{id.}}, \hat{2}_{\bar{q}})$	
$\mathcal{A}_{4,g}^{0,\text{id}.g}$	$A_4^0(1_q, 3_g^{\text{id.}}, \hat{4}_g, 2_{\bar{q}})$	
$\tilde{\mathcal{A}}_{4,q}^{0,\text{id}.q}$	$\tilde{A}_4^0(1_q^{\text{id.}}, 3_g, 4_g, \hat{2}_{\bar{q}})$	
$\tilde{\mathcal{A}}_{4,g}^{0,\text{id}.q}$	$\tilde{A}_4^0(1_q^{\text{id.}}, \hat{3}_g, 4_g, 2_{\bar{q}})$	$1 \leftrightarrow 2, 3 \leftrightarrow 4$
$\tilde{\mathcal{A}}_{4,q}^{0,\text{id}.g}$	$\tilde{A}_4^0(1_q, 3_g^{\text{id.}}, 4_g, \hat{2}_{\bar{q}})$	
$\tilde{\mathcal{A}}_{4,g}^{0,\text{id}.g}$	$\tilde{A}_4^0(1_q, 3_g^{\text{id.}}, \hat{4}_g, 2_{\bar{q}})$	
$\mathcal{B}_{4,q}^{0,\text{id}.q}$	$B_4^0(1_q^{\text{id.}}, 3_{q'}, 4_{\bar{q}'}, \hat{2}_{\bar{q}})$	
$\mathcal{B}_{4,q'}^{0,\text{id}.q}$	$B_4^0(1_q^{\text{id.}}, 3_{q'}, \hat{4}_{\bar{q}'}, 2_{\bar{q}})$	$1 \leftrightarrow 2, 3 \leftrightarrow 4$
$\mathcal{B}_{4,q}^{0,\text{id}.q'}$	$B_4^0(1_q, 3_{q'}^{\text{id.}}, 4_{\bar{q}'}, \hat{2}_{\bar{q}})$	
$\mathcal{C}_{4,q_2}^{0,\text{id}.q_1}$	$C_4^0(1_q^{\text{id.}}, 3_q, 4_{\bar{q}}, \hat{2}_{\bar{q}})$	
$\mathcal{C}_{4,q_3}^{0,\text{id}.q_1}$	$C_4^0(1_q^{\text{id.}}, \hat{3}_q, 4_{\bar{q}}, 2_{\bar{q}})$	$2 \leftrightarrow 4$
$\mathcal{C}_{4,\bar{q}_1}^{0,\text{id}.q_2}$	$C_4^0(\hat{1}_q, 3_q, 4_{\bar{q}}, 2_{\bar{q}}^{\text{id.}})$	
$\mathcal{C}_{4,\bar{q}_1}^{0,\text{id}.q_3}$	$C_4^0(\hat{1}_q, 3_q^{\text{id.}}, 4_{\bar{q}}, 2_{\bar{q}})$	

Table 6. Integrated quark-quark tree-level four-parton antenna functions. The notation $a \leftrightarrow b + c \leftrightarrow d$ indicates that the antenna is symmetric under *simultaneous* exchange of a with b and c with d .

Notation	Integral of	Integrand symm.
Hard radiators: quark-gluon		
$\mathcal{D}_{4,g_2}^{0,\text{id}.q}$	$D_4^0(1_q^{\text{id.}}, \hat{2}_g, 3_g, 4_g)$	
$\mathcal{D}_{4,g_3}^{0,\text{id}.q}$	$D_4^0(1_q^{\text{id.}}, 2_g, \hat{3}_g, 4_g)$	
$\mathcal{D}_{4,q}^{0,\text{id}.g_2}$	$D_4^0(\hat{1}_{\bar{q}}, 2_g^{\text{id.}}, 3_g, 4_g)$	
$\mathcal{D}_{4,g_3}^{0,\text{id}.g_2}$	$D_4^0(1_q, 2_g^{\text{id.}}, \hat{3}_g, 4_g)$	$2 \leftrightarrow 4$
$\mathcal{D}_{4,g_4}^{0,\text{id}.g_2}$	$D_4^0(1_q, 2_g^{\text{id.}}, 3_g, \hat{4}_g)$	
$\mathcal{D}_{4,q}^{0,\text{id}.g_3}$	$D_4^0(\hat{1}_{\bar{q}}, 2_g, 3_g^{\text{id.}}, 4_g)$	
$\mathcal{D}_{4,g_2}^{0,\text{id}.g_3}$	$D_4^0(1_q, \hat{2}_g, 3_g^{\text{id.}}, 4_g)$	
$\mathcal{E}_{4,q'}^{0,\text{id}.q}$	$E_4^0(1_q^{\text{id.}}, \hat{2}_{q'}, 3_{q'}, 4_g)$	
$\mathcal{E}_{4,q'}^{0,\text{id}.q}$	$E_4^0(1_q^{\text{id.}}, 2_{q'}, \hat{3}_{q'}, 4_g)$	
$\mathcal{E}_{4,g}^{0,\text{id}.q}$	$E_4^0(1_q^{\text{id.}}, 2_{q'}, 3_{q'}, \hat{4}_g)$	
$\mathcal{E}_{4,q}^{0,\text{id}.q'}$	$E_4^0(\hat{1}_{\bar{q}}, 2_{q'}^{\text{id.}}, 3_{q'}, 4_g)$	
$\mathcal{E}_{4,q'}^{0,\text{id}.q'}$	$E_4^0(1_q, 2_{q'}^{\text{id.}}, \hat{3}_{q'}, 4_g)$	
$\mathcal{E}_{4,g}^{0,\text{id}.q'}$	$E_4^0(1_q, 2_{q'}^{\text{id.}}, 3_{q'}, \hat{4}_g)$	No symm.
$\mathcal{E}_{4,q}^{0,\text{id}.q'}$	$E_4^0(\hat{1}_{\bar{q}}, 2_{q'}, 3_{q'}^{\text{id.}}, 4_g)$	
$\mathcal{E}_{4,q'}^{0,\text{id}.q'}$	$E_4^0(1_q, \hat{2}_{q'}, 3_{q'}^{\text{id.}}, 4_g)$	
$\mathcal{E}_{4,g}^{0,\text{id}.q'}$	$E_4^0(1_q, 2_{q'}, 3_{q'}^{\text{id.}}, \hat{4}_g)$	
$\mathcal{E}_{4,q}^{0,\text{id}.g}$	$E_4^0(\hat{1}_{\bar{q}}, 2_{q'}, 3_{q'}, 4_g^{\text{id.}})$	
$\mathcal{E}_{4,q'}^{0,\text{id}.g}$	$E_4^0(1_q, \hat{2}_{q'}, 3_{q'}, 4_g^{\text{id.}})$	
$\mathcal{E}_{4,q'}^{0,\text{id}.g}$	$E_4^0(1_q, 2_{q'}, \hat{3}_{q'}, 4_g^{\text{id.}})$	
$\tilde{\mathcal{E}}_{4,q'}^{0,\text{id}.q}$	$\tilde{E}_4^0(1_q^{\text{id.}}, 2_{q'}, \hat{3}_{q'}, 4_g)$	
$\tilde{\mathcal{E}}_{4,g}^{0,\text{id}.q}$	$\tilde{E}_4^0(1_q^{\text{id.}}, 2_{q'}, 3_{q'}, \hat{4}_g)$	
$\tilde{\mathcal{E}}_{4,q}^{0,\text{id}.q'}$	$\tilde{E}_4^0(\hat{1}_{\bar{q}}, 2_{q'}^{\text{id.}}, 3_{q'}, 4_g)$	
$\tilde{\mathcal{E}}_{4,q'}^{0,\text{id}.q'}$	$\tilde{E}_4^0(1_q, 2_{q'}^{\text{id.}}, \hat{3}_{q'}, 4_g)$	$2 \leftrightarrow 3$
$\tilde{\mathcal{E}}_{4,g}^{0,\text{id}.q'}$	$\tilde{E}_4^0(1_q, 2_{q'}^{\text{id.}}, 3_{q'}, \hat{4}_g)$	
$\tilde{\mathcal{E}}_{4,q}^{0,\text{id}.g}$	$\tilde{E}_4^0(\hat{1}_{\bar{q}}, 2_{q'}, 3_{q'}, 4_g^{\text{id.}})$	
$\tilde{\mathcal{E}}_{4,q'}^{0,\text{id}.g}$	$\tilde{E}_4^0(1_q, 2_{q'}, \hat{3}_{q'}, 4_g^{\text{id.}})$	

Table 7. Integrated quark-gluon tree-level four-parton antenna functions.

Notation	Integral of	Integrand symm.	
Hard radiators: gluon-gluon			
$\mathcal{F}_{4,g_2}^{0,\text{id}.g_1}$	$F_4^0(1_g^{\text{id.}}, \hat{2}_g, 3_g, 4_g)$	$1 \leftrightarrow 3, 2 \leftrightarrow 4, 1 \leftrightarrow 2 + 3 \leftrightarrow 4$	
$\mathcal{F}_{4,g_3}^{0,\text{id}.g_1}$	$F_4^0(1_g^{\text{id.}}, 2_g, \hat{3}_g, 4_g)$		
$\mathcal{G}_{4,g}^{0,\text{id}.g}$	$G_4^0(1_g^{\text{id.}}, 3_q, 4_{\bar{q}}, \hat{2}_g)$	$1 \leftrightarrow 2 + 3 \leftrightarrow 4$	
$\mathcal{G}_{4,q}^{0,\text{id}.g_1}$	$G_4^0(1_g^{\text{id.}}, 3_q, \hat{4}_{\bar{q}}, 2_g)$		
$\mathcal{G}_{4,q}^{0,\text{id}.g_2}$	$G_4^0(1_g, 3_q, \hat{4}_{\bar{q}}, 2_g^{\text{id.}})$		
$\mathcal{G}_{4,g_1}^{0,\text{id}.q}$	$G_4^0(\hat{1}_g, 3_q^{\text{id.}}, 4_{\bar{q}}, 2_g)$		
$\mathcal{G}_{4,q}^{0,\text{id}.q}$	$G_4^0(1_g, 3_q^{\text{id.}}, \hat{4}_{\bar{q}}, 2_g)$		
$\mathcal{G}_{4,g_2}^{0,\text{id}.q}$	$G_4^0(1_g, 3_q^{\text{id.}}, 4_{\bar{q}}, \hat{2}_g)$		
$\tilde{\mathcal{G}}_{4,q}^{0,\text{id}.g}$	$\tilde{G}_4^0(1_g^{\text{id.}}, 3_q, \hat{4}_{\bar{q}}, 2_g)$		$1 \leftrightarrow 2, 3 \leftrightarrow 4$
$\tilde{\mathcal{G}}_{4,g}^{0,\text{id}.g}$	$\tilde{G}_4^0(\hat{1}_g, 3_q, 4_{\bar{q}}, 2_g^{\text{id.}})$		
$\tilde{\mathcal{G}}_{4,g}^{0,\text{id}.q}$	$\tilde{G}_4^0(\hat{1}_g, 3_q^{\text{id.}}, 4_{\bar{q}}, 2_g)$		
$\tilde{\mathcal{G}}_{4,q}^{0,\text{id}.q}$	$\tilde{G}_4^0(1_g, 3_q^{\text{id.}}, \hat{4}_{\bar{q}}, 2_g)$		
$\mathcal{H}_{4,q}^{0,\text{id}.q}$	$H_4^0(1_q^{\text{id.}}, \hat{2}_{\bar{q}}, 3_{q'}, 4_{\bar{q}'})$	$1 \leftrightarrow 2, 3 \leftrightarrow 4, 1 \leftrightarrow 3 + 2 \leftrightarrow 4$	
$\mathcal{H}_{4,q'}^{0,\text{id}.q}$	$H_4^0(1_q^{\text{id.}}, 2_{\bar{q}}, 3_{q'}, \hat{4}_{\bar{q}'})$		

Table 8. Integrated gluon-gluon tree-level four-parton antenna functions.

4 Mass factorization of initial- and final-state collinear singularities

The integrated fragmentation antenna functions can be collected into so-called integrated dipoles, which allow for a natural organization of infrared singularities at the double-virtual level [27, 43, 56, 57]. It is noticeable that the assembly and the properties of integrated dipoles in the context on fragmentation antenna functions fully match the ones of standard antenna functions, suggesting that the overall antenna subtraction infrastructure in the presence of identified particles in the final state should not differ from the one without fragmentation. For this reason, the presentation of the integrated dipoles closely aligns with the discussion in [43], to which we refer for an in-depth illustration of characteristic structures which are summarized in the following.

We distinguish between identity-preserving (IP) integrated dipoles, reproducing the infrared singularity structure of virtual corrections, and identity-changing (IC) integrated dipoles, addressing identity-changing initial and final-state collinear singularities. For fragmentation processes, we are interested in FF and IF integrated dipoles. To resort to a more compact notation, in the following we relabel z , the momentum fraction carried by the identified final-state parton, as x_3 . Consistently, we label with 3 the identified final-state parton in the list of external momenta, to distinguish from 1 and 2 used for initial-state partons. All integrated antenna functions depend therefore on the three momentum fractions x_1 , x_2 and $x_3 (= z)$, but here we keep this dependence implicit. We note that for FF fragmentation antenna functions, the dependence on x_1 and x_2 is trivially given by $\delta(1-x_1)\delta(1-x_2)$ and for IF fragmentation antenna function the x_2 dependence is trivially given by $\delta(1-x_2)$. We also introduce the shorthand notation $\delta_i = \delta(1-x_i)$. In the presence of identity-changing configurations we use the subscripts $(a \rightarrow b)$ and $(a \leftarrow b)$ to indicate a change of parton species from a to b respectively in the initial or final state.

4.1 One-loop integrated dipoles

We first consider one-loop integrated dipoles. They are given as a combination of integrated three-parton tree-level fragmentation antenna functions and NLO mass factorisation kernels. The general structure in the IP case reads:

$$\mathcal{J}_2^{(1)}(q, \bar{q}) = J_2^{(1)}(q, \bar{q}), \quad (4.1)$$

$$\mathcal{J}_2^{(1)}(i, g) = J_2^{(1)}(i, g) + \frac{N_f}{N_c} \hat{J}_2^{(1)}(i, g), \quad i = q, g. \quad (4.2)$$

The colour decomposition of the IC dipoles follows the one of splitting kernels at NLO:

$$\mathcal{J}_{2,q \rightarrow g}^{(1)}(g, i) = \left(\frac{N_c^2 - 1}{N_c} \right) J_{2,q \rightarrow g}^{(1)}(g, i), \quad i = q, g, \quad (4.3)$$

$$\mathcal{J}_{2,g \rightarrow q}^{(1)}(q, i) = J_{2,g \rightarrow q}^{(1)}(q, i), \quad i = q, g, \quad (4.4)$$

and analogously for IC configurations on the final-state identified leg, simply obtained by inverting the direction of the arrow, for any kinematical configuration (FF or IF). To properly adjust the spin-averaging factor for initial-state identity-changing collinear limits, we introduce [27, 43]:

$$S_{g \rightarrow q} = \frac{2 - 2\epsilon}{2} = 1 - \epsilon, \quad S_{q \rightarrow g} = \frac{2}{2 - 2\epsilon} = \frac{1}{1 - \epsilon}. \quad (4.5)$$

	Integrated dipoles
FF	$J_2^{(1)}(3_q, i_{\bar{q}}) = \mathcal{A}_3^{0,\text{id},q}(s_{3i}) - \Gamma_{qq}^{(1)}(x_3) \delta_1$
IF	$J_2^{(1)}(1_q, 3_q) = \mathcal{A}_{3,q}^{0,\text{id},q}(s_{13}) - \Gamma_{qq}^{(1)}(x_1) \delta_3 - \Gamma_{qq}^{(1)}(x_3) \delta_1$

Table 9. Identity-preserving quark-antiquark one-loop colour-stripped integrated dipoles. The subscripts indicate the identified final-state parton.

	Integrated dipoles
FF ^q	$J_2^{(1)}(3_q, i_g) = \frac{1}{2} \mathcal{D}_3^{0,\text{id},q}(s_{3i}) - \Gamma_{qq}^{(1)}(x_3) \delta_1$
	$\hat{J}_2^{(1)}(3_q, i_g) = \frac{1}{2} \mathcal{E}_3^{0,\text{id},q}(s_{3i})$
FF ^g	$J_2^{(1)}(3_g, i_q) = \mathcal{D}_{3,g \rightarrow g}^{0,\text{id},g}(s_{3i}) - \frac{1}{2} \Gamma_{gg}^{(1)}(x_3) \delta_1$
	$\hat{J}_2^{(1)}(3_g, i_q) = -\frac{1}{2} \hat{\Gamma}_{gg}^{(1)}(x_3) \delta_1$
IF ^q	$J_2^{(1)}(1_q, 3_g) = \mathcal{D}_{3,q}^{0,\text{id},g}(s_{13}) - \Gamma_{qq}^{(1)}(x_1) \delta_3 - \frac{1}{2} \Gamma_{gg}^{(1)}(x_3) \delta_1$
	$\hat{J}_2^{(1)}(1_q, 3_g) = -\frac{1}{2} \hat{\Gamma}_{gg}^{(1)}(x_3) \delta_1$
IF ^g	$J_2^{(1)}(3_q, 1_g) = \mathcal{D}_{3,g}^{0,\text{id},q}(s_{13}) - \Gamma_{qq}^{(1)}(x_3) \delta_1 - \frac{1}{2} \Gamma_{gg}^{(1)}(x_1) \delta_3$
	$\hat{J}_2^{(1)}(3_q, 1_g) = -\frac{1}{2} \hat{\Gamma}_{gg}^{(1)}(x_1) \delta_3$

Table 10. Identity-preserving quark-gluon and gluon-quark one-loop colour-stripped integrated dipoles.

	Integrated dipoles
FF	$J_2^{(1)}(3_g, i_g) = \frac{1}{2} \mathcal{F}_3^{0,\text{id},g}(s_{3i}) - \frac{1}{2} \Gamma_{gg}^{(1)}(x_3) \delta_1$
	$\hat{J}_2^{(1)}(3_g, i_g) = \frac{1}{2} \mathcal{G}_3^{0,\text{id},g}(s_{3i}) - \frac{1}{2} \hat{\Gamma}_{gg}^{(1)}(x_3) \delta_1$
IF	$J_2^{(1)}(1_g, 3_g) = \mathcal{F}_{3,g}^{0,\text{id},g}(s_{13}) - \frac{1}{2} \Gamma_{gg}^{(1)}(x_1) \delta_3 - \frac{1}{2} \Gamma_{gg}^{(1)}(x_3) \delta_1$
	$\hat{J}_2^{(1)}(1_g, 3_g) = -\frac{1}{2} \hat{\Gamma}_{gg}^{(1)}(x_1) \delta_3 - \frac{1}{2} \hat{\Gamma}_{gg}^{(1)}(x_3) \delta_1$

Table 11. Identity-preserving gluon-gluon one-loop colour-stripped integrated dipoles.

In tables 9–11 we list the identity-preserving integrated dipoles and in tables 12–14 the identity-changing ones.

In complete analogy to what happens for integrated dipoles without fragmentation [43], within IP one-loop integrated dipoles the mass factorization kernels absorb the poles of integrated antenna functions coming from initial-state PDF and final-state FF collinear singularities. The residual infrared poles reproduce the ones of one-loop virtual corrections. In particular one can write [43]:

$$\mathcal{Poles} \left[\mathcal{J}_2^{(1)}(i, j) \right] = \mathcal{Poles} \left[\text{Re} \left(\mathcal{I}_{ij}^{(1)} \left(\epsilon, \mu_r^2 \right) \right) \right], \quad (4.6)$$

	Integrated dipoles
$\text{FF}^{g\leftarrow q}$	$J_{2,g\leftarrow q}^{(1)}(3_q, i_{\bar{q}}) = -\frac{1}{2}\mathcal{A}_3^{0,\text{id},g}(s_{3i}) + \Gamma_{gq}^{(1)}(x_3)\delta_1$
IF_g^q	$J_{2,g\rightarrow q}^{(1)}(1_q, 3_q) = \mathcal{A}_{3,g}^{0,\text{id},q}(s_{13}) - S_{g\rightarrow q}\Gamma_{qq}^{(1)}(x_1)\delta_3$
$\text{IF}_q^{g\leftarrow q}$	$J_{2,g\leftarrow q}^{(1)}(1_q, 3_q) = -\mathcal{A}_{3,q}^{0,\text{id},g}(s_{13}) + \Gamma_{gq}^{(1)}(x_3)\delta_1$

Table 12. Identity-changing quark-antiquark one-loop colour-stripped integrated dipoles. The subscripts indicate different choices of identified final-state partons.

	Integrated dipoles
$\text{FF}^{g\leftarrow q}$	$J_{2,g\leftarrow q}^{(1)}(3_q, i_g) = -\mathcal{D}_{3,g\leftarrow q}^{0,\text{id},g}(s_{3i}) + \Gamma_{gq}^{(1)}(x_3)\delta_1$
$\text{FF}^{q\leftarrow g}$	$J_{2,q\leftarrow g}^{(1)}(3_g, i_q) = -\mathcal{E}_3^{0,\text{id},q'}(s_{3i}) + \Gamma_{qq}^{(1)}(x_3)\delta_1$
$\text{IF}_g^{g\leftarrow q}$	$J_{2,g\leftarrow q}^{(1)}(1_g, 3_q) = -\mathcal{D}_{3,g}^{0,\text{id},g}(s_{13}) + \Gamma_{gq}^{(1)}(x_3)\delta_1$
IF_g^g	$J_{2,g\rightarrow q}^{(1)}(1_q, 3_g) = -\mathcal{D}_{3,g\rightarrow q}^0(s_{13}) - S_{g\rightarrow q}\Gamma_{qq}^{(1)}(x_1)\delta_3$
$\text{IF}_q^{q\leftarrow g}$	$J_{2,q\leftarrow g}^{(1)}(1_q, 3_g) = -\mathcal{E}_{3,q}^{0,\text{id},q'}(s_{13}) + \Gamma_{qq}^{(1)}(x_3)\delta_1$
IF_q^g	$J_{2,q\rightarrow g}^{(1)}(1_g, 3_q) = -\mathcal{E}_{3,q'}^{0,\text{id},q}(s_{13}) - S_{q\rightarrow g}\Gamma_{qq}^{(1)}(x_1)\delta_3$

Table 13. Identity-changing quark-gluon one-loop colour-stripped integrated dipoles.

	Integrated dipoles
$\text{FF}^{q\leftarrow g}$	$J_{2,q\leftarrow g}^{(1)}(3_g, i_g) = -\mathcal{G}_3^{0,\text{id},q'}(s_{3i}) + \Gamma_{qq}^{(1)}(x_3)\delta_1$
IF_g^g	$J_{2,q\rightarrow g}^{(1)}(1_g, 3_g) = -\mathcal{G}_{3,q'}^{0,\text{id},g}(s_{13}) - S_{q\rightarrow g}\Gamma_{qq}^{(1)}(x_1)\delta_3$
$\text{IF}_g^{q\leftarrow g}$	$J_{2,q\leftarrow g}^{(1)}(3_g, 1_g) = -\mathcal{G}_{3,g}^{0,\text{id},q'}(s_{13}) + \Gamma_{qq}^{(1)}(x_3)\delta_1$

Table 14. Identity-changing gluon-gluon one-loop colour-stripped integrated dipoles.

where the quantity on the right-hand-side is closely related to Catani’s one-loop infrared insertion operators [62] and is explicitly given in [43]. The relation above holds for any kinematical and partonic configuration. On the other hand, IC one-loop integrated dipoles are free of poles, given that the singularities of integrated IC fragmentation antenna functions completely cancel against mass factorization kernels:

$$\text{Poles} \left[\mathcal{J}_{2,a\rightarrow b}^{(1)}(i, j) \right] = \text{Poles} \left[\mathcal{J}_{2,a\leftarrow b}^{(1)}(i, j) \right] = 0, \tag{4.7}$$

which again holds for any kinematical and partonic configuration.

4.2 Two-loop integrated dipoles

The ingredients needed to build the two-loop integrated dipoles in fragmentation are: four-parton tree-level integrated antenna functions, three-parton one-loop antenna functions, convolutions of two three-partons tree-level integrated antenna functions, convolutions of three-parton tree-level integrated antenna functions with NLO mass factorisation kernels and NNLO mass factorisation kernels (see appendix B).

The general decomposition of IP two-loop integrated dipoles reads:

$$\mathcal{J}_2^{(2)}(q, \bar{q}) = J_2^{(2)}(q, \bar{q}) - \frac{1}{N_c^2} \tilde{J}_2^{(2)}(q, \bar{q}) + \frac{N_f}{N_c} \hat{J}_2^{(2)}(q, \bar{q}) \quad (4.8)$$

$$\begin{aligned} \mathcal{J}_2^{(2)}(g, i) &= J_2^{(2)}(g, i) + \frac{N_f}{N_c} \hat{J}_2^{(2)}(g, i) \\ &\quad - \frac{N_f}{N_c^3} \hat{\tilde{J}}_2^{(2)}(g, i) + \frac{N_f^2}{N_c^2} \hat{\hat{J}}_2^{(2)}(g, i), \quad i = g, q. \end{aligned} \quad (4.9)$$

As extensively explained in [43], the construction of two-loop integrated dipoles is significantly less straightforward than for their one-loop counterparts. In particular, the presence of spurious singularities in some of the NNLO antenna functions requires the introduction of so-called *corrective*, *auxiliary* and *flip-flopping* integrated dipoles, to properly remove unwanted terms. Here we fully present these types of dipoles in the context of fragmentation antenna functions, but we direct the reader to [43] for additional details on their definition.

The colour decomposition of identity-changing two-loop integrated dipoles follows the one of two-loop identity-changing splitting kernels:

$$\begin{aligned} \mathcal{J}_{2,q \rightarrow g}^{(2)}(g, i) &= \left(\frac{N_c^2 - 1}{N_c} \right) \left[N_c J_{2,q \rightarrow g}^{(2)}(g, i) + \frac{1}{N_c} \tilde{J}_{2,q \rightarrow g}^{(2)}(g, i) \right. \\ &\quad \left. + N_f \hat{J}_{2,q \rightarrow g}^{(2)}(g, i) \right], \end{aligned} \quad (4.10)$$

$$\mathcal{J}_{2,g \rightarrow q}^{(2)}(q, i) = N_c J_{2,g \rightarrow q}^{(2)}(q, i) + \frac{1}{N_c} \tilde{J}_{2,g \rightarrow q}^{(2)}(q, i) + N_f \hat{J}_{2,g \rightarrow q}^{(2)}(q, i), \quad (4.11)$$

$$\mathcal{J}_{2,q \rightarrow \bar{q}}^{(2)}(\bar{q}, i) = \left(\frac{N_c^2 - 1}{N_c} \right) \left[J_{2,q \rightarrow \bar{q}}^{(2)}(\bar{q}, i) + \frac{1}{N_c} \tilde{J}_{2,q \rightarrow \bar{q}}^{(2)}(\bar{q}, i) \right], \quad (4.12)$$

$$\mathcal{J}_{2,q \rightarrow q'}^{(2)}(q', i) = \left(\frac{N_c^2 - 1}{N_c} \right) J_{2,q \rightarrow q'}^{(2)}(q', i), \quad (4.13)$$

and analogously reverting the arrow for a change of identity happening on the final state. At NNLO we can also have a change of identity for both an initial- and a final-state parton. These configurations are captured by the following integrated dipoles

$$\mathcal{J}_{2,g \rightarrow q, g \leftarrow q}^{(2)}(q, \bar{q}) = N_c J_{2,g \rightarrow q, g \leftarrow q}^{(2)}(q, \bar{q}) + \frac{1}{N_c} \tilde{J}_{2,g \rightarrow q, g \leftarrow q}^{(2)}(q, \bar{q}), \quad (4.14)$$

$$\mathcal{J}_{2,q \rightarrow g, q \leftarrow g}^{(2)}(g, g) = \left(\frac{N_c^2 - 1}{N_c} \right) J_{2,q \rightarrow g, q \leftarrow g}^{(2)}(g, g), \quad (4.15)$$

$$\mathcal{J}_{2,q' \rightarrow g, g \leftarrow q}^{(2)}(g, q) = N_c J_{2,q' \rightarrow g, g \leftarrow q}^{(2)}(g, q) + \frac{1}{N_c} \tilde{J}_{2,q' \rightarrow g, g \leftarrow q}^{(2)}(g, q), \quad (4.16)$$

and analogous ones where the order of identity changes is swapped. Finally, a single parton can undergo a double change of identity (flip-flopping):

$$\mathcal{J}_{2,q \rightarrow g \rightarrow q}^{(2)}(q, i) = \left(\frac{N_c^2 - 1}{N_c} \right) \mathcal{J}_{2,q \rightarrow g \rightarrow q}^{(2)}(q, i), \quad (4.17)$$

for a flip-flop in the initial state and analogously by inverting the direction of the arrows for a final-state flip-flop.

The IP quark-antiquark two-loop integrated dipoles are listed in table 15 and the IP quark-gluon two-loop colour stripped ones are given in table 16. The flip-flopping terms are given in table 17 while the corrective terms required to remove spurious identity-changing singularities from integrated identity-preserving gluon-initiated quark-gluon antenna functions are given in table 18. Auxiliary quark-antiquark two-loop integrated dipoles needed to remove spurious poles present in integrated quark-gluon antenna functions, which are not present in physical matrix elements, are listed in table 19. The identity-preserving gluon-gluon two-loop colour-stripped integrated dipoles are given in table 20 and the required flip-flopping terms are given in table 21. The IC quark-antiquark, quark-gluon and gluon-gluon two-loop integrated dipoles are listed in tables 22–27.

As for the one-loop case, it is possible to relate the ϵ -poles of IP two-loop colour-stripped integrated dipoles to the infrared singularities of two-loop matrix elements [43]:

$$\begin{aligned} \mathcal{Poles} \left[N_c \mathcal{J}_2^{(2)}(q, \bar{q}) - \frac{\beta_0}{\epsilon} \mathcal{J}_2^{(1)}(q, \bar{q}) \right] \\ = \mathcal{Poles} \left[\text{Re} \left(\mathcal{I}_{q\bar{q}}^{(2)}(\epsilon, \mu_r^2) - \frac{\beta_0}{\epsilon} \mathcal{I}_{q\bar{q}}^{(1)}(\epsilon, \mu_r^2) \right) \right], \end{aligned} \quad (4.18)$$

$$\begin{aligned} \mathcal{Poles} \left[N_c \mathcal{J}_2^{(2)}(g, g) - \frac{\beta_0}{\epsilon} \mathcal{J}_2^{(1)}(g, g) \right] \\ = \mathcal{Poles} \left[\text{Re} \left(\mathcal{I}_{gg}^{(2)}(\epsilon, \mu_r^2) - \frac{\beta_0}{\epsilon} \mathcal{I}_{gg}^{(1)}(\epsilon, \mu_r^2) \right) \right], \end{aligned} \quad (4.19)$$

$$\begin{aligned} \mathcal{Poles} \left[N_c \left(\mathcal{J}_2^{(2)}(q, g) + \mathcal{J}_2^{(2)}(g, \bar{q}) - 2\overline{\mathcal{J}}_2^{(2)}(q, \bar{q}) \right) \right. \\ \left. - \frac{\beta_0}{\epsilon} \left(\mathcal{J}_2^{(1)}(q, g) + \mathcal{J}_2^{(1)}(g, \bar{q}) \right) \right] \\ = \mathcal{Poles} \left[\text{Re} \left(\mathcal{I}_{qg}^{(2)}(\epsilon, \mu_r^2) + \mathcal{I}_{g\bar{q}}^{(2)}(\epsilon, \mu_r^2) - \frac{\beta_0}{\epsilon} \left(\mathcal{I}_{gg}^{(1)}(\epsilon, \mu_r^2) + \mathcal{I}_{g\bar{q}}^{(1)}(\epsilon, \mu_r^2) \right) \right) \right], \end{aligned} \quad (4.20)$$

where the quantities in right-hand-side are related to Catani's one- and two-loop infrared insertion operators [62] and are given in [43]. The relations above hold for any kinematical and partonic configuration. The IC two-loop dipoles satisfy

$$\mathcal{Poles} \left[\mathcal{J}_{2,a \rightarrow b}^{(2)}(i, j) - \frac{\beta_0}{\epsilon} \mathcal{J}_{2,a \rightarrow b}^{(1)}(i, j) \right] = 0, \quad (4.21)$$

$$\mathcal{Poles} \left[\mathcal{J}_{2,a \leftarrow b}^{(2)}(i, j) - \frac{\beta_0}{\epsilon} \mathcal{J}_{2,a \leftarrow b}^{(1)}(i, j) \right] = 0, \quad (4.22)$$

where we notice that the one-loop integrated dipoles present in this formulae can vanish if the considered change of identity is not allowed at one-loop, for example $q \rightarrow q'$. The full

	Integrated dipoles
FF	$J_2^{(2)}(3_q, i_{\bar{q}}) = \mathcal{A}_4^{0,\text{id}.q} + \mathcal{A}_3^{1,\text{id}.q} + \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{A}_3^{0,\text{id}.q} - \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id}.q} \otimes \mathcal{A}_3^{0,\text{id}.q} \right] - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
	$\tilde{J}_2^{(2)}(3_q, i_{\bar{q}}) = \frac{1}{2} \tilde{\mathcal{A}}_4^{0,\text{id}.q} + 2\mathcal{C}_4^{0,\text{id}.\bar{q}} + \mathcal{C}_4^{0,\text{id}.q_1} + \tilde{\mathcal{A}}_3^{1,\text{id}.q} - \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id}.q} \otimes \mathcal{A}_3^{0,\text{id}.q} \right] + \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_2^{(2)}(3_q, i_{\bar{q}}) = \mathcal{B}_4^{0,\text{id}.q} + \tilde{\mathcal{A}}_3^{1,\text{id}.q} + \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{A}_3^{0,\text{id}.q} - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
IF	$J_2^{(2)}(1_q, 3_q) = \mathcal{A}_{4,q}^{0,\text{id}.q} + \mathcal{A}_{3,q}^{1,\text{id}.q} + \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{A}_{3,q}^{0,\text{id}.q} - \frac{1}{2} \left[\mathcal{A}_{3,q}^{0,\text{id}.q} \otimes \mathcal{A}_{3,q}^{0,\text{id}.q} \right] - \widehat{\Gamma}_{qq}^{(2)}(x_1) \delta_3 - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
	$\tilde{J}_2^{(2)}(1_q, 3_q) = \frac{1}{2} \tilde{\mathcal{A}}_{4,q}^{0,\text{id}.q} + 2\mathcal{C}_{4,\bar{q}_1}^{0,\text{id}.\bar{q}_2} + 2\mathcal{C}_{4,q_4}^{0,\text{id}.q_1} + \tilde{\mathcal{A}}_{3,q}^{1,\text{id}.q} - \frac{1}{2} \left[\mathcal{A}_{3,q}^{0,\text{id}.q} \otimes \mathcal{A}_{3,q}^{0,\text{id}.q} \right] + \widehat{\Gamma}_{qq}^{(2)}(x_1) \delta_3 + \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_2^{(2)}(1_q, 3_q) = \mathcal{B}_{4,q}^{0,\text{id}.q} + \tilde{\mathcal{A}}_{3,q}^{1,\text{id}.q} + \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{A}_{3,q}^{0,\text{id}.q} - \widehat{\Gamma}_{qq}^{(2)}(x_1) \delta_3 - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$

Table 15. Identity-preserving quark-antiquark two-loop colour-stripped integrated dipoles. The subscripts indicate different choices of initial-state partons.

set of final-final and initial-final fragmentation colour-stripped integrated dipoles is provided as ancillary files in the supplementary material.

	Integrated dipoles
FF ^q	$J_2^{(2)}(3_q, i_g) = \frac{1}{2} \mathcal{D}_4^{0,\text{id},q} + \frac{1}{2} \mathcal{D}_3^{1,\text{id},q} + \frac{1}{2} \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_3^{0,\text{id},q} - \frac{1}{4} \left[\mathcal{D}_3^{0,\text{id},q} \otimes \mathcal{D}_3^{0,\text{id},q} \right] - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_2^{(2)}(3_q, i_g) = \mathcal{E}_4^{0,\text{id},q_1} + \frac{1}{2} \widehat{\mathcal{D}}_3^{1,\text{id},q} + \frac{1}{2} \mathcal{E}_3^{1,\text{id},q} + \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_3^{0,\text{id},q} + \frac{1}{2} \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_3^{0,\text{id},q} - \frac{1}{2} \left[\mathcal{E}_3^{0,\text{id},q} \otimes \mathcal{D}_3^{0,\text{id},q} \right] - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{\hat{J}}_2^{(2)}(3_q, i_g) = \frac{1}{2} \widehat{\mathcal{E}}_4^{0,\text{id},q_1} + \frac{1}{2} \widehat{\mathcal{E}}_3^{1,\text{id},q}$
	$\hat{\hat{J}}_2^{(2)}(3_q, i_g) = \frac{1}{2} \widehat{\mathcal{E}}_3^{1,\text{id},q} + \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_3^{0,\text{id},q} - \frac{1}{4} \left[\mathcal{E}_3^{0,\text{id},q} \otimes \mathcal{E}_3^{0,\text{id},q} \right]$
FF ^g	$J_2^{(2)}(3_g, i_q) = \mathcal{D}_4^{0,\text{id},g_2} + \frac{1}{2} \mathcal{D}_4^{0,\text{id},g_3} + \mathcal{D}_3^{1,\text{id},g} + \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_3^{0,\text{id},g} - \left[\mathcal{D}_{3,g \leftarrow g}^{0,\text{id},g} \otimes \mathcal{D}_{3,g \leftarrow g}^{0,\text{id},g} \right] - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + J_{2,\text{IC corr.}}^{(2)}(3_q, i_q)$
	$\hat{J}_2^{(2)}(3_g, i_q) = \mathcal{E}_4^{0,\text{id},g} + \widehat{\mathcal{D}}_3^{1,\text{id},g} + \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_3^{0,\text{id},g} - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(3_g, i_q) + \hat{J}_{2,\text{IC corr.}}^{(2)}(3_q, i_q)$
	$\hat{\hat{J}}_2^{(2)}(3_g, i_q) = \frac{1}{2} \widehat{\mathcal{E}}_4^{0,\text{id},g} + \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(3_g, i_q)$
	$\hat{\hat{J}}_2^{(2)}(3_g, i_q) = -\frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
IF _q ^g	$J_2^{(2)}(1_q, 3_g) = \mathcal{D}_{4,q}^{0,\text{id},g_2} + \frac{1}{2} \mathcal{D}_{4,q}^{0,\text{id},g_3} + \mathcal{D}_{3,q}^{1,\text{id},g} + \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_{3,q}^{0,\text{id},g} - \left[\mathcal{D}_{3,q}^{0,\text{id},g} \otimes \mathcal{D}_{3,q}^{0,\text{id},g} \right] - \widehat{\Gamma}_{qq}^{(2)}(x_1) \delta_3 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_2^{(2)}(1_q, 3_g) = \mathcal{E}_{4,q}^{0,\text{id},g} + \frac{1}{2} \widehat{\mathcal{D}}_{3,q}^{1,\text{id},g} + \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_{3,q}^{0,\text{id},g} - \widehat{\Gamma}_{qq}^{(2)}(x_1) \delta_3 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(1_q, 3_g)$
	$\hat{\hat{J}}_2^{(2)}(1_q, 3_g) = \frac{1}{2} \widehat{\mathcal{E}}_{4,q}^{0,\text{id},g} + \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(1_q, 3_g)$
	$\hat{\hat{J}}_2^{(2)}(1_q, 3_g) = -\frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
IF _g ^q	$J_2^{(2)}(1_g, 3_q) = \mathcal{D}_{4,g_2}^{0,\text{id},q} + \frac{1}{2} \mathcal{D}_{4,g_3}^{0,\text{id},q} + \mathcal{D}_{3,g}^{1,\text{id},q} + \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_{3,g}^{0,\text{id},q} - \left[\mathcal{D}_{3,g}^{0,\text{id},q} \otimes \mathcal{D}_{3,g}^{0,\text{id},q} \right] - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_1) \delta_3$
	$\hat{J}_2^{(2)}(1_g, 3_q) = \mathcal{E}_{4,g}^{0,\text{id},q} + \frac{1}{2} \widehat{\mathcal{D}}_{3,g}^{1,\text{id},q} + \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{D}_{3,g}^{0,\text{id},q} - \widehat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_1) \delta_3 + \hat{J}_{2,f/f}^{(2)}(3_q, 1_g)$
	$\hat{\hat{J}}_2^{(2)}(1_g, 3_q) = \frac{1}{2} \widehat{\mathcal{E}}_{4,g}^{0,\text{id},q} + \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_1) \delta_3 + \hat{J}_{2,f/f}^{(2)}(3_q, 1_g)$
	$\hat{\hat{J}}_2^{(2)}(1_g, 3_q) = -\frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_1) \delta_3$

Table 16. Identity-preserving quark-gluon and gluon-quark two-loop colour-stripped integrated dipoles.

	Integrated dipoles
FF ^g	$\hat{J}_{2,f/f}^{(2)}(3_g, i_q) = - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{gq}^{(1)}(x_3) \right] \delta_1$
	$\hat{\tilde{J}}_{2,f/f}^{(2)}(3_g, i_q) = - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{gq}^{(1)}(x_3) \right] \delta_1$
IF _q ^g	$\hat{J}_{2,f/f}^{(2)}(3_g, 1_q) = - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{gq}^{(1)}(x_3) \right] \delta_1$
	$\hat{\tilde{J}}_{2,f/f}^{(2)}(3_g, 1_q) = - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{gq}^{(1)}(x_3) \right] \delta_1$
IF _g ^q	$\hat{J}_{2,f/f}^{(2)}(1_g, 3_q) = S_{g \rightarrow q} \left[\Gamma_{gq}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3$
	$\hat{\tilde{J}}_{2,f/f}^{(2)}(1_g, 3_q) = S_{g \rightarrow q} \left[\Gamma_{gq}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3$

Table 17. Flip-flopping contributions to identity-preserving quark-gluon two-loop integrated dipoles.

	Integrated dipoles
FF	$ \begin{aligned} J_{2,\text{IC corr}}^{(2)}(3_q, i_q) = & - \left[\mathcal{D}_3^{0,\text{id},q} \otimes \mathcal{D}_{3,g \leftarrow q}^{0,\text{id},g} \right] - \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{D}_{3,g \leftarrow q}^{0,\text{id},g} \right] \\ & + 2 \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{D}_{3,g \leftarrow q}^{0,\text{id},g} \right] - \mathcal{A}_4^{0,\text{id},g} - \frac{1}{2} \tilde{\mathcal{A}}_4^{0,\text{id},g} - \frac{1}{2} \mathcal{A}_3^{1,\text{id},g} \\ & - \frac{1}{2} \tilde{\mathcal{A}}_3^{1,\text{id},g} - \frac{1}{2} \frac{b_0}{\epsilon} \left(\left(\frac{ s_{3i} }{\mu^2} \right)^{-\epsilon} - 1 \right) \mathcal{A}_3^{0,\text{id},g} + \left[\mathcal{A}_3^{0,\text{id},q} \otimes \mathcal{A}_3^{0,\text{id},g} \right] \\ & + \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{A}_3^{0,\text{id},g} \right] - \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{A}_3^{0,\text{id},g} \right] \\ & - \frac{b_0}{\epsilon} \left(\mathcal{D}_3^{0,\text{id},g} - \mathcal{D}_{3,g \leftarrow q}^{0,\text{id},g} \right) \end{aligned} $
	$ \begin{aligned} \hat{J}_{2,\text{IC corr}}^{(2)}(3_q, i_q) = & - \left[\mathcal{E}_3^{0,\text{id},q} \otimes \mathcal{D}_{3,g \leftarrow q}^{0,\text{id},g} \right] - \left[\hat{\Gamma}_{gg}^{(1)}(x_3) \otimes \mathcal{D}_{3,g \leftarrow q}^{0,\text{id},g} \right] \\ & - \frac{1}{2} \hat{\mathcal{A}}_3^{1,\text{id},g} - \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\left(\frac{ s_{3i} }{\mu^2} \right)^{-\epsilon} - 1 \right) \mathcal{A}_3^{0,\text{id},g} \\ & + \frac{1}{2} \left[\hat{\Gamma}_{gg}^{(1)}(x_3) \otimes \mathcal{A}_3^{0,\text{id},g} \right] - \frac{b_{0,F}}{\epsilon} \left(\mathcal{D}_3^{0,\text{id},g} - \mathcal{D}_{3,g \rightarrow g}^{0,\text{id},g} \right) \end{aligned} $

Table 18. Corrective terms required to remove spurious identity-changing singularities from integrated identity-preserving gluon-initiated quark-gluon antenna functions.

	Integrated dipoles
FF	$\bar{J}_2^{(2)}(3_q, i_{\bar{q}}) = \frac{1}{2} \tilde{\mathcal{A}}_4^{0,\text{id},q} + \tilde{\mathcal{A}}_3^{1,\text{id},q} - \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id},q} \otimes \mathcal{A}_3^{0,\text{id},q} \right]$
	$\tilde{\tilde{J}}_2^{(2)}(3_q, i_{\bar{q}}) = \tilde{J}_2^{(2)}(3_q, j_{\bar{q}})$
IF	$\bar{J}_2^{(2)}(1_q, 3_q) = \frac{1}{2} \tilde{\mathcal{A}}_{4,q}^{0,\text{id},q} + \tilde{\mathcal{A}}_{3,q}^{1,\text{id},q} - \frac{1}{2} \left[\mathcal{A}_{3,q}^{0,\text{id},q} \otimes \mathcal{A}_{3,q}^{0,\text{id},q} \right]$
	$\tilde{\tilde{J}}_2^{(2)}(1_q, 3_q) = \tilde{J}_2^{(2)}(1_q, 3_q)$

Table 19. Auxiliary quark-antiquark two-loop integrated dipoles needed to remove spurious poles present in integrated quark-gluon antenna functions, which are not present in physical matrix elements. $\tilde{\tilde{J}}_2^{(2)}(3_q, j_{\bar{q}})$ is the FF fragmentation dipoles given in table 15.

	Integrated dipoles
FF	$J_2^{(2)}(3_g, i_g) = \frac{1}{2} \mathcal{F}_4^{0,\text{id}.g} + \frac{1}{2} \mathcal{F}_3^{1,\text{id}.g} + \frac{1}{2} \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{F}_3^{0,\text{id}.g} - \frac{1}{4} \left[\mathcal{F}_3^{0,\text{id}.g} \otimes \mathcal{F}_3^{0,\text{id}.g} \right] - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_2^{(2)}(3_g, i_g) = \mathcal{G}_4^{0,\text{id}.g} + \frac{1}{2} \widehat{\mathcal{F}}_3^{1,\text{id}.g} + \frac{1}{2} \mathcal{G}_3^{1,\text{id}.g} + \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{F}_3^{0,\text{id}.g} + \frac{1}{2} \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_3^{0,\text{id}.g} - \frac{1}{2} \left[\mathcal{G}_3^{0,\text{id}.g} \otimes \mathcal{F}_3^{0,\text{id}.g} \right] - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(3_g, i_g)$
	$\hat{\hat{J}}_2^{(2)}(3_g, i_g) = \frac{1}{2} \widehat{\mathcal{G}}_4^{0,\text{id}.g} + \frac{1}{2} \widehat{\mathcal{G}}_3^{1,\text{id}.g} + \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(3_g, i_g)$
	$\hat{\hat{J}}_2^{(2)}(3_g, i_g) = \frac{1}{2} \widehat{\mathcal{G}}_3^{1,\text{id}.g} + \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{1i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_3^{0,\text{id}.g} - \frac{1}{4} \left[\mathcal{G}_3^{0,\text{id}.g} \otimes \mathcal{G}_3^{0,\text{id}.g} \right] - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
IF	$J_2^{(2)}(1_g, 3_g) = \mathcal{F}_{4,g_2}^{0,\text{id}.g_1} + \frac{1}{2} \mathcal{F}_{4,g_3}^{0,\text{id}.g_1} + \mathcal{F}_{3,g}^{1,\text{id}.g} + \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{F}_{3,g}^{0,\text{id}.g} - \left[\mathcal{F}_{3,g}^{0,\text{id}.g} \otimes \mathcal{F}_{3,g}^{0,\text{id}.g} \right] - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)}(x_1) \delta_3 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_2^{(2)}(1_g, 3_g) = \mathcal{G}_{4,g}^{0,\text{id}.g} + \widehat{\mathcal{F}}_{3,g}^{1,\text{id}.g} + \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{F}_{3,g}^{0,\text{id}.g} - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)}(x_1) \delta_3 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 + \hat{J}_{2,f/f}^{(2)}(1_g, 3_g)$
	$\hat{\hat{J}}_2^{(2)}(1_g, 3_g) = \frac{1}{2} \mathcal{G}_{4,g}^{0,\text{id}.g} + \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)}(x_1) \delta_3 + \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_1) \delta_3 + \hat{J}_{2,f/f}^{(2)}(1_g, 3_g)$
	$\hat{\hat{J}}_2^{(2)}(1_g, 3_g) = -\frac{1}{2} \widehat{\Gamma}_{gg}^{(2)}(x_1) \delta_3 - \frac{1}{2} \widehat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$

Table 20. Identity-preserving gluon-gluon two-loop colour-stripped integrated dipoles.

	Integrated dipoles
FF	$\hat{J}_{2,f/f}^{(2)}(3_g, i_g) = - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{G}_3^{0,\text{id}.q'} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] \delta_1$
	$\hat{\hat{J}}_{2,f/f}^{(2)}(3_g, i_g) = - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{G}_3^{0,\text{id}.q'} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] \delta_1$
IF	$\hat{J}_{2,f/f}^{(2)}(1_g, 2_g) = S_{g \rightarrow q} \left[\Gamma_{qg}^{(1)}(x_1) \otimes \mathcal{G}_{3,q'}^{0,\text{id}.g} \right] + \frac{1}{2} \left[\Gamma_{qg}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3 - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{G}_{3,g}^{0,\text{id}.q} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] \delta_1$
	$\hat{\hat{J}}_{2,f/f}^{(2)}(1_g, 2_g) = S_{g \rightarrow q} \left[\Gamma_{qg}^{(1)}(x_1) \otimes \mathcal{G}_{3,q'}^{0,\text{id}.g} \right] + \frac{1}{2} \left[\Gamma_{qg}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3 - \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{G}_{3,g}^{0,\text{id}.q} \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] \delta_1$

Table 21. Flip-flopping contributions to identity-preserving gluon-gluon two-loop integrated dipoles.

	Integrated dipoles
FF $g\leftarrow q$	$ \begin{aligned} J_{2,g\leftarrow q}^{(2)}(3_q, i_{\bar{q}}) &= -\mathcal{A}_4^{0,\text{id},g} - \frac{1}{2}\mathcal{A}_3^{1,\text{id},g} - \frac{1}{2}\frac{b_0}{\epsilon} \left(\frac{ s_{3\bar{1}} }{\mu_r^2}\right)^{-\epsilon} \mathcal{A}_3^{0,\text{id},g} \\ &+ \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id},g} \otimes \mathcal{A}_3^{0,\text{id},q} \right] + \bar{\Gamma}_{gq}^{(2)\text{id.}}(x_3) \delta_1 \\ &+ \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id},g} \otimes \Gamma_{gg}^{(1)}(x_3) \right] - \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{gg}^{(1)}(x_3) \right] \delta_1 \\ &- \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id},g} \otimes \Gamma_{qq}^{(1)}(x_3) \right] + \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 \end{aligned} $
	$ \begin{aligned} \hat{J}_{2,g\leftarrow q}^{(2)}(3_q, i_{\bar{q}}) &= -\frac{1}{2}\hat{\mathcal{A}}_3^{1,\text{id},g} - \frac{1}{2}\frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3\bar{1}} }{\mu_r^2}\right)^{-\epsilon} \mathcal{A}_3^{0,\text{id},g} + \hat{\Gamma}_{gq}^{(2)\text{id.}}(x_3) \delta_1 \\ &+ \frac{1}{2} \left[\mathcal{A}_3^{0,\text{id},g} \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] - \frac{1}{2} \left[\Gamma_{gq}^{(1)}(x_3) \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] \delta_1 \end{aligned} $
	$ \begin{aligned} \tilde{J}_{2,g\leftarrow q}^{(2)}(3_q, i_{\bar{q}}) &= -\tilde{\mathcal{A}}_4^{0,\text{id},g} - \tilde{\mathcal{A}}_3^{1,\text{id},g} + \left[\mathcal{A}_3^{0,\text{id},q} \otimes \mathcal{A}_3^{0,\text{id},g} \right] - \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{A}_3^{0,\text{id},g} \right] \\ &+ \left[\Gamma_{gq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 - 2\tilde{\Gamma}_{gq}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $
FF $q\leftarrow\bar{q}$	$ \tilde{J}_{2,\bar{q}\leftarrow q}^{(2)}(3_{\bar{q}}, i_q) = C_4^{0,\text{id},q_3} + \tilde{\Gamma}_{q\bar{q}}^{(2)\text{id.}}(x_3) \delta_1 $
FF $q\leftarrow g\leftarrow q$	$ \begin{aligned} J_{2,q\leftarrow g\leftarrow q}^{(2)}(3_q, i_g) &= \mathcal{B}_4^{0,\text{id},q'} - \left[\Gamma_{qg}^{(1)}(x_3) \otimes \mathcal{A}_3^{0,\text{id},g} \right] \\ &+ \left[\Gamma_{qg}^{(1)}(x_3) \otimes \Gamma_{gq}^{(1)}(x_3) \right] \delta_1 - 2\tilde{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $

Table 22. Final-final identity-changing quark-antiquark two-loop colour-stripped integrated dipoles.

	Integrated dipoles
$\text{IF}_{g \rightarrow q}^q$	$ \begin{aligned} J_{2,g \rightarrow q}^{(2)}(1_q, 3_q) &= -\mathcal{A}_{4,g_3}^{0,\text{id},q} - \mathcal{A}_{4,g_4}^{0,\text{id},q} - \mathcal{A}_{3,g}^{1,\text{id},q} + \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu^2} \right)^{-\epsilon} \mathcal{A}_{3,g}^{0,\text{id},q} \\ &\quad - \left[\mathcal{A}_{3,g}^{0,\text{id},q} \otimes \mathcal{A}_{3,q}^{0,\text{id},q} \right] - S_{g \rightarrow q} \bar{\Gamma}_{qq}^{(2)}(x_1) \delta_3 \\ &\quad - \left[\mathcal{A}_{3,g}^{0,\text{id},q} \otimes \Gamma_{gg}^{(1)}(x_1) \right] + \frac{1}{2} S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \delta_3 \\ &\quad + \left[\mathcal{A}_{3,g}^{0,\text{id},q} \otimes \Gamma_{qq}^{(1)}(x_1) \right] - \frac{1}{2} S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{qq}^{(1)}(x_1) \right] \delta_3 \\ \hat{J}_{2,g \rightarrow q}^{(2)}(1_q, 3_q) &= -\frac{1}{2} \hat{\mathcal{A}}_{3,g}^{1,\text{id},q} + \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu^2} \right)^{-\epsilon} \mathcal{A}_{3,g}^{0,\text{id},q} - S_{g \rightarrow q} \hat{\Gamma}_{qq}^{(2)}(x_1) \delta_3 \\ &\quad - \left[\mathcal{A}_{3,g}^{0,\text{id},q} \otimes \hat{\Gamma}_{gg}^{(1)}(x_1) \right] + \frac{1}{2} S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \hat{\Gamma}_{gg}^{(1)}(x_1) \right] \delta_3 \\ \tilde{J}_{2,g \rightarrow q}^{(2)}(1_q, 3_q) &= -\tilde{\mathcal{A}}_{4,g}^{0,\text{id},q} - \tilde{\mathcal{A}}_{3,g}^{1,\text{id},q} \\ &\quad + \left[\mathcal{A}_{3,q}^{0,\text{id},q} \otimes \mathcal{A}_{4,g}^{0,\text{id},q} \right] + \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{A}_{3,g}^{0,\text{id},q} \right] \\ &\quad - \frac{1}{2} S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{qq}^{(1)}(x_1) \right] \delta_3 + S_{g \rightarrow q} \tilde{\Gamma}_{qq}^{(2)}(x_1) \delta_3 \end{aligned} $
$\text{IF}_{q \leftarrow q}^q$	$ \begin{aligned} J_{2,g \leftarrow q}^{(2)}(3_q, 1_q) &= -\mathcal{A}_{4,q}^{0,\text{id},g_3} - \mathcal{A}_{4,q}^{0,\text{id},g_4} - \mathcal{A}_{3,q}^{1,\text{id},g} - \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu^2} \right)^{-\epsilon} \mathcal{A}_{3,q}^{0,\text{id},g} \\ &\quad + \left[\mathcal{A}_{3,q}^{0,\text{id},g} \otimes \mathcal{A}_{3,q}^{0,\text{id},q} \right] + \bar{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 \\ &\quad + \left[\mathcal{A}_{3,q}^{0,\text{id},g} \otimes \Gamma_{gg}^{(1)}(x_3) \right] - \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{gg}^{(1)}(x_3) \right] \delta_1 \\ &\quad - \left[\mathcal{A}_{3,q}^{0,\text{id},g} \otimes \Gamma_{qq}^{(1)}(x_3) \right] + \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 \\ \hat{J}_{2,g \leftarrow q}^{(2)}(3_q, 1_q) &= -\frac{1}{2} \hat{\mathcal{A}}_{3,q}^{1,\text{id},g} - \frac{1}{2} \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu^2} \right)^{-\epsilon} \mathcal{A}_{3,q}^{0,\text{id},g} + \hat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 \\ &\quad + \left[\mathcal{A}_{3,q}^{0,\text{id},g} \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] - \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] \delta_1 \\ \tilde{J}_{2,g \leftarrow q}^{(2)}(3_q, 1_q) &= -\tilde{\mathcal{A}}_{4,q}^{0,\text{id},g} - \tilde{\mathcal{A}}_{3,q}^{1,\text{id},g} \\ &\quad + \left[\mathcal{A}_{3,q}^{0,\text{id},q} \otimes \mathcal{A}_{4,q}^{0,\text{id},g} \right] - \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{A}_{3,q}^{0,\text{id},g} \right] \\ &\quad + \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 - \tilde{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $
$\text{IF}_{q' \rightarrow q}^q$	$ \begin{aligned} J_{2,q' \rightarrow q}^{(2)}(1_{q'}, 3_q) &= \mathcal{B}_{4,q'}^{0,\text{id},q} - S_{q \rightarrow q} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{A}_{3,g}^{0,\text{id},q} \right] \\ &\quad + \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{qq}^{(1)}(x_1) \right] \delta_3 - \bar{\Gamma}_{qq'}^{(2)}(x_1) \delta_3 \end{aligned} $
$\text{IF}_{q \leftarrow q'}^q$	$ \begin{aligned} J_{2,q \leftarrow q'}^{(2)}(3_{q'}, 1_q) &= \mathcal{B}_{4,q}^{0,\text{id},q'} - \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{A}_{3,q}^{0,\text{id},g} \right] \\ &\quad + \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 - \bar{\Gamma}_{qq'}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $
$\text{IF}_{\bar{q} \rightarrow q}^q$	$ J_{2,\bar{q} \rightarrow q}^{(2)}(1_{\bar{q}}, 3_q) = \mathcal{C}_{4,\bar{q}_3}^{0,\text{id},q_1} + \bar{\Gamma}_{q\bar{q}}^{(2)}(x_1) \delta_3 $
$\text{IF}_{q \leftarrow \bar{q}}^q$	$ J_{2,\bar{q} \leftarrow q}^{(2)}(3_{\bar{q}}, 1_q) = \mathcal{C}_{4,\bar{q}_1}^{0,\text{id},q_3} + \bar{\Gamma}_{q\bar{q}}^{(2)\text{id.}}(x_3) \delta_1 $
$\text{IF}_{g \rightarrow q, g \leftarrow q}^q$	$ \begin{aligned} J_{2,g \rightarrow q, g \leftarrow q}^{(2)}(1_q, 3_{\bar{q}}) &= \mathcal{A}_{4,g}^{0,\text{id},g} + S_{g \rightarrow q} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{A}_{3,q}^{0,\text{id},g} \right] \\ &\quad + \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{A}_{3,g}^{0,\text{id},q} \right] - 2S_{g \rightarrow q} \Gamma_{qq}^{(1)}(x_1) \Gamma_{gg}^{(1)}(x_3) \\ \tilde{J}_{2,g \rightarrow q, g \leftarrow q}^{(2)}(1_q, 3_{\bar{q}}) &= \tilde{\mathcal{A}}_{4,g}^{0,\text{id},g} + S_{g \rightarrow q} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{A}_{3,q}^{0,\text{id},g} \right] \\ &\quad + \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{A}_{3,g}^{0,\text{id},q} \right] - 2S_{g \rightarrow q} \Gamma_{qq}^{(1)}(x_1) \Gamma_{gg}^{(1)}(x_3) \end{aligned} $

Table 23. Initial-final identity-changing quark-antiquark two-loop colour-stripped integrated dipoles.

	Integrated dipoles
FF $q \leftarrow g$	$J_{2,q \leftarrow g}^{(2)}(\mathfrak{3}_g, i_q) = -\mathcal{E}_4^{0,\text{id},q_2} - \mathcal{E}_3^{1,\text{id},q'} - \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_3^{0,\text{id},q'}$ $+ 2 \left[\mathcal{D}_{3,g \leftarrow g}^{0,\text{id},g} \otimes \mathcal{E}_3^{0,\text{id},q'} \right] + \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right]$ $- \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right] - \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{gg}^{(1)}(x_3) \right] \delta_1$ $+ \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 + \bar{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
	$\hat{J}_{2,q \leftarrow g}^{(2)}(\mathfrak{3}_g, i_q) = -\hat{\mathcal{E}}_3^{1,\text{id},q'} - \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_3^{0,\text{id},q'} - \left[\hat{\Gamma}_{gg}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right]$ $+ \frac{1}{2} \left[\hat{\Gamma}_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 + \hat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$
	$\tilde{J}_{2,q \leftarrow g}^{(2)}(\mathfrak{3}_g, i_q) = -\tilde{\mathcal{E}}_4^{0,\text{id},q_2} - \tilde{\mathcal{E}}_3^{1,\text{id},q'} + \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right]$ $- \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 - \tilde{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1$

Table 24. Identity-changing final-final quark-gluon two-loop colour-stripped integrated dipoles.

	Integrated dipoles
$\text{IF}_{q \rightarrow g}^q$	$ \begin{aligned} J_{2,q \rightarrow g}^{(2)}(1_g, 3_q) &= -\mathcal{E}_{4,q'}^{0,\text{id},q} - \mathcal{E}_{4,q'}^{0,\text{id},q} - \mathcal{E}_{3,q'}^{1,\text{id},q} - \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_{3,q'}^{0,\text{id},q} \\ &+ 2 \left[\mathcal{D}_{3,g}^{0,\text{id},q} \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] + \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] \\ &- \left[\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] + \frac{1}{2} S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \delta_3 \\ &- \frac{1}{2} S_{q \rightarrow g} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \delta_3 \\ &- S_{q \rightarrow g} \bar{\Gamma}_{gg}^{(2)}(x_1) \delta_3 \end{aligned} $
	$ \begin{aligned} \hat{J}_{2,q \rightarrow g}^{(2)}(1_g, 3_q) &= -\hat{\mathcal{E}}_{3,q'}^{1,\text{id},q} - \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_{3,q'}^{0,\text{id},q} \\ &- \left[\hat{\Gamma}_{gg}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] \\ &- \frac{1}{2} S_{q \rightarrow g} \left[\hat{\Gamma}_{gg}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \delta_3 - S_{q \rightarrow g} \hat{\Gamma}_{gg}^{(2)}(x_1) \delta_3 \end{aligned} $
	$ \begin{aligned} \tilde{J}_{2,q \rightarrow g}^{(2)}(1_g, 3_q) &= -\tilde{\mathcal{E}}_{4,q'}^{0,\text{id},q} - \mathcal{E}_{3,q'}^{1,\text{id},q} + \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] \\ &+ \frac{1}{2} S_{q \rightarrow g} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_1) \right] \delta_3 + S_{q \rightarrow g} \tilde{\Gamma}_{gg}^{(2)}(x_1) \delta_3 \end{aligned} $
$\text{IF}_q^{q \leftarrow g}$	$ \begin{aligned} J_{2,q \leftarrow g}^{(2)}(1_q, 3_g) &= -\mathcal{E}_{4,q}^{0,\text{id},q'} - \mathcal{E}_{4,q}^{0,\text{id},q'} - \mathcal{E}_{3,q}^{1,\text{id},q'} - \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_{3,q}^{0,\text{id},q'} \\ &+ 2 \left[\mathcal{D}_{3,q}^{0,\text{id},g} \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] + \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] \\ &- \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] - \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{gg}^{(1)}(x_3) \right] \delta_1 \\ &+ \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 \\ &+ \bar{\Gamma}_{qq}^{(2)}(x_3) \delta_1 \end{aligned} $
	$ \begin{aligned} \hat{J}_{2,q \leftarrow g}^{(2)}(1_q, 3_g) &= -\hat{\mathcal{E}}_{3,q}^{1,\text{id},q'} - \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{E}_{3,q}^{0,\text{id},q'} \\ &- \left[\hat{\Gamma}_{gg}^{(1)}(x_3) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] \\ &+ \frac{1}{2} \left[\hat{\Gamma}_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 + \hat{\Gamma}_{qq}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $
	$ \begin{aligned} \tilde{J}_{2,q \leftarrow g}^{(2)}(1_q, 3_g) &= -\tilde{\mathcal{E}}_{4,q}^{0,\text{id},q'} - \mathcal{E}_{3,q}^{1,\text{id},q'} + \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] \\ &- \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 - \tilde{\Gamma}_{qq}^{(2)}(x_3) \delta_1 \end{aligned} $
$\text{IF}_{q' \rightarrow g}^{g \leftarrow q}$	$ \begin{aligned} J_{2,q' \rightarrow g, g \leftarrow q}^{(2)}(1_g, 3_q) &= 2\mathcal{E}_{4,q'}^{0,\text{id},g} + 2\mathcal{E}_{4,q'}^{0,\text{id},g} - 2S_{q \rightarrow g} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{D}_{3,g}^{0,\text{id},g} \right] \\ &- \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{E}_{3,q'}^{0,\text{id},q} \right] - 2S_{q \rightarrow g} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{gg}^{(1)}(x_3) \right] \\ &- \mathcal{B}_{4,q'}^0 + \left[\mathcal{E}_{3,q'}^0 \otimes \mathcal{A}_{3,g}^0 \right] \\ &+ 2S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{E}_{3,q'}^0 \right] + 2 \left[\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{qq}^{(1)}(x_1) \right] \delta_3 \end{aligned} $
$\text{IF}_{g \rightarrow q}^{q' \leftarrow g}$	$ \begin{aligned} J_{2,g \rightarrow q, q' \leftarrow g}^{(2)}(3_g, 1_q) &= 2\mathcal{E}_{4,g}^{0,\text{id},q'} + 2\mathcal{E}_{4,g}^{0,\text{id},q'} - 2 \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{D}_{3,g}^{0,\text{id},g} \right] \\ &+ 2S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{E}_{3,q}^{0,\text{id},q'} \right] - 2S_{q \rightarrow g} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_1) \right] \\ &- \mathcal{B}_4^{0,\text{id},q'} + \left[\mathcal{E}_3^{0,\text{id},q'} \otimes \mathcal{A}_3^{0,\text{id},g} \right] \\ &- 2 \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{E}_3^{0,\text{id},q'} \right] + 2 \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 \end{aligned} $

Table 25. Identity-changing initial-final quark-gluon two-loop colour-stripped integrated dipoles.

	Integrated dipoles
FF $q \leftarrow g$	$ \begin{aligned} J_{2,q \leftarrow g}^{(2)}(3g, i_g) &= -\mathcal{G}_4^{0,\text{id},q} - \mathcal{G}_3^{1,\text{id},q} - \frac{b_0}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_3^{0,\text{id},q'} \\ &+ \left[\mathcal{G}_3^{0,\text{id},q'} \otimes \mathcal{F}_3^{0,\text{id},g} \right] + \overline{\Gamma}_{qg}^{(2)\text{id.}}(x_3) \delta_1 \\ &+ \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{G}_3^{0,\text{id},q'} \right] - \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{G}_3^{0,\text{id},q'} \right] \\ &- \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] + \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] \delta_1 \end{aligned} $
	$ \begin{aligned} \hat{J}_{2,q \leftarrow g}^{(2)}(3g, i_g) &= \mathcal{H}_4^{0,\text{id},q} - \hat{\mathcal{G}}_3^{1,\text{id},q} - \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{3i} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_3^{0,\text{id},q'} \\ &- \left[\Gamma_{qg}^{(1)}(x_3) \otimes \mathcal{G}_3^{0,\text{id},g} \right] - \frac{1}{2} \left[\Gamma_{qg}^{(1)}(x_3) \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] \delta_1 \\ &+ \hat{\overline{\Gamma}}_{qg}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $
	$ \begin{aligned} \tilde{J}_{2,q \leftarrow g}^{(2)}(3g, i_g) &= -\frac{1}{2} \tilde{\mathcal{G}}_4^{0,\text{id},q} - \tilde{\mathcal{G}}_3^{1,\text{id},q} + \left[\mathcal{G}_3^{0,\text{id},q'} \otimes \Gamma_{qq}^{(1)}(x_3) \right] \\ &- \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qg}^{(1)}(x_3) \right] \delta_1 - \tilde{\overline{\Gamma}}_{qg}^{(2)\text{id.}}(x_3) \delta_1 \end{aligned} $

Table 26. Identity-changing final-final gluon-gluon two-loop colour-stripped integrated dipoles.

	Integrated dipoles
$\mathbb{IF}_{q \rightarrow g}^g$	$ \begin{aligned} J_{2,q \rightarrow g}^{(2)}(1_g, 3_g) &= -\mathcal{G}_{4,q}^{0,\text{id}.g_1} - \mathcal{G}_{4,q}^{0,\text{id}.g_2} - \mathcal{G}_{3,q'}^{1,\text{id}.g} - \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_{3,q'}^{0,\text{id}.g} \\ &\quad + 2 \left[\mathcal{G}_{3,q'}^{0,\text{id}.g} \otimes \mathcal{F}_{3,g}^{0,\text{id}.g} \right] - S_{q \rightarrow g} \bar{\Gamma}_{gq}^{(2)}(x_1) \delta_3 \\ &\quad + \left[\Gamma_{gq}^{(1)}(x_3) \otimes \mathcal{G}_{3,q}^{0,\text{id}.q} \right] + \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{G}_{3,q'}^{0,\text{id}.g} \right] - \left[\Gamma_{gg}^{(1)}(x_1) \otimes \mathcal{G}_{3,q'}^{0,\text{id}.g} \right] \\ &\quad + \frac{1}{2} S_{q \rightarrow g} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3 \\ &\quad - \frac{1}{2} S_{q \rightarrow g} \left[\Gamma_{gg}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3 \\ \hat{J}_{2,q \rightarrow g}^{(2)}(1_g, 3_g) &= -\hat{\mathcal{G}}_{3,q'}^{1,\text{id}.g} - \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_{3,q'}^{0,\text{id}.g} - S_{q \rightarrow g} \hat{\Gamma}_{gq}^{(2)}(x_1) \delta_3 \\ &\quad - \left[\mathcal{G}_{3,q'}^{0,\text{id}.g} \otimes \hat{\Gamma}_{gg}^{(1)}(x_1) \right] - \frac{1}{2} S_{q \rightarrow g} \left[\Gamma_{gq}^{(1)}(x_1) \otimes \hat{\Gamma}_{gg}^{(1)}(x_1) \right] \delta_3 \\ \tilde{J}_{2,q \rightarrow g}^{(2)}(1_g, 3_g) &= -\mathcal{G}_{4,q}^{0,\text{id}.g} - \mathcal{G}_{3,q'}^{1,\text{id}.g} + \left[\mathcal{G}_{3,q'}^{0,\text{id}.g} \otimes \Gamma_{qq}^{(1)}(x_1) \right] \\ &\quad + \frac{1}{2} S_{q \rightarrow g} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \Gamma_{gq}^{(1)}(x_1) \right] \delta_3 + S_{q \rightarrow g} \tilde{\Gamma}_{gq}^{(2)}(x_1) \delta_3 \\ &\quad - \left[\mathcal{G}_{3,q}^{0,\text{id}.q} \otimes \Gamma_{gq}^{(1)}(x_3) \right] \end{aligned} $
$\mathbb{IF}_g^{q \leftarrow g}$	$ \begin{aligned} J_{2,q \leftarrow g}^{(2)}(3_g, 1_g) &= -\mathcal{G}_{4,g_1}^{0,\text{id}.q} - \mathcal{G}_{4,g_2}^{0,\text{id}.q} - \mathcal{G}_{3,g}^{1,\text{id}.q'} - \frac{b_0}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_{3,g}^{0,\text{id}.q} \\ &\quad + 2 \left[\mathcal{G}_{3,g}^{0,\text{id}.q} \otimes \mathcal{F}_{3,g}^{0,\text{id}.g} \right] + \bar{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 \\ &\quad + S_{g \rightarrow q} \left[\Gamma_{qq}^{(1)}(x_1) \otimes \mathcal{G}_{3,q}^{0,\text{id}.q} \right] + \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{G}_{3,g}^{0,\text{id}.q} \right] - \left[\Gamma_{gg}^{(1)}(x_3) \otimes \mathcal{G}_{3,g}^{0,\text{id}.q} \right] \\ &\quad - \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 \\ &\quad + \frac{1}{2} \left[\Gamma_{gg}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 \\ \hat{J}_{2,q \leftarrow g}^{(2)}(3_g, 1_g) &= -\hat{\mathcal{G}}_{3,g}^{1,\text{id}.q'} - \frac{b_{0,F}}{\epsilon} \left(\frac{ s_{13} }{\mu_r^2} \right)^{-\epsilon} \mathcal{G}_{3,g}^{0,\text{id}.q} + \hat{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 \\ &\quad - \left[\mathcal{G}_{3,g}^{0,\text{id}.q'} \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] + \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \hat{\Gamma}_{gg}^{(1)}(x_3) \right] \delta_1 \\ \tilde{J}_{2,q \leftarrow g}^{(2)}(3_g, 1_g) &= -\mathcal{G}_{4,g}^{0,\text{id}.q} - \mathcal{G}_{3,g}^{1,\text{id}.q'} + \left[\mathcal{G}_{3,g}^{0,\text{id}.q} \otimes \Gamma_{qq}^{(1)}(x_3) \right] \\ &\quad - \frac{1}{2} \left[\Gamma_{qq}^{(1)}(x_3) \otimes \Gamma_{qq}^{(1)}(x_3) \right] \delta_1 - \tilde{\Gamma}_{gg}^{(2)\text{id.}}(x_3) \delta_1 \\ &\quad + S_{g \rightarrow q} \left[\mathcal{G}_{3,q}^{0,\text{id}.q} \otimes \Gamma_{qq}^{(1)}(x_1) \right] \end{aligned} $
$\mathbb{IF}_{q \rightarrow g}^{q \leftarrow g}$	$ \begin{aligned} J_{2,q \rightarrow g, q \leftarrow g}^{(2)}(1_g, 3_g) &= \mathcal{H}_{4,q'}^{0,\text{id}.q} + S_{q \rightarrow g} \left[\Gamma_{gq}^{(1)}(x_1) \otimes \mathcal{G}_{3,g}^{0,\text{id}.q} \right] \\ &\quad - \left[\Gamma_{qq}^{(1)}(x_3) \otimes \mathcal{G}_{3,q'}^{0,\text{id}.g} \right] - S_{q \rightarrow g} \Gamma_{gq}^{(1)}(x_1) \Gamma_{qq}^{(1)}(x_3) \end{aligned} $

Table 27. Identity-changing initial-final gluon-gluon two-loop colour-stripped integrated dipoles.

5 Conclusions

Processes with identified final-state hadrons play an increasingly important role in precision studies at the LHC. To compute NNLO QCD corrections to these observables requires an extension of existing NNLO subtraction methods, in order to keep track of the momentum fraction of the final-state fragmenting parton. In the antenna subtraction method, this is accomplished through fragmentation antenna functions, which were derived previously at NNLO only for final-final kinematics [42], relevant to e^+e^- colliders. In the present work, we complete the set of fragmentation antenna functions for hadron colliders at NNLO by computing them in initial-final kinematics. To enable the efficient and systematic construction of the relevant antenna subtraction terms, we combine the integrated fragmentation antenna functions in all kinematical settings with the respective mass factorization counterterms into dipole operators [27, 43]. These operators mirror the infrared singularity structure of the corresponding virtual loop amplitudes.

Our results enable the extension of antenna subtraction at NNLO to important hadron collider processes such as single-inclusive hadron production [23] or vector-boson-plus-hadron production [63], which were previously accessible only to NLO QCD. Likewise, hadron production processes inside jets at hadron-hadron or lepton-hadron colliders can now be computed to NNLO.

The kinematical setting of the integrated initial-final fragmentation antenna functions is identical to semi-inclusive hadron production in deep inelastic scattering (SIDIS). We could thus employ the same integration techniques as described in [39] and in section 3 to compute the NNLO QCD coefficient functions for unpolarized and polarized SIDIS, which we already presented elsewhere [20, 21].

The ability to derive precise predictions for identified hadron cross sections has important phenomenological implications. It will for the first time enable global NNLO determinations of fragmentation functions for a variety of hadron species, thereby bringing the description of fragmentation processes to a new level of quantitative accuracy. Specific final-state hadron observables also allow to determine important aspects of the structure of the colliding hadrons, such as the flavour decomposition of the quark sea. The identification of heavy-flavoured hadrons is moreover a common approach to jet flavour tagging, which can now be mirrored in a direct manner in theory calculations, thus allowing a potential alternative to the application of a flavoured jet algorithm [64–67].

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A Master integrals for double-real radiation

In the following, we document all master integrals relevant for the integrated double-real radiation fragmentation antenna functions in initial-final kinematics $\mathcal{X}_{4,i}^{0,\text{id},j}(x, z)$. The Laurent expansions of these integrals are required for generic (x, z) up to transcendental weight 2, while they are required for generic z at $x = 1$ and for generic x at $z = 1$ up to transcendental weight 3 and at the soft endpoint $x = z = 1$ up to weight 4.

A.1 Normalization of integrals

The full set of master integrals reads:

$$\begin{aligned}
 I[0](Q^2, x, z) &= N_\Gamma(Q^2)^{1-2\epsilon} (1-x)^{1-2\epsilon} (1-z)^{1-2\epsilon} x^{-1} r_0(x, z), \\
 I[1](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon} r_1(x, z), \\
 I[5](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon} (1-x)^{-2\epsilon} (1-z)^{1-2\epsilon} r_5(x, z), \\
 I[7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon} (1-x)^{1-2\epsilon} (1-z)^{1-2\epsilon} r_7(x, z), \\
 I[-2, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{1-2\epsilon} (1-x)^{2-2\epsilon} (1-z)^{1-2\epsilon} r_{m27}(x, z), \\
 I[-3, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{1-2\epsilon} (1-x)^{1-2\epsilon} (1-z)^{1-2\epsilon} r_{m37}(x, z), \\
 I[1, 4](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-1} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon} r_{14}(x, z), \\
 I[4, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-1} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon} r_{47}(x, z), \\
 I[5, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-1} (1-x)^{-2\epsilon} (1-z)^{1-2\epsilon} r_{57}(x, z), \\
 I[1, 3, 4](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon-1} 2x^2(1+x)^{-1} r_{134}(x, z), \\
 I[1, 3, 5](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon-1} x r_{135}(x, z), \\
 I[1, 3, 8](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon-1} x^2 z^{-1} r_{138}(x, z), \\
 I[1, 4, 5](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon-1} (1-z)^{-2\epsilon} x^2 z^{-1} r_{145}(x, z), \\
 I[2, 3, 5](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon-1} (1-z)^{-2\epsilon-1} x^2 r_{235}(x, z), \\
 I[2, 3, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon-1} x^2 z^{-1} r_{237}(x, z), \\
 I[2, 4, 5](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon-1} (1-z)^{-2\epsilon} x^2 z^{-1} r_{245}(x, z), \\
 I[3, 4, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon} 2x^2(1+x)^{-1} z^{-1} r_{347}(x, z), \\
 I[3, 5, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon} (1-z)^{-2\epsilon} x z^{-1} r_{357}(x, z), \\
 I[3, 5, 8](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon-1} (1-z)^{-2\epsilon} x^2 z^{-1} r_{358}(x, z), \\
 I[4, 5, 7](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon-1} (1-z)^{-2\epsilon} x^2 z^{-1} r_{457}(x, z), \\
 I[4, 5, 8](Q^2, x, z) &= N_\Gamma(Q^2)^{-2\epsilon-2} (1-x)^{-2\epsilon-1} (1-z)^{-2\epsilon} x^2 z^{-1} r_{458}(x, z). \tag{A.1}
 \end{aligned}$$

The dimensionless functions $r_i(x, z)$ can be expressed in Laurent expansions in ϵ . The prefactors are chosen to properly isolate the dominant behaviour in the $x = 1$ and $z = 1$ endpoints, and to account for common rational prefactors where appropriate.

The $r_i(x, z)$ are documented below, truncated in ϵ to the required orders for the respective kinematical regions.

A.2 Hard region

The master integrals in the hard region ($x < 1$, $z < 1$) read:

$$\begin{aligned}
 r_0(x, y) &= 1 + \epsilon(2\ln(x) - \ln(z)) + \epsilon^2 \left(2\ln^2(x) - 2\ln(x)\ln(z) + \frac{\ln^2(z)}{2} \right) + \mathcal{O}(\epsilon^3), \\
 r_1(x, z) &= \ln(x) + \epsilon \left(2\text{Li}_2(x) + \ln(x)(2\ln(1-x) - \ln(z) - 2) + \frac{3\ln^2(x)}{2} - \frac{\pi^2}{3} \right) \\
 &\quad + \mathcal{O}(\epsilon^2), \\
 r_5(x, z) &= \frac{\ln(z)}{\epsilon(1-z)} - \frac{-12\text{Li}_2(z) - 12\ln(z)(\ln(x) + \ln(1-z) - 2) + 9\ln^2(z) + 2\pi^2}{6(1-z)} \\
 &\quad + \mathcal{O}(\epsilon), \\
 r_7(x, z) &= -\frac{\ln(x)\ln(z)}{(1-x)(1-z)} + \mathcal{O}(\epsilon), \\
 r_{m27}(x, z) &= \frac{(1-x)(2-2z+(z+1)\ln(z)) + \ln(x)((x+1)(1-z) + (x+z)\ln(z))}{(1-x)^2x(1-z)} \\
 &\quad + \mathcal{O}(\epsilon), \\
 r_{m37}(x, z) &= \frac{\ln(x)(z - z\ln(z) - 1)}{(1-x)x(1-z)} + \mathcal{O}(\epsilon), \\
 r_{14}(x, z) &= \frac{2x}{u} \left(-\text{Li}_2\left(\frac{1-u-x}{2}\right) + \text{Li}_2\left(\frac{1+u-x}{2}\right) - \text{Li}_2\left(\frac{u+x-1}{2x}\right) \right. \\
 &\quad \left. + \text{Li}_2\left(-\frac{u-x+1}{2x}\right) + \ln(x)\ln(1-u+x) - \ln(x)\ln(1+u+x) \right) + \mathcal{O}(\epsilon), \\
 r_{47}(x, z) &= -2\sqrt{\frac{x}{z}} \left(\text{Ti}_2(\sqrt{xz}) - \text{Ti}_2(-\sqrt{xz}) - \text{Ti}_2\left(\sqrt{\frac{x}{z}}\right) + \text{Ti}_2\left(-\sqrt{\frac{x}{z}}\right) \right. \\
 &\quad \left. - \ln(xz)\arctan(\sqrt{xz}) + \ln\left(\frac{x}{z}\right)\arctan\left(\sqrt{\frac{x}{z}}\right) \right) + \mathcal{O}(\epsilon), \\
 r_{57}(x, z) &= \frac{x}{(1-z)v} \left(\frac{2(\ln(v-z+1) - \ln(v+z-1))}{\epsilon} + 8\text{Li}_2\left(\frac{1-z}{v}\right) \right. \\
 &\quad - 4\text{Li}_2\left(\frac{1}{2}(-v-z+1)\right) + 4\text{Li}_2\left(\frac{v+z-1}{v-z+1}\right) + 8\text{Li}_2\left(\frac{v}{z-1}\right) \\
 &\quad + 4\text{Li}_2\left(\frac{-v+z+1}{2z}\right) + 4\text{Li}_2\left(-\frac{v-z+1}{2z}\right) - 4\text{Li}_2\left(\frac{1}{2}(-v+z+1)\right) \\
 &\quad + 3\ln^2(v-z+1) - \ln^2(v+z-1) + 2\ln(x)\ln(v-z+1) \\
 &\quad + 16\ln(2)\ln(v+z-1) - 4\ln(1-z)\ln(v-z+1) - 8\ln(v-z+1) \\
 &\quad - 2\ln(z)\ln(v-z+1) - 2\ln(v-z+1)\ln(v+z-1) + 6\ln(z)\ln(v+z-1) \\
 &\quad + 8\ln(v+z-1) + 4\ln(v-z+1)\ln(v+z+1) - 4\ln(z)\ln(v+z+1) \\
 &\quad - 4\ln(v+z-1)\ln(v+z+1) - 8\ln(v)\ln(1-z) + 4\ln(1-z)\ln(v+z-1) \\
 &\quad - 2\ln(x)\ln(v+z-1) + 4\ln^2(v) + 4\ln^2(1-z) - 8\ln(2)\ln(x) \\
 &\quad \left. - 8\ln(2)\ln(z) - 4\ln(z)\ln(x) + \frac{2\pi^2}{3} - 16\ln^2(2) \right) + \mathcal{O}(\epsilon),
 \end{aligned}$$

$$\begin{aligned}
r_{134}(x, z) &= -\frac{\ln(x)}{\epsilon} + 2\text{Li}_2(-x) - 2\text{Li}_2(x) - 2\ln^2(x) - 2\ln(1-x)\ln(x) \\
&\quad + 2\ln(x+1)\ln(x) + 4\ln(x) + \frac{\pi^2}{2} + \mathcal{O}(\epsilon), \\
r_{135}(x, z) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 2\text{Li}_2\left(1 - \frac{x}{z}\right) - 4\text{Li}_2(x) - 4\ln(1-x)\ln(x) + \frac{2\pi^2}{3} + 4 + \mathcal{O}(\epsilon), \\
r_{138}(x, z) &= \frac{1}{\epsilon^2} + \frac{2(2\ln(x) - \ln(z) - 2)}{\epsilon} - 2\text{Li}_2\left(1 - \frac{x}{z}\right) + 4\text{Li}_2(x) + 5\ln^2(x) - 16\ln(x) \\
&\quad - 6\ln(x)\ln(z) + 4\ln(1-x)\ln(x) + \ln^2(z) + 8\ln(z) - \frac{2\pi^2}{3} + 4 + \mathcal{O}(\epsilon), \\
r_{145}(x, z) &= \frac{\ln(x)}{\epsilon} + 2\text{Li}_2\left(1 - \frac{x}{z}\right) - 2\text{Li}_2(z) - \frac{1}{2}5\ln^2(x) + 4\ln(x) + \ln^2(z) \\
&\quad - 2\ln(1-z)\ln(z) + \frac{\pi^2}{3} + \mathcal{O}(\epsilon), \\
r_{235}(x, z) &= -\frac{3}{\epsilon^2} + \frac{2(-3\ln(x) + \ln(z) + 6)}{\epsilon} + 4\ln(x)\ln(z) - 6\ln^2(x) + 24\ln(x) \\
&\quad - 8\ln(z) + \frac{\pi^2}{3} - 12 + \mathcal{O}(\epsilon), \\
r_{237}(x, z) &= r_{138}(x, z), \\
r_{245}(x, z) &= -\frac{1}{\epsilon^2} + \frac{-3\ln(x) + 2\ln(z) + 4}{\epsilon} + 2\text{Li}_2\left(1 - \frac{x}{z}\right) + 2\text{Li}_2(z) + 4\ln(x)\ln(z) \\
&\quad - \frac{7}{2}\ln^2(x) + 12\ln(x) - \ln^2(z) + 2\ln(1-z)\ln(z) - 8\ln(z) \\
&\quad - \frac{\pi^2}{3} - 4 + \mathcal{O}(\epsilon), \\
r_{347}(x, z) &= -\frac{\ln(x)}{\epsilon} + -\text{Li}_2\left(-\frac{x}{z}\right) - \text{Li}_2(-xz) + 2\text{Li}_2(-x) - 2\text{Li}_2(x) - \ln(x)\ln\left(\frac{x+z}{z}\right) \\
&\quad + \ln(x)\ln(z) - \ln(x)\ln(1+xz) + \ln(z)\ln\left(\frac{x+z}{z}\right) - \ln(z)\ln(1+xz) \\
&\quad - \frac{3}{2}\ln^2(x) - 2\ln(1-x)\ln(x) + 2\ln(1+x)\ln(x) + 4\ln(x) + \frac{\ln^2(z)}{2} \\
&\quad + \frac{\pi^2}{3} + \mathcal{O}(\epsilon), \\
r_{357}(x, z) &= \frac{(1-z)^2}{2x}r_{57}(x, z) + \frac{1}{\epsilon^2} + \frac{\ln(x) - \ln(z) - 4}{\epsilon} - 2\text{Li}_2\left(\frac{1}{2}(-v-z+1)\right) \\
&\quad + 2\text{Li}_2\left(\frac{-v+z+1}{2z}\right) - 2\text{Li}_2\left(-\frac{v-z+1}{2z}\right) + 2\text{Li}_2\left(\frac{1}{2}(-v+z+1)\right) \\
&\quad + \frac{3}{2}\ln^2(v-z+1) + \frac{3}{2}\ln^2(v+z-1) + 4\ln(2)\ln(v+z+1) \\
&\quad - 4\ln(v-z+1) - \ln(v-z+1)\ln(v+z-1) - 2\ln(z)\ln(v+z-1) \\
&\quad - 4\ln(v+z-1) - 2\ln(v-z+1)\ln(v+z+1) + 2\ln(z)\ln(v+z+1) \\
&\quad - 2\ln(v+z-1)\ln(v+z+1) + 8\ln(z) - \frac{\pi^2}{3} \\
&\quad + 4 - 2\ln^2(2) + 8\ln(2) + \mathcal{O}(\epsilon),
\end{aligned}$$

$$\begin{aligned}
 r_{358}(x, z) &= -\frac{2}{\epsilon^2} + \frac{2(-2\ln(x) + \ln(z) + 4)}{\epsilon} + 4\ln(x)\ln(z) - 4\ln^2(x) + 16\ln(x) \\
 &\quad - 8\ln(z) - 8 + \mathcal{O}(\epsilon), \\
 r_{457}(x, z) &= \frac{1}{\epsilon} \left[-\ln(z) + \frac{1+z}{v} (\ln(4xz) - 2\ln(1-z+v)) \right] \\
 &\quad + 2\text{Li}_2\left(\frac{-v-z+1}{2}\right) + 2\text{Li}_2\left(\frac{-v+z+1}{2z}\right) - 2\text{Li}_2\left(-\frac{v-z+1}{2z}\right) \\
 &\quad - 2\text{Li}_2\left(\frac{-v+z+1}{2}\right) + 2\text{Li}_2\left(-\frac{x}{z}\right) - 2\text{Li}_2(-xz) - 2\text{Li}_2(z) \\
 &\quad + 2\ln(x)\ln(v+z+1) + 4\ln(2)\ln(v-z+1) + 4\ln(2)\ln(v+z+1) \\
 &\quad + 2\ln(z)\ln(v-z+1) - 4\ln(v-z+1)\ln(v+z+1) + 4\ln(z)\ln(v+z+1) \\
 &\quad - 3\ln(x)\ln(z) + 2\ln(x)\ln(x+z) - 2\ln(z)\ln(x+z) - 2\ln(x)\ln(xz+1) \\
 &\quad - 2\ln(z)\ln(xz+1) - 2\ln(2)\ln(x) + \frac{3\ln^2(z)}{2} - 6\ln(2)\ln(z) \\
 &\quad - 2\ln(1-z)\ln(z) + 4\ln(z) + \frac{\pi^2}{3} - 4\ln^2(2) \\
 &\quad + \frac{1+z}{v} \left[-4\text{Li}_2\left(\frac{1-z}{v}\right) + 2\text{Li}_2\left(\frac{-v-z+1}{2}\right) - 2\text{Li}_2\left(\frac{v+z-1}{v-z+1}\right) \right. \\
 &\quad - 4\text{Li}_2\left(\frac{v}{z-1}\right) - 2\text{Li}_2\left(\frac{-v+z+1}{2z}\right) - 2\text{Li}_2\left(-\frac{v-z+1}{2z}\right) \\
 &\quad \left. + 2\text{Li}_2\left(\frac{-v+z+1}{2}\right) - 2\ln(x)\ln(v-z+1) + 2\ln(x)\ln(v+z+1) \right. \\
 &\quad - 2\ln^2(v-z+1) + 8\ln(2)\ln(v-z+1) + 4\ln(2)\ln(v+z+1) \\
 &\quad + 4\ln(v)\ln(1-z) + 4\ln(1-z)\ln(v-z+1) + 8\ln(v-z+1) \\
 &\quad + 4\ln(z)\ln(v-z+1) - 4\ln(v-z+1)\ln(v+z+1) + 4\ln(z)\ln(v+z+1) \\
 &\quad - 2\ln^2(v) - 2\ln(x)\ln(1-z) + \ln(x)\ln(z) + \frac{3\ln^2(x)}{2} - 4\ln(x) - 2\ln^2(1-z) \\
 &\quad - \frac{5\ln^2(z)}{2} - 4\ln(2)\ln(1-z) - 8\ln(2)\ln(z) - 2\ln(1-z)\ln(z) - 4\ln(z) \\
 &\quad \left. - \frac{\pi^2}{3} - 6\ln^2(2) - 8\ln(2) \right] + \mathcal{O}(\epsilon), \\
 r_{458}(x, z) &= -r_{457}(x, z) - \frac{2\ln(z)}{\epsilon} + 2\text{Li}_2\left(-\frac{x}{z}\right) - 2\text{Li}_2(-xz) - 4\text{Li}_2(z) - 4\ln(x)\ln(z) \\
 &\quad - 2\ln(z)\ln(x+z) - 2\ln(z)\ln(xz+1) + 2\ln(x)\ln(x+z) \\
 &\quad - 2\ln(x)\ln(xz+1) + 5\ln^2(z) - 4\ln(1-z)\ln(z) + 8\ln(z) + \frac{2\pi^2}{3} + \mathcal{O}(\epsilon),
 \end{aligned} \tag{A.2}$$

where we introduced the abbreviations

$$u = u(x, z) = \sqrt{(1+x)^2 - 4xz}, \quad v = v(x, z) = \sqrt{(1-z)^2 + 4xz}. \tag{A.3}$$

In $r_{47}(x, z)$, the inverse tangent integral function

$$\text{Ti}_2(y) = \int_0^y \frac{\arctan x}{x} dx \tag{A.4}$$

appears. It is not commonly encountered in higher-order perturbative calculations and is related to the dilogarithm for purely imaginary argument as [68]:

$$\text{Li}_2(iy) = \frac{1}{4}\text{Li}_2(-y^2) + i\text{Ti}_2(y) \tag{A.5}$$

A.3 Initial state endpoint region

The master integrals in the initial-state endpoint region ($x = 1, z < 1$) become:

$$\begin{aligned} r_0(1, z) &= 1 - \epsilon \ln(z) + \epsilon^2 \frac{\ln^2(z)}{2} + \mathcal{O}(\epsilon^3), \\ r_1(1, z) &= \mathcal{O}(\epsilon^3), \\ r_5(1, z) &= \frac{\ln(z)}{\epsilon(1-z)} + \frac{12\text{Li}_2(z) - 9\ln^2(z) + 12(\ln(1-z) - 2)\ln(z) - 2\pi^2}{6(1-z)} \\ &\quad + \frac{\epsilon}{6(1-z)} \left(-48\text{Li}_2(z) - 24\text{Li}_3(1-z) - 12\text{Li}_3(z) - 12\text{Li}_2(z)\ln(z) \right. \\ &\quad + 24\text{Li}_2(1-z)\ln(1-z) + 24\text{Li}_2(z)\ln(1-z) + 7\ln^3(z) - 18\ln(1-z)\ln^2(z) \\ &\quad + 36\ln^2(z) + 24\ln^2(1-z)\ln(z) + 4\pi^2\ln(z) - 48\ln(1-z)\ln(z) + 24\ln(z) \\ &\quad \left. - 4\pi^2\ln(1-z) + 12\zeta(3) + 8\pi^2 \right) + \mathcal{O}(\epsilon^2), \\ r_7(1, z) &= \frac{\ln(z)}{1-z} + \frac{\epsilon}{(1-z)} \left(2\text{Li}_2(z) - \frac{3}{2}\ln^2(z) + 2(\ln(1-z) - 1)\ln(z) - \frac{\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\ r_{m27}(1, z) &= \frac{\ln(z)}{2} + \frac{\epsilon}{12} \left(12\text{Li}_2(z) - 9\ln^2(z) + 12\ln(1-z)\ln(z) - 12\ln(z) - 2\pi^2 \right) \\ &\quad + \mathcal{O}(\epsilon^2), \\ r_{m37}(1, z) &= \frac{1-z+z\ln(z)}{1-z} + \frac{\epsilon}{1-z} \left(2z\text{Li}_2(z) - \frac{\pi^2 z}{3} - \frac{1}{2}3z\ln^2(z) - z\ln(z) \right. \\ &\quad \left. + 2z\ln(1-z)\ln(z) - \ln(z) \right) + \mathcal{O}(\epsilon^2), \\ r_{14}(1, z) &= \mathcal{O}(\epsilon^2), \\ r_{47}(1, z) &= \mathcal{O}(\epsilon^2), \\ r_{57}(1, z) &= -\frac{2\ln(z)}{\epsilon(1-z^2)} + \frac{1}{1-z^2} \left(4\ln^2(1-z) + 8\ln(z) + 4\ln(1-z)\ln(z) + 3\ln^2(z) \right. \\ &\quad - 8\ln(1-z)\ln(z+1) - 8\ln(z)\ln(z+1) + 4\ln^2(z+1) - 8\text{Li}_2(-z) \\ &\quad \left. + 4\text{Li}_2(z) + 8\text{Li}_2\left(\frac{z+1}{z-1}\right) + 8\text{Li}_2\left(\frac{1-z}{z+1}\right) \right) \\ &\quad + \frac{\epsilon}{1-z^2} \left(16\text{Li}_2(z) + 8\text{Li}_3(1-z) + 4\text{Li}_3(z) + 4\text{Li}_2(z)\ln(z) \right. \\ &\quad - 32\text{Li}_2\left(\frac{z+1}{z-1}\right)\ln(z+1) + 32\text{Li}_2(-z)\ln(z+1) - 32\text{Li}_2(z)\ln(z+1) \\ &\quad - 32\text{Li}_2\left(\frac{1-z}{z+1}\right)\ln(z+1) - \frac{1}{3}7\ln^3(z) - 16\ln^3(z+1) + 6\ln(1-z)\ln^2(z) \\ &\quad - 12\ln^2(z) + 32\ln^2(z+1)\ln(z) + 32\ln(1-z)\ln^2(z+1) \\ &\quad \left. - 16\ln^2(1-z)\ln(z+1) - \frac{4}{3}\pi^2\ln(z) + 16\ln(1-z)\ln(z) \right) \end{aligned}$$

$$\begin{aligned}
 & -32 \ln(1-z) \ln(z+1) \ln(z) - 8 \ln(z) + \frac{8}{3} \pi^2 \ln(z+1) - 4\zeta(3) - \frac{8\pi^2}{3} \\
 & + \mathcal{O}(\epsilon^2), \\
 r_{134}(1, z) &= \mathcal{O}(\epsilon^2), \\
 r_{135}(1, z) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 2\text{Li}_2(z) - \ln^2(z) + 2 \ln(1-z) \ln(z) - \frac{\pi^2}{3} + 4 \\
 & + \epsilon \left(-8\text{Li}_2(z) - 4\text{Li}_3(1-z) - 2\text{Li}_3(z) - 2\text{Li}_2(z) \ln(z) \right. \\
 & + 4\text{Li}_2(1-z) \ln(1-z) + 4\text{Li}_2(z) \ln(1-z) + \ln^3(z) - 3 \ln(1-z) \ln^2(z) \\
 & + 4 \ln^2(z) + 4 \ln^2(1-z) \ln(z) + \frac{2}{3} \pi^2 \ln(z) - 8 \ln(1-z) \ln(z) \\
 & \left. - \frac{2}{3} \pi^2 \ln(1-z) + 2\zeta(3) + \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
 r_{138}(1, z) &= \frac{1}{\epsilon^2} - \frac{2(\ln(z)+2)}{\epsilon} - 2\text{Li}_2(z) + 2 \ln^2(z) - 2 \ln(1-z) \ln(z) + 8 \ln(z) + \frac{\pi^2}{3} + 4 \\
 & + \epsilon \left(8\text{Li}_2(z) + 4\text{Li}_3(1-z) + 2\text{Li}_3(z) + 2\text{Li}_2(z) \ln(z) - 4\text{Li}_2(1-z) \ln(1-z) \right. \\
 & - 4\text{Li}_2(z) \ln(1-z) - \frac{1}{3} 4 \ln^3(z) + 3 \ln(1-z) \ln^2(z) - 8 \ln^2(z) \\
 & - 4 \ln^2(1-z) \ln(z) - \frac{2}{3} \pi^2 \ln(z) + 8 \ln(1-z) \ln(z) - 8 \ln(z) \\
 & \left. + \frac{2}{3} \pi^2 \ln(1-z) - 2\zeta(3) - \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
 r_{145}(1, z) &= \mathcal{O}(\epsilon^2), \\
 r_{235}(1, z) &= -\frac{3}{\epsilon^2} + \frac{2(\ln(z)+6)}{\epsilon} - 8 \ln(z) + \frac{\pi^2}{3} - 12 + \epsilon \left(8\text{Li}_3(z) - 4\text{Li}_2(z) \ln(z) \right. \\
 & \left. - \frac{2}{3} \ln^3(z) - \frac{2}{3} \pi^2 \ln(z) + 8 \ln(z) + 2\zeta(3) - \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
 r_{237}(1, z) &= r_{138}(1, z), \\
 r_{245}(1, z) &= -\frac{1}{\epsilon^2} + \frac{2(\ln(z)+2)}{\epsilon} + 4\text{Li}_2(z) - 2 \ln^2(z) + 4 \ln(1-z) \ln(z) - 8 \ln(z) \\
 & - \frac{2\pi^2}{3} - 4 + \epsilon \left(-16\text{Li}_2(z) - 12\text{Li}_3(1-z) - 4\text{Li}_3(z) - 4\text{Li}_2(z) \ln(z) \right. \\
 & + 12\text{Li}_2(1-z) \ln(1-z) + 12\text{Li}_2(z) \ln(1-z) + \frac{4 \ln^3(z)}{3} - 6 \ln(1-z) \ln^2(z) \\
 & + 8 \ln^2(z) + 12 \ln^2(1-z) \ln(z) + \frac{4}{3} \pi^2 \ln(z) - 16 \ln(1-z) \ln(z) \\
 & \left. + 8 \ln(z) - 2\pi^2 \ln(1-z) + 4\zeta(3) + \frac{8\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
 r_{347}(1, z) &= \mathcal{O}(\epsilon^2), \\
 r_{357}(1, z) &= \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \left(-4 - \frac{2 \ln(z)}{1+z} \right) + 4 + 4 \ln(z) + \frac{\ln^2(z)}{2} \\
 & + \frac{1-z}{1+z} \left(-2\text{Li}_2(z^2) + 4\text{Li}_2\left(\frac{z+1}{z-1}\right) + 6\text{Li}_2(z) + 4\text{Li}_2\left(\frac{1-z}{z+1}\right) \right)
 \end{aligned}$$

$$\begin{aligned}
 & +2\ln^2(1-z) + \frac{3\ln^2(z)}{2} + 2\ln^2(z+1) + 2\ln(z)\ln(1-z) \\
 & -4\ln(z+1)\ln(1-z) + 4\ln(z) - 4\ln(z)\ln(z+1) \\
 & +\epsilon\left[-\frac{1}{6}\ln^3(z) - 2\ln^2(z) - 4\ln(z) + \frac{1-z}{1+z}\left(4\text{Li}_2(z^2)\ln(z) + 8\text{Li}_2(z)\right.\right. \\
 & +4\text{Li}_3(1-z) + 2\text{Li}_3(z) - 8\text{Li}_2\left(\frac{z+1}{z-1}\right)\ln(z) - 14\text{Li}_2(z)\ln(z) \\
 & -8\text{Li}_2\left(\frac{1-z}{z+1}\right)\ln(z) - \frac{7}{6}\ln^3(z) - 5\ln(1-z)\ln^2(z) + 8\ln(z+1)\ln^2(z) \\
 & -6\ln^2(z) - 4\ln^2(1-z)\ln(z) - 4\ln^2(z+1)\ln(z) + 8\ln(1-z)\ln(z) \\
 & \left.\left.+8\ln(1-z)\ln(z+1)\ln(z) - 4\ln(z) - 2\zeta(3) - \frac{4\pi^2}{3}\right)\right] + \mathcal{O}(\epsilon^2), \\
 r_{358}(1, z) &= -\frac{2}{\epsilon^2} + \frac{2\ln(z)+8}{\epsilon} - 8 - 8\ln(z) + \epsilon\left(4\text{Li}_3(z) - \frac{4}{3}\ln^3(z)\right. \\
 & \left.+2\ln(1-z)\ln^2(z) - \frac{2}{3}\pi^2\ln(z) + 8\ln(z) - 4\zeta(3)\right) + \mathcal{O}(\epsilon^2), \\
 r_{457}(1, z) &= \mathcal{O}(\epsilon^2), \\
 r_{458}(1, z) &= -\frac{2\ln(z)}{\epsilon} - 2\text{Li}_2(z^2) - 4\ln(1-z^2)\ln(z) + 4\ln^2(z) + 8\ln(z) + \frac{\pi^2}{3} \\
 & +\epsilon\left[8\text{Li}_2(z^2) + 2\text{Li}_2(z^2)\log(z) - 8\text{Li}_3\left(\frac{1-z}{2}\right) - 8\text{Li}_3\left(\frac{z+1}{2}\right)\right. \\
 & +16\text{Li}_3(1-z) + 8\text{Li}_3\left(\frac{2z}{z-1}\right) - 4\text{Li}_3(-z) + 8\text{Li}_3(z) - 8\text{Li}_3\left(\frac{z}{z+1}\right) \\
 & +8\text{Li}_3\left(\frac{2z}{z+1}\right) - 4\log^2(2)\log(1-z^2) + 16\log(z)\log(1-z^2) \\
 & -\frac{4}{3}\log^3(1-z) - 4\log^3(z) + 4\log(2)\log^2(1-z) + 4\log(2)\log^2(z+1) \\
 & +8\log(1-z)\log^2(z) - 16\log^2(z) + 4\log^2(1-z)\log(z) + 6\log^2(z)\log(z+1) \\
 & -\frac{4}{3}\pi^2\log(1-z) - \frac{4}{3}\pi^2\log(z) - 8\log(z) + 2\pi^2\log(z+1) \\
 & \left.-8\log(1-z)\log(z)\log(z+1) - 4\zeta(3) - \frac{4\pi^2}{3} + \frac{8\log^3(2)}{3} - \frac{4}{3}\pi^2\log(2)\right] \\
 & +\mathcal{O}(\epsilon^2). \tag{A.6}
 \end{aligned}$$

A.4 Final state endpoint region

The master integrals in the final-state endpoint region ($x < 1$, $z = 1$) become:

$$\begin{aligned}
 r_0(x, 1) &= 1 + \epsilon(2\ln(x)) + \epsilon^2(2\ln^2(x)) + \mathcal{O}(\epsilon^3), \\
 r_1(x, 1) &= \ln(x) + \epsilon\left(2\text{Li}_2(x) + \frac{3\ln^2(x)}{2} + 2(\ln(1-x) - 1)\ln(x) - \frac{\pi^2}{3}\right) \\
 & +\epsilon^2\left(-4\text{Li}_2(x) - 4\text{Li}_3(1-x) - 2\text{Li}_3(x) + 4\text{Li}_2(x)\ln(x)\right)
 \end{aligned}$$

$$\begin{aligned}
 & +4\text{Li}_2(1-x)\ln(1-x) + 4\text{Li}_2(x)\ln(1-x) + \frac{7\ln^3(x)}{6} \\
 & +3\ln(1-x)\ln^2(x) - 3\ln^2(x) + 4\ln^2(1-x)\ln(x) - \frac{1}{3}\pi^2\ln(x) \\
 & -4\ln(1-x)\ln(x) - \frac{2}{3}\pi^2\ln(1-x) + 2\zeta(3) + \frac{2\pi^2}{3} \Big) + \mathcal{O}(\epsilon^3), \\
 r_5(x, 1) &= -\frac{1}{\epsilon} - 2(\ln(x) - 1) - 2\epsilon \left(\ln^2(x) - 2\ln(x) \right) + \mathcal{O}(\epsilon^2), \\
 r_7(x, 1) &= \frac{\ln(x)}{1-x} + \frac{\epsilon \left(12\text{Li}_2(x) + 9\ln^2(x) + 12\ln(1-x)\ln(x) - 12\ln(x) - 2\pi^2 \right)}{6(1-x)} \\
 & + \mathcal{O}(\epsilon^2), \\
 r_{m27}(x, 1) &= \mathcal{O}(\epsilon^2), \\
 r_{m37}(x, 1) &= \mathcal{O}(\epsilon^2), \\
 r_{14}(x, 1) &= \frac{x\ln^2(x)}{1-x} + \epsilon \frac{x\ln(x)}{3(1-x)} \left(12\text{Li}_2(x) + 3\ln^2(x) + 12\ln(1-x)\ln(x) \right. \\
 & \left. - 12\ln(x) - 2\pi^2 \right) + \mathcal{O}(\epsilon^2), \\
 r_{47}(x, 1) &= \mathcal{O}(\epsilon^2), \\
 r_{57}(x, 1) &= \frac{1}{\epsilon} + \ln(x) - 2 + \epsilon \left(-2\text{Li}_2(x) + \frac{1}{2}\ln(x)(-4\ln(1-x) + \ln(x) - 4) \right. \\
 & \left. + \frac{\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
 r_{134}(x, 1) &= -\frac{\ln(x)}{\epsilon} + 2\text{Li}_2(-x) - 2\text{Li}_2(x) - 2\ln^2(x) - 2\ln(1-x)\ln(x) \\
 & + 2\ln(x+1)\ln(x) + 4\ln(x) + \frac{\pi^2}{2} - \epsilon \left(8\text{Li}_2(-x) - 8\text{Li}_2(x) - 4\text{Li}_3\left(\frac{1-x}{2}\right) \right. \\
 & - 4\text{Li}_3\left(\frac{x+1}{2}\right) + 4\text{Li}_3\left(\frac{2x}{x-1}\right) + 4\text{Li}_3\left(\frac{2x}{x+1}\right) - 4\text{Li}_2(-x)\ln(x) \\
 & + 4\text{Li}_2(x)\ln(x) - \frac{2}{3}\ln^3(1-x) + \frac{5\ln^3(x)}{3} - \frac{2}{3}\ln^3(x+1) - 2\ln^2(2)\ln(1-x) \\
 & - 2\ln^2(2)\ln(x+1) + 2\ln(2)\ln^2(1-x) + 2\ln(2)\ln^2(x+1) \\
 & + 4\ln(1-x)\ln^2(x) - 8\ln^2(x) + 2\ln(x)\ln^2(x+1) + 2\ln^2(1-x)\ln(x) \\
 & - 4\ln^2(x)\ln(x+1) - \frac{2}{3}\pi^2\ln(1-x) - \pi^2\ln(x) - 8\ln(1-x)\ln(x) \\
 & + 4\ln(x) + \frac{4}{3}\pi^2\ln(x+1) - 4\ln(1-x)\ln(x)\ln(x+1) + 8\ln(x)\ln(x+1) \\
 & \left. + 2\pi^2 + \frac{4\ln^3(2)}{3} - \frac{1}{6}\pi^2\ln(16) \right) + \mathcal{O}(\epsilon^2), \\
 r_{135}(x, 1) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 4 + \pi^2 - 6\ln(1-x)\ln(x) - 6\text{Li}_2(x) \\
 & + \epsilon \left(24\text{Li}_2(x) + 16\text{Li}_3(1-x) + 14\text{Li}_3(x) - 16\text{Li}_2(1-x)\ln(1-x) \right. \\
 & \left. - 16\text{Li}_2(x)\ln(1-x) - 14\text{Li}_2(x)\ln(x) - 16\ln(x)\ln^2(1-x) \right)
 \end{aligned}$$

$$\begin{aligned}
& -7\ln^2(x)\ln(1-x) + \frac{8}{3}\pi^2\ln(1-x) + 24\ln(x)\ln(1-x) \\
& -14\zeta(3) - 4\pi^2) + \mathcal{O}(\epsilon^2), \\
r_{138}(x, 1) = & \frac{1}{\epsilon^2} + \frac{4(\ln(x) - 1)}{\epsilon} + 6\text{Li}_2(x) + 5\ln^2(x) + 6\ln(1-x)\ln(x) - 16\ln(x) \\
& + 4 - \pi^2 + \epsilon \left(-2\text{Li}_3(x) - 16\text{Li}_3(1-x) + 10\text{Li}_2(x)\ln(x) \right. \\
& + 16\text{Li}_2(x)\ln(1-x) - 24\text{Li}_2(x) + 16\text{Li}_2(1-x)\ln(1-x) + \frac{11}{3}\ln^3(x) \\
& + 9\ln(1-x)\ln^2(x) - 20\ln^2(x) + 16\ln^2(1-x)\ln(x) - 24\ln(1-x)\ln(x) \\
& \left. - \frac{4\pi^2}{3}\ln(x) + 16\ln(x) - \frac{8\pi^2}{3}\ln(1-x) + 4\pi^2 + 2\zeta(3) \right) + \mathcal{O}(\epsilon^2) \\
r_{145}(x, 1) = & \frac{\ln(x)}{\epsilon} + 2\text{Li}_2(x) + \frac{5\ln^2(x)}{2} + 2\ln(1-x)\ln(x) - 4\ln(x) - \frac{\pi^2}{3} \\
& + \epsilon \left(-8\text{Li}_2(x) - 4\text{Li}_3(1-x) - 2\text{Li}_3(x) + 8\text{Li}_2(x)\ln(x) + \frac{13\ln^3(x)}{6} \right. \\
& + 7\ln(1-x)\ln^2(x) - 10\ln^2(x) - \pi^2\ln(x) - 8\ln(1-x)\ln(x) \\
& \left. + 4\ln(x) + 2\zeta(3) + \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
r_{235}(x, 1) = & -\frac{3}{\epsilon^2} + \frac{12 - 6\ln(x)}{\epsilon} - 6\ln^2(x) + 24\ln(x) + \frac{\pi^2}{3} - 12 \\
& + \epsilon \left(-4\ln^3(x) + 24\ln^2(x) + \frac{2}{3}\pi^2\ln(x) - 24\ln(x) + 10\zeta(3) - \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
r_{237}(x, 1) = & r_{138}(x, 1), \\
r_{245}(x, 1) = & -\frac{1}{\epsilon^2} + \frac{4 - 3\ln(x)}{\epsilon} - 2\text{Li}_2(x) - \frac{7}{2}\ln^2(x) - 2\ln(1-x)\ln(x) + 12\ln(x) \\
& + \frac{\pi^2}{3} - 4 + \epsilon \left(8\text{Li}_2(x) + 4\text{Li}_3(1-x) + 2\text{Li}_3(x) - 4\text{Li}_2(x)\ln(x) \right. \\
& - 4\text{Li}_2(1-x)\ln(1-x) - 4\text{Li}_2(x)\ln(1-x) - \frac{5}{2}\ln^3(x) \\
& - 3\ln(1-x)\ln^2(x) + 14\ln^2(x) - 4\ln^2(1-x)\ln(x) + \frac{1}{3}\pi^2\ln(x) \\
& \left. + 8\ln(1-x)\ln(x) - 12\ln(x) + \frac{2}{3}\pi^2\ln(1-x) - 2\zeta(3) - \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2), \\
r_{347}(x, 1) = & -\frac{\ln(x)}{\epsilon} - 2\text{Li}_2(x) - \frac{1}{2}3\ln^2(x) - 2\ln(1-x)\ln(x) + 4\ln(x) + \frac{\pi^2}{3} \\
& + \epsilon \left(8\text{Li}_2(x) + 4\text{Li}_3(1-x) + 2\text{Li}_3(x) - 4\text{Li}_2(x)\ln(x) - \frac{7}{6}\ln^3(x) \right. \\
& - 3\ln(1-x)\ln^2(x) + 6\ln^2(x) + \frac{1}{3}\pi^2\ln(x) + 8\ln(1-x)\ln(x) \\
& \left. - 4\ln(x) - 2\zeta(3) - \frac{4\pi^2}{3} \right) + \mathcal{O}(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
r_{357}(x, 1) &= \frac{1}{\epsilon^2} + \frac{\ln(x) - 4}{\epsilon} - 2\text{Li}_2(x) + \frac{\ln^2(x)}{2} - 2\ln(1-x)\ln(x) - 4\ln(x) + \frac{\pi^2}{3} + 4 \\
&\quad + \epsilon \left(8\text{Li}_2(x) + 4\text{Li}_3(1-x) + 2\text{Li}_3(x) - 4\text{Li}_2(x)\ln(x) + \frac{\ln^3(x)}{6} \right. \\
&\quad \left. - 3\ln(1-x)\ln^2(x) - 2\ln^2(x) + \frac{1}{3}\pi^2\ln(x) + 8\ln(1-x)\ln(x) \right. \\
&\quad \left. + 4\ln(x) - 2\zeta(3) - \frac{4\pi^2}{3} \right) \\
r_{358}(x, 1) &= -\frac{2}{\epsilon^2} + \frac{8 - 4\ln(x)}{\epsilon} - 4\ln^2(x) + 16\ln(x) - 8 \\
&\quad + \epsilon \left(-\frac{8}{3}\ln^3(x) + 16\ln^2(x) - 16\ln(x) \right) + \mathcal{O}(\epsilon^2), \\
r_{457}(x, 1) &= \mathcal{O}(\epsilon^2), \\
r_{458}(x, 1) &= \mathcal{O}(\epsilon^2).
\end{aligned} \tag{A.7}$$

A.5 Soft region

In the soft endpoint ($x = 1, z = 1$) the master integrals become:

$$\begin{aligned}
r_0(1, 1) &= 1, \\
r_1(1, 1) &= \mathcal{O}(\epsilon^4), \\
r_5(1, 1) &= -\frac{1}{\epsilon} + 2 + \mathcal{O}(\epsilon^3), \\
r_7(1, 1) &= -1 + \mathcal{O}(\epsilon^3), \\
r_{m27}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{m37}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{14}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{47}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{57}(1, 1) &= \frac{1}{\epsilon} - 2 + \mathcal{O}(\epsilon^3), \\
r_{134}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{135}(1, 1) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 4 + \mathcal{O}(\epsilon^3), \\
r_{138}(1, 1) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 4 + \mathcal{O}(\epsilon^3), \\
r_{145}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{235}(1, 1) &= -\frac{3}{\epsilon^2} + \frac{12}{\epsilon} + \frac{1}{3}(\pi^2 - 36) + \frac{2}{3}\epsilon(15\zeta(3) - 2\pi^2) \\
&\quad + \frac{1}{90}\epsilon^2(-3600\zeta(3) + 31\pi^4 + 120\pi^2) + \mathcal{O}(\epsilon^3), \\
r_{237}(1, 1) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 4 + \mathcal{O}(\epsilon^3), \\
r_{245}(1, 1) &= -\frac{1}{\epsilon^2} + \frac{4}{\epsilon} - 4 + \mathcal{O}(\epsilon^3), \\
r_{347}(1, 1) &= \mathcal{O}(\epsilon^3), \\
r_{357}(1, 1) &= \frac{1}{\epsilon^2} - \frac{4}{\epsilon} + 4 + \mathcal{O}(\epsilon^3),
\end{aligned}$$

$$\begin{aligned}
 r_{358}(1, 1) &= -\frac{2}{\epsilon^2} + \frac{8}{\epsilon} - 8 + \mathcal{O}(\epsilon^3), \\
 r_{457}(1, 1) &= \mathcal{O}(\epsilon^3), \\
 r_{458}(1, 1) &= \mathcal{O}(\epsilon^3),
 \end{aligned}
 \tag{A.8}$$

B NNLO time-like mass factorisation kernels

While the NLO mass factorisation kernels are identical for time-like and space-like kinematics, the NNLO ones are sensitive on whether the identified particle is in the initial or final state. In the following we list the NNLO time-like mass factorisation kernels. The reduced time-like two-loop mass factorization kernel is defined similarly to the space-like case [27]:

$$\bar{\Gamma}_{ab;cd}^{(2)\text{id.}}(z, x_1, x_2) = \bar{\Gamma}_{ca,\text{full}}^{(2)\text{id.}}(z) \delta_{ab} \delta(1-x_1) \delta(1-x_2) + \bar{\Gamma}_{db,\text{full}}^{(2)\text{id.}}(z) \delta_{ca} \delta(1-x_1) \delta(1-x_2), \tag{B.1}$$

where $\bar{\Gamma}_{ca,\text{full}}^{(2)\text{id.}}(z)$ is directly related to the LO and NLO time-like Altarelli-Parisi spitting kernels [3, 13, 16, 69, 70]

$$\bar{\Gamma}_{ab,\text{full}}^{(2)\text{id.}}(z) = -\frac{1}{2\epsilon} \left(P_{ab}^1(z) + \frac{\beta_0}{\epsilon} P_{ab}^0(z) \right). \tag{B.2}$$

We can decompose $\bar{\Gamma}_{ca,\text{full}}^{(2)\text{id.}}(z)$ into colour layers as discussed in [27]

$$\bar{\Gamma}_{qq}^{(2)\text{id.}}(z) = \left(\frac{N_c^2 - 1}{N_c} \right) \left[N_c \bar{\Gamma}_{qq}^{(2)\text{id.}}(z) + \tilde{\Gamma}_{qq}^{(2)\text{id.}}(z) + \frac{1}{N_c} \tilde{\tilde{\Gamma}}_{qq}^{(2)\text{id.}}(z) + N_f \hat{\Gamma}_{qq}^{(2)\text{id.}}(z) \right], \tag{B.3}$$

$$\bar{\Gamma}_{q\bar{q}}^{(2)\text{id.}}(z) = \left(\frac{N_c^2 - 1}{N_c} \right) \left[\bar{\Gamma}_{q\bar{q}}^{(2)\text{id.}}(z) + \frac{1}{N_c} \tilde{\tilde{\Gamma}}_{q\bar{q}}^{(2)\text{id.}}(z) \right], \tag{B.4}$$

$$\bar{\Gamma}_{qq'}^{(2)\text{id.}}(z) = \left(\frac{N_c^2 - 1}{N_c} \right) \bar{\Gamma}_{qq'}^{(2)\text{id.}}(z), \tag{B.5}$$

$$\bar{\Gamma}_{q\bar{q}'}^{(2)\text{id.}}(z) = \left(\frac{N_c^2 - 1}{N_c} \right) \bar{\Gamma}_{q\bar{q}'}^{(2)\text{id.}}(z), \tag{B.6}$$

$$\bar{\Gamma}_{gq}^{(2)\text{id.}}(z) = \left(\frac{N_c^2 - 1}{N_c} \right) \left[N_c \bar{\Gamma}_{gq}^{(2)\text{id.}}(z) + \frac{1}{N_c} \tilde{\tilde{\Gamma}}_{gq}^{(2)\text{id.}}(z) + N_f \hat{\Gamma}_{gq}^{(2)\text{id.}}(z) \right], \tag{B.7}$$

$$\bar{\Gamma}_{g\bar{q}}^{(2)\text{id.}}(z) = N_c \bar{\Gamma}_{g\bar{q}}^{(2)\text{id.}}(z) + \frac{1}{N_c} \tilde{\tilde{\Gamma}}_{g\bar{q}}^{(2)\text{id.}}(z) + N_f \hat{\Gamma}_{g\bar{q}}^{(2)\text{id.}}(z), \tag{B.8}$$

$$\bar{\Gamma}_{gg}^{(2)\text{id.}}(z) = N_c^2 \bar{\Gamma}_{gg}^{(2)\text{id.}}(z) + N_c N_f \hat{\Gamma}_{gg}^{(2)\text{id.}}(z) + \frac{N_f}{N_c} \tilde{\tilde{\Gamma}}_{gg}^{(2)\text{id.}}(z) + N_f^2 \hat{\hat{\Gamma}}_{gg}^{(2)\text{id.}}(z). \tag{B.9}$$

We report for completeness the explicit form of each colour stripped function

$$\begin{aligned}
 \bar{\Gamma}_{qq}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon^2} \left[-\frac{11}{12} p_{qq}^0(z) \right] \\
 &+ \frac{1}{\epsilon} \left[\left(\frac{\pi^2}{12} - \frac{67}{36} + \frac{1}{4} \log^2(z) - \frac{31}{24} \log(z) - \frac{1}{2} \log(1-z) \log(z) \right) \mathcal{D}_0(z) \right. \\
 &\left. - \left(\frac{43}{192} + \frac{13\pi^2}{144} \right) \delta(1-z) - \frac{\pi^2 z}{24} + \frac{71z}{36} + \left(-\frac{3z}{16} - \frac{3}{16} \right) \log^2(z) \right]
 \end{aligned}$$

$$+ \left(\frac{7z}{12} + \frac{5}{6} \right) \log(z) + \left(\frac{z}{4} + \frac{1}{4} \right) \log(1-z) \log(z) - \frac{\pi^2}{24} - \frac{1}{9} \Big], \quad (\text{B.10})$$

$$\begin{aligned} \widetilde{\Gamma}_{qq}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon} \left[-\frac{7z^2}{9} + \left(\frac{z^2}{3} + \frac{9z}{8} + \frac{5}{8} \right) \log(z) - \frac{z}{2} \right. \\ &\quad \left. + \frac{5}{18z} + \left(-\frac{z}{8} - \frac{1}{8} \right) \log^2(z) + 1 \right], \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} \widetilde{\Gamma}_{qq}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon} \left[\left(-\frac{1}{2} \log^2(z) + \frac{3}{8} \log(z) + \frac{1}{2} \log(1-z) \log(z) \right) \mathcal{D}_0(z) \right. \\ &\quad + \left(\frac{3\zeta(3)}{4} + \frac{3}{64} - \frac{\pi^2}{16} \right) \delta(1-z) + \frac{5z}{8} + \left(\frac{5z}{16} + \frac{5}{16} \right) \log^2(z) \\ &\quad \left. + \left(-\frac{3z}{8} - \frac{5}{8} \right) \log(z) - \left(\frac{z}{4} + \frac{1}{4} \right) \log(1-z) \log(z) - \frac{5}{8} \right], \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} \widehat{\Gamma}_{qq}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon^2} \left[\frac{1}{6} \mathcal{D}_0(z) + \frac{\delta(1-z)}{8} - \frac{z}{12} - \frac{1}{12} \right] \\ &\quad + \frac{1}{\epsilon} \left[\left(\frac{5}{18} + \frac{1}{6} \log(z) \right) \mathcal{D}_0(z) + \left(\frac{\pi^2}{36} + \frac{1}{48} \right) \delta(1-z) \right. \\ &\quad \left. - \frac{11z}{36} + \left(-\frac{z}{12} - \frac{1}{12} \right) \log(z) + \frac{1}{36} \right], \end{aligned} \quad (\text{B.13})$$

$$\overline{\Gamma}_{q\bar{q}}^{(2)\text{id.}}(z) = \widetilde{\Gamma}_{qq}^{(2)\text{id.}}(z), \quad (\text{B.14})$$

$$\begin{aligned} \widetilde{\Gamma}_{q\bar{q}}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon} \left[\left(\frac{1}{2} + \frac{\pi^2}{24} \right) (1-z) - \frac{\pi^2}{12(1+z)} \right. \\ &\quad + (\log(1+z) \log(z) + \text{Li}_2(z)) \left(\frac{1}{2} - \frac{1}{2}z - \frac{1}{1-z} \right) + \log(z) \frac{1}{4} (1+z) \\ &\quad \left. + \log^2(z) \left(-\frac{1}{8} + \frac{1}{8}z + \frac{1}{4} \frac{1}{1-z} \right) \right], \end{aligned} \quad (\text{B.15})$$

$$\overline{\Gamma}_{q\bar{q}'}^{(2)\text{id.}}(z) = \widetilde{\Gamma}_{qq}^{(2)\text{id.}}(z), \quad (\text{B.16})$$

$$\overline{\Gamma}_{q\bar{q}'}^{(2)\text{id.}}(z) = \widetilde{\Gamma}_{qq}^{(2)\text{id.}}(z), \quad (\text{B.17})$$

$$\begin{aligned} \overline{\Gamma}_{gq}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon^2} \left[-\frac{11}{12} p_{gq}^0(z) \right] \\ &\quad + \frac{1}{\epsilon} \left[\left(-\frac{z}{2} - \frac{1}{z} - 1 \right) \text{Li}_2(-z) + \left(-z - \frac{2}{z} + 2 \right) \text{Li}_2(z) + \frac{11z^2}{9} \right. \\ &\quad + \left(-\frac{2z^2}{3} - \frac{37z}{16} + \frac{3}{2z} - 1 \right) \log(z) - \frac{5z}{16} - \frac{17}{36z} + \left(\frac{z}{8} + \frac{1}{4z} - \frac{1}{4} \right) \log^2(1-z) \\ &\quad + \left(\frac{13z}{16} + \frac{1}{z} + \frac{3}{8} \right) \log^2(z) + \frac{1}{4} z \log(1-z) + \left(-z - \frac{2}{z} + 2 \right) \log(1-z) \log(z) \\ &\quad \left. + \left(-\frac{z}{2} - \frac{1}{z} - 1 \right) \log(z) \log(z+1) - \frac{\pi^2}{6} - \frac{19}{16} \right], \end{aligned} \quad (\text{B.18})$$

$$\widetilde{\Gamma}_{gq}^{(2)\text{id.}}(z) = \frac{1}{\epsilon} \left[\left(-z - \frac{2}{z} + 2 \right) \text{Li}_2(z) + \frac{9z}{16} + \left(\frac{z}{8} + \frac{1}{4z} - \frac{1}{4} \right) \log^2(1-z) \right]$$

$$\begin{aligned}
& + \left(\frac{1}{8} - \frac{z}{16} \right) \log^2(z) + \frac{1}{4} z \log(1-z) - \left(\frac{z}{2} + \frac{1}{z} - 1 \right) \log(z) \log(1-z) \\
& + \left(\frac{z}{16} - 1 \right) \log(z) - \frac{1}{16} \Big], \tag{B.19}
\end{aligned}$$

$$\widehat{\Gamma}_{gq}^{(2)\text{id.}}(z) = \frac{1}{\epsilon^2} \left[\frac{1}{6} p_{gq}^0(z) \right], \tag{B.20}$$

$$\begin{aligned}
\overline{\Gamma}_{gg}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon^2} \left[-\frac{11}{12} p_{gg}^0(z) \right] \\
& + \frac{1}{\epsilon} \left[\left(z^2 + z + \frac{1}{2} \right) \text{Li}_2(-z) + (2z^2 - 2z + 1) \text{Li}_2(z) + \frac{\pi^2 z^2}{12} - \frac{133z^2}{36} \right. \\
& + \left(\frac{1}{4} z^2 - \frac{1}{4} z + \frac{1}{8} \right) \log^2(z - z^2) + \left(-\frac{z^2}{2} + \frac{z}{2} - \frac{1}{4} \right) \log^2(1-z) \\
& + \left(-\frac{7z^2}{12} + \frac{7z}{12} - \frac{13}{24} \right) \log(1-z) + \left(\frac{z^2}{12} + \frac{31z}{12} + \frac{31}{48} \right) \log(z) \\
& + \left(z^2 + z + \frac{1}{2} \right) \log(z) \log(z+1) + \frac{\pi^2 z}{12} + \frac{173z}{144} + \frac{5}{9z} \\
& \left. + \left(-\frac{11z}{8} - \frac{5}{16} \right) \log^2(z) + \frac{\pi^2}{24} + \frac{7}{18} \right], \tag{B.21}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\Gamma}_{gg}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon} \left[(2z^2 - 2z + 1) \text{Li}_2(z) - \frac{\pi^2 z^2}{12} - \frac{5z^2}{4} \right. \\
& + \left(-\frac{1}{4} z^2 + \frac{1}{4} z - \frac{1}{8} \right) \log^2(z - z^2) \\
& - \left(\frac{z^2}{4} - \frac{z}{4} + \frac{3}{8} \right) \log(1-z) + \left(\frac{z^2}{4} + \frac{z}{4} - \frac{5}{16} \right) \log(z) \\
& - (-2z^2 + 2z - 1) \log(1-z) \log(z) \\
& \left. + \frac{\pi^2 z}{12} + \frac{23z}{16} + \left(\frac{1}{16} - \frac{z}{8} \right) \log^2(z) - \frac{\pi^2}{24} - \frac{3}{4} \right], \tag{B.22}
\end{aligned}$$

$$\begin{aligned}
\widehat{\Gamma}_{gg}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon^2} \left[\frac{1}{6} p_{gg}^0(z) \right] \\
& + \frac{1}{\epsilon} \left[\frac{2z^2}{9} + \left(\frac{z^2}{3} - \frac{z}{3} + \frac{1}{6} \right) (\log(1-z) + \log(z)) - \frac{2z}{9} + \frac{5}{18} \right], \tag{B.23}
\end{aligned}$$

$$\begin{aligned}
\overline{\Gamma}_{gg}^{(2)\text{id.}}(z) &= \frac{1}{\epsilon^2} \left[-\frac{11}{12} p_{gg}^0(z) \right] \\
& + \frac{1}{\epsilon} \left[\left(\frac{\pi^2}{6} - \frac{67}{18} + \frac{3}{2} \log^2(z) - \frac{11}{3} \log(z) + 2 \log(1-z) \log(z) \right) \mathcal{D}_0(z) \right. \\
& - \left(\frac{3\zeta_3}{2} + \frac{4}{3} \right) \delta(1-z) + \left(-2z^2 - 2z + \frac{2}{z+1} - \frac{2}{z} - 4 \right) \text{Li}_2(-z) - \frac{1}{3} \pi^2 z^2 \\
& + \left(-z^2 + 4z - \frac{1}{2(z+1)} + \frac{2}{z} \right) \log^2(z) + \left(\frac{11z^2}{3} + \frac{z}{2} + \frac{11}{3z} + \frac{11}{2} \right) \log(z) \\
& \left. + \left(2z^2 - 2z - \frac{2}{z} + 4 \right) \log(1-z) \log(z) \right]
\end{aligned}$$

$$\begin{aligned}
 & + \left(-2z^2 - 2z + \frac{2}{z+1} - \frac{2}{z} - 4 \right) \log(z) \log(z+1) \\
 & + \frac{109z}{36} + \frac{\pi^2}{6(z+1)} - \frac{2\pi^2}{3} + \frac{25}{36} \Big], \tag{B.24}
 \end{aligned}$$

$$\begin{aligned}
 \widehat{\Gamma}_{gg}^{(2)}(z) &= \frac{1}{\epsilon^2} \left[\frac{1}{6} p_{gg}^0(z) - \frac{11}{12} p_{gg,F}^0(z) \right] \\
 & + \frac{1}{\epsilon} \left[\left(\frac{5}{9} + \frac{2}{3} \log(z) \right) \mathcal{D}_0(z) + z^2 + \left(-\frac{4z^2}{3} - \frac{3z}{4} - \frac{9}{4} \right) \log(z) \right. \\
 & \left. - \frac{4z}{9} + \frac{11\delta(1-z)}{24} + \left(-\frac{z}{4} - \frac{1}{4} \right) \log^2(z) - \frac{10}{9} \right], \tag{B.25}
 \end{aligned}$$

$$\begin{aligned}
 \widetilde{\Gamma}_{gg}^{(2)}(z) &= \frac{1}{\epsilon} \left[-\frac{\delta(1-z)}{8} - \frac{41z^2}{18} + \left(\frac{2z^2}{3} + \frac{7z}{4} + \frac{2}{3z} + \frac{5}{4} \right) \log(z) \right. \\
 & \left. + \frac{3z}{2} + \frac{23}{18z} + \left(\frac{z}{4} + \frac{1}{4} \right) \log^2(z) - \frac{1}{2} \right],
 \end{aligned}$$

$$\widehat{\widehat{\Gamma}}_{gg}^{(2)}(z) = \frac{1}{\epsilon^2} \left[-\frac{\delta(1-z)}{18} \right], \tag{B.26}$$

where

$$\begin{aligned}
 p_{qq}^0(z) &= \mathcal{D}_0(z) - \frac{(1+z)}{2} + \frac{3}{4} \delta(1-z), \\
 p_{gq}^0(z) &= \frac{1}{z} - 1 + \frac{z}{2}, \\
 p_{qg}^0(z) &= \frac{1}{2} - z + z^2, \\
 p_{gg}^0(z) &= 2\mathcal{D}_0(z) + \frac{2}{z} - 4 + 2z - 2z^2 + b_0 \delta(1-z), \\
 p_{gg,F}^0(z) &= b_{0,F} \delta(1-z), \tag{B.27}
 \end{aligned}$$

with $b_0 = 11/6$ and $b_{0,F} = -1/3$. Here we have labelled with x_1 and x_2 the momentum fractions carried by the initial-state partons and with z the momentum fraction carried by the identified final-state parton.

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